Supplementary Materials for: Bias Amplification and Bias Unmasking

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A. OMITTED VARIABLE BIAS

To define the bias, start with a generic linear model,

$$Y = S\beta^s + O\beta^o + \epsilon^y, \tag{A.1}$$

where S and O are matrices of specified and omitted covariates, respectively. With respect to the error term, ϵ^y, assume $\mathrm{E}[\epsilon^y|S,O] = 0$.

Imagine the regression of Y on a set of covariates S only. This leads to the well known expression for omitted variable bias (for example see Greene 2000, p. 334)

$$\text{Bias}\left[\widehat{\beta}^s\right] = \left(S'S\right)^{-1} S'O\beta^o. \tag{A.2}$$

From this generic equation we can derive biases for particular sets of conditioning variables, S, under an assumed model.

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To derive omitted variable bias, first collect variables into two groups: omitted variables, O, and included (specified) variables, S. Then we can write (in matrix notation) the general case $Y = S\beta^s + O\beta^o + \epsilon^y$. Now substitute Y in $[S'S]^{-1} S'Y$ and take the expected value.

$$\begin{aligned} \mathrm{E}\left[\ \widehat{\beta}^s\ \right] &= \mathrm{E}\left[(S'S)^{-1} S'Y\right] \\ &= \mathrm{E}\left[(S'S)^{-1} S'\left(S\beta^s + O\beta^o + \epsilon_y\right)\right] \\ &= \beta^s + \mathrm{E}\left[(S'S)^{-1} S'\left(O\beta^o + \epsilon_y\right)\right] \\ &= \beta^s + (S'S)^{-1} S'O\beta^o \end{aligned} \tag{A.3}$$

The last line follows from the fact that ϵ_y is independent of S and O. Therefore, the bais is the last term,

$$\text{Bias}\left[\widehat{\beta}^s\right] = (S'S)^{-1} S'O\beta^o. \tag{A.4}$$

For the bias when conditioning on $S = [Z, X]$ and $O = [U]$, use the inverse of the partition matrix (cf. Greene 2000, section 2.6.3) to arrive at

$$\begin{aligned} &\text{Bias}\begin{bmatrix} \widehat{\tau} \\ \widehat{\beta}^y \end{bmatrix} \\ &= \begin{bmatrix} \left(Z'Z - Z'X\,[X'X]^{-1} X'Z\right)^{-1} & -[Z'Z]^{-1} Z'X\left(X'X - X'Z\,[Z'Z]\,Z'X\right)^{-1} \\ -[X'X]^{-1} X'Z\left(Z'Z - Z'X\,[X'X]^{-1} X'Z\right)^{-1} & \left(X'X - X'Z\,[Z'Z]^{-1} Z'X\right)^{-1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} Z'U \\ X'U \end{bmatrix} \zeta^y \\ &= \begin{bmatrix} \left(Z'Z - Z'X\,[X'X]^{-1} X'Z\right)^{-1} Z'U\zeta^y \\ -[X'X]^{-1} X'Z\left(Z'Z - Z'X\,[X'X]^{-1} X'Z\right)^{-1} Z'U\zeta^y \end{bmatrix} \end{aligned} \tag{A.5}$$

The last line follows from the fact that, by construction, $U \perp X$ and $\overline{U} = 0$.

B. ILLUSTRATIVE EXAMPLE

In this section we provide a simple numerical example to illustrate these biases. Suppose a researcher has city level data and is interested in the effect of (standardized) per capita real income, Z, on (standardized) proportion voting for the legislative party in power, Y. Suppose the proportion voting for the party in power is also affected by whether or not the local mayor is a member of the the same party (a reverse coat-tails effect) such that

$$Y = \frac{1}{2}Z + \frac{1}{2}U + \epsilon_y$$
$$\epsilon_y \sim N\left(0, \frac{1}{2}\right).$$

where U is an indicator coded -1 if the mayor is not of the incumbent party and 1 if the mayor is of the incumbent party and ϵ_y represents idiosyncratic factors. For simplicity say half of mayors are members of the party in power.

Now in turn suppose the treatment, (standardized) per capita income, is affected by whether the mayor is the same party as the party in power in the legislature (because the legislature rewards mayors of the same party with pork spending) and also by a development project aimed at increasing the incomes of the poor that was randomly assigned to half of the cities. The model for the treatment variable, Z (per capita income), is

$$Z = \frac{1}{2}X + \frac{1}{2}U + \epsilon_z$$
$$\epsilon_x \sim N\left(0, \frac{1}{\sqrt{2}}\right)$$

where X is an indicator of whether the development project took place in the district (coded -1 for not treated and 1 for treated).

Note that the example has been contrived such that $\mathrm{E}(Y) = \mathrm{E}(X) = \mathrm{E}(Z) = \mathrm{E}(U) = 0$ and $V(Y) = V(X) = V(Z) = V(U) = 1$. Note also that X is an instrument because it was randomly

Variable	Measure	Scale
Y	Standardized proportion voting for legislative party in power	$\mu = 0, \sigma = 1$
Z	Standardized per capita income	$\mu = 0, \sigma = 1$
U	Mayor member of party in power	1 if yes, -1 if no
X	Development project in city	1 if yes, -1 if no

Table 1: Variables in illustrative example

assigned to districts and only affects Y through its effect on Z.

Now suppose the researcher observes whether or not the development project occurs in each city but neglects to collect data on the party of the mayors. The researcher estimates the effect of (standardized) income per capita on (standardized) proportion voting for the political party in power in two ways. The first approach is to regress Y (proportion voting for party in power) on Z (per capita income). The second approach is to regress Y (proportion voting for party in power) on both Z (per capita income) and X (development project instrument).

We can compute the bias components for these specifications. The bias due to omitting X (development project instrument) is

$$\begin{aligned}\chi \equiv & \left(Z'Z\right)^{-1} Z'X\beta^y \\ = & \text{Cov}(Z, X)0 \\ = & 0\end{aligned}$$

as expected since X is an instrument. Meanwhile, the bias due to omitting U (mayor is of incumbent party) is

$$
\begin{aligned}
\upsilon \equiv & \left(Z'Z\right)^{-1} Z'U\zeta^{y} \\
= & \text{Cov}(Z,U)\frac{1}{2} \\
= & \text{Cov}\left(\frac{1}{2}X + \frac{1}{2}U + \epsilon_z, U\right)\frac{1}{2} \\
= & \frac{1}{4}V(U) \\
= & 0.25.
\end{aligned}
$$

The bias due to amplification is

$$
\begin{aligned}
\alpha \equiv & \left(\frac{r^2_{Z|X}}{1-r^2_{Z|X}}\right)\upsilon \\
= & \left(\frac{\text{Cov}(X,Z)^2}{1-\text{Cov}(X,Z)^2}\right)0.25 \\
= & \frac{1/4}{1-1/4}0.25 \\
= & 0.0833
\end{aligned}
$$

So the bias when omitting the instrumental variable X will be $(\chi + \upsilon) = (0 + 0.25) = 0.25$ while the bias when including X in the conditioning set will be $(\alpha+\upsilon) = (0.0833+0.25) = 0.333$. Thus in this case the unadjusted estimator is less biased than the estimator that includes the instrument in the conditioning set.

Why does this make sense intuitively? Controlling for X, there is a stronger (partial) correlation between Z and U, exacerbating the bias due to U. Another way to say it is that the “exogenous” (good) variability in Z is controlled for when X is included in the conditioning set.

The phenomenon of bias amplification is similar in the case of fixed effects in that conditioning on this additional variable sets up within-group comparisons that induce a negative relationship between U and X that exacerbates the bias.

C. SENSITIVITY ANALYSIS OF WATER OUTCOME

In the sensitivity plot, Figure 2, we examine the Water Infrastructure outcome. The interpretation of the plot is the same as in the case of the GOTV study. The preponderance of covariates whose benchmarking values fall in the bias-inducing region suggest that we should be concerned with including fixed effects in our analysis. Conducting sensitivity analysis on this data would have alerted the researcher for signs of potential trouble.

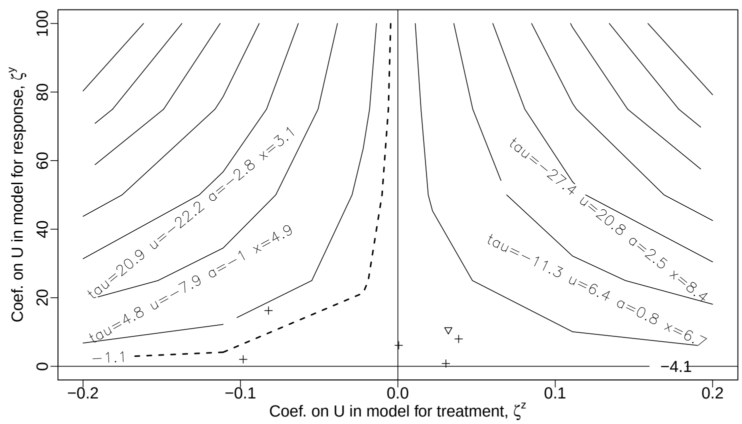

Figure 1: Sensitivity Plot of Water Outcome

REFERENCES

Greene, WH. (2000). *Econometric Analysis* Prentice Hall (4th Edition).

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