9 Appendix

Section 3: First Best Allocations.

Proof of Lemma 1. a. $\Omega(x^1, \mu)$ is (at least once) continuously differentiable in x^1, with $\Omega(0, \mu) = +\infty$, and $\Omega(1, \mu) = -\infty$. By the Intermediate Value Theorem a value $z \in (0, 1)$ such that $\Omega(z, \mu) = 0$ exists and is unique, since $\frac{\partial \Omega\left(x^1, \mu\right)}{\partial x^1} = \mu u''(x^1) + (1 - \mu) u''(1 - x^1) < 0$ for all $\mu \in (0, 1)$ and $x^1 \in [0, 1]$. b. By the Implicit Function Theorem, $z(\mu)$ is (at least once) continuously differentiable in μ, with $\frac{\partial z(\mu)}{\partial \mu} = -\frac{\frac{\partial \Omega\left(x^1, \mu\right)}{\partial \mu}}{\frac{\partial \Omega\left(x^1, \mu\right)}{\partial x^1}} \|_{x^1 = z} > 0$, since $\frac{\partial \Omega\left(x^1, \mu\right)}{\partial \mu} = u'(x^1) + u'(1 - x^1) > 0$ for all $\mu \in (0, 1)$ and $x^1 \in [0, 1]$; c. $z(0) = 0$ and $z(1) = 1$, since a type with zero weight in the objective function should be assigned zero consumption at an optimum; $\Omega\left(x_t^1, \frac{1}{2}\right) = \frac{1}{2}\left[u'(x_t^1) - u'(1 - x_t^1)\right] = 0 \Leftrightarrow x_t^1 = 1 - x_t^1$, hence, $z\left(\frac{1}{2}\right) = \frac{1}{2}$.■

Section 4: Non-Monetary Regime.

The proof of Lemma 2 requires some definitions and three preparatory Lemmas. Although only values of $y \geq \frac{1}{2}$ are feasible, it will be more convenient to consider $y \in [0, 1]$, as a first step. Then, we will restrict y to the feasible interval $\left[\frac{1}{2}, 1\right]$. Let the function $\Psi_T(y, \beta) : [0, 1] \times (0, 1) \to \mathbb{R}$ be defined as

$$\Psi_T(y, \beta) \equiv f_T(\beta)\left[u(1 - y) - u(0)\right] - g_T(\beta)\left[u(1) - u(y)\right], \qquad (24)$$

i.e. as the difference between the LHS and the RHS of (9). Define also $y_T(\beta) : (0, 1) \to [0, 1]$ as the function that explicitly relates pairs of values $(y, \beta) \in [0, 1] \times (0, 1)$ such that $\Psi_T(y, \beta) = 0$, for any given T. Define the function $\widehat{y}_T(\beta) : (0, 1) \to \left[\frac{1}{2}, 1\right]$, as $\widehat{y}_T(\beta) \equiv \max\left\{y_T(\beta), \frac{1}{2}\right\}$, for any $\beta \in (0, 1)$ and T, i.e. the restriction of $y_T(\beta)$ to feasible values in $\left[\frac{1}{2}, 1\right]$.

Lemma A1. *a. For any $\beta \in (0,1)$ and $T > 0$, there exists exactly one value $y \in [0,1)$ s.t. $\Psi_T(y,\beta) = 0$. b. For any $T > 0$, the function $y_T(\beta)$ is (at least twice) continuously differentiable in β, with $\frac{\partial y_T(\beta)}{\partial \beta} < 0$ for all $\beta \in (0,1)$.*

Proof. a. The function $\Psi_T(y,\beta) : [0,1] \times (0,1) \to \mathbb{R}$ is (at least twice) continuously differentiable in y. Observe that, for any $T > 0$, the following are true: i. $\Psi_T(0,\beta) = -\frac{1+\beta^{T+1}}{1+\beta}[u(1) - u(0)] < 0$, ii. $\Psi_T(1,\beta) = 0$, iii. $\frac{\partial \Psi_T(y,\beta)}{\partial y} = -\frac{\beta(1-\beta^T)}{1-\beta^2} u'(1-y) + \frac{1-\beta^{T+2}}{1-\beta^2} u'(y)$, iv. $\frac{\partial \Psi_T(0,\beta)}{\partial y} = +\infty$, $\frac{\partial \Psi_T(1,\beta)}{\partial y} = -\infty$, v. $\frac{\partial^2 \Psi_T(y,\beta)}{\partial y^2} = \frac{\beta(1-\beta^T)}{1-\beta^2} u''(1-y) + \frac{1-\beta^{T+2}}{1-\beta^2} u''(y) < 0$. Hence, for any $\beta \in (0,1)$ and $T > 0$, there is exactly one $y \in [0,1)$ s.t. $\Psi_T(y,\beta) = 0$. b. The derivative $\frac{\partial \Psi_T(y,\beta)}{\partial y}$ evaluated at any $(y,\beta) \in (0,1) \times (0,1)$ such that $\Psi_T(y,\beta) = 0$ is given by

$$g_T(\beta)[u(1) - u(y)]\left[\frac{u'(y)}{u(1) - u(y)} - \frac{u'(1-y)}{[u(1-y) - u(0)]}\right] > 0, \tag{25}$$

since $\frac{u'(y)}{u(1)-u(y)} > \frac{1}{1-y} > \frac{u'(1-y)}{[u(1-y)-u(0)]}$ for any $y \in (0,1)$ by strict concavity of the utility function. Therefore, the Implicit Function Theorem applies and $y_T(\beta)$ is (at least twice) continuously differentiable in β. The derivative $\frac{\partial \Psi_T(y,\beta)}{\partial \beta}$ evaluated at any $(y,\beta) \in (0,1) \times (0,1)$ such that $\Psi_T(y,\beta) = 0$ is given by

$$g_T(\beta)[u(1) - u(y)]\left[\frac{f_T'(\beta)}{f_T(\beta)} - \frac{g_T'(\beta)}{g_T(\beta)}\right] > 0, \tag{26}$$

since $f_T'(\beta) = \sum_{j=1}^{\frac{T}{2}} (2j-1)\beta^{2j-2} > \sum_{j=0}^{\frac{T}{2}} 2j\beta^{2j-1} = g_T'(\beta)$ and $g_T(\beta) = \frac{1-\beta^{T+2}}{1-\beta^2} > \frac{\beta(1-\beta^T)}{1-\beta^2} = f_T(\beta)$. Thus, $\frac{\partial y_T(\beta)}{\partial \beta} < 0$, for any $\beta \in (0,1)$ and $T > 0$, since it is given by the ratio (26) to (25) changed of sign, as an application of the Implicit Function

Theorem.■

There are two possible cases, T infinite or finite. Case 1. $T = \infty$. Define $\underline{\beta} \equiv \frac{u(1)-u\left(\frac{1}{2}\right)}{u\left(\frac{1}{2}\right)-u(0)} \in (0,1)$.

Lemma A2. $\Gamma_\infty(\beta) = [\widehat{y}_\infty(\beta), 1]$ *for all* $\beta \in (0,1)$, *where* $\widehat{y}_\infty(\beta)$ *is continuous, and*

i. $\widehat{y}_\infty(\beta) = \frac{1}{2}$, *if* $\beta \geq \underline{\beta}$;

ii. $\widehat{y}_\infty(\beta) = y_\infty(\beta) \in \left(\frac{1}{2}, 1\right)$, *if* $\beta < \underline{\beta}$.

Proof. The value $\underline{\beta}$ satisfies

$$\Psi_\infty\left(\frac{1}{2}, \beta\right) = \frac{1}{1-\beta^2}\left[\beta\left[u\left(\frac{1}{2}\right) - u(0)\right] - \left[u(1) - u\left(\frac{1}{2}\right)\right]\right] = 0.$$

Thus, $y_\infty\left(\underline{\beta}\right) = \frac{1}{2}$. By Lemma A1, $y_\infty(\beta)$ is (at least twice) continuously differentiable and strictly decreasing function for any $\beta \in (0,1)$. Hence, we have that for $\beta \in \left(0, \underline{\beta}\right)$, $y_\infty(\beta) > \frac{1}{2}$, $\lim_{\beta \to \underline{\beta}^-} y_\infty(\beta) = \frac{1}{2}$ and for $\beta \in \left[\underline{\beta}, 1\right)$, $y_\infty(\beta) \leq \frac{1}{2}$. By definition $\widehat{y}_T(\beta) \equiv \max\left\{y_T(\beta), \frac{1}{2}\right\}$. Thus, we have

$$\widehat{y}_\infty(\beta) = \begin{cases} \frac{1}{2}, & \text{if } \beta \geq \underline{\beta} \\ y_\infty(\beta), & \text{if } \beta < \underline{\beta} \end{cases},$$

which is continuous in $\beta \in (0,1)$, since $\lim_{\beta \to \underline{\beta}^-} y_\infty(\beta) = \frac{1}{2}$.■

Case 2. T finite, $0 < T < \infty$. Notice that $\overline{T}$ is defined in the text as $\overline{T} \equiv \left\lfloor \frac{2\left[u(1)-u\left(\frac{1}{2}\right)\right]}{2u\left(\frac{1}{2}\right)-u(1)-u(0)} \right\rfloor \in \mathbb{N}$, where, for any $w \in \mathbb{R}_+$, $\lfloor w \rfloor$ denotes the largest natural number not greater than w.

Lemma A3. $\Gamma_T(\beta) = [\widehat{y}_T(\beta), 1]$ *for all* $\beta \in (0,1)$, *where* $\widehat{y}_T(\beta)$ *is continuous, and* *a. if* $T > \overline{T}$*, there exists a unique* $\underline{\beta}_T \in (0,1)$ *such that:*

i. $\widehat{y}_T(\beta) = \frac{1}{2}$*, for* $\beta \geq \underline{\beta}_T$*;*

ii. $\widehat{y}_T(\beta) = y_T(\beta) \in \left(\frac{1}{2}, 1\right)$*, for* $\beta < \underline{\beta}_T$*;*

b. if $T \leq \overline{T}$*, for all* $\beta \in (0,1)$*,* $\widehat{y}_T(\beta) = y_T(\beta) \in \left(\frac{1}{2}, 1\right)$*.*

Proof. a. Evaluate $\Psi_T(y,\beta)$ at $y = \frac{1}{2}$ and $\beta \to 1$, obtaining $\lim_{\beta \to 1} \Psi_T\left(\frac{1}{2},\beta\right) = \left(\frac{T}{2}\right)\left[u\left(\frac{1}{2}\right) - u(0)\right] - \left(\frac{T}{2}+1\right)\left[u(1) - u\left(\frac{1}{2}\right)\right]$. Since $T > \overline{T}$, $\lim_{\beta \to 1} \Psi_T\left(\frac{1}{2},\beta\right) > 0$. Evaluate $\Psi_T(y,\beta)$ at $y = \frac{1}{2}$ and $\beta \to 0$, obtaining $\lim_{\beta \to 0} \Psi_T\left(\frac{1}{2},\beta\right) = -\left[u(1) - u\left(\frac{1}{2}\right)\right] < 0$. Since $\Psi_T\left(\frac{1}{2},\beta\right)$ is continuous in β, by the Intermediate Value Theorem there exists a value $\underline{\beta}_T \in (0,1)$ that solves $\Psi_T\left(\frac{1}{2},\beta\right) = 0$. The derivative $\frac{\partial \Psi_T\left(\frac{1}{2},\beta\right)}{\partial \beta} = f_T'(\beta)\left[u\left(\frac{1}{2}\right) - u(0)\right] - g_T'(\beta)\left[u(1) - u\left(\frac{1}{2}\right)\right] > 0$, since $f_T'(\beta) > g_T'(\beta)$ and $u\left(\frac{1}{2}\right) - u(0) > u(1) - u\left(\frac{1}{2}\right)$ by strict concavity of the utility function. Hence, $\underline{\beta}_T$ is unique. i. By Lemma A1, $y_T(\beta)$ is continuously differentiable with $\lim_{\beta \to \underline{\beta}_T^-} y_T(\beta) = \frac{1}{2}$, $\lim_{\beta \to 0^+} y_T(\beta) = 1$ and $\frac{\partial y_T(\beta)}{\partial \beta} < 0$. Hence, for $\beta \in \left[\underline{\beta}_T, 1\right)$, $y_T(\beta) \leq \frac{1}{2}$ and $\widehat{y}_T(\beta) = \frac{1}{2}$. ii. If $\beta \in \left(0, \underline{\beta}_T\right)$, once again by Lemma A1, $y_T(\beta) \in \left(\frac{1}{2}, 1\right)$ and $\widehat{y}_T(\beta) = y_T(\beta)$. Therefore, by definition of $\widehat{y}_T(\beta)$,

$$\widehat{y}_T(\beta) = \begin{cases} \frac{1}{2}, & \text{if } \beta \geq \underline{\beta}_T \\ y_T(\beta), & \text{if } \beta < \underline{\beta}_T \end{cases},$$

which is continuous in $\beta \in (0,1)$, since $\lim_{\beta \to \underline{\beta}_T^-} y_T(\beta) = \frac{1}{2}$. b. Since $T \leq \overline{T}$, $y = \frac{1}{2}$ never satisfies the participation constraint for any $\beta \in (0,1)$. A solution of $\Psi_T(y,\beta) = 0$ in $y \in \left(\frac{1}{2}, 1\right)$ exists for any $\beta \in (0,1)$, by the Intermediate Value Theorem, and is unique by the same argument used in part a. By Lemma A1, $y_T(\beta)$ is continuously differentiable in β with $\lim_{\beta \to 1^-} y_T(\beta) \geq \frac{1}{2}$, $\lim_{\beta \to 0^+} y_T(\beta) = 1$ and $\frac{\partial y_T(\beta)}{\partial \beta} < 0$. Hence, for all $\beta \in (0,1)$, $y_T(\beta) \in \left(\frac{1}{2}, 1\right)$. Therefore, by definition of $\widehat{y}_T(\beta)$, $\widehat{y}_T(\beta) = y_T(\beta)$, for all $\beta \in (0,1)$.■

Proof of Lemma 2. With $T = 0$, $\Gamma_0(\beta) = \{1\}$, hence, in this case the statement follows immediately. Consider $T > 0$. By Lemmas A2-A3, for any $\beta \in (0,1)$ and $T > 0$, $\Gamma_T(\beta) = [\widehat{y}_T(\beta), 1]$ is a non-empty, closed and bounded interval of the real line, hence, $\Gamma_T(\beta)$ is non-empty, compact and convex-valued. For any $T > 0$, the upper boundary of $\Gamma_T(\beta)$ is constant and the lower boundary, $\widehat{y}_T(\beta)$, varies continuously with β, by the previous Lemmas A2-A3, hence the correspondence $\Gamma_T(\beta)$ is continuous in β.■

The next Lemma is the formal proof of the statement made in the text that the set of sustainable allocations becomes larger for larger values of T. Define, for given T, $Gr(\Gamma_T) \equiv \left\{(y,\beta) \in \left[\frac{1}{2}, 1\right] \times (0,1) \mid y \in \Gamma_T(\beta)\right\}$, the graph of the correspondence Γ_T.

Lemma A4. *$Gr(\Gamma_T) \subset Gr(\Gamma_{T'}) \subset Gr(\Gamma_\infty)$, for any finite T', T with $T' > T$.*

Proof. (24) can be rewritten as $\Psi_T(y,\beta) =$

$$g_T(\beta)\left\{\frac{f_T(\beta)}{g_T(\beta)}\left[u(1-y) - u(0)\right] - \left[u(1) - u(y)\right]\right\} \geq 0. \tag{27}$$

The term $g_T(\beta) = \frac{1-\beta^{T+2}}{1-\beta^2}$ is clearly increasing in T. The term $\frac{f_T(\beta)}{g_T(\beta)} = \beta\left(\frac{1-\beta^T}{1-\beta^{T+2}}\right)$ $\leq \beta$, and approaches β when $T \to \infty$. Moreover, for any $\beta \in (0,1)$ and any T', T such that $T' > T \geq 0$, $\frac{f_T(\beta)}{g_T(\beta)} < \frac{f_{T'}(\beta)}{g_{T'}(\beta)}$, since $\beta\left(\frac{1-\beta^T}{1-\beta^{T+2}}\right) < \beta\left(\frac{1-\beta^{T'}}{1-\beta^{T'+2}}\right) \Leftrightarrow$ $\beta^T\left(1-\beta^2\right)\left(1-\beta^{T'-T}\right) > 0$. Hence, the LHS of (27) is strictly higher for larger T, for any given β and y.■

Section 5: Monetary Regime.

Lemma A5. *An allocation $\widetilde{x} \in \left[\frac{1}{2}, 1\right)$ that solves (18) exists and is unique for every $\pi \in [\beta - 1, \infty)$ and $\beta \in (0, 1)$.*

Proof. $\Phi(x, \pi, \beta)$ is (at least once) continuously differentiable in x, with $\Phi(1, \pi, \beta) = -\infty$, and $\Phi\left(\frac{1}{2}, \pi, \beta\right) = u'(x)\left(1 - \frac{\beta}{1+\pi}\right) \geq 0$. Hence, by the Intermediate Value Theorem, there exists a value $\widetilde{x} \in \left[\frac{1}{2}, 1\right)$ that solves (18) for any $\pi \in [\beta - 1, \infty)$ and $\beta \in (0, 1)$. Moreover, $\widetilde{x}$ is unique for any π and β, since $\frac{\partial \Phi(x,\pi,\beta)}{\partial x} = u''(x) + \frac{\beta}{1+\pi} u''(1 - x) < 0$ for all $\pi \in [\beta - 1, \infty)$, $\beta \in (0, 1)$ and $x \in \left[\frac{1}{2}, 1\right)$.■

Lemma A6. *a. The function $\widetilde{x}(\pi, \beta)$ is at least once continuously differentiable in π; b. i. $\widetilde{x}(\beta - 1, \beta) = \frac{1}{2}$, ii. $\lim_{\pi \to \infty} \widetilde{x}(\pi, \beta) = 1$ for any $\beta \in (0, 1)$; c. the derivative $\frac{\partial \widetilde{x}(\pi,\beta)}{\partial \pi} > 0$ for any $\beta \in (0, 1)$.*

Proof. Part a. and part c., follow from the Implicit Function Theorem, since $\frac{\partial \Phi(x,\pi,\beta)}{\partial x}\|_{x=\widetilde{x}} < 0$ from Lemma A5 and $\frac{\partial \Phi(x,\pi,\beta)}{\partial \pi}\|_{x=\widetilde{x}} = \frac{\beta}{(1+\pi)^2} u'(1 - \widetilde{x}) > 0$. Part b.i. is obvious from inspection of (18) and b. ii. from the Inada condition.■

Proof of Lemma 3. The set $\widetilde{\Gamma}(0, \beta) = [\widetilde{x}(0, \beta), 1]$ is non-empty, since $\widetilde{x}(0, \beta) < 1$ for any $\beta \in (0, 1)$, compact, convex-valued and continuous in β since $\widetilde{x}(0, \beta)$ is continuous in β by Lemma A6. The set $\left(\widetilde{\Gamma}(\beta - 1, \beta) \setminus \widetilde{\Gamma}(0, \beta)\right) = \left[\frac{1}{2}, 1\right] \setminus [\widetilde{x}(0, \beta), 1] = \left[\frac{1}{2}, \widetilde{x}(0, \beta)\right)$ is non-empty, since $\widetilde{x}(0, \beta) > \frac{1}{2}$, by (18) with $\pi = 0$, for any $\beta \in (0, 1)$. The set $\Gamma_T(\beta) = [\widehat{y}_T(\beta), 1]$ is non-empty, compact, convex-valued and continuous in β for any $T \geq 0$ by Lemma 2. The set $\left(\widetilde{\Gamma}(\beta - 1, \beta) \setminus \widetilde{\Gamma}(0, \beta)\right) \cap \Gamma_T(\beta) =$

$\left[\frac{1}{2}, \widetilde{x}(0,\beta)\right) \cap \left[\widehat{y}_T(\beta), 1\right]$ could be: 1. empty, if $\widehat{y}_T(\beta) \geq \widetilde{x}(0,\beta)$; or 2. equal to $\left[\widehat{y}_T(\beta), \widetilde{x}(0,\beta)\right)$, if $\widehat{y}_T(\beta) < \widetilde{x}(0,\beta)$. The set $\Gamma_T^M(\beta) = \left[\frac{1}{2}, \widetilde{x}(0,\beta)\right) \cap \left[\widehat{y}_T(\beta), 1\right] \cup \left[\widetilde{x}(0,\beta), 1\right]$ is equal to $\left[\widetilde{x}(0,\beta), 1\right]$ in case 1. and $\left[\widehat{y}_T(\beta), \widetilde{x}(0,\beta)\right) \cup \left[\widetilde{x}(0,\beta), 1\right] = \left[\widehat{y}_T(\beta), 1\right]$ in case 2. In either case, $\Gamma_T^M(\beta)$ is non-empty, compact, convex-valued and continuous in β for any $T \geq 0$.■

Section 6: Comparison of the Regimes.

Proof of Proposition 1. $\Gamma_T^M(\beta) \equiv \left(\left(\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta)\right) \cap \Gamma_T(\beta)\right) \cup \widetilde{\Gamma}(0,\beta)$ by definition. For any given $\beta \in (0,1)$ and $T \geq 0$, there are two possible cases: the intersection is empty or not. 1. $\left(\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta)\right) \cap \Gamma_T(\beta) = \varnothing$. Since $\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta) = \left[\frac{1}{2}, 1\right] \setminus \left[\widetilde{x}(0,\beta), 1\right] = \left[\frac{1}{2}, \widetilde{x}(0,\beta)\right)$ and $\Gamma_T(\beta) = \left[\widehat{y}_T(\beta), 1\right]$ for the intersection to be empty it must be the case that $\widehat{y}_T(\beta) \geq \widetilde{x}(0,\beta)$, therefore $\Gamma_T^M(\beta) = \left(\varnothing \cup \left[\widetilde{x}(0,\beta), 1\right]\right) = \left[\widetilde{x}(0,\beta), 1\right] \supseteq \left[\widehat{y}_T(\beta), 1\right] = \Gamma_T(\beta)$. Clearly, the inclusion is strict if $\widehat{y}_T(\beta) > \widetilde{x}(0,\beta)$, while the two sets coincide if $\widehat{y}_T(\beta) = \widetilde{x}(0,\beta)$. 2. $\left(\widetilde{\Gamma}(\beta-1,\beta) \setminus \widetilde{\Gamma}(0,\beta)\right) \cap \Gamma_T(\beta) \neq \varnothing$. For the intersection to be non-empty it must be the case that $\widehat{y}_T(\beta) < \widetilde{x}(0,\beta)$, therefore $\Gamma_T^M(\beta) = \left(\left[\frac{1}{2}, \widetilde{x}(0,\beta)\right) \cap \left[\widehat{y}_T(\beta), 1\right]\right) \cup \left[\widetilde{x}(0,\beta), 1\right] = \left[\widehat{y}_T(\beta), \widetilde{x}(0,\beta)\right) \cup \left[\widetilde{x}(0,\beta), 1\right] = \left[\widehat{y}_T(\beta), 1\right] = \Gamma_T(\beta)$.■

The proof of Proposition 2 requires some definitions and a preparatory Lemma. The ex-ante welfare functions in the non-monetary and monetary regimes are the same, given by

$$\frac{1}{1-\beta^2}\left\{\mu\left[u(h)+\beta u(1-h)\right]+(1-\mu)\left[u(1-h)+\beta u(h)\right]\right\}. \tag{28}$$

with $h \in \mathbb{R}_+$. Consider the problem of maximizing the ex-ante welfare function with the only constraint that the choice should be feasible, i.e. maximize (28) in $h \in \left[\frac{1}{2}, 1\right]$. The objective function (28) is (at least twice) continuously differentiable, strictly increasing and strictly concave in the choice variable, h. Hence, for any (μ, β) there exists a unique, global maximizer, which is characterized by the following necessary and sufficient conditions

$$\mu\left[u'(h) - \beta u'(1-h)\right] + (1-\mu)\left[-u'(1-h) + \beta u'(h)\right] - \rho + \nu = 0, \tag{29}$$

$$\rho(1-h) = 0, \tag{30}$$

$$\nu\left(h - \frac{1}{2}\right) = 0, \tag{31}$$

where $\rho \geq 0$ and $\nu \geq 0$ are the multipliers for the boundary conditions on h. Define $h^*(\mu, \beta)$:$[0,1] \times (0,1) \to \left[\frac{1}{2}, 1\right]$ as the function that satisfies (29), (30), (31). Define also $\widetilde{h}(\beta)$:$(0,1) \to \left[\frac{1}{2}, 1\right]$ as the function that satisfies

$$u'(h) - \beta u'(1-h) = 0, \tag{32}$$

for any $\beta \in (0,1)$. Such a function is continuous in $\beta \in (0,1)$, by the same argument used in Lemma A6 with $\pi = 0$.

Lemma A7. *$h^*(\mu, \beta) \leq \widetilde{h}(\beta)$ for all $\mu \in [0,1]$ at any $\beta \in (0,1)$.*

Proof. First, observe that $\rho\nu = 0$. Second, $\rho = 0$ always. Suppose $\rho > 0$, instead. By (30), $h = 1$ and (29) gives $\rho = -\infty$, which contradicts $\rho > 0$. Define

$$\Phi(h, \mu, \beta) \equiv \mu\left[u'(h) - \beta u'(1-h)\right] + (1-\mu)\left[-u'(1-h) + \beta u'(h)\right] + \nu = 0,$$

where $\Phi(1, \mu, \beta) = -\infty$, $\Phi\left(\frac{1}{2}, \mu, \beta\right) = (1-\beta)\, u'\left(\frac{1}{2}\right)(2\mu - 1) + \nu$, and

$$\frac{\partial \Phi(h, \mu, \beta)}{\partial h} = u''(h)(\mu + \beta - \mu\beta) + u''(1-h)(1 - \mu + \mu\beta) < 0.$$

Hence, for $\mu \in \left[0, \frac{1}{2}\right)$, $\nu > 0$ and $h^*(\mu, \beta) = \frac{1}{2} < \widetilde{h}(\beta)$ by (31) and (32) for any $\beta \in (0,1)$. For $\mu \in \left[\frac{1}{2}, 1\right]$, we have $\nu = 0$. Observe that $\Phi(h, 1, \beta) = u'(h) - \beta u'(1-h) = 0$, which gives $h^*(1, \beta) = \widetilde{h}(\beta)$ for any $\beta \in (0,1)$, and $\Phi\left(h, \frac{1}{2}, \beta\right) = \frac{1}{2}(1+\beta)\left[u'(h) - u'(1-h)\right] = 0$, which gives $h^*\left(\frac{1}{2}, \beta\right) = \frac{1}{2} < \widetilde{h}(\beta)$ for any $\beta \in (0,1)$. The derivative

$$\frac{\partial h^*(\mu, \beta)}{\partial \mu} = -\frac{(1-\beta)\left[u'(h) + u'(1-h)\right]}{u''(h)(\mu + \beta - \mu\beta) + u''(1-h)(1 - \mu + \mu\beta)} > 0.$$

The statement follows.■

Proof of Proposition 2. i. "if" part. From the Proof of Proposition 1, $\Gamma_T(\beta) \subset \Gamma_T^M(\beta) \Leftrightarrow \widehat{y}_T(\beta) > \widetilde{x}(0, \beta)$ for any given $\beta \in (0,1)$ and $T \geq 0$. From Lemma A7, $h^*(\mu, \beta) \leq \widetilde{h}(\beta)$ for all $\mu \in [0,1]$ at any given $\beta \in (0,1)$. By definition, $\widetilde{x}(0, \beta) \equiv \widetilde{h}(\beta)$ for any given $\beta \in (0,1)$. If $\widehat{y}_T(\beta) > \widetilde{x}(0, \beta)$ for some $\beta \in (0,1)$ and $T \geq 0$, we have $h^*(\mu, \beta) \leq \widetilde{h}(\beta) = \widetilde{x}(0, \beta) < \widehat{y}_T(\beta)$, for any given $\mu \in [0,1]$, at those values of $\beta \in (0,1)$ and $T \geq 0$. Since (28) is strictly concave in h and $h^*(\mu, \beta)$ is the global maximum for any given $\mu \in [0,1]$ and $\beta \in (0,1)$, the function (28) is strictly decreasing in h for any $h > h^*(\mu, \beta)$, for given $\mu \in [0,1]$ and $\beta \in (0,1)$. By definition, $W_T^*(\mu, \beta) = \max\left\{(28) \mid h \in \left[\widehat{y}_T(\beta), 1\right]\right\}$ and $W_T^{M*}(\mu, \beta) = \max\left\{(28) \mid h \in \left[\widetilde{x}(0, \beta), 1\right]\right\}$. The statement follows. ii. "only if" part. Suppose, $\Gamma_T(\beta) = \Gamma_T^M(\beta)$ for some $\beta \in (0,1)$ and $T \geq 0$. The objective functions (11) and (21) are identical. The statement follows by definition of $W_T^*(\mu, \beta)$ and $W_T^{M*}(\mu, \beta)$.■

Proof of Proposition 3. For any β, $\widetilde{x}(0, \beta)$ satisfies $\beta = \frac{u'(\widetilde{x})}{u'(1-\widetilde{x})}$. Moreover,

$\frac{u'(\widetilde{x})}{u'(1-\widetilde{x})} > \frac{u(1)-u(\widetilde{x})}{u(1-\widetilde{x})-u(0)}$ for any $\widetilde{x}$ by strict concavity of the utility function. Hence, $\Psi_\infty(\widetilde{x},\beta) > 0$, for any $\beta \in (0,1)$. Thus, for any $\beta \in (0,1)$, $\Gamma^M_\infty(\beta) = [\widehat{y}_\infty(\beta), 1] = \Gamma_\infty(\beta)$.■

Proof of Proposition 4. For any $T < \infty$, $\widehat{y}_T(\beta)$ and $\widetilde{x}(0,\beta)$ are continuous in $\beta \in (0,1)$, by Lemmas A2-A3 and A6 respectively. When $T < \overline{T}$, we have $\lim_{\beta\to 1}\widehat{y}_T(\beta) = \overline{y}_T > \frac{1}{2}$; moreover, $\lim_{\beta\to 1}\widetilde{x}(0,\beta) = \frac{1}{2}$. Therefore, by continuity, there exists an interval $B_T \subseteq (0,1)$ with non-empty interior, such that $\widehat{y}_T(\beta) > \widetilde{x}(0,\beta)$, for $\beta \in B_T$, and, thus, $\Gamma_T(\beta) = [\widehat{y}_T(\beta), 1] \subset [\widetilde{x}(0,\beta), 1] = \Gamma^M_T(\beta)$, for $\beta \in B_T$.■

Section 7: Discriminatory Transfers.

The proof that the set of allocations that satisfies (22) and (23) simultaneously is not empty requires some definitions. Let $\sigma(\beta,T) \equiv \frac{g_T(\beta)}{h_T(\beta)} = \frac{1-\beta^{T+2}}{(1+\beta)\left(1-\beta^{T+1}\right)}$. Notice that $\frac{f_T(\beta)}{h_T(\beta)} = \frac{\beta\left(1-\beta^T\right)}{(1+\beta)\left(1-\beta^{T+1}\right)} = 1-\sigma(\beta,T)$. Define also $\widehat{v}(\beta,T) \equiv \sigma(\beta,T)\,u(1) + (1-\sigma(\beta,T))\,u(0)$ and $\widetilde{v}(\beta,T) \equiv (1-\sigma(\beta,T))\,u(1) + \sigma(\beta,T)\,u(0)$. Let

$$Z(\beta,T) \equiv \left\{z \in [0,1] : z \geq u^{-1}(\widehat{v}(\beta,T)) \text{ and } z \leq 1 - u^{-1}(\widetilde{v}(\beta,T))\right\},$$

which identifies the allocations that can be sustained as a monetary equilibrium with discriminatory transfers. Define $Int(Z(\beta,T)) \equiv (u^{-1}(\widehat{v}(\beta,T)), 1-u^{-1}(\widetilde{v}(\beta,T)))$, the interior of $Z(\beta,T)$.

Lemma A8. *$Int(Z(\beta,T)) \neq \varnothing$ for any $\beta \in (0,1)$ and $T > 0$.*

Proof. By strict concavity of the utility function, $\widehat{v}(\beta,T) < u(\sigma(\beta,T))$, and $\widetilde{v}(\beta,T) < u(1-\sigma(\beta,T))$, for any $\beta \in (0,1)$ and $T > 0$. Since the utility function is strictly increasing, we can invert it and obtain

$$u^{-1}\left(\widehat{v}\left(\beta,T\right)\right)<\sigma\left(\beta,T\right)=1-\left(1-\sigma\left(\beta,T\right)\right)<1-u^{-1}\left(\widetilde{v}\left(\beta,T\right)\right),$$

for any $\beta\in(0,1)$ and $T>0$, which proves our statement.■