Delimited control and computational effects

Appendix

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A Correctness of the semantics of control

We now show that semantics for the various languages of control are correct, in the sense that the different presentations of the semantics for a language correspond with one another. The reduction theories and operational semantics are already closely related, and similarly for the abstract machines and CPS transforms. Therefore, we focus on the relationship between the reduction theories and CPS transforms of the languages, since they are the furthest removed from one another. The first property that we consider is the *soundness* of the reduction theory with respect to the CPS transform, so that the reductions "make sense" according to the meaning given by transform. This property ensures that every reduction in the source language relates equal programs in the target language.

Property A.1 (*Soundness*)
If $M \twoheadrightarrow M'$ then $[\![M]\!] = [\![M']\!]$.

In addition, we would also like to know that the reductions are *complete* in some sense, so that we have "enough" reductions to reach a final answer if there is one. This is an operational notion of completeness, which guarantees that whenever the CPS transform finds a final answer then the operational semantics is strong enough to also reach a final answer in the source language.

Property A.2 (*Evaluation*)
If $[\![M]\!] = V$ then there is a final answer M' such that $M \mapsto\!\!\!\!\twoheadrightarrow M'$ and $[\![M']\!] = V$.

Note that we are considering languages with control effects that allow a program to jump in and out of a context during evaluation, and may appear to "return" a value multiple times to the same location. Therefore, we emphasize the fact that the answer is *final*, so that the program is committed to a single answer and may not change its mind by returning a different value at some later point in time. Finally, to finish the comparison between the semantics, we would also like to know the fact that the operational semantics defines a subset of the reduction theory.

Property A.3
If $M \mapsto\!\!\!\!\twoheadrightarrow M'$ then $M \twoheadrightarrow M'$.

Together, these two properties ensure that the reduction theory is strong enough to find a final answer if one is given by the CPS transform.

$$\mathscr{C}_{\lambda\mu}[\![[q]t]\!] = \mathscr{C}_{\lambda\mu}[\![t]\!]\ \mathscr{C}_{\lambda\mu}[\![q]\!] \qquad \mathscr{C}_{\lambda\mu}[\![*]\!] = \overline{\lambda}x.x$$
$$\mathscr{C}_{\lambda\mu}[\![\alpha]\!] = \alpha$$
$$\mathscr{C}_{\lambda\mu}[\![V]\!] = \overline{\lambda}k.k\ \mathscr{C}_{\lambda\mu}[\![V]\!]^V$$
$$\mathscr{C}_{\lambda\mu}[\![t_1\ t_2]\!] = \overline{\lambda}k.\mathscr{C}_{\lambda\mu}[\![t_1]\!]\overline{\lambda}f.\mathscr{C}_{\lambda\mu}[\![t_2]\!]\overline{\lambda}s.f\ s\ k \qquad \mathscr{C}_{\lambda\mu}[\![x]\!]^V = x$$
$$\mathscr{C}_{\lambda\mu}[\![\mu\alpha.c]\!] = \overline{\lambda}k.(\lambda\alpha.\mathscr{C}_{\lambda\mu}[\![c]\!])\ k \qquad \mathscr{C}_{\lambda\mu}[\![\lambda x.t]\!]^V = \lambda x.\mathscr{C}_{\lambda\mu}[\![t]\!]$$

Fig. A 1. The $\mathscr{C}_{\lambda\mu}$ transform with annotations on administrative $\overline{\lambda}$-abstractions.

Since we are considering many different languages, each of which extends a previous one in some way, we develop the proofs of correctness according to a pattern that is repeated for each language. We therefore give the most attention the $\lambda\mu$-calculus, the simplest language of control, establishing the general pattern for the proofs. Since the syntax, reduction theory, and CPS transform of the pure λ-calculus is a closed subset of the $\lambda\mu$-calculus, Theorems 2.1 and 2.2 are a consequence of Theorems 3.1 and 3.2 for the $\lambda\mu$-calculus. From there, we show how the previous proofs may be extended to show the correctness of the remaining languages of control, giving details where the language under consideration differs.

A.1 Soundness of the $\lambda\mu$ reduction theory

Not all reductions in the $\mathscr{C}_{\lambda\mu}$ transform correspond to a step in the $\lambda\mu$ calculus. Some of the additional steps performed in the CPS transformed program deal with more administrative concerns such as plugging a value into an evaluation context or returning the final answer, which are hidden at the level of the source language. In terms of the abstract machine for $\lambda\mu$, refocus and apply steps are administrative reductions in the CPS transform and reduce steps are non-administrative reductions.

We can keep track of which reductions are administrative in nature by marking the functions with a $\overline{\lambda}$, in the style of Sabry & Felleisen (1993), so that the administrative reductions are only those β-reductions of the following form:

$$\overline{\beta} : \quad (\overline{\lambda}x.M)\ V \Rightarrow M\{V/x\}$$

Notation: We use $\mapsto$ to denote an administrative reduction performed in an evaluation context, and use $\Rrightarrow$ and $\mapsto\!\!\!\!\Rightarrow$ to denote the reflexive, transitive closure of $\Rightarrow$ and $\mapsto$, respectively.

We determine the administrative $\overline{\lambda}$-abstractions by finding the abstractions that introduce new variables that were not present in the original source program. In the CPS transform for $\lambda\mu$ from Figure 12, four clauses introduce administrative $\overline{\lambda}$-abstractions, and the annotated transform is given in Figure A 1. Observe that the translation $\mathscr{C}_{\lambda\mu}[\![\mu\alpha.c]\!]$ introduces a non-administrative binding for its continuation k, since the co-variable naming it is present in the source program. However, many freshly transformed $\lambda\mu$ programs contain active administrative reductions that mask the real work that needs to be done.

For the $\mathscr{C}_{\lambda\mu}$ transform, we have the following administrative reductions:

$$\begin{aligned}
\mathscr{C}_{\lambda\mu}[\![[*]V]\!] &= (\overline{\lambda}x.x)\ \mathscr{C}_{\lambda\mu}[\![V]\!]^V \Rightarrow \mathscr{C}_{\lambda\mu}[\![V]\!]^V \\
\mathscr{C}_{\lambda\mu}[\![V\ V']\!] &= \overline{\lambda}k.(\overline{\lambda}k'.k'\ \mathscr{C}_{\lambda\mu}[\![V]\!]^V)\ (\overline{\lambda}f.(\overline{\lambda}k''.k''\ \mathscr{C}_{\lambda\mu}[\![V']\!]^V)\ (\overline{\lambda}s.f\ s\ k)) \\
&\twoheadrightarrow \overline{\lambda}k.\mathscr{C}_{\lambda\mu}[\![V]\!]^V\ \mathscr{C}_{\lambda\mu}[\![V']\!]^V\ k
\end{aligned}$$

When tracking reduction in the CPS program, it's useful to have a transform that is free of administrative reductions. That is to say, an alternate CPS transform where immediately after transforming a $\lambda\mu$ program, no administrative reductions apply. The alternate transform, $\overline{\mathscr{C}}_{\lambda\mu}$, is achieved by taking the administrative-normal form of $\mathscr{C}_{\lambda\mu}$, by performing all available administrative reductions until there are none left.

$$\mathscr{C}_{\lambda\mu}[\![c]\!] \twoheadrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c]\!] \not\Rightarrow$$

A one-pass, administrative-normal CPS transform for the pure λ-calculus has been given by Sabry & Felleisen (1993) as well as Danvy & Nielsen (2003). However, in contrast to some previous administrative-normal CPS transforms like the one by Sabry & Felleisen (1992), the transform we describe above is unable to eliminate every single administrative reduction throughout the future of the program. After performing some real computation, some administrative steps may later become available. For example, since we follow Plotkin's (1975) call-by-value CPS transform, functions take their argument first and continuation second. Therefore, in the CPS program:

$$\overline{\mathscr{C}}_{\lambda\mu}[\![(\lambda x.x)\ y]\!]k = (\lambda x.\overline{\lambda}k.k\ x)\ y\ k$$

the λ-abstraction for the argument x blocks the administrative $\overline{\beta}$ reduction on the continuation k. Furthermore, we have the more fundamental issue that μ-abstractions in the source calculus introduce non-administrative bindings for continuations in the CPS program. This fact alone prevents us from eliminating certain future administrative reductions during the transformation, since any μ-abstraction in a source program blocks a continuation from seeing the value returned to it. For instance, consider the CPS transformation of the command $[\alpha](f\ (\mu\beta.[\beta]x))$:

$$\overline{\mathscr{C}}_{\lambda\mu}[\![[\alpha](f\ (\mu\beta.[\beta]x))]\!] = (\lambda\beta.\beta\ x)\ (\overline{\lambda}y.f\ y\ \alpha)$$

We must first perform a non-administrative reduction to move the continuation to the place it is used before we are able to substitute x for y.

In order to show the soundness and completeness of evaluation of $\lambda\mu$ reductions with respect to the $\mathscr{C}_{\lambda\mu}$ transform, we need a couple properties of the transform. The first is a formal statement of the notion that in a CPS transformed program, the standard redex is lifted out of its evaluation context to the top of the term. Consider the CPS transform of evaluation contexts in $\lambda\mu$ shown in Figure A 2. We have that in the CPS transform, terms are lifted out of their evaluation context, after performing any necessary administrative reductions.

Lemma A.1.1 (*Lifting*)
$\mathscr{C}_{\lambda\mu}[\![D[t]]\!] \mapsto\!\!\!\!\twoheadrightarrow \mathscr{C}_{\lambda\mu}[\![t]\!]\ \mathscr{C}_{\lambda\mu}[\![D]\!]$ and $\mathscr{C}_{\lambda\mu}[\![E[t]]\!]\ k \mapsto\!\!\!\!\twoheadrightarrow \mathscr{C}_{\lambda\mu}[\![t]\!]\ (\mathscr{C}_{\lambda\mu}[\![E]\!]k)$

Proof

$$\begin{aligned}
\mathscr{C}_{\lambda\mu}[\![[q]E]\!] &= \mathscr{C}_{\lambda\mu}[\![E]\!]\ \mathscr{C}_{\lambda\mu}[\![q]\!] \\
\mathscr{C}_{\lambda\mu}[\![\Box]\!]k &= k \\
\mathscr{C}_{\lambda\mu}[\![E\ t]\!]k &= \mathscr{C}_{\lambda\mu}[\![E]\!]\overline{\lambda} f.\mathscr{C}_{\lambda\mu}[\![t]\!]\overline{\lambda} s.f\ s\ k \\
\mathscr{C}_{\lambda\mu}[\![V\ E]\!]k &= \mathscr{C}_{\lambda\mu}[\![E]\!]\overline{\lambda} s.\mathscr{C}_{\lambda\mu}[\![V]\!]^V\ s\ k
\end{aligned}$$

Fig. A 2. CPS transform of call-by-value evaluation contexts in the $\lambda\mu$-calculus.

By induction on D and E.

- $D = [q]E$:

$$\mathscr{C}_{\lambda\mu}[\![[q](E[t])]\!] = \mathscr{C}_{\lambda\mu}[\![E[t]]\!]\ \mathscr{C}_{\lambda\mu}[\![q]\!] \Mapsto\!\!\!\!\!\rightarrow \mathscr{C}_{\lambda\mu}[\![t]\!]\ (\mathscr{C}_{\lambda\mu}[\![[q]E]\!])$$

- $E = \Box$:

$$\mathscr{C}_{\lambda\mu}[\![t]\!]\ k = \mathscr{C}_{\lambda\mu}[\![t]\!]\ (\mathscr{C}_{\lambda\mu}[\![\Box]\!]k)$$

- $E = E'\ t'$:

$$\mathscr{C}_{\lambda\mu}[\![(E'[t])\ t']\!]\ k \mapsto \mathscr{C}_{\lambda\mu}[\![E'[t]]\!]\ (\overline{\lambda} f.\mathscr{C}_{\lambda\mu}[\![t']\!]\ \overline{\lambda} s.f\ s\ k) \Mapsto\!\!\!\!\!\rightarrow \mathscr{C}_{\lambda\mu}[\![t]\!]\ (\mathscr{C}_{\lambda\mu}[\![E'\ t']\!]k)$$

- $E = V\ E'$:

$$\begin{aligned}
\mathscr{C}_{\lambda\mu}[\![V\ (E'[t])]\!]\ k &\mapsto \mathscr{C}_{\lambda\mu}[\![V]\!]\ (\overline{\lambda} f.\mathscr{C}_{\lambda\mu}[\![E'[t]]\!]\ \overline{\lambda} s.f\ s\ k) \\
&\mapsto \mathscr{C}_{\lambda\mu}[\![E'[t]]\!]\ (\overline{\lambda} s.\mathscr{C}_{\lambda\mu}[\![V]\!]^V\ s\ k) \\
&\Mapsto\!\!\!\!\!\rightarrow \mathscr{C}_{\lambda\mu}[\![t]\!]\ (\mathscr{C}_{\lambda\mu}[\![V\ E']\!]k)
\end{aligned}$$

□

The second property we need for the $\mathscr{C}_{\lambda\mu}$ transform is that substitution in the $\lambda\mu$ calculus corresponds to substitution performed in the CPS language. Since we have three different notions of substitution in the $\lambda\mu$ calculus, we have three correlations. From now on, the meta-syntactic variable M ranges over terms and commands.

Lemma A.1.2 (*Substitution*)

- $\mathscr{C}_{\lambda\mu}[\![M\{V/x\}]\!] = \mathscr{C}_{\lambda\mu}[\![M]\!]\{\mathscr{C}_{\lambda\mu}[\![V]\!]^V/x\}$
- $\mathscr{C}_{\lambda\mu}[\![M\{q/\alpha\}]\!] = \mathscr{C}_{\lambda\mu}[\![M]\!]\{\mathscr{C}_{\lambda\mu}[\![q]\!]/\alpha\}$
- $\mathscr{C}_{\lambda\mu}[\![M\{D[t]/[\alpha]t\}]\!] \Rightarrow\!\!\!\!\!\rightarrow \mathscr{C}_{\lambda\mu}[\![M]\!]\{\mathscr{C}_{\lambda\mu}[\![D]\!]/\alpha\}$

Proof

By induction on M. The only interesting cases are the ones in which a (co-)variable is being replaced by substitution.

- $M = x$, $\{V/x\}$:

$$\mathscr{C}_{\lambda\mu}[\![x\{V/x\}]\!]^V = \mathscr{C}_{\lambda\mu}[\![V]\!]^V = \mathscr{C}_{\lambda\mu}[\![x]\!]^V\{\mathscr{C}_{\lambda\mu}[\![V]\!]^V/x\}$$

- $M = \alpha$, $\{q/\alpha\}$:

$$\mathscr{C}_{\lambda\mu}[\![\alpha\{q/\alpha\}]\!] = \mathscr{C}_{\lambda\mu}[\![q]\!] = \mathscr{C}_{\lambda\mu}[\![\alpha]\!]\{\mathscr{C}_{\lambda\mu}[\![q]\!]/\alpha\}$$

- $M = [\alpha]t$, $\{D[t]/[\alpha]t\}$:

$$\begin{aligned}\mathscr{C}_{\lambda\mu}[\![([\alpha]t)\{D[t']/[\alpha]t'\}]\!] &= \mathscr{C}_{\lambda\mu}[\![D[t\{D[t']/[\alpha]t'\}]]\!]\\ &\Rrightarrow \mathscr{C}_{\lambda\mu}[\![t]\!]\{\mathscr{C}_{\lambda\mu}[\![D]\!]/\alpha\}\ \mathscr{C}_{\lambda\mu}[\![D]\!]\\ &= \mathscr{C}_{\lambda\mu}[\![[\alpha]t]\!]\{\mathscr{C}_{\lambda\mu}[\![D]\!]/\alpha\}\end{aligned}$$

□

With the lifting and substitution lemmas, proving the soundness of a single step of the $\lambda\mu$ reduction rules with respect to $\mathscr{C}_{\lambda\mu}$ is a straightforward calculation in the λ-calculus, using the usual notion of β equality.

Lemma A.1.3 (*One-step soundness*)
If $M \to M'$ then $\mathscr{C}_{\lambda\mu}[\![M]\!] =_\beta \mathscr{C}_{\lambda\mu}[\![M']\!]$.

Proof
By cases on the reduction $M \to M'$ in the $\lambda\mu$-calculus. Recall that the transform $\mathscr{C}_{\lambda\mu}[\![t]\!]$ is always an administrative $\overline{\lambda}$-abstraction of the form $\overline{\lambda}k.M$, for some CPS program M.

- $(\lambda x.t)\ V \to t\{V/x\}$:

$$\begin{aligned}\mathscr{C}_{\lambda\mu}[\![(\lambda x.t)\ V]\!] &\Rrightarrow \overline{\lambda}k.(\lambda x.\mathscr{C}_{\lambda\mu}[\![t]\!])\ \mathscr{C}_{\lambda\mu}[\![V]\!]^V\ k\\ &\to \overline{\lambda}k.\mathscr{C}_{\lambda\mu}[\![t]\!]\{\mathscr{C}_{\lambda\mu}[\![V]\!]^V/x\}\ k\\ &= \overline{\lambda}k.\mathscr{C}_{\lambda\mu}[\![t\{V/x\}]\!]\ k && \text{(by Lemma A.1.2)}\\ &\Rightarrow \mathscr{C}_{\lambda\mu}[\![t\{V/x\}]\!]\end{aligned}$$

- $F[\mu\alpha.c] \to \mu\alpha'.c\{[\alpha'](F[t])/[\alpha]t\}$:

$$\begin{aligned}\mathscr{C}_{\lambda\mu}[\![F[\mu\alpha.c]]\!] &\Rrightarrow \overline{\lambda}k.(\lambda\alpha.\mathscr{C}_{\lambda\mu}[\![c]\!])\ (\mathscr{C}_{\lambda\mu}[\![F]\!]k) && \text{(by Lemma A.1.1)}\\ &\to \overline{\lambda}k.\mathscr{C}_{\lambda\mu}[\![c]\!]\{(\mathscr{C}_{\lambda\mu}[\![F]\!]k)/\alpha\}\\ &\leftarrow \overline{\lambda}k.(\lambda\alpha'.\mathscr{C}_{\lambda\mu}[\![c]\!]\{\mathscr{C}_{\lambda\mu}[\![[\alpha']F]\!]/\alpha\})\ k\\ &\Lleftarrow \overline{\lambda}k.(\lambda\alpha'.\mathscr{C}_{\lambda\mu}[\![c\{[\alpha'](F[t])/[\alpha]t\}]\!])\ k && \text{(by Lemma A.1.2)}\\ &= \mathscr{C}_{\lambda\mu}[\![\mu\alpha'.c\{[\alpha']F[t]/[\alpha]t\}]\!]\end{aligned}$$

- $[q]\mu\alpha.c \to c\{q/\alpha\}$:

$$\begin{aligned}\mathscr{C}_{\lambda\mu}[\![[q]\mu\alpha.c]\!] &\mapsto (\lambda\alpha.\mathscr{C}_{\lambda\mu}[\![c]\!])\ \mathscr{C}_{\lambda\mu}[\![q]\!]\\ &\to \mathscr{C}_{\lambda\mu}[\![c]\!]\{\mathscr{C}_{\lambda\mu}[\![q]\!]/\alpha\}\\ &= \mathscr{C}_{\lambda\mu}[\![c\{q/\alpha\}]\!] && \text{(by Lemma A.1.2)}\end{aligned}$$

Note that the administrative reduction:

$$\overline{\lambda}k.\mathscr{C}_{\lambda\mu}[\![t]\!]\ k \Rightarrow \mathscr{C}_{\lambda\mu}[\![t]\!]$$

follows from the $\overline{\beta}$ rule since the transformation of every term t, shown in Figure A 1, has the form $\overline{\lambda}k.M$, for some CPS term M.

Since the CPS transform $\mathscr{C}_{\lambda\mu}$ is compositional, every context in $\lambda\mu$ is transformed into a context in the resulting λ-calculus term, and therefore the $\lambda\mu$ reduction rules apply in every context. □

$$c_{fa} \in FinalAnswer ::= [*]V$$
$$c_{whnf} \in WHNF ::= c_{fa} \mid [\alpha]V \mid D[x\,V]$$

Fig. A 3. Final answers and weak head-normal forms in the $\lambda\mu$-calculus.

Observe that the reductions for the $\lambda\mu$-calculus are conceptually more fine-grained than the β reductions produced by the $\mathscr{C}_{\lambda\mu}$ transform. Specifically, using the $\lambda\mu$ reduction rules, the term $\mu\alpha.c$ captures its evaluation context one small piece at a time. However, in the CPS program, the translation of $\mu\alpha.c$ captures the continuation representing its complete evaluation context in one step. Therefore, to simulate the more fine-grained reduction in the CPS program, we must "backtrack" capturing the tail of the continuation, so that only the first piece is taken. Soundness for the general $\lambda\mu$ reduction theory, as stated in Theorem 3.1, follows by straightforward induction on the reduction sequence.

Theorem A.1.1 (*Soundness*)
If $M \twoheadrightarrow M'$ then $\mathscr{C}_{\lambda\mu}[\![M]\!] =_\beta \mathscr{C}_{\lambda\mu}[\![M']\!]$.

Proof
By Lemma A.1.3 and induction on the definition of $M \twoheadrightarrow M'$ as the reflexive, transitive closure of $\rightarrow$. □

A.2 Operational completeness of the $\lambda\mu$ reduction theory

We would also like to show that the operational semantics (and therefore the reduction rules) are strong enough to reach any answer that the CPS is capable of producing. The set of commands that constitutes some form of "answer" is given in Figure A 3. In $\lambda\mu$, a *final answer* is a command of the form $[*]V$. Intuitively, the answer V is considered "final" because the program is now committed, and cannot change its mind by using control effects to jump somewhere else and produce a different answer. Additionally, a *weak head-normal form* (whnf) is either a final answer or a stuck command of the form $[\alpha]V$ or $D[x\,V]$, where x and α are free (co-)variables. Observe that with this definition of weak head-normal forms and complete evaluation contexts, we have the unique decomposition property for the $\lambda\mu$-calculus.

Property A.2.1 (*Unique decomposition*)
Every $\lambda\mu$ command is either a weak head-normal form, or it decomposes into a complete evaluation context D surrounding a redex, *i.e.* $D[(\lambda x.t)\ V]$, or a μ-abstraction, *i.e.* $D[\mu\alpha.c]$.

To start, we need a few facts about the administrative subset of the λ-calculus.

Lemma A.2.1

1. Administrative reduction in the λ-calculus is confluent.
2. If $M_1 \Rrightarrow M_2$ and $M_1 \mapsto M_1'$ by a non-administrative reduction, then there is a M_2' such that $M_2 \mapsto M_2'$ by a non-administrative reduction and $M_1' \Rrightarrow M_2'$.

Proof

For the first lemma, confluence follows from confluence of basic λ-calculus, see Sabry & Felleisen (1993) for details. For the second lemma, we have that $M_1 = E[(\lambda x.M)\ V]$ and the following diagram commutes (purely administrative reduction sequences are labeled as $\overline{\beta}$):

$$\begin{array}{ccc}
M_1 = E[(\lambda x.M)\ V] & \longmapsto & M_1' = E[M\{V/x\}] \\
\downarrow\!\!\downarrow \overline{\beta} & & \downarrow\!\!\downarrow \overline{\beta} \\
M_2 = E'[(\lambda x.M')\ V'] & \longmapsto & M_2' = E'[M'\{V'/x\}]
\end{array}$$

□

We also need a stronger correspondence of substitution in the source and target programs according to the administrative-normal CPS transform $\overline{\mathscr{C}}_{\lambda\mu}$.

Lemma A.2.2 (*Administrative-normal substitution*)

- $\overline{\mathscr{C}}_{\lambda\mu}[\![M\{V/x\}]\!] \twoheadleftarrow \overline{\mathscr{C}}_{\lambda\mu}[\![M]\!]\{\overline{\mathscr{C}}_{\lambda\mu}[\![V]\!]^V/x\}$
- $\overline{\mathscr{C}}_{\lambda\mu}[\![M\{q/\alpha\}]\!] \twoheadleftarrow \overline{\mathscr{C}}_{\lambda\mu}[\![M]\!]\{\overline{\mathscr{C}}_{\lambda\mu}[\![q]\!]/\alpha\}$
- $\overline{\mathscr{C}}_{\lambda\mu}[\![M\{D[t]/[\alpha]t\}]\!] \twoheadleftarrow \overline{\mathscr{C}}_{\lambda\mu}[\![M]\!]\{\overline{\mathscr{C}}_{\lambda\mu}[\![D]\!]/\alpha\}$

Proof

We first show the case for structural substitution of evaluation contexts. By definition of $\overline{\mathscr{C}}_{\lambda\mu}$, Lemma A.1.2, and confluence of administrative reduction (Lemma A.2.1), there is an M' such that the following diagram commutes (where all reductions are administrative only):

$$\begin{array}{ccccc}
\mathscr{C}_{\lambda\mu}[\![M\{D[t]/[\alpha]t\}]\!] & & \longrightarrow\!\!\!\!\!\rightarrow & & \mathscr{C}_{\lambda\mu}[\![M]\!]\{\mathscr{C}_{\lambda\mu}[\![D]\!]/\alpha\} \\
\downarrow\!\!\downarrow & & & & \downarrow\!\!\downarrow \\
\overline{\mathscr{C}}_{\lambda\mu}[\![M\{D[t]/[\alpha]t\}]\!] & \twoheadrightarrow & M' & \twoheadleftarrow & \overline{\mathscr{C}}_{\lambda\mu}[\![M]\!]\{\overline{\mathscr{C}}_{\lambda\mu}[\![D]\!]/\alpha\}
\end{array}$$

Furthermore, since $\overline{\mathscr{C}}_{\lambda\mu}[\![M\{D[t]/[\alpha]t\}]\!]$ is an administrative normal form by definition, we have that $\overline{\mathscr{C}}_{\lambda\mu}[\![M\{D[t]/[\alpha]t\}]\!] = M' \twoheadleftarrow \overline{\mathscr{C}}_{\lambda\mu}[\![M]\!]\{\overline{\mathscr{C}}_{\lambda\mu}[\![D]\!]/\alpha\}$.

The cases for substitution of values and co-terms follow analogously. □

We now show that non-administrative reductions in the CPS actually correspond to reductions in $\lambda\mu$, and that end states (values and whnfs) in the CPS correspond to end states in $\lambda\mu$.

Lemma A.2.3 (*One-step evaluation*)

- If $\overline{\mathscr{C}}_{\lambda\mu}[\![c]\!]$ is a value then c is a final answer.
- If $\overline{\mathscr{C}}_{\lambda\mu}[\![c]\!]$ is a whnf then c is a whnf.
- If $\overline{\mathscr{C}}_{\lambda\mu}[\![c]\!] \mapsto M'$ then there is a $\lambda\mu$ command c' such that $c \mapsto c'$ and $M' \Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c']\!]$.

Proof

By cases on c, using the unique decomposition property.

- $c = [*]V$:

$$\overline{\mathscr{C}}_{\lambda\mu}[\![[*]V]\!] = \overline{\mathscr{C}}_{\lambda\mu}[\![V]\!]^V$$

- $c = [\alpha]V$:

$$\overline{\mathscr{C}}_{\lambda\mu}[\![[\alpha]V]\!] = \alpha\, \overline{\mathscr{C}}_{\lambda\mu}[\![V]\!]^V$$

- $c = D[x\, V]$:

$$\overline{\mathscr{C}}_{\lambda\mu}[\![D[x\, V]]\!] = x\, \overline{\mathscr{C}}_{\lambda\mu}[\![V]\!]^V\, \overline{\mathscr{C}}_{\lambda\mu}[\![D]\!] \qquad \text{(by Lemma A.1.1)}$$

- $c = D[(\lambda x.t)\, V]$:

$$\begin{aligned}\overline{\mathscr{C}}_{\lambda\mu}[\![D[(\lambda x.t)\, V]]\!] &= (\lambda x.\overline{\mathscr{C}}_{\lambda\mu}[\![t]\!])\, \overline{\mathscr{C}}_{\lambda\mu}[\![V]\!]^V\, \overline{\mathscr{C}}_{\lambda\mu}[\![D]\!] && \text{(by Lemma A.1.1)}\\ &\mapsto \overline{\mathscr{C}}_{\lambda\mu}[\![t]\!]\{\overline{\mathscr{C}}_{\lambda\mu}[\![V]\!]^V/x\}\, \overline{\mathscr{C}}_{\lambda\mu}[\![D]\!] \\ &\Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![t\{V/x\}]\!]\, \overline{\mathscr{C}}_{\lambda\mu}[\![D]\!] && \text{(by Lemma A.2.2)}\\ &\Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![D[t\{V/x\}]]\!]\end{aligned}$$

$$D[(\lambda x.t)\, V] \mapsto D[t\{V/x\}]$$

- $c = D[\mu\alpha.c]$:

$$\begin{aligned}\overline{\mathscr{C}}_{\lambda\mu}[\![D[\mu\alpha.c]]\!] &= (\lambda\alpha.\overline{\mathscr{C}}_{\lambda\mu}[\![c]\!])\, \overline{\mathscr{C}}_{\lambda\mu}[\![D]\!] && \text{(by Lemma A.1.1)}\\ &\mapsto \overline{\mathscr{C}}_{\lambda\mu}[\![c]\!]\{\overline{\mathscr{C}}_{\lambda\mu}[\![D]\!]/\alpha\} \\ &\Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c\{D[t]/[\alpha]t\}]\!] && \text{(by Lemma A.2.2)}\end{aligned}$$

$$D[\mu\alpha.c] \mapsto c\{D[t]/[\alpha]t\}$$

□

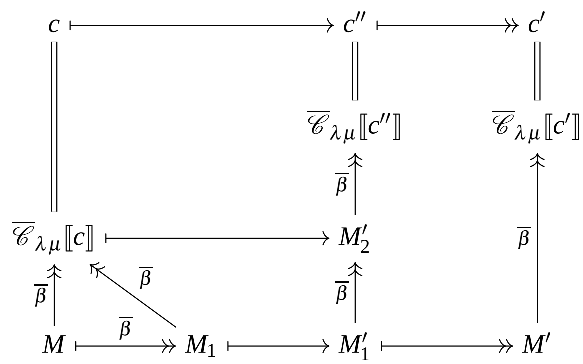
From the fact that steps and end states in the CPS program correspond properly in the source program, we can prove the main evaluation lemma for operational completeness of the call-by-value $\lambda\mu$-calculus.

Lemma A.2.4
If $M \Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c]\!]$ and $M \mapsto\!\!\!\!\rightarrow M'$, then there is a $\lambda\mu$ command c' such that $c \mapsto\!\!\!\!\rightarrow c'$ and $M' \Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c']\!]$. Furthermore, if M' is a whnf then c' is a whnf, and if M' is a value then c' is a final answer.

Proof
By induction on the non-administrative reductions in $M \mapsto\!\!\!\!\rightarrow M'$.

- In the case that $M \mapsto\!\!\!\Rrightarrow M'$ by administrative reductions only, then we have $c' = c$ and $M' \Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c']\!]$.
  Additionally by Lemma A.2.3, we have that if M' is a whnf then $\overline{\mathscr{C}}_{\lambda\mu}[\![c']\!]$ is also a whnf, and so c' is a whnf. Also, if M' is a value then $\overline{\mathscr{C}}_{\lambda\mu}[\![c']\!]$ is also a value, and so c' is a final answer.
- Otherwise, we have the case where $M \mapsto\!\!\!\Rrightarrow M_1$ and $M_1 \mapsto M_1'$ by a non-administrative reduction. We have that $M_1 \Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c]\!]$ by Lemma A.2.1 (confluence) and the fact that $\overline{\mathscr{C}}_{\lambda\mu}[\![c]\!]$ is an administrative-normal form. Also by Lemma A.2.1, we have that there is an M_2' such that $\overline{\mathscr{C}}_{\lambda\mu}[\![c]\!] \mapsto M_2'$ and $M_1' \Rrightarrow M_2'$. By Lemma A.2.3, we know

that there is a c'' such that $c \mapsto c''$ and $M_2' \Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c'']\!]$. Applying the induction hypothesis to $M_1' \Rrightarrow M_2' \Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c']\!]$ and $M_1' \mapsto\!\!\!\twoheadrightarrow M'$ finishes the proof.
This reasoning is summed up by the following commutative diagram (purely administrative reduction sequences are labeled as $\overline{\beta}$):



□

Theorem A.2.1 (*Evaluation*)
If $\mathscr{C}_{\lambda\mu}[\![c]\!] =_\beta V$ then there is a final answer c' such that $c \mapsto\!\!\!\twoheadrightarrow c'$ and $\mathscr{C}_{\lambda\mu}[\![c']\!] =_\beta V$.

Proof
First, recall that $\mathscr{C}_{\lambda\mu}[\![c]\!] \Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c]\!]$, so that $\overline{\mathscr{C}}_{\lambda\mu}[\![c]\!] =_\beta V$. By confluence and the fact that values are closed under reduction in the λ-calculus, we know that there is a V' such that $\overline{\mathscr{C}}_{\lambda\mu}[\![c]\!] \twoheadrightarrow V'$. Furthermore, by standardization of reduction in the λ-calculus (both call-by-name or call-by-value standard reduction is the same for CPS programs), there is a V'' such that $\overline{\mathscr{C}}_{\lambda\mu}[\![c]\!] \mapsto\!\!\!\twoheadrightarrow V''$. From applying Lemma A.2.4 to $\mathscr{C}_{\lambda\mu}[\![c]\!] \Rrightarrow \overline{\mathscr{C}}_{\lambda\mu}[\![c]\!]$ and $\overline{\mathscr{C}}_{\lambda\mu}[\![c]\!] \mapsto\!\!\!\twoheadrightarrow V''$, we have a final answer c' such that $c \mapsto\!\!\!\twoheadrightarrow c'$ and $V'' \Rrightarrow \mathscr{C}_{\lambda\mu}[\![c']\!]$. Finally, by transitivity of β-equivalence in the λ-calculus we have that $\mathscr{C}_{\lambda\mu}[\![c']\!] =_\beta V'' =_\beta V$. □

Finally, the fine-grained reductions presented in Figure 10 are capable of simulating the operational semantics of $\lambda\mu$ given in Figure 11.

Theorem A.2.2
If $c \mapsto\!\!\!\twoheadrightarrow c'$ then $c \twoheadrightarrow c'$.

Proof
By induction on reduction sequence $c \mapsto\!\!\!\twoheadrightarrow c'$ as the reflexive, transitive closure of $\mapsto$. The result is immediate for β-reduction on functions by applying the reduction rule inside the evaluation context. In the case that a μ-abstraction captures its context,

$$D[\mu\alpha.c] \mapsto c\{D[t]/[\alpha]t\}$$

the simulation is obtained by induction on D. □

Therefore, from Theorems A.2.1 and A.2.2, we have that the reduction theory is capable of finding any final answer given by the CPS transform.

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\mu\widehat{\mathsf{tp}}.c]\!] = \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c]\!]\lambda x.k\ x\ \gamma \qquad \begin{aligned}\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\widehat{\mathsf{tp}}]\!] &= \overline{\lambda}x.\overline{\lambda}\gamma.\gamma\ x\\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![*]\!] &= \overline{\lambda}x.\overline{\lambda}\gamma.x\end{aligned}$$

Fig. A 4. The administrative $\overline{\lambda}$-abstractions of the $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}$ transform.

A.3 Soundness and operational completeness of the $\lambda\mu\widehat{\mathsf{tp}}$ reduction theory

The administrative $\overline{\lambda}$-abstractions in the $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}$ transformation are those from Figure A 1 as well as the new ones shown in Figure A 4. Observe that on the one hand, the clauses of the $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}$ transform inherited from the $\mathscr{C}_{\lambda\mu}$ transform in Figure A 1 do not make use of meta-continuation, γ, but just implicitly pass it along. On the other hand, the clauses of $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}$ from Figure A 4 explicitly introduce and manipulate a meta-continuation due to the behavior of the dynamic $\widehat{\mathsf{tp}}$. We could η-expand the $\mathscr{C}_{\lambda\mu}$ transform to remove this difference in CPS programs, but choose not to so that the CPS transform of purely $\lambda\mu$ programs remains unchanged. Instead, in order to mediate between implicit and explicit uses of the meta-continuation, we use η-reduction while reasoning about CPS programs. In fact, we only need a very restricted form of usual notion of η-reduction that eliminates trivial $\overline{\lambda}$-abstractions in the place of a meta-continuation:

$$\overline{\eta}: \quad \overline{\lambda}k.\overline{\lambda}\gamma.M\ \gamma \Rightarrow \overline{\lambda}k.M$$

where γ is not a free variable of M. It follows that the reduction rules of $\lambda\mu\widehat{\mathsf{tp}}$, which include the reductions from Figure 16 along with the additional rule:

$$\mu\widehat{\mathsf{tp}}.[*]V \to \mu\_.[*]V$$

are sound with respect to $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}$ up to $\beta\eta$-equality in the λ-calculus.

Lemma A.3.1 (*One-step soundness*)
If $M \to M'$ then $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![M]\!] =_{\beta\eta} \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![M']\!]$.

Proof
By cases on the reduction $M \to M'$ in the $\lambda\mu\widehat{\mathsf{tp}}$-calculus:

- $\mu\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]V \to V$:

$$\begin{aligned}\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\mu\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]V]\!] &\Rrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\widehat{\mathsf{tp}}]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V\ (\lambda x.k\ x\ \gamma)\\ &\Rrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\lambda x.k\ x\ \gamma)\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V\\ &\to \overline{\lambda}k.\overline{\lambda}\gamma.k\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V\ \gamma\\ &\Rightarrow_{\overline{\eta}} \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]\end{aligned}$$

$$c_{fa} \in \mathit{FinalAnswer} ::= D^2[[*]V]$$
$$c_{whnf} \in \mathit{WHNF} ::= c_{fa} \mid [\bullet][\widehat{\mathsf{tp}}]V \mid D^2[[\alpha]V] \mid D^2[D[x\,V]]$$

Fig. A 5. Final answers and weak head-normal forms in the $\lambda\mu\widehat{\mathsf{tp}}$-calculus.

- $\mu\widehat{\mathsf{tp}}.[*]V \to \mu\_.[*]V$:

$$\begin{aligned}
\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\mu\widehat{\mathsf{tp}}.[*]V]\!] &\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![*]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V\ (\lambda x.k\ x\ \gamma) \\
&\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V \\
&\twoheadleftarrow \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![*]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V\ \gamma \\
&\leftarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\lambda_.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![*]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V)\ k\ \gamma \\
&\Rightarrow_{\overline{\eta}} \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\mu_.[*]V]\!]
\end{aligned}$$

The rest of the cases are the same as for the proof of Lemma A.1.3 for $\lambda\mu$. □

Note how $\overline{\eta}$ reduction is used to prove soundness of the $\lambda\mu\widehat{\mathsf{tp}}$ reduction theory. This allows us to equate CPS terms that do not mention a meta-continuation with those that pass along their meta-continuation unchanged, as in $\overline{\lambda}\gamma.k\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V\ \gamma$. Soundness of the reduction theory, as stated in Theorem 4.1, follows by induction.

Theorem A.3.1 (*Soundness*)
If $M \twoheadrightarrow M'$ then $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![M]\!] =_{\beta\eta} \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![M']\!]$.

Since $\lambda\mu\widehat{\mathsf{tp}}$ operates on meta-commands, final answers in $\lambda\mu\widehat{\mathsf{tp}}$ are correspondingly meta-commands of the form $D^2[[*]V]$, as shown in Figure A 5. We must also expand the set of weak head normal forms to accommodate the new syntax, so that the whnfs of $\lambda\mu\widehat{\mathsf{tp}}$ are final answers or meta-commands of the form $[\bullet][\widehat{\mathsf{tp}}]V$, $D^2[[\alpha]V]$, or $D^2[D[x\,V]]$, where α and x are free (co-)variables. Likewise, the unique decomposition property is also lifted into meta-commands in the $\lambda\mu\widehat{\mathsf{tp}}$-calculus.

Property A.3.1 (*Unique decomposition*)
Every $\lambda\mu\widehat{\mathsf{tp}}$ meta-command is either a weak head-normal form, or it decomposes into a complete evaluation meta-context and complete evaluation context, D^2 and D, surrounding a redex, *i.e.* $D^2[D[(\lambda x.t)\ V]]$, or a μ-abstraction, *i.e.* $D^2[D[\mu\alpha.c]]$.

Note that the reduction

$$\mu\widehat{\mathsf{tp}}.[*]V \to \mu\_.[*]V$$

does not occur during standard reduction, but instead equates final answers by discarding unnecessary information in the meta-context.

Just like we needed to use the fact that terms are lifted out of their evaluation context in the $\mathscr{C}_{\lambda\mu}$ transform, we also need to show that commands are lifted out of their evaluation meta-contexts in the $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}$ transform. Evaluation meta-context are transformed as shown in Figure A 6. This gives us a similar result for evaluation meta-contexts in the $\lambda\mu\widehat{\mathsf{tp}}$-calculus.

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![[q^2]E^2]\!] = \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![E^2]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![q^2]\!]$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\Box]\!]\gamma = \gamma$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D[\mu\widehat{\mathsf{tp}}.E^2]]\!]\gamma = \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![E^2]\!]\lambda x.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D]\!]\ x\ \gamma$$

Fig. A 6. CPS transform of call-by-value evaluation meta-contexts in the $\lambda\mu\widehat{\mathsf{tp}}$-calculus.

Lemma A.3.2 (*Meta-lifting*)
$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D^2[c]]\!] \mapsto\!\!\!\!\to \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D^2]\!]$ and $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![E^2[c]]\!]\gamma \mapsto\!\!\!\!\to \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c]\!]\ (\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![E^2]\!]\gamma)$

Proof
By induction on D^2 and E^2.

- $D^2 = [q^2]E^2$:

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![[q^2]E^2[c]]\!] = \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![E^2[c]]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![q^2]\!] \mapsto\!\!\!\!\to \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![[q^2]E^2]\!]$$

- $E^2 = \Box$:

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c]\!]\gamma = \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c]\!]\ (\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\Box]\!]\gamma)$$

- $E^2 = D[\mu\widehat{\mathsf{tp}}.E'^2]$:

$$\begin{aligned}\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D[\mu\widehat{\mathsf{tp}}.E'^2[c]]]\!]\gamma &\mapsto\!\!\!\!\to \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\mu\widehat{\mathsf{tp}}.E'^2[c]]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D]\!]\ \gamma \\ &\mapsto\!\!\!\!\to \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![E'^2[c]]\!]\lambda x.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D]\!]\ x\ \gamma \\ &\mapsto\!\!\!\!\to \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c]\!]\ (\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D[\mu\widehat{\mathsf{tp}}.E'^2]]\!]\gamma)\end{aligned}$$

□

The proof that the $\lambda\mu\widehat{\mathsf{tp}}$ operational and reduction rules are complete with respect to evaluation of $\mathscr{C}_{\lambda\mu\widehat{\mathsf{tp}}}$ follows the same strategy as for $\lambda\mu$.

Lemma A.3.3 (*One-step evaluation*)

- If $\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c^2]\!]$ is a value then c^2 is a final answer.
- If $\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c^2]\!]$ is a whnf then c^2 is a whnf.
- If $\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c^2]\!] \mapsto M'$ by a non-administrative reduction then there is a $\lambda\mu\widehat{\mathsf{tp}}$ meta-command c'^2 such that $c^2 \mapsto c'^2$ and $M' \Rightarrow\!\!\!\!\to \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c'^2]\!]$.

Proof
By cases on c^2, using the unique decomposition property.

- $c^2 = D^2[[*]V]$:

$$\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D^2[[*]V]]\!] = \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V$$

- $c^2 = [\bullet][\widehat{\mathsf{tp}}]V$:

$$\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![[\bullet][\widehat{\mathsf{tp}}]V]\!] = \gamma_0\ \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V$$

- $c^2 = D^2[D[\mu\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]V]]$:

$$\begin{aligned}
&\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D^2[D[\mu\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]V]]]\!] \\
&= (\lambda x.\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D[x]]\!]\ \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D^2]\!])\ \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V && \text{(by Lemmas A.1.1 and A.3.2)} \\
&= (\lambda x.\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D^2[D[x]]]\!])\ \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V && \text{(by Lemma A.3.2)} \\
&\mapsto \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D^2[D[x]]]\!]\{\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V/x\} \\
&\twoheadrightarrow \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![D^2[D[V]]]\!] && \text{(by Lemma A.2.2)}
\end{aligned}$$

$$D^2[D[\mu\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]V]] \mapsto D^2[D[V]]$$

The rest of the cases are the same as the proof of Lemma A.2.3 for $\lambda\mu$, except lifted into an arbitrary evaluation meta-context D^2. □

Lemma A.3.4
If $M \twoheadrightarrow \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c^2]\!]$ and $M \mapsto\!\!\!\!\rightarrow M'$, then there is a $\lambda\mu\widehat{\mathsf{tp}}$ meta-command c'^2 such that $c^2 \mapsto\!\!\!\!\rightarrow c'^2$ and $M' \twoheadrightarrow \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c'^2]\!]$. Furthermore, if M' is a whnf then c'^2 is a whnf, and if M' is a value then c'^2 is a final answer.

Proof
The same as the proof of Lemma A.2.4 for $\lambda\mu$. □

Finally, we have a proof of the operational completeness of $\lambda\mu\widehat{\mathsf{tp}}$ as a strengthened version of Theorem 4.2.

Theorem A.3.2 (*Evaluation*)
If $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c^2]\!] =_\beta V$ then there is a final answer c'^2 such that $c^2 \mapsto\!\!\!\!\rightarrow c'^2$ and $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c'^2]\!] =_\beta V$.

Proof
The same as the proof of Theorem A.2.1 for $\lambda\mu$. □

Theorem A.3.3
If $c^2 \mapsto\!\!\!\!\rightarrow c'^2$ then $c^2 \twoheadrightarrow c'^2$.

Proof
By induction on reduction sequence $c^2 \mapsto\!\!\!\!\rightarrow c'^2$ as the reflexive, transitive closure of $\mapsto$. The result is immediate for reduction of $\mu\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]V$. The rest of the proof is the same as the proof of Theorem A.2.2 for $\lambda\mu$. □

A.4 *Soundness and operational completeness of the $\lambda\widehat{\mu}$ reduction theory*

The administrative $\overline{\lambda}$-abstractions in the $\lambda\widehat{\mu}$ two-pass, $\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}$ transform are the same ones given for $\lambda\mu\widehat{\mathsf{tp}}$ in Figure A 4 with the changes shown in Figure A 7. In addition, for purposes of classifying reductions in the CPS program, reducing the if-then-else expression is also considered administrative.

The reduction rules for $\lambda\widehat{\mu}$ are sound with respect to the transform $\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}$, and the operational semantics is complete with respect to evaluation of the transform.

Lemma A.4.1 (*One-step soundness*)

$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![\mu\widehat{\alpha}.c]\!] = \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![c]\!]\gamma[\widehat{\alpha}\mapsto k] \qquad \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![\widehat{\alpha}]\!] = \overline{\lambda}x.\overline{\lambda}\gamma.\gamma(\widehat{\alpha})\ x$$
$$\gamma[\widehat{\alpha}\mapsto k] = \overline{\lambda}p.\,\textbf{if}\, p \equiv \ulcorner\widehat{\alpha}\urcorner\,\textbf{then}\,\lambda x.k\ x\ \gamma\,\textbf{else}\,\gamma\ p$$

Fig. A 7. The administrative $\overline{\lambda}$-abstractions in the $\lambda\widehat{\mu}$ transform.

If $M \to M'$ then $\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![M]\!] =_{\beta\eta} \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![M']\!]$.

Proof
By cases on the reduction $M \to M'$ in the $\lambda\widehat{\mu}$-calculus. For the purpose of conciseness, in this proof the transform $\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![M]\!]$ is referred to as just $[\![M]\!]$.

- $\mu\widehat{\alpha}.[\widehat{\alpha}]V \to V$:

$$\begin{aligned}
[\![\mu\widehat{\alpha}.[\widehat{\alpha}]V]\!] &\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\gamma[\widehat{\alpha}\mapsto k])(\widehat{\alpha})\ [\![V]\!]^V \\
&\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\lambda x.k\ x\ \gamma)\ [\![V]\!]^V \\
&\to \overline{\lambda}k.\overline{\lambda}\gamma.k\ [\![V]\!]^V\ \gamma \\
&\Rightarrow_{\overline{\eta}} [\![V]\!]
\end{aligned}$$

- $\mu\widehat{\alpha}.[\widehat{\beta}]V \to \mu\_.[\widehat{\beta}]V$:

$$\begin{aligned}
[\![\mu\widehat{\alpha}.[\widehat{\beta}]V]\!] &\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.[\![\widehat{\beta}]\!]\ [\![V]\!]^V\ (\gamma[\widehat{\alpha}\mapsto k]) \\
&\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\gamma[\widehat{\alpha}\mapsto k])(\widehat{\beta})\ [\![V]\!]^V \\
&\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.\gamma(\widehat{\beta})\ [\![V]\!]^V \\
&\twoheadleftarrow \overline{\lambda}k.\overline{\lambda}\gamma.[\![\widehat{\beta}]\!]\ [\![V]\!]^V\ \gamma \\
&\leftarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\lambda_.[\![\widehat{\beta}]\!]\ [\![V]\!]^V)\ k\ \gamma \\
&\Rightarrow_{\overline{\eta}} [\![\mu_.[\widehat{\beta}]V]\!]
\end{aligned}$$

- $\mu\widehat{\alpha}.[*]V \to \mu\_.[*]V$:

$$\begin{aligned}
[\![\mu\widehat{\alpha}.[*]V]\!] &\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.[\![*]\!]\ [\![V]\!]^V\ (\gamma[\widehat{\alpha}\mapsto k]) \\
&\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.[\![V]\!]^V \\
&\twoheadleftarrow \overline{\lambda}k.\overline{\lambda}\gamma.[\![*]\!]\ [\![V]\!]^V\ \gamma \\
&\leftarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\lambda_.[\![*]\!]\ [\![V]\!]^V)\ k\ \gamma \\
&\Rightarrow_{\overline{\eta}} [\![\mu_.[*]V]\!]k\gamma
\end{aligned}$$

The rest of the cases are the same as for the proof of Lemma A.3.1 for $\lambda\mu\widehat{\mathrm{tp}}$. □

Theorem A.4.1 (*Soundness*)
If $M \twoheadrightarrow M'$ then $\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![M]\!] =_{\beta\eta} \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![M']\!]$.

The final answers and weak head-normal forms of $\lambda\widehat{\mu}$ are shown in Figure A 8 and are the same as those from Figure A 5 for $\lambda\mu\widehat{\mathrm{tp}}$, except that whnfs $D^2_{\widehat{\alpha}}[[\widehat{\alpha}]V]$ are generalized to any meta-context $D^2_{\widehat{\alpha}}$ which does not bind $\widehat{\alpha}$.

To accommodate the addition of multiple dynamic co-variables in the $\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}$ transform, we need to generalize the transform of meta-contexts to dynamic binding environments, as

$$c_{fa} \in FinalAnswer ::= D^2[[*]V]$$
$$c_{whnf} \in WHNF ::= c_{fa} \mid D^2_{\widehat{\alpha}}[[\widehat{\alpha}]V] \mid D^2[[\alpha]V] \mid D^2[D[x\,V]]$$

Fig. A 8. Final answers and weak head-normal forms in the $\lambda\widehat{\mu}$-calculus.

$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![[q^2]E^2]\!] = \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![E^2]\!]\ \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![q^2]\!]$$
$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![\Box]\!]\gamma = \gamma$$
$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![D[\mu\widehat{\alpha}.E^2]]\!]\gamma = \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![E^2]\!]\ \gamma[\widehat{\alpha} \mapsto \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![D]\!]]$$

Fig. A 9. CPS transform of call-by-value evaluation contexts of the $\lambda\widehat{\mu}$-calculus.

shown in Figure A 9. With this change, we still have the normal lifting property, as well as the fact that dynamic co-variable lookup behaves as expected.

Lemma A.4.2 (*Meta-lifting*)
For the transform $\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}$, $[\![D^2[c]]\!] \mapsto\!\!\!\!\!\to [\![c]\!]\ [\![D^2]\!]$ and $[\![E^2[c]]\!]\gamma \mapsto\!\!\!\!\!\to [\![c]\!]\ ([\![E^2]\!]\gamma)$.

Proof
By induction on D^2 and E^2. The proof follows the same form as the meta-lifting lemma for $\lambda\mu\widehat{\mathrm{tp}}$. □

Lemma A.4.3 (*Lookup*)
If E^2 does not contain a binding for $\widehat{\alpha}$, then $\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![D^2[E^2]]\!](\widehat{\alpha}) \mapsto\!\!\!\!\!\to \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![D^2]\!](\widehat{\alpha})$.

Proof
By induction on E^2. □

With these lemmas, proving completeness of the operational rules with respect to evaluation in the transform follows as it did for $\lambda\mu\widehat{\mathrm{tp}}$.

Lemma A.4.4 (*One-step evaluation*)

- If $\overline{\mathscr{D}_{\widehat{\lambda}}}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}}[\![c^2]\!]$ is a value then c^2 is a final answer.
- If $\overline{\mathscr{D}_{\widehat{\lambda}}}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}}[\![c^2]\!]$ is a whnf then c^2 is a whnf.
- If $\overline{\mathscr{D}_{\widehat{\lambda}}}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}}[\![c^2]\!] \mapsto M'$ by a non-administrative reduction then there is a $\lambda\widehat{\mu}$ meta-command c'^2 such that $c^2 \mapsto c'^2$ and $M' \twoheadrightarrow \overline{\mathscr{D}_{\widehat{\lambda}}}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}}[\![c'^2]\!]$.

Proof
By cases on c^2, using the unique decomposition property. For the purpose of conciseness, in this proof the transform $\overline{\mathscr{D}_{\widehat{\lambda}}}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}}[\![M]\!]$ is referred to as just $[\![M]\!]$.

- $c^2 = [\bullet]E^2[[\widehat{\alpha}]V]$, where E^2 does not contain a binding for $\widehat{\alpha}$:

$$[\![[\bullet]E^2[[\widehat{\alpha}]V]]\!] = \gamma_0(\widehat{\alpha})\ [\![V]\!]^V \qquad \text{(by Lemmas A.4.2 and A.4.3)}$$

- $c^2 = D^2[D[\mu\widehat{\alpha}.E^2[[\widehat{\alpha}]V]]$, where E^2 does not contain a binding for $\widehat{\alpha}$:

$$
\begin{aligned}
&[\![D^2[D[\mu\widehat{\alpha}.E^2[[\widehat{\alpha}]V]]]]\!]\\
&= [\![\widehat{\alpha}]\!]\ [\![V]\!]^V\ ([\![D^2]\!][\widehat{\alpha} \mapsto [\![D]\!]])\\
&= ([\![D^2]\!][\widehat{\alpha} \mapsto [\![D]\!]])(\widehat{\alpha})\ [\![V]\!]^V && \text{(by Lemmas A.1.1 and A.4.2)}\\
&= (\lambda x.[\![D]\!]\ x\ [\![D^2]\!])\ [\![V]\!]^V\\
&= (\lambda x.[\![D^2[D[x]]]\!])\ [\![V]\!]^V && \text{(by Lemmas A.1.1 and A.4.2)}\\
&\mapsto [\![D^2[D[x]]]\!]\{[\![V]\!]^V/x\}\\
&\Rrightarrow [\![D^2[D[V]]]\!] && \text{(by Lemma A.1.2)}
\end{aligned}
$$

$$D^2[D[\mu\widehat{\alpha}.E^2[[\widehat{\alpha}]V]] \mapsto D^2[D[V]]$$

The rest of the cases are the same as for the proof of Lemma A.3.3 for $\lambda\mu\widehat{\mathsf{tp}}$. □

Lemma A.4.5
If $M \Rrightarrow \overline{\mathscr{D}}_{\widehat{\lambda}}\overline{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}}[\![c^2]\!]$ and $M \mapsto\!\!\!\!\rightarrow M'$, then there is a $\lambda\mu\widehat{\mathsf{tp}}$ meta-command c'^2 such that $c^2 \mapsto\!\!\!\!\rightarrow c'^2$ and $M' \Rrightarrow \overline{\mathscr{D}}_{\widehat{\lambda}}\overline{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}}[\![c'^2]\!]$. Furthermore, if M' is a whnf then c'^2 is a whnf, and if M' is a value then c'^2 is a final answer.

Proof
The same as the proof of Lemma A.2.4 for $\lambda\mu$. □

Theorem A.4.2 (*Evaluation*)
If $\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![c^2]\!] =_\beta V$ then there is a final answer c'^2 such that $c^2 \mapsto\!\!\!\!\rightarrow c'^2$ and $\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![c'^2]\!] =_\beta V$.

Proof
The same as the proof of Theorem A.2.1 for $\lambda\mu$. □

Theorem A.4.3
If $c^2 \mapsto\!\!\!\!\rightarrow c'^2$ then $c^2 \twoheadrightarrow c'^2$.

Proof
By induction on reduction sequence $c^2 \mapsto\!\!\!\!\rightarrow c'^2$ as the reflexive, transitive closure of $\mapsto$. In the case that a value is given to a dynamic co-variable,

$$D^2[D[\mu\widehat{\alpha}.E^2_{\widehat{\alpha}}[[\widehat{\alpha}]V]]] \mapsto D^2[D[V]]$$

the simulation is obtained by induction on $E^2_{\widehat{\alpha}}$. The rest of the proof is the same as the proof of Theorem A.2.2 for $\lambda\mu$. □

A.5 Soundness and operational completeness of the $\lambda\mu\widehat{\mathsf{tp}}_0$ reduction theory

The new administrative $\overline{\lambda}$-abstractions introduced in the $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}$ transform are those of the $\mathscr{C}_{\lambda\mu}$ transform from Figure A 1 along with the new ones shown in Figure A 10. The reduction rules of $\lambda\mu\widehat{\mathsf{tp}}_0$ are sound with respect to the transform $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}$.

$$\begin{aligned}
\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![\mu_0\widehat{\mathsf{tp}}.c]\!] &= \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c]\!]\lambda u.u\ k\ \gamma \\
\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![[\widehat{\mathsf{tp}}]_0 t]\!] &= \overline{\lambda}\gamma.\gamma\,\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!] \\
\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![*]\!] &= \overline{\lambda}x.\overline{\lambda}\gamma.x
\end{aligned}$$

Fig. A 10. The administrative $\overline{\lambda}$-abstractions of the $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}$ transform.

$$\begin{aligned}
c_{fa} \in \mathit{FinalAnswer} &::= D^2[[*]V] \\
c_{whnf} \in \mathit{WHNF} &::= c_{fa} \mid [\bullet][\widehat{\mathsf{tp}}]_0 V \mid D^2[[\alpha]V] \mid D^2[D[x\ V]]
\end{aligned}$$

Fig. A 11. Final answers and weak head-normal forms in the $\lambda\mu\widehat{\mathsf{tp}}_0$-calculus.

Lemma A.5.1 (*One-step soundness*)
If $M \to M'$ then $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![M]\!] =_{\beta\eta} \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![M']\!]$.

Proof
By cases on the reduction $M \to M'$ in the $\lambda\mu\widehat{\mathsf{tp}}_0$-calculus. Recall that the transform $\mathscr{C}_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!]$ is always an administrative $\overline{\lambda}$-abstraction of the form $\overline{\lambda}k.M$, for some CPS program M.

- $\mu_0\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]_0 t \to t$:

$$\begin{aligned}
\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![\mu_0\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]_0 t]\!] &\Rrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\lambda u.u\ k\ \gamma)\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!] \\
&\to \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!]k\gamma \\
&\Rightarrow_{\overline{\eta}} \lambda k.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!]k \\
&\Rightarrow \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!]
\end{aligned}$$

- $\mu_0\widehat{\mathsf{tp}}.[*]V \to \mu\_.[*]V$:

$$\begin{aligned}
\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![\mu_0\widehat{\mathsf{tp}}.[*]V]\!] &\Rrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![*]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![V]\!]^V\ (\lambda u.u\ k\ \gamma) \\
&\Rrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![V]\!]^V \\
&\Lleftarrow \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![*]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![V]\!]^V\ \gamma \\
&\leftarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\lambda_.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![*]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![V]\!]^V)\ k\ \gamma \\
&\Rightarrow_{\overline{\eta}} \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![\mu_.[*]V]\!]
\end{aligned}$$

The rest of the cases are the same as for the proof of Lemma A.3.1 for $\lambda\mu\widehat{\mathsf{tp}}$. □

Theorem A.5.1 (*Soundness*)
If $M \twoheadrightarrow M'$ then $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![M]\!] =_{\beta\eta} \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![M']\!]$.

In $\lambda\mu\widehat{\mathsf{tp}}_0$, the final answers are the same as from Figure A 5 for $\lambda\mu\widehat{\mathsf{tp}}$, and the set of weak head normal forms instead includes meta-commands of the form $[\bullet][\widehat{\mathsf{tp}}]_0 t$, as shown in Figure A 11. The transform of evaluation meta-contexts is correspondingly modified from Figure A 6 to match the change in the CPS transform for $\mu\widehat{\mathsf{tp}}.c$, as given in Figure A 12.

Lemma A.5.2 (*Meta-lifting*)

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![[q^2]E^2]\!] = \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![E^2]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![q^2]\!]$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![\Box]\!]\gamma = \gamma$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![D[\mu_0\widehat{\mathsf{tp}}.E^2]]\!]\gamma = \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![E^2]\!]\lambda t.t\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![D]\!]\ \gamma$$

Fig. A 12. CPS transform of call-by-value evaluation contexts of $\lambda\mu\widehat{\mathsf{tp}}_0$.

$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![D^2[c]]\!] \mapsto\!\!\!\!\to \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![D^2]\!]$ and $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![E^2[c]]\!]\gamma \mapsto\!\!\!\!\to [\![c]\!]\ (\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![E^2]\!]\gamma)$

Proof
By induction on D^2 and E^2. The proof follows the same form as the proof of Lemma A.3.2 for $\lambda\mu\widehat{\mathsf{tp}}$. □

Lemma A.5.3 (*One-step evaluation*)

- If $\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c]\!]$ is a value then c^2 is a final answer.
- If $\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c]\!]$ is a whnf then c^2 is a whnf.
- If $\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c]\!] \mapsto M'$ by a non-administrative reduction then there is a $\lambda\mu\widehat{\mathsf{tp}}_0$ meta-command c'^2 such that $c^2 \mapsto c'^2$ and $M' \twoheadrightarrow \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c'^2]\!]$.

Proof
By cases on c^2, using the unique decomposition property.

- $c^2 = [\bullet][\widehat{\mathsf{tp}}]_0 t$:
$$\overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![[\bullet][\widehat{\mathsf{tp}}]_0 t]\!] = \gamma_0\ \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!]$$
- $c^2 = D^2[D[\mu_0\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]_0 t]]$:
$$\begin{aligned}\overline{\mathscr{C}}^2[\![D^2[D[\mu_0\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]_0 t]]]\!] &= (\lambda u.u\ \overline{\mathscr{C}}^2[\![D]\!]\ \overline{\mathscr{C}}^2[\![D^2]\!])\ \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!] \\ &\quad \text{(by Lemmas A.1.1 and A.5.2)} \\ &\mapsto \overline{\mathscr{C}}^2[\![t]\!]\ \overline{\mathscr{C}}^2[\![D]\!]\ \overline{\mathscr{C}}^2[\![D^2]\!] \\ &\twoheadrightarrow \overline{\mathscr{C}}^2[\![D^2[D[t]]]\!]\end{aligned}$$

$$D^2[D[\mu_0\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]_0 t]] \mapsto D^2[D[t]]$$

The rest of the cases are the same as for the proof of Lemma A.3.3 for $\lambda\mu\widehat{\mathsf{tp}}$. □

Lemma A.5.4
If $M \twoheadrightarrow \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c^2]\!]$ and $M \mapsto\!\!\!\!\to M'$, then there is a $\lambda\mu\widehat{\mathsf{tp}}_0$ meta-command c'^2 such that $c^2 \mapsto\!\!\!\!\to c'^2$ and $M' \twoheadrightarrow \overline{\mathscr{C}}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c'^2]\!]$. Furthermore, if M' is a whnf then c'^2 is a whnf, and if M' is a value then c'^2 is a final answer.

Proof
The same as the proof of Lemma A.2.4 for $\lambda\mu$. □

Theorem A.5.2 (*Evaluation*)
If $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c^2]\!] =_\beta V$ then there is a final answer c'^2 such that $c^2 \mapsto\!\!\!\!\to c'^2$ and $\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c'^2]\!] =_\beta V$.

$$\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![[\widehat{\alpha}]_0\Delta.t]\!] = \overline{\lambda}\gamma.\gamma(\widehat{\alpha})\ (\lambda\Delta.\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![t]\!])$$
$$\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![\mu_0\widehat{\alpha}.c]\!] = \overline{\lambda}k.\overline{\lambda}\gamma.\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![c]\!]\gamma[\widehat{\alpha}\mapsto k]$$
$$\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![*]\!] = \overline{\lambda}x.\overline{\lambda}\gamma.x$$

$$\begin{aligned}\gamma[\widehat{\alpha}\mapsto k] = \overline{\lambda}p.\,&\mathbf{if}\ p\equiv\ulcorner\widehat{\alpha}\urcorner\\ &\mathbf{then}\,(\overline{\lambda}g.g\ (\overline{\lambda}\gamma'.\gamma')\ k\ \gamma)\\ &\mathbf{else}\,(\overline{\lambda}g.\gamma\ p\ (\overline{\lambda}\Delta.g\ (\overline{\lambda}\gamma'.(\Delta\ \gamma')[\widehat{\alpha}\mapsto k])))\end{aligned}$$

Fig. A 13. The administrative $\overline{\lambda}$-abstractions of the $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}$ transform.

$$\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![[\Delta]E^2]\!]\gamma = \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![E^2]\!](\Delta\ \gamma)$$

Fig. A 14. CPS transform of call-by-value evaluation contexts of the $\lambda\widehat{\mu}_0$-calculus.

Proof
The same as the proof of Lemma A.2.1 for $\lambda\mu$. □

Theorem A.5.3
If $c^2 \mapsto\!\!\!\rightarrow c'^2$ then $c^2 \twoheadrightarrow c'^2$.

Proof
By induction on reduction sequence $c^2 \mapsto\!\!\!\rightarrow c'^2$ as the reflexive, transitive closure of $\mapsto$. The result is immediate for reduction of $\mu_0\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]_0 t$. The rest of the proof is the same as the proof of Theorem A.2.2 for $\lambda\mu$. □

A.6 Soundness and operational completeness of the $\lambda\widehat{\mu}_0$ reduction theory

Like with $\lambda\widehat{\mu}$ the reduction rules and operational semantics are sound and complete with respect to the transform $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}$. The administrative $\overline{\lambda}$-abstractions for $\lambda\widehat{\mu}_0$ are those of $\lambda\mu$ from Figure A 1 along with the ones shown in Figure A 13. For the purpose of classifying reductions, reducing the if-then-else expression is an administrative reduction.

Since $\lambda\widehat{\mu}_0$ introduces a new notion of substitution (structural substitution for a meta-context variable Δ), we need to check that our substitution lemma for $\lambda\widehat{\mu}_0$ still holds for the $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}$ transform. The new syntax for extending the meta-context, $[\Delta]c$, adds another case, shown in Figure A 14, to the transform of meta-context given for $\lambda\widehat{\mu}$ from Figure A 9.

Lemma A.6.1 (*Meta-lifting*)
For the transform $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}$, $[\![D^2[c]]\!] \Mapsto [\![c]\!]\ [\![D^2]\!]$ and $[\![E^2[c]]\!]\gamma \Mapsto [\![c]\!]\ ([\![E^2]\!]\gamma)$

Proof
Analogous to the proof of Lemma A.4.2 for $\lambda\widehat{\mu}$. □

This allows us to show the correspondence of structural substitution of the meta-context between the source and target languages.

Lemma A.6.2 (*Substitution*)
$\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![M\{E^2[c]/[\Delta]c\}]\!] \twoheadrightarrow \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![M]\!]\{([\![E^2]\!]\gamma)/(\Delta\,\gamma)\}$

Proof
By induction on M. The only new interesting case is for structural substitution on Δ. For conciseness, the transform $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![M]\!]$ is written as just $[\![M]\!]$.

$$
\begin{aligned}
[\![([\Delta]c)\{E^2[c']/[\Delta]c'\}]\!]\gamma &= [\![E^2[c\{E^2[c']/[\Delta]c'\}]]\!]\gamma \\
&\twoheadrightarrow [\![c\{E^2[c']/[\Delta]c'\}]\!]\ ([\![E^2]\!]\gamma) && \text{(by Lemma A.6.1)} \\
&\twoheadrightarrow [\![c]\!]\{([\![E^2]\!]\gamma')/(\Delta\,\gamma')\}\ ([\![E^2]\!]\gamma) && \text{(by Lemma A.6.1 and I.H.)} \\
&= ([\![c]\!]\ (\Delta\,\gamma))\{([\![E^2]\!]\gamma')/(\Delta\,\gamma')\} \\
&= ([\![[\Delta]c]\!]\gamma)\{([\![E^2]\!]\gamma')/(\Delta\,\gamma')\}
\end{aligned}
$$

□

Likewise, we have a stronger substitution lemma for the administrative-normal CPS transform $\overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\overline{\mathscr{C}}}_{\lambda\widehat{\mu}_0}$.

Lemma A.6.3
$\overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\overline{\mathscr{C}}}_{\lambda\widehat{\mu}_0}[\![M\{E^2[c]/[\Delta]c\}]\!] \twoheadleftarrow \overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\overline{\mathscr{C}}}_{\lambda\widehat{\mu}_0}[\![M]\!]\{([\![E^2]\!]\gamma)/(\Delta\,\gamma)\}$

Proof
Same as the proof of Lemma A.2.2 for $\lambda\mu$. □

The proof of soundness of the $\lambda\widehat{\mu}_0$ reduction rules follows from the extended substitution Lemma A.6.2.

Lemma A.6.4 (*One-step soundness*)
If $M \to M'$ then $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![M]\!] =_{\beta\eta} \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![M']\!]$.

Proof
By cases on the reduction $M \to M'$ in the $\lambda\widehat{\mu}_0$-calculus. Recall that the transform $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![t]\!]$ is always an administrative $\overline{\lambda}$-abstraction of the form $\overline{\lambda}k.M$, for some CPS program M. For conciseness, in this proof the transform $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![M]\!]$ is written as just $[\![M]\!]$.

- $\mu_0\widehat{\alpha}.[\widehat{\alpha}]_0\Delta.t \to t\{c/[\Delta]c\}$:

$$
\begin{aligned}
[\![\mu_0\widehat{\alpha}.[\widehat{\alpha}]_0\Delta.t]\!] &\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\gamma[\widehat{\alpha}\mapsto k])(\widehat{\alpha})\ (\lambda\Delta.[\![t]\!]) \\
&\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\lambda\Delta.[\![t]\!])\ (\overline{\lambda}\gamma'.\gamma')\ k\ \gamma \\
&\to \overline{\lambda}k.\overline{\lambda}\gamma.[\![t]\!]\{(\overline{\lambda}\gamma'.\gamma')/\Delta\}k\gamma \\
&\twoheadrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.[\![t]\!]\{\gamma'/(\Delta\,\gamma')\}k\gamma \\
&\twoheadleftarrow \overline{\lambda}k.\overline{\lambda}\gamma.[\![t\{c/[\Delta]c\}]\!]k\gamma && \text{(by Lemma A.6.2)} \\
&\Rightarrow_{\overline{\eta}} \overline{\lambda}k.[\![t\{c/[\Delta]c\}]\!]k \\
&\Rightarrow [\![t\{c/[\Delta]c\}]\!]
\end{aligned}
$$

$$c_{fa} \in FinalAnswer ::= D^2[[*]V]$$
$$c_{whnf} \in WHNF ::= c_{fa} \mid [\bullet]E^2_{\widehat{\alpha}}[[\widehat{\alpha}]V] \mid D^2[[\Delta']E^2_{\widehat{\alpha}}[[\widehat{\alpha}]V]] \mid D^2[[\alpha]V] \mid D^2[D[x\ V]]$$

Fig. A 15. Final answers and weak head-normal forms in the $\lambda\widehat{\mu}_0$-calculus.

- $\mu_0\widehat{\beta}.[\widehat{\alpha}]_0\Delta.t \to \mu\beta.[\widehat{\alpha}]_0\Delta.t\{[\Delta]([\beta]\mu_0\widehat{\beta}.c)/[\Delta]c\}$:

$$\begin{aligned}
[\![\mu_0\widehat{\beta}.[\widehat{\alpha}]_0\Delta.t]\!] &\Rrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\gamma[\widehat{\beta}\mapsto k])(\widehat{\alpha})\ (\lambda\Delta.[\![t]\!])\\
&\Rrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.\gamma(\widehat{\alpha})\ (\overline{\lambda}\Delta'.(\lambda\Delta.[\![t]\!])\ (\overline{\lambda}\gamma'.(\Delta'\ \gamma')[\widehat{\beta}\mapsto k]))\\
&\to \overline{\lambda}k.\overline{\lambda}\gamma.\gamma(\widehat{\alpha})\ (\overline{\lambda}\Delta'.[\![t]\!]\{(\overline{\lambda}\gamma'.(\Delta'\ \gamma')[\widehat{\beta}\mapsto k])/\Delta\})\\
&\Rrightarrow \overline{\lambda}k.\overline{\lambda}\gamma.\gamma(\widehat{\alpha})\ (\overline{\lambda}\Delta'.[\![t]\!]\{((\Delta'\ \gamma')[\widehat{\beta}\mapsto k])/(\Delta\ \gamma')\})\\
&\leftarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\lambda\beta.\overline{\lambda}\gamma.\gamma(\widehat{\alpha})\ (\overline{\lambda}\Delta'.[\![t]\!]\{((\Delta'\ \gamma')[\widehat{\beta}\mapsto \beta])/(\Delta\ \gamma')\}))\ k\ \gamma\\
&\Lleftarrow \overline{\lambda}k.\overline{\lambda}\gamma.(\lambda\beta.\overline{\lambda}\gamma.\gamma(\widehat{\alpha})\ (\overline{\lambda}\Delta'.[\![t\{[\Delta'][\beta]\mu_0\widehat{\beta}.c/[\Delta]c\}]\!]))\ k\ \gamma\\
&\quad\text{(by Lemma A.6.2)}\\
&\Rightarrow_\eta [\![\mu\beta.[\widehat{\alpha}]_0\Delta'.t\{[\Delta']([\beta]\mu_0\widehat{\beta}.c)/[\Delta]c\}]\!]
\end{aligned}$$

The rest of the cases are the same as for the proofs of Lemmas A.4.4 and A.5.3 for $\lambda\widehat{\mu}$ and $\lambda\mu\widehat{\mathsf{tp}}_0$. □

Theorem A.6.1 (*Soundness*)
If $M \twoheadrightarrow M'$ then $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![M]\!] =_{\beta\eta} \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![M']\!]$.

The final answers of $\lambda\widehat{\mu}_0$ are the same as the final answers from Figure A 8 for $\lambda\widehat{\mu}$. The weak head-normal forms of $\lambda\widehat{\mu}_0$ still include $D^2[D[x\ V]]$ and $D^2[[\alpha]V]$, as well as $D^2[E^2_{\widehat{\alpha}}[[\widehat{\alpha}]_0\Delta.t]]$ with $D^2 = [\bullet]\square$ or $D^2 = D'^2[[\Delta']\square]$ and where $E^2_{\widehat{\alpha}}$ does not bind $\widehat{\alpha}$. Looking up a dynamic co-variable still behaves as expected, except that now the prefix of the environment that is skipped over is collected and returned to the calling continuation.

Lemma A.6.5 (*Lookup*)
For the transform $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}$, $[\![D^2[E^2_{\widehat{\alpha}}]]\!](\widehat{\alpha}) \Rrightarrow \overline{\lambda}g.[\![D^2]\!](\widehat{\alpha})\ (\overline{\lambda}\Delta.g\ \overline{\lambda}\gamma.[\![[\Delta]E^2_{\widehat{\alpha}}]\!]\gamma)$ when $E^2_{\widehat{\alpha}} \neq \square$.

Proof
By induction on $E^2_{\widehat{\alpha}}$. For conciseness, the transform $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![M]\!]$ is written as just $[\![M]\!]$.

- $E^2_{\widehat{\alpha}} = D[\mu_0\widehat{\beta}.\square]$:

$$\begin{aligned}
[\![D^2[D[\mu_0\widehat{\beta}.\square]]]\!](\widehat{\alpha}) &= [\![D^2]\!][\widehat{\beta}\mapsto [\![D]\!]](\widehat{\alpha})\\
&\Rrightarrow \overline{\lambda}g.[\![D^2]\!](\widehat{\alpha})\ \overline{\lambda}\Delta.g\ \overline{\lambda}\gamma.(\Delta\ \gamma)[\widehat{\beta}\mapsto [\![D]\!]]\\
&= \overline{\lambda}g.[\![D^2]\!](\widehat{\alpha})\ \overline{\lambda}\Delta.g\ \overline{\lambda}\gamma.[\![[\Delta]D[\mu_0\widehat{\beta}.\square]]\!]\gamma
\end{aligned}$$

- $E^2_{\widehat{\alpha}} = D[\mu_0\widehat{\beta}.E'^2_{\widehat{\alpha}}]$, where $E'^2_{\widehat{\alpha}} \neq \Box$:

$$\begin{aligned}\llbracket D^2[D[\mu_0\widehat{\beta}.E'^2_{\widehat{\alpha}}]]\rrbracket(\widehat{\alpha}) &\twoheadrightarrow \overline{\lambda}g.\llbracket D^2[D[\mu_0\widehat{\beta}.\Box]]\rrbracket(\widehat{\alpha})\ (\overline{\lambda}\Delta.g\ \overline{\lambda}\gamma.\llbracket[\Delta]E'^2_{\widehat{\alpha}}\rrbracket\gamma) \qquad \text{(by I.H.)}\\ &= \overline{\lambda}g.\llbracket D^2\rrbracket[\widehat{\beta}\mapsto\llbracket D\rrbracket](\widehat{\alpha})\ (\overline{\lambda}\Delta.g\ \overline{\lambda}\gamma.\llbracket E'^2_{\widehat{\alpha}}\rrbracket(\Delta\ \gamma))\\ &\twoheadrightarrow \overline{\lambda}g.\llbracket D^2\rrbracket(\widehat{\alpha})\ (\overline{\lambda}\Delta.g\ \overline{\lambda}\gamma.\llbracket E'^2_{\widehat{\alpha}}\rrbracket(\Delta\ \gamma[\widehat{\beta}\mapsto\llbracket D\rrbracket]))\\ &= \overline{\lambda}g.\llbracket D^2\rrbracket(\widehat{\alpha})\ (\overline{\lambda}\Delta.g\ \overline{\lambda}\gamma.\llbracket[\Delta]D[\mu_0\widehat{\beta}.E'^2_{\widehat{\alpha}}]\rrbracket)\end{aligned}$$

□

Finally, completeness with respect to evaluation in the CPS transform follows as expected.

Lemma A.6.6 (*One-step evaluation*)
If $\overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}_0}\llbracket c^2\rrbracket \mapsto\!\!\!\twoheadrightarrow M$ then

- If $\overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}_0}\llbracket c^2\rrbracket$ is a value then c^2 is a final answer.
- If $\overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}_0}\llbracket c^2\rrbracket$ is a whnf then c^2 is a whnf.
- If $\overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}_0}\llbracket c^2\rrbracket \mapsto M'$ by a non-administrative reduction then there is a $\lambda\widehat{\mu}_0$ meta-command c'^2 such that $c^2 \mapsto c'^2$ and $M' \twoheadrightarrow \overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}_0}\llbracket c'^2\rrbracket$.

Proof
By cases on c^2 using the unique decomposition property. For conciseness, in this proof the transform $\overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\widehat{\mathscr{C}}}_{\lambda\widehat{\mu}_0}\llbracket M\rrbracket$ is written as just $\llbracket M\rrbracket$.

- $c^2 = [\bullet]E^2_{\widehat{\alpha}}[[\widehat{\alpha}]_0\Delta.t]$:

$$\llbracket[\bullet]E^2_{\widehat{\alpha}}[[\widehat{\alpha}]_0\Delta.t]\rrbracket = \gamma_0(\widehat{\alpha})\ g \qquad \text{(by Lemmas A.6.1 and A.6.5)}$$
$$\textbf{where}\, g = \begin{cases}\lambda\Delta.\llbracket t\rrbracket & \text{if } E^2_{\widehat{\alpha}} = \Box\\ \overline{\lambda}\Delta'.(\lambda\Delta.\llbracket t\rrbracket)\ \llbracket[\Delta']E^2_{\widehat{\alpha}}\rrbracket & \text{otherwise}\end{cases}$$

- $c^2 = D^2[[\Delta]E^2_{\widehat{\alpha}}[[\widehat{\alpha}]_0\Delta.t]]$:

$$\llbracket D^2[[\Delta]E^2_{\widehat{\alpha}}[[\widehat{\alpha}]_0\Delta.t]]\rrbracket = (\Delta\ \llbracket D^2\rrbracket)(\widehat{\alpha})\ g \qquad \text{(by Lemmas A.6.1 and A.6.5)}$$
$$\textbf{where}\, g = \begin{cases}\lambda\Delta.\llbracket t\rrbracket & \text{if } E^2_{\widehat{\alpha}} = \Box\\ \overline{\lambda}\Delta'.(\lambda\Delta.\llbracket t\rrbracket)\ \llbracket[\Delta']E^2_{\widehat{\alpha}}\rrbracket & \text{otherwise}\end{cases}$$

- $c^2 = D^2[D[\mu_0\widehat{\alpha}.E^2_{\widehat{\alpha}}[[\widehat{\alpha}]_0\Delta.t]]]$, when $E^2_{\widehat{\alpha}} \neq \Box$:

$$\begin{aligned}&\llbracket D^2[D[\mu_0\widehat{\alpha}.E^2_{\widehat{\alpha}}[[\widehat{\alpha}]_0\Delta.t]]]\rrbracket\\ &= (D^2[\widehat{\alpha}\mapsto\llbracket D\rrbracket])(\widehat{\alpha})\ (\overline{\lambda}\Delta'.(\lambda\Delta.\llbracket t\rrbracket)\ \llbracket[\Delta']E^2_{\widehat{\alpha}}\rrbracket) \quad \text{(by Lemmas A.1.1, A.6.1 and A.6.5)}\\ &= (\lambda\Delta.\llbracket t\rrbracket)\ \llbracket E^2_{\widehat{\alpha}}\rrbracket\ \llbracket D\rrbracket\ \llbracket D^2\rrbracket\\ &\mapsto \llbracket t\rrbracket\{\llbracket E^2_{\widehat{\alpha}}\rrbracket/\Delta\}\ \llbracket D\rrbracket\ \llbracket D^2\rrbracket\\ &\twoheadrightarrow \llbracket t\rrbracket\{(\llbracket E^2_{\widehat{\alpha}}\rrbracket\gamma)/(\Delta\ \gamma)\}\ \llbracket D\rrbracket\ \llbracket D^2\rrbracket\\ &\twoheadrightarrow \llbracket t\{E^2_{\widehat{\alpha}}[c]/[\Delta]c\}\rrbracket\ \llbracket D\rrbracket\ \llbracket D^2\rrbracket \qquad \text{(by Lemma A.6.3)}\\ &= \llbracket D^2[D[t\{E^2_{\widehat{\alpha}}[c]/[\Delta]c\}]]\rrbracket \qquad \text{(by Lemmas A.1.1 and A.6.1)}\end{aligned}$$

$$D^2[D[\mu_0\widehat{\alpha}.E^2_{\widehat{\alpha}}[[\widehat{\alpha}]_0\Delta.t]]] \mapsto D^2[D[t\{E^2_{\widehat{\alpha}}[c]/[\Delta]c\}]]$$

And similarly when $E^2_{\widehat{\alpha}} = \square$.

The rest of the cases are the same as for the proofs of Lemmas A.4.4 and A.5.3 for $\lambda\widehat{\mu}$ and $\lambda\mu\widehat{\mathsf{tp}}_0$. $\square$

Lemma A.6.7
If $M \Rrightarrow \overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\overline{\mathscr{C}}}_{\lambda\widehat{\mu}_0}[\![c^2]\!]$ and $M \mapsto\!\!\!\rightarrow M'$, then there is a $\lambda\widehat{\mu}_0$ meta-command c'^2 such that $c^2 \mapsto\!\!\!\rightarrow c'^2$ and $M' \Rrightarrow \overline{\mathscr{D}}_{\widehat{\lambda}_0}\widehat{\overline{\mathscr{C}}}_{\lambda\widehat{\mu}_0}[\![c'^2]\!]$. Furthermore, if M' is a whnf then c'^2 is a whnf, and if M' is a value then c'^2 is a final answer.

Proof
The same as the proof of Lemma A.2.4 for $\lambda\mu$. $\square$

Theorem A.6.2 (*Evaluation*)
If $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![c^2]\!] =_\beta V$ then there is a final answer c'^2 such that $c^2 \mapsto\!\!\!\rightarrow c'^2$ and $\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![c'^2]\!] =_\beta V$.

Proof
The same as the proof of Lemma A.2.1 for $\lambda\mu$. $\square$

Theorem A.6.3
If $c^2 \mapsto\!\!\!\rightarrow c'^2$ then $c^2 \twoheadrightarrow c'^2$.

Proof
By induction on reduction sequence $c^2 \mapsto\!\!\!\rightarrow c'^2$ as the reflexive, transitive closure of $\mapsto$. In the case that a prefix of the meta-context is captured by a command of the form $[\widehat{\alpha}]_0\Delta.t$,

$$D^2[D[\mu_0\widehat{\alpha}.E^2_{\widehat{\alpha}}[[\widehat{\alpha}]_0\Delta.t]]] \mapsto D^2[D[t\{E^2_{\widehat{\alpha}}[c]/[\Delta]c\}]]$$

the simulation is obtained by induction on the context $E^2_{\widehat{\alpha}}$, where reduction is lifted into the context $D^2[D]$.

- $E^2_{\widehat{\alpha}} = \square$:

$$\mu_0\widehat{\alpha}.[\widehat{\alpha}]_0\Delta.t \to t\{c/[\Delta]c\}$$

- $E^2_{\widehat{\alpha}} = E'^2_{\widehat{\alpha}}[D'[\mu\widehat{\beta}.\square]]$:

$$\begin{aligned}\mu_0\widehat{\alpha}.E'^2_{\widehat{\alpha}}[D'[\mu\widehat{\beta}.[\widehat{\alpha}]_0\Delta.t]] &\to \mu_0\widehat{\alpha}.E'^2_{\widehat{\alpha}}[D'[\mu\gamma.[\widehat{\alpha}]_0\Delta.t\{[\Delta][\gamma]\mu\widehat{\beta}.c/[\Delta]c\}]] \\ &\twoheadrightarrow \mu_0\widehat{\alpha}.E'^2_{\widehat{\alpha}}[[\widehat{\alpha}]_0\Delta.t\{[\Delta]D'[\mu\widehat{\beta}.c]/[\Delta]c\}] && \text{(by Theorem A.2.2)} \\ &\twoheadrightarrow t\{E'^2_{\widehat{\alpha}}[D'[\mu\widehat{\beta}.c]]/[\Delta]c\} && \text{(by I.H.)}\end{aligned}$$

The rest of the proof is the same as the proof of Theorem A.2.2 for $\lambda\mu$. $\square$

B List of definitions

B.1 Languages of Control

In the following, we summarize the syntax and semantics of the various languages of control by listing their reduction theories and CPS transforms in their entirety.

Parigot's $\lambda\mu$-calculus:

$$t \in Term ::= V \mid t_1\ t_2 \mid \mu\alpha.c \qquad c \in Command ::= [q]t$$
$$V \in Value ::= x \mid \lambda x.t \qquad q \in CoTerm ::= \alpha \mid *$$

Fig. B 1. The syntax of the $\lambda\mu$-calculus.

$$F \in Frame ::= \Box\ t \mid V\ \Box$$

$$(\lambda x.t)\ V \to t\{V/x\}$$
$$F[\mu\alpha.c] \to \mu\alpha.c\{[\alpha](F[t])/[\alpha]t\}$$
$$[q]\mu\alpha.c \to c\{q/\alpha\}$$

Fig. B 2. Call-by-value reduction theory of the $\lambda\mu$-calculus.

$$\mathscr{C}_{\lambda\mu}[\![[q]t]\!] = \mathscr{C}_{\lambda\mu}[\![t]\!]\ \mathscr{C}_{\lambda\mu}[\![q]\!] \qquad \mathscr{C}_{\lambda\mu}[\![*]\!] = \lambda x.x$$
$$\mathscr{C}_{\lambda\mu}[\![\alpha]\!] = \alpha$$
$$\mathscr{C}_{\lambda\mu}[\![V]\!] = \lambda k.k\ \mathscr{C}_{\lambda\mu}[\![V]\!]^V$$
$$\mathscr{C}_{\lambda\mu}[\![t_1\ t_2]\!] = \lambda k.\mathscr{C}_{\lambda\mu}[\![t_1]\!]\lambda f.\mathscr{C}_{\lambda\mu}[\![t_2]\!]\lambda s.f\ s\ k \qquad \mathscr{C}_{\lambda\mu}[\![x]\!]^V = x$$
$$\mathscr{C}_{\lambda\mu}[\![\mu\alpha.c]\!] = \lambda k.(\lambda\alpha.\mathscr{C}_{\lambda\mu}[\![c]\!])\ k \qquad \mathscr{C}_{\lambda\mu}[\![\lambda x.t]\!]^V = \lambda x.\mathscr{C}_{\lambda\mu}[\![t]\!]$$

Fig. B 3. Call-by-value CPS transform for the $\lambda\mu$-calculus.

$\lambda\mu\widehat{\mathsf{tp}}$-calculus:

$$c^2 \in Command^2 ::= [q^2]c$$
$$c \in Command ::= [q]t$$
$$t \in Term ::= V \mid t_1\ t_2 \mid \mu\alpha.c \mid \mu\widehat{\mathsf{tp}}.c \qquad q^2 \in CoTerm^2 ::= \bullet \mid \circledast$$
$$V \in Value ::= x \mid \lambda x.t \qquad q \in CoTerm ::= \alpha \mid \widehat{\mathsf{tp}} \mid *$$

Fig. B 4. The syntax of the $\lambda\mu\widehat{\mathsf{tp}}$-calculus.

$$(\lambda x.t)\ V \to t\{V/x\} \qquad \mu\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]V \to V$$
$$F[\mu\alpha.c] \to \mu\alpha.c\{[\alpha](F[t])/[\alpha]t\} \qquad \mu\widehat{\mathsf{tp}}.[*]V \to \mu\_.[*]V$$
$$[q]\mu\alpha.c \to c\{q/\alpha\}$$

Fig. B 5. Call-by-value reduction theory of the $\lambda\mu\widehat{\mathsf{tp}}$-calculus.

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\bullet]\!] = \gamma_0 \qquad \textbf{where } \gamma_0 \text{ free}$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![[q^2]c]\!] = \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![q^2]\!] \qquad \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\circledast]\!] = \lambda x.x$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![[q]t]\!] = \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![t]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![q]\!]$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\alpha]\!] = \alpha$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!] = \lambda k.k\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![V]\!]^V \qquad \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\widehat{\mathsf{tp}}]\!] = \lambda x.\lambda\gamma.\gamma\, x$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![t_1\ t_2]\!] = \lambda k.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![t_1]\!]\lambda f.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![t_2]\!]\lambda s.f\ s\ k \qquad \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![*]\!] = \lambda x.\lambda\gamma.x$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\mu\alpha.c]\!] = \lambda k.(\lambda\alpha.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c]\!])\ k$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\mu\widehat{\mathsf{tp}}.c]\!] = \lambda k.\lambda\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![c]\!]\lambda x.k\ x\ \gamma \qquad \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![x]\!]^V = x$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![\lambda x.t]\!]^V = \lambda x.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}}[\![t]\!]$$

Fig. B 6. Call-by-value double CPS transform of the $\lambda\mu\widehat{\mathsf{tp}}$-calculus.

$\lambda\widehat{\mu}$-calculus:

$$c^2 \in \mathit{Command}^2 ::= [q^2]c$$
$$c \in \mathit{Command} ::= [q]t$$
$$t \in \mathit{Term} ::= V \mid t_1\ t_2 \mid \mu\alpha.c \mid \mu\widehat{\alpha}.c \qquad q^2 \in \mathit{CoTerm}^2 ::= \bullet$$
$$V \in \mathit{Value} ::= x \mid \lambda x.t \qquad q \in \mathit{CoTerm} ::= \alpha \mid \widehat{\alpha} \mid *$$

Fig. B 7. The syntax of the $\lambda\widehat{\mu}$-calculus.

$$(\lambda x.t)\ V \to t\{V/x\} \qquad \mu\widehat{\alpha}.[\widehat{\alpha}]V \to V$$
$$F[\mu\alpha.c] \to \mu\alpha.c\{[\alpha](F[t])/[\alpha]t\} \qquad \mu\widehat{\alpha}.[\widehat{\beta}]V \to \mu\_.[\widehat{\beta}]V \qquad \textbf{where } \widehat{\alpha} \neq \widehat{\beta}$$
$$[q]\mu\alpha.c \to c\{q/\alpha\} \qquad \mu\widehat{\alpha}.[*]V \to \mu\_.[*]V$$

Fig. B 8. Call-by-value reduction theory for dynamic co-variables in the $\lambda\widehat{\mu}$-calculus.

$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![\bullet]\!] = \gamma_0 \qquad \textbf{where } \gamma_0 \text{ free}$$

$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![[q^2]c]\!] = \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![c]\!]\ \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![q^2]\!]$$

$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![[q]t]\!] = \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![t]\!]\ \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![q]\!] \qquad \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![\alpha]\!] = \alpha$$

$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![\widehat{\alpha}]\!] = \lambda x.\lambda\gamma.\gamma(\widehat{\alpha})\ x$$

$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![V]\!] = \lambda k.k\ \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![V]\!]^V \qquad \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![*]\!] = \lambda x.\lambda\gamma.x$$

$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![t_1\ t_2]\!] = \lambda k.\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![t_1]\!]\lambda f.\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![t_2]\!]\lambda s.f\ s\ k$$

$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![\mu\alpha.c]\!] = \lambda k.(\lambda\alpha.\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![c]\!])\ k \qquad \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![x]\!]^V = x$$

$$\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![\mu\widehat{\alpha}.c]\!] = \lambda k.\lambda\gamma.\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![c]\!](\gamma[\widehat{\alpha} \mapsto k]) \qquad \mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![\lambda x.t]\!]^V = \lambda x.\mathscr{D}_{\widehat{\lambda}}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}}[\![t]\!]$$

$$\gamma(\widehat{\alpha}) = \gamma \ulcorner\widehat{\alpha}\urcorner$$
$$\gamma[\widehat{\alpha} \mapsto k] = \lambda p.\,\textbf{if}\ p \equiv \ulcorner\widehat{\alpha}\urcorner\ \textbf{then}\ (\lambda x.k\ x\ \gamma)\ \textbf{else}\ \gamma\ p$$

Fig. B 9. Call-by-value composed CPS transform of the $\lambda\widehat{\mu}$-calculus.

$\lambda\mu\widehat{\mathsf{tp}}_0$-calculus:

$$c^2 \in Command^2 ::= [q^2]c$$
$$c \in Command ::= [q]t \mid [\widehat{\mathsf{tp}}]_0 t$$
$$t \in Term ::= V \mid t_1\ t_2 \mid \mu\alpha.c \mid \mu_0\widehat{\mathsf{tp}}.c \qquad q^2 \in CoTerm^2 ::= \bullet$$
$$V \in Value ::= x \mid \lambda x.t \qquad q \in CoTerm ::= \alpha \mid *$$

Fig. B 10. The syntax of the $\lambda\mu\widehat{\mathsf{tp}}_0$-calculus.

$$(\lambda x.t)\ V \to t\{V/x\}$$
$$F[\mu\alpha.c] \to \mu\alpha.c\{[\alpha](F[t])/[\alpha]t\}$$
$$[q]\mu\alpha.c \to c\{q/\alpha\}$$
$$\mu_0\widehat{\mathsf{tp}}.[\widehat{\mathsf{tp}}]_0 t \to t$$
$$\mu_0\widehat{\mathsf{tp}}.[*]V \to \mu\_.[*]V$$

Fig. B 11. Call-by-value reduction theory of the $\lambda\mu\widehat{\mathsf{tp}}_0$-calculus.

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![[q^2]c]\!] = \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![q^2]\!]$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![[q]t]\!] = \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!]\ \mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![q]\!]$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![[\widehat{\mathsf{tp}}]_0 t]\!] = \lambda\gamma.\gamma\,\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!]$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![V]\!] = \lambda k.k\,\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![V]\!]^V$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t_1\ t_2]\!] = \lambda k.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t_1]\!]\lambda f.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t_2]\!]\lambda s.f\ s\ k$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![\mu\alpha.c]\!] = \lambda k.(\lambda\alpha.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c]\!])\ k$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![\mu_0\widehat{\mathsf{tp}}.c]\!] = \lambda k.\lambda\gamma.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![c]\!]\lambda u.u\ k\ \gamma$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![\bullet]\!] = \gamma_0 \qquad \textbf{where } \gamma_0 \text{ free}$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![\alpha]\!] = \alpha$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![*]\!] = \lambda x.\lambda\gamma.x$$

$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![x]\!]^V = x$$
$$\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![\lambda x.t]\!]^V = \lambda x.\mathscr{C}^2_{\lambda\mu\widehat{\mathsf{tp}}_0}[\![t]\!]$$

Fig. B 12. Call-by-value double CPS transform of the $\lambda\mu\widehat{\mathsf{tp}}_0$-calculus.

$\lambda\widehat{\mu}_0$-calculus:

$$c^2 \in Command^2 ::= [q^2]c$$
$$c \in Command ::= [q]t \mid [\widehat{\alpha}]_0\Delta.t \mid [\Delta]c$$
$$t \in Term ::= V \mid t_1\ t_2 \mid \mu\alpha.c \mid \mu_0\widehat{\alpha}.c \qquad q^2 \in CoTerm^2 ::= \bullet$$
$$V \in Value ::= x \mid \lambda x.t \qquad q \in CoTerm ::= \alpha \mid *$$

Fig. B 13. The syntax of the $\lambda\widehat{\mu}_0$-calculus.

$$(\lambda x.t)\ V \to t\{V/x\}$$
$$F[\mu\alpha.c] \to \mu\alpha.c\{[\alpha](F[t])/[\alpha]t\}$$
$$[q]\mu\alpha.c \to c\{q/\alpha\}$$
$$\mu_0\widehat{\alpha}.[\widehat{\alpha}]_0\Delta.t \to t\{c/[\Delta]c\}$$
$$\mu_0\widehat{\alpha}.[\widehat{\beta}]_0\Delta.t \to \mu\alpha.[\widehat{\beta}]_0\Delta.t\{[\Delta][\alpha](\mu_0\widehat{\alpha}.c)/[\Delta]c\}$$
$$\textbf{where } \widehat{\alpha} \neq \widehat{\beta}$$
$$\mu_0\widehat{\alpha}.[*]V \to \mu\_.[*]V$$

Fig. B 14. Call-by-value reduction theory of the $\lambda\widehat{\mu}_0$-calculus.

$$\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![x]\!]^V = x \qquad \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![\alpha]\!] = \alpha \qquad \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![\bullet]\!] = \gamma_0$$

$$\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![\lambda x.t]\!]^V = \lambda x.\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![t]\!] \qquad \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![*]\!] = \lambda x.\lambda\gamma.x \qquad \textbf{where}\ \gamma_0 \text{ free}$$

$$\begin{aligned}
\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![V]\!] &= \lambda k.k\ \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![V]\!]^V \\
\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![t_1\ t_2]\!] &= \lambda k.\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![t_1]\!]\lambda f.\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![t_2]\!]\lambda s.f\ s\ k \\
\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![\mu\alpha.c]\!] &= \lambda k.(\lambda\alpha.\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![c]\!])\ k \\
\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![\mu_0\widehat{\alpha}.c]\!] &= \lambda k.\lambda\gamma.\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![c]\!]\ (\gamma[\widehat{\alpha}\mapsto k])
\end{aligned}$$

$$\begin{aligned}
\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![[q^2]c]\!] &= \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![c]\!]\ \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![q^2]\!] \\
\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![[q]t]\!] &= \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![t]\!]\ \mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![q]\!] \\
\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![[\widehat{\alpha}]_0\Delta.t]\!] &= \lambda\gamma.\gamma(\widehat{\alpha})\ (\lambda\Delta.\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![t]\!]) \\
\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![[\Delta]c]\!] &= \lambda\gamma.\mathscr{D}_{\widehat{\lambda}_0}\widehat{\mathscr{C}}_{\lambda\widehat{\mu}_0}[\![c]\!]\ (\Delta\ \gamma)
\end{aligned}$$

$$\begin{aligned}
\gamma(\widehat{\alpha}) &= \gamma\ulcorner\widehat{\alpha}\urcorner \\
\gamma[\widehat{\alpha}\mapsto k] &= \lambda p.\,\textbf{if}\ p\equiv\ulcorner\widehat{\alpha}\urcorner \\
&\qquad\qquad \textbf{then}\ (\lambda g.g\ (\lambda\gamma'.\gamma')\ k\ \gamma) \\
&\qquad\qquad \textbf{else}\ (\lambda g.\gamma\ p\ (\lambda\Delta.g\ (\lambda\gamma'.(\Delta\ \gamma')[\widehat{\alpha}\mapsto k])))
\end{aligned}$$

Fig. B 15. Call-by-value composed CPS transform of the $\widehat{\lambda}_0$-calculus.

B.2 Languages of Dynamic Binding

In the following, we summarize the syntax and semantics, in terms of environment-passing style transforms, of the intermediate languages of dynamic binding.

$\lambda\widehat{\mathsf{tp}}$-calculus:

$$\begin{aligned}
&c\in Closure ::= [e]t \\
t\in Term ::= V \mid t_1\ t_2 \mid \widehat{\mathsf{tp}}\ t \qquad & e\in Environment ::= \bullet \\
V\in Value ::= x \mid \lambda\widetilde{x}.t \qquad & \widetilde{x}\in Var ::= x \mid \widehat{\mathsf{tp}}
\end{aligned}$$

Fig. B 16. The syntax of the $\lambda\widehat{\mathsf{tp}}$-calculus.

$$\mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![[e]t]\!] = \mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![t]\!]\ \mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![e]\!] \qquad \mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![\bullet]\!] = \gamma_0 \quad \textbf{where } \gamma_0 \text{ free}$$

$$\mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![V]\!] = \lambda\gamma.\mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![V]\!]^V \qquad \mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![x]\!]^V = x$$
$$\mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![t_1\ t_2]\!] = \lambda\gamma.(\mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![t_1]\!]\gamma)\ (\mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![t_2]\!]\gamma)\ \gamma \qquad \mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![\lambda x.t]\!]^V = \lambda x.\mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![t]\!]$$
$$\mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![\widehat{\mathsf{tp}}\ t]\!] = \lambda\gamma.\gamma(\widehat{\mathsf{tp}})\ (\mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![t]\!]\gamma) \qquad \mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![\lambda\widehat{\mathsf{tp}}.t]\!]^V = \lambda v.\lambda\gamma.\mathscr{D}_{\lambda\widehat{\mathsf{tp}}}[\![t]\!]\ (\gamma[\widehat{\mathsf{tp}} \mapsto v])$$

$$\gamma(\widehat{\mathsf{tp}}) = \gamma \qquad \gamma[\widehat{\mathsf{tp}} \mapsto v] = \lambda x.v\ x\ \gamma$$

Fig. B 17. Environment-passing style transform of the $\lambda\widehat{\mathsf{tp}}$-calculus.

$\widehat{\lambda}$-calculus:

$$c \in Closure ::= [e]t$$
$$t \in Term ::= V \mid t_1\ t_2 \mid \widehat{\mathsf{tp}}\ t \qquad e \in Environment ::= \bullet$$
$$V \in Value ::= x \mid \lambda\widetilde{x}.t \qquad \widetilde{x} \in Var ::= x \mid \widehat{x}$$

Fig. B 18. The syntax of the $\widehat{\lambda}$-calculus.

$$\mathscr{D}_{\widehat{\lambda}}[\![[e]t]\!] = \mathscr{D}_{\widehat{\lambda}}[\![t]\!]\ \mathscr{D}_{\widehat{\lambda}}[\![e]\!] \qquad \mathscr{D}_{\widehat{\lambda}}[\![\bullet]\!] = \gamma_0 \quad \textbf{where } \gamma_0 \text{ free}$$

$$\mathscr{D}_{\widehat{\lambda}}[\![V]\!] = \lambda\gamma.\mathscr{D}_{\widehat{\lambda}}[\![V]\!]^V \qquad \mathscr{D}_{\widehat{\lambda}}[\![x]\!]^V = x$$
$$\mathscr{D}_{\widehat{\lambda}}[\![t_1\ t_2]\!] = \lambda\gamma.(\mathscr{D}_{\widehat{\lambda}}[\![t_1]\!]\gamma)\ (\mathscr{D}_{\widehat{\lambda}}[\![t_2]\!]\gamma)\ \gamma \qquad \mathscr{D}_{\widehat{\lambda}}[\![\lambda x.t]\!]^V = \lambda x.\mathscr{D}_{\widehat{\lambda}}[\![t]\!]$$
$$\mathscr{D}_{\widehat{\lambda}}[\![\widehat{\mathsf{tp}}\ t]\!] = \lambda\gamma.\gamma(\widehat{\mathsf{tp}})\ (\mathscr{D}_{\widehat{\lambda}}[\![t]\!]\gamma) \qquad \mathscr{D}_{\widehat{\lambda}}[\![\lambda\widehat{\mathsf{tp}}.t]\!]^V = \lambda v.\lambda\gamma.\mathscr{D}_{\widehat{\lambda}}[\![t]\!]\ (\gamma[\widehat{\mathsf{tp}} \mapsto v])$$

$$\gamma(\widehat{x}) = \gamma\ \ulcorner\widehat{x}\urcorner \qquad \gamma[\widehat{x} \mapsto v] = \lambda p.\,\textbf{if}\ p \equiv \ulcorner\widehat{x}\urcorner\ \textbf{then}\ (\lambda x.v\ x\ \gamma)\ \textbf{else}\ \gamma\ p$$

Fig. B 19. Environment-passing style transform of the $\widehat{\lambda}$-calculus.

$\lambda\widehat{\mathsf{tp}}_0$-calculus:

$$c \in Closure ::= [e]t$$
$$t \in Term ::= V \mid t_1\ t_2 \mid t\ \widehat{\mathsf{tp}} \qquad e \in Environment ::= \bullet$$
$$V \in Value ::= x \mid \lambda\widetilde{x}.t \qquad \widetilde{x} \in Var ::= x \mid \widehat{\mathsf{tp}}$$

Fig. B 20. The syntax of the $\lambda\widehat{\mathsf{tp}}_0$-calculus.

$$\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![[e]t]\!] = \mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![t]\!]\ \mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![e]\!] \qquad \mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![\bullet]\!] = \gamma_0 \quad \textbf{where}\ \gamma_0 \text{ free}$$

$$\begin{aligned}
\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![V]\!] &= \lambda\gamma.\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![V]\!]^V \\
\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![t_1\ t_2]\!] &= \lambda\gamma.(\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![t_1]\!]\gamma)\ (\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![t_2]\!]\gamma)\ \gamma \\
\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![t\ \widehat{\mathsf{tp}}]\!] &= \lambda\gamma.\gamma(\widehat{\mathsf{tp}})\ (\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![t]\!]\gamma)
\end{aligned}
\qquad
\begin{aligned}
\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![x]\!]^V &= x \\
\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![\lambda x.t]\!]^V &= \lambda x.\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![t]\!] \\
\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![\lambda\widehat{\mathsf{tp}}.t]\!]^V &= \lambda v.\lambda\gamma.\mathscr{D}_{\lambda\widehat{\mathsf{tp}}_0}[\![t]\!]\ (\gamma[\widehat{\mathsf{tp}} \mapsto v])
\end{aligned}$$

$$\gamma(\widehat{\mathsf{tp}}) = \gamma \qquad \gamma[\widehat{\mathsf{tp}} \mapsto v] = \lambda g.g\ v\ \gamma$$

Fig. B 21. Environment-passing style transform of the $\lambda\widehat{\mathsf{tp}}$-calculus.

$\widehat{\lambda}_0$-calculus:

$$\begin{aligned}
&c \in Closure ::= [e]t \\
&t \in Term ::= V \mid t_1\ t_2 \mid t\ \widehat{x} \mid [\Delta]t \qquad e \in Environment ::= \bullet \\
&V \in Value ::= x \mid \lambda\widetilde{x}.t \mid \lambda\langle\Delta, x\rangle.t \qquad \widetilde{x} \in Var ::= x \mid \widehat{x}
\end{aligned}$$

Fig. B 22. The syntax of the $\widehat{\lambda}_0$-calculus.

$$\mathscr{D}_{\widehat{\lambda}_0}[\![[e]t]\!] = \mathscr{D}_{\widehat{\lambda}_0}[\![t]\!]\ \mathscr{D}_{\widehat{\lambda}_0}[\![e]\!] \qquad \mathscr{D}_{\widehat{\lambda}_0}[\![\bullet]\!] = \gamma_0 \quad \textbf{where}\ \gamma_0 \text{ free}$$

$$\begin{aligned}
\mathscr{D}_{\widehat{\lambda}_0}[\![V]\!] &= \lambda\gamma.\mathscr{D}_{\widehat{\lambda}_0}[\![V]\!]^V \\
\mathscr{D}_{\widehat{\lambda}_0}[\![t_1\ t_2]\!] &= \lambda\gamma.(\mathscr{D}_{\widehat{\lambda}_0}[\![t_1]\!]\gamma)\ (\mathscr{D}_{\widehat{\lambda}_0}[\![t_2]\!]\gamma)\ \gamma \\
\mathscr{D}_{\widehat{\lambda}_0}[\![t\ \widehat{x}]\!] &= \lambda\gamma.\gamma(\widehat{x})\ (\mathscr{D}_{\widehat{\lambda}_0}[\![t]\!]\gamma) \\
\mathscr{D}_{\widehat{\lambda}_0}[\![[\Delta]t]\!] &= \lambda\gamma.\mathscr{D}_{\widehat{\lambda}_0}[\![t]\!]\ (\Delta\ \gamma)
\end{aligned}
\qquad
\begin{aligned}
\mathscr{D}_{\widehat{\lambda}_0}[\![x]\!]^V &= x \\
\mathscr{D}_{\widehat{\lambda}_0}[\![\lambda x.t]\!]^V &= \lambda x.\mathscr{D}_{\widehat{\lambda}_0}[\![t]\!] \\
\mathscr{D}_{\widehat{\lambda}_0}[\![\lambda\widehat{x}.t]\!]^V &= \lambda v.\lambda\gamma.\mathscr{D}_{\widehat{\lambda}_0}[\![t]\!]\ (\gamma[\widehat{x} \mapsto v])
\end{aligned}$$

$$\begin{aligned}
\gamma(\widehat{x}) &= \gamma\ \ulcorner\widehat{x}\urcorner \\
\gamma[\widehat{x} \mapsto v] &= \lambda p.\,\textbf{if}\ p \equiv \ulcorner\widehat{x}\urcorner \\
&\qquad \textbf{then}\ (\lambda g.g\ (\lambda\gamma'.\gamma')\ v\ \gamma) \\
&\qquad \textbf{else}\ (\lambda g.\gamma\ p\ (\lambda\Delta.g\ (\lambda\gamma'.(\Delta\ \gamma')[\widehat{x} \mapsto v])))
\end{aligned}$$

Fig. B 23. Environment-passing style transform of the $\widehat{\lambda}_0$-calculus.