Supplementary Material for the Paper "Endogenous Jurisprudential Regimes"

ABSTRACT

This supplementary material contains the MCMC algorithm for estimating the multivariate probit multiple change-point model and the simulation algorithm for approximating the Bayes factor using the marginal likelihood method.

1. MODEL FITTING USING MCMC

We use Markov chain Monte Carlo methods to simulate from the posterior distribution of the parameters of the model. The parameters to be estimated are $\mathbf{Z}, \beta_1, ..., \beta_K, P, \mathbf{S}$. Using the priors

$$p_{ii} \sim \text{Beta}(a, b); \; \beta_k \sim N(\bar{\beta}_{0,k}, \mathbf{B}_{0,k}), \tag{1}$$

we construct the following recursive sampling scheme to update the parameters:

1. $z_{i_t,t}|\beta_{s_t}, y_{i_t,t} \sim TN(\mathbf{x}'_{i_t,t}\beta_{s_t}, 1)$, following the data augmentation approach of Albert and Chib (1993)

2. $\beta_\lambda|\mathbf{S}, \mathbf{Z} \sim N(\bar{\beta}_\lambda, \bar{\mathbf{B}}_\lambda)$, where $\bar{\mathbf{B}}_\lambda = (\mathbf{X}_{D_\lambda}\mathbf{X}'_{D_\lambda} + \mathbf{B}^{-1}_{0,\lambda})^{-1}$, $\bar{\beta}_\lambda = \bar{\mathbf{B}}_\lambda(\mathbf{X}'_{D_\lambda}\mathbf{Z}_{D_\lambda} + \mathbf{B}^{-1}_{0,\lambda}\bar{\beta}_{0,\lambda})$ and $D_\lambda = \{d : s = \lambda\}$.

3. $p_{ii}|\mathbf{S} \sim \text{Beta}(a + n_{ii}, b + 1), i = 1, 2, ..., K$, where n_{ii} is the number of one-step transitions from state i to state i in the sequence $\mathbf{S}$.

4. To update $\mathbf{S}$, we use the algorithm in Chib (1996) and Chib (1998):

   - Define: $\mathbf{s}_t = (s_1, s_2, ..., s_{t-1}); \mathbf{s}^{t+1} = (s_{t+1}, s_{t+2}, ..., s_T)$ and $\mathbf{Z}_t = (\mathbf{z}_1, \mathbf{z}_2, ..., \mathbf{z}_{t-1}); \mathbf{Z}^{t+1} = (\mathbf{z}_{t+1}, \mathbf{z}_{t+2}, ..., \mathbf{z}_T)$
   - Write the joint distribution as follows:

$$p(\mathbf{S}|\mathbf{Z}, \beta, P) = p(s_{T-1}|\mathbf{Z}, s_T, \beta, P) \times \cdots \times p(s_t|\mathbf{Z}, \mathbf{s}^{t+1}, \beta, P) \times \cdots \times p(s_1|\mathbf{Z}, \mathbf{s}^2, \beta, P) \tag{2}$$

- The sampling scheme is:

$$\begin{aligned}
s_{T-1} :& p(s_{T_1}|\mathbf{Z}, s_n = K+1, \beta, P) \\
s_{T-2} :& p(s_{T-2}|\mathbf{Z}, \mathbf{s}^{T-1}, \beta, P) \\
& \cdots\cdots \\
& \cdots\cdots \\
s_1 :& p(s_1|\mathbf{Z}, \mathbf{s}^2, \beta, P)
\end{aligned} \tag{3}$$

- The first state $s_1 = 1$ by definition. To implement this sampling, we should know the full conditionals of $p(s_t|\mathbf{Z}, \mathbf{s}^{t+1}, \beta, P)$. For this, we can take the following steps:
  - $p(s_t|\mathbf{Z}, \mathbf{s}^{t+1}, \beta, P) \propto p(s_t|\mathbf{Z}_t, \beta, P)p(s_{t+1}|s_t, P)$
  - The probability $p(s_{t+1}|s_t, P)$ is easy to obtain once we have P.
  - Sampling from the filtering density: $p(s_t|\mathbf{Z}_t, \beta, P)$

$$p(s_t = k|\mathbf{Z}_t, \beta, P) = \frac{p(s_t = k|\mathbf{Z}_{t-1}, \beta, P) \times f(\mathbf{z}_t|\mathbf{Z}_{t-1}, \beta_k)}{\sum_{l=k-1}^{k} p(s_t = l|\mathbf{Z}_{t-1}, \beta, P) \times f(\mathbf{z}_t|\mathbf{Z}_{t-1}, \beta_l)} \tag{4}$$

$$p(s_t = k|\mathbf{Z}_{t-1}, \beta, P) = \sum_{l=k-1}^{k} p_{lk} \times p(s_{t-1} = l|\mathbf{Z}_{t-1}, \beta, P) \tag{5}$$

    * Initialize $p(s_1 = 1|\mathbf{Z}_0, \beta) = 1$
    * Obtain the mass filtering probabilities $p(s_t|\mathbf{Z}_t, \beta, P)$ by recursively sampling from equation (4) and (5)
    * With the mass filtering probabilities in hand, do backward sampling of the states from $p(s_{T-1}|\mathbf{Z}, \mathbf{s}^T, \beta, P)$ to $p(s_1|\mathbf{Z}, \mathbf{s}^2, \beta, P)$

2. MARGINAL LIKELIHOOD COMPUTATION USING MCMC

We use Markov chain Monte Carlo methods to compute the marginal likelihood for each model. Chib (1995) shows that the marginal likelihood can be expresses as follows:

$$\begin{aligned}
\log p(\mathbf{Y}|\mathscr{M}) =& \log f(\mathbf{Y}|\mathscr{M}, \beta^*, P^*) + \log \pi(\beta^*, P^*|\mathscr{M}) \\
& - \log \pi(\beta^*, P^*|\mathbf{Y}, \mathscr{M}),
\end{aligned} \tag{6}$$

where α^* means that we fix the value of α at α^*. The likelihood ordinate $\log f(\mathbf{Y}|\beta^*, P^*)$ can be approximated:

$$\begin{aligned}
p(\mathbf{Y}|\beta^*, P^*) =& \prod_{t=1}^{T} p(\mathbf{y}_t|\mathbf{Y}_{t-1}, \beta^*, P^*) \\
=& \sum_{k=1}^{K+1} \Phi(\mathbf{X}'_{D_k}\beta_k)^{\mathbf{Y}_{D_k}} (\mathbf{1} - \Phi(\mathbf{X}'_{D_k}\beta_k)^{\mathbf{1}-\mathbf{Y}_{D_k}} p(s_t = k|\mathbf{Z}, \beta^*, P^*) \\
\approx& \frac{1}{G} \sum_{g=1}^{G} \sum_{k=1}^{K+1} \Phi(\mathbf{X}'_{D_k}\beta_k^{(g)})^{\mathbf{Y}_{D_k}} (\mathbf{1} - \Phi(\mathbf{X}'_{D_k}\beta_k^{(g)})^{\mathbf{1}-\mathbf{Y}_{D_k}} \\
& \times p(s_t^{(g)} = k|\beta^*, P^*, \mathbf{Z}^{(g)}).
\end{aligned} \tag{7}$$

To do this, we need a reduced run by recursively sampling from $\pi(\mathbf{S}|\beta^*, P^*, \mathbf{Z})$ and $f(\mathbf{Z}|\beta^*, P^*, \mathbf{S})$, and plug the draws of $\mathbf{S}$ and $\mathbf{Z}$ in the above formula.

The posterior ordinate is $\pi(\beta^*, P^*|\mathbf{Y}) = \pi(P^*|\mathbf{Y})\pi(\beta^*|\mathbf{Y}, P^*)$ and can be approximated in two steps:

- $\pi(P^*|\mathbf{Y}) \approx \frac{1}{G}\sum_{g=1}^{G} \pi(P^*|S_T^{(g)})\pi(S_T^{(g)}|\mathbf{Y}, \mathbf{Z}^{(g)})$

- $\pi(\beta^*|\mathbf{Y}, P^*) \approx \frac{1}{G}\sum_{g=1}^{G} \pi(\beta^*|S_T^{(g)}, \mathbf{Z}^{(g)})p(S_T^{(g)}, \mathbf{Z}^{(g)}|\mathbf{Y}, P^*)$
  In the second step, we do one reduced run by recursively sampling from: $\pi(\mathbf{S}, \beta|P^*, \mathbf{Z})$ and $f(\mathbf{Z}|P^*, \beta, \mathbf{S})$

References

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