

FIFTY YEARS OF THE SPECTRUM PROBLEM:
SURVEY AND NEW RESULTS

APPENDIX

A. DURAND, N. D. JONES, J. A. MAKOWSKY, AND M. MORE

A review of some not easily accessible references

This appendix contains a detailed presentation of the material of Section 4. Note that the proofs sketched here do not necessarily correspond to the original proofs.

Asser's paper. In chronological ordering, the first paper related to spectra is [1], in German, due to Asser in 1955. Though it does not use the name “spectrum”, nor refer to Scholz in its title or in the text, the long introduction clarifies the context in which the concept of spectrum was born. The author addresses the general question of classes of cardinal numbers (not only natural numbers) so-called “representable” by a sentence of first-order logic with equality, both in the framework of satisfiability theory and validity theory. Here a first-order sentence φ represents a given (finite or infinite) cardinality m regarding satisfiability if there is a structure whose domain has cardinality m that is a model of φ (i.e., for finite m , it means $m \in \text{SPEC}(\varphi)$), and regarding validity, if φ holds in every structure with cardinality m . Asser first notices that φ represents m regarding satisfiability if and only if $\neg\varphi$ does not represent m regarding validity, so that validity reduces to satisfiability via complement. Then, he remarks that, from Löwenheim–Skolem Theorem [3], the representation question in satisfiability theory for infinite cardinalities is trivial: the first-order sentence φ either has no infinite model (and in this case it has finite models in finitely many finite cardinalities only) or has models in every infinite cardinality. Hence, the problem actually is about exactly which sets of natural numbers are the set of cardinalities of finite models of first-order sentences, i.e., what we would call spectra. In a footnote, one reads “this question was also asked by Scholz as a problem in [9]”.

With this background, Asser's aim is to give a purely arithmetical characterization of spectra. This is done via an arithmetical encoding of finite structures, first-order sentences and satisfiability. Let us make precise Asser's construction.

Note that in the sequel “characteristic functions” (of sets or relations) are not taken in the usual way: a unary function f is said to be the characteristic function of the set of integers n such that $f(n) = 0$. It is only a technical matter to come back to the usual definition with little machinery, for instance use $\chi(n) = 1 - f(n)$ (so-called *modified subtraction* i.e., $x \dot{-} y = x - y$ if $x \geq y$ and 0 otherwise). Using this alternative definition, characteristic functions are not required to be 0–1-valued.

Without loss of generality, let φ be a sentence in relational Skolem normal form, i.e., $\varphi \equiv \forall x_1 \dots \forall x_r \exists x_{r+1} \dots \exists x_s \psi(x_1, \dots, x_s)$, where $\psi(x_1, \dots, x_s)$ is a Boolean combination of atomic formulas $R_i^{(a_i)}(x_{j_1}, \dots, x_{j_{a_i}})$ with $i = 1, \dots, t$ and of atoms $x_{l_1} = x_{l_2}$. Assume that ψ contains u different atoms of type $R_i^{(a_i)}(x_{j_1}, \dots, x_{j_{a_i}})$ and v different atoms of type $x_{l_1} = x_{l_2}$. Let $\Psi: \{0, 1\}^{u+v} \rightarrow \{0, 1\}$ be the Boolean function associated to the propositional version of ψ (using the convention that 0 encodes true and 1 encodes false).

Denote by $Bit_k(y, z_1, \dots, z_k, n)$ the binary digit of y of rank $\sum_{l=1}^k z_l \cdot n^{l-1}$, assuming $y < 2^{n^k}$, $z_1 < n, \dots, z_k < n$. Encode the k -ary relation R on the domain $\{0, \dots, n-1\}$ by the number $y < 2^{n^k}$ such that $Bit_k(y, z_1, \dots, z_k, n) = 0$ if and only if $R(z_1, \dots, z_k)$ holds. Let $\delta(z_1, z_2) = 0$ if $z_1 = z_2$ and 1 otherwise. Obtain $\Psi^*(y_1, \dots, y_t, x_1, \dots, x_s, n)$ from Ψ by replacing each atom $R_i^{(a_i)}(x_{j_1}, \dots, x_{j_{a_i}})$ by $Bit_{a_i}(y_i, x_{j_1}, \dots, x_{j_{a_i}}, n)$ and every atom $x_{l_1} = x_{l_2}$ by $\delta(x_{l_1}, x_{l_2})$.

The first-order quantifiers $\forall x_1 \dots \forall x_r \exists x_{r+1} \dots \exists x_s$ are dealt with by defining

$$\Psi^{**}(y_1, \dots, y_t, n) = \sum_{x_1=0}^{n-1} \dots \sum_{x_r=0}^{n-1} \prod_{x_{r+1}=0}^{n-1} \dots \prod_{x_s=0}^{n-1} \Psi^*(\bar{y}, \bar{x}, n).$$

Note the non-standard use of \sum for \forall and \prod for \exists , due to the fact that 0 encodes true and 1 encodes false. Finally the characteristic function of the spectrum of the sentence $\varphi \equiv \forall x_1 \dots \forall x_r \exists x_{r+1} \dots \exists x_s \psi(x_1, \dots, x_s)$ is $\chi(n) = \prod_{y_1=0}^{2^{n^{a_1}}-1} \dots \prod_{y_t=0}^{2^{n^{a_t}}-1} \Psi^{**}(y_1, \dots, y_t, n)$. This construction is clearly elementary. Conversely, it is also easy to verify that any function defined as $\chi(n) = \prod_{y_1=0}^{2^{n^{a_1}}-1} \dots \prod_{y_t=0}^{2^{n^{a_t}}-1} \Psi^{**}(y_1, \dots, y_t, n)$, where Ψ^{**} is obtained from some Boolean function Ψ by the same type of construction, is the characteristic function of the spectrum of the corresponding first-order sentence. Hence we have the following result.

THEOREM 1. A set \mathcal{S} is a spectrum iff its characteristic function χ has the form $\chi(n) = \prod_{y_1=0}^{2^{n^{a_1}}-1} \dots \prod_{y_t=0}^{2^{n^{a_t}}-1} \Psi^{**}(y_1, \dots, y_t, n)$, where Ψ^{**} is obtained from some Boolean function Ψ by the above construction.

Note that Asser judges his result “non satisfactory”, in particular because this paraphrastic characterization is of no help in proving that a given set is

or not a spectrum, or in providing any concrete spectrum. However, Asser's characterization is enough to prove the following theorem (rephrasing Theorem 4.10 of the paper), that we restate here for sake of self-containment.

THEOREM 2. $\text{SPEC} \subsetneq \mathcal{E}_*^3$

The inclusion follows from the fact that Theorem 1 provides elementary characteristic functions for spectra. The properness is obtained by diagonalization.

As a conclusion, Asser asks some questions, that have essentially remained open up to now. First, he asked for a recursive characterization of spectra. He notes that there are actually two different problems. The first one asks for a recursively defined class of functions, i.e., a class of functions defined via some basis functions and closure under some functional operations, such that the unary functions in this class are exactly the characteristic functions of spectra. Second, he asks for a recursively defined class of functions, but now such that the unary functions in it enumerate exactly the spectra, i.e., a set S is a spectrum if and only if $S = f(\mathbb{N})$ for some f in the class. Note that this is not the most commonly admitted meaning for enumeration, because the enumeration functions are usually required to be strictly increasing, which is not the case here.

Next, Asser refers to “work in progress” that proves that a large class of unary functions are characteristic functions of spectra, among which the following arithmetically defined sets: prime numbers, multiples of a given integer k , powers of a given k , k -th powers, composite numbers.

Finally, the third and most famous open question proposed in this paper is usually known as Asser's Problem (Open Question 2) and asks whether spectra are closed under complement.

Mostowski's paper. A paper almost simultaneous with Asser's is [6], due to A. Mostowski in 1956. It also addresses recursive characterization of spectra, and explicitly uses the name “spectrum”. It is noticed that “The results of Asser overlap in part with results which I have found in 1953 while attempting (unsuccessfully) to solve Scholz's problem (cf. *Roczniki Polskiego Towarzystwa Matematycznego*, series I, vol. 1 (1955), p. 427). I shall give here proofs of my results which do not overlap with Asser's.”¹

A. Mostowski defines a class of functions denoted by K as follows.

DEFINITION 3. The class K is the least class

- containing the functions Z_k, U_k^i, S, C respectively defined by:
 - $Z_k(x_1, \dots, x_k, n) = 0$,
 - $U_k^i(x_1, \dots, x_k, n) = \min(x_i, n)$, for $i = 1, \dots, k$,

¹Thanks to J. Tomasik, we have seen a translation of the Polish reference. It is the abstract of a seminar given by Andrzej Mostowski on October 16, 1953. In addition to the following material, it is also stated that spectra form a strict subclass of primitive recursive sets, a result which indeed overlaps with Asser's.

- $S(x, n) = \min(x + 1, n)$,
- $C(x) = n$,
- closed under composition:
 $f(x_1, \dots, x_{j-1}, g(y_1, \dots, y_p, n), x_{j+1}, \dots, x_k, n)$,
- closed under recursion:

$$\begin{cases} f(0, \vec{x}, n) = g(\vec{x}, n), \\ f(x + 1, \vec{x}, n) = \min(h(x, f(x, \vec{x}, n), \vec{x}, n), n). \end{cases}$$

The basis functions Z_k , U_k^i and S are intended as the classical zero, projections and successor functions, but the special variable n always bounds their values. The function C is intended as a maximum function. The functional operations composition and recursion are also bounded by n . The main result of Mostowski's paper is the following theorem.

THEOREM 4. For any unary function $f \in K$, the set $\{n + 1 \mid f(n) = 0\}$ is a spectrum.

Let us give an idea of the proof via an example. Consider the functions f , g and h defined as follows:

- $f(x, n) = 1$ if $x = 0$ and $f(x, n) = 0$ otherwise.
 I.e.,
$$\begin{cases} f(0, n) = S(Z(C(n), n), n), \\ f(x + 1, n) = \min(Z(f(x, n), n), n). \end{cases}$$
- $g(x, n) = 0$ if x is even and $g(x, n) = 1$ otherwise.
 I.e.,
$$\begin{cases} g(0, n) = Z(C(n), n), \\ g(x + 1, n) = \min(f(g(x, n), n), n). \end{cases}$$
- $h(n) = g(C(n), n)$.

Clearly we have $h \in K$ and $h(n) = 0$ if and only if n is even. Let us derive from the definition of h a sentence ψ in the vocabulary

$$\sigma = \{\leq, \min, \max, Succ^{(2)}, R_f^{(3)}, R_g^{(3)}, R_h^{(2)}\}$$

such that ψ has a model with $n + 1$ elements if and only if $h(n) = 0$ (i.e., $SPEC(\psi)$ is the set of odd numbers). The key point of the construction is that the functions in the class K can be interpreted as functions on finite structures (e.g., from $\{0, \dots, n\}^k$ to $\{0, \dots, n\}$) without loss of generality, because of the special variable n that bounds all their values.

The sentence ψ first expresses the fact that \leq is a linear ordering, \min and \max are its first and last elements and $Succ$ its successor relation. Then, ψ describes the behavior of the predicates R_f , R_g and R_h corresponding to the graphs of the functions f , g and h . For instance, R_g obeys the conjunction of the following sentences:

- g is functional in its first variable:
 $\forall x, y (y = \max \longrightarrow \exists! z R_g(x, y, z))$,

- the second variable in g is always n :
 $\forall x, y, z (R_g(x, y, z) \longrightarrow y = \max)$,
- description of the base case of the definition of g :
 $\forall x, y, z [(R_g(x, y, z) \wedge x = \min) \longrightarrow z = \min]$,
- description of the recursive recursive case of the definition of g :

$$\forall x, y, z [(R_g(x, y, z) \wedge \neg x = \min) \longrightarrow \exists t, u (Succ(t, x) \wedge R_g(t, y, u) \wedge R_f(u, y, z))].$$

Our goal is then achieved by adding to ψ the following condition:

$$\forall x, y (R_h(x, y) \longrightarrow (x = \max \wedge y = \min)).$$

Finally, it is clear that $\text{SPEC}(\psi)$ is the set of odd numbers as required.

Mostowski asks if the converse is true, i.e.,

OPEN QUESTION 1. *Is every spectrum representable as $\{n + 1 \mid f(n) = 0\}$ for some function $f \in K$?*

No answer is known up to now.

As a conclusion, new examples of spectra are presented: the set of integers having the form $n!$ for some n , and the set $\{n \mid n^2 + 1 \text{ is prime}\}$. Also, Mostowski asks whether Fermat's prime numbers, i.e., primes of the form $2^{2^n} + 1$, form a spectrum. This question can be understood in two different ways, as noticed by Bennett: which one of the sets $A = \{p \mid p \text{ is prime and } p = 2^{2^n} + 1 \text{ for some integer } n\}$ and $B = \{n \mid 2^{2^n} + 1 \text{ is prime}\}$ is intended? Using rudimentary relations, the set A is easily proved to be a spectrum, whereas it is still not known for the set B .

Finally, we remark that it is ordinarily considered that what Mostowski proved is that the unary relations in \mathcal{E}_*^2 are spectra. This is not exactly the case, but the reason is most probably the fact that Bennett attributes this result to Mostowski. However, Bennett also notes that, even if it is easy to prove that $K \subseteq \mathcal{E}^2$, it is not clear that the bounded version of any function in \mathcal{E}^2 (i.e., $f_b(x_1, \dots, x_k, n) = \min(f(x_1, \dots, x_k), n)$) is in K . Mostowski's construction crucially relies on the fact that the functions in K are bounded by their last variable, and does not generalize to functions in \mathcal{E}^2 . In contrast, it is not difficult to verify that the bounded versions of addition and multiplication are in K , and consequently that the rudimentary relations have their characteristic functions in K . Whatever, it is true that the unary relations in \mathcal{E}_*^2 are indeed spectra, see Corollary 6.

Bennett's thesis. This is a huge work titled "On spectra" [2], but which also deals with a lot of other subjects. Bennett's thesis is unpublished, and only available via library services. It is one of the remarkable early texts anticipating later developments in finite model theory, definability theory and complexity theory. It contains a characterization (and various definitions) of rudimentary sets and already relates spectra to space bounded Turing

machines, thus catching a glimpse of many of the results concerning spectra that were formulated and proved in more modern language after 1970.

Not only first-order spectra are considered by Bennett, but also spectra of higher order logics, and not only sets, but also many-sorted sets, all in all spectra of the whole theory of types. This full generality makes the notations quite clumsy. The use of many-sorted structures corresponds to relations with arity greater than one, and the use of higher order logics provides more complicated relations.

We shall limit ourselves to the cases of one-sorted (i.e., ordinary) spectra of orders one and two. Note that the first item of Theorem 5 is also partially stated as Theorem 4.13.

- THEOREM 5** (Bennett, 1962 [2]). (i) A set $S \subseteq \mathbb{N}$ is a first-order spectrum iff it can be defined by a formula of the form $\exists y \leq 2^{x^j} R(x, y)$ for some $j \geq 1$, where R is strictly rudimentary.
- (ii) A set $S \subseteq \mathbb{N}$ is a second-order spectrum iff it can be defined by a formula of the form $\exists y \leq 2^{x^j} R(x, y)$ for some $j \geq 1$, where R is rudimentary.

Spectra of higher order are characterized by similar features: spectra of order $2n$ correspond to rudimentary relations prefixed by an existential quantifier bounded by an iterated exponential $2^{2^{\dots^{2^x}}}$ of height n , and spectra of order $2n - 1$ correspond to strictly rudimentary relations prefixed by an existential quantifier bounded by an iterated exponential of height n . Spectra of sentences over a d -sorted universe have the same types of characterizations, using $\exists y \leq 2^{\max(x_1, \dots, x_d)^j} R(x_1, \dots, x_d, y)$. Finally, the spectra of the entire type theory are characterized as the elementary relations.

Bennett also introduces several other subrudimentary classes, respectively called “strongly”, “positive” and “extended” rudimentary relations, which yield a bunch of slightly different characterizations of spectra, which may witness various unsuccessful attempts to design a truly satisfactory characterization. In this survey, we shall limit ourselves to RUD and SRUD.

Some consequences of the characterization theorem (not all of them are immediate):

- COROLLARY 6.** (i) For each $n \geq 1$, the class of spectra of order n is closed under \wedge, \vee , bounded quantifications, substitution of rudimentary functions, explicit transformations and finite modifications.
- (ii) For each $n \geq 1$, the class of spectra of order $2n$ is closed under \neg .
- (iii) The class of first-order spectra contains the rudimentary relations and \mathcal{E}_*^2 .
- (iv) The class of second-order spectra strictly contains the rudimentary relations.
- (v) For each $n \geq 1$, spectra of order n form a subset of spectra of order $n + 1$ and a strict subset of spectra of order $n + 2$.

We sketch below a proof of Bennett's theorem.

PROOF OF THEOREM 5. (ii) We first present the second-order case, because it has fewer technical difficulties.

– First inclusion: $\{\text{SPEC}(\varphi) \mid \varphi \in \text{SO}\} \subseteq \{\exists y \leq 2^{x^j} R(x, y) \mid j \geq 1 \text{ and } R \in \text{RUD}\}$ i.e., φ has a model with x elements iff $\exists y \leq 2^{x^j} R(x, y)$ is true.

W.l.o.g. we may assume that φ has no first-order or second-order free variables (just quantify existentially in case there are any). Assume the second-order variables appearing in φ have arities strictly less than j . Then we take $y = 2^{x^j}$. We encode a second-order variable Z with arity $a < j$ by the number $z < 2^{x^a} < y$ in the usual way. Hence, every second-order quantification $QZ^{(a)}$ in φ is translated into the first-order bounded quantification $Qz < 2^{x^a} < y$. Recall that $\text{Bit}(a, b)$ is true iff the bit of rank b of a is 1. Now, every atomic formula $Z(z_1, \dots, z_a)$ is translated into $\text{Bit}(z, z_1 + z_2 \cdot x + \dots + z_a \cdot x^{a-1})$. Every first-order quantification qz in φ is translated into the bounded quantification $qz < x$. The atomic formulas $z = z'$ in φ remain unchanged. Let φ' denote the obtained formula. Finally, let $R \equiv (y = 2^{x^j}) \wedge \varphi'$.

– Second inclusion: $\{\text{SPEC}(\varphi) \mid \varphi \in \text{SO}\} \supseteq \{\exists y \leq 2^{x^j} R(x, y) \mid j \geq 1 \text{ and } R \in \text{RUD}\}$ i.e., $\exists y \leq 2^{x^j} R(x, y)$ is true iff φ has a model with x elements.

First, we use three existentially quantified relations, namely $\leq^{(2j)}$ which is bound to be a linear ordering over the j -tuples of vertices, $+^{(3j)}$ which is bound to be the associated addition and $\times^{(3j)}$ which is bound to be the associated multiplication. Let us denote by $\text{Arithm}(\leq, +, \times)$ the first-order sentence expressing this requirement. Note that we may now use for free any usual arithmetic predicate on numbers bounded by x^j (written in x -ary notation, i.e., seen as j -tuples of integers in $\{0, \dots, x-1\}$). Next, all variables in R , including x and y , are encoded by j -ary second-order variables in φ in the usual way. For instance if $x = \sum_{l=0}^p 2^{xl}$, we let $X = \{(i_0, 0, \dots, 0), \dots, (i_p, 0, \dots, 0)\}$.

W.l.o.g. we may assume that all the atomic formulas in R are of type $u \cdot v = w$ (concatenation), which we translate into

$$\begin{aligned} \text{Concat}(U, V, W) \equiv & \\ & \exists \bar{t} V(\bar{t}) \wedge \forall \bar{z} (V(\bar{z}) \longrightarrow \bar{z} \leq \bar{t}) \wedge \forall \bar{z} (U(\bar{z}) \longrightarrow \bar{z} \leq (\overline{\max} - \bar{t})) \wedge \\ & \forall \bar{z} (W(\bar{z}) \longleftrightarrow ((\bar{z} \leq \bar{t} \wedge V(\bar{z})) \vee (\bar{z} > \bar{t} \wedge U(\bar{z} - \bar{t}))). \end{aligned}$$

Note that this sentence would be cleaner in dyadic than it is in binary, but the whole encoding would also be more complicated because two unary relations are needed to encode an integer in dyadic (the set of 1s and the set of 2s) because its length is fixed.

In order to translate the bounded quantifications in R , we also need the following first-order sentence, which expresses the fact that the integers u and v respectively encoded by U and V are such that $u < v$.

$$\begin{aligned} \text{Smaller}(U, V) \equiv & \exists \bar{z} (V(\bar{z}) \wedge \neg U(\bar{z}) \wedge \forall \bar{z}' > \bar{z} \neg U(\bar{z}')) \vee \\ & \exists \bar{z} (V(\bar{z}) \wedge U(\bar{z}) \wedge \forall \bar{z}' > \bar{z} (\neg V(\bar{z}') \wedge \neg U(\bar{z}')) \wedge \\ & \exists \bar{z}' < \bar{z} (V(\bar{z}') \wedge \neg U(\bar{z}'))). \end{aligned}$$

Now, let R' be obtained from R by applying the following rules: every bounded first-order quantification $\forall z < z' \dots$ is translated into the second order quantification $\forall Z^{(j)} \text{Smaller}(Z, Z') \longrightarrow \dots$, and accordingly for $\exists z < z' \dots$; and every atomic formula $u \cdot v = w$ is translated into $\text{Concat}(U, V, W)$.

It remains to express that X encodes the size x of the domain, which is done using the binary notation of $(\max, 0, \dots, 0)$, which represents the largest element of the domain. More precisely, we have $\max + 1 = x$, which translates in binary as follows:

$$\begin{aligned} \text{Dom}(X) \equiv & \forall \bar{z} (((X(\bar{z}) \wedge \forall \bar{z}' < \bar{z} \neg X(\bar{z}')) \longrightarrow \neg \text{Bit}((\max, 0, \dots, 0), \bar{z})) \wedge \\ & ((\exists \bar{z}' < \bar{z} X(\bar{z}')) \longrightarrow (X(\bar{z}) \longleftrightarrow \text{Bit}((\max, 0, \dots, 0), \bar{z}))) \wedge \\ & ((\exists \bar{z}' > \bar{z} (X(\bar{z}') \wedge \forall \bar{z}'' < \bar{z}' \neg X(\bar{z}'')))) \\ & \longrightarrow \text{Bit}((\max, 0, \dots, 0), \bar{z}))). \end{aligned}$$

Finally, φ is

$$\exists \leq^{(2j)} \exists +^{(3j)} \exists \times^{(3j)} \exists Y^{(j)} \exists X^{(j)} (\text{Arithm}(\leq, +, \times) \wedge \text{Dom}(X) \wedge R').$$

(i) Next we turn to the first-order case. We consider the proof of the second-order case and show how it has to be modified in order to fit to the first-order case. Note that the proof is now more tricky, and we use dyadic notation because we have to be more precise.

– First inclusion: $\{\text{SPEC}(\varphi) \mid \varphi \in \text{FO}\} \subseteq \{\exists y \leq 2^{x^j} R(x, y) \mid j \geq 1 \text{ and } R \in \text{SRUD}\}$

The main difference concerning φ is that it contains no second-order quantifications. Concerning R , we have to deal with two differences: bounded quantifications are now replaced by part-of quantifications ($\forall z_1 \upharpoonright z_2$ and $\exists z_1 \upharpoonright z_2$) on the one hand and we have to use concatenation instead of arithmetic on the other hand.

However, φ does contain free second-order variables, say $Z_1^{(a_1)}, \dots, Z_k^{(a_k)}$, which we do not encode in the usual way because SRUD does not allow to use arithmetical predicates, hence the *Bit* predicate is not available. Instead, we assume for now that the alphabet is $\{1, 2, *, \star, \bullet\}$ and we first define a provisional predicate $R'(x, y)$. We shall explain later how to get rid of the extra symbols $*$, \star and \bullet to obtain the expected $R(x, y)$.

We use the following encoding: if $Z = \{(x_1^1, \dots, x_a^1), \dots, (x_1^p, \dots, x_a^p)\}$, with $p \leq x^a$, then let $z = \star * x_1^1 * \dots * x_a^1 * \star \dots * \star * x_1^p * \dots * x_a^p * \star$. Note that we have $|z| \leq x^a \cdot a \cdot (|x| + 2)$.

Let us define $x_0 = * x * (x - 1) * \dots * 1 *$, i.e., the dyadic representation of x_0 is the concatenation of the dyadic representations of all integers in $\{1, \dots, x\}$, separated by $*$ s. Note that $|x_0| \leq x \cdot (|x + 2|) < x^2$. Finally, let $y = \bullet z_1 \bullet \dots \bullet z_k \bullet x_0 \bullet$. Clearly we have $y \leq 2^{x^j}$ for some $j \geq 1$.

Now, $R'(x, y)$ will begin with $\exists z_1 \uparrow y \dots \exists z_k \uparrow y \exists x_0 \uparrow y ((y = \bullet z_1 \bullet \dots \bullet z_k \bullet x_0 \bullet) \wedge \neg(\bullet \uparrow z_1) \wedge \dots \wedge \neg(\bullet \uparrow z_k) \wedge \neg(\bullet \uparrow x_0) \wedge \dots)$, in order to retrieve the significant parts of y .

We use x_0 to replace every first-order quantification $\forall u \dots$ appearing in φ by a part-of quantification $\forall u \uparrow x_0 (Int(u, x_0) \longrightarrow \dots)$ in R' , and similarly for $\exists u \dots$, where $Int(u, x_0)$ means that u is a maximal non-empty string of 1s and 2s in x_0 . The most technical part of the proof is to write a strictly rudimentary formula $Dom(x_0, x)$ which is true iff x_0 has the expected form, but for sake of brevity, we do not show this formula explicitly. In particular, note that we now consider the domain as $\{1, \dots, x\}$ instead of $\{0, \dots, x - 1\}$ as we did previously. Finally it is not difficult to write a formula $Verif(x_0, z)$ expressing the fact that z has the expected form $\star * x_1^1 * \dots * x_a^1 * \star \dots * \star * x_1^p * \dots * x_a^p * \star$. Namely, take

$$\begin{aligned} Verif(x_0, z) &\equiv \exists u \uparrow z \\ &(\star u \star \uparrow z) \wedge \forall u \uparrow z (((\star u \star \uparrow z) \wedge \neg(\star \uparrow u) \wedge u \neq \epsilon) \longrightarrow \exists v_1 \uparrow u \dots \exists v_a \uparrow u \\ &\quad (Int(v_1, x_0) \wedge \dots \wedge Int(v_a, x_0) \wedge u = * v_1 * \dots * v_a *)) \\ &\wedge \forall u_1, u_2, \alpha, \beta, \gamma \uparrow z (((z = \alpha * u_1 * \beta * u_2 * \gamma \vee z = \alpha * u_1 * u_2 * \gamma) \\ &\quad \wedge \neg(\star \uparrow u_1) \wedge \neg(\star \uparrow u_2)) \longrightarrow u_1 \neq u_2). \end{aligned}$$

There are two types of atomic formulas in φ : equalities $z_1 = z_2$ and atoms $Z(z_1, \dots, z_a)$. Equalities remain unchanged and $Z(z_1, \dots, z_a)$ is changed into $\star * z_1 * \dots * z_a * \star \uparrow z$. These operations lead to the strictly rudimentary formula φ' .

Finally, $R'(x, y)$ is $\exists z_1 \uparrow y \dots \exists z_k \uparrow y \exists x_0 \uparrow y ((y = \bullet z_1 \bullet \dots \bullet z_k \bullet x_0 \bullet) \wedge \neg(\bullet \uparrow z_1) \wedge \dots \wedge \neg(\bullet \uparrow z_k) \wedge \neg(\bullet \uparrow x_0) \wedge Dom(x_0, x) \wedge Verif(x_0, z_1) \wedge \dots \wedge Verif(x_0, z_k) \wedge \varphi')$.

To obtain R , it remains to get rid of the alphabet $\{1, 2, *, \star, \bullet\}$. Let $*$ be a string of 1s which is not a subword of $x, x - 1, \dots, 2$ and 1. For instance, $*$ could be of length $|x| + 1$. Let $\star = 2 * 2$ and $\bullet = 22 * 22$. The final length of y is polynomially longer than it used to be, which remains acceptable. Finally, take $R(x, y) \equiv \exists * \uparrow y \exists \star \uparrow y \exists \bullet \uparrow y ((\forall u \uparrow * (u = 1)) \wedge * \neq \epsilon \wedge \star = 2 * 2 \wedge \bullet = 22 * 22 \wedge R')$. Note that strictly rudimentary relations do not define predicates referring to the length of integers, so that $*$ cannot be bound to be some specific word like $1^{|x|+1}$.

– Second inclusion: $\{\text{SPEC}(\varphi) \mid \varphi \in \text{FO}\} \supseteq \{\exists y \leq 2^{x^j} R(x, y) \mid j \geq 1 \text{ and } R \in \text{SRUD}\}$

The main difference with the second-order case concerning φ is that it only contains first-order quantifications. However, we are still free to choose as many free second-order variables as we may need. In particular, we still use usual arithmetic predicates on the (j -tuples of) elements of the domain, and the previous first-order sentence *Arithm*($\leq, +, \times$) is still required to hold for this purpose. In addition, we introduce the second-order variables X_1, X_2 and Y_1, Y_2 , both of arity j , respectively representing the set of positions where x and y have 1s and 2s and no other second-order variables are introduced. Let $Word(X_1, X_2)$ be the sentence expressing the fact that X_1 and X_2 (and similarly Y_1, Y_2) do represent a dyadic word, namely

$$\begin{aligned} Word(X_1, X_2) \equiv & \forall \bar{z} \neg (X_1(\bar{z}) \wedge X_2(\bar{z})) \wedge \\ & \exists \bar{z} \forall \bar{t} ((\bar{t} > \bar{z} \longrightarrow (\neg X_1(\bar{t}) \wedge \neg X_2(\bar{t}))) \wedge \\ & (\bar{t} \leq \bar{z} \longrightarrow (X_1(\bar{t}) \vee X_2(\bar{t}))). \end{aligned}$$

Concerning R , we may assume w.l.o.g. that it only contains part-of quantifications $qz \upharpoonright x$ and $qz \upharpoonright y$ and no $qz \upharpoonright z'$ for $z' \notin \{x, y\}$.

The main trick is that a part-of quantification $\exists z \upharpoonright y \dots$ (for instance) will be replaced by $2j$ first-order quantifications $\exists \bar{z}_1 \exists \bar{z}_2 (\bar{z}_1 \leq \bar{z}_2 \wedge \dots)$, where \bar{z}_1 and \bar{z}_2 encode the positions where z begins and ends, as a subword of y .

We have to translate the atomic formulas $u \cdot v = w$. W.l.o.g. we may rewrite R in an equivalent formula by replacing everywhere $u \cdot v = w$ with $(u \cdot v = w \wedge u \upharpoonright y \wedge v \upharpoonright y \wedge w \upharpoonright y) \vee (u \cdot v = w \wedge u \upharpoonright y \wedge v \upharpoonright y \wedge w \upharpoonright x) \vee \dots \vee (u \cdot v = w \wedge u \upharpoonright x \wedge v \upharpoonright x \wedge w \upharpoonright x)$. Hence, there are 8 slightly different cases to be taken care of. We limit ourselves with the case $u \cdot v \cdot w \wedge u \upharpoonright y \wedge v \upharpoonright y \wedge w \upharpoonright y$. The corresponding formula $Concat_{yyy}(\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2, \bar{w}_1, \bar{w}_2)$ is as follows:

$$\begin{aligned} \bar{w}_2 = & \bar{w}_1 + \bar{u}_2 - \bar{u}_1 + \bar{v}_2 - \bar{v}_1 \wedge \\ \forall \bar{z} (\bar{w}_1 \leq \bar{z} < \bar{w}_1 + \bar{u}_2 - \bar{u}_1 \longrightarrow & \\ & ((Y_1(\bar{z}) \longleftrightarrow Y_1(\bar{z} - \bar{w}_1)) \wedge (Y_2(\bar{z}) \longleftrightarrow Y_2(\bar{z} - \bar{w}_1)))) \wedge \\ \forall \bar{z} (\bar{w}_1 + \bar{u}_2 - \bar{u}_1 \leq \bar{z} < \bar{w}_2 \longrightarrow & ((Y_1(\bar{z}) \longleftrightarrow Y_1(\bar{z} - \bar{w}_1 - \bar{u}_2 + \bar{u}_1)) \wedge \\ & (Y_2(\bar{z}) \longleftrightarrow Y_2(\bar{z} - \bar{w}_1 - \bar{u}_2 + \bar{u}_1))))). \end{aligned}$$

Let us denote by R' the obtained sentence.

The last remaining part is to write out a sentence $Dom'(X_1, X_2)$ expressing the fact that X_1, X_2 encodes (in dyadic) the cardinality of the domain, i.e., the successor of the j -tuple $(\max, 0, \dots, 0)$. This is a bit more technical than the sentence $Dom(X)$ we used for the binary notation and we do not spell it out here. Finally, take $\varphi \equiv \text{Arithm} \wedge Word(X_1, X_2) \wedge Word(Y_1, Y_2) \wedge Dom(X_1, X_2) \wedge R'$. \dashv

Connections with complexity classes. At the beginning of complexity theory, the usual complexity classes such as the polynomial hierarchy had not yet emerged. So the classes used by Bennett are not standard ones. He considers two hierarchies based on space-bounded deterministic Turing machines defined in a recursive fashion: the base class is of type $FDSpace(f(n))$, and the next class has a space bound which is a function in the previous class.

Let us denote by $(\mathcal{R}^i)_{i \geq 1}$ the first hierarchy, introduced in Ritchie's 1963 paper [8], which comes from his Ph.D. thesis [7].

- DEFINITION 7 (Ritchie's classes). – Let \mathcal{R}^1 be the class of functions computable by some (deterministic) Turing machine in space bounded by $b \cdot \max(\vec{x})$ on input \vec{x} , where $b \geq 1$ is some integer fixed for each machine, i.e., $\mathcal{R}^1 = FDSpace(\mathcal{O}(2^n))$ in modern notation.
- For each $i \geq 1$, let us denote by \mathcal{R}^{i+1} the class of functions computable by a Turing machine in space bounded by $B(\vec{x})$, where B is some function in \mathcal{R}^i , fixed for each machine.
 - For each $i \geq 1$, let us denote by \mathcal{R}_*^i the class of relations whose characteristic functions are in \mathcal{R}^i .

It is proved in [8] that this hierarchy $(\mathcal{R}_*^i)_{i \geq 1}$ is strict and that its union corresponds to elementary relations.

Using the same pattern, Bennett introduces a second hierarchy, that we denote by $(\mathcal{B}^i)_{i \geq 1}$.

- DEFINITION 8 (Bennett's classes). – Let \mathcal{B}^1 be the class of functions computable by some (deterministic) Turing machine in space bounded by $P(\vec{x})$ on input \vec{x} , where P is some arithmetical polynomial fixed for each machine, i.e., $\mathcal{B}^1 = FDSpace((2^{\mathcal{O}(n)}))$ in modern notation.
- For each $i \geq 1$, let us denote by \mathcal{B}^{i+1} the class of functions computable by a Turing machine in space bounded by $B(\vec{x})$, where B is some function in \mathcal{B}^i , fixed for each machine.
 - For each $i \geq 1$, let us denote by \mathcal{B}_*^i the class of relations whose characteristic functions are in \mathcal{B}^i .

Bennett shows that Ritchie's classes \mathcal{R}_*^i come in between spectra of various orders, but not in a very nice way. In contrast, he proves nice closure properties and an exact intercalation between the classes of spectra of consecutive orders for the classes \mathcal{B}_*^i . However, all these classes are too big to be informative concerning relationship between first-order spectra and complexity classes.

In order to state the next theorem, let us denote by \mathcal{S}^i the class of (many-sorted) spectra of formulas of order i . For instance SPEC is the class of unary relations in \mathcal{S}^1 .

- THEOREM 9. (i) $\mathcal{R}_*^1 \subseteq \mathcal{S}^3$ and for each $i \geq 2$, $\mathcal{S}^{2i-2} \subseteq \mathcal{R}_*^i \subseteq \mathcal{S}^{2i+1}$.
 Moreover, for no $i, j \geq 1$ does $\mathcal{R}_*^i = \mathcal{S}^j$.

- (ii) For each $i \geq 1$, $\mathcal{S}^{2i} \subseteq \mathcal{B}_*^i \subseteq \mathcal{S}^{2i+1}$ (equality or strictness is unknown) and $\mathcal{R}_*^i \subsetneq \mathcal{B}_*^i \subsetneq \mathcal{R}_*^{i+1}$. Moreover, \mathcal{B}_*^i is closed with respect to union, intersection, bounded quantifications, substitution of rudimentary functions, explicit transformations and finite modifications.

The proof of item (i) is based on recursive characterizations of the classes \mathcal{R}_*^i , whereas item (ii) is stated without proof.

Mo's paper. There is a late paper on the recursive aspect of spectra, namely [5], due to the Chinese logician Mo Shaokui in 1991, only available in Chinese (see the author's English abstract in Mathematics Abstracts of Zentralblatt [4]). With the help of Zhu Ping [10], we have been able to state Mo's result, and we propose a proof sketch.

DEFINITION 10. Let x, x_1, x_2, \dots and y, y_1, y_2, \dots be two disjoint sets of variables. Let **MO** be the smallest class of predicates over integers containing the relations $x_1 + x_2 = x_3$, $x_1 \times x_2 = x_3$ (both for variables of type x only) and $Bit(y, x)$ (where the first variable is of type y , the second of type x) and closed under Boolean operations and (polynomially) bounded quantifications for variables of type x only.

Note that a predicate in **MO** has two types of variables, which do not play similar roles, and that **MO** extends the rudimentary relations by the use of $Bit(y, x)$ atoms, which are not definable because y variables are not allowed in the atomic formulas for addition and multiplication.

THEOREM 11.

$$\{\text{SPEC}(\varphi) \mid \varphi \in FO\} = \{\exists y_1 \leq 2^{x^{j_1}} \dots \exists y_k \leq 2^{x^{j_k}} \\ R(x, y_1, \dots, y_k) \mid k, j_1, \dots, j_k \geq 1 \text{ and } R \in \text{MO}\}$$

PROOF. It is a slightly modified version of the proof of Bennett's theorem for second-order spectra.

– First inclusion: φ has a model with x elements iff $\exists y_1 \leq 2^{x^{j_1}} \dots \exists y_k \leq 2^{x^{j_k}} R(x, y_1, \dots, y_k)$ is true.

We encode a predicate symbol Y with arity j by the number $y < 2^{x^j}$ in the usual way. Hence, every atomic formula $Y(x_1, \dots, x_j)$ is translated into $\exists x' < x^j (Bit(y, x') \wedge x' = x_1 + x_2 \cdot x + \dots + x_j \cdot x^{j-1})$. Every first-order quantification qx_i in φ is translated into the bounded quantification $qx_i < x$. The atomic formulas $x_1 = x_2$ in φ remain unchanged. Let R denote the obtained formula with free variables x, y_1, \dots, y_k .

– Second inclusion: $\exists y_1 \leq 2^{x^{j_1}} \dots \exists y_k \leq 2^{x^{j_k}} R(x, y_1, \dots, y_k)$ is true iff φ has a model with x elements.

First, we use three predicate symbols, namely $\leq^{(2)}$ which is bound to be a linear ordering on the vertices, $+^{(3)}$ which is bound to be the associated addition and $\times^{(3)}$ which is bound to be the associated multiplication. Let us

denote by $Arithm_1(\leq, +, \times)$ the first-order sentence expressing this requirement. Note that we may now use for free any usual arithmetic predicate on numbers bounded by x .

Next, every free variable of type y in R and bounded by 2^{x^j} is translated into a predicate symbol Y of arity j .

W.l.o.g. we may assume that all the bounded quantifications in R are of type $qx' < x^i$ for some $i \geq 1$. The bounded quantification $qx' < x^i$ in R is simply translated into $qx'_1 \dots qx'_i$ and x' is represented by the i -tuple (x'_1, \dots, x'_i) .

There are three types of atomic formulas in R . Let us first consider formulas $x_1 + x_2 = x_3$ and $x_1 \times x_2 = x_3$. Assume we have $x_1 < x^{i_1} \leq x^j, x_2 < x^{i_2} \leq x^j, x_3 < x^{i_3} \leq x^j$, with $j = \max(i_1, i_2, i_3)$. The variables x_1, x_2, x_3 correspond to the tuples $(x_1^1, \dots, x_1^j), (x_2^1, \dots, x_2^j), (x_3^1, \dots, x_3^j)$ (padding with as many 0s as necessary). This includes the case $x < x^2$ so that x corresponds to $(0, 1, 0, \dots, 0)$. Then $x_1 + x_2 = x_3$ is changed into $Add_j(x_1^1, \dots, x_1^j, x_2^1, \dots, x_2^j, x_3^1, \dots, x_3^j)$ and $x_1 \times x_2 = x_3$ is changed into $Mult_j(x_1^1, \dots, x_1^j, x_2^1, \dots, x_2^j, x_3^1, \dots, x_3^j)$, where the formulas Add_j and $Mult_j$ express addition and multiplication on j -tuples in x -ary notation. The case of atomic formulas $Bit(y, x')$ is dealt with similarly. Assume we have $y < 2^{x^j}$, and $x' < x^i$ then there are three possibilities. If $i < j$, then $Bit(y, x')$ is changed into $Y(x'_1, \dots, x'_i, 0, \dots, 0)$. If $i = j$, then $Bit(y, x')$ is changed into $Y(x'_1, \dots, x'_j)$. If $i > j$, then $Bit(y, x')$ is changed into $Y(x'_1, \dots, x'_j) \wedge x_{j+1} = 0 \wedge \dots \wedge x_i = 0$. Similarly, $Bit(y, x)$ (which may also occur in R because x is a free variable of type x) translates into $Y(0, 1, 0, \dots, 0)$ if Y has arity 2 at least and into $0 \neq 0$ (false) if Y is unary. Let us denote by R' the first-order sentence thus obtained.

Finally, φ is $Arithm_1(\leq, +, \times) \wedge R'$. -†

Note that, in order to make the proofs more similar to those of Bennett's theorem and help comparison, we have slightly modified the original statement in two points. First, Mo uses functional vocabularies, which yields bounds of type x^{x^j} for y type variables and the use of atoms $Digit_x(y, x') = x''$ (meaning "the digit of rank x' of y in x -ary notation is x'' ") instead of $Bit(y, x)$. Second, the relation R is originally described using Grzegorzcyk's classes $\mathcal{E}_*^0, \mathcal{E}_*^1$ or \mathcal{E}_*^2 instead of RUD .

Finally, concerning Asser's problem (so-called second Scholz problem here), the author's abstract [4] asserts that:

It is also shown that if all the functions in \mathcal{E}^0 can be enumerated by a function in \mathcal{E}^2 , then the complement of a certain finite spectrum cannot be any finite spectrum. Hence, under such a condition, the answer to the second Scholz problem is negative.

Hence, the conditional negative solution proposed here seems to be linked to some separation of \mathcal{E}^0 and \mathcal{E}^2 via diagonalization, which seems unlikely

(since the classical proof of separation of \mathcal{E}^i and \mathcal{E}^{i+1} uses the bound on the growth of the functions in \mathcal{E}^i).

REFERENCES

- [1] G. ASSER, *Das Repräsentantenproblem in Prädikatenkalkül der ersten Stufe mit Identität*, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 1 (1955), pp. 252–263.
- [2] J. H. BENNETT, *On spectra*, Ph.D. thesis, Princeton University, Princeton, New Jersey, USA, 1962.
- [3] L. LÖWENHEIM, *Über Möglichkeiten im Relativkalkül*, *Mathematische Annalen*, vol. 76 (1915), no. 4, pp. 447–470.
- [4] S. Mo, *Abstract of [5]*, Zbl 0729.03004.
- [5] ———, *The solution of Scholz problems*, *Chinese Annals of Mathematics. Series A*, vol. 12 (1991), no. 1, pp. 89–97.
- [6] A. MOSTOWSKI, *Concerning a problem of H. Scholz*, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 12 (1956), pp. 210–214.
- [7] R. W. RITCHIE, *Classes of recursive functions of predictable complexity*, Ph.D. thesis, Princeton University, 1960.
- [8] ———, *Classes of predictably computable functions*, *Transactions of the American Mathematical Society*, vol. 106 (1963), pp. 139–173.
- [9] H. SCHOLZ, *Ein ungelöstes Problem in der symbolischen Logik*, *The Journal of Symbolic Logic*, vol. 17 (1952), p. 160.
- [10] P. ZHU, *Translation of [5]*, Personal communication.

IMJ – PROJET LOGIQUE
 UNIVERSITÉ PARIS DIDEROT
 PARIS 7, FRANCE
E-mail: durand@logique.jussieu.fr

DIKU
 UNIVERSITY OF COPENHAGEN
 DENMARK
E-mail: neil@diku.dk

FACULTY OF COMPUTER SCIENCE
 TECHNION
 HAIFA, ISRAEL
E-mail: janos@cs.technion.ac.il

UNIV CLERMONT 1
 LAIC
 FRANCE
E-mail: malika.more@u-clermont1.fr