APPENDICES FOR: REVIVING LEGISLATIVE AVENUES FOR GERRYMANDERING REFORM WITH A FLEXIBLE, AUTOMATED TOOL

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Appendix A. Expected Seats for Additional States

This appendix presents the numbers of seats expected to accrue to each party, for several elections in nine additional states: Florida, Illinois, Louisiana, Maryland, Minnesota, North Carolina, Tennessee, Texas, and Wisconsin. These are shown in Tables A.1-A.9

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Date: November 23, 2019.

	2008 Presidential
107th Congress	14.1
111th Congress	10.8
114th Congress	10.0
Areal Distance	13.3
Axis Ratio	12.8
Circumscribing Circles	13.5
Distance to Perimeter	13.2
Dynamic Radius	13.1
Exchange	13.2
Harmonic Radius	13.4
Hull Area	12.9
Hull Population	12.9
Inertia Area	13.3
Inertia Population	12.9
Inscribed Circles	12.7
Isoperimeter Quotient	13.0
Mean Radius	13.1
Path Fraction	13.3
Population Distance	13.1
Power Diagram	13.6
Split-Line	13.0

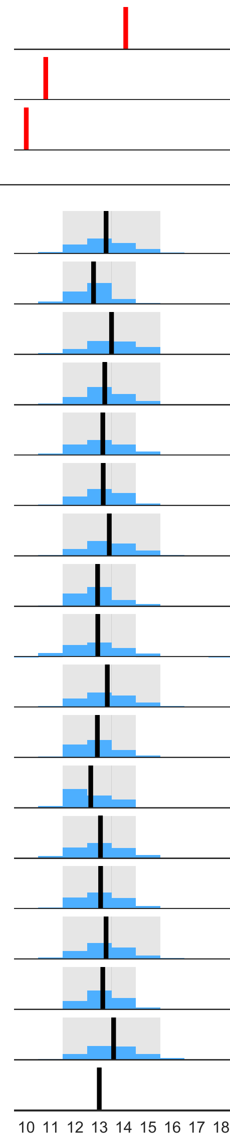


Table A.1. Votes from presidential elections in Florida are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 27 assigned after the 2010 Census.

	2008 Presidential	2016 Presidential
107th Congress	15.3	11.7
111th Congress	15.2	11.4
114th Congress	16.0	11.0
Areal Distance	15.9	11.3
Axis Ratio	16.1	11.7
Circumscribing Circles	16.3	11.5
Distance to Perimeter	16.1	11.7
Dynamic Radius	16.4	11.5
Exchange	16.4	11.6
Harmonic Radius	16.3	11.5
Hull Area	16.4	11.7
Hull Population	16.3	11.4
Inertia Area	16.3	11.5
Inertia Population	16.4	11.6
Inscribed Circles	16.4	11.6
Isoperimeter Quotient	16.3	11.5
Mean Radius	16.3	11.5
Path Fraction	16.3	11.5
Population Distance	16.1	11.5
Power Diagram	16.3	11.6
Split-Line	16.0	12.0
	14 15 16 17 18	10 11 12 13

TABLE A.2. Votes from presidential elections in Illinois are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 18 assigned after the 2010 Census.

	2012 Presidential	2016 Presidential
107th Congress	0.9	0.9
111th Congress	0.9	0.9
114th Congress	1.0	1.0
Areal Distance	0.8	0.8
Axis Ratio	0.9	0.9
Circumscribing Circles	1.0	0.9
Distance to Perimeter	0.7	0.7
Dynamic Radius	0.8	0.8
Exchange	0.9	1.0
Harmonic Radius	0.8	0.8
Hull Area	0.8	0.8
Hull Population	0.9	0.8
Inertia Area	0.8	0.8
Inertia Population	0.9	0.9
Inscribed Circles	0.8	0.8
Isoperimeter Quotient	0.9	0.9
Mean Radius	0.9	0.9
Path Fraction	0.8	0.8
Population Distance	0.8	0.8
Power Diagram	0.9	0.9
Split-Line	1.0	1.0

Table A.3. Votes from presidential elections in Louisiana are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 6 assigned after the 2010 Census.

	2008 Presidential		2016 Presidential	
107th Congress	5.0		5.0	
111th Congress	6.0		6.0	
114th Congress	7.0		7.0	
Areal Distance	6.1		5.9	
Axis Ratio	6.1		5.9	
Circumscribing Circles	6.2		5.9	
Distance to Perimeter	6.2		5.8	
Dynamic Radius	6.2		5.9	
Exchange	6.2		5.9	
Harmonic Radius	6.3		5.9	
Hull Area	6.2		5.8	
Hull Population	6.3		5.8	
Inertia Area	6.2		5.9	
Inertia Population	6.2		5.9	
Inscribed Circles	6.2		5.9	
Isoperimeter Quotient	6.2		5.9	
Mean Radius	6.2		5.9	
Path Fraction	6.3		5.7	
Population Distance	6.1		5.9	
Power Diagram	6.0		5.8	
Split-Line	6.0		6.0	
		5 6 7		4 5 6 7

Table A.4. Votes from presidential elections in Maryland are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 8 assigned after the 2010 Census.

	2008 Presidential	2012 Presidential	2016 Presidential
107th Congress	6.0	5.0	3.0
111th Congress	5.0	5.0	3.0
114th Congress	6.0	6.0	3.0
Areal Distance	5.3	5.2	3.2
Axis Ratio	5.8	5.3	3.6
Circumscribing Circles	5.7	5.4	3.5
Distance to Perimeter	5.9	5.5	3.8
Dynamic Radius	5.6	5.2	3.4
Exchange	5.6	5.3	3.5
Harmonic Radius	5.4	5.0	3.2
Hull Area	6.1	5.8	4.0
Hull Population	5.5	5.1	3.4
Inertia Area	5.5	5.2	3.4
Inertia Population	5.8	5.5	3.6
Inscribed Circles	5.7	5.0	3.4
Isoperimeter Quotient	5.7	5.2	3.5
Mean Radius	5.6	5.2	3.4
Path Fraction	5.7	5.1	3.5
Population Distance	5.7	5.4	3.5
Power Diagram	5.4	5.2	3.2
Split-Line	6.0	6.0	4.0

TABLE A.5. Votes from presidential elections in Minnesota are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 8 assigned after the 2010 Census.

	2012 Presidential	2016 Presidential
107th Congress	3.2	4.3
111th Congress	6.0	5.0
114th Congress	3.0	3.0
Areal Distance	6.0	6.0
Axis Ratio	5.7	5.3
Circumscribing Circles	5.6	5.2
Distance to Perimeter	5.9	5.5
Dynamic Radius	5.8	5.5
Exchange	5.8	5.3
Harmonic Radius	6.1	5.3
Hull Area	5.7	5.3
Hull Population	5.8	5.5
Inertia Area	5.7	5.4
Inertia Population	5.7	5.9
Inscribed Circles	5.9	5.3
Isoperimeter Quotient	5.7	5.3
Mean Radius	5.9	5.5
Path Fraction	5.7	5.3
Population Distance	6.2	6.0
Power Diagram	6.1	6.1
Split-Line	5.0	5.0

Table A.6. Votes from presidential elections in North Carolina are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 13 assigned after the 2010 Census.

	2016 Presidential
107th Congress	2.0
111th Congress	2.0
114th Congress	2.0
Areal Distance	1.4
Axis Ratio	1.3
Circumscribing Circles	1.3
Distance to Perimeter	1.3
Dynamic Radius	1.5
Exchange	1.5
Harmonic Radius	1.7
Hull Area	1.3
Hull Population	1.5
Inertia Area	1.5
Inertia Population	1.5
Inscribed Circles	1.7
Isoperimeter Quotient	1.4
Mean Radius	1.6
Path Fraction	1.5
Population Distance	1.5
Power Diagram	1.3
Split-Line	2.0

Table A.7. Votes from presidential elections in Tennessee are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 9 assigned after the 2010 Census.

	2004 Presidential	2008 Presidential	2012 Presidential	2016 Presidential
107th Congress	10.8	14.4	13.2	15.6
111th Congress	7.9	12.4	11.2	15.8
114th Congress	8.0	12.0	11.0	14.0
Areal Distance	8.3	12.2	11.6	15.0
Axis Ratio	8.0	12.4	11.6	14.9
Circumscribing Circles	8.2	12.4	11.6	15.0
Distance to Perimeter	8.3	12.3	11.6	14.6
Dynamic Radius	8.2	12.6	11.9	15.4
Exchange	8.2	12.6	11.9	15.2
Harmonic Radius	8.2	12.4	11.8	15.1
Hull Area	8.1	12.5	11.4	14.5
Hull Population	8.2	12.7	11.9	15.3
Inertia Area	8.2	12.4	11.7	15.2
Inertia Population	8.2	11.9	11.2	15.1
Inscribed Circles	8.3	13.1	12.4	15.4
Isoperimeter Quotient	8.1	12.5	11.7	15.2
Mean Radius	8.2	12.6	11.9	15.3
Path Fraction	8.1	12.5	11.7	15.1
Population Distance	8.5	12.4	11.6	15.4
Power Diagram	8.8	11.8	11.3	14.9
Split-Line	11.0	15.0	15.0	17.0
	4 5 6 7 8 9 10 11 12	8 9 10 11 12 13 14 15 16 17	7 8 9 10 11 12 13 14 15 16	11 12 13 14 15 16 17 18 19 20

Table A.8. Votes from presidential elections in Texas are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 36 assigned after the 2010 Census.

	2004 Presidential	2008 Presidential	2012 Presidential	2016 Presidential
107th Congress	4.4	7.1	4.4	2.7
111th Congress	4.0	7.0	4.0	2.0
114th Congress	3.0	7.0	3.0	2.0
Areal Distance	3.6	6.8	4.7	2.6
Axis Ratio	3.8	6.9	4.7	2.9
Circumscribing Circles	3.6	6.8	4.5	2.7
Distance to Perimeter	3.7	7.0	4.7	2.7
Dynamic Radius	3.8	6.8	4.7	3.0
Exchange	3.8	6.9	4.6	2.9
Harmonic Radius	3.7	6.7	4.3	2.7
Hull Area	3.5	7.1	4.6	2.7
Hull Population	3.6	6.8	4.4	2.7
Inertia Area	3.7	6.8	4.7	3.0
Inertia Population	3.8	6.8	4.7	2.8
Inscribed Circles	3.7	6.8	4.7	2.7
Isoperimeter Quotient	3.7	6.9	4.7	2.8
Mean Radius	3.8	6.8	4.7	3.0
Path Fraction	3.6	7.0	4.6	2.9
Population Distance	3.6	6.6	4.3	2.4
Power Diagram	4.1	6.9	4.8	2.7
Split-Line	3.0	7.0	5.0	2.0
	1 2 3 4 5 6	5 6 7 8	3 4 5 6 7	2 3 4 5

Table A.9. Votes from presidential elections in Wisconsin are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 8 assigned after the 2010 Census.

Appendix B. Automated Districting Procedures

Automation has long held promise as a "solution" to gerrymandering. As early as 1961, Vickrey wrote that "elimination of gerrymandering would seem to require the establishment of an automatic and impersonal procedure for carrying out a redistricting." By just 1963, Weaver and Hess began applying infrastructure built for the "warehouse problem" to districting in small cases (Delaware).

In the decades since, political observers presumed that with the political will, the districting problem would wilt before the burgeoning capacity of computing. In *Karcher v. Daggett* for instance, Brennan opined optimistically that "rapid advances in computer technology [...] make it relatively simple to draw contiguous districts of equal population and at the same time to further whatever secondary goals the State has." (462 U.S. 725, 1983) This has taken longer than expected. Districting is exactly the type of (NP-hard) categorization problem that computers struggle with. So while computers quickly proved useful for tabulation and error checks, humans have until very recently performed much of the partitioning by hand. (Altman et al. 2005) Yet with appropriate metaheuristics and approximations, solutions are now in reach. In the past few years, a number of scholars have implemented serviceable automated procedures for individual metrics or algorithms. (Chen and Rodden 2013, 2015; Cho and Liu 2016; Chou et al. 2012; Fryer and Holden 2011; Li et al. 2014; Spann et al. 2007)

It is critical to recognize that none of the software here terminates with an exact solution – *the* optimal result – and that the software proposed in Section 3.2 does not either.[1] In other words, there is no guarantee that the partition generated corresponds to the highest possible value of the objective function. Set in the context of the partitioning of Pennsylvania discussed earlier, none of the methods here consider every one of the 10^{3800} possibilities. That is the nature of NP-hard problems. Instead, the algorithms begin are seeded with an initial configuration and iteratively move towards a "better" solutions. By initializing the algorithm many times with different starting points, it produces a variety of different "good" outputs – what I will call a "population" of maps. Fifield et al. (2017) criticize this approach and have sought to situate it on firmer theoretical foundations using a Markov Chain Monte Carlo algorithm. But they demonstrate full-state simulation only for New Hampshire, and generate local modifications to the districting plan in Pennsylvania. To be clear about the statistical meaning of this project, the initialization (Section C.1 is a bona fide random draw of points (census tracts) in the state; the final output of the optimization is of course *not* random. Depending on the stability of the minima, a few maps may be represented many times. Independent seeds may have lead to similar or even identical results.

Altman and McDonald published the first modern salvo at a comprehensive "solution" to automated districting in 2011, with BARD (Better Automated ReDistricting). They implemented several objective functions and metaheuristic search procedures but did not demonstrate the results from the program and claimed only to have had "moderate success with moderate size plans." The package has not been maintained, but Liu et al. (2016) did attempt to use it and found it slow. The issues seem to have been the choice of language (R), inefficient objective functions, and the use of a contiguity score instead of a constraint.

The same year that Altman and McDonald produced BARD, Fryer and Holden (2011) proposed to evaluate non-compactness via a "relative proximity index" (RPI). They defined the RPI as the average distance between people in a state's districts, divided by the average

[1] Excepted is the split-line algorithm, which generates a unique solution, but does not make any claims about "optimality" by any measure.

distance for an optimal plan. Assuming that an "optimal plan" would (i) respect anonymity, (ii) minimize distances between voters and (iii) be scale invariant, they prove that it is a power diagram and implement an algorithm to generate solutions (see Section 2.2.1).

In the last several years, Chen and Rodden (2013; 2015) and Cho and Liu (2016) have proposed to measure gerrymandering by comparing a population of "potential plans" to enacted ones. These too have required functional algorithms. Chen and Rodden implemented a fairly straightforward algorithm, beginning from random seeds, merging cells to form an (agglomerative) initial solution, and then trading neighbors based on proximity until the population equality is satisfied. Their initial analysis focussed on Florida, but they did apply the algorithm to twenty states, and Chen and Cottrell subsequently applied it to the full country. (2016)

Cho and Liu implemented a genetic algorithm, but allow that their crossover mechanism is "sufficiently disruptive" that it is used infrequently, and may not be much better than re-initialization. It is thus effectively a GRASP algorithm: greedy with random mutations. What distinguishes their work is the care given to the preservation of contiguity and the diversity of mutations that they allow. In short, contiguity is built-in – it cannot be violated. Given a set of cells to be removed from a region, their algorithm checks that the set's external neighbors in the source region are themselves connected. This purely-local check only works however, because of their choices to preclude holes in their plans and use queen instead of rook weights for contiguity.[2] Though they discuss the extensibility of their framework, they implement only the IPQ measure, which is computationally extremely simple. They use the National Center for Supercomputing Applications to generate millions of solutions, which is impressive but statistically unnecessary and hampers reproducibility. They only demonstrate plans for Maryland and do not give a good sense of the overall quality of their plans.

Kimbrough et al. (2011) and Chou et al. (2012, 2014) also implement a genetic algorithm with no crossover, to optimize an extremely simple measure of compactness: the maximum intradistrict distance. The crux of their series of papers is a lovely suggestion to use Interactive Evolutionary Computation (IEC) with what they call solution pluralism. In other words: they ask actual people to compare plans, and use these to validate the performance of the IPQ, moment of inertia, and perimeter measures. They do this with subjects both in-person and through Amazon Mechanical Turk. This work would then ideally lead to a collection of solutions for debate.

Of course, the regionalization literature extends far beyond districting. Duque et al. (2011) provides an outstanding review of regionalization methods, with a useful classification of heuristics. Though this project focusses on trading optimizations, the graph theoretic and hierarchical/agglomerative strategies he reviews could usefully extend this work. From a practical perspective, Li et al. (2013, 2014) propose a generic tool for compact regionalization with goals similar to the present work. Their tool optimizes a normalized moment of inertia, and they compare the performance of greedy, GRASP, and tabu searches. They obtain their best results from the randomization in GRASP.

Finally, a number of regionalization algorithms are available in clusterPy (Duque et al. 2011). Unfortunately, the feature set is not well-suited to the districting problem. First, the objective function is fixed (maximum in-region homogeneity on intensive variables). Further, though many of the included algorithms allow the user to specify the number of regions, they

[2]With "queen" contiguity, elements touching at a single point are considered contiguous, whereas "rook" quantity requires the contiguous elements to share non-zero perimeter. Cho et al state that their choice of Queen weights and no holes is legal rather than computational, but do not provide sources for this.

do not provide the possibility of enforcing a minimum population deviation among the regions. The notable exception is the max-p algorithm (Duque et al. 2012), which is also available in PySAL. (Rey and Anselin 2010) Though max-p generates the number of regions endogenously, one can easily obtain the desired number by requiring each district's population to match the target. The algorithm does not intrinsically privilege geographical compactness, but this can be achieved by providing latitude and longitude as clustering variables.

Appendix C. The *C4* Software

C4 is a multi-objective application for contiguity-constrained clustering. It is implemented in c++, with bindings to python with cython. This allows it to draw on the strengths of the two languages: c++ for speed and python for easy configurability and post-processing (mapping, analysis, etc.). I use PostGIS/Postgresql geographic database software to prepare the inputs in two ways: I simplify the boundaries between census tracts, and I extract the contiguity matrix among the cells in a state. This pre-processing stage can be cached, so there is no dependence on a private database. The software is open-sourced and freely available on GitHub.

The core code is implemented as three classes: (1) a universe which owns a collection of (2) regions and (3) cells. In the present application, regions are congressional districts and cells are census tracts. Speed requires explicit memory management: the cells are instantiated once by the universe and never copied. Each region maintains a vectors of pointers to the cells currently assigned to it, and additional vectors for the cells along its internal and external borders. The cells hold their properties (location, area, perimeter, population, etc.) as well as pointers to their neighbors.

To start, the user loads the cells and adjacencies, and starts an initialization round that creates a contiguous solution from which to iterate. The iterative search then begins, trading cells between regions to equalize populations and improve one of the compactness metrics described above. Upon convergence, the program returns a list of assignments of cells (census tracts) to regions (congressional districts).

C.1. Initialization strategies.

A number of initialization strategies are implemented. The split-line and power diagram algorithms discussed earlier are stand-alone, and can be used to initialize a statewide plan. A simple k-means clustering approach is also provided, as well as with random contiguous growth. An existing plan can also be loaded.

However, for the studies presented in this paper, initialization consists simply of a random draw of a cell (census tract) for each region. This ensures that the single strategy under study is completely responsible for the outcomes presented.

C.2. Contiguity.

Implementing contiguity as a constraint instead of an objective significantly improves performance. To do this, the contiguity graph of the state is extracted from its topology in Postgres. In this graph, each cell (census tract) is a node, and edges are drawn between adjacent cells (i.e., neighbors). To preserve contiguity in the search, the graph must be modified in two ways:

(1) The statewide graph must be connected in the first place. *C4* provides an automated procedure that connects islands to their closest neighbors until the graph is connected, but it also allows the user to specify explicit connections. In practice, I connect islands and disconnected subgraphs to land based on bridges, ferry routes, and shared jurisdictions. For example, the Eastern Shore of Virginia is connected to Virginia Beach by the Bay Bridge, and

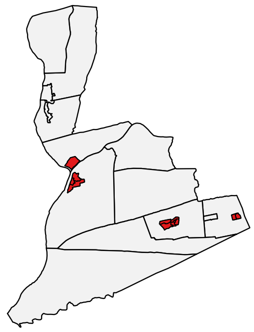


Figure C. Census tracts of Northumberland County, Pennsylvania are shown with enclaves highlighted. For a trading algorithm to work consistently with a contiguity requirement, enclaves and islands must be assigned to their external neighbor or nearest land neighbor.

San Nicolas Island, California is connected to Naval Base Ventura County, of which it is a detachment.

(2) "Enclave" census tracts must be assigned to their external neighbor. An example of this is shown in Figure C. Without merging the highlighted cells, it would be impossible to move their outer neighbors without violating contiguity. This argument also applies to coastal tracts that are not enclaves of another tract, but whose entire land border is shared with a single tract. The topology is different, but the graph structure is the same: they are across a "cut vertex" from the main body of the district. It is worth noting that this procedure modifies slightly the definitions described above: the compactness scores and optimizations are evaluated on this merged (or "pruned") topology.

After these modifications, any move can be required to preserve the source region's contiguity check: a cell's neighbors in the source region must be members of a connected subgraph of the source region. This check resembles Cho and Liu's local check, but it differs in that the neighbors do not have to be directly connected. If they are not, the search extends through the neighbors' neighbors until it is shown that the removal of the cell would result in disconnected subgraphs in the source region. The result of this distinction is that both rook and queen weights are allowed, as are holes. In practice, I have used rook contiguity.

C.3. **Hill Climbing.** Each iteration of *C4* consists of a loop over regions. For each region, a loop over the neighboring (external) cells identifies those cells whose removal from their current region would not break that region's contiguity. For each such cell, the change in the combined spatial and population objective function from reassignment is evaluated, as described in Section 3.1. If, among all possible moves for a region, the best one is an improvement, that cell is removed from the source and added to the destination.

C.4. **Metaheuristics.** To escape local minima, I have implemented several of the touchstone metaheuristic strategies – greedy, GRASP (Feo and Resende 1989), and tabu lists (Glover 1989) – but have found in practice that naïve greedy works. However, I have also implemented a 'de-stranding' procedure and a high-level 'restart' procedure, that do prove important.

C.4.1. *De-stranding.* "Strands" or "tentacles" occur somewhat frequently with the inscribed and circumscribing circle methods, where the objective function does not prioritize closer over further cells unless they affect the reference circle. Other algorithms occasionally also fall into a Catch-22 where the end of a long strand is not an "optimal candidate" for a trade, but no cell but the last one can be moved without violating contiguity. Since the last cell is never removed, the strand cannot be removed. I target this behavior directly, by including a configurable "de-strand-ing" procedure that trims away subgraphs connected to the main region at a single node (cell). This method is admittedly ad hoc, but as long strands and singly-connected subgraphs are not consistent with any of the methods near global optima, I believe its sparse use is justifiable.

C.4.2. *Split-restart.* When an optimization does not improve for a configurable number of cycles, it terminates. If activated, the split-restart procedure then (a) splits the district with the worst score on the current metric, and then (b) merges a (random) district with one of its neighbors. This usually results in one district with twice the target population and two with half the target. The interest of this is to give the system a very large shock that nevertheless leaves it in a fairly good starting solution to generate a new partitioning. The shocks are large enough to yield distinct maps in each cycle, but leave the better (more-compact) districts are intact. The starting solutions thus typically improve with each cycle. The method is philosophically appealing since it is entirely based on the reference measure; unlike "de-stranding," it does not import outside prejudices about "good" behavior.

C.5. **Approximations.** Despite significant care to implement the objective functions efficiently, a number of approximations are required to achieve adequate performance. Foremost among these is the use of cell centroids in calculating the convex hull and circumscribing circles, and the various distances (to the perimeter or center). For the circumscribing circles, an approximation of the perimeter is made from the topology, tracing the trijunctions of census tracts along the border. This method was first suggested by Schwartzberg (1966) for a variant of the IPQ measure. For the fraction of shortest internal paths that are themselves in the district, I constrain the path to run between centroids of adjacent cells, in order to simplify the network problem.[3]

Appendix D. Results with State Level Races

This paper presents seat share results for multiple (presidential) elections per state, when data permit. This shows that with geographically realistic vote distributions, the fundamental result on the consistency of seat shares between methods is robust to dramatic changes in statewide vote shares. In Illinois, Minnesota, and Wisonsin for instance, Barack Obama garnered far more votes in 2008 than Hillary Clinton did in 2016, but the seat shares within years remain consistent across districting methods (Tables A.2, A.5, and A.9). Using national races ensures that all races are contested by competitive candidates, but it is an approximation because local candidates might be better able to respond to local preferences.

Table D presents aggregated seat shares in Texas, replacing presidential contests (Appendix A, Table A.8) with elections to the US Senate. Texas is presented simply because many years of data are readily available. As in Illinois betweeen 2008 and 2016, the vote shares vaccilate wildly in the four elections shown. John Cornyn garnered 64% of the two-party share in 2014 while Ted Cruz won just 51% in 2018. Despite this swing, the table shows that the empirical consistency in seat shares among methods is preserved when shifting to state level races.

[3]The exact solution, of crossing a visibility graph is not feasible since the graph must be reconstructed for every proposed move, and the shortest paths reevaluated.

	2008 US Senate	2012 US Senate	2014 US Senate	2018 US Senate
107th Congress	14.4	13.2	12.0	19.2
111th Congress	12.4	11.2	9.0	19.1
114th Congress	12.0	11.0	10.0	17.0
Areal Distance	12.2	11.4	8.6	17.6
Axis Ratio	12.0	11.3	8.4	17.0
Circumscribing Circles	11.9	11.3	8.5	17.3
Distance to Perimeter	12.1	11.2	8.8	17.1
Dynamic Radius	12.3	11.7	8.5	17.7
Exchange	12.1	11.6	8.5	17.6
Harmonic Radius	12.2	11.5	8.6	17.5
Hull Area	12.2	11.2	8.5	17.2
Hull Population	12.4	11.6	8.5	18.1
Inertia Area	12.1	11.5	8.5	17.5
Inertia Population	11.8	11.0	8.2	17.2
Inscribed Circles	12.8	12.2	8.6	17.8
Isoperimeter Quotient	12.2	11.4	8.4	17.8
Mean Radius	12.3	11.7	8.5	17.7
Path Fraction	12.2	11.4	8.4	17.7
Population Distance	12.1	11.2	8.8	17.8
Power Diagram	11.6	11.1	9.4	17.3
Split-Line	15.0	15.0	9.0	18.0
	8 9 10 11 12 13 14 15 16 17	7 8 9 10 11 12 13 14 15 16	5 6 7 8 9 10 11 12	13 14 15 16 17 18 19 20 21 22

TABLE D. Votes from presidential elections in Texas are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 36 assigned after the 2010 Census.

Appendix E. Preserving Communities: Objectives and Evaluations

This paper focusses narrowly on the spatial compactness of Congressional districts, since it was the last objective enacted by Congress. It is not the only districting criterion, however. This Appendix illustrates how alternative objectives may be included in the framework developed in this paper. The Supreme Court has itemized traditional districting principles regularly and consistently for over half a century, as "including but not limited to compactness, contiguity, respect for political subdivisions or communities defined by actual shared interests." (*Miller v. Johnson* 515 U.S. 900, 1995, 916)[4] This respect for existing political subdivisions stands foremost among the other objectives.

It can be incorporated in the framework of this paper in a number of ways: as an optimization objective, as a criterion for evaluation and comparison, or as a hard requirement on permissible maps. This Appendix develops the first two strategies. (The hard binary requirements are used in Appendix F in the context of minority representation.) I define first an objective for community preservation (E.1), and then a measure of split communities (E.2), before presenting and discussing results (E.3). Note that my definitions are intended only to illustrate how such conditions might be written mathematically and algorithmically; they are by no means generally accepted or legal doctrine.

E.1. Modified objective functions.

There are a number of ways of implementing a "respect for existing political subdivisions" as an objective function. The first is to add an explicit and separate measure of community preservation in the objective function for each district. This component would be weighed against compactness and equipopulation. The second strategy is to modify the compactness-based functions to account for existing boundaries. I take the second approach.

The Isoperimeter Quotient (IPQ, or Polsby-Popper) method considers the ratio of a district's area to the square of its perimeter (see subsection 2.1.1). To account for existing political subdivisions, perimeters can be downweighted – made less costly – when they fall along the extant boundaries between subdivsions. In the context of district optimization, this will privilege district borders that coincide with existing lines. For this Appendix, I define a county-weighted IPQ by weighting perimeters along county lines by a factor of 2/3. This reweighting could be extended with overlapping communities of varying importance.

Many of the other compactness measures can be modified along similar lines. For example, for methods derived from distances among residents (power diagrams) or to the district center (moments of inertia), components of a community can be "shrunk" towards its center. For county preservation, this simply means moving the census tracts in each county towards the center of the county.

E.2. Evaluating the performance of existing maps.

A common way of evaluating how well a districting plan respects existing communities is to count the number of times those communities are split between districts. This method is insensitive to variation in how "grievously" the community is divided. It may be worse to split a county in two than to shave a bit from its edge. Naïve counts also ignore the fact that some communities exceed the scale of a legislative district and *must* be divided.

[4]Consistent language can be found in *Reynolds v. Sims* (377 U.S. 533, 1964, 580-581), *Davis v. Bandemer* (478 U.S. 109, 1986, 116-117, 138-39, 162, 173), *Shaw v. Reno* (509 U.S. 630, 1993, 16, 47), *Karcher v. Daggett* (462 U.S. 725, 1983, 755), *Bush v. Vera* (517 U.S. 952, 1996, 977), or *Vieth v. Jubelirer* (541 U.S. 267, 2004, 273, 322), among many others.

I define a measure of community-splits as the likelihood of two residents of a community having the same legislator. More formally, it is the quotient of the population of shared (intersecting) county residence and Congressional representation, divided by its maximum value. The denominator is then the lesser of the community (county) population and that of the legislative district. Denote the legislative district of individual i by $\mathrm{LD}(i)$, his or her community by $\mathrm{comm}(i)$, the intersection of the two by $(\mathrm{LD}(i) \cap \mathrm{comm}(i))$, and their populations by $p(\cdot)$. The "shared legislator" value for constituent i can then be written

$$s_i = p(\mathrm{LD}(i) \cap \mathrm{comm}(i)) / \min(p(\mathrm{LD}(i)), p(\mathrm{comm}(i))).$$

As defined, this measure lies in the range from 0 to 1. Higher values indicate better respect for existing communities. This is one definition among many that could be defined, and over which reasoned debate might prove useful.

E.3. **Results for community-based objectives and criteria.** Table E shows the likelihood of sharing a legislator with county co-residents, in three states whose 2012 Congressional districts have been extensively litigated: Maryland, North Carolina, and Pennsylvania. The first three rows of that Table show enacted plans. The division of counties has gotten worse over time in all three states. This is true even though Pennsylvania lost seats in Congress in both 2000 and 2010, and therefore has mechanically fewer divisions to make. North Carolina, on the other hand, gained one seat in 2000. On the whole, citizens share legislators with about 70% of the residents of their county, out of the maximum value already discussed.

Plans enacted for the 1992 redistricting generally outperform simulated maps in the states shown, but this has ceased to be true as the enacted plans have gotten worse. The simulated maps outperform the plans used for the 114th Congress in Maryland and North Carolina, and in Pennsylvania they are comparable. The table also includes results for maps derived with the "county-weighted IPQ" as the objective function. With respect to the nominal IPQ, this simple and conservative modification increases likelihood of co-residents sharing a legislator by a few percent, for each of the three states. By contrast, the path fraction consistently underperforms its peers. It is interesting to note that power diagram districts (which are always convex), have high path fraction, but that path districts with good path fraction may have poor interpersonal distance (the measure corresponding to the path diagram algorithm – see Section 2.2.1). The space of high-scoring districts is different between the two methods, and this affects performance on this measure of divided constituencies.

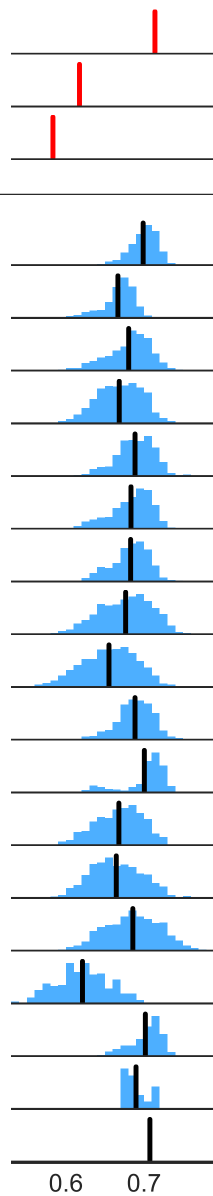
There are two conclusions to draw from this work. The substantive conclusion is that automated optimization of spatial objective functions does not significantly exacerbate the division of counties. At least in some states, compactness does better. This might not be a surprise, since counties are usually quite compact. But it shows that alternative objectives cannot reliably be depended on to absolve spatial monstrosities of their partisan intent.

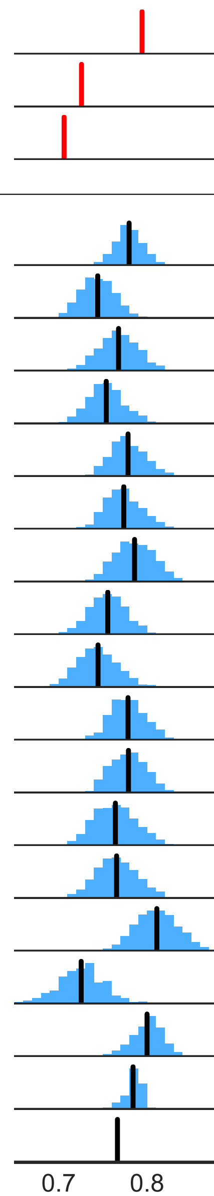
The methodological point is that it is straightforward to extend the methods of this paper to districting objectives beyond compactness.

Appendix F. Post-Selection of Minority Majority Districts

The main text of this paper affects distributions of maps in states by changing how they are generated. It is also trivial to alter these generated distributions by selecting subsamples with specific properties. For observables correlated with the selection criteria, the distributions in the subsample will differ from the full population. This Appendix illustrates this selection method, as applied to minority representation in North Carolina.

	Maryland	North Carolina	Pennsylvania
107th Congress	0.71	0.79	0.87
111th Congress	0.62	0.73	0.77
114th Congress	0.58	0.71	0.72
Areal Distance	0.70	0.78	0.70
Axis Ratio	0.67	0.74	0.69
Circumscribing Circles	0.68	0.77	0.69
Distance to Perimeter	0.67	0.75	0.68
Dynamic Radius	0.69	0.78	0.71
Exchange	0.68	0.77	0.70
Harmonic Radius	0.68	0.79	0.71
Hull Area	0.68	0.76	0.67
Hull Population	0.66	0.74	0.67
Inertia Area	0.69	0.78	0.70
Inertia Population	0.70	0.78	0.72
Inscribed Circles	0.67	0.76	0.69
Isoperimeter Quotient	0.66	0.77	0.69
County-Weighted IPQ	0.69	0.81	0.73
Path Fraction	0.62	0.73	0.64
Population Distance	0.70	0.80	0.73
Power Diagram	0.69	0.78	0.71
Split-Line	0.71	0.77	0.70





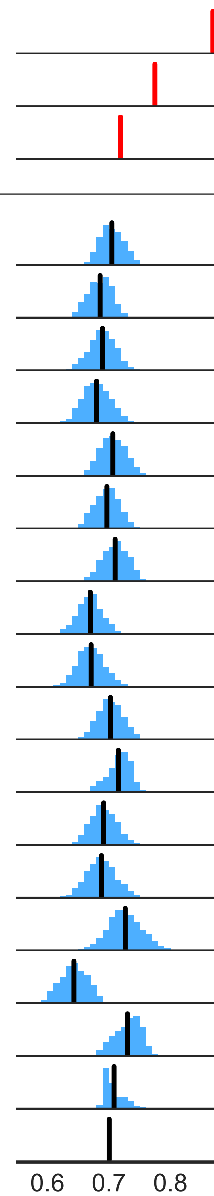


Table E. Probability of a citizen of a county living in the same congressional district as another randomly-selected citizen of their county, for distributions of maps derived for various compactness definitions. The "probability" is modified to account for counties larger than Congressional Districts (see text). North Carolina gained a seat after the 2000 Census, whereas Pennsylvania lost seats in both 2002 and 2012. The probabilities do not not correct for this effect (first three rows). There is a progression towards more-dividied counties among the enacted maps. For Maryland and North Carolina, the automated procedures outperform the enacted maps. The county-weighted isoperimeter quotient measure outperforms baseline IPQ by a few percent.

Section 5.3 showed that a compactness requirement would engender small reductions in minority seat shares, nationally. I argued that these reductions would in practice likely be more severe than indicated in the simple model. That model was based on race/ethnicity alone, and did not ensure the party composition necessary for minority candidates to be successful. How then, to ensure better outcomes for minority voters? Perhaps the easiest way is to select statewide plans that favor them. To do this, I select statewide maps from the samples generated for North Carolina, that have at least two seats where over 30% of the voting age population is Black, *and* with a Democratic two-party share over half (averaged between 2012 and 2016). This is a rough requirement, intended to match or exceed the Black seat share of the state's current delegation. More-precise and involved selection mechanisms for minority seats could easily be concocted.

This requirement rejects a large fraction of the generated maps, including the single split-line map. One might expect that selecting on the (imperfectly) correlated observables of party and race would increase the Democratic two-party share above the 50% required to win. This might lead Democratic votes statewide to be "less efficient," and result in their capturing fewer seats. The actual behavior depends on the spatial configurations of the maps. For example, the correlation between the Black share in the second most Black Democratic district and the number of Democratic seats is negative for power diagrams (-0.20) but positive for isoperimeter quotient (0.28). Table F shows the Democratic seat shares for this subsample. With the cuts defined, the seat distributions from the subsample are remarkably consistent with the full populations (Cf Table A.6).

For power diagrams, however, this consistency turns out to be "lucky." The minima of the power diagram method are more stable than many of the other methods, and plans optimized with different seeds can terminate with the same geographic configuration. This means fewer substantially-different maps, and more bunching in any derived distribution. Figure F shows the share of the voting age population that is Black, in the Democratic district with the second-highest share, for maps derived using power diagrams or isoperimeter quotient. To simplify the graphic, I define party based simply on the vote shares in 2012. Requiring two Democratic districts with a voting age population more than one third Black leaves no power diagram maps at all. Lowering the threshold for the Black share to 0.32 rejects all maps with less than six Democratic seats, and raises the expected Democratic seat share. But lowering it yet again to 0.29 does the reverse: it allows the maps with four Democratic seats, and removes more of the maps that *favor* Democrats.

The other way to read this plot is that the full population of maps shows a high degree of consistency in the values simulated for a rather abstruse observable. For the IPQ and power diagram methods, the range in the Black share of the second-most Black district is largely contained between 0.25 and 0.35. The ranges are even tighter, within methods. To alter the distributions of these observables, one could of course include them as objectives in the optimization, as per Appendix E. If two Democratic districts were desired with a voting age population greater than 40% Black, intervention would be necessary at the optimization stage.

This post-selection strategy is trivial to implement for any manner of political, demographic, or spatial conditions. Selection affects other observables, including the representation of the parties, in non-trivial ways. These impacts are not necessarily consistent across methods.

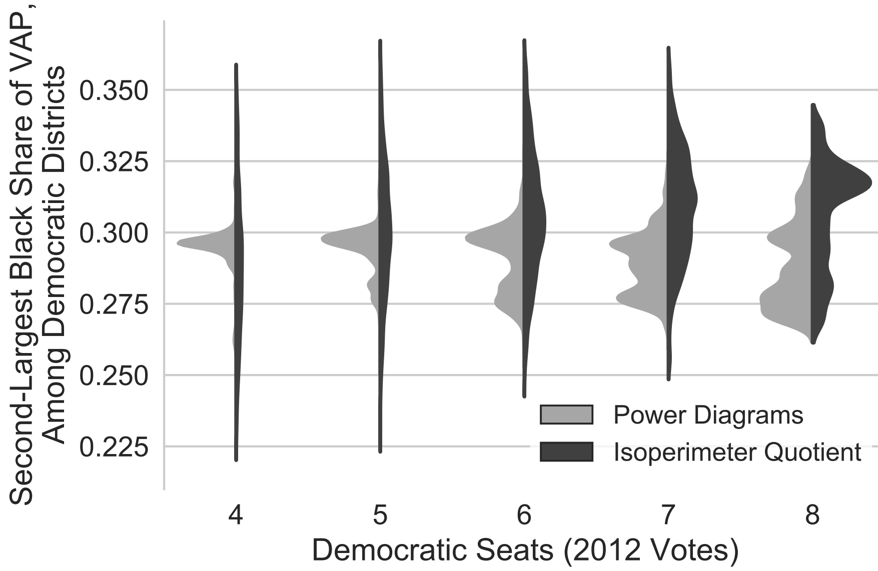
Appendix G. A Legal History of Compactness

Much of the recent academic, legal, and public discussion of gerrymandering has focussed on remedies through the Supreme Court. I suggest that this focus is misdirected: for much of

	2012 Presidential	2016 Presidential
107th Congress	3.2	4.3
111th Congress	6.0	5.0
114th Congress	3.0	3.0
Areal Distance	6.0	5.8
Axis Ratio	5.9	5.4
Circumscribing Circles	5.7	5.2
Distance to Perimeter	6.0	5.5
Dynamic Radius	5.9	5.3
Exchange	5.8	5.2
Harmonic Radius	6.2	5.3
Hull Area	5.8	5.4
Hull Population	5.9	5.5
Inertia Area	5.7	5.3
Inertia Population	5.9	6.1
Inscribed Circles	5.9	5.3
Isoperimeter Quotient	5.8	5.4
Mean Radius	5.9	5.3
Path Fraction	5.8	5.2
Population Distance	6.6	6.2
Power Diagram	5.9	5.9
	3 4 5 6 7 8 9	2 3 4 5 6 7 8
Split-Line	5.0	5.0
	3 4 5 6 7 8 9	2 3 4 5 6 7 8

TABLE F. Votes from presidential elections in North Carolina are aggregated from precinct-level returns, into maps simulated with each algorithm or compactness metric. The seats expected to accrue to Democrats (mean across maps) are displayed numerically as well as by a solid black line. The normalized distribution of seats per metric/algorithm is shown in blue and the 10-90% range of possible seats is highlighted in gray. The same re-aggregation is performed for enacted maps used for the 107th, 111th, and 114th Congresses and shown in red. Since reapportionment shifts the number of seats per state, the entries for the 107th and 111th Congresses are the Democratic share, times the 13 assigned after the 2010 Census.

Figure F. The fraction of the voting age population that is black, in the second most-black Democratic district is plotted against the number of Democratic districts in the map, based on votes from 2012. Distributions are shown for maps derived with both power diagrams and the isoperimeter quotient compactness definition. Bunching in the distribution of power diagram districts makes the number of expected Democratic seats sensitive to the cut on the Black vote share. On the other hand, though the distributions differ substantially, the numerical consistency is very high between the two methods, for this (somewhat contrived) variable.



American history, debates over apportionment and districting centered instead on Congress. It is useful to recall the arc of the historical debates to situate the possible legislative and judicial uses of compactness.

G.1. **Congress.** The Constitution gives Congress tremendous latitude over the composition of the House and the election of its members. It stipulates that Congress reapportion the House of Representatives after each decennial census according to the relative populations of the states, but leaves the details of that procedure open. Although authority over the "Times, Places and Manner of holding Elections" is tentatively delegated to state legislatures, Congress retains supervisory power and "may at any time by Law make or alter such Regulations." Congress is the Judge of its own elections. (U.S. Const., art. 1 §2, 4, 5)

In the state ratifying conventions, some delegates feared that Congress's supervisory power over elections might be used to circumvent terms of office. The Federalists defended it on two main grounds. The first was practical and broadly conceded: the Federal government needed direct access to its electors, so as not to be beholden to the state legislatures. In Federalist 59, Hamilton wrote that "every government ought to contain in itself the means of its own preservation." This was no idle worry: Rhode Island had recently boycotted the Constitutional Convention and would not ratify for another two years. The second justification reflected the same concern as the Guarantee Clause (art. 4, §4): that Congress has a responsibility to ensure just elections and equal representation to its citizens. At the Virginia convention, Madison argued that the citizens of the several states ought to be treated uniformly in the election of their representatives and remarked that "diversity [of regulations] would be obviously unjust." Of particular interest to the present discussion, he drew for his example the inchoate gerrymander of South Carolina, where Charleston had seized more than its fair share of members. (Kurland, *et al.* 2000)

While the Founders clearly anticipated that the members of the House would represent geographically defined single-member districts, they did not require it in the Constitution. The ultimate scale of the House – actively defended in Federalist 9 and 10 – was similarly left to Congress in posterity. In discussing the role of the representatives in the House in Federalist 56 and 57, Madison makes it clear that each member was to represent a place-based district of "thirty thousand inhabitants" with which he would be well-acquainted. New York and Albany were each entitled to one representative, Philadelphia to two, and some of the other Pennsylvania counties to nearly one. Nevertheless, "general ticket" and multi-member districts were commonplace through the country's history. In the first House elections, Pennsylvania, New Jersey, and Georgia elected members at large (Cong. Quarterly 1994), and the practice formally ended only in the 91st Congress. (US Code, Title 2, §2c Amended 1967)

Congress began to flex its powers of self-definition with the Apportionment Act of 1842. Previous Acts had evaluated the proper scale of the House, but the 1842 debate raised the question of the form of its constituencies. It was broadly agreed that single-member districts were preferred in principle, but a number of smaller states had adopted (statewide) "general ticket" elections as a means of ensuring a "bloc vote" that would enhance their power. (Zagarri 1987) So along with the regular debate over the scale of the House, congressmen sparred over the stipulation that representatives be elected from single-member districts of contiguous territory. Critics of the proposal argued that Congress was either entirely powerless over the shape of districts or else that the supervisory powers of art. 1, §4 gave Congress the power to district – but not to direct the state legislatures *how* to district. They forecast doom should Congress overstep this bound: it would unleash war between the Federal and state governments. (Cong. Globe 1842) It is true that after the Act passed with the requirement intact, several multi-member states sent members from at-large districts, who were seated. (Cong. Quarterly 1994) But the Union held, and the states bowed to the new norm within a few years. Though the measure was not reenacted after the 1850 Census, it elicited a tame debate when it was revived and reenacted as a separate (non-apportionment) act in 1862. (Cong. Globe 1862)

After subsequent censuses, Congress continued to elaborate its conception of fair representation. In 1872 it added the requirement that districts be equipopulous, which persisted in 1882. (17 Stat. 28 1872; 22 Stat. 5 1882) In 1901, an amendment was introduced with the stated goal of avoiding "shoestring" districts, and requiring yet further that the districts be "compact." (Cong. Rec. 1901) That amendment passed over concerns that the new standard was ill-defined and would prove difficult of application when judging the qualifications of Representatives. The 1911 act reiterated the accumulated requirements: single member districts "composed of a contiguous and compact territory, and containing as nearly as practicable an equal number of inhabitants." (31 Stat. 733 1901; 37 Stat. 13 1911) But following a botched census after the First World War, no act was passed in 1922 and a frantic Congress abandoned the requirements with little debate for the Apportionment Act of 1929. (Cong. Rec. 1929) The Supreme Court ruled in *Wood v. Broom* (287 U.S. 1, 1932) that Congress's refusal to reiterate these rules amounted to their annulment.

In the aftermath of the Court's initial assumption of responsibility for apportionment (below), Congress failed to reassert itself. In 1967 the House and Senate revived the contiguity and compactness debate, and passed a bill requiring them along with equipopulation at the

10% level. The measure failed after its second conference over disagreements on the time allowed to the states for its implementation. In its place, Congress required only that members be elected from single-member districts, which is today the only federal statute on the form of districts. (Cong. Quarterly 1968; US Code, Title 2, §2c Amended 1967)

The interest of these proceedings is that Congress has tremendous powers over the scale of the House and the form of its constituencies – and that it has exercised these powers throughout history. The statutory requirements of compactness, contiguity, and equality of population have been lost, but Congress would be well within the letter and spirit of its constitutional mandate to revive them. Should it do so, the Court would have very little recourse since under art. 1, §5, "Each House shall be the Judge of the Elections, Returns and Qualifications of its own Members."

G.2. **The Courts.** Congress' retreat was the Court's ascendancy. This proceeded in two stages: the recognition of the justiciability first of malapportionment and then of political gerrymandering.

G.2.1. *The Reapportionment Revolution.* Rapid urbanization of the Country through the early 1900s paired with unresponsive state legislators resulted in galling malapportionment of legislative districts by mid-century. After refusing to enter the "political thicket" of apportionment in 1946 with *Colegrove v. Green*, the Court dramatically revised the division of judicial and legislative authority with *Baker v. Carr* (369 U.S. 186, 1962). Ruling on a badly-malapportioned Tennessee statehouse, Brennan wrote for the majority that the Court had jurisdiction on equal protection grounds, and further that the question was justiciable and *not* a "political question" per *Colegrove*. He classified political questions as those that are *inter alia* constitutionally committed to another coequal branch of government or characterized by a "lack of judicially discoverable and manageable standards." Neither condition applied in *Baker*, and the Court ordered the state to reapportion.

Warren and Brennan had won Stewart's vote in *Baker* by convincing him that the decision was narrow, but the "Reapportionment Revolution" was a cataclysm. Justice Whittaker had a nervous breakdown over *Baker*, recused himself from the case, and retired. Frankfurter suffered a stroke a week after delivering his masterful dissent, and resigned as well.[5] Within two years, the Court formalized its "one person, one vote" standard (*Gray v. Sanders* 372 U.S. 368, 1963) and applied it to congressional districts (*Wesberry v. Sanders* 376 U.S. 1, 1964) and both houses of state legislatures (*Reynolds v. Sims* 377 U.S. 533, 1964). Fighting *Reynolds*, Senate Minority Leader Everett Dirksen brought the country within one vote of a Constitutional Convention. Reflecting in retirement on a tenure punctuated by *Brown* and

[5]Frankfurter argued that the definition of the level of equipopulation was a fundamentally political question. Drawing examples from England and the early American states, he argued moreover that it had admitted political solutions. (*Baker v. Carr* 369 U.S. 186, 1962, p. 301-310) Second he argued with almost uniquely pertinent originalist sources that the 14th Amendment was not intended to grant Congress power to regulate the suffrage within the states. This argument used the fact that the Joint Committee on the Reconstruction that drafted the 14th Amendment was also charged with restoring the southern states to the Union. This resulted in debates that directly touched on the concern. He argued further that the Union consisted of three categories of states: (*i*) Union states that approved the Amendment, (*ii*) Southern states readmitted to the Union under the Joint Committee, and (*iii*) states that have joined since the Amendment's ratification. The large majority of these states that did not apportion representatives equally on population, established a legal precedent for unequal apportionment congenital with the Amendment itself. Finally, Frankfurter assessed whether the Court had any say over a state matter (capricious state action), and found that it did not.

Miranda, Chief Justice Warren called *Baker* the most consequential decision of his Court. (Smith 2014)

The Court eliminated malapportionment in just two years, but the success was not unalloyed. With *Kirkpatrick v. Preisler* (394 U.S. 526, 1969), the Court clarified that the "as nearly as practicable" population equality of *Gray* and *Wesberry* was less than 10%. In the majority opinion for *Karcher v. Daggett* (462 U.S. 725, 1983), Brennan rejected even sub-percent levels of inequality and indeed any *de minimis* threshold. This had two ill effects. The 'supremacy' of the equipopulation requirement was absolute. In his dissent to *Baker*, Frankfurter argued that the balance of equipopulation with other principles was the very core of the political problem to be avoided by the Court. The Court ruled at one extreme. Other countries require looser, 10-40% level of equality, which they balance with other representational principles: minority representation, preservation of existing subdivisions, etc. (Stephanopoulos 2013) But one need not appeal to foreign examples: when Congress debated reasserting its authority over districts in 1967, the proposed threshold on deviations was similarly just 10%. (Cong. Quarterly 1968) Further, the Court's absolute adherence to equipopulation required fine manipulation of district lines. It instigated a skyrocketing level of split communities (Altman 1998), and "normalized" the fingery divisions now familiar in district maps. The second problem was that after the Court signalled its assumption of responsibility for districting, it has struggled to deliver beyond apportionment, and Congress has failed to contribute.

G.2.2. *Political Gerrymandering.* Political gerrymandering, the manipulation of district lines for partisan advantage, has of late received significant national attention. (Burns and Martin 2017; McIntee 2016; Oliver 2017) Rodden and Chen have conclusively demonstrated that demographic effects like Democrats' tendency to cluster in cities currently favors Republicans. Nevertheless, they and others have also shown that a number of states' maps show persistent, significant, and growing seat advantages for partisan mapmakers. I reproduce and elaborate both of these effects in Section 5. In Pennsylvania, Maryland, and North Carolina, the party controlling the legislature enacted maps for the 114th Congress that yield seats shares for that party significantly larger than those seen in the compactness-based simulations. Beyond seat shares, political gerrymandering has been widely demonstrated to reduce trust in government and voter turn-out. (Stephanopoulos 2013)

The harms perceived in political gerrymandering are thus real, and the Court since *Davis v. Bandemer* (478 U.S. 109, 1986) has held them to be justiciable. However, it has failed to enunciate a standard for adjudicating it, and therefore declined to provide relief. This is a curious position, since Brennan's criterion for justiciability in *Baker* was the very existence of a "clear and discoverable standard" for the question at hand. The Court in *Davis* set a high bar for such a standard: plaintiffs must show intentional discrimination and actual discriminatory effect that persistently frustrates a group's efforts to influence the political process as a whole.

In *Davis* and each subsequent case before the Court, a slender majority of justices has tendered proposals or held out hope, and in each case a slender majority has rejected every standard or justiciability itself. In his dissent (in judgment) to *Davis*, Powell wrote that standards from *Reynolds* ought to be adopted to political gerrymandering. "The most important of these factors are the shapes of voting districts and adherence to established political subdivision boundaries."

With *Vieth v. Jubelirer* in 2004 (541 U.S. 267, 2004), four justices argued in favor of relief according to three standards. Stevens wrote that the "totality of the circumstances" approach used in racial gerrymandering cases had proven manageable and could be applied to political gerrymandering. The argument is that racial cases provide an "alternate universe" where gerrymandering is regulated. Racial gerrymandering is unconstitutional under the 15th Amendment and regulated under the 1965 Voting Rights Act (as amended in 1970, 1975, 1982, and 2006). (US Code §52 1965) Confronted with this clear charge, the Court has enunciated consistent "traditional districting principles," of compactness, contiguity, and respect for existing political subdivisions. The avowed stumbling block in political gerrymandering cases had not been the harm or the jurisdiction but the standard. The racial gerrymandering cases were proof positive of a standard. Stevens pointedly remarked, "the plurality does not argue that the judicially manageable standards that have been used to adjudicate racial gerrymandering claims would not be equally manageable in political gerrymandering cases." (324)

Souter proposed that a plaintiff who was a member of an identifiable group could make a successful complaint by demonstrating that the group had suffered intentional harm from a plan enacted in disregard of traditional principles. As usual, the principles itemized were "contiguity, compactness, respect for political subdivisions, and conformity with geographic features." Plaintiffs would have to show that plans that better-respected those principles were possible. (347-351)

For his part, Breyer penned a meditation beginning with "We the People" and seeking "workable form of government that is . . . basically democratic." He ended by declaiming "*unjustified* entrenchment . . . purely the result of partisan manipulation and not other factors." (356, 360, italics original)

Kennedy rejected each of these standards, and went out of his way to reject compactness: "even those criteria that might seem promising at the outset (e.g., contiguity and compactness) are not altogether sound as independent judicial standards for measuring a burden on representational rights. They [...] would unavoidably have significant political effect, whether intended or not." He nevertheless declined to foreclose hope that a standard could eventually emerge.

Two other themes emerge in the Court's deliberations in *Vieth*. The first is that a comfortable majority (the plurality, with Kennedy and Breyer) expressly endorsed politics as a "generally permissible" component of districting. (307) The second theme is the majority's insistence on a pithy standard. In the plurality opinion, Scalia criticised Powell's "flabby" approach which Kennedy echoed, calling for a "limited and precise rationale."

Two years after *Vieth*, the Court in *League of United Latin American Citizens v. Perry* (548 U.S. 399, 2006) again failed to identify a standard, but Stevens (joined by Breyer) and Souter (joined by Ginsburg) expressed some interest in developing a standard on partisan symmetry.

The Court's entry into congressional districting in the Reapportionment Revolution had enormous, immediate, and overwhelmingly positive impact on fair representation in America. Yet its legacy in political gerrymandering has ultimately mirrored Congress' – a broad recognition of the harm, paired with a toothless consensus over the use of compactness as a component of fair districting laws. Powell's prophecy for *Davis* has been fulfilled: it has signalled a "'constitutional green light' to would-be gerrymanderers" while simultaneously "inviting further litigation."

What is still required – for Congress or the Court – is a "limited and precise" constraint on Congressional districting. Can compactness fill that need?

Appendix H. Compactness as a Legal Standard

The legal history reviewed in Appendix G shows that both Congress and the Court have the potential to eliminate political gerrymandering. Both have recognized compactness as a part of the solution. Section 5 argued that various notions of compactness line up with each other, and that implementing them results in consistent practical effects. Put boldly, there is only one compactness; it is a meaningful standard. Yet the House districts in Figure I.1 suggest even without the math, that current House districts are not uniformly compact. Why has compactness failed to gain traction?

Although American history suggests that Congress is at least as well suited to districting reform as the courts, the modern debate has centered on the judiciary. Three broad arguments stand out from that debate, as defenses of the status quo or critiques of compactness:

(1) districting is a legitimate and even quintessentially political activity,
(2) equations are an inadequate palette for expressing political communities, and
(3) the selection of one definition of compactness over another amounts to a political choice.

The first argument – that districting is a political activity – is broadly agreed by the Court. In her concurrence (in judgment) to *Davis*, O'Connor rejected the Court's grounds and wrote that districting is a "political question in the truest sense." (*Davis v. Bandemer* 478 U.S. 109, 1986, p. 145) As remarked above, a majority affirmed this position in *Vieth*. (p. 307) The initial weakness of this argument is that the same defense was marshalled in support of malapportioned districts. The Reapportionment Revolution also recalibrated urban and rural power, and it is the nature of democracy to limit manipulation of the ballot box.

Moreover, inverting the question – asking whether politicians in practice use their authority to express an idealized 'social value' – makes a defense of the status quo less credible. The national parties' activities and avowed motives are partisan, not political, and do not create or elevate cognizable political communities. Unlike town or school district borders, the shifting lines of congressional (let alone state legislator) districts are hardly a standard 'source of identity.' Only around half of Americans know the political party of their representatives in Congress (Pew Research Center 2014), let alone his or her name or the district lines. In a defense of incumbent protections in districting, Nathaniel Persily presents some anecdotes of earnest legislators. He writes that through the districting process, "they create service relationships between representatives and constituents that fit into larger public policy program." (Persily 2002, p. 32) But his argument is stuck in reverse: in a Democracy this is precisely what is supposed to be what is decided in the ballot box – not the other way around.

The Republican State Leadership Committee's REDistricting MAjority Project (REDMAP) forwards explicitly partisan goals, which it has executed successfully. Democrats have signalled less-than-beneficent intent to compete on the same terms. (Burns and Martin 2017) In *Coooper v. Harris*, North Carolina Republicans even claimed partisan intent as a defense. And even less-noble justifications like "winner's bonuses" as for patronage fall flat in the context of elections: entrenchment between elections is anathema to Democracy. The practice is not in short, so high-minded as the principle that the justices appear to defend.

Several authors criticize automation as inadequately nuanced and human. Since it's impossible to subsume competing conceptions of public interests, communities, and identities in a single pithy metric, any effort is misplaced. Altman writes, "automated redistricting is significantly more complicated than gerrymandering" because the latter only requires "the maximization of one simple goal. Optimal redistricting may involve many simultaneous, complicated, and conflicting goals." Capturing the "social value" of "subtle patterns of community" in an equation or automated procedure is chasing leprechauns. (Altman 1997, pp. 82, 112) The geometric approach that I have presented is guilty as charged: it does not seek to encode every human interaction and adjacency. The fault of Altman's argument is that gerrymandering is presently real and pernicious. Progress is possible without perfection; we ought not wallow in a swamp for the impossibility of reaching the end of the rainbow.

The final common critique of automation is that naming an algorithm is naming a winner. In past papers, the meat of this argument has been that choosing metric or algorithm was simply a re-labelled political choice. Altman wrote, "the idea that automated redistricting is not inherently objective seems both correct and unavoidable." As cited above, Kennedy wrote in his equivocation for *Vieth* that contiguity and compactness "would unavoidably have significant political effect, whether intended or not." Since I have shown that a broad array of metrics yields consistent seat shares, the political choice is more constrained than one might have naïvely believed.[6] So it is true that the choice of an algorithm will affect electoral outcomes, but the fact that rules result in winners and losers hardly precludes their fairness or neutrality. The question then reduces to compactness's suitability and fairness. While a criterion of partisan bias might generate different baseline seat shares, the historical and constitutional justification is weak. Compactness on the other hand, has remained central to the debate and woven through the law for over a century. It is a recognized, non-partisan criterion for the principle of geographic representation.

Appendix I. The Consistency of Compactness Measures on Historical Maps

In addition to understanding the political consistency of maps optimized for various measures it is interesting to know whether the various compactness scores or ranks agree on enacted maps. Asking whether a district is compact or not is an important practical question, quite distinct from the possibility of generating compact districts. It could well be that away from the "extrema" chosen by the automated procedures, the compactness measures would rank districts differently. This Section evaluates the consistency of compactness scores and ranks on congressional districts and finds that the spatial "measurements" agree well in practice.

This analysis requires data beyond that used for the main results. Populations are drawn from the 1990, 2000, and 2010 Decennial Censuses. (US Cenus Bureau 2016) Additional, block-group (1990) and block-level (2000, 2010) geometries are also needed to calculate population distributions. (US Cenus Bureau 2012b,c, 2013b)

I then turn to the population of maps enacted in the last three districting cycles, in states with more than one representative. I evaluate all of the compactness measures aside path fraction, which has specific technical issues. The power diagrams, split-line procedure, and areal and population distance algorithms (Section 2.2) do not yield scores, and are not included in this analysis. However, I do replace power diagrams with the average interpersonal distance, with the normalization from Section 2.1.9. This gives me 14 measures of compactness (see Figure 3), and 1284 enacted districts.

I perform a principal component analysis (PCA) of this population using Scikit-learn.[7] (Pedregosa et al. 2011) The explained variance of the first component is 69% and that of the second component is 14%. Excluding axis ratio, which has dubious merit, these numbers

[6]The fact that they give consistent results in automation does not mean that they are equally effective as *constraints* on gerrymandering. That is, when manipulating lines with partisan intent, it may be easy to keep some compactness scores high and harder for others. This point underscores the strength of a process-based, instead of results-based solution.

[7]A PCA reduces the dimensionality of a collection of measurements; within a population, it collapses as much of the variation as possible on to a single axis, or component. Of the remaining variation, it collapses as much as possible onto a second axis, and so forth. Each successive component is orthogonal to earlier ones.

Figure I.1. Presented are the five districts of the 114th Congress scored least compact by the first component of the compactness principal component analysis (see Section 5). The North Carolina, Florida, and Texas districts have since been struck down.

change to 73% and 15%. Renormalizing the measures to have equal variance before the PCA, the loading on the first component falls to 60% (10% on the second).

The interpretation is thus again that despite the different shapes shown in Figure I.2, compactness in the context of districting reduces to a single concept: it is one-dimensional. The five districts of the 114th Congress that were least compact according to the first principal component are presented in Figure I.1. It is notable that of these five, three have already been overturned. The sixth least-compact district is Hawaii's second, which encompasses the entire 1500-mile long island chain, save the urban core of Honolulu. That district shows that non-compact districts are not necessarily gerrymandered, but it is the exception that proves the rule.

Figure I.2 presents the score and rank correlations among the fourteen measures and the PCA factors. Darker shading (closer to 1) represents higher correlation. As expected, all correlations are positive except between the two components of the PCA in the Spearman's rank analysis. Axis ratio is notably less-correlated with the other measures. The remaining thirteen have fairly high score and rank correlations; a poor compactness score or rank by one definition generally predicts a poor score or rank on another definition. By construction, the first principal component is highly correlated with the other measures. It is a good proxy measure of compactness. The correlation with interpersonal distance is particularly high ($\rho = 0.95$); this means that power diagrams will result in good scores across measures. Although the PCA suggests that a large part of compactness is explained by a single axis, I have block-diagonalized the matrices to highlight the various "notions" of compactness.

This analysis suggests that the question of whether or not a real district is compact can meaningfully be answered: it is not just a question of the choice of ruler. From a practical perspective, the first principal component from the analysis represents a robust – though opaque – combined measure of compactness. Optimizing plans for interpersonal distance (or generating power diagrams) will produce districts that score well by all measures.

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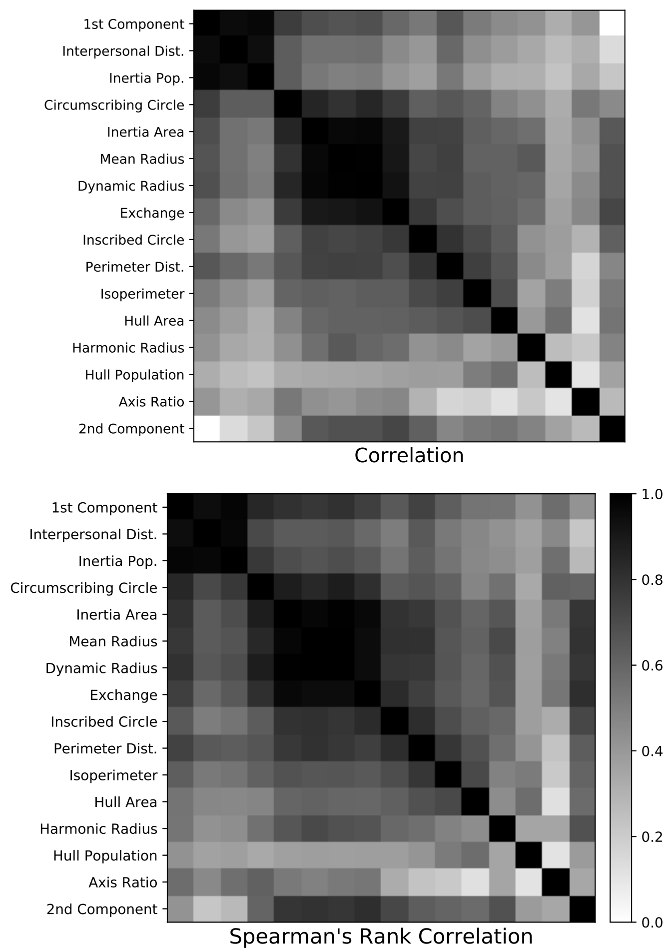
FIGURE I.2. Correlations among measures of compactness, and with the first two principal components. The correlations are all positive except between the first two primary components in the Spearman's rank correlation, and so the absolute value is taken.



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