Appendices of *Consistency of the Theory of Contexts* by Bucalo, Hofmann, Honsell, Miculan, Scagnetto

A Category-theoretic preliminaries

As one of the aims of this paper is to present categorical methods to non-categorically minded readers, in the following we briefly review some standard notions and important results we will need. This also allows us to fix notation and to give more complete references to the involved topics.

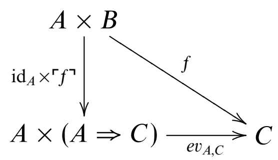
Let us start with some basic notation: in the following we will write $X \in \mathscr{C}$ to mean that X is an object of the category $\mathscr{C}$ and we will denote with $\mathscr{C}(X, Y)$ the family of arrows in $\mathscr{C}$ from X to Y.

We will assume fixed a universe of sets, whose elements are called *small sets*. A category $\mathscr{C}$ is *locally small* if for all $X, Y \in \mathscr{C}$, the family $\mathscr{C}(X, Y)$ is a small set, and *small* if, moreover, the class of objects is a small set. In the following, we will refer to small sets simply as sets.

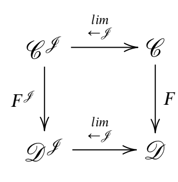
Next, we will present some basic results about functor categories, so it is useful a quick review on some standard notions.

Definition A.1

A category $\mathscr{C}$ with terminal object and binary products is *cartesian closed* if for every $A, C \in \mathscr{C}$ there is an object $A \Rightarrow C$ and a morphism $ev_{A,C} : A \times (A \Rightarrow C) \longrightarrow C$ such that for each morphism $f : A \times B \longrightarrow C$ there is a unique morphism $\ulcorner f \urcorner : B \longrightarrow A \Rightarrow C$, the exponential transpose of f, such that the following diagram commutes:



A functor $F : \mathscr{C} \to \mathscr{D}$ is said to be *faithful* if, for all $A, B \in \mathscr{C}$, F is injective on $\mathscr{C}(A, B)$, it is said to be *full* if for each $A, B \in \mathscr{C}$, F carries $\mathscr{C}(A, B)$ onto $\mathscr{D}(F(A), F(B))$. Finally it is an *embedding* if it is injective on objects and faithful. given a small index category $\mathscr{I}$, F induces a functor $F^{\mathscr{I}} : \mathscr{C}^{\mathscr{I}} \to \mathscr{D}^{\mathscr{I}}$ such that, if limits exist both in $\mathscr{C}$ and in $\mathscr{D}$, we have the following diagram:



where $\overset{lim}{\leftarrow_{\mathscr{I}}}$ is the limit functor. By the universal property of limits, we can infer the existence of a natural transformation $\alpha : F \circ \overset{lim}{\leftarrow_{\mathscr{I}}} \longrightarrow \overset{lim}{\leftarrow_{\mathscr{I}}} \circ F^{\mathscr{I}}$. If α is a natural isomorphism, then F is said to preserve limits. In this case, it will in particular preserve cartesian products.

In order to improve the readability of formulas and diagrams, we may denote the application of functors in three different ways: for instance, for $F : \mathscr{C} \longrightarrow \mathscr{D}$ and A object of $\mathscr{C}$, the notations "FA", "$F(A)$" and "F_A" are equivalent.

Let $\mathscr{S}et$ be the category whose objects are sets and whose morphisms are functions between sets. Given a locally small category $\mathscr{C}$, we will denote with $\check{\mathscr{C}}$ the category $\mathscr{S}et^{\mathscr{C}}$ whose objects are the functors from $\mathscr{C}$ to $\mathscr{S}et$ and whose morphisms are natural transformations between them. More precisely:

- an object A of $\check{\mathscr{C}}$ consists of a family of sets $\{A_X\}_{X\in\mathscr{C}}$, together with a family of functions $\{A_f\}_{f\in\mathscr{C}(X,Y),\ X,Y\in\mathscr{C}}$ such that $A_f : A_X \longrightarrow A_Y$, $A_{\mathrm{id}_X} = \mathrm{id}_{A_X}$ and $A_{f\circ g} = A_f \circ A_g$;
- a morphism $m \in \check{\mathscr{C}}(A,B)$ is a family of functions $\{m_X\}_{X\in\mathscr{C}}$, such that $m_X : A_X \longrightarrow B_X$ and for each $f : X \longrightarrow Y$, $m_Y \circ A_f = B_f \circ m_X$.

If $\mathscr{C}$ is small, it is known that the category $\check{\mathscr{C}}$ is cartesian closed with finite products given by

$$\mathbf{1}_X \triangleq \{\star\} \text{ and } \mathbf{1}_f \triangleq \mathrm{id}_{\{\star\}} \text{ (empty product)}$$
$$(A \times B)_X \triangleq A_X \times B_X \text{ and } (A \times B)_f \triangleq A_f \times B_f\,,$$

moreover $(A \Rightarrow B)$ is given by

$$(A \Rightarrow B)_X \triangleq \check{\mathscr{C}}(A \times \mathscr{C}(X,\_), B)$$
$$(A \Rightarrow B)_f(m) \triangleq m \circ (\mathrm{id}_A \times (\_ \circ f)),\ \text{for } f : Y \longrightarrow Z \text{ and } m \in \check{\mathscr{C}}(A \times \mathscr{C}(Y,\_), B)$$

and finally $ev_{A,C}$ and $\ulcorner f \urcorner : B \to A \Rightarrow C$ are given by

$$(ev_{A,C})_X(a,m) \triangleq m_X(a, \mathrm{id}_X),\ \text{for all } X \in \mathscr{C}, a \in A_X,\ \text{and } m \in (A \Rightarrow C)_X$$
$$(\ulcorner f \urcorner_X(b))_Y : A_Y \times \mathscr{V}(X,Y) \longrightarrow C_Y$$
$$(\ulcorner f \urcorner_X(b))_Y(a,h) \triangleq f_Y(a, B_h(b))$$

Let us consider the functor $\check{\mathscr{Y}} : \mathscr{C}^{op} \longrightarrow \check{\mathscr{C}}$, defined as follows:

- for $X \in \mathscr{C}$, $\check{\mathscr{Y}}(X) : \mathscr{C} \to \mathscr{S}et$ is the Homset functor $\mathscr{C}(X,\_)$, i.e.: $\check{\mathscr{Y}}(X)_Z \triangleq \mathscr{C}(X,Z)$ and, given $f : Y \longrightarrow Z$, for all $g \in \mathscr{C}(X,Y)$, $\check{\mathscr{Y}}(X)_f(g) \triangleq f \circ g$;
- for $f : X \to Y$, $\check{\mathscr{Y}}(f) : \check{\mathscr{Y}}(X) \to \check{\mathscr{Y}}(Y)$ is the natural transformation such that, for all $Z \in \mathscr{C}$ and $g \in \mathscr{C}(Y,Z)$, $(\check{\mathscr{Y}}(f))_X(g) \triangleq g \circ f$.

Then, the following fundamental lemma holds:

Proposition A.1 (*Yoneda Lemma*)
For each $A \in \check{\mathscr{C}}$ and $X \in \mathscr{C}$ there is a bijective correspondence between $\check{\mathscr{C}}(\check{\mathscr{Y}}(X), A)$ and A_X, and moreover the correspondence is natural in A and X.

We give the definition of this bijective correspondence between $\check{\mathscr{C}}(\check{\mathscr{Y}}(X), A)$ and A_X: $\Phi_{X,A}(m) = m_X(\mathrm{id}_X)$, for $m \in \check{\mathscr{C}}(\check{\mathscr{Y}}(X), A)$; the inverse is the natural transformation defined on $a \in A_X$ by $(\Phi^{-1}_{X,A}(a))_Z(f) \triangleq A_f(a)$, for $f \in \check{\mathscr{Y}}(X)_Z$.

An immediate and important consequence of the previous result is that the category $\mathscr{C}^{op}$ fully embeds in $\check{\mathscr{C}}$ by means of $\check{\mathscr{Y}}$, which is called, therefore, *Yoneda embedding*.

When an object in $\check{\mathscr{C}}$ is isomorphic to an object in the image of $\check{\mathscr{Y}}$ it is said to be *representable*. Notice, for example, that, if $\mathscr{C}$ has an initial object $\mathbf{0}$, then the terminal object $\mathbf{1}$ is representable since $\mathbf{1} \cong \check{\mathscr{Y}}(\mathbf{0})$.

Another useful notion to recall is the concept of adjunction; for our purposes the following definition suffices.

Definition A.2
Given categories $\mathscr{C}$ and $\mathscr{D}$, an *adjunction* from $\mathscr{C}$ to $\mathscr{D}$ is a triple (F, G, ϕ), where F, G are functors, $F : \mathscr{C} \longrightarrow \mathscr{D}$, $G : \mathscr{D} \longrightarrow \mathscr{C}$ and ϕ is a function which maps every $A \in \mathscr{C}$ and $B \in \mathscr{D}$ to a bijection $\phi_{A,B} : \mathscr{C}(A, G_B) \cong \mathscr{D}(F_A, B)$, natural in A and B.

F and G are respectively called the left and the right adjoint of the adjunction and this is denoted by $F \dashv G$ or $G \vdash F$.

We will use the known property that a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ with a right (left) adjoint preserves colimits (limits). For the proof see, e.g., (Mac Lane, 1971). Theorem 1.27.

Now we introduce some notions and a result about algebras of functors.

Definition A.3
Given a functor $T : \mathscr{C} \longrightarrow \mathscr{C}$, a T-*algebra* is a pair $\langle A, \alpha\rangle$, with $A \in \mathscr{C}$ and $\phi : TA \longrightarrow A$ morphism of $\mathscr{C}$. A T-algebra morphism from $\langle A, \alpha\rangle$ to $\langle B, \beta\rangle$ is an arrow $f \in C(A, B)$ such that the following diagram commutes:

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{f} & B \end{array}$$

T-algebras and T-algebra morphisms form a category, whose initial object, if it exists, is said an initial T-algebra.

Theorem A.1 ((Hofmann, 1999))
Let $\mathscr{C}, \mathscr{D}$ be two categories and $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor with a right adjoint F^*. Let $T : \mathscr{C} \longrightarrow \mathscr{C}$ and $T' : \mathscr{D} \longrightarrow \mathscr{D}$ be two functors such that $T' \circ F \cong F \circ T$ for some natural isomorphism $\phi : T' \circ F \longrightarrow F \circ T$. If $(A, a : TA \to A)$ is an initial T-algebra in $\mathscr{C}$, then $(F_A, F_a \circ \phi_A : T'(F_A) \to F_A)$ is an initial T'-algebra in $\mathscr{D}$.

Proof
The adjoint pair $F \dashv F^*$ can be lifted to a pair of adjoint functors between the categories of T- and T'- algebras. Since any functor with a right adjoint preserves colimits and the initial object is a colimit, then the initial object of the former category is preserved in the latter. □

Another useful technique for building initial algebras is based on the notions of *simple slice category* and *strong functor*. We recall here the basic definitions and related properties from (Jacobs, 1995).

Definition A.4
Given a category $\mathscr{C}$ with binary products and $G \in \mathscr{C}$, the *simple slice category* $\mathscr{C}/\!\!/G$ is defined as follows:

1. $Obj(\mathscr{C}/\!\!/G) \triangleq Obj(\mathscr{C})$,
2. $\mathscr{C}/\!\!/G(A,B) \triangleq \mathscr{C}(G \times A, B)$,
3. the identity map on A in $\mathscr{C}/\!\!/G$ is the second projection $\pi' : G \times A \longrightarrow A$ in $\mathscr{C}$,
4. the composition of $f : A \longrightarrow B$ and $g : B \longrightarrow C$ is defined as follows:

$$g \bullet f \triangleq g \circ \langle \pi, f \rangle : G \times A \longrightarrow G \times B \longrightarrow C,$$

where $\bullet$ denotes the composition in $\mathscr{C}/\!\!/G$ and $\circ$ the composition in $\mathscr{C}$.

Given $G \in \mathscr{C}$, there is a functor $G^* : \mathscr{C} \longrightarrow \mathscr{C}/\!\!/G$ defined as follows:

1. $G^*(A) \triangleq A$ for every $A \in \mathscr{C}$,
2. $G^*(f) \triangleq f \circ \pi'$ for every $f \in \mathscr{C}(A,B)$.

Definition A.5 (2.6.7)
An endofunctor $T : \mathscr{C} \longrightarrow \mathscr{C}$ on a category $\mathscr{C}$ with finite products is called *strong* if it comes equipped with a natural transformation, called *strength*, with components $st_{A,B} : A \times TB \longrightarrow T(A \times B)$ making the following two diagrams commute:

$$\begin{array}{ccc} A \times TB & \xrightarrow{st} & T(A \times B) \\ & \searrow^{\pi'} & \downarrow{\scriptstyle T\pi'} \\ & & TB \end{array} \qquad \begin{array}{ccccc} A \times (C \times TB) & \xrightarrow{id \times st} & A \times T(C \times B) & \xrightarrow{st} & T(A \times (C \times B)) \\ {\scriptstyle \beta}\downarrow & & & & \downarrow{\scriptstyle T\beta} \\ (A \times C) \times TB & & \xrightarrow{st} & & T((A \times C) \times B) \end{array}$$

where β is the obvious isomorphism $\langle \langle \pi, \pi \circ \pi' \rangle, \pi' \circ \pi' \rangle$.

As proved in (Jacobs, 1995), if T is a strong functor, we can define, for every $A \in \mathscr{C}$, a functor $T/\!\!/A : \mathscr{C}/\!\!/A \longrightarrow \mathscr{C}/\!\!/A$ as follows:

- $(T/\!\!/A)_B \triangleq TB$,
- $(T/\!\!/A)_f \triangleq Tf \circ st_{A,B}$ (for every $f \in \mathscr{C}/\!\!/A(B,C)$).

It turns out that also this new functor is strong.

B Proofs

B.1 Proof of Proposition 3.2

For $U, V \in \mathbf{Pred}_{\mathscr{J}}(F)$, we put

$$(U \vee V)_X \triangleq U_X \cup V_X \qquad (U \wedge V)_X \triangleq U_X \cap V_X$$
$$(\overline{U})_X \triangleq F_X \setminus U_X \qquad 0_X \triangleq \emptyset \qquad 1_X \triangleq F_X.$$

Now we prove that these objects are indeed predicates, by checking the three conditions of Definition 3.1:

$(U \vee V) \in \mathbf{Pred}_{\mathscr{I}}(F)$:

Sub Since, by hypothesis, $U, V \in \mathbf{Pred}(F)$, it follows that $U_X \subseteq F_X$ and $V_X \subseteq F_X$ for $X \in \mathscr{I}$; then $(U \vee V)_X \triangleq U_X \cup V_X \subseteq F_X$;

Func given $h \in \mathscr{I}(X, Y)$ and $t \in (U \vee V)_X$, we can infer that either $t \in U_X$ or $t \in V_X$ (since $(U \vee V)_X \triangleq U_X \cup V_X$); in the former case we have that $F_h(t) \in U_Y$ by hypothesis, hence $F_h(t) \in U_Y \cup V_Y \triangleq (U \vee V)_Y$ (in the latter case we can conclude by a similar argument);

Closure given $t \in F_X$ and $F_h(t) \in (U \vee V)_Y$ for some $h \in \mathscr{I}(X, Y)$, we can infer that either $F_h(t) \in U_Y$ or $F_h(t) \in V_Y$ (since $(U \vee V)_Y \triangleq U_Y \cup V_Y$); in the former case we can conclude that $t \in U_X$, hence $t \in U_X \cup V_X \triangleq (U \vee V)_X$ (in the latter case we can conclude by a similar argument).

$(U \wedge V) \in \mathbf{Pred}_{\mathscr{I}}(F)$:

Sub Since, by hypothesis, $U, V \in \mathbf{Pred}(F)$, it follows that $U_X \subseteq F_X$ and $V_X \subseteq F_X$ for $X \in \mathscr{I}$; then $(U \wedge V)_X \triangleq U_X \cap V_X \subseteq F_X$;

Func given $h \in \mathscr{I}(X, Y)$ and $t \in (U \wedge V)_X$, we can infer that $t \in U_X$ and $t \in V_X$ (since $(U \wedge V)_X \triangleq U_X \cap V_X$); then, by hypothesis, $F_h(t) \in U_Y$ and $F_h(t) \in V_Y$, hence we can conclude that $F_h(t) \in (U_Y \cap V_Y) \triangleq (U \wedge V)_Y$;

Closure given $t \in F_X$ and $F_h(t) \in (U \wedge V)_Y$ for some $h \in \mathscr{I}(X, Y)$, we can infer that $F_h(t) \in U_Y$ and $F_h(t) \in V_Y$ (since $(U \wedge V)_Y \triangleq U_Y \cap V_Y$); then, by hypothesis, we have that $t \in U_X$ and $t \in V_X$, hence we can conclude $t \in U_X \cap V_X \triangleq (U \wedge V)_X$.

$\overline{U} \in \mathbf{Pred}_{\mathscr{I}}(F)$:

Sub Condition (Sub) trivially holds by definition of $(\overline{U})_X$;

Func given $h \in \mathscr{I}(X, Y)$ and $t \in (\overline{U})_X$, by definition of $\overline{U}$ we have that $t \in F_X$ and $t \notin U_X$; then, as $U \in \mathbf{Pred}(F)$, we can apply condition (Closure) to conclude that $F_h(t) \notin U_Y$, hence $F_h(t) \in (\overline{U})_Y$;

Closure given $t \in F_X$ and $F_h(t) \in (\overline{U})_Y$ for some $h \in \mathscr{I}(X, Y)$, we can infer that $F_h(t) \in F_Y$ and $F_h(t) \notin U_Y$ (by definition of $\overline{U}$); then, as $U \in \mathbf{Pred}(F)$, we can apply condition (Closure) to conclude that $t \notin U_X$, hence $t \in (\overline{U})_X$.

$0 \in \mathbf{Pred}_{\mathscr{I}}(F)$:

Sub We trivially have $0_X \triangleq \emptyset \subseteq F_X$ for $X \in \mathscr{I}$;

Func this condition trivially holds since the premise $t \in 0_X \triangleq \emptyset$ is false;

Closure similarly to the previous case this condition is also trivially verified, since the premise $F_h(t) \in 0_Y \triangleq \emptyset$ cannot be fulfilled.

$1 \in \mathbf{Pred}_{\mathscr{I}}(F)$:

Sub We trivially have $1_X \triangleq F_X \subseteq F_X$ for $X \in \mathscr{I}$;

Func given $h \in \mathscr{I}(X, Y)$ and $t \in 1_X \triangleq F_X$, we trivially have $F_h(t) \in F_Y$ by functoriality of F, hence we can immediately conclude since $1_Y \triangleq F_Y$;

Closure given $t \in F_X$ and $F_h(t) \in 1_Y \triangleq F_Y$ for some $h \in \mathscr{I}(X, Y)$, we have by hypothesis that $t \in F_X$, hence we can immediately conclude since $1_X \triangleq F_X$.

One can easily check that $\mathbf{Pred}_{\check{\mathscr{I}}}(F)$ endowed with these operations can indeed be turned into a complemented distributive lattice.

B.2 Proof of Proposition 3.3

Given $\eta : F \longrightarrow G$, $U \in \mathbf{Pred}_{\check{\mathscr{I}}}(G)$ and $X \in \mathscr{I}$, we have that $(\mathbf{Pred}_{\check{\mathscr{I}}}(\eta)(U))_X \triangleq \eta_X^{-1}(U_X)$, hence

$$\chi_F^{\check{\mathscr{I}}}(\mathbf{Pred}_{\check{\mathscr{I}}}(\eta)(U))_X \triangleq \lambda t \in F_X.(\{f : X \to Y \mid F_f(t) \in (\mathbf{Pred}_{\check{\mathscr{I}}}(\eta)(U))_Y\}\}_{Y \in \mathscr{I}}.$$

On the other hand, we have $(\chi_G^{\check{\mathscr{I}}}(U))_X \triangleq \lambda t \in G_X.(\{f : X \to Y \mid G_f(t) \in U_Y\})_{Y \in \mathscr{I}}$, hence

$$\begin{aligned} (\check{\mathscr{I}}(\eta, \Omega)(\chi_G^{\check{\mathscr{I}}}(U)))_X &= (\chi_G^{\check{\mathscr{I}}}(U) \circ \eta)_X \\ &= (\chi_G^{\check{\mathscr{I}}}(U))_X \circ \eta_X \\ &\triangleq \lambda t \in F_X.(\{f : X \to Y \mid G_f(\eta_X(t)) \in U_Y\})_{Y \in \mathscr{I}}, \end{aligned}$$

but, by naturality of η, we have that $G_f(\eta_X(t)) = \eta_Y(F_f(t))$, hence $G_f(\eta_X(t)) \in U_Y$ if and only if $F_f(t) \in \eta_Y^{-1}(U_Y) \triangleq (\mathbf{Pred}_{\check{\mathscr{I}}}(\eta)(U))_Y$, i.e.,

$$\chi_F^{\check{\mathscr{I}}}(\mathbf{Pred}_{\check{\mathscr{I}}}(\eta)(U))_X = (\check{\mathscr{I}}(\eta, \Omega)(\chi_G^{\check{\mathscr{I}}}(U)))_X.$$

Thus, naturality of $\chi^{\check{\mathscr{I}}}$ is proved. Now, it remains to show that $\chi^{\check{\mathscr{I}}}$ is a natural isomorphism, i.e., that $\chi_F^{\check{\mathscr{I}}}$ has an inverse for each $F \in \check{\mathscr{V}}$. We will prove that this inverse is indeed $\kappa_F^{\check{\mathscr{I}}}$. First let us verify that $\kappa_F^{\check{\mathscr{I}}}(\chi_F^{\check{\mathscr{I}}}(V)) = V$ for $V \in \mathbf{Pred}_{\check{\mathscr{I}}}(F)$ (i.e. $\kappa_F^{\check{\mathscr{I}}} \circ \chi_F^{\check{\mathscr{I}}} = \mathrm{id}_{\mathbf{Pred}_{\check{\mathscr{I}}}(F)}$):

$$\begin{aligned} \kappa_F^{\check{\mathscr{I}}}(\chi_F^{\check{\mathscr{I}}}(V)) &\triangleq (\{t \in F_X \mid (\chi_F^{\check{\mathscr{I}}}(V))_X(f) = \check{\mathscr{Y}}_{\check{\mathscr{I}}}(X)\})_{X \in \mathscr{I}} \\ &\triangleq \left(\{t \in F_X \mid (\{g : X \to Y \mid F_g(t) \in V_Y\})_{Y \in \mathscr{I}} = \mathscr{I}(X, \_)\}\right)_{X \in \mathscr{I}} \\ &= (V_X)_{X \in \mathscr{I}} \text{ (because of property 2 of predicates)} \\ &\triangleq V \end{aligned}$$

Now we have to prove that $\chi_F^{\check{\mathscr{I}}}(\kappa_F^{\check{\mathscr{I}}}(m)) = m$ (i.e. $\chi_F^{\check{\mathscr{I}}} \circ \kappa_F^{\check{\mathscr{I}}} = \mathrm{id}_{\check{\mathscr{V}}(F,\Omega)}$):

$$\chi_F^{\check{\mathscr{I}}}(\kappa_F^{\check{\mathscr{I}}}(m)) \triangleq (\lambda t \in F_X.\{f : X \to Y \mid F_f(t) \in (\kappa_F^{\check{\mathscr{I}}}(m))_Y\}_{Y \in \mathscr{I}})_{X \in \mathscr{I}}$$

By definition of $(\kappa_F^{\check{\mathscr{I}}}(m))_Y$, we have that $F_f(t) \in (\kappa_F^{\check{\mathscr{I}}}(m))_Y$ if and only if $m_Y(F_f(t)) = \mathscr{I}(Y, \_)$. By naturality of m, it follows that

$$m_Y(F_f(t)) = \Omega_f(m_X(t)) \triangleq \mathbf{Pred}_{\check{\mathscr{I}}}(\check{\mathscr{Y}}_{\check{\mathscr{I}}}(f))(m_X(t))\ .$$

Hence for any $Z \in \mathscr{I}$,

$$(\mathbf{Pred}_{\check{\mathscr{I}}}(\check{\mathscr{Y}}_{\check{\mathscr{I}}}(f))(m_X(t)))_Z \triangleq (\check{\mathscr{Y}}_{\check{\mathscr{I}}}(f))_Z^{-1}((m_X(t))_Z) = \mathscr{I}(Y, Z),$$

i.e., for all $g \in \mathscr{I}(Y, Z)$, $(\check{\mathscr{Y}}_{\check{\mathscr{I}}}(f))_Z(g) \in (m_X(t))_Z$ holds. Since $(\check{\mathscr{Y}}_{\check{\mathscr{I}}}(f))_Z(g) = g \circ f \triangleq \mathscr{I}(X, g)(f)$, we have, by properties (Func) and (Closure) of predicates (remember that $m_X(t) \in \mathbf{Pred}_{\check{\mathscr{I}}}(\mathscr{I}(X, \_))$), that $m_Y(F_f(t)) = \mathscr{I}(Y, \_)$ if and only if $f \in (m_X(t))_Y$ holds. Hence, we can conclude that

$$\chi_F^{\check{\mathscr{I}}}(\kappa_F^{\check{\mathscr{I}}}(m)) = (\lambda t \in F_X.\{f : X \to Y \mid f \in (m_X(t))_Y\}_{Y \in \mathscr{I}})_{X \in \mathscr{I}},$$

i.e., $\chi_F^{\check{\mathscr{I}}}(\kappa_F^{\check{\mathscr{I}}}(m)) = m$.

B.3 Proof of Theorem 4.1

1. First, notice that $[\![\Gamma \vdash_\Sigma \forall x^\sigma.p : o]\!]_X(\eta) =$

$$\begin{aligned}
&= (forall_\sigma)_X([\![\Gamma \vdash_\Sigma \lambda x^\sigma.p : \sigma \to o]\!]_X(\eta)) \\
&= \{u : X \longrightarrow Y \mid \forall g \in \mathscr{I}(Y,Z).\forall t \in [\![\sigma]\!]_Z. \\
&\qquad \langle t, g \circ u\rangle \in \kappa_{[\![\sigma]\!]\times\check{\mathscr{Y}}(X)}([\![\Gamma \vdash_\Sigma \lambda x^\sigma.p : \sigma \to o]\!]_X(\eta))_Z\}_{Y\in\mathscr{V}} \\
&= \{u : X \longrightarrow Y \mid \forall g \in \mathscr{I}(Y,Z).\forall t \in [\![\sigma]\!]_Z. \\
&\qquad ([\![\Gamma \vdash_\Sigma \lambda x^\sigma.p : \sigma \to o]\!]_X(\eta))_Z(\langle t, g \circ u\rangle) \geqslant \mathscr{I}(Z, _)\}_{Y\in\mathscr{V}} \\
&= \{u : X \longrightarrow Y \mid \forall g \in \mathscr{I}(Y,Z).\forall t \in [\![\sigma]\!]_Z.(\lambda\langle b, f\rangle \in [\![\sigma]\!]_Z \times \mathscr{I}(X,Z). \\
&\qquad [\![\Gamma, x : \sigma \vdash_\Sigma p : o]\!]_Z(\langle [\![\Gamma]\!]_f(\eta), b\rangle))(\langle t, g \circ u\rangle) \geqslant \mathscr{I}(Z, _)\}_{Y\in\mathscr{V}} \\
&= \{u : X \longrightarrow Y \mid \forall g \in \mathscr{I}(Y,Z).\forall t \in [\![\sigma]\!]_Z. \\
&\qquad [\![\Gamma, x : \sigma \vdash_\Sigma p : o]\!]_Z(\langle [\![\Gamma]\!]_{(g\circ u)}(\eta), t\rangle) \geqslant \mathscr{I}(Z, _)\}_{Y\in\mathscr{V}}
\end{aligned}$$

($\Rightarrow$) By hypothesis we have that $X \Vdash_{\Gamma,\eta} \forall x^\sigma.p$, i.e., $\eta \in \kappa_{[\![\Gamma]\!]}([\![\Gamma \vdash_\Sigma \forall x^\sigma.p : o]\!])_X$ which, in turn, is equivalent to $[\![\Gamma \vdash_\Sigma \forall x^\sigma.p : o]\!]_X(\eta) \geqslant \mathscr{I}(X, \_)$. In particular we have that $h \in \mathscr{I}(X, Y)$ belongs to $([\![\Gamma \vdash_\Sigma \forall x^\sigma.p : o]\!]_X(\eta))_Y$. Then, taking $g = \mathrm{id}_Y$ and $t = a$, we have that $[\![\Gamma, x : \sigma \vdash_\Sigma p : o]\!]_Y(\langle [\![\Gamma]\!]_{(\mathrm{id}_Y\circ h)}(\eta), a\rangle) = [\![\Gamma, x : \sigma \vdash_\Sigma p : o]\!]_Y(\langle [\![\Gamma]\!]_h(\eta), a\rangle) \geqslant \mathscr{I}(Y, \_)$, i.e., $Y \Vdash_{(\Gamma,x:\sigma),\ \langle[\![\Gamma]\!]_h(\eta),a\rangle} p$.

($\Leftarrow$) By hypothesis for all Y and $h \in \mathscr{I}(X, Y)$, and for all $a \in [\![\sigma]\!]_Y$ we have that $Y \Vdash_{(\Gamma,x:\sigma),\ \langle[\![\Gamma]\!]_h(\eta),a\rangle} p$, i.e., $[\![\Gamma, x : \sigma \vdash_\Sigma p : o]\!]_Y(\langle [\![\Gamma]\!]_h(\eta), a\rangle) \geqslant \mathscr{I}(Y, \_)$. Then, take any $u \in \mathscr{I}(X, Y)$, $g \in \mathscr{I}(Y,Z)$ and $t \in [\![\sigma]\!]_Z$; it follows that $h = g \circ u \in \mathscr{I}(X,Z)$. Hence, we can apply the hypothesis and conclude that

$$[\![\Gamma, x : \sigma \vdash_\Sigma p : o]\!]_Z(\langle [\![\Gamma]\!]_{(g\circ u)}(\eta), t\rangle) \geqslant \mathscr{I}(Z, \_)$$

holds. Since the latter holds for every Y and $u \in \mathscr{I}(X, Y)$, we have that $[\![\Gamma \vdash_\Sigma \forall x^\sigma.p : o]\!]_X(\eta) \geqslant \mathscr{I}(X, \_)$, i.e, $X \Vdash_{\Gamma,\eta} \forall x^\sigma.p$.

2. First we note that $X \Vdash_{\Gamma,\eta} p \Rightarrow q$ if and only if $\eta \in \kappa_{[\![\Gamma]\!]}([\![\Gamma \vdash_\Sigma p \Rightarrow q : o]\!])_X$ if and only if $[\![\Gamma \vdash_\Sigma p \Rightarrow q : o]\!]_X(\eta) \geqslant \mathscr{I}(X, \_)$. Then, since we have that $[\![\Gamma \vdash_\Sigma p \Rightarrow q : o]\!] = imp \circ \langle [\![\Gamma \vdash_\Sigma p : o]\!], [\![\Gamma \vdash_\Sigma q : o]\!]\rangle$, the latter condition is equivalent to $\overline{[\![\Gamma \vdash_\Sigma p : o]\!]_X(\eta)} \vee [\![\Gamma \vdash_\Sigma q : o]\!]_X(\eta) \geqslant \mathscr{I}(X, \_)$, i.e., for all Y $(\overline{[\![\Gamma \vdash_\Sigma p : o]\!]_X(\eta)})_Y \cup ([\![\Gamma \vdash_\Sigma q : o]\!]_X(\eta))_Y \supseteq \mathscr{I}(X, Y)$.

($\Rightarrow$) By hypothesis we have that $X \Vdash_{\Gamma,\eta} p \Rightarrow q$ and $X \Vdash_{\Gamma,\eta} p$ hold and the latter is equivalent to $[\![\Gamma \vdash_\Sigma p : o]\!]_X(\eta) \geqslant \mathscr{I}(X, \_)$, i.e., for all Y

$$([\![\Gamma \vdash_\Sigma p : o]\!]_X(\eta))_Y \supseteq \mathscr{I}(X, Y).$$

It follows that $(\overline{[\![\Gamma \vdash_\Sigma p : o]\!]_X(\eta)})_Y = \mathscr{V}(X, Y) \setminus \mathscr{I}(X, Y)$, hence, by the preliminary observation, $([\![\Gamma \vdash_\Sigma q : o]\!]_X(\eta))_Y \supseteq \mathscr{I}(X, Y)$. So we proved that $[\![\Gamma \vdash_\Sigma q : o]\!]_X(\eta) \geqslant \mathscr{I}(X, \_)$, i.e., that $X \Vdash_{\Gamma,\eta} q$.

($\Leftarrow$) By hypothesis we have that either $X \Vdash_{\Gamma,\eta} p$ does not hold or $X \Vdash_{\Gamma,\eta} q$ holds. In the former case for all Y $([\![\Gamma \vdash_\Sigma p : o]\!]_X(\eta))_Y \not\supseteq \mathscr{I}(X, Y)$, hence $(\overline{[\![\Gamma \vdash_\Sigma p : o]\!]_X(\eta)})_Y \supseteq \mathscr{I}(X, Y)$. So, by the preliminary observation, we also have that for all Y $([\![\Gamma \vdash_\Sigma p \Rightarrow q : o]\!]_X(\eta))_Y \supseteq \mathscr{I}(X, Y)$, hence $X \Vdash_{\Gamma,\eta} p \Rightarrow q$.

The other case is even easier, since we have that for all Y

$$([\![\Gamma \vdash_\Sigma q : o]\!]_X(\eta))_Y \supseteq \mathscr{I}(X, Y)$$

and we can conlude again by the preliminary observation.

3. By definition, $X \Vdash_{\Gamma,\eta} PM$ if and only if $\eta \in \kappa_{[\![\Gamma]\!]}([\![\Gamma \vdash_\Sigma PM : o]\!])_X$, i.e., if and only if $[\![\Gamma \vdash_\Sigma PM : o]\!]_X(\eta) \geqslant \mathscr{I}(X, \_)$. Then the thesis is a direct consequence of the following argument:

$$\begin{aligned}[\![\Gamma \vdash_\Sigma PM : o]\!]_X(\eta) &= (ev_{[\![\sigma]\!],Prop} \circ \langle [\![\Gamma \vdash_\Sigma M : \sigma]\!], [\![\Gamma \vdash_\Sigma P : \sigma \to o]\!] \rangle)_X(\eta) \\ &= (ev_{[\![\sigma]\!],Prop})_X(\langle [\![\Gamma \vdash_\Sigma M : \sigma]\!]_X(\eta), [\![\Gamma \vdash_\Sigma P : \sigma \to o]\!]_X(\eta) \rangle) \\ &= ([\![\Gamma \vdash_\Sigma P : \sigma \to o]\!]_X(\eta))_X(\mathrm{id}_X, [\![\Gamma \vdash_\Sigma M : \sigma]\!]_X(\eta))\end{aligned}$$

4. By definition of $\perp$ the thesis is equivalent to $X \Vdash_{\Gamma,\eta} \forall r^o.r$. It follows, by the first item of this theorem, that we have to prove that there exist Y, $h \in \mathscr{I}(X, Y)$ and $a \in Prop_Y$ such that it is not the case that $Y \Vdash_{(\Gamma,r:o),\langle [\![\Gamma]\!]_h(\eta),a\rangle} r$, i.e., that $[\![\Gamma, r : o \vdash_\Sigma r : o]\!]_Y(\langle [\![\Gamma]\!]_h(\eta), a\rangle) = a \not\geqslant \mathscr{I}(Y, \_)$. Hence, it is sufficient to take $a = \mathbf{0}$ (i.e., the initial object of $\check{\mathscr{I}}$) to obtain the thesis.

B.4 Proof of Corollary 4.1

1. First of all we have that $X \Vdash_{\Gamma,\eta} \neg p$ stands for $X \Vdash_{\Gamma,\eta} p \Rightarrow \perp$, which is equivalent (by Theorem 4.1) to $X \Vdash_{\Gamma,\eta} p$ implies $X \Vdash_{\Gamma,\eta} \perp$. Obviously, this is true if and only if $X \Vdash_{\Gamma,\eta} \perp$ or it is not the case that $X \Vdash_{\Gamma,\eta} p$.

   (⇒) Since by Proposition 4.1 it is not the case that $X \Vdash_{\Gamma,\eta} \perp$, it must be not the case that $X \Vdash_{\Gamma,\eta} p$ (by the preliminary observation), i.e., the thesis.

   (⇐) Since, by hypothesis, it is not the case that $X \Vdash_{\Gamma,\eta} p$, we automatically have (by the preliminary observation) that $X \Vdash_{\Gamma,\eta} \neg p$.

2. By definition of $\wedge$, the previous item and Theorem 4.1, we have:

$$\begin{aligned}X \Vdash_{\Gamma,\eta} p \wedge q \quad &\text{iff} \quad X \Vdash_{\Gamma,\eta} \neg(p \Rightarrow \neg q) \\ &\text{iff} \quad \text{it is not the case that } X \Vdash_{\Gamma,\eta} p \Rightarrow \neg q \\ &\text{iff} \quad X \Vdash_{\Gamma,\eta} p \text{ and it is not the case that } X \Vdash_{\Gamma,\eta} \neg q \\ &\text{iff} \quad X \Vdash_{\Gamma,\eta} p \text{ and } X \Vdash_{\Gamma,\eta} q\end{aligned}$$

3. By definition of $\vee$, point 1 and Theorem 4.1, we have:

$$\begin{aligned}X \Vdash_{\Gamma,\eta} p \vee q \quad &\text{iff} \quad X \Vdash_{\Gamma,\eta} \neg p \Rightarrow q \\ &\text{iff} \quad X \Vdash_{\Gamma,\eta} \neg p \text{ implies } X \Vdash_{\Gamma,\eta} q \\ &\text{iff} \quad \text{it is not the case that } X \Vdash_{\Gamma,\eta} \neg p \text{ or } X \Vdash_{\Gamma,\eta} q \\ &\text{iff} \quad X \Vdash_{\Gamma,\eta} p \text{ or } X \Vdash_{\Gamma,\eta} q\end{aligned}$$

4. By definition of $\exists$, point 1 and Theorem 4.1, we have:

$$\begin{aligned}X \Vdash_{\Gamma,\eta} \exists x^\sigma.p \quad &\text{iff} \quad X \Vdash_{\Gamma,\eta} \neg\forall x^\sigma.\neg p \\ &\text{iff} \quad \text{it is not the case that } X \Vdash_{\Gamma,\eta} \forall x^\sigma.\neg p \\ &\text{iff} \quad \text{there are } Y, h \in \mathscr{I}(X, Y) \text{ and } a \in [\![\sigma]\!]_Y \text{ such that} \\ &\qquad \text{it is not the case that } Y \Vdash_{(\Gamma,x:\sigma),\langle [\![\Gamma]\!]_h(\eta),a\rangle} \neg p \\ &\text{iff} \quad \text{there are } Y, h \in \mathscr{I}(X, Y) \text{ and } a \in [\![\sigma]\!]_Y \text{ such that} \\ &\qquad Y \Vdash_{(\Gamma,x:\sigma),\langle [\![\Gamma]\!]_h(\eta),a\rangle} p\end{aligned}$$

5. The proof will proceed by induction on n:

   (Base case) $n = 1$: we have to prove that $X \Vdash_{\Gamma,\eta} \forall x_1^{\sigma_1}.p$ if and only if for all Y, $f \in \mathscr{I}(X,Y)$ and $\eta_1 \in [\![\sigma_1]\!]_Y$, we have that $Y \Vdash_{(\Gamma,x_1:\sigma_1),\langle [\![\Gamma]\!]_f(\eta),\eta_1\rangle}\ p$ holds. This is straightforward by point 1 of Theorem 4.1.

   (Inductive case) let us suppose that the thesis holds for n; we will prove that it also holds for $n+1$. First of all we apply point 1 of Theorem 4.1 to obtain the following: $X \Vdash_{\Gamma,\eta} \forall x_1^{\sigma_1}.\forall x_2^{\sigma_2}.\ldots\forall x_{n+1}^{\sigma_{n+1}}p$ if and only if for all Y, $f \in \mathscr{I}(X,Y)$, $\eta_1 \in [\![\sigma_1]\!]_Y$ $Y \Vdash_{(\Gamma,x_1:\sigma_1),\langle [\![\Gamma]\!]_f(\eta),\eta_1\rangle} \forall x_2^{\sigma_2}.\ldots\forall x_{n+1}^{\sigma_{n+1}}p$ holds. Then we may apply the inductive hypothesis to deduce that the previous forcing statement holds if and only if for all Z, $g \in \mathscr{I}(Y,Z)$, $\eta_2 \in [\![\sigma_2]\!]_Z,\ldots,\eta_{n+1} \in [\![\sigma_{n+1}]\!]_Z$ we have that the following holds:

$$Z \Vdash_{(\Gamma,x_1:\sigma_1,x_2:\sigma_2,\ldots,x_{n+1}:\sigma_{n+1}),\langle [\![\Gamma,x_1:\sigma_1]\!]_g(\langle [\![\Gamma]\!]_f(\eta),\eta_1\rangle),\eta_2,\ldots,\eta_{n+1}\rangle}\ p.$$

   Then we observe that

$$[\![\Gamma, x_1 : \sigma_1]\!]_g(\langle [\![\Gamma]\!]_f(\eta),\eta_1\rangle) = \langle [\![\Gamma]\!]_{g\circ f}(\eta), [\![x_1 : \sigma_1]\!]_g(\eta_1)\rangle).$$

   Hence we can easily conclude by taking $Z = Y$ and $g = \mathrm{id}_Y$:

$$Y \Vdash_{(\Gamma,x_1:\sigma_1,\ldots,x_{n+1}:\sigma_{n+1}),\langle [\![\Gamma]\!]_f(\eta),\eta_1,\eta_2,\ldots,\eta_{n+1}\rangle}\ p.$$

B.5 Proof of Theorem 4.4

1. In this case we have to prove that $\Gamma \rhd_\Sigma (p \Rightarrow q \Rightarrow r) \Rightarrow (p \Rightarrow q) \Rightarrow p \Rightarrow r$ holds, i.e., that for all X, $\eta \in [\![\Gamma]\!]_X$ we have

$$X \Vdash_{\Gamma,\eta} (p \Rightarrow q \Rightarrow r) \Rightarrow (p \Rightarrow q) \Rightarrow p \Rightarrow r.$$

   By Theorem 4.1, this is equivalent to prove that $X \Vdash_{\Gamma,\eta} (p \Rightarrow q \Rightarrow r)$, $X \Vdash_{\Gamma,\eta} (p \Rightarrow q)$ and $X \Vdash_{\Gamma,\eta} p$ imply $X \Vdash_{\Gamma,\eta} r$. Hence, applying repeatedly Theorem 4.1, we can easily deduce that $X \Vdash_{\Gamma,\eta} q$ holds from $X \Vdash_{\Gamma,\eta} (p \Rightarrow q)$, since we know that $X \Vdash_{\Gamma,\eta} p$ holds. At this point we can easily conclude, applying again Theorem 4.1, since $X \Vdash_{\Gamma,\eta} r$ derives from $X \Vdash_{\Gamma,\eta} (p \Rightarrow q \Rightarrow r)$, $X \Vdash_{\Gamma,\eta} p$ and $X \Vdash_{\Gamma,\eta} q$.
2. By definition we have to prove that for all X, $\eta \in [\![\Gamma]\!]_X$ we have $X \Vdash_{\Gamma,\eta} p \Rightarrow q \Rightarrow p$. By Theorem 4.1, this is equivalent to proving that $X \Vdash_{\Gamma,\eta} p$ and $X \Vdash_{\Gamma,\eta} q$ imply $X \Vdash_{\Gamma,\eta} q$. Hence the conclusion is trivial.
3. By definition we have to prove that for all X, $\eta \in [\![\Gamma]\!]_X$ we have $X \Vdash_{\Gamma,\eta} \forall_\sigma(P) \Rightarrow PM$. By Theorem 4.1, this is equivalent to prove that $X \Vdash_{\Gamma,\eta} \forall_\sigma(P)$ implies $X \Vdash_{\Gamma,\eta} PM$. But $X \Vdash_{\Gamma,\eta} \forall_\sigma(P)$ is equivalent to saying that, for all Y, $f \in \mathscr{I}(X,Y)$ and $a \in [\![\sigma]\!]_Y$, $Y \Vdash_{(\Gamma,x:\sigma),\langle [\![\Gamma]\!]_f(\eta),a\rangle} Px$ holds. Hence, taking $Y \triangleq X$, $f \triangleq \mathrm{id}_X$ and $a \triangleq [\![\Gamma \vdash_\Sigma M : \sigma]\!]_X(\eta)$, we have that $X \Vdash_{(\Gamma,x:\sigma),\langle \eta,[\![\Gamma\vdash_\Sigma M:\sigma]\!]_X(\eta)\rangle} Px$ holds. By Theorem 4.1, this is equivalent to say that

$$\begin{aligned}&([\![\Gamma, x : \sigma \vdash_\Sigma P : \sigma \to o]\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma M : \sigma]\!]_X(\eta)\rangle))_X\\ &\qquad(\langle \mathrm{id}_X, [\![\Gamma, x : \sigma \vdash_\Sigma x : \sigma]\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma M : \sigma]\!]_X(\eta)\rangle)\rangle) \geqslant \mathscr{I}(X, \_).\end{aligned}$$

Now we observe that

$$\begin{aligned}&([\![\Gamma, x:\sigma \vdash_\Sigma P : \sigma \to o]\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma M : \sigma]\!]_X(\eta)\rangle))_X\\ &\quad(\langle \mathrm{id}_X, [\![\Gamma, x:\sigma \vdash_\Sigma x : \sigma]\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma M : \sigma]\!]_X(\eta)\rangle)\rangle)\\ &= ([\![\Gamma, x:\sigma \vdash_\Sigma P : \sigma \to o]\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma M : \sigma]\!]_X(\eta)\rangle))_X(\langle \mathrm{id}_X, [\![\Gamma \vdash_\Sigma M : \sigma]\!]_X(\eta)\rangle)\\ &= ([\![\Gamma \vdash_\Sigma P : \sigma \to o]\!]_X(\eta))_X(\langle \mathrm{id}_X, [\![\Gamma \vdash_\Sigma M : \sigma]\!]_X(\eta)\rangle).\end{aligned}$$

Hence, applying again Theorem 4.1, we have proved that $X \Vdash_{\Gamma,\eta} PM$ holds.

4. By definition we have to prove that for all X, $\eta \in [\![\Gamma]\!]_X$ we have $X \Vdash_{\Gamma,\eta} (\lambda x^\sigma.M)N =^{\sigma'} M[N/x]$. First of all we notice that the following holds:

$$\begin{aligned}&[\![\Gamma \vdash_\Sigma (\lambda x^\sigma.M)N]\!]_X(\eta) =\\ &= (ev_{[\![\sigma]\!],[\![\sigma']\!]})_X(\langle [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta), [\![\Gamma \vdash_\Sigma \lambda x^\sigma.M : \sigma \to \sigma']\!]_X(\eta)\rangle)\\ &= ([\![\Gamma \vdash_\Sigma \lambda x^\sigma.M : \sigma \to \sigma']\!]_X(\eta))_X(\langle \mathrm{id}_X, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle)\\ &= [\![\Gamma, x:\sigma \vdash_\Sigma M : \sigma']\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle)\end{aligned}$$

Now, we can proceed by structural induction on M:
($M \equiv y \neq x$): Trivial.
($M \equiv x$): Trivial.
($M \equiv PQ$): The following holds:

$$\begin{aligned}&[\![\Gamma \vdash_\Sigma (PQ)[N/x] : \sigma']\!]_X(\eta) =\\ &= [\![\Gamma \vdash_\Sigma P[N/x]Q[N/x] : \sigma']\!]_X(\eta)\\ &= (ev_{[\![\gamma]\!],[\![\sigma']\!]})_X(\langle [\![\Gamma \vdash_\Sigma Q[N/x] : \gamma]\!]_X(\eta), [\![\Gamma \vdash_\Sigma P[N/x] : \gamma \to \sigma']\!]_X(\eta)\rangle)\\ &= ([\![\Gamma \vdash_\Sigma P[N/x] : \gamma \to \sigma']\!]_X(\eta))_X(\langle \mathrm{id}_X, [\![\Gamma \vdash_\Sigma Q[N/x] : \gamma]\!]_X(\eta)\rangle)\\ &\overset{\text{(I.H.)}}{=} ([\![\Gamma, \vdash_\Sigma (\lambda x^\sigma.P)N : \gamma \to \sigma']\!]_X(\eta))_X(\langle \mathrm{id}_X, [\![\Gamma \vdash_\Sigma (\lambda x^\sigma.Q)N : \gamma]\!]_X(\eta)\rangle)\end{aligned}$$

Moreover, we have that:

$$\begin{aligned}&[\![\Gamma, x:\sigma \vdash_\Sigma M : \sigma']\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle) =\\ &= [\![\Gamma, x:\sigma \vdash_\Sigma PQ : \sigma']\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle)\\ &= (ev_{[\![\gamma]\!],[\![\sigma']\!]})_X(\langle A, B\rangle)\\ &= (B)_X(\langle \mathrm{id}_X, A\rangle)\end{aligned}$$

where $A \triangleq [\![\Gamma, x:\sigma \vdash_\Sigma Q : \gamma]\!]_X(\eta)(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle)$ and $B \triangleq [\![\Gamma, x:\sigma \vdash_\Sigma P : \gamma \to \sigma']\!]_X(\eta)(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle)$. Hence we may conclude since we have

$$\begin{aligned}&[\![\Gamma \vdash_\Sigma (\lambda x^\sigma.Q)N : \gamma]\!]_X(\eta) =\\ &= [\![\Gamma, x:\sigma \vdash_\Sigma Q : \gamma]\!]_X(\eta)(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle) = A\end{aligned}$$

and

$$\begin{aligned}&[\![\Gamma, \vdash_\Sigma (\lambda x^\sigma.P)N : \gamma \to \sigma']\!]_X(\eta) =\\ &= [\![\Gamma, x:\sigma \vdash_\Sigma P : \gamma \to \sigma']\!]_X(\eta)(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle) = B\end{aligned}$$

($M \equiv \lambda z^\gamma.P$ with $x \neq z$): In this case $\sigma' \equiv \gamma \to \delta$; hence the following holds (since we identify terms up-to α-conversion, without loss of generality, we can assume that z does not occur in N):

$$
\begin{aligned}
&[\![\Gamma \vdash_\Sigma (\lambda z^\gamma.P)[N/x] : \sigma']\!]_X(\eta) = \\
&= [\![\Gamma \vdash_\Sigma (\lambda z^\gamma.P[N/x]) : \sigma']\!]_X(\eta) \\
&= \{\lambda\langle f,b\rangle \in \mathscr{V}(X,Y) \times [\![\gamma]\!]_Y.[\![\Gamma, z : \gamma \vdash_\Sigma P[N/x] : \delta]\!]_Y(\mu)\}_{Y\in\mathscr{V}} \\
&\overset{\text{(I.H.)}}{=} \{\lambda\langle f,b\rangle \in \mathscr{V}(X,Y) \times [\![\gamma]\!]_Y.[\![\Gamma, z : \gamma \vdash_\Sigma (\lambda x^\sigma.P)N : \delta]\!]_Y(\mu)\}_{Y\in\mathscr{V}}
\end{aligned}
$$

where $\mu \triangleq \langle [\![\Gamma]\!]_f(\eta), b\rangle$. Moreover, we have

$$
\begin{aligned}
&[\![\Gamma, z : \gamma \vdash_\Sigma (\lambda x^\sigma.P)N : \delta]\!]_Y(\mu) = \\
&= (ev_{[\![\sigma]\!],[\![\delta]\!]})_Y(\langle [\![\Gamma, z : \gamma \vdash_\Sigma N : \sigma]\!]_Y(\mu), [\![\Gamma, z : \gamma \vdash_\Sigma \lambda x^\sigma.P : \sigma \to \delta]\!]_Y(\mu)\rangle) \\
&= ([\![\Gamma, z : \gamma \vdash_\Sigma \lambda x^\sigma.P : \sigma \to \delta]\!]_Y(\mu))_Y(\langle \mathrm{id}_Y, [\![\Gamma, z : \gamma \vdash_\Sigma N : \sigma]\!]_Y(\mu)\rangle) \\
&= [\![\Gamma, z : \gamma, x : \sigma \vdash_\Sigma P : \delta]\!]_Y(\langle [\![\Gamma]\!]_f(\eta), b, [\![\Gamma, z : \gamma \vdash_\Sigma N : \sigma]\!]_Y(\mu)\rangle) \\
&= [\![\Gamma, x : \sigma, z : \gamma \vdash_\Sigma P : \delta]\!]_Y(\langle [\![\Gamma]\!]_f(\eta), [\![\Gamma, z : \gamma \vdash_\Sigma N : \sigma]\!]_Y(\mu), b\rangle)
\end{aligned}
$$

For what concerns $[\![\Gamma, x : \sigma \vdash_\Sigma M : \sigma']\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle)$, we have the following:

$$
\begin{aligned}
&[\![\Gamma, x : \sigma \vdash_\Sigma M : \sigma']\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle) = \\
&= [\![\Gamma, x : \sigma \vdash_\Sigma \lambda z^\gamma.P : \sigma']\!]_X(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle) \\
&= \{\lambda\langle f,b\rangle \in \mathscr{V}(X,Y) \times [\![\gamma]\!]_Y.m_Y(\langle [\![\Gamma, x : \sigma]\!]_f(\langle \eta, [\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)\rangle), b\rangle)\}_{Y\in\mathscr{V}} \\
&= \{\lambda\langle f,b\rangle \in \mathscr{V}(X,Y) \times [\![\gamma]\!]_Y.m_Y(\langle [\![\Gamma]\!]_f(\eta), [\![\Gamma \vdash_\Sigma N : \sigma]\!]_Y([\![\Gamma]\!]_f(\eta)), b\rangle)\}_{Y\in\mathscr{V}} \\
&= \{\lambda\langle f,b\rangle \in \mathscr{V}(X,Y) \times [\![\gamma]\!]_Y.m_Y(\langle [\![\Gamma]\!]_f(\eta), \beta, b\rangle)\}_{Y\in\mathscr{V}}
\end{aligned}
$$

where $m \triangleq [\![\Gamma, x : \sigma, z : \gamma \vdash_\Sigma P : \delta]\!]$ and $\beta \triangleq [\![\Gamma, z : \gamma \vdash_\Sigma N : \sigma]\!]_Y([\![\Gamma]\!]_f(\eta), b)$; in the fourth step we exploited the naturality of $[\![\Gamma \vdash_\Sigma N : \sigma]\!]$ since $[\![x : \sigma]\!]_f([\![\Gamma \vdash_\Sigma N : \sigma]\!]_X(\eta)) = [\![\Gamma \vdash_\Sigma N : \sigma]\!]_Y([\![\Gamma]\!]_f(\eta))$ and the weakening rule. Hence we have the thesis.

5. In this case we have to prove that for all X, $\eta \in [\![\Gamma]\!]_X$ we have

$$X \Vdash_{\Gamma,\eta} (\forall x^\sigma.M =^{\sigma'} N) \Rightarrow \lambda x^\sigma.M = \lambda x^{\sigma'}.N,$$

i.e., by Corollary 4.1, that $X \Vdash_{\Gamma,\eta} (\forall x^\sigma.M =^{\sigma'} N)$ implies

$$X \Vdash_{\Gamma,\eta} \lambda x^\sigma.M =^{\sigma\to\sigma'} \lambda x^\sigma.N.$$

First, we observe the following:

$$[\![\Gamma \vdash_\Sigma \lambda x^\sigma.M]\!]_X(\eta) = \{\lambda\langle f,b\rangle \in \mathscr{V}(X,Y) \times [\![\sigma]\!]_Y.m_Y(\langle [\![\Gamma]\!]_f(\eta), b\rangle)\}_{Y\in\mathscr{V}},$$

where $m \triangleq [\![\Gamma, x : \sigma \vdash_\Sigma M : \sigma']\!]$. Similarly, we have:

$$[\![\Gamma \vdash_\Sigma \lambda x^\sigma.N]\!]_X(\eta) = \{\lambda\langle f,b\rangle \in \mathscr{V}(X,Y) \times [\![\sigma]\!]_Y.n_Y(\langle [\![\Gamma]\!]_f(\eta), b\rangle)\}_{Y\in\mathscr{V}},$$

where $n \triangleq [\![\Gamma, x : \sigma \vdash_\Sigma N : \sigma']\!]$. Hence, in order to conclude, it is sufficient to show that $m = n$, i.e., that, for every $Y \in \mathscr{V}$, $f \in \mathscr{V}(X,Y)$ and $b \in [\![\sigma]\!]_Y$, $[\![\Gamma, x : \sigma \vdash_\Sigma M : \sigma']\!](\langle f,b\rangle) = [\![\Gamma, x : \sigma \vdash_\Sigma N : \sigma']\!]_Y(\langle f,b\rangle)$. Hence, observing that our hypothesis is equivalent (by Theorem 4.1) to say that for all $Y \in \mathscr{V}$, $h \in \mathscr{I}(X,Y)$, $\eta_x \in [\![\sigma]\!]_Y$, $Y \Vdash_{(\Gamma,x:\sigma),\langle [\![\Gamma]\!]_h(\eta),\eta_x\rangle} M =^{\sigma'} N$ holds, we can conclude by the same argument used in the proof of Theorem 4.3.

6. By Theorem 4.2 we have to show that for all X, $\eta \in [\![\Gamma]\!]_X$ the following holds:

$$[\![\Gamma \vdash_\Sigma \lambda x^\sigma.Mx : \sigma \to \sigma']\!]_X(\eta) = [\![\Gamma \vdash_\Sigma M : \sigma \to \sigma']\!]_X(\eta).$$

Since the members of the latter equation are natural transformations between the functors $\mathscr{V}(X, \_) \times [\![\sigma]\!]$ and $[\![\sigma']\!]$, the thesis is equivalent to prove that the following holds for every Y, $f \in \mathscr{V}(X, Y)$ and $b \in [\![\sigma]\!]_Y$:

$$([\![\Gamma \vdash_\Sigma \lambda x^\sigma.Mx : \sigma \to \sigma']\!]_X(\eta))_Y(\langle f, b\rangle) = ([\![\Gamma \vdash_\Sigma M : \sigma \to \sigma']\!]_X(\eta))_Y(\langle f, b\rangle).$$

Indeed, we have:

$$\begin{aligned}
&([\![\Gamma \vdash_\Sigma \lambda x^\sigma.Mx : \sigma \to \sigma']\!]_X(\eta))_Y(\langle f, b\rangle) = \\
&= [\![\Gamma, x : \sigma \vdash_\Sigma Mx : \sigma']\!]_Y(\langle [\![\Gamma]\!]_f(\eta), b\rangle) \\
&= (ev_{[\![\sigma]\!],[\![\sigma']\!]})_Y(\langle b, [\![\Gamma, x : \sigma \vdash_\Sigma M : \sigma \to \sigma']\!]_Y(\langle [\![\Gamma]\!]_f(\eta), b\rangle)\rangle) \\
&= ([\![\Gamma, x : \sigma \vdash_\Sigma M : \sigma \to \sigma']\!]_Y(\langle [\![\Gamma]\!]_f(\eta), b\rangle))_Y(\langle \mathrm{id}_Y, b\rangle) \\
&= ([\![\Gamma \vdash_\Sigma M : \sigma \to \sigma']\!]_Y([\![\Gamma]\!]_f(\eta)))_Y(\langle \mathrm{id}_Y, b\rangle) \\
&= (([\![\sigma]\!] \Rightarrow [\![\sigma']\!])_f([\![\Gamma \vdash_\Sigma M : \sigma \to \sigma']\!]_X(\eta)))_Y(\langle \mathrm{id}_Y, b\rangle) \\
&= ([\![\Gamma \vdash_\Sigma M : \sigma \to \sigma']\!]_X(\eta))_Y(\langle f, b\rangle).
\end{aligned}$$

7. We have to show that for all X, $\eta \in [\![\Gamma]\!]_X$ $X \Vdash_{\Gamma,\eta} \neg\neg p \Rightarrow p$ holds. By Theorem 4.1, this is equivelent to prove that $X \Vdash_{\Gamma,\eta} \neg\neg p$ implies $X \Vdash_{\Gamma,\eta} p$. By Corollary 4.1, the premise means that it is not the case that $X \Vdash_{\Gamma,\eta} \neg p$ holds. Applying again the same corollary, we have that it is not the case that $X \Vdash_{\Gamma,\eta} p$ does not hold, i.e., the thesis.
8. In this case the thesis follows directly from Theorem 4.1.
9. By Theorem 4.1, the premise is equivalent to say that for all X and $\eta \in [\![\Gamma, x : \sigma]\!]_X$ $X \Vdash_{(\Gamma,x:\sigma),\eta} p$ implies $X \Vdash_{(\Gamma,x:\sigma),\eta} q$. To prove that the thesis holds it suffices to show, by Theorem 4.1, that for all Y and $\mu \in [\![\Gamma]\!]_Y$ $Y \Vdash_{\Gamma,\mu} p$ implies $Y \Vdash_{\Gamma,\mu} \forall x^\sigma.q$. The latter, again by Theorem 4.1, is equivalent to show that for all Z, $f \in \mathscr{I}(Y, Z)$ and $a \in [\![\sigma]\!]_Z$
$Z \Vdash_{(\Gamma,x:\sigma),\langle [\![\Gamma]\!]_f(\mu),a\rangle} q$ holds. From the validity of $Y \Vdash_{\Gamma,\mu} p$, by the monotonicity of forcing, we can deduce that, for all Z and $f \in \mathscr{I}(Y, Z)$, $Z \Vdash_{\Gamma,[\![\Gamma]\!]_f(\mu)} p$ holds. By the weakening rule, we also have that, for all $a \in [\![\sigma]\!]_Z$,
$Z \Vdash_{(\Gamma,x:\sigma),\langle [\![\Gamma]\!]_f(\mu),a\rangle} p$ holds. Hence we can apply the premise to conclude that
$Z \Vdash_{(\Gamma,x:\sigma),\langle [\![\Gamma]\!]_f(\mu),a\rangle} q$ holds.

B.6 Proof of Theorem 4.5

($\Rightarrow$) By structural induction on the derivation of $\Gamma \vdash_\Sigma M : \iota$:

($\Gamma \vdash_\Sigma 0 : \iota$) Since we have $[\![\Gamma \vdash_\Sigma 0 : \iota]\!]_X(\eta) = 0$, we can easily conclude observing that $FV(0) = \emptyset$.

($\Gamma \vdash_\Sigma \tau.P : \iota$) Hence the previous derivation step yields $\Gamma \vdash_\Sigma P : \iota$. By inductive hypothesis we have that $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV([\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta))$. Hence we can deduce that $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV(\tau.[\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta))$. The thesis is an easy consequence observing the following:

$$\begin{aligned}\tau.[\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta) &= tau_X([\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta))\\ &= (tau \circ [\![\Gamma \vdash_\Sigma P : \iota]\!])_X(\eta) \triangleq [\![\Gamma \vdash_\Sigma \tau.P : \iota]\!]_X(\eta).\end{aligned}$$

($\Gamma \vdash_\Sigma P \mid Q : \iota$) Hence the previous derivation step yields $\Gamma \vdash_\Sigma P_1 : \iota$ and $\Gamma \vdash_\Sigma P_2 : \iota$. By inductive hypothesis we have $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV([\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta))$ and $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV([\![\Gamma \vdash_\Sigma Q : \iota]\!]_X(\eta))$. Hence we can deduce that $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV([\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta) \mid [\![\Gamma \vdash_\Sigma Q : \iota]\!]_X(\eta))$. The thesis is an easy consequence observing the following:

$$\begin{aligned}[\![\Gamma \vdash_\Sigma P{:}\iota]\!]_X(\eta) | [\![\Gamma \vdash_\Sigma Q{:}\iota]\!]_X(\eta) &= par_X(\langle [\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta), [\![\Gamma \vdash_\Sigma Q : \iota]\!]_X(\eta)\rangle)\\ &= (par \circ \langle [\![\Gamma \vdash_\Sigma P : \iota]\!], [\![\Gamma \vdash_\Sigma Q : \iota]\!]\rangle)_X(\eta)\\ &= [\![\Gamma \vdash_\Sigma P \mid Q : \iota]\!]_X(\eta).\end{aligned}$$

($\Gamma \vdash_\Sigma [u \neq v]P : \iota$) Hence the previous derivation step yields $\Gamma \vdash_\Sigma P : \iota$. By inductive hypothesis $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV([\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta))$; moreover, $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \neq [\![\Gamma \vdash_\Sigma u : \upsilon]\!]_X(\eta)$ and $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \neq [\![\Gamma \vdash_\Sigma v : \upsilon]\!]_X(\eta)$. Hence we can deduce that $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV([\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta)) \cup \{[\![\Gamma \vdash_\Sigma u : \upsilon]\!]_X(\eta), [\![\Gamma \vdash_\Sigma v : \upsilon]\!]_X(\eta)\}$. The thesis is an easy consequence observing the following:

$$\begin{aligned}&[[\![\Gamma \vdash_\Sigma u : \upsilon]\!]_X(\eta) \neq [\![\Gamma \vdash_\Sigma v : \upsilon]\!]_X(\eta)][\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta) =\\ &= mismatch_X(\langle [\![\Gamma \vdash_\Sigma u : \upsilon]\!]_X(\eta), [\![\Gamma \vdash_\Sigma v : \upsilon]\!]_X(\eta), [\![\Gamma \vdash_\Sigma P : \iota]\!]_X(\eta)\rangle)\\ &= (mismatch \circ \langle [\![\Gamma \vdash_\Sigma u : \upsilon]\!], [\![\Gamma \vdash_\Sigma v : \upsilon]\!], [\![\Gamma \vdash_\Sigma P : \iota]\!]\rangle)_X(\eta)\\ &\triangleq [\![\Gamma \vdash_\Sigma [u \neq v]P : \iota]\!]_X(\eta).\end{aligned}$$

($\Gamma \vdash_\Sigma \nu\lambda x^\upsilon.P : \iota$) Hence a preceding derivation step yields $\Gamma, x : \upsilon \vdash_\Sigma P : \iota$. By inductive hypothesis $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV([\![\Gamma, x : \upsilon \vdash_\Sigma P : \iota]\!]_X(\langle \eta, \eta_x\rangle))$ for all $\eta_x \neq [\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta)$. Hence we can deduce that $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV((\nu\eta_x)([\![\Gamma, x : \upsilon \vdash_\Sigma P : \iota]\!]_X(\langle \eta, \eta_x\rangle)))$, where $\eta_x \in [\![x : \upsilon]\!]_X$. Again, the thesis is a direct consequence of the following:

$$\begin{aligned}&(\nu\eta_x)([\![\Gamma, x : \upsilon \vdash_\Sigma P : \iota]\!]_X(\langle \eta, \eta_x\rangle))\\ &= (\nu\eta_x)([\![\Gamma \vdash_\Sigma \lambda x^\upsilon.P : \upsilon \to \iota]\!]_X(\eta))_{X \uplus \{x\}}(\langle \mathrm{id}_X, \eta_x\rangle))\\ &= new_X([\![\Gamma \vdash_\Sigma \lambda x^\upsilon.P : \upsilon \to \iota]\!]_X(\eta))\\ &= (new \circ [\![\Gamma \vdash_\Sigma \lambda x^\upsilon.P : \upsilon \to \iota]\!])_X(\eta)\\ &\triangleq [\![\Gamma \vdash_\Sigma \nu\lambda x^\upsilon.P : \iota]\!]_X(\eta)\end{aligned}$$

($\Leftarrow$) **Preliminary observation**: we recall that $y \notin M$ is an abbreviation for

$$\forall p^{\upsilon \to \iota \to o}.(\forall z^\upsilon.\forall Q^\iota.(T_{\notin}\ p\ z\ Q) \Rightarrow (p\ z\ Q)) \Rightarrow (p\ y\ M).$$

Hence, by point 1 of Theorem 4.1, in order to prove that $X \Vdash_{\Gamma,\eta} y \notin M$, we must show that for all $Y, f \in \mathscr{I}(X, Y)$ and $\eta_p \in [\![\upsilon \to \iota \to o]\!]_Y = (Var \Rightarrow Proc \Rightarrow Prop)_Y$,

$$Y \Vdash_{(\Gamma, p:\upsilon\to\iota\to o), \langle [\![\Gamma]\!]_f(\eta), \eta_p\rangle} (\forall z^\upsilon.\forall Q^\iota.(T_{\notin}\ p\ z\ Q) \Rightarrow (p\ z\ Q)) \Rightarrow\ (p\ y\ M)$$

holds, i.e., by point 2 of Theorem 4.1, if and only if

$$Y \Vdash_{(\Gamma,p:\upsilon\to\iota\to o),\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle} \forall z^\upsilon.\ \forall Q^\iota.(T_{\notin}\ p\ z\ Q) \Rightarrow (p\ z\ Q)$$
$$\text{implies}\quad Y \Vdash_{(\Gamma,p:\upsilon\to\iota\to o),\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle} (p\ y\ M).$$

So, we suppose that the premise is true and we show that the consequence also holds; by point 5 of Corollary 4.1, we can deduce that the premise is true if and only if for all Z, $g \in \mathscr{I}(Y,Z)$, $\eta_z \in Var_Z \triangleq Z$ and $\eta_Q \in Proc_Z$, $Z \Vdash_{\Delta,\mu} (T_{\notin}\ p\ z\ Q) \Rightarrow (p\ z\ Q)$ holds, where $\Delta \triangleq (\Gamma, p : \upsilon \to \iota \to o, z : \upsilon, Q : \iota)$ and $\mu \triangleq \langle[\![\Gamma]\!]_{g\circ f}(\eta), [\![p : \upsilon \to \iota \to o]\!]_g(\eta_p), \eta_z, \eta_Q\rangle$. In particular, taking $Z \triangleq Y$, $g \triangleq \mathrm{id}_Y$, $\eta_z \triangleq [\![\Gamma, p : \upsilon \to \iota \to o \vdash_\Sigma y : \upsilon]\!]_Y(\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle)$ and $\eta_Q \triangleq [\![\Gamma, p : \upsilon \to \iota \to o \vdash_\Sigma M : \iota]\!]_Y(\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle)$, we have that the following holds:

$$Y \Vdash_{(\Gamma,p:\upsilon\to\iota\to o,z:\upsilon,Q:\iota),\langle[\![\Gamma]\!]_f(\eta),\eta_p,\eta_z,\eta_Q\rangle} (T_{\notin}\ p\ z\ Q) \Rightarrow (p\ z\ Q)$$

This is equivalent, by Theorem 4.1, to say that

$$Y \Vdash_{(\Gamma,p:\upsilon\to\iota\to o,z:\upsilon,Q:\iota),\langle[\![\Gamma]\!]_f(\eta),\eta_p,\eta_z,\eta_Q\rangle} (T_{\notin}\ p\ z\ Q)$$
$$\text{implies}\quad Y \Vdash_{(\Gamma,p:\upsilon\to\iota\to o,z:\upsilon,Q:\iota),\langle[\![\Gamma]\!]_f(\eta),\eta_p,\eta_z,\eta_Q\rangle} (p\ z\ Q).$$

Since $[\![\Gamma, p : \upsilon \to \iota \to o, z : \upsilon, Q : \iota \vdash_\Sigma (p\ z\ Q)]\!]_Y(\langle[\![\Gamma]\!]_f(\eta),\eta_p,\eta_z,\eta_Q\rangle) = [\![\Gamma, p : \upsilon \to \iota \to o \vdash_\Sigma (p\ y\ M)]\!]_Y(\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle)$, to conclude, it suffices to prove that $Y \Vdash_{(\Gamma,p:\upsilon\to\iota\to o,z:\upsilon,Q:\iota),\langle[\![\Gamma]\!]_f(\eta),\eta_p,\eta_z,\eta_Q\rangle} (T_{\notin}\ p\ z\ Q)$ holds.

By definition of $T_{\notin}$, $(T_{\notin}\ p\ z\ Q)$ is the following λ-term:

$$\begin{aligned}
&Q = 0\ \vee\\
&(\exists P^\iota.Q = \sigma.P \wedge (p\ z\ P))\ \vee\\
&(\exists P_1^\iota.\exists P_2^\iota.Q = P_1 \mid P_2 \wedge (p\ z\ P_1) \wedge (p\ z\ P_2))\ \vee\\
&(\exists P^\iota.\exists y^\upsilon.\exists u^\upsilon.Q = [y \neq u]P \wedge \neg z =^\upsilon y \wedge \neg z =^\upsilon u \wedge (p\ z\ P))\ \vee\\
&(\exists P^{\upsilon\to\iota}.Q = \nu P \wedge (\forall y^\upsilon.\neg z =^\upsilon y \Rightarrow (p\ z\ (P\ \ y))))
\end{aligned}$$

Hence (by Corollary 4.1), to prove the premise, it suffices to show that one of the disjunctions holds. At this point we can proceed by structural induction on the derivation of $\Gamma \vdash_\Sigma M : \iota$:

($\Gamma \vdash_\Sigma 0 : \iota$) Since $M \equiv 0$, we can immediately conclude by the preliminary observation, since η_Q was chosen as $[\![\Gamma, p : \upsilon \to \iota \to o \vdash_\Sigma M : \iota]\!]_Y(\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle)$, whence

$$Y \Vdash_{(\Gamma,p:\upsilon\to\iota\to o,z:\upsilon,Q:\iota),\langle[\![\Gamma]\!]_f(\eta),\eta_p,\eta_z,\eta_Q\rangle} Q = 0.$$

($\Gamma \vdash_\Sigma \tau.P : \iota$) By inductive hypothesis, we know that $X \Vdash_{\Gamma,\eta} y \notin P$ holds. Hence, by an argument similar to that used in the preliminary observation, we have that

$$Y \Vdash_{(\Gamma,p:\upsilon\to\iota\to o),\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle} \forall z^\upsilon.\ \forall Q^\iota.(T_{\notin}\ p\ z\ Q) \Rightarrow (p\ z\ Q)$$
$$\text{implies}\quad Y \Vdash_{(\Gamma,p:\upsilon\to\iota\to o),\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle} (p\ y\ P).$$

But, since the premise is true (by the preliminary observation), we have that $Y \Vdash_{(\Gamma,p:\upsilon\to\iota\to o),\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle} (p\ y\ P)$ holds. At this point we may easily conclude observing that the second disjunction holds (remember that $\eta_z \triangleq [\![\Gamma, p : \upsilon \rightarrow \iota \rightarrow o \vdash_\Sigma y : \upsilon]\!]_Y(\langle [\![\Gamma]\!]_f(\eta), \eta_p\rangle)$ and $\eta_Q \triangleq [\![\Gamma, p : \upsilon \rightarrow \iota \rightarrow o \vdash_\Sigma M : \iota]\!]_Y(\langle [\![\Gamma]\!]_f(\eta), \eta_p\rangle)$, where $M \equiv \sigma.P$).

($\Gamma \vdash_\Sigma P_1 \mid P_2 : \iota$) $X \Vdash_{\Gamma,\eta} y \notin P_1$ and $X \Vdash_{\Gamma,\eta} y \notin P_2$ hold by inductive hypothesis. Hence, like in the previous case, we can deduce that

$$Y \Vdash_{(\Gamma,p:\upsilon\rightarrow\iota\rightarrow o),\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle} (p\ y\ P_1) \quad \text{and} \quad Y \Vdash_{(\Gamma,p:\upsilon\rightarrow\iota\rightarrow o),\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle} (p\ y\ P_2)$$

hold. At this point we may easily conclude observing that the third disjunction holds (remember that $\eta_z \triangleq [\![\Gamma, p : \upsilon \rightarrow \iota \rightarrow o \vdash_\Sigma y : \upsilon]\!]_Y(\langle [\![\Gamma]\!]_f(\eta), \eta_p\rangle)$ and $\eta_Q \triangleq [\![\Gamma, p : \upsilon \rightarrow \iota \rightarrow o \vdash_\Sigma M : \iota]\!]_Y(\langle [\![\Gamma]\!]_f(\eta), \eta_p\rangle)$, where $M \equiv P_1 \mid P_2$).

($\Gamma \vdash_\Sigma [u \neq v]P : \iota$) By inductive hypothesis we know that $X \Vdash_{\Gamma,\eta} y \notin P$. Hence, as in the previous cases, we can deduce that

$$Y \Vdash_{(\Gamma,p:\upsilon\rightarrow\iota\rightarrow o),\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle} (p\ y\ P)$$

holds. Moreover from the hypothesis that $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV([\![\Gamma \vdash_\Sigma M : \iota]\!]_X(\eta))$ we have that $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \neq [\![\Gamma \vdash_\Sigma u : \upsilon]\!]_X(\eta)$ and $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \neq [\![\Gamma \vdash_\Sigma v : \upsilon]\!]_X(\eta)$ and consequently that the statements $X \Vdash_{\Gamma,\eta} y =^\upsilon u$ and $X \Vdash_{\Gamma,\eta} y =^\upsilon v$ do not hold. By Corollary 4.1 this is equivalent to say that $X \Vdash_{\Gamma,\eta} \neg y =^\upsilon u$ and $X \Vdash_{\Gamma,\eta} \neg y =^\upsilon v$ hold. Whence, by the weakening rule and the monotonicity of forcing, we have that

$$Y \Vdash_{(\Gamma,p:\upsilon\rightarrow\iota\rightarrow o),\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle} \neg y =^\upsilon u \quad \text{and} \quad Y \Vdash_{(\Gamma,p:\upsilon\rightarrow\iota\rightarrow o),\langle[\![\Gamma]\!]_f(\eta),\eta_p\rangle} \neg y =^\upsilon v$$

hold. Again, we may easily conclude by the preliminary observation since the fourth disjunction holds (remember that $\eta_z \triangleq [\![\Gamma, p : \upsilon \rightarrow \iota \rightarrow o \vdash_\Sigma y : \upsilon]\!]_Y(\langle [\![\Gamma]\!]_f(\eta), \eta_p\rangle)$ and $\eta_Q \triangleq [\![\Gamma, p : \upsilon \rightarrow \iota \rightarrow o \vdash_\Sigma M : \iota]\!]_Y(\langle [\![\Gamma]\!]_f(\eta), \eta_p\rangle)$, where $M \equiv [u \neq v]P$).

($\Gamma \vdash_\Sigma \nu\lambda x^\upsilon.P$) Since we know that $[\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta) \notin FV([\![\Gamma \vdash_\Sigma \nu\lambda x^\upsilon.P]\!]_X(\eta))$ and $[\![\Gamma \vdash_\Sigma \nu\lambda x^\upsilon.P]\!]_X(\eta) \triangleq (\nu\eta_x)([\![\Gamma, x : \upsilon \vdash_\Sigma P : \iota]\!]_X(\langle \eta, \eta_x\rangle))$, by inductive hypothesis we deduce that $X \Vdash_{(\Gamma,x:\upsilon),\langle\eta,\eta_x\rangle} y \notin P$ holds for all $\eta_x \neq [\![\Gamma \vdash_\Sigma y : \upsilon]\!]_X(\eta)$; hence, proceeding as in the previous cases and applying the weakening rule, we have that

$$Y \Vdash_{(\Gamma,x:\upsilon,p:\upsilon\rightarrow\iota\rightarrow o),\langle[\![\Gamma]\!]_f(\eta),f(\eta_x),\eta_p\rangle} (p\ y\ P)$$

holds. Moreover we have that $Y \Vdash_{(\Gamma,x:\upsilon,p:\upsilon\rightarrow\iota\rightarrow o),\langle[\![\Gamma]\!]_f(\eta),f(\eta_x),\eta_p\rangle} \neg y =^\upsilon x$. At this point we may easily conclude by the preliminary observation since the fifth disjunction holds.

B.7 Proof of Theorem 6.1

We will show only the base case (rule $Rec^\iota_\sigma\_red_1$) and the case of higher-order constructor (rule $Rec^\iota_\sigma\_red_5$), the others being similar.

Let $G \triangleq [\![\Gamma]\!]$, $A \triangleq [\![\sigma]\!]$, and

$$\begin{array}{ll} g_1 = [\![\Gamma \vdash f_1 : \sigma]\!] & : G \longrightarrow A \\ g_2 = [\![\Gamma \vdash f_2 : \sigma \to \sigma]\!] & : G \longrightarrow A \Rightarrow A \\ g_3 = [\![\Gamma \vdash f_3 : \sigma \to \sigma \to \sigma]\!] & : G \longrightarrow A \Rightarrow A \Rightarrow A \\ g_4 = [\![\Gamma \vdash f_4 : \upsilon \to \upsilon \to \sigma \to \sigma]\!] & : G \longrightarrow Var \Rightarrow Var \Rightarrow A \Rightarrow A \\ g_5 = [\![\Gamma \vdash f_5 : (\upsilon \to \sigma) \to \sigma]\!] & : G \longrightarrow (Var \Rightarrow A) \Rightarrow A \end{array}$$

For proving the soundness of $Rec^{\iota}_{\sigma}\_red_1$ and $Rec^{\iota}_{\sigma}\_red_5$, we have to prove that for all X and $\eta \in [\![\Gamma]\!]_X$ the following properties hold:

$$X \Vdash_{\Gamma,\eta} (R\ 0) =^{\sigma} f_1 \tag{B 1}$$

$$X \Vdash_{\Gamma,\eta} \forall P^{\upsilon \to \iota}.(R\ \nu P) =^{\sigma} (f_5\ \lambda x^{\upsilon}.(R\ (P\ x))) \tag{B 2}$$

where R is a syntactic shorthand for $(Rec^{\iota}_{\sigma}\ f_1\ f_2\ f_3\ f_4\ f_5)$.

We prove equivalence (B 1). By Theorem 4.2, this is equivalent to proving that

$$[\![\Gamma \vdash_\Sigma (R\ 0) : \sigma]\!]_X(\eta) = [\![\Gamma \vdash_\Sigma f_1 : \sigma]\!]_X(\eta)$$

In fact, the following equalities hold, where $[\![R]\!]$ is a syntactic shorthand for the interpretation of R, and $m : T(G \Rightarrow A) \longrightarrow G \Rightarrow A$, $\bar{m} : Proc \longrightarrow G \Rightarrow A$ are the natural transformations used in the interpretation of R above:

$$\begin{array}{ll} [\![\Gamma \vdash_\Sigma (R\ 0) : \sigma]\!]_X(\eta) = ev_X(\langle [\![\Gamma \vdash_\Sigma 0 : \iota]\!]_X(\eta), [\![R]\!]_X(\eta)\rangle) & \\ \quad = ([\![R]\!]_X(\eta))_X(\langle [\![\Gamma \vdash_\Sigma 0 : \iota]\!]_X(\eta), \mathrm{id}_X\rangle) & \\ \quad = (\bar{m}_X(0))_X(\eta, \mathrm{id}_X) & \text{by definition of } [\![R]\!] \text{ and since} \\ & [\![\Gamma \vdash_\Sigma 0 : \iota]\!]_X(\eta) = 0 \\ \quad = ((\bar{m} \circ \alpha)_X(in_1(*)))_X(\eta, \mathrm{id}_X) & \text{since } \alpha_X(in_1(*)) = 0 \\ \quad = ((m \circ T\bar{m})_X(in_1(*)))_X(\eta, \mathrm{id}_X) & \text{by the initial algebra property} \\ \quad = (m_X(in_1(*)))_X(\eta, \mathrm{id}_X) & \text{since } (T\bar{m})_X(in_1(*)) = in_1(*) \\ \quad = {g_1}_X(\eta) & \text{by definition of } m \\ \quad = [\![\Gamma \vdash_\Sigma f_1 : \sigma]\!]_X(\eta) & \end{array}$$

We prove equivalence (B 2). By Theorem 4.2, this is equivalent to proving that for all Y stage, $h \in \mathscr{I}(X, Y)$, $p \in (Var \Rightarrow Proc)_Y$:

$$[\![\Gamma, P : \upsilon \to \iota \vdash_\Sigma (R\ \nu P) : \sigma]\!]_Y(\eta[h], p) =$$
$$= [\![\Gamma, P : \iota \vdash_\Sigma (f_5\ \lambda x.(R\ (P\ x))) : \sigma]\!]_Y(\eta[h], p)$$

In fact, the following equalities hold:

$$\begin{array}{ll} [\![\Gamma, P : \upsilon \to \iota \vdash_\Sigma (R\ \nu P) : \sigma]\!]_Y(\eta[h], p) = & \\ = ev_Y(\langle [\![\Gamma, P : \upsilon \to \iota \vdash_\Sigma \nu P : \iota]\!]_Y(\eta[h], p), [\![R]\!]_Y(\eta[h], p)\rangle) & \\ = ([\![R]\!]_Y(\eta[h], p))_Y(\langle [\![\Gamma, P : \upsilon \to \iota \vdash_\Sigma \nu P : \iota]\!]_Y(\eta[h], p), \mathrm{id}_Y\rangle) & \\ = (\bar{m}_Y(\nu\lambda x.p)_Y(\eta, \mathrm{id}_Y) & \text{by definition of } [\![R]\!] \text{ and since} \\ & [\![\Gamma, P : \upsilon \to \iota \vdash_\Sigma \nu P : \iota]\!]_Y(\eta[h], p) = \nu\lambda x.p \\ = ((\bar{m} \circ \alpha)_Y(in_5(p)))_Y(\eta[h], \mathrm{id}_Y) & \text{since } \alpha_Y(in_5(p)) = \nu\lambda x.p \\ = ((m \circ T\bar{m})_Y(in_5(p)))_Y(\eta[h], \mathrm{id}_Y) & \text{by the initial algebra property} \\ = \ldots & \end{array}$$

Now, it is not hard to see that $(T\bar{m})_Y(in_5(p)) = in_5(\bar{m} \circ p)$, where $\bar{m} \circ p : Var \times \mathscr{V}(Y, \_) \longrightarrow G \Rightarrow A$; thus, let $r' \in (Var \Rightarrow A)_Y$ be the natural transformation defined as

$$\begin{aligned} r' &: Var \times \mathscr{V}(Y, \_) \longrightarrow A \\ r'_Z &: Z \times \mathscr{V}(Y, Z) \longrightarrow A_Z \\ &\quad \langle z, k\rangle \longmapsto ((\bar{m} \circ p)_Z(z,k))_Z(\eta[k \circ h], \mathrm{id}_Z) \end{aligned}$$

We have then

$$\begin{aligned} \ldots &= (m_Y(in_5(\bar{m} \circ p))_Y(\eta[h], \mathrm{id}_Y) \\ &= (g_{5\,Y}(\eta[h]))_Y(\langle r', \mathrm{id}_Y\rangle) && \text{by definition of } m \\ &= ev_Y(\langle r', [\![\Gamma \vdash_\Sigma f_5 : (\upsilon \to \sigma) \to \sigma]\!]_Y(\eta[h], p)\rangle) \\ &= ev_Y(\langle [\![\Gamma, P : \upsilon \to \iota \vdash_\Sigma \lambda x^\upsilon.(R\ (P\ x)) : \upsilon \to \sigma]\!]_Y(\eta[h], p), && (*) \\ &\qquad [\![\Gamma \vdash_\Sigma f_5 : (\upsilon \to \sigma) \to \sigma]\!]_Y(\eta[h], p)\rangle) \\ &= [\![\Gamma \vdash_\Sigma (f_5\ \lambda x^\upsilon.(R\ (P\ x)) : \sigma]\!]_Y(\eta[h], p) \end{aligned}$$

The equality $(*)$ holds because

$$[\![\Gamma, P : \upsilon \to \iota \vdash_\Sigma \lambda x^\upsilon.(R\ (P\ x)) : \upsilon \to \sigma]\!]_Y(\eta[h], p) = r'.$$

Indeed, for all stage Z, $z \in Z$, $k \in \mathscr{V}(Y, Z)$, and let $\eta' \triangleq \langle \eta[k \circ h], p[k], z\rangle$:

$$\begin{aligned} &([\![\Gamma, P : \upsilon \to \iota \vdash_\Sigma \lambda x^\upsilon.(R\ (P\ x)) : \upsilon \to \sigma]\!]_Y(\eta[h], p))_Z\,(z,k) = \\ &= [\![\Gamma, P : \upsilon \to \iota, x : \upsilon \vdash_\Sigma (R\ (P\ x)) : \sigma]\!]_Z(\eta') \\ &= ([\![R]\!]_Z(\eta'))_Z(\langle [\![\Gamma, P : \upsilon \to \iota, x : \upsilon \vdash_\Sigma (P\ x) : \iota]\!]_Z(\eta'), \mathrm{id}_Z\rangle) \\ &= (\bar{m}_Z([\![\Gamma, P : \upsilon \to \iota, x : \upsilon \vdash_\Sigma (P\ x) : \iota]\!]_Z(\eta')))_Z(\eta', \mathrm{id}_Z) \\ &= (\bar{m}_Z(p[k]_Z(z, \mathrm{id}_Z)))_Z(\eta', \mathrm{id}_Z) \\ &= ((\bar{m} \circ p[k])_Z(z, \mathrm{id}_Z))_Z(\eta', \mathrm{id}_Z) \\ &= ((\bar{m} \circ p)_Z(z,k))_Z(\eta', \mathrm{id}_Z) = r'_Z(z,k) \end{aligned}$$

B.8 Proof of Proposition 6.3

Let us check that the first diagram of Definition A.5 commutes, i.e., that for every $A, B \in \check{\mathscr{V}}$, $X \in \mathscr{V}$, $a \in A_X$ and $b \in (TB)_X$ we have

$$(T\pi')_X((st_{A,B})_X(\langle a, b\rangle)) = \pi'_X(\langle a, b\rangle) = b$$

This is proved by cases over b:

($b = in_1(*)$) $(T\pi')_X((st_{A,B})_X(\langle a, in_1(*)\rangle)) = (T\pi')_X(in_1(*)) \triangleq in_1(*)$.
($b = in_2(b')$) $(T\pi')_X((st_{A,B})_X(\langle a, in_2(b')\rangle)) = (T\pi')_X(in_2(a, b'))$
$\triangleq in_2(\pi'(\langle a, b'\rangle)) = in_2(b')$.
($b = in_3(\langle b', b''\rangle)$) $(T\pi')_X((st_{A,B})_X(\langle a, in_3(\langle b', b''\rangle)\rangle)) =$
$(T\pi')_X(in_3(\langle a, b', a, b''\rangle)) \triangleq in_3(\langle \pi'(\langle a, b'\rangle), \pi'(\langle a, b''\rangle)\rangle) = in_3(\langle b', b''\rangle)$.
($b = in_4(\langle x, y, b'\rangle)$) $(T\pi')_X((st_{A,B})_X(\langle a, in_4(\langle x, y, b'\rangle)\rangle)) =$
$(T\pi')_X(in_4(\langle x, y, a, b'\rangle)) \triangleq in_4(\langle x, y, \pi'(\langle a, b'\rangle)\rangle) = in_3(\langle x, y, b'\rangle)$.
($b = in_5(b')$) $(T\pi')_X((st_{A,B})_X(\langle a, in_5(b')\rangle)) = (T\pi')_X(in_5(\overline{b}_a)) \triangleq$
$in_5(\gamma_{B,X}(\pi'(\langle G_{in_X}(g), b_{X \uplus \{x\}}(x, in_X)\rangle))) = in_5(\gamma_{B,X}(b_{X \uplus \{x\}}(x, in_X))) = in_5(b)$.

For what concerns the commutativity of the second diagram of Definition A.5, we have to show that for every $A, B, C \in \check{\mathscr{V}}$, $X \in \mathscr{V}$, $a \in A_X$, $b \in (TB)_X$ and $c \in C_X$ we have

$$(T\beta)_X((st_{A,C\times B})_X((\mathrm{id}_A \times st_{C,B})_X(\langle a, \langle c, b\rangle\rangle))) = (st_{A\times C,B})_X(\beta_X(\langle a, \langle c, b\rangle\rangle))$$

where $\beta \triangleq \langle\langle \pi, \pi \circ \pi'\rangle, \pi' \circ \pi'\rangle$; it follows that the second member can be simplified to $(st_{A\times C,B})_X(\langle\langle a, c\rangle, b\rangle)$. We prove the thesis, again by cases on b:

($b = in_1(*)$) $(T\beta)_X((st_{A,C\times B})_X((\mathrm{id}_A \times st_{C,B})_X(\langle a, \langle c, in_1(*)\rangle\rangle))) =$
$= (T\beta)_X((st_{A,C\times B})_X(\langle a, in_1(*)\rangle)) = (T\beta)_X(in_1(*)) = in_1(*) =$
$= (st_{A\times C,B})_X(\langle\langle a, c\rangle, in_1(*)\rangle)$.

($b = in_2(b')$) $(T\beta)_X((st_{A,C\times B})_X((\mathrm{id}_A \times st_{C,B})_X(\langle a, \langle c, in_2(b')\rangle\rangle))) =$
$= (T\beta)_X((st_{A,C\times B})_X(\langle a, in_2(\langle c, b'\rangle)\rangle)) = (T\beta)_X(in_2(\langle a, \langle c, b'\rangle\rangle)) =$
$= in_2(\beta_X(\langle a, \langle c, b'\rangle\rangle)) = in_2(\langle\langle a, c\rangle, b'\rangle) = (st_{A\times C,B})_X(\langle\langle a, c\rangle, in_2(b')\rangle)$.

($b = in_3(\langle b', b''\rangle)$) $(T\beta)_X((st_{A,C\times B})_X((\mathrm{id}_A \times st_{C,B})_X(\langle a, \langle c, in_3(\langle b', b''\rangle)\rangle\rangle))) =$
$= (T\beta)_X((st_{A,C\times B})_X(\langle a, in_3(\langle c, b', c, b''\rangle)\rangle)) = (T\beta)_X(in_3(\langle a, \langle c, b', c, b''\rangle\rangle)) =$
$= in_3(\beta_X(\langle a, \langle c, b'\rangle, a, \langle c, b''\rangle\rangle)) = in_3(\langle\langle a, c\rangle, b', \langle a, c\rangle, b''\rangle) =$
$= (st_{A\times C,B})_X(\langle\langle a, c\rangle, in_3(\langle b', b''\rangle)\rangle)$.

($b = in_4(\langle x, y, b'\rangle)$) $(T\beta)_X((st_{A,C\times B})_X((\mathrm{id}_A \times st_{C,B})_X(\langle a, \langle c, in_4(\langle x, y, b'\rangle)\rangle\rangle))) =$
$= (T\beta)_X((st_{A,C\times B})_X(\langle a, in_4(\langle x, y, c, b'\rangle)\rangle)) = (T\beta)_X(in_4(\langle x, y, a, \langle c, b'\rangle\rangle)) =$
$= in_4(\langle x, y, \beta_X(\langle a, \langle c, b'\rangle\rangle)) = in_4(\langle x, y, \langle a, c\rangle, b'\rangle) =$
$= (st_{A\times C,B})_X(\langle\langle a, c\rangle, in_4(\langle x, y, b'\rangle)\rangle)$.

($b = in_5(b')$) $(T\beta)_X((st_{A,C\times B})_X((\mathrm{id}_A \times st_{C,B})_X(\langle a, \langle c, in_5(b')\rangle\rangle))) =$
$= (T\beta)_X((st_{A,C\times B})_X(\langle a, in_5(\overline{b'}_c)\rangle)) = (T\beta)_X(in_5(\overline{(\overline{b'}_c)}_a)) =$
$= in_5(\gamma_{(A\times C)\times B,X}(\beta_{X\uplus\{x\}}(\langle A_{in_X}(a), \langle C_{in_X}(c), b'_{X\uplus\{x\}}(\langle x, in_X\rangle)\rangle\rangle))) =$
$= in_5(\gamma_{(A\times C)\times B,X}(\langle A_{in_X}(a), C_{in_X}(c)\rangle, b'_{X\uplus\{x\}}(\langle x, in_X\rangle))) = in_5(\overline{b'}_{\langle a,c\rangle}) =$
$= (st_{A\times C,B})_X(\langle\langle a, c\rangle, in_5(b')\rangle)$.

B.9 Proof of Proposition 6.4

In order to prove the commutativity of the diagram we must show that, for every $X \in \mathscr{V}$, $g \in G_X$ and $P \in (TProc)_X$, we have $f_X((\mathrm{id}_G \times \alpha)_X(\langle g, P\rangle)) = \beta_X((\langle \pi, Tf \circ st_{G,Proc}\rangle)_X(\langle g, P\rangle))$. First of all we notice that the second member can be simplified to $\beta_X(\langle g, (Tf)_X((st_{G,Proc})_X(\langle g, P\rangle))\rangle)$, then we proceed by cases on P:

($P = in_1(*)$) we have $f_X((\mathrm{id}_G\times\alpha)_X(\langle g, in_1(*)\rangle)) = f_X(\langle g, 0\rangle) \triangleq \beta_X(\langle g, in_1(*\ )\rangle)$, whence the thesis since $\beta_X(\langle g, (Tf)_X((st_{G,Proc})_X(\langle g, in_1(*)\rangle))\rangle) =$
$= \beta_X(\langle g, (Tf)_X(in_1(*))\rangle) = \beta_X(\langle g, in_1(*)\rangle)$.

($P = in_2(P')$) $f_X((\mathrm{id}_G \times \alpha)_X(\langle g, in_2(P')\rangle)) = f_X(\langle g, \tau.P'\rangle) \triangleq \beta_X(\langle g, in_2(f_X(\langle g,\ P'\rangle))\rangle)$, whence the thesis since $\beta_X(\langle g, (Tf)_X((st_{G,Proc})_X(\langle g, in_2(P')\rangle))\rangle) =$
$= \beta_X(\langle g,\ (Tf)_X(in_2(\langle g, P'\rangle))\rangle) = \beta_X(\langle g, in_2(f_X(\langle g, P'\rangle))\rangle)$.

($P = in_3(\langle P', P''\rangle)$) we have $f_X((\mathrm{id}_G \times \alpha)_X(\langle g, in_3(\langle P', P''\rangle)\rangle)) = f_X(\langle g, P'|P''\rangle) \triangleq \beta_X(\langle g, in_3(\langle f_X(\langle g, P'\rangle), f_X(\langle g, P''\rangle)\rangle)\rangle)$, whence the thesis since
$\beta_X(\langle g, (Tf)_X((st_{G,Proc})_X(\langle g, in_3(\langle P', P''\rangle)\rangle))\rangle) =$
$= \beta_X(\langle g, (Tf)_X(in_3(\langle g, P', g, P''\rangle))\rangle) = \beta_X(\langle g, in_3(\langle f_X(\langle g, P'\rangle), f_X(\langle g, P''\rangle)\rangle)\rangle)$.

($P = in_4(\langle x, y, P'\rangle)$) we have $f_X((\mathrm{id}_G\times\alpha)_X(\langle g, in_4(\langle x, y, P'\rangle)\rangle)) = f_X(\langle g, [x \neq y]P'\rangle) \triangleq$
$\beta_X(\langle g, in_4(\langle x, y, f_X(\langle g, P'\rangle)\rangle)\rangle)$, whence the thesis since
$\beta_X(\langle g, (Tf)_X((st_{G,Proc})_X(\langle g, in_4(\langle x, y, P'\rangle)\rangle))\rangle) =$
$= \beta_X(\langle g, (Tf)_X(in_4(\langle x, y, g, P'\rangle))\rangle) = \beta_X(\langle g, in_4(\langle x, y, f_X(\langle g, P'\rangle)\rangle)\rangle)$.
($P = in_5(P')$) : $f_X((\mathrm{id}_G \times \alpha)_X(\langle g, in_5(P')\rangle)) = f_X(\langle g, (\nu x)P'_{X\uplus\{x\}}(x, in_X)\rangle)$
$\triangleq \beta_X(\langle g, in_5(\gamma_{B,X}(f_{X\uplus\{x\}}(\langle G_{in_X}(g), P'\rangle)))\rangle)$, whence the thesis since
$\beta_X(\langle g, (Tf)_X((st_{G,Proc})_X(\langle g, in_5(P')\rangle))\rangle) = \beta_X(\langle g, (Tf)_X(in_5(\overline{P'}_g))\rangle) =$
$= \beta_X(\langle g, in_5(\gamma_{B,X}(f_{X\uplus\{x\}}(P'_{X\uplus\{x\}}(\langle x, in_X\rangle))))\rangle)$.

B.10 Proof of Theorem 6.4

Suppose that

$$Y \Vdash_{R:\iota\to o,\eta_R} (R\ 0), \tag{B 3}$$
$$Y \Vdash_{R:\iota\to o,\eta_R} (\forall P^\iota.(R\ P) \Rightarrow (R\ \tau.P)), \tag{B 4}$$
$$Y \Vdash_{R:\iota\to o,\eta_R} (\forall P^\iota.(R\ P) \Rightarrow \forall Q^\iota.(R\ Q) \Rightarrow (R\ P|Q)), \tag{B 5}$$
$$Y \Vdash_{R:\iota\to o,\eta_R} (\forall y^\upsilon.\forall z^\upsilon.\forall P^\iota.(R\ P) \Rightarrow (R\ [y \neq z]P)), \tag{B 6}$$
$$Y \Vdash_{R:\iota\to o,\eta_R} (\forall P^{\upsilon\to\iota}.(\forall x^\upsilon.(R\ (P\ x))) \Rightarrow (R\ \nu P)), \tag{B 7}$$

We prove that $G^*(\top) \bullet G^*(!_{TU}) = p \bullet G^*(\alpha) \bullet T/\!\!/G(h)$. We first translate the latter equation in terms of composition in the category $\check{\mathscr{V}}$ and we obtain the following:

$$G^*(\top) \circ \langle \pi, G^*(!_{TU})\rangle = p \circ \langle \pi, G^*(\alpha)\rangle \circ \langle \pi, (T/\!\!/G)_h\rangle.$$

Then, unfolding the definitions of G^* and $T/\!\!/G$, we get:

$$\top \circ \pi' \circ \langle \pi, !_{TU} \circ \pi'\rangle = p \circ \langle \pi, \alpha \circ \pi'\rangle \circ \langle \pi, Th \circ st_{G,U}\rangle,$$

i.e., we have to prove that $\top\circ !_{TU} \circ \pi' = p \circ \langle \pi, \alpha \circ Th \circ st_{G,U}\rangle$. So, taken any $Z \in \mathscr{V}$, $g \in G_Z$ and $u \in (TU)_Z$, we have that $\top_Z((!_{TU})_Z(\pi'_Z(\langle g, u\rangle))) = \top_Z((!_{TU})_Z(u)) = \top_Z(*) = \mathscr{I}(Z, \_)$, while for the second member of the equation we have the following:

($u = in_1(*)$)

$$p_Z(\langle \pi_Z(\langle g, in_1(*)\rangle), \alpha_Z((Th)_Z((st_{G,U})_Z(\langle g, in_1(*)\rangle)))\rangle) =$$
$$= p_Z(\langle g, \alpha_Z((Th)_Z(in_1(*)))\rangle) = p_Z(\langle g, \alpha_Z(in_1(*))\rangle) = p_Z(\langle g, 0\rangle)$$

Hence, $p_Z(\langle g, 0\rangle) = (ev_{Proc,Prop})_Z(\langle 0, g\rangle) \wedge \mathscr{I}(Z, \_) = g_Z(\langle 0, \mathrm{id}_Z\rangle) \wedge \mathscr{I}(Z, \_)$. Since we know that for all $Y \in \mathscr{V}$, and $\eta_R \in (Proc \Rightarrow Prop)_Y$, $Y \Vdash_{R:\iota\to o,\eta_R} (R\ 0)$ holds, we can deduce, by point 3 of Theorem 4.1, that $[\![R : \iota \to o, P : \iota \vdash_\Sigma (R\ P) : o]\!]_Z(\langle g, 0\rangle) = ([\![R : \iota \to o, P : \iota \vdash_\Sigma R : \iota \to o]\!]_Z(\langle g, 0\rangle))_Z(\langle [\![R : \iota \to o, P : \iota \vdash_\Sigma 0 : \iota]\!]_Z(\langle g, 0\rangle), \mathrm{id}_Z\rangle) = g_Z(\langle 0, \mathrm{id}_Z\rangle) \geqslant \mathscr{I}(Z, \_)$, whence the thesis.
($u = in_2(q)$)

$$p_Z(\langle \pi_Z(\langle g, in_2(q)\rangle), \alpha_Z((Th)_Z((st_{G,U})_Z(\langle g, in_2(q)\rangle)))\rangle) =$$
$$= p_Z(\langle g, \alpha_Z((Th)_Z(in_2(\langle g, q\rangle)))\rangle) = p_Z(\langle g, \alpha_Z(in_2(h_Z(\langle g, q\rangle)))\rangle) =$$
$$= p_Z(\langle g, \tau.h_Z(\langle g, q\rangle)\rangle)$$

At this point we know, by equation B 4, that for all $Y \in \mathscr{V}$, and $\eta_R \in (Proc \Rightarrow Prop)_Y$, $Y \Vdash_{R:\iota\to o,\eta_R} \forall P^\iota.(R\ P) \Rightarrow (R\ \tau P)$ holds. By points 1 and 2 of Theorem 4.1, this amounts to say that, for all $V \in \mathscr{V}$, $l \in \mathscr{I}(Y,V)$ and $\eta_P \in Proc_V$,

$$V \Vdash_{(R:\iota\to o,P:\iota),\langle (Proc\Rightarrow Prop)_l(\eta_R),\eta_P\rangle} (R\ P) \quad \text{implies} \quad V \Vdash_{(R:\iota\to o,P:\iota),\langle (Proc\Rightarrow Prop)_l(\eta_R),\eta_P\rangle} (R\ \tau P).$$

Then we notice the following facts:

1. $p_Z(\langle g, h_Z(\langle g,q\rangle)\rangle) = \mathscr{I}(Z,\_)$;
2. $p_Z(\langle g, h_Z(\langle g,q\rangle)\rangle) =$
   $= (ev_{Proc,Prop})_Z(\langle h_Z(\langle g,q\rangle), g\rangle) \wedge \mathscr{I}(Z,\_) = g_Z(\langle h_Z(\langle g,q\rangle), \mathrm{id}_Z\rangle) \wedge \mathscr{I}(Z,\_)$;
3. $[\![R:\iota\to o, P:\iota \vdash_\Sigma (R\ P):o]\!]_Z(\langle g, h_Z(\langle g,q\rangle)\rangle) = ([\![R:\iota\to o, P:\iota \vdash_\Sigma R:\iota\to o]\!]_Z(\langle g, h_Z(\langle g,q\rangle)\rangle))_Z(\langle [\![R:\iota\to o, P:\iota\vdash_\Sigma P:\iota]\!]_Z(\langle g,h_Z(\langle g,q\rangle)\rangle), \mathrm{id}_Z\rangle) = g_Z(\langle h_Z(\langle g,q\rangle), \mathrm{id}_Z\rangle)$ (by point 3 of Theorem 4.1); it follows from the previous two facts that $g_Z(\langle h_Z(\langle g,q\rangle),\ \mathrm{id}_Z\rangle) \geqslant \mathscr{I}(Z,\_)$; hence $Z \Vdash_{(R:\iota\to o,P:\iota),\langle g,h_Z(\langle g,q\rangle)\rangle)} (R\ P)$ holds;
4. from the previous fact and the inductive hypothesis we can deduce that
   $$Z \Vdash_{(R:\iota\to o,P:\iota),\langle g,h_Z(\langle g,q\rangle)\rangle} (R\ \tau P)$$
   holds, i.e., $[\![R:\iota\to o, P:\iota\vdash_\Sigma (R\ \tau P):o]\!]_Z(\langle g,h_Z(\langle g,q\rangle)\rangle) \geqslant \mathscr{I}(Z,\_)$;
5. by Theorem 4.1(3), we have $[\![R{:}\iota\to o, P{:}\iota \vdash_\Sigma (R\ \tau P){:}o]\!]_Z(\langle g, h_Z(\langle g,q\rangle)\rangle) = ([\![R{:}\iota\to o, P{:}\iota\vdash_\Sigma R{:}\iota\to o]\!]_Z(\langle g,h_Z(\langle g,q\rangle)\rangle))_Z(\langle [\![R:\iota\to o, P:\iota\vdash_\Sigma \tau P:\iota]\!]_Z(\langle g,h_Z(\langle g,q\rangle)\rangle), \mathrm{id}_Z\rangle) =$
   $g_Z(\langle tau([\![R{:}\iota\to o, P:\iota\vdash_\Sigma P{:}\iota]\!]_Z(\langle g,h_Z(\langle g,q\rangle)), \mathrm{id}_Z\rangle) =$
   $g_Z(\langle tau(h_Z(\langle g,q\rangle)), \mathrm{id}_Z\rangle) = g_Z(\tau.h_Z(\langle g,q\rangle), \mathrm{id}_Z) = p_Z(\langle g, \tau.h_Z(\langle g,q\rangle)\rangle) \wedge \mathscr{I}(Z,\_)$
   whence the thesis.

($u = in_3(q,r)$)

$$\begin{aligned}
&p_Z(\langle \pi_Z(\langle g, in_3(q,r)\rangle), \alpha_Z((Th)_Z((st_{G,U})_Z(\langle g, in_3(q,r)\rangle)))\rangle) = \\
&= p_Z(\langle g, \alpha_Z((Th)_Z(in_3(\langle g,q,g,r\rangle)))\rangle) = \\
&= p_Z(\langle g, \alpha_Z(in_3(h_Z(\langle g,q\rangle), in_3(h_Z(\langle g,r\rangle))))\rangle) = \\
&= p_Z(\langle g, h_Z(\langle g,q\rangle) | h_Z(\langle g,r\rangle)\rangle)
\end{aligned}$$

Equation B 5, states that for all $Y \in \mathscr{V}$, and $\eta_R \in (Proc \Rightarrow Prop)_Y$, $Y \Vdash_{R:\iota\to o,\eta_R} \forall P^\iota.(R\ P) \Rightarrow \forall Q^\iota.(R\ Q) \Rightarrow (R\ P|Q)$ holds. By points 1 and 2 of Theorem 4.1, this amounts to say that, for all $V \in \mathscr{V}$, $l \in \mathscr{I}(Y,V)$ and $\eta_P \in Proc_V$,

$$V \Vdash_{(R:\iota\to o,P:\iota),\langle (Proc\Rightarrow Prop)_l(\eta_R),\eta_P\rangle} (R\ P) \quad \text{implies} \quad V \Vdash_{(R:\iota\to o,P:\iota),\langle (Proc\Rightarrow Prop)_l(\eta_R),\eta_P\rangle} \forall Q^\iota.(R\ Q) \Rightarrow (R\ P|Q).$$

Applying again the same theorem, the latter judgment is in turn equivalent to say that, for all $W \in \mathscr{V}$, $m \in \mathscr{I}(V,W)$ and $\eta_Q \in Proc_W$,

$$W \Vdash_{(R:\iota\to o,P:\iota,Q:\iota),\langle (Proc\Rightarrow Prop)_{m\circ l}(\eta_R), Proc_m(\eta_P),\eta_Q\rangle} (R\ Q) \quad \text{implies} \quad W \Vdash_{(R:\iota\to o,P:\iota,Q:\iota),\langle (Proc\Rightarrow Prop)_{m\circ l}(\eta_R), Proc_m(\eta_P),\eta_Q\rangle} (R\ P|Q).$$

Then we notice the following facts:

1. $p_Z(\langle g, h_Z(\langle g,q\rangle)\rangle) = \mathscr{I}(Z,\_)$ and $p_Z(\langle g, h_Z(\langle g,r\rangle)\rangle) = \mathscr{I}(Z,\_)$;
2. $p_Z(\langle g, h_Z(\langle g,q\rangle)\rangle) = (ev_{Proc,Prop})_Z(\langle h_Z(\langle g,q\rangle), g\rangle) \wedge \mathscr{I}(Z,\_) = g_Z(\langle h_Z(\langle g,q\rangle), \mathrm{id}_Z\rangle) \wedge \mathscr{I}(Z,\_)$ and analogously $p_Z(\langle g, h_Z(\langle g,r\rangle)\rangle) = g_Z(\langle h_Z(\langle g,r\rangle), \mathrm{id}_Z\rangle) \wedge \mathscr{I}(Z,\_)$;
3. $[\![R : \iota \to o, P : \iota \vdash_\Sigma (R\ P) : o]\!]_Z(\langle g, h_Z(\langle g,q\rangle)\rangle) = ([\![R : \iota \to o, P : \iota \vdash_\Sigma R : \iota \to o]\!]_Z(\langle g, h_Z(\langle g,q\rangle)\rangle))_Z(\langle [\![R : \iota \to o, P : \iota \vdash_\Sigma P : \iota]\!]_Z(\langle g, h_Z(\langle g,q\rangle)\rangle), \mathrm{id}_Z\rangle) = g_Z(\langle h_Z(\langle g,q\rangle), \mathrm{id}_Z\rangle)$ (by point 3 of Theorem 4.1); it follows from the previous two facts that $g_Z(\langle h_Z(\langle g,q\rangle), \mathrm{id}_Z\rangle) \geqslant \mathscr{I}(Z,\_)$; hence $Z \Vdash_{(R:\iota\to o, P:\iota), \langle g, h_Z(\langle g,q\rangle)\rangle}$ $(R\ P)$ holds;
4. similarly we have that $[\![R{:}\iota \to o, P{:}\iota, Q{:}\iota \vdash_\Sigma (R\ Q) : o]\!]_Z(\langle g, h_Z(\langle g,q\rangle), h_Z(\langle g,r\rangle)\rangle) = g_Z(\langle h_Z(\langle g,r\rangle), \mathrm{id}_Z\rangle) \geqslant \mathscr{I}(Z,\_)$; hence $Z \Vdash_{(R:\iota\to o, P:\iota), \langle g, h_Z(\langle g,r\rangle)\rangle}$ $(R\ Q)$ holds;
5. from the previous facts and the inductive hypothesis we can deduce that

   $$Z \Vdash_{(R:\iota\to o, P:\iota, Q:\iota), \langle g, h_Z(\langle g,q\rangle), h_Z(\langle g,r\rangle)\rangle} (R\ P|Q)$$

   holds, i.e.,

   $$[\![R : \iota \to o, P : \iota, Q : \iota \vdash_\Sigma (R\ P|Q) : o]\!]_Z(\langle g, h_Z(\langle g,q\rangle), h_Z(\langle g,r\rangle)\rangle) \geqslant \mathscr{I}(Z,\_);$$

6. by Theorem 4.1(3), we have
   $[\![R{:}\iota \to o, P{:}\iota, Q{:}\iota \vdash_\Sigma (R\ P|Q) : o]\!]_Z(\langle g, h_Z(\langle g,q\rangle), h_Z(\langle g,r\rangle)\rangle) =$
   $= ([\![R{:}\iota \to o, P{:}\iota, Q{:}\iota \vdash_\Sigma R : \iota \to o]\!]_Z(\langle g, h_Z(\langle g,q\rangle), h_Z(\langle g,r\rangle)\rangle))_Z(\langle [\![R{:}\iota \to o, P{:}\iota, Q{:}\iota \vdash_\Sigma P|Q{:}\iota]\!]_Z(\langle g, h_Z(\langle g,q\rangle), h_Z(\langle g,r\rangle)\rangle), \mathrm{id}_Z\rangle) =$
   $= g_Z(\langle par(\langle [\![R : \iota \to o, P{:}\iota, Q{:}\iota \vdash_\Sigma P{:}\iota]\!]_Z(\langle g, h_Z(\langle g,q\rangle), h_Z(\langle g,r\rangle)\rangle), [\![R{:}\iota \to o, P{:}\iota, Q{:}\iota \vdash_\Sigma Q{:}\iota]\!]_Z(\langle g, h_Z(\langle g,q\rangle), h_Z(\langle g,r\rangle)\rangle)\rangle), \mathrm{id}_Z\rangle) =$
   $= g_Z(\langle par(\langle h_Z(\langle g,q\rangle), h_Z(\langle g,r\rangle)\rangle), \mathrm{id}_Z\rangle) = g_Z(h_Z(\langle g,q\rangle)|h_Z(\langle g,r\rangle), \mathrm{id}_Z) =$
   $p_Z(\langle g, h_Z(\langle g,q\rangle)|h_Z(\langle g,r\rangle)\rangle) \wedge \mathscr{I}(Z,\_)$, whence the thesis.

($u = in_4(v,w,q)$)

$$\begin{aligned} &p_Z(\langle \pi_Z(\langle g, in_4(v,w,q)\rangle), \alpha_Z((Th)_Z((st_{G,U})_Z(\langle g, in_4(v,w,q)\rangle)))\rangle) = \\ &= p_Z(\langle g, \alpha_Z((Th)_Z(in_4(\langle v,w,g,q\rangle)))\rangle) = \\ &= p_Z(\langle g, \alpha_Z(in_4(\langle v,w,h_Z(\langle g,q\rangle)\rangle))\rangle) = p_Z(\langle g, [v \neq w]h_Z(\langle g,q\rangle)\rangle) \end{aligned}$$

At this point we know, by equation B 6, that for all $Y \in \mathscr{V}$, and $\eta_R \in (Proc \Rightarrow Prop)_Y$, $Y \Vdash_{R:\iota\to o, \eta_R} \forall x^\upsilon.\forall y^\upsilon.\forall P^\iota.(R\ P) \Rightarrow (R\ [x \neq y]P)$ holds. By point 2 of Theorem 4.1 and point 5 of Corollary 4.1, this amounts to say that, for all $V \in \mathscr{V}$, $l \in \mathscr{I}(Y,V)$ $\eta_x, \eta_y \in V$ and $\eta_P \in Proc_V$,

$$V \Vdash_{(R:\iota\to o, x:\upsilon, y:\upsilon, P:\iota), \langle (Proc\Rightarrow Prop)_l(\eta_R), \eta_x, \eta_y, \eta_P\rangle} (R\ P)$$
$$\text{implies} \quad V \Vdash_{(R:\iota\to o, x:\upsilon, y:\upsilon, P:\iota), \langle (Proc\Rightarrow Prop)_l(\eta_R), \eta_x, \eta_y, \eta_P\rangle} (R\ [x \neq y]P).$$

Then we notice the following facts:

1. $p_Z(\langle g, h_Z(\langle g,q\rangle)\rangle) = \mathscr{I}(Z,\_)$;
2. $p_Z(\langle g, h_Z(\langle g,q\rangle)\rangle) = (ev_{Proc,Prop})_Z(\langle h_Z(\langle g,q\rangle), g\rangle) \wedge \mathscr{I}(Z,\_) =$
   $= g_Z(\langle h_Z(\langle g,q\rangle), \mathrm{id}_Z\rangle) \wedge \mathscr{I}(Z,\_)$;
3. $[\![R : \iota \to o, P : \iota \vdash_\Sigma (R\ P) : o]\!]_Z(\langle g, h_Z(\langle g,q\rangle)\rangle) = ([\![R : \iota \to o, P : \iota \vdash_\Sigma R : \iota \to o]\!]_Z(\langle g, h_Z(\langle g,q\rangle)\rangle))_Z(\langle [\![R : \iota \to o, P : \iota \vdash_\Sigma P : \iota]\!]_Z(\langle g, h_Z(\langle g,q\rangle)\rangle), \mathrm{id}_Z\rangle) =$

$g_Z(\langle h_Z(\langle g, q\rangle), \mathrm{id}_Z\rangle)$ (by point 3 of Theorem 4.1); it follows from the previous two facts that $g_Z(\langle h_Z(\langle g, q\rangle), \mathrm{id}_Z\rangle) \geqslant \mathscr{I}(Z, \_)$; hence $Z \Vdash_{(R:\iota\to o, P:\iota), \langle g, h_Z(\langle g,q\rangle)\rangle)}$ $(R\ P)$ holds;

4. from the previous fact and the inductive hypothesis we can deduce that

$$Z \Vdash_{(R:\iota\to o, x:\upsilon, y:\upsilon, P:\iota), \langle g, v, w, h_Z(\langle g,q\rangle)\rangle} (R\ [x \neq y]P)$$

holds, i.e.,

$$[\![R:\iota\to o, x:\upsilon, y:\upsilon, P:\iota \vdash_\Sigma (R\,[x\neq y]P):o]\!]_Z(\langle g, v, w, h_Z(\langle g,q\rangle)\rangle) \geqslant \mathscr{I}(Z, \_);$$

5. by Theorem 4.1(3), we have $[\![R{:}\iota\to o, x{:}\upsilon, y{:}\upsilon, P{:}\iota \vdash_\Sigma (R\ [x\neq y]P){:}o]\!]_Z(\langle g, v, w, h_Z(\langle g,q\rangle)\rangle) = ([\![R{:}\iota\to o, x{:}\upsilon, y{:}\upsilon, P{:}\iota \vdash_\Sigma R : \iota\to o]\!]_Z(\langle g, v, w, h_Z(\langle g,q\rangle)\rangle))_Z$ $(\langle [\![R:\iota\to o, x:\upsilon, y:\upsilon, P:\iota \vdash_\Sigma [x\neq y]P:\iota]\!]_Z(\langle g, v, w, h_Z(\langle g,q\rangle)\rangle), \mathrm{id}_Z\rangle) =$ $g_Z(\langle \mathit{mismatch}(\langle [\![R{:}\iota\to o, x{:}\upsilon, y{:}\upsilon, P{:}\iota \vdash_\Sigma x{:}\upsilon]\!]_Z(\langle g, v, w, h_Z(\langle g,q\rangle)\rangle), [\![R:\iota\to o, x{:}\upsilon, y{:}\upsilon, P{:}\iota \vdash_\Sigma y{:}\upsilon]\!]_Z(\langle g, v, w, h_Z(\langle g,q\rangle)\rangle), [\![R{:}\iota\to o, x{:}\upsilon, y{:}\upsilon, P{:}\iota \vdash_\Sigma P:\iota]\!]_Z(\langle g, v, w, h_Z(\langle g,q\rangle)\rangle)\rangle), \mathrm{id}_Z\rangle) = g_Z(\langle \mathit{mismatch}(\langle v, w, h_Z(\langle g,q\rangle)\rangle), \mathrm{id}_Z\rangle) =$ $g_Z([v\neq w]h_Z(\langle g,q\rangle), \mathrm{id}_Z) = p_Z(\langle g, [v\neq w]h_Z(\langle g,q\rangle)\rangle) \wedge \mathscr{I}(Z, \_)$, whence the thesis.

$(u = in_5(q))$

$$\begin{aligned} &p_Z(\langle \pi_Z(\langle g, in_5(q)\rangle), \alpha_Z((Th)_Z((st_{G,U})_Z(\langle g, in_5(q)\rangle)))\rangle) = \\ &= p_Z(\langle g, \alpha_Z((Th)_Z(in_5(\overline{q}_g)))\rangle) = \\ &= p_Z(\langle g, \alpha_Z(in_5(h_{Z\uplus z}((\overline{q}_g)_{Z\uplus z}(z, in_Z))))\rangle) = \\ &= p_Z(\langle g, (\nu z)h_{Z\uplus z}((\overline{q}_g)_{Z\uplus z}(z, in_Z))\rangle), \end{aligned}$$

where $\overline{q}_g : Var \times \mathscr{V}(Z, \_) \longrightarrow G\times U$ is the natural transformation such that, for all $Y\in\mathscr{V}$, $y\in Y$ and $f\in\mathscr{V}(Z,Y)$, $(\overline{q}_g)_Y(y,f) = \langle G_f(g), q_Y(\langle y, f\rangle)\rangle$.
At this point we know, by equation B 7, that for all $Y\in\mathscr{V}$, and $\eta_R \in (Proc \Rightarrow Prop)_Y$, $Y \Vdash_{R:\iota\to o, \eta_R} \forall P^{\upsilon\to\iota}.(\forall x^\upsilon.(R\ (P\ x))) \Rightarrow (R\ \nu P)$ holds. By points 1 and 2 of Theorem 4.1, this amounts to say that, for all $V\in\mathscr{V}$, $l\in\mathscr{I}(Y,V)$ and $\eta_P \in (Var \Rightarrow Proc)_V$,

$$V \Vdash_{(R:\iota\to o, P:\upsilon\to\iota), \langle (Proc\Rightarrow Prop)_l(\eta_R), \eta_P\rangle} \forall x^\upsilon.(R\ (P\ x)) \quad \text{implies} \quad V \Vdash_{(R:\iota\to o, P:\upsilon\to\iota), \langle (Proc\Rightarrow Prop)_l(\eta_R), \eta_P\rangle} (R\ \nu P).$$

Then we notice the following facts:

1. $V \Vdash_{(R:\iota\to o, P:\upsilon\to\iota), \langle (Proc\Rightarrow Prop)_l(\eta_R), \eta_P\rangle} \forall x^\upsilon.(R\ (P\ x))$ iff, for all $W\in\mathscr{W}$, $m\in\mathscr{I}(V,W)$ and $\eta_x\in W$, the following holds:

$$W \Vdash_{(R:\iota\to o, P:\upsilon\to\iota, x:\upsilon), \langle (Proc\Rightarrow Prop)_{m\circ l}(\eta_R), \eta_P, \eta_x\rangle} (R\ (P\ x)),$$

i.e., iff

$$[\![\Delta \vdash_\Sigma (R\ (P\ x)) : o]\!]_W(\eta) \geqslant \mathscr{I}(W, \_),$$

where $\Delta \triangleq R:\iota\to o, P:\upsilon\to\iota, x:\upsilon$ and $\eta \triangleq \langle (Proc\Rightarrow Prop)_{m\circ l}(\eta_R), (Var\Rightarrow Proc)_m(\eta_P), \eta_x\rangle$. The first member of the preceding inequality can be simplified as follows according to Theorem 4.1:

$$\begin{aligned}&[\![\Delta \vdash_\Sigma (R\ (P\ \ x)) : o]\!]_W(\eta) \geqslant \mathscr{I}(W, _)\\
=&([\![\Delta \vdash_\Sigma R : \iota \to o]\!]_W(\eta))_W(\langle [\![\Delta \vdash_\Sigma (P\ \ x) : \iota]\!]_W(\eta), \mathrm{id}_W\rangle)\\
=&((Proc \Rightarrow Prop)_{m\circ l}(\eta_R))_W(\langle([\![\Delta \vdash_\Sigma P : \upsilon \to \iota]\!]_W(\eta))_W\\
&\quad(\langle [\![\Delta \vdash_\Sigma x : \upsilon]\!]_W(\eta), \mathrm{id}_W\rangle), \mathrm{id}_W\rangle)\\
=&((Proc \Rightarrow Prop)_{m\circ l}(\eta_R))_W(\langle(((Var \Rightarrow Proc)_m(\eta_P))_W(\langle \eta_x, \mathrm{id}_W\rangle), \mathrm{id}_W\rangle)\end{aligned}$$

2. in particular, when $V \triangleq Z$, $l \triangleq \mathrm{id}_Z$, $\eta_R \triangleq g$ and $\eta_P \triangleq h \circ \overline{q}_g$, we have that the following holds:

$$\begin{aligned}&((Proc \Rightarrow Prop)_m(g))_W(\langle(((Var \Rightarrow Proc)_m(h\circ \overline{q}_g))_W(\langle \eta_x, \mathrm{id}_W\rangle), \mathrm{id}_W\rangle)\\
=&((Proc \Rightarrow Prop)_m(g))_W(\langle (h\circ \overline{q}_g)_W(\langle \eta_x, m\rangle), \mathrm{id}_W\rangle)\\
=&((Proc \Rightarrow Prop)_m(g))_W(\langle h_W(\langle (Proc \Rightarrow Prop)_m(g), q_W(\langle \eta_X, m\rangle)\rangle), \mathrm{id}_W\rangle)\end{aligned}$$

3. $p_W(\langle (Proc \Rightarrow Prop)_m(g), h_W(\langle (Proc \Rightarrow Prop)_m(g), q_W(\langle \eta_X, m\rangle)\rangle), \mathrm{id}_W\rangle\rangle) = \mathscr{I}(W, \_)$;
4. $p_W(\langle (Proc \Rightarrow Prop)_m(g), h_W(\langle (Proc \Rightarrow Prop)_m(g), q_W(\langle \eta_X, m\rangle)\rangle), \mathrm{id}_W\rangle\rangle) = ((Proc \Rightarrow Prop)_m(g))_W(\langle h_W(\langle (Proc \Rightarrow Prop)_m(g), q_W(\langle \eta_X, m\rangle)\rangle), \mathrm{id}_W\rangle) \wedge \mathscr{I}(W, \_)$; hence, for all W, $m \in \mathscr{I}(Z, W)$ and $\eta_x \in W$ we have

$$W \Vdash_{(R:\iota\to o, P:\upsilon\to\iota, x:\upsilon), \langle (Proc\Rightarrow Prop)_m(g), h\circ\overline{q}_g, \eta_x\rangle} (R\ (P\ \ x));$$

5. it follows that $Z \Vdash_{(R:\iota\to o, P:\upsilon\to\iota), \langle g, h\circ\overline{q}_g\rangle} (R\ \nu P)$ holds by the previous point and the inductive hypothesis, i.e, $[\![R{:}\iota \to o, P{:}\upsilon \to \iota \vdash_\Sigma (R\ \nu P) : \iota]\!]_Z(\langle g, h\circ \overline{q}_g\rangle) = ([\![R{:}\iota \to o, P{:}\upsilon \to \iota \vdash_\Sigma R{:}\iota \to o]\!]_Z(\langle g, h\circ\overline{q}_g\rangle))_Z(\langle [\![R{:}\iota \to o, P{:}\upsilon \to \iota \vdash_\Sigma \nu P : \iota]\!]_Z(\langle g, h\circ\overline{q}_g\rangle), \mathrm{id}_Z\rangle) = g_Z(\langle new_Z(\langle h\circ\overline{q}_g\rangle), \mathrm{id}_Z\rangle) = g_Z(\langle(\nu z)((h\circ\overline{q}_g)_{Z\uplus z}(\langle z, in_Z\rangle)), \mathrm{id}_Z\rangle) = g_Z(\langle (\nu z)(h_{Z\uplus z}((\overline{q}_g)_{Z\uplus z}(\langle z, in_Z\rangle))), \mathrm{id}_Z\rangle) \geqslant \mathscr{I}(Z, \_)$ holds. The thesis follows since

$$p_Z(\langle g, (\nu z)h_{Z\uplus z}((\overline{q}_g)_{Z\uplus z}(z, in_Z))\rangle) = g_Z(\langle (\nu z)(h_{Z\uplus z}((\overline{q}_g)_{Z\uplus z}(\langle z, in_Z\rangle))), \mathrm{id}_Z\rangle) \wedge \mathscr{I}(Z, \_).$$