Resource augmenting R&D with heterogeneous labor supply
Appendix

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August 29, 2007

Keywords: Exhaustible resources, R&D, Labor allocation, Education policy, Adjustment costs

JEL Classification Codes: Q010, Q300, I200, J000

This note is an appendix to the paper with the same title published on *Environment and Development Economics* (special issue on sustainable development, forthcoming 2007). The first two sections of the appendix provide detailed proofs of propositions 1 and 2. the third section presents the examples of effective labor possibilities frontier (ELPF) that are presented in expressions (1)-(2) at the end of section 2 in the paper. It then goes on presenting another class of special an interesting cases, and showing how it is possible to obtain the ELPF. The fourth section presents in detail the analysis of the dynamic system given by equations (14) and (15) resulting from the solution of the welfare maximization problem set up in section 3 of the paper. The last section presents the formal analysis of the impact of a skill-neutral demographic expansion, as considered in section 4.1 in the paper.

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1 Proof of Proposition 1

To prove the efficiency of the allocation rule consider the following perturbation. Take any R&D skill $\tilde{\nu} \in (0, \bar{\nu})$ and two different production skills $\lambda_1, \lambda_2 \in \left(0, \bar{\lambda}\right)$ with $\lambda_1 < \hat{\theta}(n)\tilde{\nu} < \lambda_2$. Draw a circle of area Δ_1 and diameter δ_1 around point $\{\tilde{\nu}, \lambda_1\}$, small enough so that all the workers with productivity bundles within the circle are initially assigned to R&D, producing $n_1 = \int_{\nu\in\Delta_1} \nu \left(\int_{\lambda\in\Delta_1} \tilde{g}(\nu, \lambda)\, d\lambda\right) dv$. Relocate them to production to obtain $l_1 = \int_{\lambda\in\Delta_1} \lambda \left(\int_{\nu\in\Delta_1} \tilde{g}(\nu, \lambda)\, d\nu\right) d\lambda$ instead. In order to maintain R&D inputs unchanged a sufficient mass of workers initially assigned to production must be relocated from the circle of area Δ_2 around point $\{\tilde{\nu}, \lambda_2\}$, by choosing its diameter δ_2 just large enough to satisfy $\int_{\nu\in\Delta_2} \nu \left(\int_{\lambda\in\Delta_2} \tilde{g}(\nu, \lambda)\, d\lambda\right) dv = n_1$. This reallocation implies a reduction of labor inputs to production by $l_2 = \int_{\lambda\in\Delta_2} \lambda \left(\int_{\nu\in\Delta_2} \tilde{g}(\nu, \lambda)\, d\nu\right) d\lambda$. Overall the reallocation will, by definition of δ_2, maintain the R&D inputs constant. However the reallocation will reduce the labor inputs to production, i.e., $l_2 < l_1$. In fact relocated individuals have similar R&D skill (about $\tilde{\nu}$) but those with higher production skill (about λ_2) specialize in R&D and those with relatively low production skill (about λ_1) specialize in production.

2 Proof of Proposition 2

The frontier is decreasing because

$$\frac{d\hat{l}(n)}{dn} = \frac{\partial\tilde{l}(\theta)/\partial\theta}{\partial\tilde{n}(\theta)/\partial\theta} < 0$$

The sign is established using results obtained in the paragraph preceding the proposition.

We first obtain the explicit expression of the ELPF in the case of a one-point mass distributions. Then we do the same for a two-points mass distributions. Finally we consider the case of a continuous distribution.

One point distribution. Assume a degenerate population density, $\tilde{g}(\nu, \lambda) = g > 0$ only at one skill bundle $\{\bar{\nu}, \bar{\lambda}\} \in \Gamma$, with $\bar{\nu}, \bar{\lambda} > 0$, and $\tilde{g}(\nu, \lambda) = 0$ over the rest of Γ. We have $P = g$, $\bar{n} = g\bar{\nu}$ and $\bar{l} = g\bar{\lambda}$. Since $\forall i \in [0, P]$ $\theta_i = \bar{\theta} \equiv \bar{\lambda}/\bar{\nu}$ according to Proposition 1 a worker can indifferently be assigned to any sector. The ELPF is linear

$$\hat{l}(n) = g\bar{\lambda} - \bar{\theta}n \quad ; \quad n \in [0, \bar{n}]$$

Two-points distribution. Assume $\tilde{g}(\nu_1, \lambda_1) = g_1 > 0$ and $\tilde{g}(\nu_2, \lambda_2) = g_2 > 0$ only at $\{\nu_1, \lambda_1\}, \{\nu_2, \lambda_2\} \in \Gamma$, with $\nu_1, \lambda_1, \nu_2, \lambda_2 > 0$, and $\tilde{g}(\nu, \lambda) = 0$ over the rest of Γ. It follows that $P = g_1 + g_2$, $\bar{n} = g_1\nu_1 + g_2\nu_2$ and $\bar{l} = g_1\lambda_1 + g_2\lambda_2$. Individuals are characterized by one of the two possible relative skill indices $\theta_1 = \lambda_1/\nu_1$ or $\theta_2 = \lambda_2/\nu_2$. According to Proposition 1, first workers characterized by the lowest relative skill index

$\theta_a = \arg\min\{\theta_1, \theta_2\}$ should be assigned to R&D and only once all of them are employed those characterized by $\theta_b = \arg\max\{\theta_1, \theta_2\}$ can be employed in R&D. This allocation rule defines two possible thresholds, $\tilde{n} = g_1\nu_1$ and $\tilde{l} = g_2\lambda_2$ if $\theta_a = \theta_1$ (i.e. $\theta_1 < \theta_2$), while $\tilde{n} = g_2\nu_2$ and $\tilde{l} = g_1\lambda_1$ if $\theta_a = \theta_2$ (i.e. $\theta_2 < \theta_1$). This leads to the ELPF

$$\hat{l}(n) = \begin{cases} \tilde{l} + \theta_a(\tilde{n} - n) & \text{for} \quad n \in [0, \tilde{n}] \\ \tilde{l} + \theta_b(\tilde{n} - n) & \text{for} \quad n \in [\tilde{n}, \bar{n}] \end{cases}$$

The ELPF is linear if $\theta_1 = \theta_2$, otherwise it is piecewise linear and quasi-concave.

The same procedure applies to the case of more than two-points in a discontinuous distribution with any finite number of points with positive population mass. The efficient allocation rule implies an ELPF that is piece-wise linear and quasi-concave.

Continuous distribution of the population over Γ. In the special case of a distribution admitting positive mass only on a subset of a ray going through the origin, $\forall i \in [0, P]$ $\theta_i = \bar{\theta}$ so that the ELPF is linear. Otherwise $\theta_i \in [\theta_a, \theta_b]$ where $\theta_a = \arg\min_{i\in[0,P]}\{\theta_i\}$. To increase effective labor inputs in R&D it is necessary to raise the amount of raw labor assigned to R&D, and according to the rule of efficient labor allocation this requires a larger cut-off ray $\hat{\theta}(n)$. Since $\hat{\theta}(n)$ is a continuous function, the slope of the ELPF increases continuously with n.

3 Examples of effective labor possibilities frontiers

3.1 The ELPF with uniform population distribution over Γ

Using the results obtained in the first part of section 2 we get

$$\bar{n} = \int_0^{\bar{\lambda}} \int_0^{\bar{\nu}} \tilde{g}(\nu, \lambda)\, \nu d\lambda d\nu = g\bar{\lambda} \left| \frac{\nu^2}{2} \right|_0^{\bar{\nu}} = g\bar{\lambda}\bar{\nu}^2/2 = P\bar{\nu}/2$$

$$\bar{l} = \int_0^{\bar{\lambda}} \int_0^{\bar{\nu}} \tilde{g}(\nu, \lambda)\, \lambda d\lambda d\nu = g\bar{\nu} \left| \frac{\lambda^2}{2} \right|_0^{\bar{\lambda}} = g\bar{\lambda}^2\bar{\nu}/2 = P\bar{\lambda}/2$$

where population size is $P = \int_0^{\bar{\nu}} \int_0^{\bar{\lambda}} g d\lambda d\nu = g\bar{\lambda}\bar{\nu}$. For $\theta \in \left[0, \bar{\theta}\right]$, we have

$$n_1(\theta) = \int_0^{\bar{\nu}} \nu \left(\int_0^{\theta\nu} \tilde{g}(\nu, \lambda)\, d\lambda \right) d\nu = \int_0^{\bar{\nu}} g\theta\nu^2 d\nu = g\theta\frac{\bar{\nu}^3}{3}$$

$$\begin{aligned} l_1(\theta) &= \int_0^{\tilde{\lambda}=\theta\bar{\nu}} \lambda \left(\int_0^{\lambda/\theta} \tilde{g}(\nu, \lambda)\, d\nu \right) d\lambda + \int_{\tilde{\lambda}=\theta\bar{\nu}}^{\bar{\lambda}} \lambda \left(\int_0^{\bar{\nu}} \tilde{g}(\nu, \lambda)\, d\nu \right) d\lambda \\ &= \int_0^{\tilde{\lambda}=\theta\bar{\nu}} g\lambda^2/\theta d\lambda + \int_{\tilde{\lambda}=\theta\bar{\nu}}^{\bar{\lambda}} g\lambda\bar{\nu} d\lambda = g\bar{\lambda}^2\bar{\nu}/2 - g\theta^2\bar{\nu}^3/6 \end{aligned}$$

and for $\theta \in \left[\bar{\theta}, \infty\right]$ we have

$$\begin{aligned} n_2(\theta) &= \int_0^{\tilde{\nu}=\bar{\lambda}/\theta} \nu \left(\int_0^{\theta\nu} \tilde{g}(\nu,\lambda)\, d\lambda \right) d\nu + \int_{\tilde{\nu}=\bar{\lambda}/\theta}^{\bar{\nu}} \nu \left(\int_0^{\bar{\lambda}} \tilde{g}(\nu,\lambda)\, d\lambda \right) d\nu \\ &= \int_0^{\tilde{\nu}=\bar{\lambda}/\theta} g\theta\nu^2 d\nu + \int_{\tilde{\nu}=\bar{\lambda}/\theta}^{\bar{\nu}} \nu g \bar{\lambda} d\nu = g\bar{\lambda}\bar{\nu}^2/2 - g\bar{\lambda}^3/(6\theta^2) \end{aligned}$$

$$l_2(\theta) = \int_0^{\bar{\lambda}} \lambda \left(\int_0^{\lambda/\theta} \tilde{g}(\nu,\lambda)\, d\nu \right) d\lambda = \int_0^{\bar{\lambda}} g\lambda^2/\theta d\lambda = g\bar{\lambda}^3/(3\theta)$$

The threshold levels are computed for $\theta = \bar{\theta}$

$$\begin{aligned} \tilde{n} \equiv n_1\left(\bar{\theta}\right) &= \int_0^{\bar{\nu}} \nu \left(\int_0^{\bar{\theta}\nu} \tilde{g}(\nu,\lambda)\, d\lambda \right) d\nu = \int_0^{\bar{\nu}} g\bar{\theta}\nu^2 d\nu = 2\bar{n}/3 \\ \tilde{l} \equiv l_2\left(\bar{\theta}\right) &= \int_0^{\bar{\lambda}} \lambda \left(\int_0^{\lambda/\bar{\theta}} \tilde{g}(\nu,\lambda)\, d\nu \right) d\lambda = \int_0^{\bar{\lambda}} g\lambda^2/\bar{\theta} d\lambda = 2\bar{l}/3 \end{aligned}$$

Substituting for θ we obtain the ELPF

$$\hat{l}(n) = \begin{cases} \bar{l} - \frac{3}{2}n^2/\left(g\bar{\nu}^3\right) & \text{for} \quad n \in [0, \tilde{n}] \\ \left(\frac{2}{3}g\bar{\lambda}^3\right)^{1/2} (\bar{n}-n)^{1/2} & \text{for} \quad n \in [\tilde{n}, \bar{n}] \end{cases}$$

The ELPF is strictly concave since $\hat{l}'(n) = -3n/\left(g\bar{\nu}^3\right) < 0$ and $\hat{l}''(n) = -3/\left(g\bar{\nu}^3\right) < 0$ for $n \leq \tilde{n}$, and $\hat{l}'(n) = -\left(\frac{2}{3}g\bar{\lambda}^3\right)^{1/2} (\bar{n}-n)^{-1/2}/2 < 0$ and $\hat{l}''(n) = -\left(\frac{2}{3}g\bar{\lambda}^3\right)^{1/2} (\bar{n}-n)^{-3/2}/4 < 0$ for $n \geq \tilde{n}$.

3.2 The ELPF with heterogeneous distribution due to specialization

Let us set two labor-allocation thresholds $\tilde{\theta}_1 \in \left(0, \bar{\theta}\right)$ and $\tilde{\theta}_2 \in \left(\bar{\theta}, \infty\right)$, which define $\lambda_1 = \tilde{\theta}_1\bar{\nu}$ and $\nu_2 = \bar{\lambda}/\tilde{\theta}_2$ (see the right panel of Figure 1). We assume that population density is reduced to $g_1 < g$ in the area between the two rays $\tilde{\theta}_1\nu$ and $\tilde{\theta}_2\nu$, and increased to $g_2 > g_1$ in the rest of the rectangle. The choice of g_1 and of g_2 are constrained because of the population size. To compute population size in the heterogeneous case we add to the uniform population with density g_1 over the whole rectangle, the increment by $g_2 - g_1$ over the two regions North-West of the $\tilde{\theta}_2\nu$ ray and South-East of the $\tilde{\theta}_1\nu$ ray and get

$$P = g_1\bar{\lambda}\bar{\nu} + (g_2 - g_1)\frac{1}{2}\left(\lambda_1\bar{\nu} + \bar{\lambda}\nu_2\right)$$

The maximum amount of effective labor in R&D is

$$\begin{aligned}\bar{n} &= \int_0^{\bar{\nu}} g_1 \bar{\lambda} \cdot \nu d\nu + \int_0^{\bar{\nu}} (g_2 - g_1) \tilde{\theta}_1 \nu \cdot \nu d\nu + \int_0^{\nu_2 = \bar{\lambda}/\tilde{\theta}_2} (g_2 - g_1) \tilde{\theta}_2 \nu \cdot (\nu_2 - \nu) d\nu \\ &= g_1 \bar{\lambda} \left| \frac{\nu^2}{2} \right|_0^{\bar{\nu}} + (g_2 - g_1) \tilde{\theta}_1 \left| \frac{\nu^3}{3} \right|_0^{\bar{\nu}} + (g_2 - g_1) \tilde{\theta}_2 \nu_2 \left| \frac{\nu^2}{2} \right|_0^{\nu_2 = \bar{\lambda}/\tilde{\theta}_2} - (g_2 - g_1) \tilde{\theta}_2 \left| \frac{\nu^3}{3} \right|_0^{\nu_2 = \bar{\lambda}/\tilde{\theta}_2} \\ &= g_1 \frac{1}{2} \bar{\lambda} \bar{\nu}^2 + (g_2 - g_1) \frac{1}{3} \tilde{\theta}_1 \bar{\nu}^3 + (g_2 - g_1) \frac{1}{2} \tilde{\theta}_2 \nu_2^3 - (g_2 - g_1) \frac{1}{3} \tilde{\theta}_2 \nu_2^3 \\ &= g_1 \frac{1}{2} \bar{\lambda} \bar{\nu}^2 + (g_2 - g_1) \frac{1}{3} \lambda_1 \bar{\nu}^2 + (g_2 - g_1) \frac{1}{6} \bar{\lambda} \nu_2^2\end{aligned}$$

and the maximum amount of effective labor in production is

$$\begin{aligned}\bar{l} &= \int_0^{\bar{\lambda}} g_1 \bar{\nu} \cdot \lambda d\lambda + \int_0^{\bar{\lambda}} (g_2 - g_1) \frac{\lambda}{\tilde{\theta}_2} \lambda d\lambda + \int_0^{\lambda_1 = \tilde{\theta}_1 \bar{\nu}} (g_2 - g_1) \frac{\lambda}{\tilde{\theta}_1} (\lambda_1 - \lambda) d\lambda \\ &= g_1 \bar{\nu} \left| \frac{\lambda^2}{2} \right|_0^{\bar{\lambda}} + (g_2 - g_1) \frac{1}{\tilde{\theta}_2} \left| \frac{\lambda^3}{3} \right|_0^{\bar{\lambda}} + (g_2 - g_1) \frac{\lambda_1}{\tilde{\theta}_1} \left| \frac{\lambda^2}{2} \right|_0^{\lambda_1 = \tilde{\theta}_1 \bar{\nu}} - (g_2 - g_1) \frac{1}{\tilde{\theta}_1} \left| \frac{\lambda^3}{3} \right|_0^{\lambda_1 = \tilde{\theta}_1 \bar{\nu}} \\ &= g_1 \frac{1}{2} \bar{\lambda}^2 \bar{\nu} + (g_2 - g_1) \frac{1}{3} \frac{\bar{\lambda}^3}{\tilde{\theta}_2} + (g_2 - g_1) \frac{1}{2} \frac{\lambda_1^3}{\tilde{\theta}_1} - (g_2 - g_1) \frac{1}{3} \frac{\lambda_1^3}{\tilde{\theta}_1} \\ &= g_1 \frac{1}{2} \bar{\lambda}^2 \bar{\nu} + (g_2 - g_1) \frac{1}{3} \bar{\lambda}^2 \nu_2 + (g_2 - g_1) \frac{1}{6} \lambda_1^2 \bar{\nu}\end{aligned}$$

To built the ELPF we establish the amount of effective labor in each sector as a function of the labor-allocation cut-off θ, $n(\theta)$ and $l(\theta)$, using the rule of efficient labor allocation of Proposition 1. Next we obtain the frontier $l = \hat{l}(n)$ by substituting for θ. The procedure is applied to each of the four different regions as θ varies from 0 to ∞:

- For $\theta \in [0, \tilde{\theta}_1]$

$$n(\theta) = \int_0^{\bar{\nu}} g_2 \theta \nu \cdot \nu d\nu = g_2 \theta \left| \frac{\nu^3}{3} \right|_0^{\bar{\nu}} = g_2 \frac{1}{3} \theta \bar{\nu}^3$$

$$\begin{aligned}l(\theta) &= \bar{l} - \int_0^{\tilde{\lambda} = \theta \bar{\nu}} g_2 \frac{\lambda}{\theta} \left(\tilde{\lambda} - \lambda \right) d\lambda = \bar{l} - g_2 \frac{\tilde{\lambda}}{\theta} \left| \frac{\lambda^2}{2} \right|_0^{\tilde{\lambda} = \theta \bar{\nu}} + g_2 \frac{1}{\theta} \left| \frac{\lambda^3}{3} \right|_0^{\tilde{\lambda} = \theta \bar{\nu}} \\ &= \bar{l} - g_2 \frac{\tilde{\lambda}^3}{\theta} \left(\frac{1}{2} - \frac{1}{3} \right) = \bar{l} - g_2 \frac{1}{6} \theta^2 \bar{\nu}^3\end{aligned}$$

so that

$$\hat{l}(n) = \bar{l} - \frac{3}{2} \frac{1}{g_2 \bar{\nu}^3} n^2$$

and n bounded between $n(0) = 0$ and $n\left(\tilde{\theta}_1\right) = g_2 \tilde{\theta}_1 \bar{\nu}^3 / 3 = g_2 \lambda_1 \bar{\nu}^2 / 3 \equiv n_1$;

- For $\theta \in [\tilde{\theta}_1, \bar{\theta}]$

$$\begin{aligned} n(\theta) &= \int_0^{\bar{\nu}} g_1 \theta \nu \cdot \nu d\nu + \int_0^{\bar{\nu}} (g_2 - g_1) \tilde{\theta}_1 \nu \cdot \nu d\nu \\ &= g_1 \theta \left| \frac{\nu^3}{3} \right|_0^{\bar{\nu}} + (g_2 - g_1) \tilde{\theta}_1 \left| \frac{\nu^3}{3} \right|_0^{\bar{\nu}} \\ &= g_1 \frac{1}{3} \theta \bar{\nu}^3 + (g_2 - g_1) \frac{1}{3} \lambda_1 \bar{\nu}^2 \end{aligned}$$

$$\begin{aligned} l(\theta) &= \bar{l} - \int_0^{\tilde{\lambda}=\theta\bar{\nu}} g_1 \left(\bar{\nu} - \frac{\lambda}{\theta} \right) \lambda d\lambda - \int_0^{\lambda_1=\tilde{\theta}_1\bar{\nu}} (g_2 - g_1) \left(\bar{\nu} - \frac{\lambda}{\tilde{\theta}_1} \right) \lambda d\lambda \\ &= \bar{l} - g_1 \left[\frac{1}{2} \tilde{\lambda}^2 \bar{\nu} - \frac{1}{3} \frac{\tilde{\lambda}^3}{\theta} \right] - (g_2 - g_1) \left[\frac{1}{2} \lambda_1^2 \bar{\nu} - \frac{1}{3} \frac{\lambda_1^3}{\tilde{\theta}_1} \right] \\ &= \bar{l} - g_1 \frac{1}{6} \theta^2 \bar{\nu}^3 - (g_2 - g_1) \frac{1}{6} \lambda_1^2 \bar{\nu} \end{aligned}$$

Rearranging $n(\theta)$ we have:

$$\theta = 3 \frac{1}{g_1 \bar{\nu}^3} \left[n - (g_2 - g_1) \frac{1}{3} \lambda_1 \bar{\nu}^2 \right]$$

Substituting in $l(\theta)$ we get:

$$\hat{l}(n) = \bar{l} - (g_2 - g_1) \frac{1}{6} \lambda_1^2 \bar{\nu} - \frac{3}{2} \frac{1}{g_1 \bar{\nu}^3} \left[n - g_2 \frac{1}{3} \lambda_1 \bar{\nu}^2 + g_1 \frac{1}{3} \lambda_1 \bar{\nu}^2 \right]^2$$

n is bounded between n_1 and $n\left(\bar{\theta}\right) = g_1 \bar{\nu}^3 \bar{\theta}/3 + (g_2 - g_1) \lambda_1 \bar{\nu}^2/3 = g_1 \bar{\lambda} \bar{\nu}^2/3 + g_2 \lambda_1 \bar{\nu}^2/3 - g_1 \lambda_1 \bar{\nu}^2/3 = g_2 \lambda_1 \bar{\nu}^2/3 + g_1 \left(\bar{\lambda} - \lambda_1\right) \bar{\nu}^2/3 \equiv n_2$;

- For $\theta \in [\bar{\theta}, \tilde{\theta}_2]$

$$\begin{aligned} n(\theta) &= \int_0^{\tilde{\nu}=\bar{\lambda}/\theta} g_1 \theta \nu \cdot \nu d\nu + \int_{\tilde{\nu}=\bar{\lambda}/\theta}^{\bar{\nu}} g_1 \bar{\lambda} \cdot \nu d\nu + \int_0^{\bar{\nu}} (g_2 - g_1) \tilde{\theta}_1 \nu \cdot \nu d\nu \\ &= g_1 \theta \left| \frac{\nu^3}{3} \right|_0^{\tilde{\nu}=\bar{\lambda}/\theta} + g_1 \bar{\lambda} \left| \frac{\nu^2}{2} \right|_{\tilde{\nu}=\bar{\lambda}/\theta}^{\bar{\nu}} + (g_2 - g_1) \tilde{\theta}_1 \left| \frac{\nu^3}{3} \right|_0^{\bar{\nu}} \\ &= g_1 \frac{1}{3} \frac{\bar{\lambda}^3}{\theta^2} + g_1 \frac{1}{2} \bar{\lambda} \bar{\nu}^2 - g_1 \frac{1}{2} \frac{\bar{\lambda}^3}{\theta^2} + (g_2 - g_1) \frac{1}{3} \tilde{\theta}_1 \bar{\nu}^3 \\ &= g_1 \frac{1}{2} \bar{\lambda} \bar{\nu}^2 - g_1 \frac{1}{6} \frac{\bar{\lambda}^3}{\theta^2} + (g_2 - g_1) \frac{1}{3} \lambda_1 \bar{\nu}^2 \end{aligned}$$

$$\begin{aligned} l(\theta) &= \int_0^{\bar{\lambda}} g_1 \frac{\lambda}{\theta} \cdot \lambda d\lambda + \int_0^{\bar{\lambda}} (g_2 - g_1) \frac{\lambda}{\tilde{\theta}_2} \cdot \lambda d\lambda \\ &= g_1 \frac{1}{3} \frac{\bar{\lambda}^3}{\theta} + (g_2 - g_1) \frac{1}{3} \frac{\bar{\lambda}^3}{\tilde{\theta}_2} \\ &= g_1 \frac{1}{3} \frac{\bar{\lambda}^3}{\theta} + (g_2 - g_1) \frac{1}{3} \bar{\lambda}^2 \nu_2 \end{aligned}$$

Rearranging $n(\theta)$ we have:

$$\frac{1}{\theta} = \left(\frac{6}{g_1 \bar{\lambda}^3} \right)^{1/2} \left[g_1 \frac{1}{2} \bar{\lambda} \bar{\nu}^2 + (g_2 - g_1) \frac{1}{3} \lambda_1 \bar{\nu}^2 - n \right]^{1/2}$$

Substituting in $l(\theta)$ we get:

$$\hat{l}(n) = (g_2 - g_1) \frac{1}{3} \bar{\lambda}^2 \nu_2 + \left(\frac{2}{3} g_1 \bar{\lambda}^3 \right)^{1/2} \left[g_2 \frac{1}{3} \lambda_1 \bar{\nu}^2 + g_1 \frac{1}{2} \left(\bar{\lambda} - \lambda_1 \right) \bar{\nu}^2 - n \right]^{1/2}$$

n is bounded between n_2 and $n\left(\tilde{\theta}_2\right) = g_1 \bar{\lambda} \bar{\nu}^2 / 2 - g_1 \bar{\lambda}^3 / \left(6 \tilde{\theta}_2^2\right) + (g_2 - g_1) \lambda_1 \bar{\nu}^2 / 3 = g_1 \bar{\lambda} \bar{\nu}^2 / 2 - g_1 \bar{\lambda} \nu_2^2 / 6 + (g_2 - g_1) \lambda_1 \bar{\nu}^2 / 3 = g_2 \lambda_1 \bar{\nu}^2 / 3 + g_1 \left(\bar{\lambda} - \lambda_1 \right) \bar{\nu}^2 / 3 + g_1 \bar{\lambda} \left(\bar{\nu}^2 - \nu_2^2 \right) / 6 \equiv n_3$;

- For $\theta \in [\tilde{\theta}_2, \infty]$

$$\begin{aligned} n(\theta) &= \bar{n} - \int_0^{\tilde{\nu} = \bar{\lambda}/\theta} g_2 \left(\bar{\lambda} - \theta \nu \right) \cdot \nu d\nu \\ &= \bar{n} - g_2 \bar{\lambda} \left| \frac{\nu^2}{2} \right|_0^{\tilde{\nu} = \bar{\lambda}/\theta} + g_2 \theta \left| \frac{\nu^3}{3} \right|_0^{\tilde{\nu} = \bar{\lambda}/\theta} \\ &= \bar{n} - g_2 \bar{\lambda} \frac{\tilde{\nu}^2}{2} + g_2 \theta \frac{\tilde{\nu}^3}{3} \\ &= \bar{n} - g_2 \frac{1}{6} \frac{\bar{\lambda}^3}{\theta^2} \end{aligned}$$

$$l(\theta) = \int_0^{\bar{\lambda}} g_2 \frac{\lambda}{\theta} \cdot \lambda d\lambda = g_2 \frac{1}{\theta} \left| \frac{\lambda^3}{3} \right|_0^{\bar{\lambda}} = g_2 \frac{1}{3} \frac{\bar{\lambda}^3}{\theta}$$

Rearranging $n(\theta)$ we have:

$$\frac{1}{\theta} = \left(\frac{6}{g_2 \bar{\lambda}^3} \right)^{1/2} (\bar{n} - n)^{1/2}$$

Substituting in $l(\theta)$ we get:

$$\hat{l}(n) = \left(\frac{2}{3}g_2\bar{\lambda}^3\right)^{1/2}(\bar{n}-n)^{1/2}$$

n is bounded between n_3 and $\bar{n}$.

The ELPF is now the envelope of four concave functions

$$\hat{l}(n) = \begin{cases} \bar{l} - \frac{3}{2}\left(g_2\bar{\nu}^3\right)^{-1} n^2 & \forall n \in [0, n_1] \\ \bar{l} - \frac{g_2-g_1}{6}\lambda_1^2\bar{\nu} - \frac{3}{2}\left(g_1\bar{\nu}^3\right)^{-1}\left(n - n_1 + \frac{g_1}{3}\lambda_1\bar{\nu}^2\right)^2 & \forall n \in [n_1, n_2] \\ \frac{g_2-g_1}{3}\bar{\lambda}^2\nu_2 + \left(\frac{2}{3}g_1\bar{\lambda}^3\right)^{1/2}\left(n_2 - n + \frac{g_1}{6}\bar{\lambda}\bar{\nu}^2\right)^{1/2} & \forall n \in [n_2, n_3] \\ \left(\frac{2}{3}g_2\bar{\lambda}^3\right)^{1/2}(\bar{n}-n)^{1/2} & \forall n \in [n_3, \bar{n}] \end{cases}$$

where $n_1 \equiv \frac{g_2}{3}\lambda_1\bar{\nu}^2$, $n_2 \equiv n_1 + \frac{g_1}{3}\left(\bar{\lambda} - \lambda_1\right)\bar{\nu}^2$, $n_3 \equiv n_2 + \frac{g_1}{6}\bar{\lambda}\left(\bar{\nu}^2 - \nu_2^2\right)$, $\bar{n} = n_3 + \frac{g_2}{6}\bar{\lambda}\nu_2^2$, and $\bar{l} = \frac{g_1}{2}\bar{\lambda}^2\bar{\nu} + \frac{g_2-g_1}{3}\bar{\lambda}^2\nu_2 + \frac{g_2-g_1}{6}\lambda_1^2\bar{\nu}$.

When comparing this case with the case of uniform population density g over Γ, we impose the following constraint on g_1 and g_2 to maintain the population size constant:

$$\frac{g-g_1}{g_2-g_1} = \frac{1}{2}\left(\frac{\lambda_1}{\bar{\lambda}} + \frac{\nu_2}{\bar{\nu}}\right)$$

3.3 The ELPF with uniform distribution over a segment

Individual sector-specific skills are a function of individual ability $a_i \sim U[0,1]$ as

$$\nu_i = \alpha_\nu + \beta_\nu a_i \text{ and } \lambda_i = \alpha_\lambda + \beta_\lambda a_i$$

This representation is equivalent to constraining the domain to $\Gamma_\alpha \equiv \{\nu, \tilde{\lambda}(\nu) | \nu \in [\alpha_\nu, \bar{\nu}]\} \subset \Gamma$ where $\lambda_i = \tilde{\lambda}(\nu) \equiv \alpha + \beta\nu_i$, with $\alpha \equiv \alpha_\lambda - \alpha_\nu\beta_\lambda/\beta_\nu$ and $\beta \equiv \beta_\lambda/\beta_v$, and $\bar{\nu} = \alpha_\nu + \beta_\nu$ so that $\bar{\lambda} = \alpha_\lambda + \beta_\lambda$, implying $\lambda \in \left[\alpha_\lambda, \bar{\lambda}\right]$. Define $\underline{\theta} \equiv \alpha_\lambda/\alpha_\nu$ and $\bar{\theta} \equiv (\alpha_\lambda + \beta_\lambda)/(\alpha_\nu + \beta_\nu)$. Denote by $x \in [0,1]$ the fraction of the population P that is employed in R&D. The units of effective labor input in each sector are computed as the product of the mass of individuals and the average productivity, as function of x. First we consider the case of positive correlation between individual skills (from case a to c), then the case of negative correlation (case d). These different cases are illustrated in Figure **??**.

Case (a): If $\underline{\theta} = \bar{\theta}$, then $\beta = \bar{\theta}$ and all individuals are characterized by the same relative skill index independently of their ability. The opportunity cost of providing effective labor inputs to R&D is therefore independent of the relative size of the R&D sector. This is exactly the same situation as in the one-point distribution case. The ELPF is linear.

Case (b): $\underline{\theta} > \bar{\theta}$ according to Proposition 1 in the R&D sector individuals with higher a_i are employed first. Hence the average productivity of workers decreases with the size of the R&D sector

$$
\begin{aligned}
n(x) &= xP\left(\alpha_\nu + \beta_\nu - \beta_\nu \frac{x}{2}\right) \\
l(x) &= (1-x)\,P\left(\alpha_\lambda + \beta_\lambda \frac{1-x}{2}\right)
\end{aligned}
$$

Implying $dn/dx = P(\alpha_\nu + \beta_\nu - \beta_\nu x) > 0$, $d^2n/dx^2 = -\beta_\nu P < 0$, $dl/dx = -P[\alpha_\lambda + \beta_\lambda(1-x)] < 0$ and $d^2l/dx^2 = \beta_\lambda P > 0$. Hence

$$
\frac{dl}{dn} = \frac{dl}{dx}\frac{dx}{dn} = -\frac{\alpha_\lambda + \beta_\lambda (1-x)}{\alpha_\nu + \beta_\nu (1-x)} < 0
$$

$$
\frac{d^2l}{dn^2} = \frac{d\left(\frac{dl}{dx}\right)}{dx}\frac{dx}{dn} = \frac{\alpha_\nu\beta_\lambda - \alpha_\lambda\beta_\nu}{P\left[\alpha_\nu + \beta_\nu (1-x)\right]^3} < 0
$$

where the sign is established using $\underline{\theta} \equiv \alpha_\lambda/\alpha_\nu > (\alpha_\lambda + \beta_\lambda)/(\alpha_\nu + \beta_\nu) \equiv \bar{\theta}$, implying that $\alpha_\lambda/\alpha_\nu > \beta_\lambda/\beta_\nu \equiv \beta$. This is the special case considered in O. Galor and D. Tsiddon's paper 'Technological progress, mobility and economic growth' (*American Economic Review* 87(3), 363-382, 1997).

Case (*c*): $\underline{\theta} < \bar{\theta}$ according to Proposition 1 in the R&D sector individuals with lower a_i are employed first. Hence the average productivity of workers increases with the size of the R&D sector

$$
\begin{aligned}
n(x) &= xP\left(\alpha_\nu + \beta_\nu \frac{x}{2}\right) \\
l(x) &= (1-x)\,P\left(\alpha_\lambda + \beta_\lambda - \beta_\lambda \frac{1-x}{2}\right)
\end{aligned}
$$

Implying $dn/dx = P(\alpha_\nu + \beta_\nu x) > 0$, $d^2n/dx^2 = \beta_\nu P > 0$, $dl/dx = -P[\alpha_\lambda + \beta_\lambda x] < 0$ and $d^2l/dx^2 = -\beta_\lambda P < 0$. Hence

$$
\frac{dl}{dn} = \frac{dl}{dx}\frac{dx}{dn} = -\frac{\alpha_\lambda + \beta_\lambda x}{\alpha_\nu + \beta_\nu x} < 0
$$

$$
\frac{d^2l}{dn^2} = \frac{d\left(\frac{dl}{dx}\right)}{dx}\frac{dx}{dn} = \frac{\alpha_\lambda\beta_\nu - \alpha_\nu\beta_\lambda}{P\left(\alpha_\nu + \beta_\nu x\right)^3} < 0
$$

where the sign is established using $\underline{\theta} \equiv \alpha_\lambda/\alpha_\nu < (\alpha_\lambda + \beta_\lambda)/(\alpha_\nu + \beta_\nu) \equiv \bar{\theta}$, implying that $\alpha_\lambda/\alpha_\nu < \beta_\lambda/\beta_\nu \equiv \beta$.

Case (*d*): If $\beta < 0$ there is negative correlation of sector-specific skills across individuals. Here a_i is not an index of absolute competence over all sectors, i.e., "ability", but

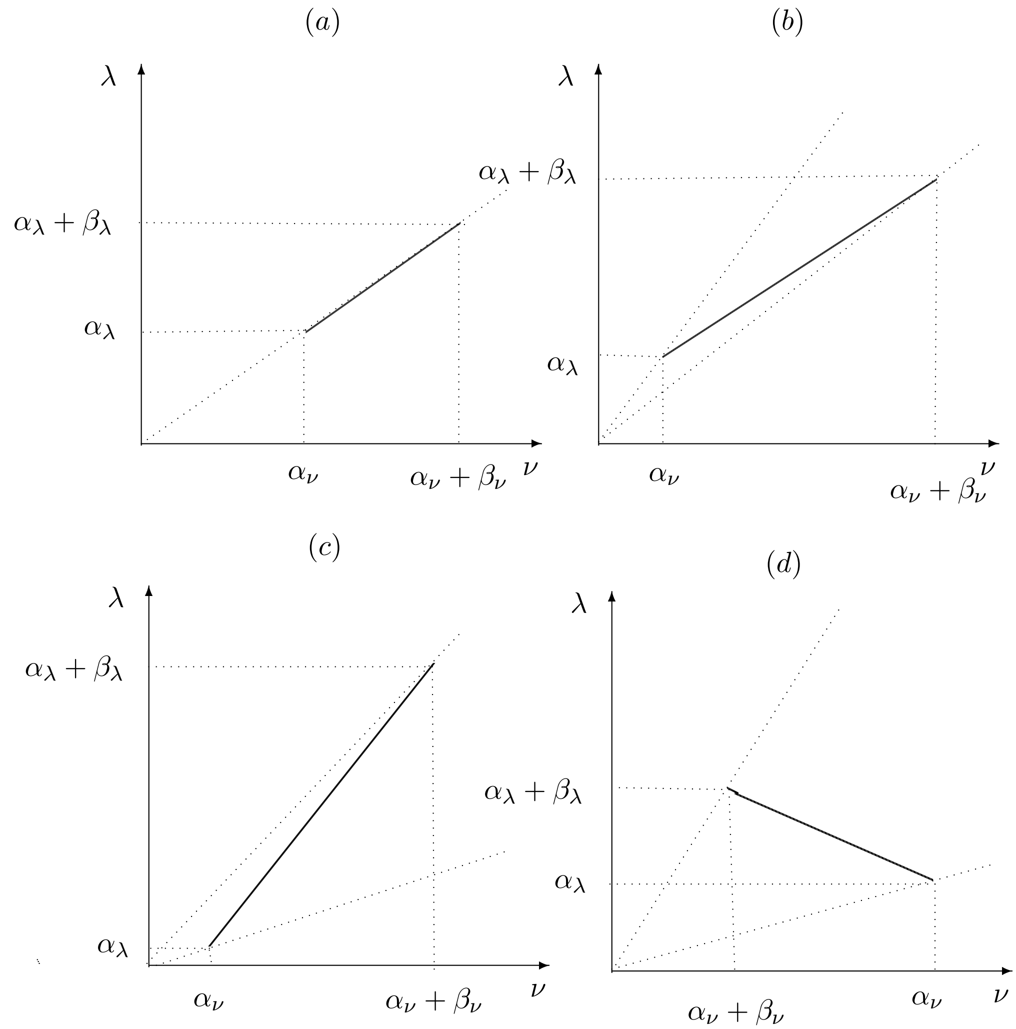

Figure 6: Uniform distributions over a segment.

rather an index of comparative advantage in R&D. Starting from no R&D activity, the first individuals to be employed are the best researchers, who are also the least effective workers in the production sector. Let x be the share of population employed in R&D. Effective labor inputs are given by

$$n(x) = xP\left(\alpha_\nu + \frac{\beta_\nu}{2}x\right)$$

$$l(x) = (1-x)\,P\left[\alpha_\lambda + \frac{\beta_\lambda}{2}(1-x)\right]$$

The two equations define implicitly a strictly concave frontier, since

$$d\hat{l}\,(n)\,/dn = (dl\,(x)\,/dx)\,(dx/dn) = -\,(\alpha_\lambda + \beta_\lambda - \beta_\lambda x)\,/\,(\alpha_\nu + \beta_\nu) < 0$$

since $\alpha_\lambda + \beta_\lambda > 0$ and $x \in [0, 1]$, while

$$d^2\hat{l}\,(n)\,/dn^2 = \left(d^2 l\,(x)\,/dx^2\right)(dx/dn) = \beta_\lambda/\,(\alpha_\nu + \beta_\nu) < 0.$$

3.4 Fully specialized individuals

Assume $\tilde{g}\,(\nu, \lambda) > 0$ only for skill bundles lying on the axes of Γ, i.e., $\forall i \in [0, P]$ $\nu_i > 0 \Rightarrow \lambda_i = 0$ and $\lambda_i > 0 \Rightarrow \nu_i = 0$. It is impossible to increase effective labor inputs in one sector by diverting raw labor from the other sector. The ELPF equals $\bar{l}$ $\forall n \in [0, \bar{n})$, can take any value $l \in [0, \bar{l}]$ for $n = \bar{n}$, and $l = 0$ $\forall n > \bar{n}$, where $\bar{n} = \int_0^{\bar{\nu}} \nu \tilde{g}\,(\nu, 0)\, d\nu$ and $\bar{l} = \int_0^{\bar{\lambda}} \lambda \tilde{g}\,(0, \lambda)\, d\lambda$. The ELPF has the shape of a Leontief production function (see the working paper version of this article available of LERNA's web site as w.p. n.06.22.215).

4 Dynamic analysis

This section presents the details of the analysis of the dynamic system obtained in section 3 of the paper from the social planner optimization problem.

In order to characterize the dynamics of the system, we need to obtain the two functions defining the phase diagram in the (R, n) plane. First we determine and analyze the schedule $\dot{R} = 0$, then we turn to the locus $\dot{n} = 0$.

Finally we linearize the dynamic system around the steady state to obtain the eigenvalues that are used in the reversed shooting procedure for the simulation.[1]

Determining the locus $\dot{R} = 0$. By definition of R_t, taking logs and differentiating with respect to time, then using (8) and (6), we have:

$$\frac{\dot{R}_t}{R_t} = bn_t - \frac{A\hat{l}\,(n_t)}{R_t} \tag{19}$$

Hence the schedule $\dot{R} = 0$ is given by the function $n^R\,(R)$, defined implicitly by:

$$G\,(R, n) \equiv bn - \frac{A\hat{l}\,(n)}{R} = 0$$

We check that $\frac{\partial G}{\partial n} = b - \frac{A\hat{l}'(n)}{R} > 0$ and $\frac{\partial G}{\partial R} = \frac{A\hat{l}(n)}{R^2} > 0$. The $\dot{R} = 0$ locus is therefore

[1] The procedure and program were adapted from M. Brunner and H. Strulik's paper 'Solution of perfect foresight saddlepoint problems: a simple method and applications' (*Journal of Economic Dynamics and Control* 26: 737-753, 2002).

downward sloping

$$\frac{dn^R}{dR} = -\frac{\partial G/\partial R}{\partial G/\partial n} = -\frac{A\hat{l}\,(n)}{bR - A\hat{l}'\,(n)} < 0$$

Furthermore along n^R, $R = (A/b)\left(\hat{l}\,(n)\,/n\right)$ (where $\hat{l}\,(n)\,/n$ is the slope of the ray from the origin to $\hat{l}\,(n)$), so that if $R \to 0$, $\hat{l}\,(n)\,/n \to 0$ and $n \to \bar{n}$ along n^R, while if $R \to \infty$, $\hat{l}\,(n)\,/n \to \infty$ and $n \to 0$ along n^R. Since $\partial G/\partial R > 0$, if R is reduced from $n^R\,(R)$, holding n constant (i.e. below the schedule) then $\dot{R} < 0$, and vice versa on the North-East of the schedule $\dot{R} > 0$.

Determining the locus $\dot{n} = 0$. We begin by substituting (13) in the F.O.C. (9) to get:

$$\begin{aligned}
\left[A\hat{l}\,(n_t)\right]^{-\varepsilon} e^{-\rho t} A\hat{l}'\,(n_t) &= \mu A B_t^{-1}\hat{l}'\,(n_t) - b\mu S_t \\
c_t^{-\varepsilon} e^{-\rho t} A\hat{l}'\,(n_t) &= \mu \frac{A}{B_t}\left[\hat{l}'\,(n_t) - \frac{b}{A} B_t S_t\right] \\
c_t^{-\varepsilon} e^{-\rho t} &= \frac{\mu}{B_t}\left[1 - \frac{b}{A}\frac{R_t}{\hat{l}'\,(n_t)}\right] \\
c_t^{-\varepsilon} e^{-\rho t} &= \frac{\mu}{B_t}\left[1 - bX\,(R_t, n_t)\right]
\end{aligned}$$

Taking logs and differentiating with respect to t:

$$-\varepsilon\frac{\dot{c}_t}{c_t} - \rho = -\frac{\dot{B}_t}{B_t} - \frac{b\dot{X}_t}{1 - bX_t} \tag{20}$$

From the Leontief technology we know that $c_t = A\hat{l}\,(n_t)$, and therefore:

$$\frac{\dot{c}_t}{c_t} = \frac{\hat{l}'\,(n_t)}{\hat{l}\,(n_t)}\dot{n}_t = \frac{\hat{l}'\,(n_t)}{\hat{l}\,(n_t)} n_t \frac{\dot{n}_t}{n_t} = -\sigma\,(n_t)\,\frac{\dot{n}_t}{n_t} \tag{21}$$

From the definition of $X\,(R_t, n_t)$ we have:

$$\frac{\dot{X}_t}{X_t} = \frac{\dot{R}_t}{R_t} - \frac{\dot{n}_t}{n_t}\frac{\hat{l}''\,(n_t)\,n_t}{\hat{l}'\,(n_t)}$$

which taking into account (**??**) and the definition of $\eta\,(n_t)$ gives

$$\frac{\dot{X}_t}{X_t} = bn_t - \frac{A\hat{l}\,(n_t)}{R_t} - \eta\,(n_t)\,\frac{\dot{n}_t}{n_t}$$

substituting into (**??**), using (**??**), we get

$$\begin{aligned} -\varepsilon\sigma\left(n_t\right)\frac{\dot{n}_t}{n_t}+\rho &= bn_t+\frac{bX_t}{1-bX_t}\left[bn_t-\frac{A\hat{l}\left(n_t\right)}{R_t}-\eta\left(n_t\right)\frac{\dot{n}_t}{n_t}\right] \\ &= \frac{1}{1-bX_t}\left[bn_t-bX_t\frac{\hat{l}\left(n_t\right)}{\hat{l}'\left(n_t\right)}\frac{1}{X_t}-bX_t\eta\left(n_t\right)\frac{\dot{n}_t}{n_t}\right] \end{aligned}$$

Taking all terms in $\dot{n}_t$ on the right-hand-side and simplifying

$$\left[\varepsilon\sigma\left(n_t\right)-\frac{bX_t}{1-bX_t}\eta\left(n_t\right)\right]\frac{\dot{n}_t}{n_t}=\frac{bn_t}{1-bX_t}\left[\frac{\hat{l}\left(n_t\right)}{\hat{l}'\left(n_t\right)n_t}\frac{X_t}{X_t}-1\right]+\rho$$

We have therefore determined the law of motion of n as function of n and R as given by

$$\frac{\dot{n}_t}{n_t}=\frac{\rho-\frac{bn_t}{1-bX_t}\left[1+\frac{1}{\sigma(n_t)}\right]}{\varepsilon\sigma\left(n_t\right)-\frac{bX_t}{1-bX_t}\eta\left(n_t\right)} \tag{22}$$

The locus $\dot{n}=0$ in the (R,n) plane is given by the function $n^n\left(R\right)$ defined implicitly by

$$\frac{\dot{n}_t}{n_t}=0 \qquad \Leftrightarrow \qquad F\left(R,n\right)=\frac{bn_t}{1-b\frac{R_t}{A\hat{l}'(n_t)}}\left[1+\frac{1}{\sigma\left(n_t\right)}\right]-\rho=0 \tag{23}$$

To study the slope of this schedule, we need to explore how F depends on n and R. We have that

$$\frac{\partial F}{\partial R}=\frac{b^2n_t}{\left[1-b\frac{R_t}{A\hat{l}'(n_t)}\right]^2}\left[1+\frac{1}{\sigma\left(n_t\right)}\right]\frac{1}{A\hat{l}'\left(n_t\right)}<0$$

which is negative because $\hat{l}'<0$ and $\sigma>0$. When differentiating F with respect to n, we need to go through some tedious algebra to determine the sign.

$$\begin{aligned} \frac{\partial F}{\partial n} &= \left[1+\frac{1}{\sigma\left(n_t\right)}\right]\frac{\partial\left(\frac{bn_t}{1-b\frac{R_t}{A\hat{l}'(n_t)}}\right)}{\partial n}+\frac{bn_t}{1-b\frac{R_t}{A\hat{l}'(n_t)}}\frac{\partial 1/\sigma\left(n_t\right)}{\partial n} \\ &= \left[1+\frac{1}{\sigma}\right]\frac{b\left(1-b\frac{R}{A\hat{l}'}\right)-bnbR\frac{A\hat{l}''}{\left[A\hat{l}'\right]^2}}{\left[1-b\frac{R}{A\hat{l}'}\right]^2}+\frac{bn}{1-b\frac{R}{A\hat{l}'}}\frac{-\partial\sigma/\partial n}{\left[\sigma\right]^2} \\ &= \left[1+\frac{1}{\sigma}\right]\frac{b\left(1-bX\right)-b^2n\frac{R}{A\hat{l}'}\frac{A}{A}\frac{\hat{l}''}{\hat{l}'}}{\left(1-bX\right)^2}+\frac{bn}{1-bX}\frac{\hat{l}'\hat{l}+\hat{l}''\hat{l}-\hat{l}'n\hat{l}'}{\left(\sigma\hat{l}\right)^2} \end{aligned}$$

and continuing

$$\begin{aligned}
\frac{\partial F}{\partial n} &= \frac{b}{1-bX}\left[1+\frac{1}{\sigma}\right]-\frac{bX}{(1-bX)^2}b\eta\left[1+\frac{1}{\sigma}\right]+\frac{bn}{1-bX}\frac{\hat{l}^2}{\left(\hat{l}'n\right)^2}\frac{\hat{l}\hat{l}'+\hat{l}\hat{l}''}{\hat{l}^2}-\frac{b}{1-bX}\frac{1}{\sigma^2}\left(\frac{\hat{l}'n}{\hat{l}}\right)^2 \\
&= \frac{b}{1-bX}\frac{1}{\sigma}-\frac{bX}{(1-bX)^2}b\eta\left[1+\frac{1}{\sigma}\right]+\frac{b}{1-bX}\left[\frac{\hat{l}}{\hat{l}'n}+\frac{\hat{l}}{\hat{l}'n}\frac{\hat{l}''n}{\hat{l}'}\frac{1}{n}\right] \\
&= \frac{b}{1-bX}\frac{1}{\sigma}-\frac{bX}{(1-bX)^2}b\eta\left[1+\frac{1}{\sigma}\right]-\frac{b}{1-bX}\frac{1}{\sigma}\left[1+\frac{\eta}{n}\right] \\
&= -\frac{bX}{(1-bX)^2}b\eta\left[1+\frac{1}{\sigma}\right]-\frac{bX}{1-bX}\frac{1}{\sigma}\frac{\eta}{n} \\
&= -\frac{bX\eta}{(1-bX)}\left[\frac{b}{1-bX}\left(1+\frac{1}{\sigma}\right)+\frac{1}{\sigma}\frac{1}{n}\right] \\
&= -\frac{bX\eta}{n(1-bX)}\left(\rho+\frac{1}{\sigma}\right)>0
\end{aligned}$$

Substituting for $\frac{bn}{1-bX}\left(1+\frac{1}{\sigma}\right)=\rho$ we obtain

$$\frac{\partial F}{\partial n}=-\frac{bX\eta}{n(1-bX)}\left(\rho+\frac{1}{\sigma}\right)>0$$

The sign is determined knowing that $\eta>0$, $X<0$, $b>0$, $\sigma>0$, $n>0$.

We conclude that the $\dot{n}=0$ schedule is upward sloping in the (R,n) plane since

$$\frac{dn^n}{dR}=-\frac{\partial F/\partial R}{\partial F/\partial n}=\frac{bn\left(1+\frac{1}{\sigma}\right)}{A\hat{l}'\eta\left[bX-\frac{1}{\sigma}\frac{1}{n}+\frac{1}{\sigma}\frac{1}{n}bX\right]}>0$$

We also have that $\dot{n}<0$ North-West of the n^n schedule and vice versa n increases South-East of the schedule. In fact, starting from a point on the n^n schedule, hold R constant and increase n. This change implies $F>0$ since $\partial F/\partial n>0$, i.e., $bn(1+1/\sigma)/(1-bX)>\rho$ which with (**??**) determines $\dot{n}<0$.

Figure **??** illustrates the phase diagram. The steady state is a saddle path stable.

Linearization. Consider the system of non-linear differential equations given by (14) and (15):

$$\begin{cases}\dot{R}\equiv f^1(R,n)=bnR-A\hat{l}(n)\\ \dot{n}\equiv f^2(R,n)=\frac{\rho-\frac{bn}{1-bX}\left(1+\frac{1}{\sigma}\right)}{\varepsilon\sigma-\frac{bX}{1-bX}\eta}n\end{cases}$$

where time subscripts have been dropped. To linearize the system around the steady state

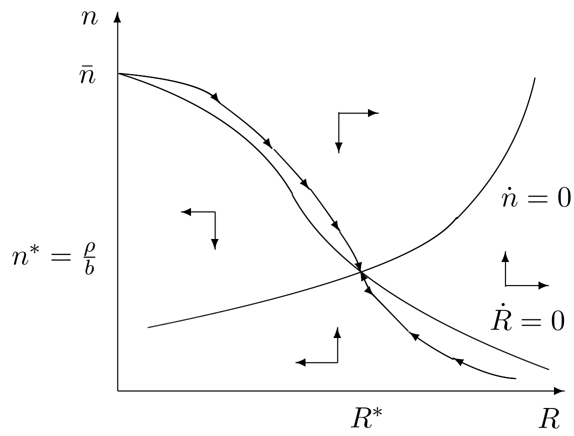

Figure 7: Phase diagram.

it is necessary to perform a Taylor expansion of the first order, i.e.

$$\begin{vmatrix} \dot{R} \\ \dot{n} \end{vmatrix} = \begin{vmatrix} f^1(R^*, n^*) \\ f^2(R^*, n^*) \end{vmatrix} + \begin{vmatrix} f^1_R(R^*, n^*) & f^1_n(R^*, n^*) \\ f^2_R(R^*, n^*) & f^2_n(R^*, n^*) \end{vmatrix} \begin{vmatrix} R - R^* \\ n - n^* \end{vmatrix} + \circ$$

Of course $f^1(R^*, n^*) = f^2(R^*, n^*) = 0$, by definition of R^* and n^*. Before computing the partial derivatives of the differential equations, let us recall a few definitions:

$$\sigma = -\frac{\hat{l}'(n)}{\hat{l}(n)} n > 0 \quad ; \quad \eta = \frac{\hat{l}''(n)}{\hat{l}'(n)} n > 0$$

$$X = \frac{R}{A\hat{l}'(n)} < 0 \quad ; \quad R^* = \frac{A}{\rho}\hat{l}\left(\frac{\rho}{b}\right) > 0$$

and $n^* = \rho/b > 0$. We have that:

$$f^1_R = bn$$

implying:

$$f^1_R(R^*, n^*) = \rho > 0$$

and

$$f^1_n = bR - A\hat{l}'(n)$$

so that

$$f^1_n(R^*, n^*) = -A\hat{l}'\left(\frac{\rho}{b}\right)\left(1 + \frac{1}{\sigma^*}\right) > 0$$

Turning to the differential equation describing the optimal evolution of the control variable, we find

$$f_R^2 = \frac{\frac{bn}{(1-bX)^2}\frac{\partial X}{\partial R}}{\varepsilon\sigma - \frac{bX}{1-bX}\eta}\left[\frac{\rho - \frac{bn}{1-bX}\left(1+\frac{1}{\sigma}\right)}{\varepsilon\sigma - \frac{bX}{1-bX}\eta}\eta - bn\left(1+\frac{1}{\sigma}\right)\right]$$

where $\partial X/\partial R = X/R = \left[A\hat{l}'(n)\right]^{-1}$. Using the fact that at steady state $bX^* = -1/\sigma^*$ and $\frac{bn^*}{1-bX^*}\left(1+\frac{1}{\sigma^*}\right) = \rho$, we get

$$\begin{aligned} f_R^2(R^*, n^*) &= -\frac{\rho^2/\left[A\hat{l}'\left(\frac{\rho}{b}\right)\left(1+\frac{1}{\sigma^*}\right)\right]}{\varepsilon\sigma^* + \frac{\eta^*}{1+\sigma^*}} \\ &= \frac{\rho^2}{\varepsilon\sigma^* + \frac{\eta^*}{1+\sigma^*}}\frac{1}{f_n^1(R^*, n^*)} > 0 \end{aligned}$$

Finally the partial derivative with respect to R&D employment is

$$\begin{aligned} f_n^2 &= \frac{1}{\varepsilon\sigma - \frac{bX}{1-bX}\eta}\left[\rho - 2\frac{bn}{1-bX}\left(1+\frac{1}{\sigma}\right) + \frac{bn}{1-bX}\frac{n}{\sigma^2}\frac{\partial\sigma}{\partial n} - \left(\frac{bn}{1-bX}\right)^2\left(1+\frac{1}{\sigma}\right)\frac{\partial X}{\partial n}\right. \\ &\quad \left. - \frac{\rho - \frac{bn}{1-bX}\left(1+\frac{1}{\sigma}\right)}{\varepsilon\sigma - \frac{bX}{1-bX}\eta} n\left(\varepsilon\frac{\partial\sigma}{\partial n} - \frac{bX}{1-bX}\frac{\partial\eta}{\partial n} - \frac{b\eta}{(1-bX)^2}\frac{\partial X}{\partial n}\right)\right] \end{aligned}$$

where $\frac{\partial\sigma}{\partial n} = \frac{\sigma}{n}(1+\sigma+\eta)$, $\frac{\partial\eta}{\partial n} = \frac{\eta}{n}\left(1-\eta+\frac{\hat{l}'''(n)}{\hat{l}''(n)}n\right)$, and $\frac{\partial X}{\partial n} = -\frac{1}{n}X\eta$. Using this and again $1-bX^* = 1+1/\sigma^*$ and $\frac{bn^*}{1-bX^*}\left(1+\frac{1}{\sigma^*}\right) = \rho$, the expression simplifies at steady state to

$$\begin{aligned} f_n^2(R^*, n^*) &= \frac{1}{\varepsilon\sigma^* + \frac{\eta^*}{1+\sigma^*}}\left[\rho - 2\frac{b\rho/b}{1+\frac{1}{\sigma}}\left(1+\frac{1}{\sigma}\right) + \frac{b\rho/b}{1+\frac{1}{\sigma}}\frac{\rho/b}{\sigma^2}\frac{\sigma}{\rho/b}(1+\sigma+\eta)\right. \\ &\quad \left. - \left(\frac{b\rho/b}{1+\frac{1}{\sigma}}\right)^2\left(1+\frac{1}{\sigma}\right)\frac{b}{\rho}\frac{1}{b\sigma^*}\eta^* - 0\cdot ...\right] \\ &= \frac{1}{\varepsilon\sigma^* + \frac{\eta^*}{1+\sigma^*}}\left[\rho - 2\rho + \rho\left(1+\frac{\eta^*}{1+\sigma^*}\right) - \rho\frac{\eta^*}{1+\sigma^*}\right] \\ &= 0 \end{aligned}$$

Hence the linearized system can be computed as

$$\begin{cases} \dot{R} = \rho(R-R^*) + -A\hat{l}'\left(\frac{\rho}{b}\right)\left(1+\frac{1}{\sigma^*}\right)(n-n^*) \\ \dot{n} = \frac{\rho^2}{\varepsilon\sigma^* + \frac{\eta^*}{1+\sigma^*}}\frac{1}{f_n^1(R^*,n^*)}(R-R^*) + 0\cdot(n-n^*) \end{cases}$$

that is

$$\begin{cases} \dot{R} = \rho R - A\hat{l}'\left(\frac{\rho}{b}\right)\left(1+\frac{1}{\sigma^*}\right)n + A\frac{\rho}{b}\hat{l}'\left(\frac{\rho}{b}\right) \\ \dot{n} = \frac{\rho^2}{\varepsilon\sigma^* + \frac{\eta^*}{1+\sigma^*}}\frac{1}{f_n^1(R^*,n^*)}R - \frac{\rho^2/b}{\varepsilon\sigma^*(1+\sigma^*)+\eta^*} \end{cases}$$

The matrix of the corresponding linear autonomous system of differential equations is

$$M \equiv \begin{vmatrix} \rho & -A\hat{l}'\left(\frac{\rho}{b}\right)\left(1+\frac{1}{\sigma^*}\right) \\ -\frac{\rho^2}{\varepsilon\sigma^* + \frac{\eta^*}{1+\sigma^*}}\frac{1}{A\hat{l}'\left(\frac{\rho}{b}\right)\left(1+\frac{1}{\sigma^*}\right)} & 0 \end{vmatrix}$$

This matrix has a negative determinant

$$\det(M) = -\frac{\rho^2}{\varepsilon\sigma^* + \frac{\eta^*}{1+\sigma^*}} < 0$$

meaning that the eigenvalues are real and of opposite sign. The steady state is characterized by saddle-path dynamics.

5 Consequences of a demographic expansion

Define $\alpha \equiv P_2/P_0 = g_2/g_0 > 0$ the factor measuring the increase in population size. The ELPF shifts outward homothetically by α, with $\bar{n}_2/\bar{n}_0 = \bar{l}_2/\bar{l}_0 = \alpha$. We use the notation $l = \hat{l}(n;\alpha) \equiv \hat{l}_2(n)$ for the ELPF with larger population, as compared to $l = \hat{l}(n;1) \equiv \hat{l}_0(n)$ for the ELPF with normalized population size $P_0 = 1$ corresponding to density g_0 in (1). We have that $\forall \tilde{n} \in [0, \bar{n}_0]$ we get $\tilde{l} = \hat{l}(\tilde{n};1) \equiv \hat{l}_0(\tilde{n})$ if $\tilde{g} = g_0$. To obtain $\alpha\tilde{l}$ when $\tilde{g} = g_2$ it is necessary and sufficient to employ $\alpha\tilde{n}$ units in R&D, i.e., $\alpha\tilde{l} = \hat{l}(\alpha\tilde{n};\alpha)$. Hence[2]

$$\hat{l}_2(\alpha\tilde{n}) = \hat{l}(\alpha\tilde{n};\alpha) = \alpha\hat{l}(\tilde{n};1) \equiv \alpha\hat{l}_0(\tilde{n})$$

We change the scale of effective labor in R&D to $x \equiv \alpha\tilde{n}$ in order to be able to express aggregate consumption as $C_t = A\hat{l}_2(x_t) = A\alpha\hat{l}_0(\tilde{n}_t)$, requiring efficient extraction of natural resources equal to $\dot{S}_t = -A\hat{l}_2(x_t)/B_t = -A\alpha\hat{l}_0(\tilde{n}_t)/B_t$, and per capita consumption $c_t = C_t/\alpha = A\hat{l}_2(x_t)/\alpha = A\hat{l}_0(\tilde{n}_t)$. We use these transformations to restate the social planner problem. The objective function is unchanged if the social planner targets per capita utility. The laws of motion (8) and (6) are scaled by parameter $\alpha > 1$, since effective labor inputs in R&D are $x = \alpha\tilde{n}$. The first order condition with respect to $\tilde{n}$ is the same as (9) apart for the scale factor α multiplying the right-hand-side of (9). The phase diagram is slightly modified since it is characterized by $\dot{R}_t = \alpha\left(b\tilde{n}_t R_t - A\hat{l}_0(\tilde{n}_t)\right) = 0$ instead of

[2] This is an application of the replication principle according to which if population increases by a factor α without altering the distribution of skills and the shares of labor allocated to each sector are left unchanged, the effective labor inputs increase by the same factor α in each sector (case illustrated by the shift from point 0 to point 4 in the North-East panel of Figure 3).

(14), and by $d\tilde{n}_t/dt = [\rho - b\alpha\tilde{n}_t(1+1/\sigma)/(1-bX_t)]\,\tilde{n}_t/[\varepsilon\sigma - \eta bX_t/(1-bX_t)] = 0$ instead of (15). The steady state of the system is defined by $\tilde{n}^* = \rho/b\alpha$, $R^* = A\hat{l}_0(\tilde{n}^*)/b\tilde{n}^*$, $c^* = A\hat{l}_0(\tilde{n}^*)$. Translating these results in effective units of labor in R&D and production, using $\tilde{n} = x/\alpha$, $\hat{l}_2(x) = \alpha\hat{l}_0(\tilde{n})$ and the notation n for x, we obtain that at steady state:

$$\begin{aligned} n^* &\equiv x^* = \alpha\tilde{n}^* = \frac{\rho}{b} \\ R^* &= \frac{A}{\rho}\alpha\hat{l}_0(\tilde{n}^*) = \frac{A}{\rho}\hat{l}_2(x^*) \equiv \frac{A}{\rho}\hat{l}_2(n^*) \\ c^* &= A\hat{l}_0(\tilde{n}^*) = \frac{A}{\alpha}\hat{l}_2(x^*) \equiv \frac{A}{\alpha}\hat{l}_2(n^*) \end{aligned}$$

R&D effort is constant, since the quantity of effective units of labor in R&D is independent of population size (density). Constant n^* entails an expansion in per capita steady state consumption. In fact, substitute for $\tilde{n}^*$ into $c^* = A\hat{l}_0(\tilde{n}^*)$, which gives $c^* = A\hat{l}_0\left(\frac{\rho}{b\alpha}\right) > A\hat{l}_0\left(\frac{\rho}{b}\right)$ since $\alpha > 1$ and $\hat{l}_0' < 0$.