Supplementary Material for

"Nonparametric Ideal-Point Estimation and Inference"

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Appendix

Contents

A Stratified bootstrap

The following summarizes the algorithm for the stratified bootstrap discussed in Section 4:

1. For i in 1 to n:

   (a) Sample a new $\tilde{A}$ of equal size to A by sampling bills from A with replacement

   (b) Sample a new $\tilde{B}$ of equal size to B by sampling bills from B with replacement

   (c) Using votes for each legislator from the resampled $\tilde{A}$ and $\tilde{B}$, compute $\tilde{\tau}_i = \sum\left(-2\log\left(\rho_{ik\ell m}\right)-1\right)$

2. Compute the test statistic

$$z = \frac{\sum\left(-2\log\left(\rho_{ik\ell m}\right)-1\right)}{\mathrm{Var}\left(\tilde{\tau}\right)^{-\frac{1}{2}}}$$

   where

$$\mathrm{Var}\left(\tilde{\tau}\right) = \frac{1}{n-1}\sum_{i=1}^{n}\left(\tilde{\tau}_i - \frac{1}{n}\sum_{j=1}^{n}\tilde{\tau}_j\right)^2.$$

3. Compute the p-value as $1-\Phi\left(z\right)$.

Note that, asymptotically, the test statistic z has distribution $\tau \sim \mathcal{N}\left(\mu, 1\right)$ with $\mu \leq 0$ under the null hypothesis hypothesis that $E\left[\sum\left(-2\log\left(\rho_{ik\ell m}\right)-1\right)\right] \leq 0$.

B Proofs

Definition 1. A vector $\boldsymbol{v}$ is **positive** if and only if $v_i > 0$ for all i. A matrix $\mathbf{M}$ is **positive** if and only if $M_{ij} > 0$ for all i and j. Note that whether a matrix is positive is distinct from whether it is positive definite.

Definition 2. A vector $\boldsymbol{v}$ is **negative** if and only if $v_i < 0$ for all i. A matrix $\mathbf{M}$ is **negative** if and only if $M_{ij} < 0$ for all i and j. Note that whether a matrix is negative is not the same as whether it is negative definite.

Definition 3. A nonnegative real number σ is a **singular value** of a real matrix $\mathbf{A}$ if and only if there exist unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ such that $\mathbf{A}\boldsymbol{v} = \sigma\boldsymbol{u}$ and $\mathbf{A}^\top\boldsymbol{u} = \sigma\boldsymbol{v}$. The unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are called **left-singular** and **right-singular vectors** corresponding to σ, respectively.

Definition 4. The vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ ($\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$) are **order equivalent** if and only if either $\forall i \neq j,\ (x_i \leq x_j) \Leftrightarrow (y_i \leq y_j)$ or $\forall i \neq j,\ (x_i \leq x_j) \Leftrightarrow (y_i \geq y_j)$.

Notation 5. Let K denote the number of legislators, and let $\mathbb{K} = \{1, \ldots, K\}$ be the set of integers between 1 and K.

Definition 6. Let ρ be a bijection from $\{(a, b) \in \mathbb{K}^2 \mid a < b\}$ to $\left\{1, \ldots, \binom{K}{2}\right\}$. Thus, ρ maps each unique ordered pair of numbers between 1 and K with the first number less than the second into a unique integer between 1 to $\binom{K}{2}$, inclusive.

Note. Such a mapping must exist since both sets have $\binom{K}{2}$ elements. One such bijection is $\rho(a, b) = \frac{(b-1)(b-2)}{2} + a$, but the choice of mapping will be irrelevant to the results.

Also, note that $\sum_{j=1}^{\binom{K}{2}} X_{j,i} = \sum_{\ell,m \in \mathbb{K}:\ell<m} X_{\rho(\ell,m),i}$. These two forms will be used interchangeably.

Definition 7. Given ideal points $\boldsymbol{x}$ and bijection ρ, the **ideal-point-rank matrix**, $\mathbf{X}$, is defined by

$$X_{\rho(\ell,m),i} = \begin{cases} 0 & \text{if } i \in \{\ell, m\} \\ \frac{1}{2}\operatorname{sgn}(x_m - x_\ell) \sum\limits_{k \in \mathbb{K} \setminus \{\ell,m,i\}} \operatorname{sgn}(x_i - x_k) & \text{otherwise} \end{cases} \tag{A1}$$

where $\operatorname{sgn}(\cdot)$ is the signum function. $\mathbf{X}$ is a **sorted ideal-point-rank matrix** if, additionally, $x_i \leq x_k$ if and only if $i \leq k$.

Note. Thus, for each integer ℓ and m ($1 \leq \ell < m \leq K$), the elements of row $\rho(\ell, m)$ are values from $-\frac{K-3}{2}$ to $\frac{K-3}{2}$ (in unit increments) in increasing or decreasing order of the ideal points except for the ℓ^{th} and m^{th} elements ($X_{\rho(\ell,m),\ell}$ and $X_{\rho(\ell,m),m}$), which are zero, with the order increasing if $x_m > x_\ell$ or decreasing otherwise. So, each row gives the rank of the ideal points, except those of legislators ℓ and m, normalized to sum to zero.

Note that each row has mean zero, as

$$\begin{aligned}
\sum_{i=1}^{K} X_{\rho(\ell,m),i} &= \sum_{i \in \mathbb{K} \setminus \{\ell,m\}} \frac{1}{2}\operatorname{sgn}(x_m - x_\ell) \sum_{k \in \mathbb{K} \setminus \{\ell,m,i\}} \operatorname{sgn}(x_i - x_k) \\
&= \frac{1}{2}\operatorname{sgn}(x_m - x_\ell) \sum_{i,k \in \mathbb{K} \setminus \{\ell,m\} : i \neq k} \operatorname{sgn}(x_i - x_k) \\
&= \frac{1}{2}\operatorname{sgn}(x_m - x_\ell) \sum_{i,k \in \mathbb{K} \setminus \{\ell,m\} : i \neq k} (-\operatorname{sgn}(x_k - x_i)) \\
&= -\frac{1}{2}\operatorname{sgn}(x_m - x_\ell) \sum_{k \in \mathbb{K} \setminus \{\ell,m\}} \left\{ \sum_{i \in \mathbb{K} \setminus \{\ell,m,k\}} \operatorname{sgn}(x_k - x_i) \right\} \\
&= -\sum_{k \in \mathbb{K} \setminus \{\ell,m\}} \frac{1}{2}\operatorname{sgn}(x_m - x_\ell) \sum_{i \in \mathbb{K} \setminus \{\ell,m,k\}} \operatorname{sgn}(x_k - x_i) \\
&= -\sum_{i=1}^{K} X_{\rho(\ell,m),i}.
\end{aligned} \tag{A2}$$

As a result, the each row of the matrix is equivalent to one found by assigning the integers from 1 to $K-2$ to all but two elements of the row, in increasing or decreasing order of the ideal-points, and then mean-centering each row.

The ordering of the rows of $\mathbf{X}$ (and, thus, the choice of ρ) will be irrelevant to the results, as the right singular vectors of $\mathbf{X}$ and their corresponding singular values do not depend on the order of the rows. For one particular choice of ρ, the matrix can be generated by the following R function:

```
ideal.point.rank.matrix <- function(K=length(theta), theta=1:K)
    t(combn(K, 2,
            function(x, theta) {
                z <- replicate(K, ifelse(1:K %in% x, NA, theta))
                return(rowSums(sign(z - t(z)), na.rm=TRUE) / 2) },
            theta=theta))
```

Definition 8. Given vote matrix $\mathbf{V}$ and bijection ρ, the **observed ideal-point-rank matrix**, $\hat{\mathbf{X}}$, is defined by

$$\hat{X}_{\rho(\ell,m),i} = \begin{cases} 0 & \text{if } i \in \{\ell, m\} \\ \frac{1}{2} \sum\limits_{k \in \mathbb{K} \setminus \{i,\ell,m\}} \operatorname{sgn}\left(\sum\limits_{j=1}^{N} v^j_{km|i\ell} - v^j_{im|k\ell} \right) & \text{otherwise} \end{cases} \tag{A3}$$

where $v^j_{im|k\ell}$ is defined as in Equation 25. As will be shown in Corollary 12, $\hat{\mathbf{X}} \xrightarrow{p} \mathbf{X}$ under the regularity condition mentioned in Footnote 10. Note that, as in Definition 7, each row is zero-centered.

B.1 Results on the inconsistency of Optimal Classification

Theorem 9. *For any ideal-point model with quadratic utility and a fixed number of legislators $K \geq 3$, there is a vector of ideal points $x_1, \ldots, x_K$ and sequence of bill parameters $(\alpha_i, \beta_i)_{i=1}^{\infty}$ such that Optimal Classification is not a consistent estimate of the order of the ideal points.*

Proof. Let v_{ij} denote the vote of legislator i on bill j and equal one for a 'yea' vote and zero for a ‘nay’ vote where the probability of a yea vote is $F(\beta_j x_i - \alpha_j)$ with $\alpha_j = (y_{j1}j)^2 - (y_{j0}j)^2$ and $\beta_j = 2(y_{j1}j - y_{j0}j)$. Let $m_j(x_i')$ denote the minimum number of misclassifications possible on vote j given ideal points x_i' and let $m(x_i') = \sum_{j=1}^{n} m_j(x_i')$ denote the minimum number of misclassifications over all votes. Note that in all cases, $0 \leq m_j \leq \frac{k}{2}$, as it is always possible to classify over half the votes correctly by classifying all votes together.

Because of the identifying constraints, $x_1 = 1$ and $x_2 = 0$. Let

$$\alpha_j = F^{-1}\left(1 - \left(1 - \frac{1}{6k^2 - 1}\right)^{\frac{1}{k-2}}\right) \tag{A4}$$

and $\beta_j = F^{-1}\left(\frac{1}{2k}\right) + \alpha_j$ for all j. Let $x_3 = \left(F^{-1}\left(\frac{1}{3k}\right) + \alpha_j\right)\beta_j^{-1}$. Finally, take any $x_i < 0$ for all $i > 3$. Note that this implies that

$$F(\beta_j x_3 - \alpha_j) = \frac{1}{3k} < \frac{1}{2k} = F(\beta_j x_1 - \alpha_j). \tag{A5}$$

Further, $F(\beta_j x_i - \alpha_j) < 1 - \left(1 - \frac{1}{6k^2-1}\right)^{\frac{1}{k-2}} = F(\beta_j x_1 - \alpha_j)$, so

$$\Pr(v_{1j} = 0 \wedge v_{ij} = 0 \forall i > 3) = \frac{6k^2 - 2}{6k^2 - 1} > \Pr(v_{3j} = 0) = \frac{3k - 1}{3k} > \Pr(v_{3j} = 0) = \frac{2k - 1}{2k}. \tag{A6}$$

Let $m_j(x_i')$ denote the minimum number of misclassifications possible on vote j given ideal points x_i' and let $m(x_i') = \sum_{j=1}^{n} m_j(x_i')$ denote the minimum number of misclassifications over all votes. Note that in all cases, $0 \leq m_j \leq \frac{k}{2}$, as it is always possible to classify over half the votes correctly by classifying all votes together.

Consider the case where the estimated ordering is $\hat{x}_{i3} < \hat{x}_{i4} < \ldots < \hat{x}_{ik} < \hat{x}_{i1} < \hat{x}_{i2}$. If $v_{1j} = 1$ and $v_{ij} = 0$ for all $i \neq 1$, then the data are perfectly separating and so $m_j(\hat{x}_i) = 0$. If $v_{3j} = 1$ and $v_{ij} = 0$ for all $i \neq 3$, then the data are again perfectly separating and so $m_j(\hat{x}_i) = 0$. And if $v_{ij} = 0$ for all i, $m_j(\hat{x}_i) = 0$. These three outcomes occur with probability

$$\Pr(v_{ij} = 0 \forall i) + \Pr(v_{1j} = 1, v_{ij} = 0 \forall i \neq 1) + \Pr(v_{3j} = 1, v_{ij} = 0 \forall i \neq 3)$$

$$
\begin{aligned}
&> \frac{2k-1}{2k}\cdot\frac{3k-1}{3k}\cdot\frac{6k^2-2}{6k^2-1}+\frac{1}{2k}\cdot\frac{3k-1}{3k}\cdot\frac{6k^2-2}{6k^2-1}+\frac{2k-1}{2k}\cdot\frac{1}{3k}\cdot\frac{6k^2-2}{6k^2-1} \\
&= \frac{3k^2-1}{3k^2} \qquad \text{(A7)}
\end{aligned}
$$

Since $m_j\left(\hat{x}_i\right) \leq \frac{k}{2}$ for all other cases and these occur with probability less than $\frac{1}{3k^2}$, we have

$$
E\left[m_j\left(\hat{x}_i\right)\right] < \frac{1}{6k}. \qquad \text{(A8)}
$$

Now, consider the case where $\tilde{x}_{i4} < \ldots < \tilde{x}_{ik} < \tilde{x}_{i1} < \tilde{x}_{i3} < \tilde{x}_{i2}$, which matches the true ordering. If $v_{3j} = 1$ and $v_{ij} = 0$ for all $i \neq 3$, then perfect classification is not possible, so $m_j\left(\tilde{x}_i\right) \geq 1$. This voting outcomes occur with probability

$$
\Pr\left(v_{3j}=1, v_{ij}=0 \forall i \neq 3\right) = \frac{2k-1}{2k}\cdot\frac{1}{3k}\cdot\frac{6k^2-2}{6k^2-1} > \frac{1}{4k} \qquad \text{(A9)}
$$

since $k \geq 3$. Since, in all other cases, $m_j \geq 0$, we have

$$
E\left[m_j\left(\tilde{x}_i\right)\right] > \frac{1}{4k}. \qquad \text{(A10)}
$$

Thus,

$$
E\left[m_j\left(\tilde{x}_i\right) - m_j\left(\hat{x}_i\right)\right] > \frac{1}{12k}. \qquad \text{(A11)}
$$

By the strong law of large numbers,

$$
\Pr\left(\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\left\{m_j\left(\tilde{x}_i\right) - m_j\left(\hat{x}_i\right)\right\} > 0\right) = 1. \qquad \text{(A12)}
$$

Thus,

$$
\begin{aligned}
1 &= \Pr\left(\lim_{n\to\infty}\frac{1}{n}\left\{\sum_{j=1}^{n} m_j\left(\tilde{x}_i\right) - \sum_{j=1}^{n} m_j\left(\hat{x}_i\right)\right\} > 0\right) \\
&= \Pr\left(\lim_{n\to\infty}\frac{1}{n}\left\{m\left(\tilde{x}_i\right) - m\left(\hat{x}_i\right)\right\} > 0\right)
\end{aligned}
$$

$$= \Pr\left(\exists N : \forall n > N, \sum_{j=1}^{n} m_j\left(\hat{x}_i\right) < \sum_{j=1}^{n} m_j\left(\tilde{x}_i\right)\right). \tag{A13}$$

So, given sufficiently many votes, the maximum number of correct classifications under $\hat{x}_i$ will almost surely exceed the maximum number under $\tilde{x}_i$, which is the correct ordering. Since optimal classification produces an estimated order which maximizes the number of correct classifications, optimal classification is not a consistent estimator of the order of the ideal points. □

B.2 Results related to Footnote 10

Lemma 10. *If*

$$0 \leq \liminf_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(\Pr\left(v^j_{km|i\ell}\right) - \Pr\left(v^j_{im|k\ell}\right)\right), \tag{A14}$$

then

$$\lim_{N\to\infty} \Pr\left(\sum_{j=1}^{N} v^j_{km|i\ell} > \sum_{j=1}^{N} v^j_{im|k\ell}\right) = 1. \tag{A15}$$

Proof. Assume

$$0 \leq \liminf_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(\Pr\left(v^j_{km|i\ell}\right) - \Pr\left(v^j_{im|k\ell}\right)\right), \tag{A16}$$

and let

$$Y_N = \frac{1}{N} \sum_{j=1}^{N} v^j_{km|i\ell} - v^j_{im|k\ell}. \tag{A17}$$

Since $0 \leq v^j_{im|k\ell} \leq 1$ and $0 \leq v^j_{km|i\ell} \leq 1$, $-1 \leq v^j_{km|i\ell} - v^j_{im|k\ell} \leq 1$. So, Popoviciu's Inequality implies that $\mathrm{Var}\left(v^j_{km|i\ell} - v^j_{im|k\ell}\right) \leq 1$. Thus,

$$\begin{aligned}\mathrm{Var}\left(Y_N\right) &= \mathrm{Var}\left(\frac{1}{N} \sum_{j=1}^{N} v^j_{km|i\ell} - v^j_{im|k\ell}\right)\\ &= \frac{1}{N^2} \sum_{j=1}^{N} \mathrm{Var}\left(v^j_{km|i\ell} - v^j_{im|k\ell}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{N^2}\sum_{j=1}^{N} 1 \\
&= \frac{1}{N} \qquad \text{(A18)}
\end{aligned}
$$

Chebyshev's inequality gives

$$\Pr\left(t \leq \left|Y_N - E\left[Y_N\right]\right|\right) \leq \frac{1}{t^2}\operatorname{Var}\left(Y_N\right) \tag{A19}$$

for all $t > 0$. Since $\operatorname{Var}\left(Y_N\right) \leq \frac{1}{N}$, $\frac{1}{t^2}\operatorname{Var}\left(Y_N\right) \leq \frac{1}{Nt^2}$ for all $t > 0$. Thus,

$$\Pr\left(t \leq \left|Y_N - E\left[Y_N\right]\right|\right) \leq \frac{1}{Nt^2} \tag{A20}$$

for all $t > 0$. And, as $\Pr\left(E\left[Y_N\right] - Y_N \geq t\right) \leq \Pr\left(\left|Y_N - E\left[Y_N\right]\right| \geq t\right)$ for all $t > 0$,

$$\Pr\left(t \leq -\left(Y_N - E\left[Y_N\right]\right)\right) \leq \frac{1}{Nt^2} \tag{A21}$$

for all $t > 0$. Setting $t = E\left[Y_N\right]$ gives

$$\Pr\left(E\left[Y_N\right] \leq E\left[Y_N\right] - Y_N\right) \leq \frac{1}{E\left[Y_N\right]^2 N}, \tag{A22}$$

which simplifies to

$$\Pr\left(Y_N \leq 0\right) \leq \frac{1}{E\left[Y_N\right]^2 N}. \tag{A23}$$

Noting that $0 \leq \Pr\left(Y_N \leq 0\right)$ and taking the limit superior of both sides thus gives

$$0 \leq \limsup_{N\to\infty} \Pr\left(Y_N \leq 0\right) \leq \limsup_{N\to\infty} \frac{1}{E\left[Y_N\right]^2 N}. \tag{A24}$$

Since $0 < \liminf_{N\to\infty} E\left[Y_N\right]^2$ by assumption, $\liminf_{N\to\infty} E\left[Y_N\right]^2 N = \infty$. Thus,

$$\limsup_{N\to\infty} \frac{1}{E\left[Y_N\right]^2 N} = 0. \tag{A25}$$

So,

$$\limsup_{N\to\infty} \Pr\left(Y_N \leq 0\right) \leq \limsup_{N\to\infty} \frac{1}{E\left[Y_N\right]^2 N} = 0. \tag{A26}$$

Therefore, $\limsup_{N\to\infty} \Pr\left(Y_N \leq 0\right) = 0$. Since $0 \leq \Pr\left(Y_N \leq 0\right)$ and $\liminf_{N\to\infty} \Pr\left(Y_N \leq 0\right) \leq \limsup_{N\to\infty} \Pr\left(Y_N \leq 0\right)$,

$$0 \leq \liminf_{N\to\infty} \Pr\left(Y_N \leq 0\right) \leq \limsup_{N\to\infty} \Pr\left(Y_N \leq 0\right) = 0. \tag{A27}$$

So,

$$\liminf_{N\to\infty} \Pr\left(Y_N \leq 0\right) = \limsup_{N\to\infty} \Pr\left(Y_N \leq 0\right) = 0. \tag{A28}$$

Thus,

$$\lim_{N\to\infty} \Pr\left(Y_N \leq 0\right) = 0 \tag{A29}$$

and

$$\lim_{N\to\infty} \Pr\left(Y_N > 0\right) = \lim_{N\to\infty} 1 - \Pr\left(Y_N \leq 0\right) = 1 - \lim_{N\to\infty} \Pr\left(Y_N \leq 0\right) = 1. \tag{A30}$$

Replacing Y_N with its definition, this last equation becomes gives

$$\lim_{N\to\infty} \Pr\left(\frac{1}{N}\sum_{j=1}^{N} v^j_{km|i\ell} - v^j_{im|k\ell} > 0\right) = 1. \tag{A31}$$

Multiplying both sides of the inequality by N, this becomes

$$\lim_{N\to\infty} \Pr\left(\sum_{j=1}^{N} v^j_{km|i\ell} - v^j_{im|k\ell} > 0\right) = 1. \tag{A32}$$

Therefore,

$$\lim_{N\to\infty} \Pr\left(\sum_{j=1}^{N} v^j_{km|i\ell} > \sum_{j=1}^{N} v^j_{im|k\ell}\right) = 1. \tag{A33}$$

$\square$

Theorem 11. *Let* $\left(\hat{\mathbf{X}}^{(N)}\right)_{N=1}^{\infty}$ *be a sequence of observed ideal-point-rank matrix after N votes generated by unique ideal points,* $\boldsymbol{x}$*. If*

$$0 \leq \liminf_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(\Pr\left(v^j_{km|i\ell}\right) - \Pr\left(v^j_{im|k\ell}\right)\right) \tag{A34}$$

for all i, k, ℓ, and m such that $x_i < x_k$ and $x_\ell < x_m$, then

$$\lim_{N\to\infty} \Pr\left(\hat{\mathbf{X}}^{(N)} = \mathbf{X}\right) = 1. \tag{A35}$$

Proof. Assume

$$0 \leq \liminf_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(\Pr\left(v^j_{km|i\ell}\right) - \Pr\left(v^j_{im|k\ell}\right)\right) \tag{A36}$$

for all i, k, ℓ, and m such that $x_i < x_k$ and $x_\ell < x_m$.

Take any N and any distinct $a, b, c, d \in \{1, 2, \ldots, K\}$. Since a and b are distinct and the elements of $\boldsymbol{x}$ are unique, $x_a \neq x_b$. Let $i = a$ and $k = b$ if $x_a < x_b$ and $i = b$ and $k = a$ otherwise. Thus, $x_i < x_k$. Similarly, since c and d are distinct, $x_c \neq x_d$. Let $\ell = c$ and $m = d$ if $x_c < x_d$ and $\ell = d$ and $m = c$ otherwise. Thus, $x_\ell < x_m$.

Note that $v^j_{bm|a\ell} = v^j_{km|i\ell}$ and $v^j_{am|b\ell} = v^j_{im|k\ell}$ if $x_a < x_b$. Similarly, $v^j_{am|b\ell} = v^j_{km|i\ell}$ and $v^j_{bm|a\ell} = v^j_{im|k\ell}$ if $x_a > x_b$. So, $v^j_{bm|a\ell} - v^j_{am|b\ell} = v^j_{km|i\ell} - v^j_{im|k\ell}$ if $x_a < x_b$ and $-\left(v^j_{bm|a\ell} - v^j_{am|b\ell}\right) = v^j_{km|i\ell} - v^j_{im|k\ell}$ if $x_a > x_b$. Thus,

$$v^j_{km|i\ell} - v^j_{im|k\ell} = \operatorname{sgn}\left(x_b - x_a\right)\left(v^j_{bm|a\ell} - v^j_{am|b\ell}\right). \tag{A37}$$

Further, note that $v^j_{bd|ac} = v^j_{bm|a\ell}$ and $v^j_{ad|bc} = v^j_{am|b\ell}$ if $x_c < x_d$. Similarly, $v^j_{bd|ac} = v^j_{b\ell|am} = v^j_{am|b\ell}$ and $v^j_{ad|bc} = v^j_{a\ell|bm} = v^j_{bm|a\ell}$ if $x_c > x_d$ (with $v^j_{b\ell|am} = v^j_{am|b\ell}$ and $v^j_{a\ell|bm} = v^j_{bm|a\ell}$ following from their definitions). So, $v^j_{bd|ac} - v^j_{ad|bc} = v^j_{bm|a\ell} - v^j_{am|b\ell}$ if $x_c < x_d$ and $-\left(v^j_{bd|ac} - v^j_{ad|bc}\right) = v^j_{bm|a\ell} - v^j_{am|b\ell}$ if $x_c > x_d$. Thus,

$$v^j_{bm|a\ell} - v^j_{am|b\ell} = \operatorname{sgn}(x_d - x_c)\left(v^j_{bd|ac} - v^j_{ac|bd}\right). \tag{A38}$$

So,

$$\begin{aligned} v^j_{km|i\ell} - v^j_{im|k\ell} &= \operatorname{sgn}(x_b - x_a)\left(v^j_{bm|a\ell} - v^j_{am|b\ell}\right) \\ &= \operatorname{sgn}(x_b - x_a)\operatorname{sgn}(x_d - x_c)\left(v^j_{bd|ac} - v^j_{ac|bd}\right). \end{aligned} \tag{A39}$$

By Lemma 10, since $x_i < x_k$ and $x_\ell < x_m$,

$$\lim_{N\to\infty} \Pr\left(\sum_{j=1}^N v^j_{km|i\ell} > \sum_{j=1}^N v^j_{im|k\ell}\right) = 1. \tag{A40}$$

Thus,

$$\lim_{N\to\infty} \Pr\left(\sum_{j=1}^N v^j_{km|i\ell} - v^j_{im|k\ell} > 0\right) = 1. \tag{A41}$$

Multiplying both sides of the equation inside $\Pr(\cdot)$ by $\operatorname{sgn}(x_b - x_a)\operatorname{sgn}(x_c - x_d)$ and replacing $\operatorname{sgn}(x_b - x_a)\operatorname{sgn}(x_c - x_d)\left(v^j_{km|i\ell} - v^j_{im|k\ell}\right)$ with $v^j_{bd|ac} - v^j_{ac|bd}$ gives

$$\lim_{N\to\infty} \Pr\left(\sum_{j=1}^N v^j_{bd|ac} - v^j_{ac|bd} = \operatorname{sgn}(x_b - x_a)\operatorname{sgn}(x_c - x_d)\right) = 1. \tag{A42}$$

So,

$$\lim_{N\to\infty} \Pr\left(\operatorname{sgn}\left(\sum_{j=1}^N v^j_{bd|ac} - v^j_{ac|bd}\right) = \operatorname{sgn}(x_b - x_a)\operatorname{sgn}(x_c - x_d)\right) = 1. \tag{A43}$$

Therefore, $\operatorname{sgn}\left(\sum_{j=1}^N v^j_{bd|ac} - v^j_{ac|bd}\right) \xrightarrow{p} \operatorname{sgn}(x_b - x_a)\operatorname{sgn}(x_c - x_d)$ for all distinct a, b, c, and d.

So, summing over $b \notin \{a, c, d\}$ and multiplying by $\frac{1}{2}$ gives

$$\lim_{N\to\infty} \Pr\left(\frac{1}{2}\sum_{b\notin\{a,c,d\}} \operatorname{sgn}\left(\sum_{j=1}^{N} v_{bd|ac}^{j} - v_{ac|bd}^{j}\right) = \frac{1}{2}\sum_{b\notin\{a,c,d\}} \operatorname{sgn}(x_b - x_a)\operatorname{sgn}(x_c - x_d)\right) = 1 \tag{A44}$$

for all distinct a, c, and d. Thus, by definition of $\hat{X}_{\rho(c,d),a}^{(N)}$ and $X_{\rho(c,d),a}$,

$$\lim_{N\to\infty} \Pr\left(\hat{X}_{\rho(c,d),a}^{(N)} = X_{\rho(c,d),a}\right) = 1 \tag{A45}$$

for all distinct a, c, and d such that $c < d$. Therefore, $\lim_{N\to\infty} \Pr\left(\hat{\mathbf{X}}^{(N)} = \mathbf{X}\right) = 1$. $\square$

Corollary 12. *Let $\left(\hat{\mathbf{X}}^{(N)}\right)_{N=1}^{\infty}$ be a sequence of observed ideal-point-rank matrix after N votes generated by unique ideal points, $\boldsymbol{x}$. If*

$$0 \leq \liminf_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\left(\Pr\left(v_{km|i\ell}^{j}\right) - \Pr\left(v_{im|k\ell}^{j}\right)\right) \tag{A46}$$

for all i, k, ℓ, and m such that $x_i < x_k$ and $x_\ell < x_m$, then

$$\hat{\mathbf{X}}^{(N)} \xrightarrow{p} \mathbf{X} \tag{A47}$$

as $N \to \infty$.

Proof. Assume $0 \leq \liminf_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N}\left(\Pr\left(v_{km|i\ell}^{j}\right) - \Pr\left(v_{im|k\ell}^{j}\right)\right)$ for all i, k, ℓ, and m such that $x_i < x_k$ and $x_\ell < x_m$. For all $\varepsilon > 0$ and all N,

$$\begin{aligned}
\Pr\left(\hat{X}_{\rho(c,d),a}^{(N)} = X_{\rho(c,d),a}\right) &= \Pr\left(\left|\hat{X}_{\rho(c,d),a}^{(N)} - X_{\rho(c,d),a}\right| = 0\right) \\
&\leq \Pr\left(\left|\hat{X}_{\rho(c,d),a}^{(N)} = X_{\rho(c,d),a}\right| < \varepsilon\right) \\
&\leq 1.
\end{aligned} \tag{A48}$$

Thus, as $\lim_{N\to\infty} \Pr\left(\hat{X}_{\rho(c,d),a}^{(N)} = X_{\rho(c,d),a}\right) = 1$ by Theorem 11, the squeeze theorem implies

$\lim_{N\to\infty} \Pr\left(\left|\hat{X}^{(N)}_{\rho(c,d),a} = X_{\rho(c,d),a}\right| < \varepsilon\right) = 1$ for all $\varepsilon > 0$ and all distinct a, c, and d such that $c < d$. So, by definition of convergence in probability, $\hat{X}^{(N)}_{\rho(c,d),a} \xrightarrow{p} X_{\rho(c,d),a}$ as $N \to \infty$ for all distinct a, c, and d such that $c < d$. Therefore, $\hat{\mathbf{X}}^{(N)} \xrightarrow{p} \mathbf{X}$ as $N \to \infty$. □

B.3 Useful linear-algebra theorems

The following are well-known results in linear algebra which will be useful in demonstrating later results. Proofs are omitted, as they can be found in many standard linear algebra texts.

Theorem 13 (Raleigh–Ritz)**.** *Let $\mathbf{A}$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then*

$$\lambda_1 = \max_{\boldsymbol{x}\neq 0} \frac{\boldsymbol{x}^\top \mathbf{A}\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}} \tag{A49}$$

(with $\mathbf{A}\boldsymbol{x} = \lambda_1 \boldsymbol{x}$ if and only if $\boldsymbol{x} \in \arg\max_{\boldsymbol{x}\neq 0} \frac{\boldsymbol{x}^\top \mathbf{A}\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}}$), and

$$\lambda_n = \min_{\boldsymbol{x}\neq 0} \frac{\boldsymbol{x}^\top \mathbf{A}\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}} \tag{A50}$$

(with $\mathbf{A}\boldsymbol{x} = \lambda_n \boldsymbol{x}$ if and only if $\boldsymbol{x} \in \arg\min_{\boldsymbol{x}\neq 0} \frac{\boldsymbol{x}^\top \mathbf{A}\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}}$).

Corollary 14. *If $\mathbf{A}$ is a $n \times n$ real symmetric matrix and $\boldsymbol{z}$ is an eigenvector of $\mathbf{A}$ corresponding to the largest eigenvalue, λ_1, then $\boldsymbol{z} \in \arg\max_{\boldsymbol{x}\neq 0} \frac{\boldsymbol{x}^\top \mathbf{A}\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}}$ and $\lambda_1 = \frac{\boldsymbol{z}^\top \mathbf{A}\boldsymbol{z}}{\boldsymbol{z}^\top \boldsymbol{z}}$.*

Theorem 15 (Singular Value Decomposition)**.** *For any $m \times n$ real matrix, $\mathbf{A}$, there exist nonnegative constants $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min(m,n)} \geq 0$; an orthogonal $m \times m$ matrix, $\mathbf{U}$; and an orthogonal $n \times n$ matrix $\mathbf{V}$ such that $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ where $\mathbf{D}$ is the $m \times n$ diagonal matrix such that $D_{ij} = \begin{cases} \sigma_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$.*

Corollary 16. *Let $\mathbf{A}$ be an $m \times n$ real matrix with singular value decomposition $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ and singular values $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{\min(m,n)}$ (where $\sigma_i = D_{ii}$). Then, the symmetric matrix $\mathbf{A}^\top\mathbf{A}$ has eigendecomposition $\mathbf{A}^\top\mathbf{A} = \mathbf{V}\mathbf{D}^2\mathbf{V}^\top$. Thus, $\mathbf{A}^\top\mathbf{A}$ has eigenvalues $\lambda_1 \geq \lambda_2 \geq$*

$\ldots \geq \lambda_n \geq 0$ *where* $\lambda_i = \sigma_i^2$ *if* $i \leq \min(m, n)$ *and* $\lambda_i = 0$ *otherwise, and a unit vector* $\boldsymbol{v}_i$ *is an eigenvector of* $\mathbf{A}^\top \mathbf{A}$ *corresponding to* λ_i *if and only if it is a right-singular vector of* $\mathbf{A}$ *corresponding to* σ_i.

Corollary 17. *Let* $\mathbf{A}$ *be an* $m \times n$ *real matrix with singular value decomposition* $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ *and singular values* $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{\min(m,n)}$ *(where* $\sigma_i = D_{ii}$*), the symmetric matrix* $\mathbf{A}\mathbf{A}^\top$ *has eigendecomposition* $\mathbf{A}\mathbf{A}^\top = \mathbf{U}\mathbf{D}^2\mathbf{U}^\top$*. Thus,* $\mathbf{A}\mathbf{A}^\top$ *has eigenvalues* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$ *where* $\lambda_i = \sigma_i^2$ *if* $i \leq \min(m, n)$ *and* $\lambda_i = 0$*, and a unit vector* $\boldsymbol{u}_i$ *is an eigenvector of* $\mathbf{A}\mathbf{A}^\top$ *corresponding to* λ_i *if and only if it is a left-singular vector of* $\mathbf{A}$ *corresponding to* σ_i.

B.4 Results regarding singular value decomposition of ideal-point-rank matrices

Theorem 23 establishes the basis for the suggested estimator based on a singular value decomposition (see page 15). Aside from the two legislators left out of each row, the values in each row are monotonic in the ideal points. Were it not necessary to leave two legislators out of each row, these results would follow more directly.

Lemma 18. *If* $\mathbf{Y}$ *is a real matrix,* $\boldsymbol{z}$ *is a left singular vector of* $\mathbf{Y}$ *corresponding to the largest singular value, and* $\left(\mathbf{Y}\mathbf{Y}^\top\right)^p$ *is positive for some* $p \geq 1$*, then either* $\boldsymbol{z}$ *is positive or* $\boldsymbol{z}$ *is negative.*

Proof. Assume $\mathbf{Y}$ is a real matrix, $\boldsymbol{z}$ is a left-singular vector of $\mathbf{Y}$ corresponding to the largest singular value, and $\left(\mathbf{Y}\mathbf{Y}^\top\right)^p$ is positive for some $p \geq 1$. Let $\mathbf{R} = \left(\mathbf{Y}\mathbf{Y}^\top\right)^p$.

Step 1 (Show $\boldsymbol{z}$ *is an eigenvector of* $\mathbf{R}$ *corresponding to the largest eigenvalue):*

By Corollary 17, $\boldsymbol{z}$ is also an eigenvector of $\mathbf{Y}\mathbf{Y}^\top$ corresponding to the largest eigenvalue, and all eigenvalues of $\mathbf{Y}\mathbf{Y}^\top$ are nonnegative. If $\mathbf{Y}\mathbf{Y}^\top$ has eigen decomposition $\mathbf{Y}\mathbf{Y}^\top = \mathbf{U}\mathbf{D}\mathbf{U}^\top$, then $\mathbf{R} = \left(\mathbf{Y}\mathbf{Y}^\top\right)^p = \left(\mathbf{U}\mathbf{D}\mathbf{U}^\top\right)^p = \mathbf{U}\mathbf{D}^p\mathbf{U}^\top$ since $\mathbf{U}$ is orthogonal. Thus, if $\lambda_1 \geq \ldots \geq \lambda_n \geq 0$ are the eigenvalues of $\mathbf{Y}\mathbf{Y}^\top$, then, since $p \geq 1$, $\lambda_1^p \geq \ldots \geq \lambda_n^p \geq 0$ are the eigenvalues of $\mathbf{R}$ with vector $\boldsymbol{z}'$ being an eigenvector of $\mathbf{R}$ corresponding to λ_1^p if and only if it is an eigenvector of $\mathbf{YY}^\top$ corresponding to λ_1. Thus, since $\boldsymbol{z}$ is an eigenvector of $\mathbf{YY}^\top$ corresponding to the largest eigenvalue, it is also an eigenvector of $\mathbf{R}$ corresponding to the largest eigenvalue.

Step 2 (Show either $z_i \geq 0$ for all i or $z_i \leq 0$ for all i):

Assume $z_k > 0$ and $z_\ell < 0$ for some k and ℓ. Take any such k and ℓ. Since $R_{k\ell} > 0$,

$$z_k R_{k\ell} z_\ell < 0 < \left| z_k R_{k\ell} z_\ell \right|. \tag{A51}$$

Since $z_i R_{ij} z_j \leq |z_i R_{ij} z_j|$ for all i and j and $z_k R_{k\ell} z_\ell < |z_k R_{k\ell} z_\ell|$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} z_i R_{ij} z_j < \sum_{i=1}^{n} \sum_{j=1}^{n} \left| z_i R_{ij} z_j \right|. \tag{A52}$$

Since $R_{ij} > 0$ for all i and j, $|z_i R_{ij} z_j| = |z_i| \, |R_{ij}| \, |z_j| = |z_i| \, (R_{ij}) \, |z_j|$. So,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} z_i R_{ij} z_j < \sum_{i=1}^{n} \sum_{j=1}^{n} |z_i| \, (R_{ij}) \, |z_j|. \tag{A53}$$

Let $\boldsymbol{z}^+$ be the vector such that $z_i^+ = |z_i|$ for all i. Thus,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} z_i R_{ij} z_j < \sum_{i=1}^{n} \sum_{j=1}^{n} z_i^+ R_{ij} z_j^+ \tag{A54}$$

or, equivalently,

$$\boldsymbol{z}^\top \mathbf{R} \boldsymbol{z} < \left(\boldsymbol{z}^+\right)^\top \mathbf{R} \boldsymbol{z}^+. \tag{A55}$$

Dividing both sides by $\boldsymbol{z}^\top \boldsymbol{z}$ and noting that $\boldsymbol{z}^\top \boldsymbol{z} \neq 0$ gives

$$\frac{\boldsymbol{z}^\top \mathbf{R} \boldsymbol{z}}{\boldsymbol{z}^\top \boldsymbol{z}} < \frac{\left(\boldsymbol{z}^+\right)^\top \mathbf{R} \boldsymbol{z}^+}{\boldsymbol{z}^\top \boldsymbol{z}}. \tag{A56}$$

Note that $\boldsymbol{z}^\top \boldsymbol{z} = \sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n} |z_i|^2 = \sum_{i=1}^{n} \left(z_i^+\right)^2 = \left(\boldsymbol{z}^+\right)^\top \boldsymbol{z}^+$. So,

$$\frac{\boldsymbol{z}^\top \mathbf{R} \boldsymbol{z}}{\boldsymbol{z}^\top \boldsymbol{z}} < \frac{\left(\boldsymbol{z}^+\right)^\top \mathbf{R} \boldsymbol{z}^+}{\left(\boldsymbol{z}^+\right)^\top \boldsymbol{z}^+}. \tag{A57}$$

But, since $\boldsymbol{z}$ is an eigenvector corresponding to the largest eigenvalue, the Rayleigh-Ritz Theorem implies that $\boldsymbol{z} \in \arg\max_{\boldsymbol{x}} \frac{\boldsymbol{x}^\top \mathbf{R} \boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}}$, which contradicts $\frac{\boldsymbol{z}^\top \mathbf{R} \boldsymbol{z}}{\boldsymbol{z}^\top \boldsymbol{z}} < \frac{(\boldsymbol{z}^+)^\top \mathbf{R} \boldsymbol{z}^+}{(\boldsymbol{z}^+)^\top \boldsymbol{z}^+}$. Therefore, either $z_i \geq 0$ for all i or $z_i \leq 0$ for all i.

Step 3 (Show $z_i \neq 0$ for all i):

Assume $z_k = 0$ for some k. As $\boldsymbol{z}$ is an eigenvector of $\mathbf{R}$ corresponding to the eigenvalue λ_1^p, $\lambda_1^p \boldsymbol{z} = \mathbf{R}\boldsymbol{z}$. So, $0 = \lambda_1^p z_k = \sum_{j=1}^{n} R_{kj} z_j$. Since $R_{kj} > 0$ and $z_j \geq 0$ for all j, $R_{kj} z_j \geq 0$ for all j. And, as the zero vector cannot be an eigenvector by definition, $z_j > 0$ for some j and, so, $R_{kj} z_j > 0$ for some j. Thus, $0 < \sum_{j=1}^{n} R_{kj} z_j$. This contradicts $0 = \sum_{j=1}^{n} R_{kj} z_j$. Therefore, $z_i \neq 0$ for all i.

Therefore, either $z_i > 0$ for all i or $z_i < 0$ for all i. So, $\boldsymbol{z}$ is either positive or negative. □

Lemma 19. *Let $\mathbf{X}$ be a sorted ideal-point-rank matrix with $K \geq 5$. If $\boldsymbol{z}$ is a left-singular vector of $\mathbf{X}$ corresponding to the largest singular value, then $\boldsymbol{z}$ is positive or $\boldsymbol{z}$ is negative.*

Proof. Let $\boldsymbol{z}$ be a left-singular vector of $\mathbf{X}$ corresponding to the largest singular value. Proceed by cases:

Case $n = 5$: As can be verified with the R code in Definition 7,

$$\mathbf{X}\mathbf{X}^\top = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & -1 \\ 1 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 2 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$
\left(\mathbf{X}\mathbf{X}^{\top}\right)^{3} = \begin{bmatrix}
66 & 75 & 75 & 26 & 85 & 85 & 36 & 85 & 36 & 19 \\
75 & 100 & 100 & 46 & 111 & 111 & 57 & 111 & 57 & 36 \\
75 & 100 & 100 & 46 & 111 & 111 & 57 & 111 & 57 & 36 \\
26 & 46 & 46 & 40 & 52 & 52 & 46 & 52 & 46 & 26 \\
85 & 111 & 111 & 52 & 170 & 170 & 111 & 170 & 111 & 85 \\
85 & 111 & 111 & 52 & 170 & 170 & 111 & 170 & 111 & 85 \\
36 & 57 & 57 & 46 & 111 & 111 & 100 & 111 & 100 & 75 \\
85 & 111 & 111 & 52 & 170 & 170 & 111 & 170 & 111 & 85 \\
36 & 57 & 57 & 46 & 111 & 111 & 100 & 111 & 100 & 75 \\
19 & 36 & 36 & 26 & 85 & 85 & 75 & 85 & 75 & 66
\end{bmatrix}
$$

Thus, $\left(\mathbf{X}\mathbf{X}^{\top}\right)^{3}$ is positive. Therefore, by Lemma 18, $\boldsymbol{z}$ is positive or $\boldsymbol{z}$ is negative.

Case $n = 6$**:** As can be verified with the R code in Definition 7,

$$
\mathbf{x}\mathbf{x}^{\top} = \begin{bmatrix}
5 & 2\frac{3}{4} & 3\frac{1}{4} & 2\frac{3}{4} & 1\frac{1}{4} & 2\frac{3}{4} & 3\frac{1}{4} & 2\frac{3}{4} & 1\frac{1}{4} & 2\frac{1}{2} & 2 & \frac{1}{2} & 1\frac{1}{2} & 0 & -1\frac{1}{2} \\
2\frac{3}{4} & 5 & 4\frac{3}{4} & 4\frac{1}{4} & 2\frac{3}{4} & 2\frac{3}{4} & 2\frac{1}{2} & 2 & \frac{1}{2} & 3\frac{1}{4} & 2\frac{3}{4} & 1\frac{1}{4} & 3 & 1\frac{1}{2} & 0 \\
3\frac{1}{4} & 4\frac{3}{4} & 5 & 4\frac{3}{4} & 3\frac{1}{4} & 2\frac{1}{2} & 2\frac{3}{4} & 2\frac{1}{2} & 1 & 3\frac{1}{4} & 3 & 1\frac{1}{2} & 2\frac{3}{4} & 1\frac{1}{4} & \frac{1}{2} \\
2\frac{3}{4} & 4\frac{1}{4} & 4\frac{3}{4} & 5 & 2\frac{3}{4} & 2 & 2\frac{1}{2} & 2\frac{3}{4} & \frac{1}{2} & 3 & 3\frac{1}{4} & 1 & 2\frac{3}{4} & \frac{1}{2} & 1\frac{1}{4} \\
1\frac{1}{4} & 2\frac{3}{4} & 3\frac{1}{4} & 2\frac{3}{4} & 5 & \frac{1}{2} & 1 & \frac{1}{2} & 2\frac{3}{4} & 1\frac{1}{2} & 1 & 3\frac{1}{4} & \frac{1}{2} & 2\frac{3}{4} & 1\frac{1}{4} \\
2\frac{3}{4} & 2\frac{3}{4} & 2\frac{1}{2} & 2 & \frac{1}{2} & 5 & 4\frac{3}{4} & 4\frac{1}{4} & 2\frac{3}{4} & 4\frac{3}{4} & 4\frac{1}{4} & 2\frac{3}{4} & 4\frac{1}{2} & 3 & 1\frac{1}{2} \\
3\frac{1}{4} & 2\frac{1}{2} & 2\frac{3}{4} & 2\frac{1}{2} & 1 & 4\frac{3}{4} & 5 & 4\frac{3}{4} & 3\frac{1}{4} & 4\frac{3}{4} & 4\frac{1}{2} & 3 & 4\frac{1}{4} & 2\frac{3}{4} & 2 \\
2\frac{3}{4} & 2 & 2\frac{1}{2} & 2\frac{3}{4} & \frac{1}{2} & 4\frac{1}{4} & 4\frac{3}{4} & 5 & 2\frac{3}{4} & 4\frac{1}{2} & 4\frac{3}{4} & 2\frac{1}{2} & 4\frac{1}{4} & 2 & 2\frac{3}{4} \\
1\frac{1}{4} & \frac{1}{2} & 1 & \frac{1}{2} & 2\frac{3}{4} & 2\frac{3}{4} & 3\frac{1}{4} & 2\frac{3}{4} & 5 & 3 & 2\frac{1}{2} & 4\frac{3}{4} & 2 & 4\frac{1}{4} & 2\frac{3}{4} \\
2\frac{1}{2} & 3\frac{1}{4} & 3\frac{1}{4} & 3 & 1\frac{1}{2} & 4\frac{3}{4} & 4\frac{3}{4} & 4\frac{1}{2} & 3 & 5 & 4\frac{3}{4} & 3\frac{1}{4} & 4\frac{3}{4} & 3\frac{1}{4} & 2\frac{1}{2} \\
2 & 2\frac{3}{4} & 3 & 3\frac{1}{4} & 1 & 4\frac{1}{4} & 4\frac{1}{2} & 4\frac{3}{4} & 2\frac{1}{2} & 4\frac{3}{4} & 5 & 2\frac{3}{4} & 4\frac{3}{4} & 2\frac{1}{2} & 3\frac{1}{4} \\
\frac{1}{2} & 1\frac{1}{4} & 1\frac{1}{2} & 1 & 3\frac{1}{4} & 2\frac{3}{4} & 3 & 2\frac{1}{2} & 4\frac{3}{4} & 3\frac{1}{4} & 2\frac{3}{4} & 5 & 2\frac{1}{2} & 4\frac{3}{4} & 3\frac{1}{4} \\
1\frac{1}{2} & 3 & 2\frac{3}{4} & 2\frac{3}{4} & \frac{1}{2} & 4\frac{1}{2} & 4\frac{1}{4} & 4\frac{1}{4} & 2 & 4\frac{3}{4} & 4\frac{3}{4} & 2\frac{1}{2} & 5 & 2\frac{3}{4} & 2\frac{3}{4} \\
0 & 1\frac{1}{2} & 1\frac{1}{4} & \frac{1}{2} & 2\frac{3}{4} & 3 & 2\frac{3}{4} & 2 & 4\frac{1}{4} & 3\frac{1}{4} & 2\frac{1}{2} & 4\frac{3}{4} & 2\frac{3}{4} & 5 & 2\frac{3}{4} \\
-1\frac{1}{2} & 0 & \frac{1}{2} & 1\frac{1}{4} & 1\frac{1}{4} & 1\frac{1}{2} & 2 & 2\frac{3}{4} & 2\frac{3}{4} & 2\frac{1}{2} & 3\frac{1}{4} & 3\frac{1}{4} & 2\frac{3}{4} & 2\frac{3}{4} & 5
\end{bmatrix}
$$

$$
\left(\mathbf{x}\mathbf{x}^{\top}\right)^{2} =
$$

$$
\begin{bmatrix}
94\frac{1}{2} & 98\frac{5}{8} & 104\frac{7}{8} & 96\frac{5}{8} & 53\frac{7}{8} & 106\frac{1}{8} & 112\frac{3}{8} & 104\frac{1}{8} & 61\frac{3}{8} & 113\frac{3}{4} & 105\frac{1}{2} & 62\frac{3}{4} & 99\frac{1}{4} & 56\frac{1}{2} & 31\frac{3}{4} \\
98\frac{5}{8} & 129\frac{3}{4} & 135\frac{5}{8} & 126\frac{5}{8} & 81\frac{5}{8} & 127\frac{1}{8} & 133 & 124 & 79 & 143\frac{3}{8} & 134\frac{3}{8} & 89\frac{3}{8} & 128\frac{1}{2} & 83\frac{1}{2} & 56\frac{1}{2} \\
104\frac{7}{8} & 135\frac{5}{8} & 143\frac{1}{2} & 134\frac{5}{8} & 88\frac{7}{8} & 133\frac{3}{4} & 141\frac{5}{8} & 132\frac{3}{4} & 87 & 151\frac{7}{8} & 143 & 97\frac{1}{4} & 135\frac{1}{8} & 89\frac{3}{8} & 62\frac{3}{4} \\
96\frac{5}{8} & 126\frac{5}{8} & 134\frac{5}{8} & 128\frac{3}{4} & 79\frac{5}{8} & 124 & 132 & 126\frac{1}{8} & 77 & 142 & 136\frac{1}{8} & 87 & 128\frac{1}{8} & 79 & 61\frac{3}{8} \\
53\frac{7}{8} & 81\frac{5}{8} & 88\frac{7}{8} & 79\frac{5}{8} & 84\frac{1}{2} & 76\frac{3}{4} & 84 & 74\frac{3}{4} & 79\frac{5}{8} & 93\frac{1}{4} & 84 & 88\frac{7}{8} & 76\frac{3}{4} & 81\frac{5}{8} & 53\frac{7}{8} \\
106\frac{1}{8} & 127\frac{1}{8} & 133\frac{3}{4} & 124 & 76\frac{3}{4} & 178\frac{1}{2} & 185\frac{1}{8} & 175\frac{3}{8} & 128\frac{1}{8} & 192\frac{1}{8} & 182\frac{3}{8} & 135\frac{1}{8} & 175\frac{3}{4} & 128\frac{1}{2} & 99\frac{1}{4} \\
112\frac{3}{8} & 133 & 141\frac{5}{8} & 132 & 84 & 185\frac{1}{8} & 193\frac{3}{4} & 184\frac{1}{8} & 136\frac{1}{8} & 200\frac{5}{8} & 191 & 143 & 182\frac{3}{8} & 134\frac{3}{8} & 105\frac{1}{2} \\
104\frac{1}{8} & 124 & 132\frac{3}{4} & 126\frac{1}{8} & 74\frac{3}{4} & 175\frac{3}{8} & 184\frac{1}{8} & 177\frac{1}{2} & 126\frac{1}{8} & 190\frac{3}{4} & 184\frac{1}{8} & 132\frac{3}{4} & 175\frac{3}{8} & 124 & 104\frac{1}{8} \\
61\frac{3}{8} & 79 & 87 & 77 & 79\frac{5}{8} & 128\frac{1}{8} & 136\frac{1}{8} & 126\frac{1}{8} & 128\frac{3}{4} & 142 & 132 & 134\frac{5}{8} & 124 & 126\frac{5}{8} & 96\frac{5}{8} \\
113\frac{3}{4} & 143\frac{3}{8} & 151\frac{7}{8} & 142 & 93\frac{1}{4} & 192\frac{1}{8} & 200\frac{5}{8} & 190\frac{3}{4} & 142 & 210\frac{1}{2} & 200\frac{5}{8} & 151\frac{7}{8} & 192\frac{1}{8} & 143\frac{3}{8} & 113\frac{3}{4} \\
105\frac{1}{2} & 134\frac{3}{8} & 143 & 136\frac{1}{8} & 84 & 182\frac{3}{8} & 191 & 184\frac{1}{8} & 132 & 200\frac{5}{8} & 193\frac{3}{4} & 141\frac{5}{8} & 185\frac{1}{8} & 133 & 112\frac{3}{8} \\
62\frac{3}{4} & 89\frac{3}{8} & 97\frac{1}{4} & 87 & 88\frac{7}{8} & 135\frac{1}{8} & 143 & 132\frac{3}{4} & 134\frac{5}{8} & 151\frac{7}{8} & 141\frac{5}{8} & 143\frac{1}{2} & 133\frac{3}{4} & 135\frac{5}{8} & 104\frac{7}{8} \\
99\frac{1}{4} & 128\frac{1}{2} & 135\frac{1}{8} & 128\frac{1}{8} & 76\frac{3}{4} & 175\frac{3}{4} & 182\frac{3}{8} & 175\frac{3}{8} & 124 & 192\frac{1}{8} & 185\frac{1}{8} & 133\frac{3}{4} & 178\frac{1}{2} & 127\frac{1}{8} & 106\frac{1}{8} \\
56\frac{1}{2} & 83\frac{1}{2} & 89\frac{3}{8} & 79 & 81\frac{5}{8} & 128\frac{1}{2} & 134\frac{3}{8} & 124 & 126\frac{5}{8} & 143\frac{3}{8} & 133 & 135\frac{5}{8} & 127\frac{1}{8} & 129\frac{3}{4} & 98\frac{5}{8} \\
31\frac{3}{4} & 56\frac{1}{2} & 62\frac{3}{4} & 61\frac{3}{8} & 53\frac{7}{8} & 99\frac{1}{4} & 105\frac{1}{2} & 104\frac{1}{8} & 96\frac{5}{8} & 113\frac{3}{4} & 112\frac{3}{8} & 104\frac{7}{8} & 106\frac{1}{8} & 98\frac{5}{8} & 94\frac{1}{2}
\end{bmatrix}
$$

Thus, $\left(\mathbf{X}\mathbf{X}^{\top}\right)^2$ is positive. Therefore, by Lemma 18, $\boldsymbol{z}$ is positive or $\boldsymbol{z}$ is negative.

Case $n = 7$**:** As can be verified with the R code in Definition 7,

$$\mathbf{x}\mathbf{x}^{\top} = \begin{bmatrix}
10 & 6 & 7 & 7 & 6 & 4 & 6 & 7 & 7 & 6 & 4 & 5 & 5 & 4 & 2 & 5 & 4 & 2 & 3 & 1 & -1 \\
6 & 10 & 9 & 9 & 8 & 6 & 6 & 5 & 5 & 4 & 2 & 7 & 7 & 6 & 4 & 7 & 6 & 4 & 5 & 3 & 1 \\
7 & 9 & 10 & 10 & 9 & 7 & 5 & 6 & 6 & 5 & 3 & 7 & 7 & 6 & 4 & 7 & 6 & 4 & 6 & 4 & 2 \\
7 & 9 & 10 & 10 & 9 & 7 & 5 & 6 & 6 & 5 & 3 & 7 & 7 & 6 & 4 & 7 & 6 & 4 & 6 & 4 & 2 \\
6 & 8 & 9 & 9 & 10 & 6 & 4 & 5 & 5 & 6 & 2 & 6 & 6 & 7 & 3 & 6 & 7 & 3 & 6 & 2 & 4 \\
4 & 6 & 7 & 7 & 6 & 10 & 2 & 3 & 3 & 2 & 6 & 4 & 4 & 3 & 7 & 4 & 3 & 7 & 2 & 6 & 4 \\
6 & 6 & 5 & 5 & 4 & 2 & 10 & 9 & 9 & 8 & 6 & 9 & 9 & 8 & 6 & 9 & 8 & 6 & 7 & 5 & 3 \\
7 & 5 & 6 & 6 & 5 & 3 & 9 & 10 & 10 & 9 & 7 & 9 & 9 & 8 & 6 & 9 & 8 & 6 & 8 & 6 & 4 \\
7 & 5 & 6 & 6 & 5 & 3 & 9 & 10 & 10 & 9 & 7 & 9 & 9 & 8 & 6 & 9 & 8 & 6 & 8 & 6 & 4 \\
6 & 4 & 5 & 5 & 6 & 2 & 8 & 9 & 9 & 10 & 6 & 8 & 8 & 9 & 5 & 8 & 9 & 5 & 8 & 4 & 6 \\
4 & 2 & 3 & 3 & 2 & 6 & 6 & 7 & 7 & 6 & 10 & 6 & 6 & 5 & 9 & 6 & 5 & 9 & 4 & 8 & 6 \\
5 & 7 & 7 & 7 & 6 & 4 & 9 & 9 & 9 & 8 & 6 & 10 & 10 & 9 & 7 & 10 & 9 & 7 & 9 & 7 & 5 \\
5 & 7 & 7 & 7 & 6 & 4 & 9 & 9 & 9 & 8 & 6 & 10 & 10 & 9 & 7 & 10 & 9 & 7 & 9 & 7 & 5 \\
4 & 6 & 6 & 6 & 7 & 3 & 8 & 8 & 8 & 9 & 5 & 9 & 9 & 10 & 6 & 9 & 10 & 6 & 9 & 5 & 7 \\
2 & 4 & 4 & 4 & 3 & 7 & 6 & 6 & 6 & 5 & 9 & 7 & 7 & 6 & 10 & 7 & 6 & 10 & 5 & 9 & 7 \\
5 & 7 & 7 & 7 & 6 & 4 & 9 & 9 & 9 & 8 & 6 & 10 & 10 & 9 & 7 & 10 & 9 & 7 & 9 & 7 & 5 \\
4 & 6 & 6 & 6 & 7 & 3 & 8 & 8 & 8 & 9 & 5 & 9 & 9 & 10 & 6 & 9 & 10 & 6 & 9 & 5 & 7 \\
2 & 4 & 4 & 4 & 3 & 7 & 6 & 6 & 6 & 5 & 9 & 7 & 7 & 6 & 10 & 7 & 6 & 10 & 5 & 9 & 7 \\
3 & 5 & 6 & 6 & 6 & 2 & 7 & 8 & 8 & 8 & 4 & 9 & 9 & 9 & 5 & 9 & 9 & 5 & 10 & 6 & 6 \\
1 & 3 & 4 & 4 & 2 & 6 & 5 & 6 & 6 & 4 & 8 & 7 & 7 & 5 & 9 & 7 & 5 & 9 & 6 & 10 & 6 \\
-1 & 1 & 2 & 2 & 4 & 4 & 3 & 4 & 4 & 6 & 6 & 5 & 5 & 7 & 7 & 5 & 7 & 7 & 6 & 6 & 10
\end{bmatrix}$$

$$\left(\mathbf{x}\mathbf{x}^{\top}\right)^2 =$$

$$\begin{bmatrix}
598 & 642 & 691 & 691 & 630 & 460 & 702 & 751 & 751 & 690 & 520 & 773 & 773 & 712 & 542 & 773 & 712 & 542 & 663 & 493 & 371 \\
642 & 790 & 837 & 837 & 770 & 588 & 810 & 857 & 857 & 790 & 608 & 931 & 931 & 864 & 682 & 931 & 864 & 682 & 809 & 627 & 493 \\
691 & 837 & 898 & 898 & 828 & 640 & 863 & 924 & 924 & 854 & 666 & 997 & 997 & 927 & 739 & 997 & 927 & 739 & 870 & 682 & 542 \\
691 & 837 & 898 & 898 & 828 & 640 & 863 & 924 & 924 & 854 & 666 & 997 & 997 & 927 & 739 & 997 & 927 & 739 & 870 & 682 & 542 \\
630 & 770 & 828 & 828 & 784 & 576 & 790 & 848 & 848 & 804 & 596 & 918 & 918 & 874 & 666 & 918 & 874 & 666 & 816 & 608 & 520 \\
460 & 588 & 640 & 640 & 576 & 568 & 596 & 648 & 648 & 584 & 576 & 712 & 712 & 648 & 640 & 712 & 648 & 640 & 596 & 588 & 460 \\
702 & 810 & 863 & 863 & 790 & 596 & 1030 & 1083 & 1083 & 1010 & 816 & 1137 & 1137 & 1064 & 870 & 1137 & 1064 & 870 & 1003 & 809 & 663 \\
751 & 857 & 924 & 924 & 848 & 648 & 1083 & 1150 & 1150 & 1074 & 874 & 1203 & 1203 & 1127 & 927 & 1203 & 1127 & 927 & 1064 & 864 & 712 \\
751 & 857 & 924 & 924 & 848 & 648 & 1083 & 1150 & 1150 & 1074 & 874 & 1203 & 1203 & 1127 & 927 & 1203 & 1127 & 927 & 1064 & 864 & 712 \\
690 & 790 & 854 & 854 & 804 & 584 & 1010 & 1074 & 1074 & 1024 & 804 & 1124 & 1124 & 1074 & 854 & 1124 & 1074 & 854 & 1010 & 790 & 690 \\
520 & 608 & 666 & 666 & 596 & 576 & 816 & 874 & 874 & 804 & 784 & 918 & 918 & 848 & 828 & 918 & 848 & 828 & 790 & 770 & 630 \\
773 & 931 & 997 & 997 & 918 & 712 & 1137 & 1203 & 1203 & 1124 & 918 & 1282 & 1282 & 1203 & 997 & 1282 & 1203 & 997 & 1137 & 931 & 773 \\
773 & 931 & 997 & 997 & 918 & 712 & 1137 & 1203 & 1203 & 1124 & 918 & 1282 & 1282 & 1203 & 997 & 1282 & 1203 & 997 & 1137 & 931 & 773 \\
712 & 864 & 927 & 927 & 874 & 648 & 1064 & 1127 & 1127 & 1074 & 848 & 1203 & 1203 & 1150 & 924 & 1203 & 1150 & 924 & 1083 & 857 & 751 \\
542 & 682 & 739 & 739 & 666 & 640 & 870 & 927 & 927 & 854 & 828 & 997 & 997 & 924 & 898 & 997 & 924 & 898 & 863 & 837 & 691 \\
773 & 931 & 997 & 997 & 918 & 712 & 1137 & 1203 & 1203 & 1124 & 918 & 1282 & 1282 & 1203 & 997 & 1282 & 1203 & 997 & 1137 & 931 & 773 \\
712 & 864 & 927 & 927 & 874 & 648 & 1064 & 1127 & 1127 & 1074 & 848 & 1203 & 1203 & 1150 & 924 & 1203 & 1150 & 924 & 1083 & 857 & 751 \\
542 & 682 & 739 & 739 & 666 & 640 & 870 & 927 & 927 & 854 & 828 & 997 & 997 & 924 & 898 & 997 & 924 & 898 & 863 & 837 & 691 \\
663 & 809 & 870 & 870 & 816 & 596 & 1003 & 1064 & 1064 & 1010 & 790 & 1137 & 1137 & 1083 & 863 & 1137 & 1083 & 863 & 1030 & 810 & 702 \\
493 & 627 & 682 & 682 & 608 & 588 & 809 & 864 & 864 & 790 & 770 & 931 & 931 & 857 & 837 & 931 & 857 & 837 & 810 & 790 & 642 \\
371 & 493 & 542 & 542 & 520 & 460 & 663 & 712 & 712 & 690 & 630 & 773 & 773 & 751 & 691 & 773 & 751 & 691 & 702 & 642 & 598
\end{bmatrix}$$

Thus, $\left(\mathbf{X}\mathbf{X}^{\top}\right)^2$ is positive. Therefore, by Lemma 18, $\boldsymbol{z}$ is positive or $\boldsymbol{z}$ is negative.

Case $K \geq 8$**:** Let $\mathbf{Y} = \mathbf{X}\mathbf{X}^\top$. Thus,

$$
\begin{aligned}
Y_{ij} &= \sum_{k=1}^{K} X_{ik} X_{jk} \\
&= \sum_{k=1}^{k} X_{ik} \left(X_{ik} - X_{ik} + X_{jk} \right) \\
&= \sum_{k=1}^{K} X_{ik}^2 - \sum_{k=1}^{K} X_{ik} \left(X_{ik} - X_{jk} \right) \\
&= \sum_{k=1}^{K} X_{ik}^2 - \sum_{k:X_{jk}=0} X_{ik} \left(X_{ik} - X_{jk} \right) - \sum_{k:X_{jk}\neq 0} X_{ik} \left(X_{ik} - X_{jk} \right) \\
&= \sum_{k=1}^{K} X_{ik}^2 - \sum_{k:X_{jk}=0} X_{ik}^2 - \sum_{k:X_{jk}\neq 0} X_{ik} \left(X_{ik} - X_{jk} \right) \\
&= \sum_{k=1}^{K-2} \left(c_1 + i \right)^2 - \sum_{k:X_{jk}=0} X_{ik}^2 - \sum_{k:X_{jk}\neq 0} X_{ik} \left(X_{ik} - X_{jk} \right) \\
Y_{ij} &\geq \frac{(K-2)(K-1)(K-3)}{12} - 2c_1^2 - \sum_{k:X_{jk}\neq 0} X_{ik} \left(X_{ik} - X_{jk} \right) \\
&= \frac{(K-2)(K-1)(K-3)}{12} - \frac{(K-1)^2}{2} - \sum_{k:X_{jk}\neq 0} X_{ik} \left(X_{ik} - X_{jk} \right) \\
Y_{ij} &\geq \frac{(K-2)(K-1)(K-3)}{12} - \frac{(K-1)^2}{2} - 2(K-2) \\
&= \frac{(K-2)(K-1)(K-3)}{12} - \left(\frac{1}{2}K^2 - K + \frac{1}{2} - 2K - 4 \right) \\
&= \frac{(K-2)(K-1)(K-3)}{12} - \frac{1}{2}\left(K^2 - 2K + 1 - 4K - 8 \right) \\
&= \frac{(K-2)(K-1)(K-3)}{12} - \frac{1}{2}\left(K^2 - 6K - 7 \right) \\
Y_{ij} &\geq \frac{(K-2)(K-1)(K-3)}{12} - \frac{1}{2}(K-2)(K-3) \\
&= \left(\frac{(K-2)(K-3)}{2} \right) \left(\frac{K-1}{6} - 1 \right) \\
&= \left(\frac{(K-2)(K-3)}{2} \right) \left(\frac{K-7}{6} \right) \\
&= \frac{(K-2)(K-3)(K-7)}{12} \\
Y_{ij} &> 0
\end{aligned}
\tag{A58}
$$

Thus, $\mathbf{X}\mathbf{X}^\top$ is positive. Therefore, by Lemma 18, $\boldsymbol{z}$ is positive or $\boldsymbol{z}$ is negative.

Therefore, in all cases, either $\boldsymbol{z}$ is positive or $\boldsymbol{z}$ is negative. $\square$

Lemma 20. *If $\mathbf{X}$ is an $m \times n$ real matrix, $\mathbf{U}\mathbf{D}\mathbf{V}^\top$ is a singular value decomposition of $\mathbf{X}$, $\mathbf{P}$ is an $m \times m$ orthogonal matrix, and $\mathbf{Q}$ is an $n \times n$ orthogonal matrix, then $(\mathbf{P}\mathbf{U})\,\mathbf{D}\,(\mathbf{Q}\mathbf{V})^\top$ is a singular value decomposition of $\mathbf{P}\mathbf{X}\mathbf{Q}^\top$.*

Proof. Let $\mathbf{X}$ be an $m \times n$ real matrix, $\mathbf{P}$ be an $m \times m$ orthogonal matrix, and $\mathbf{Q}$ be an $n \times n$ orthogonal matrix. Assume $\mathbf{U}\mathbf{D}\mathbf{V}^\top$ is a singular value decomposition of $\mathbf{X}$. Thus, by definition, $\mathbf{U}^\top\mathbf{U} = \mathbf{I}$, $\mathbf{V}^\top\mathbf{V} = \mathbf{I}$ and $\mathbf{X} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$. Thus, $\mathbf{P}\mathbf{X}\mathbf{Q}^\top = \mathbf{P}\mathbf{U}\mathbf{D}\mathbf{V}^\top\mathbf{Q}^\top = (\mathbf{P}\mathbf{U})\,\mathbf{D}\,(\mathbf{Q}\mathbf{V})^\top$.

Since $\mathbf{P}$ is orthogonal, $\mathbf{P}^\top\mathbf{P} = \mathbf{I}$. As $\mathbf{U}^\top\mathbf{U} = \mathbf{I}$, this implies $(\mathbf{P}\mathbf{U})^\top(\mathbf{P}\mathbf{U}) = \mathbf{P}^\top\mathbf{U}^\top\mathbf{U}\mathbf{P} = \mathbf{P}^\top\mathbf{P} = \mathbf{I}$. So, $\mathbf{P}\mathbf{U}$ is orthogonal. Similarly, since $\mathbf{Q}$ is orthogonal, $\mathbf{Q}^\top\mathbf{Q} = \mathbf{I}$. And, as $\mathbf{V}^\top\mathbf{V} = \mathbf{I}$, $(\mathbf{Q}\mathbf{V})^\top(\mathbf{Q}\mathbf{V}) = \mathbf{Q}^\top\mathbf{V}^\top\mathbf{V}\mathbf{Q} = \mathbf{Q}^\top\mathbf{Q} = \mathbf{I}$. So, $\mathbf{Q}\mathbf{V}$ is orthogonal.

Therefore, since $\mathbf{P}\mathbf{X}\mathbf{Q}^\top = (\mathbf{P}\mathbf{U})\,\mathbf{D}\,(\mathbf{Q}\mathbf{V})^\top$ with $\mathbf{P}\mathbf{U}$ and $\mathbf{Q}\mathbf{V}$ both orthogonal, it must be the case that $(\mathbf{P}\mathbf{U})\,\mathbf{D}\,(\mathbf{Q}\mathbf{V})^\top$ is a singular value decomposition of $\mathbf{X}$. $\square$

Lemma 21. *If $\mathbf{X}$ is a $\binom{K}{2} \times K$ sorted ideal-point-rank matrix with $K \geq 5$ for legislators with unique ideal-points $\boldsymbol{x}$ and $\boldsymbol{v}$ is a right singular vector of $\mathbf{X}$ corresponding to the largest singular value, then $\boldsymbol{x}$ and $\boldsymbol{v}$ are order equivalent.*

Proof. Assume $\mathbf{X}$ is a $\binom{K}{2} \times K$ sorted ideal-point-rank matrix with $K \geq 5$ for legislators with unique ideal-points $\boldsymbol{x}$ and $\boldsymbol{v}$ is a right singular vector of $\mathbf{X}$ corresponding to the largest singular value. By definition, since $\mathbf{X}$ is a sorted ideal-point-rank matrix,

$$X_{\rho(\ell,m),i} = \begin{cases} 0 & \text{if } i \in \{\ell, m\} \\ \frac{1}{2}\operatorname{sgn}(x_m - x_\ell) \sum\limits_{k \in \mathbb{K}\setminus\{\ell,m,i\}} \operatorname{sgn}(x_i - x_k) & \text{otherwise} \end{cases} \tag{A59}$$

and $x_\ell \leq x_m$ for all $\ell < m$. And, since the elements of $\boldsymbol{x}$ are unique, $x_\ell < x_m$ for all $\ell < m$. Thus, $\operatorname{sgn}(x_m - x_\ell) = 1$.

And since, by definition, $x_i < x_k$ if $i < k$ and $x_i > x_k$ if $i > k$ for all $i \neq k$, this leaves

$$X_{\rho(\ell,m),i} = \begin{cases} 0 & \text{if } i \in \{\ell, m\} \\ \frac{1}{2} \sum\limits_{k \in \mathbb{K} \setminus \{\ell,m,i\}} \operatorname{sgn}(i-k) & \text{otherwise} \end{cases} \tag{A60}$$

for all $\ell < m$. So, for all a, ℓ, and m such that $\ell < m$ and $\ell, m \notin \{a, a+1\}$, it follows that

$$X_{\rho(a,m),a+1} = \frac{1}{2} \sum_{k \in \mathbb{K} \setminus \{a,a+1,m\}} \operatorname{sgn}(a+1-k) = \frac{1}{2} \sum_{k \in \mathbb{K} \setminus \{a,a+1,m\}} \operatorname{sgn}(a-k) = X_{\rho(a+1,m),a} \tag{A61}$$

$$X_{\rho(\ell,a),a+1} = \frac{1}{2} \sum_{k \in \mathbb{K} \setminus \{\ell,a,a+1\}} \operatorname{sgn}(a+1-k) = \frac{1}{2} \sum_{k \in \mathbb{K} \setminus \{\ell,a,a+1\}} \operatorname{sgn}(a-k) = X_{\rho(\ell,a+1),a} \tag{A62}$$

$$X_{\rho(a,a+1),a+1} = 0 = X_{\rho(a,a+1),a} \tag{A63}$$

Further, since $\operatorname{sgn}(a-k) = \operatorname{sgn}(a+1-k)$ for all integers $k \notin \{a, a+1\}$,

$$\begin{aligned} X_{\rho(\ell,m),a+1} &= \frac{1}{2} \sum_{k \in \mathbb{K} \setminus \{\ell,m,a+1\}} \operatorname{sgn}(a+1-k) \\ &= \frac{1}{2}\operatorname{sgn}((a+1)-a) + \frac{1}{2} \sum_{k \in \mathbb{K} \setminus \{\ell,m,a,a+1\}} \operatorname{sgn}(a-k) \\ &= \frac{1}{2} + \frac{1}{2} \sum_{k \in \mathbb{K} \setminus \{\ell,m,a,a+1\}} \operatorname{sgn}(a-k) \\ &= 1 - \frac{1}{2} + \frac{1}{2} \sum_{k \in \mathbb{K} \setminus \{\ell,m,a,a+1\}} \operatorname{sgn}(a-k) \\ &= 1 + \frac{1}{2}\operatorname{sgn}(a-(a+1)) + \frac{1}{2} \sum_{k \in \mathbb{K} \setminus \{\ell,m,a,a+1\}} \operatorname{sgn}(a+1-k) \\ &= 1 + \frac{1}{2} \sum_{k \in \mathbb{K} \setminus \{\ell,m,a\}} \operatorname{sgn}(a+1-k) \\ &= 1 + X_{\rho(\ell,m),a} \end{aligned} \tag{A64}$$

Let $\boldsymbol{u}$ be the left singular vector of $\mathbf{X}$ corresponding to the right singular vector $\boldsymbol{v}$ and the largest singular value. By Lemma 19, $\boldsymbol{u}$ is positive or $\boldsymbol{u}$ is negative. Assume, without loss of generality, that $\boldsymbol{u}$ is positive (see final paragraph).

By definition, singular values must be non-negative. If all singular values of $\mathbf{X}$ are zero, then $\mathbf{X}$ has singular value decomposition $\mathbf{UDV}^\top = \mathbf{U0V}^\top = \mathbf{0}$. Since $\mathbf{X} \neq \mathbf{0}$, at least one singular value must be positive. Thus, the largest singular value of $\mathbf{X}$ must be positive. Let σ be the largest singular value of $\mathbf{X}$. So, $\sigma > 0$.

By definition of singular vectors, $\mathbf{X}\boldsymbol{v} = \sigma\boldsymbol{u}$. So, $\sum_{j=1}^{K} X_{ij}v_j = \sigma u_i$ for all i. Since $\boldsymbol{u}$ is positive, $u_i > 0$ for all i. So, since $\sigma > 0$, $\sum_{j=1}^{K} X_{ij}v_j = \sigma u_i > 0$ for all i.

Assume there exists a value a such that $v_a \geq v_{a+1}$. Take any such a.

Case $v_a > v_{a+1}$: Let $\boldsymbol{w}$ be the vector such that $w_a = v_{a+1}$, $w_{a+1} = v_a$, and $w_i = v_i$ for all $i < a$ or $i > a+1$. It follows from this definition and Equations A61–A63 that

$$
\begin{aligned}
\sum_{i=1}^{K} X_{\rho(a,m),i}w_i &= \sum_{i=1}^{a-1} X_{\rho(a,m),i}w_i + \sum_{i=a+2}^{K} X_{\rho(a,m),i}w_i + X_{\rho(a,m),a}w_a + X_{\rho(a,m),a+1}w_{a+1} \\
&= \sum_{i=1}^{a-1} X_{\rho(a+1,m),i}v_i + \sum_{i=a+2}^{K} X_{\rho(a+1,m),i}v_i + X_{\rho(a+1,m),a+1}v_{a+1} + X_{\rho(a+1,m),a}v_a \\
&= \sum_{i=1}^{K} X_{\rho(a+1,m),i}v_i \\
\sum_{i=1}^{K} X_{\rho(\ell,a),i}w_i &= \sum_{i=1}^{a-1} X_{\rho(\ell,a),i}w_i + \sum_{i=a+2}^{K} X_{\rho(\ell,a),i}w_i + X_{\rho(\ell,a),a}w_a + X_{\rho(\ell,a),a+1}w_{a+1} \\
&= \sum_{i=1}^{a-1} X_{\rho(\ell,a+1),i}v_i + \sum_{i=a+2}^{K} X_{\rho(\ell,a+1),i}v_i + X_{\rho(\ell,a+1),a+1}v_{a+1} + X_{\rho(\ell,a+1),a}v_a \\
&= \sum_{i=1}^{K} X_{\rho(\ell,a+1),i}v_i \\
\sum_{i=1}^{K} X_{\rho(a,a+1),i}w_i &= \sum_{i=1}^{a-1} X_{\rho(a,a+1),i}w_i + \sum_{i=a+2}^{K} X_{\rho(a,a+1),i}w_i + X_{\rho(a,a+1),a}w_a + X_{\rho(a,a+1),a+1}w_{a+1} \\
&= \sum_{i=1}^{a-1} X_{\rho(a,a+1),i}v_i + \sum_{i=a+2}^{K} X_{\rho(a,a+1),i}v_i + X_{\rho(a,a+1),a+1}v_{a+1} + X_{\rho(\ell,a+1),a}v_a
\end{aligned}
$$

$$= \sum_{i=1}^{K} X_{\rho(a,a+1),i} v_i \tag{A65}$$

Similarly, it follows from Equation A64 that

$$\begin{aligned}
\sum_{i=1}^{K} X_{\rho(\ell,m),i} w_i &= \sum_{i=1}^{a-1} X_{\rho(\ell,m),i} w_i + \sum_{i=a+2}^{K} X_{\rho(\ell,m),i} w_i + X_{\rho(\ell,m),a} w_a + X_{\rho(\ell,m),a+1} w_{a+1} \\
&= \sum_{i=1}^{a-1} X_{\rho(\ell,m),i} v_i + \sum_{i=a+2}^{K} X_{\rho(\ell,m),i} v_i + X_{\rho(\ell,m),a} v_{a+1} + X_{\rho(\ell,m),a+1} v_a \\
&= \sum_{i=1}^{a-1} X_{\rho(\ell,m),i} v_i + \sum_{i=a+2}^{K} X_{\rho(\ell,m),i} v_i + \left(X_{\rho(\ell,m),a+1} - 1\right) v_{a+1} + \left(1 + X_{\rho(\ell,m),a}\right) v_a \\
&= \sum_{i=1}^{a-1} X_{\rho(\ell,m),i} v_i + \sum_{i=a+2}^{K} X_{\rho(\ell,m),i} v_i + X_{\rho(\ell,m),a+1} v_{a+1} + X_{\rho(\ell,m),a} v_a + v_a - v_{a+1} \\
&= \sum_{i=1}^{K} X_{\rho(\ell,m),i} v_i + v_a - v_{a+1} \qquad \text{(A66)}
\end{aligned}$$

And, since $\mathbf{X}\boldsymbol{v} = \sigma \boldsymbol{u}$, $\sum_{i=1}^{K} X_{\rho(\ell,m),i} v_i = 2\sigma \left(v_a - v_{a+1}\right) u_{\rho(\ell,m)}$. So,

$$\begin{aligned}
\left(\sum_{i=1}^{K} X_{\rho(\ell,m),i} w_i\right)^2 &= \left(\sum_{i=1}^{K} X_{\rho(\ell,m),i} v_i + \left(v_a - v_{a+1}\right)\right)^2 \\
&= \left(\sum_{i=1}^{K} X_{\rho(\ell,m),i} v_i\right)^2 + \left(v_a - v_{a+1}\right)^2 + 2\left(v_a - v_{a+1}\right) \sum_{i=1}^{K} X_{\rho(\ell,m),i} v_i \\
&= \left(\sum_{i=1}^{K} X_{\rho(\ell,m),i} v_i\right)^2 + \left(v_a - v_{a+1}\right)^2 + 2\sigma \left(v_a - v_{a+1}\right) u_{\rho(\ell,m)} \qquad \text{(A67)}
\end{aligned}$$

Thus, summing across $\ell < m$ and dividing this sum into cases where where $\ell, m \notin \{a, a+1\}$, $\ell = a < a+1 < m$, $\ell = a+1 < m$, $\ell < m = a$, $\ell < a < m = a+1$, and $\ell = a < m = a+1$ gives

$$\sum_{\ell,m \in \mathbb{K}: \ell < m} \left(\sum_{i=1}^{K} X_{\rho(\ell,m),i} w_i\right)^2$$

$$
\begin{aligned}
&= \sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}w_i\right)^2 + \sum_{\ell,m\in\mathbb{K}:\ell=a<a+1<m}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}w_i\right)^2 \\
&\quad + \sum_{\ell,m\in\mathbb{K}:\ell=a+1<m}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}w_i\right)^2 + \sum_{\ell,m\in\mathbb{K}:\ell<m=a}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}w_i\right)^2 \\
&\quad + \sum_{\ell,m\in\mathbb{K}:\ell<a<m=a+1}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}w_i\right)^2 + \sum_{\ell,m\in\mathbb{K}:\ell=a<m=a+1}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}w_i\right)^2 \\
&= \sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}w_i\right)^2 + \sum_{m\in\mathbb{K}:a+1<m}\left(\sum_{i=1}^{K}X_{\rho(a,m),i}w_i\right)^2 \\
&\quad + \sum_{m\in\mathbb{K}:a+1<m}\left(\sum_{i=1}^{K}X_{\rho(a+1,m),i}w_i\right)^2 + \sum_{\ell,m\in\mathbb{K}:\ell<m}\left(\sum_{i=1}^{K}X_{\rho(\ell,a),i}w_i\right)^2 \\
&\qquad + \sum_{\ell,m\in\mathbb{K}:\ell<a}\left(\sum_{i=1}^{K}X_{\rho(\ell,a+1),i}w_i\right)^2 + \left(\sum_{i=1}^{K}X_{\rho(a,a+1),i}w_i\right)^2 \\
&= \sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m}2\sigma\left(v_a-v_{a+1}\right)u_{\rho(\ell,m)} + \sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m}\left(v_a-v_{a+1}\right)^2 \\
&\quad + \sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}v_i\right)^2 + \sum_{m\in\mathbb{K}:a+1<m}\left(\sum_{i=1}^{K}X_{\rho(a+1,m),i}v_i\right)^2 \\
&\quad + \sum_{m\in\mathbb{K}:a+1<m}\left(\sum_{i=1}^{K}X_{\rho(a,m),i}v_i\right)^2 + \sum_{\ell,m\in\mathbb{K}:\ell<m}\left(\sum_{i=1}^{K}X_{\rho(\ell,a+1),i}v_i\right)^2 \\
&\qquad + \sum_{\ell,m\in\mathbb{K}:\ell<a}\left(\sum_{i=1}^{K}X_{\rho(\ell,a),i}v_i\right)^2 + \left(\sum_{i=1}^{K}X_{\rho(a,a+1),i}v_i\right)^2 \\
&= 2\sigma\left(v_a-v_{a+1}\right)\sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m}u_{\rho(\ell,m)} + \binom{K-2}{2}\left(v_a-v_{a+1}\right)^2 \\
&\qquad + \sum_{\ell,m\in\mathbb{K}:\ell<m}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}v_i\right)^2
\end{aligned}
\tag{A68}
$$

Since $\sigma>0$, $v_a-v_{a+1}>0$ by assumption, and $u_{\rho(\ell,m)}>0$ since $\boldsymbol{u}$ is positive,

$$
2\sigma\left(v_a-v_{a+1}\right)\sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m}u_{\rho(\ell,m)} + \binom{K-2}{2}\left(v_a-v_{a+1}\right)^2>0. \tag{A69}
$$

Thus,

$$\sum_{\ell,m\in\mathbb{K}:\ell<m}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}w_i\right)^2 > \sum_{\ell,m\in\mathbb{K}:\ell<m}\left(\sum_{i=1}^{K}X_{\rho(\ell,m),i}v_i\right)^2 \tag{A70}$$

or, in matrix form,

$$\boldsymbol{w}^\top\mathbf{X}^\top\mathbf{X}\boldsymbol{w} > \boldsymbol{v}^\top\mathbf{X}^\top\mathbf{X}\boldsymbol{v}. \tag{A71}$$

But, since $\boldsymbol{v}$ is a right singular vector of $\mathbf{X}$ corresponding to the largest singular value, it follows from the Rayleigh-Ritz Theorem and Corollary 16 that $\boldsymbol{v} \in \arg\max_{\boldsymbol{z}} \frac{\boldsymbol{z}^\top\mathbf{X}^\top\mathbf{X}\boldsymbol{z}}{\boldsymbol{z}^\top\boldsymbol{z}}$. But, since $\boldsymbol{v}^\top\boldsymbol{v} > 0$ and

$$\boldsymbol{w}^\top\boldsymbol{w} = \sum_{i=1}^{K} w_i^2 = \sum_{i=1}^{K} v_i^2 = \boldsymbol{v}^\top\boldsymbol{v}, \tag{A72}$$

$\frac{\boldsymbol{w}^\top\mathbf{X}^\top\mathbf{X}\boldsymbol{w}}{\boldsymbol{w}^\top\boldsymbol{w}} > \frac{\boldsymbol{v}^\top\mathbf{X}^\top\mathbf{X}\boldsymbol{v}}{\boldsymbol{v}^\top\boldsymbol{v}}$. This is a contradiction.

Case $v_a = v_{a+1}$: Since $v_a = v_{a+1}$, $0 = \sigma v_{a+1} - \sigma v_a$. And since, by assumption, $\boldsymbol{u}$, $\boldsymbol{v}$, and σ are a corresponding left-singular vector, right-singular vector and singular value of $\mathbf{X}$, $\sigma\boldsymbol{v} = \mathbf{X}^\top\boldsymbol{u}$ by definition. So, $\sigma v_i = \sum_{\ell,m\in\mathbb{N}:\ell<m} X_{\rho(\ell,m),a}u_{\rho(\ell,m)}$. Thus,

$$0 = \sigma v_{a+1} - \sigma v_a = \sum_{\ell,m\in\mathbb{K}:\ell<m}\left(X_{\rho(\ell,m),a+1} - X_{\rho(\ell,m),a}\right)u_{\rho(\ell,m)}. \tag{A73}$$

Dividing this sum into the cases where $\ell, m \notin \{a, a+1\}$, $\ell = a < a+1 < m$, $\ell = a+1 < m$, $\ell < m = a$, $\ell < a < m = a+1$, and $\ell = a < m = a+1$ gives

$$
\begin{aligned}
0 = & \sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m} \left(X_{\rho(\ell,m),a+1} - X_{\rho(\ell,m),a}\right) u_{\rho(\ell,m)} \\
& + \sum_{\ell,m\in\mathbb{K}:\ell=a<a+1<m} \left(X_{\rho(\ell,m),a+1} - X_{\rho(\ell,m),a}\right) u_{\rho(\ell,m)} \\
& + \sum_{\ell,m\in\mathbb{K}:a<\ell=a+1<m} \left(X_{\rho(\ell,m),a+1} - X_{\rho(\ell,m),a}\right) u_{\rho(\ell,m)} \\
& + \sum_{\ell,m\in\mathbb{K}:\ell<m=a<a+1} \left(X_{\rho(\ell,m),a+1} - X_{\rho(\ell,m),a}\right) u_{\rho(\ell,m)} \\
& + \sum_{\ell,m\in\mathbb{K}:\ell<a<m=a+1} \left(X_{\rho(\ell,m),a+1} - X_{\rho(\ell,m),a}\right) u_{\rho(\ell,m)} \\
& + \sum_{\ell,m\in\mathbb{K}:a=\ell<m=a+1} \left(X_{\rho(\ell,m),a+1} - X_{\rho(\ell,m),a}\right) u_{\rho(\ell,m)} \\
= & \sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m} \left(X_{\rho(\ell,m),a+1} - X_{\rho(\ell,m),a}\right) u_{\rho(\ell,m)} \\
& + \sum_{m\in\mathbb{K}:a+1<m} \left(X_{\rho(a,m),a+1} - X_{\rho(a,m),a}\right) u_{\rho(a,m)} \\
& + \sum_{m\in\mathbb{K}:a+1<m} \left(X_{\rho(a+1,m),a+1} - X_{\rho(a+1,m),a}\right) u_{\rho(a+1,m)} \\
& + \sum_{\ell\in\mathbb{K}:\ell<a} \left(X_{\rho(\ell,a),a+1} - X_{\rho(\ell,a),a}\right) u_{\rho(\ell,a)} \\
& + \sum_{\ell\in\mathbb{K}:\ell<a} \left(X_{\rho(\ell,a+1),a+1} - X_{\rho(\ell,a+1),a}\right) u_{\rho(\ell,a+1)} \\
& + \left(X_{\rho(a,a+1),a+1} - X_{\rho(a,a+1),a}\right) u_{\rho(a,a+1)}
\end{aligned}
\tag{A74}
$$

Substituting in values from Equations A61–A63,

$$\begin{aligned}
0 = & \sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m} \left(1 + X_{\rho(\ell,m),a} - X_{\rho(\ell,m),a}\right) u_{\rho(\ell,m)} \\
& + \sum_{m\in\mathbb{K}:a+1<m} \left(X_{\rho(a,m),a+1} - 0\right) u_{\rho(a,m)} + \sum_{m\in\mathbb{K}:a+1<m} \left(0 - X_{\rho(a,m),a+1}\right) u_{\rho(a+1,m)} \\
& \quad + \sum_{\ell\in\mathbb{K}:\ell<a} \left(X_{\rho(\ell,a),a+1} - 0\right) u_{\rho(\ell,a)} + \sum_{\ell\in\mathbb{K}:\ell<a} \left(0 - X_{\rho(\ell,a),a+1}\right) u_{\rho(\ell,a+1)} \\
& \qquad + (0-0)\, u_{\rho(a,a+1)} \\
= & \sum_{\ell,m\in\mathbb{K}\setminus\{a,a+1\}:\ell<m} u_{\rho(\ell,m)} + \sum_{m\in\mathbb{K}:a+1<m} X_{\rho(a,m),a+1}\left(u_{\rho(a,m)} - u_{\rho(a+1,m)}\right) \\
& + \sum_{\ell\in\mathbb{K}:\ell<a} X_{\rho(\ell,a),a+1}\left(u_{\rho(\ell,a)} - u_{\rho(\ell,a+1)}\right)
\end{aligned} \tag{A75}$$

But, since $\sigma\boldsymbol{u} = \mathbf{X}$ and $v_a = v_{a+1}$,

$$\begin{aligned}
u_{\rho(a,m)} &= \sigma \sum_{i=1}^{K} X_{\rho(a,m),i} v_i \\
&= \sigma \left\{ \sum_{i=1}^{a-1} X_{\rho(a,m),i} v_i + \sum_{i=a+2}^{K} X_{\rho(a,m),i} v_i + X_{\rho(a,m),a} v_a + X_{\rho(a,m),a+1} v_{a+1} \right\} \\
&= \sigma \left\{ \sum_{i=1}^{a-1} X_{\rho(a,m),i} v_i + \sum_{i=a+2}^{K} X_{\rho(a,m),i} v_i + X_{\rho(a,m),a} v_{a+1} + X_{\rho(a,m),a+1} v_a \right\}
\end{aligned} \tag{A76}$$

So, using Equations A61–A63,

$$\begin{aligned}
u_{\rho(a,m)} &= \sigma \left\{ \sum_{i=1}^{a-1} X_{\rho(a+1,m),i} v_i + \sum_{i=a+2}^{K} X_{\rho(a+1,m),i} v_i + X_{\rho(a+1,m),a+1} v_{a+1} + X_{\rho(a+1,m),a} v_a \right\} \\
&= \sigma \sum_{i=1}^{K} X_{\rho(a+1,m),i} v_i \\
&= u_{\rho(a+1,m)}
\end{aligned} \tag{A77}$$

Similarly,

$$
\begin{aligned}
u_{\rho(\ell,a)} &= \sigma \sum_{i=1}^{K} X_{\rho(\ell,a),i} v_i \\
&= \sigma \left\{ \sum_{i=1}^{a-1} X_{\rho(\ell,a),i} v_i + \sum_{i=a+2}^{K} X_{\rho(\ell,a),i} v_i + X_{\rho(\ell,a),a} v_a + X_{\rho(\ell,a),a+1} v_{a+1} \right\} \\
&= \sigma \left\{ \sum_{i=1}^{a-1} X_{\rho(\ell,a),i} v_i + \sum_{i=a+2}^{K} X_{\rho(\ell,a),i} v_i + X_{\rho(\ell,a),a} v_{a+1} + X_{\rho(\ell,a),a+1} v_a \right\} \\
&= \sigma \left\{ \sum_{i=1}^{a-1} X_{\rho(\ell,a+1),i} v_i + \sum_{i=a+2}^{K} X_{\rho(\ell,a+1),i} v_i + X_{\rho(\ell,a+1),a+1} v_{a+1} + X_{\rho(\ell,a+1),a} v_a \right\} \\
&= \sigma \sum_{i=1}^{K} X_{\rho(\ell,a+1),i} v_i \\
&= u_{\rho(\ell,a+1)} \qquad \text{(A78)}
\end{aligned}
$$

Substituting $u_{\rho(a,m)} = u_{\rho(a+1,m)}$ and $u_{\rho(\ell,a+1)} = u_{\rho(\ell,a)}$ into Equation A75 gives

$$
\begin{aligned}
0 = & \sum_{\ell,m \in \mathbb{K} \setminus \{a,a+1\}: \ell < m} u_{\rho(\ell,m)} + \sum_{m \in \mathbb{K}: a+1 < m} X_{\rho(a,m),a+1} \left(u_{\rho(a,m)} - u_{\rho(a,m)} \right) \\
& \qquad + \sum_{\ell \in \mathbb{K}: \ell < a} X_{\rho(\ell,a),a+1} \left(u_{\rho(\ell,a)} - u_{\rho(\ell,a)} \right) \\
= & \sum_{\ell,m \in \mathbb{K} \setminus \{a,a+1\}: \ell < m} u_{\rho(\ell,m)} \qquad \text{(A79)}
\end{aligned}
$$

But, since $\boldsymbol{u}$ is positive, $u_{\rho(\ell,m)} > 0$,

$$
\sum_{\ell,m \in \mathbb{K} \setminus \{a,a+1\}: \ell < m} u_{\rho(\ell,m)} > 0, \qquad \text{(A80)}
$$

which is a contradiction.

Since we arrive at a contradiction in both cases, there does not exist a value a such that $v_a \geq v_{a+1}$. Thus, $v_i < v_j$ for all $i < j$. So, $v_i \leq v_j$ if and only if $i \leq j$. Since $x_i \leq x_j$ if and only if $i \leq j$ by assumption, $v_i \leq v_j$ if and only if $i \leq j$. Therefore, $\boldsymbol{x}$ and $\boldsymbol{v}$ are order equivalent.

The same result can be seen to hold if $\boldsymbol{u}$ is negative as follows: If $\boldsymbol{u}$, $\boldsymbol{v}$ are left- and right-singular vectors corresponding to the largest singular value of $\mathbf{X}$, then so are $-\boldsymbol{u}$ and $-\boldsymbol{v}$. If $\boldsymbol{u}$ is negative, $-\boldsymbol{u}$ is positive. So, $-v_i \leq -v_j$ if and only if $i \leq j$. Thus, $v_i \geq v_j$ if and only if $i \leq j$. Therefore, $\boldsymbol{x}$ and $\boldsymbol{v}$ are order equivalent. $\square$

Lemma 22. *If $\mathbf{X}$ is a $\binom{K}{2} \times K$ ideal-point-rank matrix with $K \geq 5$ for legislators with unique ideal-points $\boldsymbol{x}$ and $\boldsymbol{v}$ is a right singular vector of $\mathbf{X}$ corresponding to the largest singular value, then $\boldsymbol{x}$ and $\boldsymbol{v}$ are order equivalent.*

Proof. Let $\mathbf{P}$ be the $\binom{K}{2} \times \binom{K}{2}$ diagonal matrix where $P_{\rho(a,b),\rho(a,b)} = \operatorname{sgn}(x_b - x_a)$ for all natural numbers a and b such that $a < b \leq K$. Note that $x_a \neq x_b$, as the elements of $\boldsymbol{x}$ are unique and $a \neq b$. So, $P_{\rho(a,b),\rho(a,b)} = \operatorname{sgn}(x_b - x_a) = 1$ (if $x_a < x_b$) or $P_{\rho(a,b),\rho(a,b)} = \operatorname{sgn}(x_b - x_a) = -1$ (if $x_a > x_b$). Since $\mathbf{P}$ is a diagonal matrix with $(P_{ii})^2 = 1$ for all i, $\mathbf{P}$ is orthogonal.

Let π be the permutation of $\{1, \ldots, K\}$ ($\pi : \{1, \ldots, K\} \mapsto \{1, \ldots, K\}$) such that, for all i and j, $x_{\pi(i)} \leq x_{\pi(j)}$ if and only if $i \leq j$ (that is, π sorts the elements of $\boldsymbol{x}$). Since the elements of $\boldsymbol{x}$ are unique, exactly one such permutation exists. Let $\mathbf{Q}$ be the $K \times K$ permutation matrix corresponding to the permutation π. That is, $\mathbf{Q}$ is a $K \times K$ matrix such that $Q_{i,j} = \begin{cases} 1 & \text{if } j = \pi(i) \\ 0 & \text{otherwise} \end{cases}$. Thus, if $\boldsymbol{z} = \mathbf{Q}\boldsymbol{x}$, then, for all i and j, $z_i \leq z_j$ if and only if $i \leq j$. So, $\mathbf{Q}$ provides the permutation that sorts the elements of $\boldsymbol{x}$. Note that, since $\mathbf{Q}$ is a permutation matrix, $\mathbf{Q}$ is orthogonal.

Let $\mathbf{Y}$=$\mathbf{PXQ}^\top$. Thus, since $\mathbf{P}$ is diagonal,

$$Y_{\rho(a,b),j} = P_{\rho(a,b),\rho(a,b)} \sum_{k=1}^{K} X_{\rho(a,b),k} Q_{j,k} = \operatorname{sgn}(x_b - x_a)\, X_{\rho(a,b),\pi(j)}. \tag{A81}$$

So, by definition of an ideal-point-rank matrix, for all natural numbers a and b such that

$a < b \le K$, $Y_{\rho(a,b),j} = 0$ if $\pi(j) \in \{a,b\}$, and, otherwise,

$$\begin{aligned} Y_{\rho(a,b),j} &= \operatorname{sgn}(x_b - x_a)\left(\frac{1}{2}\operatorname{sgn}(x_b - x_a)\sum_{k \notin \{a,b,\pi(j)\}} \operatorname{sgn}\left(x_{\pi(j)} - x_k\right)\right) \\ &= \frac{1}{2}\operatorname{sgn}(x_b - x_a)^2 \sum_{k \notin \{a,b,j\}} \operatorname{sgn}(x_j - x_k)\,. \end{aligned} \tag{A82}$$

Since $\operatorname{sgn}(x_b - x_a)^2 = 1$, this implies

$$Y_{\rho(a,b),j} = \frac{1}{2}\operatorname{sgn}(x_b - x_a)^2 \sum_{k \notin \{a,b,j\}} \operatorname{sgn}\left(x_{\pi(j)} - x_k\right) = \frac{1}{2}\sum_{k \notin \{a,b,j\}} \operatorname{sgn}(x_j - x_k)\,. \tag{A83}$$

And, since $\pi(k) = \pi(\ell)$ if and only if $k = \ell$,

$$Y_{\rho(a,b),j} = \frac{1}{2}\sum_{k \in \mathbb{K} : \pi(k) \notin \{\pi(a),\pi(b),\pi(j)\}} \operatorname{sgn}\left(x_{\pi(j)} - x_{\pi(k)}\right). \tag{A84}$$

Let $\rho'(\min(\pi(a),\pi(b)), \max(\pi(a),\pi(b))) = \rho(a,b)$. Note that, like ρ, ρ' is also a bijection from $\{(a,b) \in \mathbb{N}^2 \mid a < b \le K\}$ to $\left\{1,\ldots,\binom{K}{2}\right\}$. Thus, setting $\ell = \min(\pi(a),\pi(b))$ and $m = \max(\pi(a),\pi(b))$ gives

$$Y_{\rho'(\ell,m),j} = \frac{1}{2}\sum_{k \in \mathbb{K} : \pi(k) \notin \{\ell,m,\pi(j)\}} \operatorname{sgn}\left(x_{\pi(j)} - x_{\pi(k)}\right) \tag{A85}$$

for all natural numbers ℓ and m such that $\ell < m \le K$.

Let $\boldsymbol{x}' = \mathbf{Q}\boldsymbol{x}$. So, $x'_j = x_{\pi(j)}$ and the elements of $\boldsymbol{x}'$ are unique. Thus,

$$Y_{\rho'(\ell,m),j} = \frac{1}{2}\sum_{k \in \mathbb{K} \setminus \{\ell,m,j\}} \operatorname{sgn}\left(x'_j - x'_k\right) \tag{A86}$$

for all natural numbers ℓ and m such that $\ell < m \le K$. Since $x'_\ell = x_{\pi(\ell)} \le x_{\pi(m)} = x'_m$ for all

$\ell \leq m$, $\operatorname{sgn}(x_m - x_\ell) = 1$ for all for all $\ell \leq m$. Thus,

$$Y_{\rho'(\ell,m),j} = \frac{1}{2}\operatorname{sgn}(x_m - x_\ell) \sum_{k \in \mathbb{K} \setminus \{\ell,m,j\}} \operatorname{sgn}\left(x'_j - x'_k\right). \tag{A87}$$

Therefore, $\mathbf{Y} = \mathbf{P}\mathbf{X}\mathbf{Q}^\top$ is a sorted ideal-point-rank matrix with unique ideal points $\boldsymbol{x}'$.

Let $\mathbf{U}\mathbf{D}\mathbf{V}^\top$ be a singular value decomposition of $\mathbf{X}$ with $\boldsymbol{v}$ as a right singular vector corresponding to the largest singular value. Since $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal, $(\mathbf{P}\mathbf{U})\,\mathbf{Y}\,(\mathbf{Q}\mathbf{V})^\top$ is a singular value decomposition of $\mathbf{P}\mathbf{X}\mathbf{Q}^\top = \mathbf{Y}$ by Lemma 20. Thus, $\mathbf{Q}\boldsymbol{v}$ is a right-singular vector of $\mathbf{Y}$ corresponding to the largest singular value. Therefore, by Lemma 21, $\boldsymbol{x}' = \mathbf{Q}\boldsymbol{x}$ and $\mathbf{Q}\boldsymbol{v}$ are order equivalent. So, $\forall i \neq j$, $\left(x_{\pi(i)} \leq x_{\pi(j)}\right) \Leftrightarrow \left(v_{\pi(i)} \leq v_{\pi(j)}\right)$ or $\forall i \neq j$, $\left(x_{\pi(i)} \leq x_{\pi(j)}\right) \Leftrightarrow \left(v_{\pi(i)} \geq v_{\pi(j)}\right)$. Thus, since $\pi(i) = \pi(j)$ if and only if $i = j$, either $\forall i \neq j$, $(x_i \leq x_j) \Leftrightarrow (v_i \leq v_j)$ or $\forall i \neq j$, $(x_i \leq x_j) \Leftrightarrow (v_i \geq v_j)$. Therefore, $\boldsymbol{x}$ and $\boldsymbol{v}$ are order equivalent. □

Theorem 23. *Let $\left(\hat{\mathbf{X}}^{(N)}\right)_{N=1}^{\infty}$ be a sequence of observed ideal-point-rank matrix after N votes generated by unique ideal points, $\boldsymbol{x}$, with $K \geq 5$ legislators. Assume*

$$0 \leq \liminf_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(\Pr\left(v^j_{km|i\ell}\right) - \Pr\left(v^j_{im|k\ell}\right)\right) \tag{A88}$$

for all i, k, ℓ, and m such that $x_i < x_k$ and $x_\ell < x_m$. If $\left(\boldsymbol{v}^{(N)}\right)_{N=1}^{\infty}$ is a sequence of vectors such that $\boldsymbol{v}^{(N)}$ is a right singular vector of $\mathbf{X}^{(N)}$ corresponding to the largest singular value, then

$$\lim_{N\to\infty} \Pr\left(\boldsymbol{x} \textit{ and } \boldsymbol{v}^{(N)} \textit{ are order equivalent}\right) = 1. \tag{A89}$$

Proof. By Theorem 11, $\lim_{N\to\infty} \Pr\left(\hat{\mathbf{X}}^{(N)} = \mathbf{X}\right) = 1$ where $\mathbf{X}$ is the ideal-point-rank matrix. By Lemma 22, if $\hat{\mathbf{X}}^{(N)} = \mathbf{X}$ and $\boldsymbol{v}$ is a right singular value of $\hat{\mathbf{X}}^{(N)}$ corresponding to the largest eigenvalue, then $\boldsymbol{x}$ and $\boldsymbol{v}$ are order equivalent. So, $\Pr\left(\hat{\mathbf{X}}^{(N)} = \mathbf{X}\right) \leq$

$\Pr\left(\boldsymbol{x} \text{ and } \boldsymbol{v}^{(N)} \text{ are order equivalent}\right)$. Since probabilities cannot be greater than one,

$$\Pr\left(\hat{\mathbf{X}}^{(N)} = \mathbf{X}\right) \leq \Pr\left(\boldsymbol{x} \text{ and } \boldsymbol{v}^{(N)} \text{ are order equivalent}\right) \leq 1. \tag{A90}$$

Therefore, as $\lim_{N\to\infty} \Pr\left(\hat{\mathbf{X}}^{(N)} = \mathbf{X}\right) = 1$ and $\lim_{N\to\infty} 1 = 1$,

$$\lim_{N\to\infty} \Pr\left(\boldsymbol{x} \text{ and } \boldsymbol{v}^{(N)} \text{ are order equivalent}\right) = 1 \tag{A91}$$

by the squeeze theorem. $\square$