Online Appendix to
Long-Run Tax Incidence in a Human Capital-based Endogenous Growth Model with Labor-Market Frictions

by

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In the Appendix, we provide mathematical details of the second-order conditions of household/firm optimization and wage bargaining, the concavity of household/firm value functions, quasi-socail planner's optimization, the centralized solution by coordinating labor matching and wage bargain, dynamic taxation, as well as the Alternative Models II (linear human capital accumulation) and III (Walrasian).

1 Second-Order Conditions

The second-order conditions of firm's optimization with respect to v and k are (to ease notation burden, we carry time subscript t only for perpetually growing variables):

$$\frac{\partial^2 \Gamma(n_t)}{\partial (v_t)^2} = \frac{\eta_t^2}{1+\bar{r}} \Gamma_{nn}(n_{t+1}) < 0$$

$$\frac{\partial^2 \Gamma(n_t)}{\partial (k_t)^2} = \frac{s_t}{x_t} \alpha(\alpha-1) A \left(\frac{s_t}{n_t x_t}\right)^{\alpha-1} (k_t)^{\alpha-2} < 0$$

which hold automatically under our functional form specifications.

The second-order conditions of household's optimization with respect to c_t, ℓ_t and s_t are:

$$\Omega_{cc}(\mathcal{H}) = U_{cc} + \frac{1}{1+\rho}\Omega_{kk}(\mathcal{H}') < 0$$

$$\Omega_{\ell\ell}(\mathcal{H}) = \frac{n_t h_t}{1+\rho}\{(1\text{-}\tau_{L_t})w_t \left\{\Omega_{kk}(\mathcal{H}')(1\text{-}\tau_{L_t})w_t n_t h_t\text{-}[\Omega_{kh}(\mathcal{H}')+\Omega_{hk}(\mathcal{H}')]\left[D+\widetilde{D}(1\text{-}\gamma)\left(q_t^H\right)^{\gamma}\right] n_t h_t\right\}$$
$$-\Omega_h(\mathcal{H}')\widetilde{D}\gamma(1\text{-}\gamma)\left(q_t^H\right)^{\gamma-1} q_{t+1}^H(\ell_t) + \Omega_{hh}(\mathcal{H}')\left[D+\widetilde{D}(1\text{-}\gamma)\left(q_t^H\right)^{\gamma}\right]^2 n_t h_t\} < 0$$

$$\Omega_{ss}(\mathcal{H}) = \frac{k_t}{1+\rho}\{(1\text{-}\tau_{K_t})r_t k_t \left[\Omega_{kk}(\mathcal{H}')(1\text{-}\tau_{K_t})r_t\text{-}[\Omega_{kh}(\mathcal{H}')+\Omega_{hk}(\mathcal{H}')]\widetilde{D}\gamma\left(q_t^H\right)^{\gamma-1}\right]$$
$$-\Omega_h(\mathcal{H}')\widetilde{D}\gamma(\gamma\text{-}1)\left(q_t^H\right)^{\gamma-2} q_{t+1}^H(s_t) + \Omega_{hh}(\mathcal{H}')k_t\left[\widetilde{D}\gamma\left(q_t^H\right)^{\gamma-1}\right]^2\} < 0$$

which also hold under our functional form specifications and parameterization in the benchmark model.

Finally, we turn to the second-order condition of wage bargaining. From (30), it is easily see that $MB_{ww} < 0$ and $MC_{ww} > 0$, thus assuring the second-order condition: $d(MB_w - MC_w)/dw < 0$.

2 Wage Bargaining

The bargained wage rate and the equilibrium wage can be derived by solving the following quadratic equation:

$$S_w(1\text{-}\zeta)F_1 w_t^2 + \{S_r q_t[(1\text{-}\zeta)F_1 + \zeta]\text{-}S_w F_2(1\text{-}\zeta)\}w_t - S_r q_t[(1\text{-}\zeta)F_2 + \zeta(1\text{-}\alpha)A q_t^{\alpha}] = 0$$

where $F_1 = \frac{(1-\tau_L)(1-\bar{b})+mn\ell S_w}{(1-\tau_L)(1-\bar{b})} > 0$ and $F_2 = \frac{-m[S_r qn\ell+(\pi+T)/h]}{(1-\tau_L)(1-\bar{b})} < 0$.

3 Concavity of Household and Firm Value Functions

The concavity of the value function $\Gamma(n_t)$ in firm's optimization is easily confirmed as:

$$\frac{\partial^2 \Gamma(n_t)}{\partial (n_t)^2} = \alpha(1-\alpha)A\left(q_t^F\right)^{\alpha-1} q_{t+1}^F(n_t) + \frac{(1-\psi)^2}{1+\bar{r}}\Gamma_{nn}\left(n_{t+1}\right) < 0$$

The concavity of the value function $\Omega(\mathcal{H})$ in household's optimization is not as trivial, as it requires the Hessian matrix of $\Omega(\mathcal{H})$

$$J^{\Omega} \equiv \begin{pmatrix} \Omega_{kk} & \Omega_{kh} & \Omega_{kn} \\ \Omega_{hk} & \Omega_{hh} & \Omega_{hn} \\ \Omega_{nk} & \Omega_{nh} & \Omega_{nn} \end{pmatrix}$$

to be negative semidefinite. We can easily show:

$$\Omega_{kk}(\mathcal{H}) = \frac{1-\delta_k+(1-\tau_{K_t})r_t}{1+\rho}\{\Omega_{kk}(\mathcal{H}')[1-\delta_k+(1-\tau_{Kt})r_t s_t]+\Omega_{kh}(\mathcal{H}')\gamma(1-s_t)\widetilde{D}\left(q_t^H\right)^{\gamma-1}\}$$

Under our parameterization in the benchmark model, $\Omega_{kk}(H) < 0$.

By exhaustive manipulations, we have:

$$\Omega_{hh}(\mathcal{H}) = \frac{1}{1+\rho}(TE1 + TE2 + TE3 + TE4)$$

where

$$\begin{aligned}
TE1 &= \Omega_{kk}(\mathcal{H}')\left\{(1\text{-}\tau_{L_t})w_t[n_t\ell_t+(1\text{-}n)\bar{b}_t]\right\}^2 < 0 \\
TE2 &= \Omega_{hh}(\mathcal{H}')\{1+n_t(1\text{-}\ell_t)[D+\widetilde{D}(1\text{-}\gamma)\left(q_t^H\right)^{\gamma}]\}^2 < 0 \\
TE3 &= \Omega_h(\mathcal{H}')n_t(1-\ell_t)\widetilde{D}(1\text{-}\gamma)\gamma\left(q_t^H\right)^{\gamma-1} q_{t+1}^H(h_t) < 0 \\
TE4 &= [\Omega_{kh}(\mathcal{H}')+\Omega_{hk}(\mathcal{H}')](1\text{-}\tau_{L_t})w_t[n_t\ell_t+(1\text{-}n_t)\bar{b}_t]\{1+n_t(1\text{-}\ell_t)[D+\widetilde{D}(1\text{-}\gamma)\left(q_t^H\right)^{\gamma}]\} > 0
\end{aligned}$$

Under our parameterization in the benchmark model, $\Omega_{hh}(\mathcal{H}) < 0$. Additional exhaustive manipulations yield:

$$\Omega_{nn}(\mathcal{H}) = \frac{1}{1+\rho}(TE5 + TE6 + TE7 + TE8 + TE9 + TE10 + TE11)$$

where

$$
\begin{aligned}
TE5 &= \Omega_{kk}(\mathcal{H}')[(1\text{-}\tau_{L_t})w_t h_t(\ell_t\text{-}\bar{b}_t)]^2 < 0 \\
TE6 &= \Omega_{hh}(\mathcal{H}')\{(1\text{-}\ell_t)h_t[D+\widetilde{D}(1\text{-}\gamma)\left(q_t^H\right)^{\gamma}]\}^2 < 0 \\
TE7 &= \Omega_h(\mathcal{H}')(1\text{-}\ell_t)h_t\widetilde{D}(1\text{-}\gamma)\gamma\left(q_t^H\right)^{\gamma-1} q_{t+1}^H(n_t) < 0 \\
TE8 &= \Omega_{nn}(\mathcal{H}')(1\text{-}\psi\text{-}\mu_t)^2 < 0 \\
TE9 &= [\Omega_{kh}(\mathcal{H}') + \Omega_{hk}(\mathcal{H}'](1\text{-}\tau_{L_t})w_t h_t(\ell_t\text{-}\bar{b}_t)(1\text{-}\ell_t)h_t[D+\widetilde{D}(1\text{-}\gamma)\left(q_t^H\right)^{\gamma}] > 0 \\
TE10 &= [\Omega_{kn}(\mathcal{H}') + \Omega_{nk}(\mathcal{H}')](1\text{-}\tau_{L_t})w_t h_t(\ell_t\text{-}\bar{b}_t)(1\text{-}\psi\text{-}\mu_t) > 0 \\
TE11 &= [\Omega_{hn}(\mathcal{H}') + \Omega_{nh}(\mathcal{H}')](1\text{-}\ell_t)h_t[D+\widetilde{D}(1\text{-}\gamma)\left(q_t^H\right)^{\gamma}](1\text{-}\psi\text{-}\mu_t) > 0
\end{aligned}
$$

Under our parameterization in the benchmark model, $\Omega_{nn}(\mathcal{H}) < 0$. The 2×2 principal minors of J^{Ω} need be all positive and the determinant $\left|J^{\Omega}\right|$ need be negative, which are too complicated to identify clean sufficient conditions; nonetheless, they all hold true under our calibrated benchmark parametrization.

4 Quasi-Social Planner's Optimization

The quasi-social planner's problem is given by,

$$
\Lambda(k_t, h_t, n_t) = \max_{c_t,\, \ell_t,\, s_t,\, v_t} U\left(c_t\right) + m\left(1 - n_t\right) + \frac{1}{1+\rho}\Lambda(k_{t+1}, h_{t+1}, n_{t+1})
$$

subject to:

$$
k_{t+1} = A(s_t k_t)^{\alpha}(n_t \ell_t h_t)^{1-\alpha} - \phi \bar{y}_t v_t - c_t
$$

$$
h_{t+1} - h_t = D n_t(1-\ell_t)h_t + \widetilde{D}[(1-s_t)k_t]^{\gamma}[n_t(1-\ell_t)h_t]^{1-\gamma}
$$

$$
n_{t+1} = (1-\psi)n_t + B(1-n_t)^{\beta}\left(v_t\right)^{1-\beta}
$$

It is noted that while the resource constraint is straightforward by replacing income with net output and the human capital accumulation equation is identical to the decentralized problem, the evolution equations of employment differs from the decentralized program now with coordinated labor matches.

The quasi-social planner's optimization satisfies the following first-order conditions (with respect to $\{c_t, \ell_t, s_t, v_t\}$),

$$
\begin{aligned}
U_c &= \frac{1}{1+\rho}\Lambda_k(\mathcal{H}') \\
\Lambda_k(\mathcal{H}')A(1-\alpha)(q_t^F)^{\alpha} &= \Lambda_h(\mathcal{H}')\left[D + \widetilde{D}(1-\gamma)\left(q_t^H\right)^{\gamma}\right] \\
\Lambda_k(\mathcal{H}')A\alpha(q_t^F)^{\alpha-1} &= \Lambda_h(\mathcal{H}')\widetilde{D}\gamma\left(q_t^H\right)^{\gamma-1} \\
\Lambda_k(\mathcal{H}')\phi A(q_t^F)^{\alpha} n_t \ell_t h_t &= \Lambda_n(\mathcal{H}')(1-\beta)B(1-n_t)^{\beta}\left(v_t\right)^{-\beta}
\end{aligned}
$$

together with the respective Benveniste-Scheinkman conditions (associated with $\{k_t, h_t, n_t\}$):

$$\begin{aligned}
\Lambda_k(\mathcal{H}) &= \frac{1}{1+\rho}\Lambda_k(\mathcal{H}')[1\text{-}\delta_k+A\alpha(q_t^F)^{\alpha-1}] \\
\Lambda_h(\mathcal{H}) &= \frac{1}{1+\rho}(\Lambda_k(\mathcal{H}')A(1\text{-}\alpha)(q_t^F)^{\alpha}n_t\ell_t+\Lambda_h(\mathcal{H}')\{1+n_t(1\text{-}\ell_t)[D+\widetilde{D}(1\text{-}\gamma)\left(q_t^H\right)^{\gamma}]\}) \\
\Lambda_n(\mathcal{H}) &= \text{-}m+\frac{1}{1+\rho}\{\Lambda_k(\mathcal{H}')A(1\text{-}\alpha)(q_t^F)^{\alpha}\ell_t h_t+\Lambda_h(\mathcal{H}')(1\text{-}\ell_t)h_t[D+\widetilde{D}(1\text{-}\gamma)\left(q_t^H\right)^{\gamma}] \\
&\quad +\Lambda_n(\mathcal{H}')(1\text{-}\psi\text{-}\beta B(1\text{-}n_t)^{\beta-1}\left(v_t\right)^{1-\beta})\}
\end{aligned}$$

5 Equilibrium

We derive algebra in Section 3. Under the logarithmic utility function: $U(c) = \ln c$, households' lifetime utility is always bounded along a BGP. Moreover, $\Gamma_n(n')$ and $\Omega_n(\mathcal{H}')$ are constant along a BGP, whereas $\Omega_k(\mathcal{H}')$ and $\Omega_h(\mathcal{H}')$ are decreasing at rate g. Then, we use (6), (8), and (9) to derive a standard Keynes-Ramsey relationship governing consumption growth and an intertemporal optimization condition governing human capital accumulation as follows.

$$g = \frac{(1-\tau_K)r - \delta_k - \rho}{1+\rho} \tag{A1}$$

$$\rho + (1+\rho)g = [D + \widetilde{D}(1-\gamma)\left(q^H\right)^{\gamma}][n + (1-n)\bar{b}] \tag{A2}$$

From the definition of π and (16), we can derive the flow profit redistribution to each household in effective units as follows.

$$\frac{\pi}{h} = n\ell\left\{A\left(q^F\right)^{\alpha}[(1-\alpha)-\phi v] - w\right\} \tag{A3}$$

From (3), the definition of q^F and the flow profit redistribution given above, we can derive effective consumption along a BGP as:

$$\begin{aligned}
\frac{c}{h} &= \left(S_w w + S_r q^F\right)n\ell + \frac{\pi}{h} + \frac{T}{h} \\
&= \left\{A\left(q^F\right)^{\alpha}[(1-\alpha)-\phi v] + S_r q^F - (1-S_w)\,w\right\}n\ell + \frac{T}{h} \qquad \text{(A4)}
\end{aligned}$$

where T is regarded as given by individuals with its equilibrium value being pinned down by the government budget constraint (21).

6 Efficiency

This Appendix derive algebra in Section 4. For the purpose of comparison, it is convenient to rewrite the conditions in Lemmas 1 and 2 in the decentralized problem as:

$$\begin{aligned}
\Omega_k(\mathcal{H}') &= (1+\rho)\,U_c \\
\Omega_k(\mathcal{H}')(1-\tau_{L_t})w_t &= \Omega_h(\mathcal{H}')\left[D+\widetilde{D}(1-\gamma)\left(q_t^H\right)^{\gamma}\right] \\
\Omega_k(\mathcal{H}')(1-\tau_{K_t})r_t &= \Omega_h(\mathcal{H}')\widetilde{D}\gamma\left(q_t^H\right)^{\gamma-1} \\
(1+\rho)\,(1+g_t) &= [1-\delta_k+(1-\tau_{K_t})r_t] \\
(1+\rho)\,(1+g_t)-1 &= \left[D+\widetilde{D}(1-\gamma)\left(q_t^H\right)^{\gamma}\right][n_t+(1-n_t)\bar{b}_t]
\end{aligned} \tag{A5}$$

$$\Omega_n = \frac{1+\rho}{\rho+\psi+\mu_t}\left[\left(1-\bar{b}_t\right)(1-\tau_{L_t})w_t h_t U_c - m\right] \tag{A6}$$

$$\Gamma_n = \frac{1+R_t}{\psi+R_t}\left[(1-\alpha)A\left(q_t^F\right)^{\alpha}-w_t\right] \tag{A7}$$

We can then differentiate the above expressions to obtain:

$$\begin{aligned}
d\Omega_n/dw_t &= \frac{1+\rho}{\rho+\psi+\mu_t}\left(1-\bar{b}_t\right)(1-\tau_{L_t})h_t U_c \\
d\Gamma_n/dw_t &= -\frac{1+R_t}{\psi+R_t}
\end{aligned}$$

The cooperative Nash wage bargaining therefore implies:

$$\begin{aligned}
\Omega_n &= -\frac{\zeta}{1-\zeta}\frac{d\Omega_n/dw_t}{d\Gamma_n/dw_t}\Gamma_n \\
&= -\frac{\zeta}{1-\zeta}\frac{d\Omega_n/dw_t}{d\Gamma_n/dw_t}\frac{1+R_t}{\psi+R_t}\left[(1-\alpha)A\left(q_t^F\right)^{\alpha}-w_t\right]
\end{aligned}$$

The above expression can be combined with (A6) to yield:

$$\begin{aligned}
(1{+}\rho)\left[\left(1\text{-}\bar{b}_t\right)(1\text{-}\tau_{L_t})w_t h_t U_c\text{-}m\right] &= -\frac{\zeta}{1\text{-}\zeta}\frac{d\Omega_n/dw_t}{d\Gamma_n/dw_t}\frac{(1{+}R_t)\,(\rho{+}\psi{+}\mu_t)}{\psi{+}R_t}\left[(1\text{-}\alpha)A\left(q_t^F\right)^{\alpha}\text{-}w_t\right] \\
&= \frac{\zeta}{1\text{-}\zeta}(1{+}\rho)\left(1\text{-}\bar{b}_t\right)(1\text{-}\tau_{L_t})h_t U_c\left[(1\text{-}\alpha)A\left(q_t^F\right)^{\alpha}\text{-}w_t\right]
\end{aligned}$$

which can be simplified to the decentralized labor-leisure-consumption trade-off as follows.

$$\left(1-\bar{b}_t\right)(1-\tau_{L_t})w_t h_t U_c-(1-\zeta)\,m=\zeta\left(1-\bar{b}_t\right)(1-\tau_{L_t})h_t(1-\alpha)A\left(q_t^F\right)^{\alpha}U_c$$

Concerning centralized solution, we rewrite the conditions in Lemma 5 to get:

$$\begin{aligned}
\Lambda_k(\mathcal{H}') &= (1+\rho)\, U_c \\
\Lambda_k(\mathcal{H}') A(1-\alpha)(q_t^F)^\alpha &= \Lambda_h(\mathcal{H}') \left[D + \widetilde{D}(1-\gamma)\left(q_t^H\right)^\gamma\right] \\
\Lambda_k(\mathcal{H}') A\alpha(q_t^F)^{\alpha-1} &= \Lambda_h(\mathcal{H}') \widetilde{D}\gamma \left(q_t^H\right)^{\gamma-1} \\
\Lambda_k(\mathcal{H}') \phi A(q_t^F)^\alpha n_t \ell_t h_t &= \Lambda_n(\mathcal{H}')(1-\beta)B(1-n_t)^\beta \left(v_t\right)^{-\beta} \\
(1+\rho)\left(1+g_t\right) &= \left[1-\delta_k + A\alpha(q_t^F)^{\alpha-1}\right] \\
(1+\rho)\left(1+g_t\right) - 1 &= \left[D + \widetilde{D}(1-\gamma)\left(q_t^H\right)^\gamma\right] n_t
\end{aligned} \tag{A8}$$

$$\begin{aligned}
\left[\rho + \psi + \beta B(1-n_t)^{\beta-1}\left(v_t\right)^{1-\beta}\right]\Lambda_n &= (1+\rho)\left[A(1-\alpha)(q_t^F)^\alpha h_t U_c - m\right] \\
(1-\beta)B(1-n_t)^\beta \left(v_t\right)^{-\beta}\Lambda_n &= \phi A(q_t^F)^\alpha n_t \ell_t h_t \left(1+\rho\right) U_c
\end{aligned}$$

where the last two expressions can be combined with $\psi n_t = \mu_t(1-n_t) = \eta_t v_t = B(1-n_t)^\beta \left(v_t\right)^{1-\beta}$ to yield:

$$\begin{aligned}
A(1-\alpha)(q_t^F)^\alpha h_t U_c - m &= \frac{\rho+\psi+\beta B(1-n_t)^{\beta-1}\left(v_t\right)^{1-\beta}}{(1-\beta)B(1-n_t)^\beta\left(v_t\right)^{-\beta}}\phi A(q_t^F)^\alpha n_t \ell_t h_t U_c \\
&= \frac{\rho+\psi+\beta\mu_t}{(1-\beta)\eta_t}\phi A(q_t^F)^\alpha n_t \ell_t h_t U_c
\end{aligned}$$

which can be simplified to the counterpart of this labor-leisure-consumption trade-off under the centralized solution as follows.

$$(1-\alpha)A(q_t^F)^\alpha h_t U_c - \frac{\rho+\psi+\beta\mu_t}{\eta_t v_t}\Phi U_c - (1-\beta)m = \beta(1-\alpha)A(q_t^F)^\alpha h_t U_c$$

where $\Phi = \phi v_t \bar{y}_t = \phi v_t A(q_t^F)^\alpha n_t \ell_t h_t$ is the vacancy creation cost.

Then, by comparing the decentralized and centralized solutions, namely (A5) and (A8), we can identify four conditions in a more straightforward manner:

$$\begin{aligned}
R_t &= r_t \\
\Omega_k(\mathcal{H}') &= \Lambda_k(\mathcal{H}') \\
\tau_{K_t} &= 0 \\
\bar{b}_t &= 0
\end{aligned}$$

Moreover, to ensure the labor-leisure-consumption trade-off under decentralization and centralization to be identical, we need to establish equivalence between the decentralized labor-leisure-consumption trade-off and the counterpart of this labor-leisure-consumption trade-off under the centralized solution, which holds true under the following conditions:

$$\begin{aligned} \zeta &= \beta \\ \tau_{L_t} &= 0 \\ w_t &= (1-\Delta_t^*)\,(1-\alpha)A(q_t^F)^{\alpha} \end{aligned}$$

where efficient wage discount Δ_t^* is given by,

$$\Delta_t^* = \frac{(\rho+\psi+\beta\mu_t)\,\phi v_t \ell_t}{\psi(1-\alpha)} = \frac{(\rho+\psi+\beta\mu_t)\,\phi n_t \ell_t}{(1-\alpha)\eta_t}$$

7 Dynamic Taxation and Dynamic Tax Incidence

This Appendix derives dynamic taxation. In deriving dynamic taxation, we maintain a BGP equilibrium with stationary matching and bargaining. This implies that g and n are constant. Then (22)-(25) indicate that μ, η, v, and θ are constant.

From (A1), we derive:

$$r_t = r(\tau_{Kt}) = \frac{1}{1-\tau_{Kt}}[g(1+\rho)+\delta_k+\rho]$$

implying $r'(\tau_{Kt}) = \frac{1}{(1-\tau_{Kt})^2}[g(1+\rho)+\delta_k+\rho] > 0$ and $r''(\tau_{Kt}) = \frac{2}{(1-\tau_{Kt})^3}[g(1+\rho)+\delta_k+\rho] > 0$.

From (A2), we derive:

$$q_t^H = q^H(\bar{b}_t) = \left\{\frac{1}{\widetilde{D}(1-\gamma)}\left[\frac{\rho+(1+\rho)g}{n+(1-n)\bar{b}_t} - D\right]\right\}^{\frac{1}{\gamma}}$$

implying $\frac{\partial q_t^H}{\partial \bar{b}_t} < 0$. Moreover, $\frac{\partial q_t^H}{\partial g} > 0$ and $\frac{\partial q_t^H}{\partial n} < 0$.

From (26), we derive:

$$\ell_t = \ell(\bar{b}_t) = 1 - \frac{g}{n\{D+\widetilde{D}[q^H(\bar{b}_t)]^{\gamma}\}}$$

implying $\ell'(\bar{b}_t) < 0$.Moreover, $\frac{\partial \ell_t}{\partial g} < 0$ and $\frac{\partial \ell_t}{\partial n} > 0$.

From (16), we derive:

$$R_t = R(\tau_{Kt}) = r(\tau_{Kt})$$

From (18), we derive:

$$q_t^F = q^F(\tau_{Kt}) = \left(\frac{\alpha A}{r(\tau_{Kt})}\right)^{\frac{1}{1-\alpha}}$$

implying $\frac{\partial q_t^F}{\partial \tau_{Kt}} = -\frac{1}{1-\alpha}\left(\frac{\alpha A}{r(\tau_{Kt})}\right)^{\frac{1}{1-\alpha}-1} r'(\tau_{Kt}) < 0$.

From (A1) and (31), we derive:

$$w_t = w(\tau_{Kt}) = \left(\frac{\alpha^\alpha A}{r(\tau_{Kt})^\alpha}\right)^{\frac{1}{1-\alpha}}\left[1-\alpha-\frac{(r(\tau_{Kt})+\psi)\phi}{\eta(n)}n\right]$$

implying $w'(\tau_{Kt}) = -(\alpha^\alpha A)^{\frac{1}{1-\alpha}} r'(\tau_{Kt})\left\{\frac{\alpha r(\tau_{Kt})^{\frac{-1}{1-\alpha}}}{1-\alpha}\frac{\eta(n)(1-\alpha)-(r(\tau_{Kt})+\psi)\phi n}{\eta(n)} + \frac{\phi n r(\tau_{Kt})^{\frac{-\alpha}{1-\alpha}}}{\eta(n)}\right\} < 0$. Although $w''(\tau_{Kt})$ may be ambiguous, as a result of factor substitution, when the direct production cost effect dominates the labor market friction effect (the last term in the square bracket of the effective wage expression), effective wage is strictly concave in the capital tax rate and thus, $w''(\tau_{Kt}) < 0$.

Equation (11) becomes:

$$\left(q^H(\bar{b}_t)\right)^{1-\gamma}\left[D+\widetilde{D}(1-\gamma)\left(q^H(\bar{b}_t)\right)^\gamma\right] = \widetilde{D}\gamma\frac{(1-\tau_{L_t})w(\tau_{Kt})}{(1-\tau_{K_t})r(\tau_{Kt})}$$

from which we can express $\bar{b}_t$ as an equation of τ_{Kt} and τ_{Lt}:

$$\bar{b}_t = \bar{b}(\tau_{Kt}, \tau_{Lt})$$

implying $\frac{\partial \bar{b}_t}{\partial \tau_{Lt}} > 0$. Moreover, $\frac{\partial \bar{b}_t}{\partial \tau_{Kt}} > 0$, because from (A1), $(1-\tau_{K_t})r(\tau_{Kt})$ is dependent on τ_{Kt} only through its BGP effect on g, which we have already proved. Thus, the time varying effect of τ_{Kt} only affect $w(\tau_{Kt})$ negatively, so τ_{Kt} also affects $\bar{b}_t$ positively. This is the IR locus.

The IR locus is negatively sloping in the (τ_{Lt}, τ_{Kt}) plane.

$$\frac{d\tau_{L_t}}{d\tau_{Kt}} = \frac{w'(\tau_{Kt})(1-\tau_{L_t})}{\frac{w(\tau_{Kt})}{(1-\tau_{K_t})r(\tau_{Kt})}} = \frac{w'(\tau_{Kt})(1-\tau_{L_t})(1-\tau_{K_t})r(\tau_{Kt})}{w(\tau_{Kt})} < 0.$$

To see whether the isoquant is concave or convex, we take a second-order derivative to get

$$\frac{d^2\tau_{L_t}}{d\tau_{Kt}^2} = \frac{\{w(\tau_{Kt})w''(\tau_{Kt}) - w'(\tau_{Kt})^2[1+(1-\tau_{K_t})r(\tau_{Kt})]\}(1-\tau_{L_t})(1-\tau_{K_t})r(\tau_{Kt})}{w(\tau_{Kt})^2}$$

If $w''(\tau_{Kt}) < 0$, then $\frac{d^2\tau_{L_t}}{d\tau_{Kt}^2} < 0$ and the IR locus is concave to the origin. Our quantitative exercises confirm $w''(\tau_{Kt}) < 0$ and thus, the IR locus is concave to the origin.

In order to quantify time-varying factor taxes, we impose the following factor tax schedules

$$\begin{aligned}\tau_{K_t} - \tau_K^* &= A_K e^{-a_K(t)\cdot t}\\ \tau_{L_t} - \tau_L^* &= -A_L e^{-a_L(t)\cdot t}\end{aligned}$$

where $A_K, a_K(t), A_L, a_L(t)$ are coefficients that are to be calibrated. Below, we characterize these coefficients. First, from (A1), we have $(1-\tau_{L_t})w(\tau_{Kt}) = g(1+\rho)+\delta_k+\rho$ which is constant. Then we rewrite (11) as

$$\left(q^H(\bar{b}_t)\right)^{1-\gamma}\left[D+\widetilde{D}(1-\gamma)\left(q^H(\bar{b}_t)\right)^{\gamma}\right] = \widetilde{D}\gamma\frac{(1-\tau_{L_t})w(\tau_{Kt})}{g(1+\rho)+\delta_k+\rho}$$

Next, using (31) to substitute $w(\tau_{Kt})$ in (11), we have

$$(1-\tau_{L_t})\left(\frac{\alpha^\alpha A}{r(\tau_{Kt})^\alpha}\right)^{\frac{1}{1-\alpha}}\left[1-\alpha-\frac{\phi n}{\eta}\left(r(\tau_{Kt})+\psi\right)\right] = \Theta$$

where $\Theta = \left(q^H(\bar{b}_t)\right)^{1-\gamma}\left[D+\widetilde{D}(1-\gamma)\left(q^H(\bar{b}_t)\right)^{\gamma}\right]\frac{g(1+\rho)+\delta_k+\rho}{\widetilde{D}\gamma}$.

Using (A1) to substitute $r(\tau_{Kt})$ in (11) gives

$$(1-\tau_{L_t})\left(\frac{g(1+\rho)+\delta_k+\rho}{1-\tau_{Kt}}\right)^{\frac{-\alpha}{1-\alpha}}\left[1-\alpha-\frac{\phi n}{\eta}\{\frac{1}{1-\tau_{Kt}}[g(1+\rho)+\delta_k+\rho]+\psi\}\right] = (\alpha^\alpha A)^{\frac{-1}{1-\alpha}}\Theta$$

At $t=0$, the initial condition is $\tau_{K_0} = \tau_K^* + A_K$ and $\tau_{L_0} = \tau_L^* - A_L$. Then (11) at $t=0$ gives the relationship between A_L and A_K as follows.

$$A_L(A_K) = -1+\tau_L^* + \frac{(\alpha^\alpha A)^{\frac{-1}{1-\alpha}}\left[\frac{g(1+\rho)+\delta_k+\rho}{1-\tau_K^*-A_K}\right]^{\frac{\alpha}{1-\alpha}}\Theta(\bar{b})}{1-\alpha-\frac{\phi n}{\eta}\left[\frac{g(1+\rho)+\delta_k+\rho}{1-\tau_K^*-A_K}+\psi\right]}$$

which depends positively on A_K when the capital income share is not too high such that $\alpha < \min\left\{\frac{1}{2}, 1-\frac{\psi\phi n}{\eta}\right\}$. Thus the two initial tax rates are negatively related, which is easily understood because the IR locus is downward sloping. Moreover, at time t, the speed of labor taxation $a_L(t)$ is governed by

$$\begin{aligned}\ln\left[1-\tau_L^*+A_L(A_K)e^{-a_L(t)\cdot t}\right] &= \ln(\alpha^\alpha A)^{\frac{-1}{1-\alpha}}\Theta(\bar{b}) + \frac{\alpha}{1-\alpha}\ln\frac{g(1+\rho)+\delta_k+\rho}{1-\tau_K^*-A_K e^{-a_K(t)\cdot t}} \\ &\quad -\ln\left\{1-\alpha-\frac{\phi n}{\eta}\left[\frac{g(1+\rho)+\delta_k+\rho}{1-\tau_K^*-A_K e^{-a_K(t)\cdot t}}+\psi\right]\right\}\end{aligned}$$

Define $\Xi = \frac{\alpha}{1-\alpha}\frac{1}{1-\tau_K-A_K\cdot e^{-a_K(t)\cdot t}} + \frac{\phi n}{\eta}\frac{g(1+\rho)+\delta_k+\rho}{\left[1-\tau_K-A_K e^{-a_K(t)\cdot t}\right]^2\left\{1-\alpha-\frac{\phi n}{\eta}\left[\frac{g(1+\rho)+\delta_k+\rho}{1-\tau_K-A_K e^{-a_K(t)\cdot t}}+\psi\right]\right\}}$.

Totally differentiating (34) gives

$$\frac{A_L t e^{-a_L(t)\cdot t}}{1-\tau_L+A_L e^{-a_L(t)\cdot t}}da_L = \left\{\frac{A_L'(A_K)e^{-a_L(t)\cdot t}}{1-\tau_L+A_L e^{-a_L(t)\cdot t}} - \Xi e^{-a_K(t)\cdot t}\right\}dA_K + \Xi A_K t e^{-a_K(t)\cdot t}da_K$$

Thus, $\frac{da_L}{da_K} > 0$ whereas $\frac{da_L}{dA_K}$ is ambiguous, but $\frac{da_L}{dA_K} > 0$ if the effect through A_L dominates (which is the case of our calibrated economy).

To evaluate the welfare, first we rewrite (2) as

$$\frac{h_{t+1}}{h_t} = 1 + Dn_t(1-\ell_t) + \widetilde{D}(q_t^H)^\gamma n_t(1-\ell_t)$$

Since n_t, q_t^H and ℓ_t are constant, then the growth rate of h_t is constant. Since the growth rate of h_t is constant, then we only need to analyze the transition of $\frac{c_t}{h_t}$ when evaluating welfare.

Then, from (27), we derive

$$s_t = s(\tau_{Kt}, \bar{b}) = \frac{q^F(\tau_{Kt})\ell(\bar{b}_t)}{[1-\ell(\bar{b})]q^H(\bar{b}) + q^F(\tau_{Kt})\ell(\bar{b})}$$

which is decreasing in τ_{Kt}.

Moreover, from (21), we have

$$\frac{T_t}{h_t} = w(\tau_{Kt})\{\tau_{Lt}\left[n\ell + (1-n)\bar{b}\right] - (1-n)\bar{b}\} + \tau_{K_t} r(\tau_{Kt})q^F(\tau_{Kt})n\ell$$

which is increasing in τ_{Lt} but the effect of τ_{Kt} is complicated. Since $\tau_{K_t} r(\tau_{Kt})q^F(\tau_{Kt}) \propto \tau_{K_t}(1-\tau_{K_t})^{\frac{\alpha}{1-\alpha}}$, straightforward differentiation implies $\tau_{K_t} r(\tau_{Kt})q^F(\tau_{Kt})$ is increasing in τ_{K_t} if $\tau_{K_t} < 1-\alpha$. Yet, a higher τ_{K_t} lowers $w(\tau_{Kt})$, thus leading to an ambiguous result. In our calibrated economy, the indirect effect via $w(\tau_{Kt})$ is dominated by the direct effect. As a result, both factor tax rates raise effective lump-sum tax. Also note that $S_{wt} = S_w(\tau_{Lt}, \bar{b}) = (1-\tau_{Lt})\left[1 + \frac{(1-n)\bar{b}}{n\ell}\right]$ and $S_{rt} = S_r(\tau_{Kt}, \bar{b}) = \left[(1-\tau_{Kt})r(\tau_{Kt}) - \frac{g+\delta_k}{s(\tau_{Kt}, \bar{b}_t)}\right]$. While S_{wt} is decreasing in τ_{Lt}, S_{rt} is decreasing in τ_{Kt}.

Furthermore, from (A4), we have

$$\frac{c_t}{h_t} = \left\{A\left(q^F(\tau_{Kt})\right)^\alpha [(1-\alpha) - \phi v] + S_r(\tau_{Kt}, \bar{b})q^F(\tau_{Kt}) - \left(1 - S_w(\tau_{Lt}, \bar{b}_t)\right)w(\tau_{Kt})\right\} n\ell + \frac{T_t}{h_t}$$

Finally, we are ready to calibrate tax schedules. The calibration is carried out in the following steps.

(i) We set A_K so to match the (given) initial τ_{K0};

(ii) Given A_K, the value of a_K is set so that the difference between τ_{K_t} and τ_K^* (the optimal capital tax rate, 16.11%) is within 1% of τ_K^* in the 50th period;

(iii) Then A_L is set so that (11) is satisfied at the 10th period ($t = 9$);

(iv) The value of a_L is set so that the difference between τ_{L_t} and τ_L^* (the optimal capital labor rate, 24.09%) is within 1% of τ_L^* at the 50th period.

8 Alternative Model II: Linear Human Capital Accumulation

In the case with a linear human capital accumulation process independent of market goods, the first-order condition of the household's optimization problem (5) is the same while (6) becomes:

$$\Omega_k(\mathcal{H}')(1-\tau_{L_t})w_t = \Omega_h(\mathcal{H}')D$$

The Benveniste-Scheinkman conditions of the household's optimization problem are now:

$$\Omega_k(\mathcal{H}) = \frac{1}{1+\rho}\Omega_k(\mathcal{H}')[(1-\delta_k)+(1-\tau_{K_t})r_t]$$

$$\Omega_h(\mathcal{H}) = \frac{1}{1+\rho}\left\{\Omega_k(\mathcal{H}')(1-\tau_{L_t})w_t[n_t\ell_t+(1-n_t)\bar{b}_t] + \Omega_h(\mathcal{H}')[1+Dn_t(1-\ell_t)]\right\}$$

$$\Omega_n(\mathcal{H}) = -m+\frac{1}{1+\rho}\left[\Omega_k(\mathcal{H}')\left(\ell_t\text{-}\bar{b}_t\right)(1\text{-}\tau_{L_t})w_th_tv_t + \Omega_h(\mathcal{H}')D(1\text{-}\ell_t)h_t + \Omega_n(\mathcal{H}')(1\text{-}\psi\text{-}\mu_t)\right]$$

The BGP equilibrium expressions follow by simply setting $\widetilde{D}=0$ and $s=1$.

9 Alternative Model III: Walrasin Model

We consider a Walrasian economy with $n=1$. Let $q_t^H = \frac{(1-s_t)k_t}{(1-\ell_t)h_t}$ and $q_t^F = \frac{s_tk_t}{\ell_th_t}$. Then the firm's optimal decisions are:

$$\begin{aligned}\alpha A\left(q_t^F\right)^{\alpha-1} &= r_t\\ (1-\alpha)A\left(q_t^F\right)^{\alpha} &= w_t\end{aligned}$$

Combining these, we have:

$$q_t^F = \frac{\alpha w_t}{(1-\alpha)r_t}$$

The household faces the following budget constraint:

$$k_{t+1} = (1-\tau_{Lt})w_t\ell_th_t + [(1-\delta_k+(1-\tau_{K_t})r_t]k_t - c_t + T_t$$

The main change is the Benveniste-Scheinkman condition with respect to h_t:

$$\Omega_h(\mathcal{H}) = \frac{1}{1+\rho}\left\{\Omega_k(\mathcal{H}')(1\text{-}\tau_{L_t})w_t\ell_t+\Omega_h(\mathcal{H}')\left[1+(1\text{-}\ell_t)\left[D+\widetilde{D}(1\text{-}\gamma)\left(q_t^H\right)^{\gamma}\right]\right]\right\}$$

By imposing a log utility function $U(c)=\ln c$, we can derive the following equations along the BGP:

$$\rho+(1+\rho)g = \left[D+\widetilde{D}(1-\gamma)\left(q^H\right)^{\gamma}\right]$$

$$\ell = 1-\frac{g}{D+\widetilde{D}\left(q^H\right)^{\gamma}}$$

The Keynes-Ramsey relationship (A1) and (11) remain unchanged. The effective consumption along a BGP becomes:

$$\frac{c}{h} = (1-\tau_L)w\ell + \left[(1-\tau_K)r - \frac{g+\delta_k}{s}\right]q^F\ell + \frac{T}{h}$$