Endogenous longevity and the joint dynamics of pollution and capital accumulation

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Online Appendix

A. The Phase Diagram (Figure 1)

The PS locus is derived from points that satisfy $\mu_t = \mathrm{M}(k_t, \mu_t)$. From (26) we get $\mu_t = \frac{pB}{1-\eta} k_t^{\beta} \equiv G(k_t)$ such that $G' > 0$, $G(0) = 0$ and $G(\infty) \to \infty$. The CS locus is derived from points that satisfy $k_t = \mathrm{K}(k_t, \mu_t)$. Using $k_{t+1} = k_t$ in (25) and rearranging yields $\frac{k_t^{1-\beta}}{\Psi\left(\frac{Bk_t^{\beta}}{\mu_t}\right)(\Theta - k_t^{1-\beta})} = 1$ or, alternatively, $\Phi(k_t, \mu_t) = 1$. First, we can check that $\Phi_{\mu_t} = \frac{k_t^{1-\beta}}{(\Theta - k_t^{1-\beta})} \frac{\Psi'}{[\Psi(\cdot)]^2} \frac{Bk_t^{\beta}}{\mu_t^2} > 0$. The next step is to analyse the derivative, $\Phi_{k_t} = \frac{(1-\beta)k_t^{-\beta}}{\Psi(\cdot)(\Theta - k_t^{1-\beta})} + \frac{k_t^{1-\beta}(1-\beta)k_t^{-\beta}}{\Psi(\cdot)(\Theta - k_t^{1-\beta})^2} - \frac{k_t^{1-\beta}\Psi'}{[\Psi(\cdot)]^2(\Theta - k_t^{1-\beta})} \frac{\beta B k_t^{\beta-1}}{\mu_t}$ or (after factorisation) $\Phi_{k_t} = \frac{k_t^{-\beta}}{\Psi(\cdot)(\Theta - k_t^{1-\beta})}\left[(1-\beta) + \frac{(1-\beta)k_t^{1-\beta}}{(\Theta - k_t^{1-\beta})} - \beta \frac{\Psi'}{\Psi(\cdot)} \frac{Bk_t^{\beta}}{\mu_t}\right]$. We know that $\frac{\Psi(x_t)}{x_t} > \Psi'(x_t)$ therefore $\frac{1}{x_t} > \frac{\Psi'(x_t)}{\Psi(x_t)}$. If we replace $\frac{1}{x_t}$ for $\frac{\Psi'(x_t)}{\Psi(x_t)}$ in the third term of the expression inside brackets, and then add the first term of the same expression, we get $1 - \beta - \frac{\beta}{x_t} \frac{Bk_t^{\beta}}{\mu_t}$.[1] After substituting (17) and $\overline{Y}_t = y_t = Bk_t^{\beta}$, this expression becomes $1 - 2\beta$ which is non-negative given that $\beta \leq \frac{1}{2}$ holds by assumption. However, if this expression is non-negative when using $\frac{1}{x_t}$ then it is certainly positive when using $\frac{\Psi'(x_t)}{\Psi(x_t)} < \frac{1}{x_t}$. Consequently, $\Phi_{k_t} > 0$ and equation (25) defines a function

[1] The second term $\frac{(1-\beta)k_t^{1-\beta}}{(\Theta - k_t^{1-\beta})}$ is obviously positive.

$\mu_t \equiv J(k_t)$ such that $J' = -\dfrac{\Phi_{k_t}}{\Phi_{\mu_t}} < 0$. In addition, $\mu_t = 0$ implies $\Psi(\cdot) = \lambda$ and $k_t = \left(\Theta \dfrac{\lambda}{1+\lambda}\right)^{\frac{1}{1-\beta}}$ while $\mu_t \to \infty$ implies $\Psi(\cdot) = 0$ and $k_t = 0$. The construction of the diagram is completed by observing that $\mathrm{K}_{\mu_t} < 0$ (see Appendix B below) and $\mathrm{M}_{k_t} > 0$. These imply that above (below) the CS schedule we have $k_{t+1} < k_t$ ($k_{t+1} > k_t$) and on the left (right) of the PS schedule we have $\mu_{t+1} < \mu_t$ ($\mu_{t+1} > \mu_t$). ■

B. Proof of Lemma 1

The Jacobian matrix associated with the dynamical system of (25) and (26) is

$$\begin{pmatrix} \mathrm{K}_{k_t}(\hat{k},\hat{\mu}) & \mathrm{K}_{\mu_t}(\hat{k},\hat{\mu}) \\ \mathrm{M}_{k_t}(\hat{k},\hat{\mu}) & \mathrm{M}_{\mu_t}(\hat{k},\hat{\mu}) \end{pmatrix}.$$

The trace and the determinant are given by $T = \mathrm{K}_{k_t}(\hat{k},\hat{\mu}) + \mathrm{M}_{\mu_t}(\hat{k},\hat{\mu})$ and $D = \mathrm{K}_{k_t}(\hat{k},\hat{\mu})\mathrm{M}_{\mu_t}(\hat{k},\hat{\mu}) - \mathrm{K}_{\mu_t}(\hat{k},\hat{\mu})\mathrm{M}_{k_t}(\hat{k},\hat{\mu})$ respectively. It is well known that the stability of the equilibrium is established when the conditions $(1+D-T) > 0$, $(1+D+T) > 0$, $|D| < 1$ and $T \in (-2,2)$ hold simultaneously.

From equation (25), we have

$$\mathrm{K}_{k_t}(\hat{k},\hat{\mu}) = \Theta\left\{\beta\hat{k}^{\beta-1}\frac{\Psi(\cdot)}{1+\Psi(\cdot)} + \hat{k}^{\beta}\frac{\Psi'(\cdot)}{[1+\Psi(\cdot)]^2}\frac{\beta B\hat{k}^{\beta-1}}{\hat{\mu}}\right\} > 0. \tag{A1}$$

Substituting (27) and (28) in (A1) yields

$$\mathrm{K}_{k_t}(\hat{k},\hat{\mu}) = \beta\Theta\left\{\Theta^{\frac{\beta-1}{1-\beta}}\left[\frac{\Psi(\cdot)}{1+\Psi(\cdot)}\right]^{\frac{\beta-1}{1-\beta}}\frac{\Psi(\cdot)}{1+\Psi(\cdot)} + \frac{\Psi'(\cdot)}{[1+\Psi(\cdot)]^2}\frac{B\hat{k}^{2\beta-1}}{\frac{pB}{1-\eta}\hat{k}^{\beta}}\right\} \Rightarrow$$

$$\mathrm{K}_{k_t}(\hat{k},\hat{\mu}) = \beta\Theta\left\{\frac{1}{\Theta} + \frac{\Psi'(\cdot)}{[1+\Psi(\cdot)]^2}\frac{(1-\eta)\hat{k}^{\beta-1}}{p}\right\} \Rightarrow$$

$$\mathrm{K}_{k_t}(\hat{k},\hat{\mu}) = \beta\Theta\left\{\frac{1}{\Theta} + \frac{\Psi'(\cdot)}{[1+\Psi(\cdot)]^2}\frac{1-\eta}{p}\Theta^{\frac{\beta-1}{1-\beta}}\left[\frac{\Psi(\cdot)}{1+\Psi(\cdot)}\right]^{\frac{\beta-1}{1-\beta}}\right\} \Rightarrow$$

$$\mathrm{K}_{k_t}(\hat{k},\hat{\mu}) = \beta\left\{1 + \frac{\Psi'(\cdot)}{\Psi(\cdot)[1+\Psi(\cdot)]}\frac{1-\eta}{p}\right\}. \tag{A2}$$

Let us consider the expression

$$\beta\left(1 + \frac{1}{\hat{x}}\frac{1-\eta}{p}\right). \tag{A3}$$

In the steady-state we have $\hat{x} = \frac{\hat{y}}{\hat{\mu}} = \frac{B\hat{k}^{\beta}}{pB\hat{k}^{\beta}/(1-\eta)} = \frac{1-\eta}{p}$, therefore (A3) becomes 2β. Of course, $2\beta \le 1$ given that $\beta \le 1/2$ by assumption. Since $\frac{\Psi(\hat{x})}{\hat{x}} > \Psi'(\hat{x})$ also holds then $\frac{1}{\hat{x}} > \frac{\Psi'(\hat{x})}{\Psi(\hat{x})} > \frac{\Psi'(\hat{x})}{\Psi(\hat{x})[1+\Psi(\hat{x})]}$. Consequently, if (A3) cannot take a value above unity then, from (A2), it is certainly $0 < \mathrm{K}_{k_t}(\hat{k},\hat{\mu}) < 1$.

Using equation (26) we get $\mathrm{M}_{\mu_t}(\hat{k},\hat{\mu}) = \eta \in (0,1)$ which implies that $0 < T = \eta + \mathrm{K}_{k_t}(\hat{k},\hat{\mu}) < 2$, thus the condition $T \in (-2,2)$ is satisfied. Furthermore, we can use (25) and (26) to derive

$$\mathrm{M}_{k_t}(\hat{k},\hat{\mu}) = p\beta B\hat{k}^{\beta-1} > 0, \tag{A4}$$

and

$$\mathrm{K}_{\mu_t}(\hat{k},\hat{\mu}) = \Theta\hat{k}^{\beta}\frac{\Psi'(\cdot)}{[1+\Psi(\cdot)]^2}\left(-\frac{B\hat{k}^{\beta}}{\hat{\mu}^2}\right) < 0. \tag{A5}$$

Thus, (A4) and (A5), combined with previous results, imply that $D = \eta K_{k_t}(\hat{k},\hat{\mu}) - K_{\mu_t}(\hat{k},\hat{\mu}) M_{k_t}(\hat{k},\hat{\mu}) > 0$ and $1 + D + T > 0$. Additionally, we can derive

$$D - T + 1 = \eta K_{k_t}(\hat{k},\hat{\mu}) - K_{\mu_t}(\hat{k},\hat{\mu}) M_{k_t}(\hat{k},\hat{\mu}) - \eta - K_{k_t}(\hat{k},\hat{\mu}) + 1 \Rightarrow$$

$$D - T + 1 = 1 - \eta - (1-\eta) K_{k_t}(\hat{k},\hat{\mu}) - K_{\mu_t}(\hat{k},\hat{\mu}) M_{k_t}(\hat{k},\hat{\mu}) \Rightarrow$$

$$D - T + 1 = (1-\eta)[1 - K_{k_t}(\hat{k},\hat{\mu})] - K_{\mu_t}(\hat{k},\hat{\mu}) M_{k_t}(\hat{k},\hat{\mu}) .$$

Given (A4), (A5) and $0 < K_{k_t}(\hat{k},\hat{\mu}) < 1$, we have $D - T + 1 > 0$. Consequently, since $D > 0$, we need to show that $D < 1$ in order to establish the stability of the equilibrium.

Substitution of (28) in (A5) yields

$$K_{\mu_t}(\hat{k},\hat{\mu}) = -\Theta \frac{\Psi'(\cdot)}{[1+\Psi(\cdot)]^2} \frac{B\hat{k}^{2\beta}}{(pB)^2 \hat{k}^{2\beta} / (1-\eta)^2} \Rightarrow$$

$$K_{\mu_t}(\hat{k},\hat{\mu}) = \frac{-\Theta(1-\eta)^2}{p^2 B} \frac{\Psi'(\cdot)}{[1+\Psi(\cdot)]^2} . \qquad \text{(A6)}$$

Using (27) in (A4) yields

$$M_{k_t}(\hat{k},\hat{\mu}) = p\beta B \Theta^{\frac{\beta-1}{1-\beta}} \left[\frac{\Psi(\cdot)}{1+\Psi(\cdot)} \right]^{\frac{\beta-1}{1-\beta}} \Rightarrow$$

$$M_{k_t}(\hat{k},\hat{\mu}) = \frac{p\beta B}{\Theta} \frac{1+\Psi(\cdot)}{\Psi(\cdot)} . \qquad \text{(A7)}$$

Combining (A6) and (A7), we can derive

$$K_{\mu_t}(\hat{k},\hat{\mu}) M_{k_t}(\hat{k},\hat{\mu}) = \frac{-\Theta(1-\eta)^2}{p^2 B} \frac{\Psi'(\cdot)}{[1+\Psi(\cdot)]^2} \frac{p\beta B}{\Theta} \frac{1+\Psi(\cdot)}{\Psi(\cdot)} \Rightarrow$$

$$K_{\mu_t}(\hat{k},\hat{\mu}) M_{k_t}(\hat{k},\hat{\mu}) = \frac{-\beta(1-\eta)^2}{p} \frac{\Psi'(\cdot)}{\Psi(\cdot)[1+\Psi(\cdot)]} . \qquad \text{(A8)}$$

Next, we can combine (A2) and (A8) to derive the determinant

$$D = \eta \mathrm{K}_{k_t}(\hat{k}, \hat{\mu}) - \mathrm{K}_{\mu_t}(\hat{k}, \hat{\mu}) \mathrm{M}_{k_t}(\hat{k}, \hat{\mu}) \Rightarrow$$

$$D = \eta\beta + \eta\beta \frac{\Psi'(\cdot)}{\Psi(\cdot)[1+\Psi(\cdot)]} \frac{1-\eta}{p} + \frac{\beta(1-\eta)^2}{p} \frac{\Psi'(\cdot)}{\Psi(\cdot)[1+\Psi(\cdot)]} \Rightarrow$$

$$D = \beta \left\{ \eta + \frac{\Psi'(\cdot)}{\Psi(\cdot)[1+\Psi(\cdot)]} \frac{1-\eta}{p} [\eta + (1-\eta)] \right\} \Rightarrow$$

$$D = \beta \left\{ \eta + \frac{\Psi'(\cdot)}{\Psi(\cdot)[1+\Psi(\cdot)]} \frac{1-\eta}{p} \right\}. \tag{A9}$$

Now, consider the expression

$$\beta \left(\eta + \frac{1}{\hat{x}} \frac{1-\eta}{p} \right). \tag{A10}$$

In the steady-state we have $\hat{x} = \frac{1-\eta}{p}$. Substituting in (A10) yields $\beta(1+\eta) < 1$ because $\beta \leq 1/2$ and $0 < \eta < 1$. However, it is $\frac{1}{\hat{x}} > \frac{\Psi'(\hat{x})}{\Psi(\hat{x})} > \frac{\Psi'(\hat{x})}{\Psi(\hat{x})[1+\Psi(\hat{x})]}$ because $\frac{\Psi(\hat{x})}{\hat{x}} > \Psi'(\hat{x})$ holds by assumption. This implies that, if (A10) is below 1, then, given (A9), we can conclude that $D < 1$ as well. Hence, we have proven that the equilibrium $\hat{k}, \hat{\mu} > 0$ is locally stable. ■

C. Proof of Proposition 1

From (27) we can derive

$$\frac{\partial \hat{k}}{\partial p} = \frac{1}{1-\beta} \left[\frac{\Theta\Psi(\cdot)}{1+\Psi(\cdot)} \right]^{\frac{1}{1-\beta}-1} \frac{\Theta\Psi'(\cdot)}{[1+\Psi(\cdot)]^2} \left(-\frac{1-\eta}{p^2} \right) < 0,$$

$$\frac{\partial \hat{k}}{\partial \eta} = \frac{1}{1-\beta} \left[\frac{\Theta\Psi(\cdot)}{1+\Psi(\cdot)} \right]^{\frac{1}{1-\beta}-1} \frac{\Theta\Psi'(\cdot)}{[1+\Psi(\cdot)]^2} \left(-\frac{1}{p} \right) < 0,$$

$$\frac{\partial \hat{k}}{\partial \sigma} = \frac{1}{1-\beta}\left[\frac{\Theta\Psi(\cdot)}{1+\Psi(\cdot)}\right]^{\frac{1}{1-\beta}-1}\left[\frac{(1-\beta)\Psi(\cdot)}{1+\Psi(\cdot)}\frac{1}{\sigma^2}\right] > 0,$$

and

$$\frac{\partial \hat{k}}{\partial B} = \frac{1}{1-\beta}B^{\frac{1}{1-\beta}-1}\left[\frac{\sigma-1}{\sigma}\frac{(1-\beta)\Psi(\cdot)}{1+\Psi(\cdot)}\right]^{\frac{1}{1-\beta}} > 0.$$

From (28) we have

$$\frac{\partial \hat{\mu}}{\partial B} = \frac{p}{1-\eta}\hat{k}^{\frac{\beta}{1-\beta}} + \frac{pB}{1-\eta}\frac{\beta}{1-\beta}\hat{k}^{\frac{\beta}{1-\beta}-1}\frac{\partial \hat{k}}{\partial B} > 0,$$

$$\frac{\partial \hat{\mu}}{\partial \sigma} = \frac{pB}{1-\eta}\frac{\beta}{1-\beta}\hat{k}^{\frac{\beta}{1-\beta}-1}\frac{\partial \hat{k}}{\partial \sigma} > 0,$$

$$\begin{aligned}\frac{\partial \hat{\mu}}{\partial p} = {} & \frac{B}{1-\eta}\left[\frac{\Theta\Psi(\cdot)}{1+\Psi(\cdot)}\right]^{\frac{\beta}{1-\beta}} + \frac{pB}{1-\eta}\times \\ & \frac{\beta}{1-\beta}\left[\frac{\Theta\Psi(\cdot)}{1+\Psi(\cdot)}\right]^{\frac{\beta}{1-\beta}-1}\frac{\Theta\Psi'(\cdot)}{[1+\Psi(\cdot)]^2}\left(-\frac{1-\eta}{p^2}\right),\end{aligned} \tag{A11}$$

and

$$\begin{aligned}\frac{\partial \hat{\mu}}{\partial \eta} = {} & \frac{Bp}{(1-\eta)^2}\left[\frac{\Theta\Psi(\cdot)}{1+\Psi(\cdot)}\right]^{\frac{\beta}{1-\beta}} + \frac{pB}{1-\eta}\times \\ & \frac{\beta}{1-\beta}\left[\frac{\Theta\Psi(\cdot)}{1+\Psi(\cdot)}\right]^{\frac{\beta}{1-\beta}-1}\frac{\Theta\Psi'(\cdot)}{[1+\Psi(\cdot)]^2}\left(-\frac{1}{p}\right)\end{aligned} \tag{A12}$$

After some manipulation, equations (A11) and (A12) can be written as

$$\frac{\partial \hat{\mu}}{\partial p} = \frac{B}{1-\eta}\left[\frac{\Theta\Psi(\cdot)}{1+\Psi(\cdot)}\right]^{\frac{\beta}{1-\beta}}\left[1-\frac{\beta}{1-\beta}\frac{1-\eta}{p}\frac{\Psi'(\cdot)}{\Psi(\cdot)[1+\Psi(\cdot)]}\right], \tag{A13}$$

and

$$\frac{\partial \hat{\mu}}{\partial \eta} = \frac{Bp}{(1-\eta)^2}\left[\frac{\Theta\Psi(\cdot)}{1+\Psi(\cdot)}\right]^{\frac{\beta}{1-\beta}}\left[1-\frac{\beta}{1-\beta}\frac{1-\eta}{p}\frac{\Psi'(\cdot)}{\Psi(\cdot)[1+\Psi(\cdot)]}\right], \tag{A14}$$

respectively. Now consider the expression $\frac{\beta}{1-\beta}\frac{1-\eta}{p}\frac{1}{\hat{x}}$ which, given $\hat{x}=\frac{1-\eta}{p}$, equals $\frac{\beta}{1-\beta}\leq 1$ because $\beta \leq 1/2$ holds. However, we know that $\frac{1}{\hat{x}} > \frac{\Psi'(\hat{x})}{\Psi(\hat{x})} > \frac{\Psi'(\hat{x})}{\Psi(\hat{x})[1+\Psi(\hat{x})]}$ holds by assumption. Taking account of equations (20) and (21), we conclude that $\frac{\partial \hat{\mu}}{\partial p} > 0$ and $\frac{\partial \hat{\mu}}{\partial \eta} > 0$. ■