Phonology **37** (2020). Supplementary materials

Notational equivalence in tonal geometry

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Supplementary materials

Appendix A: Process transductions

1 Preliminaries

Graph models of separated and bundled representations – such as those in (14) and (13) – are defined over model signatures, ζ_s and ζ_b respectively. These signatures comprise the relations and functions which define label nodes (features) and edges (association, dominance and immediate successor) in each model.

(51) $\zeta_s = \{P_\sigma, P_T, P_c, P_{+\mathrm{u}}, P_{-\mathrm{u}}, P_h, P_l; \alpha, \delta, s\}$
$\zeta_b = \{P_\sigma, P_{+\mathrm{u}}, P_{-\mathrm{u}}, P_h, P_l; \alpha, \delta, s\}$

I define two QF predicate logical languages, $\mathcal{L}_s$ and $\mathcal{L}_b$, from these signatures. Such a logical language contains atomic predicates of the form P(t) for each unary relation in the signature, which is true when a term t is in that relation for a given interpretation. Terms are either members x of a set of variables (which are assigned to a value in a domain $\mathcal{D}$) or any of the unary functions – α, δ, s – applied to a term. Atomic predicates of unary functions are of the form $f(t) \approx t$, where $\approx$ denotes a special identity relation; thus these predicates are true when the two terms denote the same value.

Each of these predicates is a well-formed formula (WFF) in the logical language. We may recurse over the atomic predicates to define the full set of WFFs in each logical language using Boolean connectives (negation $\neg$, conjunction $\wedge$, disjunction $\vee$ and material implication $\rightarrow$). For WFFs φ and ψ, we also have the WFFs $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\varphi \rightarrow \psi$. For example, $P_l(x) \vee P_h(x)$ is a WFF in $\mathcal{L}_s$ and $\mathcal{L}_b$, as is $\alpha(s(x)) \approx y$.

Mappings from input to output are defined as logical TRANSDUCTIONS, denoted τ. These are logical interpretations of an output SIGNATURE (comprising unary relations and functions) in the logical language of the input signature. Crucially, we may allow transductions which are interpretations over a finite ordered copy set $C = \{1, \ldots, n\}$. A set of formulae of the form $P^c(x)$ are defined with one free variable (x) for each unary relation in the output signature and for copy $c \in C$. Similarly, formulae of the form

$f^{n,m}(x) \approx y$ are defined with two free variables (x and y) for all unary functions in the output signature and for all logically possible pairings of copies $n, m \in C$. Thus, for a copy set of size two, the number of formulae for each function matches the four possible pairings of {1,1}, {1,2}, {2,1} and {2,2}.

The semantics of these transductions follows Engelfriet & Hoogeboom (2001). Given an input graph model $\mathcal{M}$ defined over an input signature ζ_I and a domain of elements $\mathcal{D}$, the output $\tau(\mathcal{M})$ is a graph model $\mathcal{M}'$ over an output signature ζ_O and a domain of elements $\mathcal{D}'$. For each element in the input domain $\mathcal{D}$, there is a corresponding output element in $\mathcal{D}'$ for a given copy c which belongs to a unary relation in ζ_O provided that the following conditions are met: the input model satisfies the logical formula $P^c(x)$ for an assignment of x to a domain element d, it does so for exactly one unary relation in the output signature, and it does so for exactly one copy $c \in C$.

I define the following set of auxiliary relations. The first identifies the final string position on a tier (i.e. the position which is its own successor), the second identifies the penultimate string position on a tier. The third is a general 'register node' relation (i.e. labelled either +u or −u), and the fourth is a general 'terminal tonal node' relation (i.e. labelled either h or l).

(52) a. $lst(x) \stackrel{\text{def}}{=} s(x) \approx x$ c. $P_r(x) \stackrel{\text{def}}{=} P_{+\text{u}}(x) \vee P_{-\text{u}}(x)$
b. $pnlt(x) \stackrel{\text{def}}{=} s(x) \approx lst(x)$ d. $P_t(x) \stackrel{\text{def}}{=} P_h(x) \vee P_l(x)$

2 Pingyao: separated model

τ_s^p denotes a transduction over a separated representation model signature, and models register assimilation in Pingyao. It is defined over a copy set of size one. A brief explanation of this definition and how it is satisfied by the graph mapping in (22) is provided in (53).

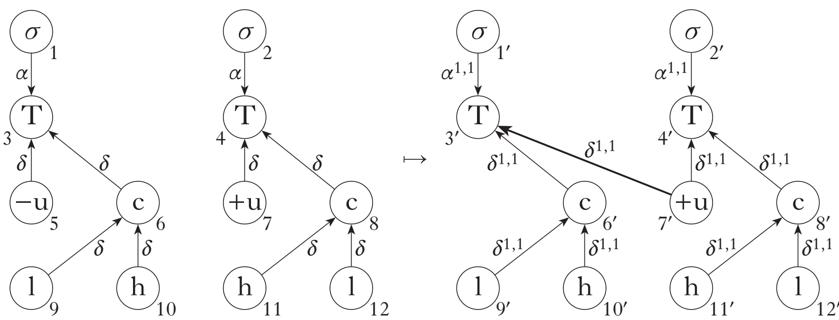
(53) $P^1_\sigma(x) \stackrel{\text{def}}{=} P_\sigma(x)$ $P^1_c(x) \stackrel{\text{def}}{=} P_c(x)$
$P^1_T(x) \stackrel{\text{def}}{=} P_T(x)$ $P^1_h(x) \stackrel{\text{def}}{=} P_h(x)$
$P^1_{+u}(x) \stackrel{\text{def}}{=} P_{+\text{u}}(x) \wedge lst(x)$ $P^1_l(x) \stackrel{\text{def}}{=} P_l(x)$
$P^1_{-u}(x) \stackrel{\text{def}}{=} P_{-\text{u}}(x) \wedge lst(x)$
$\alpha^{1,1}(x) \approx y \stackrel{\text{def}}{=} \alpha(x) \approx y$
$\delta^{1,1}(x) \approx y \stackrel{\text{def}}{=} (P_t(x) \wedge P_c(y) \wedge \delta(x) \approx y) \vee (P_c(x) \wedge P_T(y) \wedge \delta(x) \approx y) \vee (P_r(x) \wedge P_T(y) \wedge lst(x) \wedge \delta(x) \approx y) \vee (P_r(x) \wedge P_T(y) \wedge lst(x) \wedge \delta(x) \approx s(y))$
$s^{1,1}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y$

This definition preserves the following input labels via identity, i.e. definitions of the form $P^1(x) = P(x)$ for unary relations: syllable nodes, T root nodes, c contour nodes and terminal tonal nodes labelled h and l. The definitions of P_{+u} and P_{-u} preserve labels on the final register node only (and therefore penultimate register nodes are unlabelled). Association ($\alpha^{1,1}(x) \approx y$) and successor ($s^{1,1}(x) \approx y$) functions maintain input specifications, as these edges do not vary between input and output.

The definition of output δ edges over graph structures crucially modifies input edges and thus models the assimilatory pattern. It does so in the following way. The first two disjuncts of the $\delta^{1,1}(x) \approx y$ definition evaluate to true for graph structures maintaining input δ edges between tonal terminal nodes and c nodes as well as between c and T nodes. Disjunct three preserves dominance between the final register node and its tautosyllabic T node (guaranteed by the conjunct $\delta(x) \approx y$), and the final disjunct defines dominance between the final register node and a T node whose successor shares a δ edge with that node in the input ($\delta(x) \approx s(y)$); that is, the penultimate T root.

An output graph structure which satisfies this definition is therefore one for which all nodes and edges are preserved from the input, with the exception of an *additional* δ edge between the final register node and penultimate T root node. The mapping in (22) (repeated as (54)) represents such a structure.

(54)



3 Pingyao: bundled model

τ_b^p denotes a transduction over a bundled representation model signature, and models register assimilation in Pingyao. It is defined over a copy set of size two. Formulae defined as F ('False') below and in subsequent definitions indicate no labels/edges in the output structure for the given unary relation/function and copy.

(55)

$$
\begin{array}{lll}
P^1_\sigma(x) \stackrel{\text{def}}{=} P_\sigma(x) & P^2_\sigma(x) \stackrel{\text{def}}{=} \mathrm{F} & \\
P^1_{+\mathrm{u}}(x) \stackrel{\text{def}}{=} P_{+\mathrm{u}}(x) \wedge lst(x) & P^2_{+\mathrm{u}}(x) \stackrel{\text{def}}{=} P_{+\mathrm{u}}(x) \wedge lst(x) & \\
P^1_{-\mathrm{u}}(x) \stackrel{\text{def}}{=} P_{-\mathrm{u}}(x) \wedge lst(x) & P^2_{-\mathrm{u}}(x) \stackrel{\text{def}}{=} P_{-\mathrm{u}}(x) \wedge lst(x) & \\
P^1_h(x) \stackrel{\text{def}}{=} P_h(x) & P^2_h(x) \stackrel{\text{def}}{=} \mathrm{F} & \\
P^1_l(x) \stackrel{\text{def}}{=} P_l(x) & P^2_l(x) \stackrel{\text{def}}{=} \mathrm{F} & \\
\multicolumn{2}{l}{\alpha^{1,1}(x) \approx y \stackrel{\text{def}}{=} P_\sigma(x) \wedge P_r(y) \wedge lst(x) \wedge lst(y)} & \alpha^{2,1}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\multicolumn{2}{l}{\alpha^{1,2}(x) \approx y \stackrel{\text{def}}{=} P_\sigma(x) \wedge P_r(y) \wedge lst(y) \wedge \alpha(s(x)) \approx y} & \alpha^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\multicolumn{2}{l}{\delta^{1,1}(x) \approx y \stackrel{\text{def}}{=} P_t(x) \wedge P_r(y) \wedge lst(\delta(x)) \wedge lst(y)} & \delta^{2,1}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\multicolumn{2}{l}{\delta^{1,2}(x) \approx y \stackrel{\text{def}}{=} P_t(x) \wedge P_r(y) \wedge pnlt(\delta(x)) \wedge s(\delta(x)) \approx y} & \delta^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
s^{1,1}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y & &
\end{array}
$$

Graph mappings such as those in (26) satisfy this definition.

4 Zhenjiang: separated model

τ_s^z denotes a transduction over a separated representation model signature, and models contour assimilation in Zhenjiang. It is defined over a copy set of size two.

(56)
$$\begin{array}{ll}
P_\sigma^1(x) \stackrel{\text{def}}{=} P_\sigma(x) & P_\sigma^2(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P_T^1(x) \stackrel{\text{def}}{=} P_T(x) & P_T^2(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P_{+\mathrm{u}}^1(x) \stackrel{\text{def}}{=} P_{+\mathrm{u}}(x) & P_{+\mathrm{u}}^2(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P_{-\mathrm{u}}^1(x) \stackrel{\text{def}}{=} P_{-\mathrm{u}}(x) & P_{-\mathrm{u}}^2(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P_c^1(x) \stackrel{\text{def}}{=} P_c(x) \wedge lst(x) & P_c^2(x) \stackrel{\text{def}}{=} P_c(x) \wedge lst(x) \\
P_h^1(x) \stackrel{\text{def}}{=} P_h(x) \wedge lst(\delta(x)) & P_h^2(x) \stackrel{\text{def}}{=} P_h(x) \wedge lst(\delta(x)) \\
P_l^1(x) \stackrel{\text{def}}{=} \mathrm{F} & P_l^2(x) \stackrel{\text{def}}{=} \mathrm{F} \\
\alpha^{1,1}(x) \approx y \stackrel{\text{def}}{=} \alpha(x) \approx y & \alpha^{2,1}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\alpha^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \alpha^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\delta^{1,1}(x) \approx y \stackrel{\text{def}}{=} \delta(x) \approx y & \delta^{2,1}(x) \approx y \stackrel{\text{def}}{=} P_c(x) \wedge P_T(y) \wedge lst(x) \wedge \delta(x) \approx s(y) \\
\delta^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \delta^{2,2}(x) \approx y \stackrel{\text{def}}{=} P_h(x) \wedge P_c(y) \wedge lst(\delta(x)) \wedge lst(y) \\
s^{1,1}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y &
\end{array}$$

Graph mappings such as those in (27) satisfy this definition.

5 Zhenjiang: bundled model

τ_b^z denotes a transduction over a bundled representation model signature, and models contour assimilation in Zhenjiang. It is defined over a copy set of size two.

(56)
$$\begin{array}{ll}
P_\sigma^1(x) \stackrel{\text{def}}{=} P_\sigma(x) & P_\sigma^2(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P_{+\mathrm{u}}^1(x) \stackrel{\text{def}}{=} P_{+\mathrm{u}}(x) & P_{+\mathrm{u}}^2(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P_{-\mathrm{u}}^1(x) \stackrel{\text{def}}{=} P_{-\mathrm{u}}(x) & P_{-\mathrm{u}}^2(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P_h^1(x) \stackrel{\text{def}}{=} P_h(x) \wedge lst(\delta(x)) & P_h^2(x) \stackrel{\text{def}}{=} P_h(x) \wedge lst(\delta(x)) \\
P_l^1(x) \stackrel{\text{def}}{=} \mathrm{F} & P_l^2(x) \stackrel{\text{def}}{=} \mathrm{F} \\
\alpha^{1,1}(x) \approx y \stackrel{\text{def}}{=} \alpha(x) \approx y & \alpha^{2,1}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\alpha^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \alpha^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\delta^{1,1}(x) \approx y \stackrel{\text{def}}{=} P_t(x) \wedge P_r(y) \wedge & \delta^{2,1}(x) \approx y \stackrel{\text{def}}{=} P_t(x) \wedge P_r(y) \wedge \\
\quad lst(\delta(x)) \wedge lst(y) & \quad lst(\delta(x)) \wedge \delta(x) \approx s(y) \\
\delta^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \delta^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
s^{1,1}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y &
\end{array}$$

Graph mappings such as those in (35) satisfy this definition.

Appendix B: Translation transductions

The transductions Γ^{sb} and Γ^{bs} below satisfy the first component of the bi-interpretability definition. Γ^{sb} is an interpretation of separated models in terms of bundled models, while Γ^{bs} is an interpretation of bundled models in terms of separated models.

1 Separated to bundled: fusion

Γ^{sb} is a transduction defined over a bundled representation model signature which translates any separated model into an equivalent bundled model. It is defined over a copy set of size one.

(57)
$$
\begin{array}{ll}
P^1_\sigma(x) \stackrel{\text{def}}{=} P_\sigma(x) & P^1_h(x) \stackrel{\text{def}}{=} P_t(x) \\
P^1_{+\mathrm{u}}(x) \stackrel{\text{def}}{=} P_r(x) & P^1_l(x) \stackrel{\text{def}}{=} P_t(x) \\
P^1_{-\mathrm{u}}(x) \stackrel{\text{def}}{=} P_r(x) & \\
\multicolumn{2}{l}{\alpha^{1,1}(x) \approx y \stackrel{\text{def}}{=} P_\sigma(x) \wedge P_r(y) \wedge \alpha(x) \approx \delta(y)} \\
\multicolumn{2}{l}{\delta^{1,1}(x) \approx y \stackrel{\text{def}}{=} P_t(x) \wedge P_r(y) \wedge \delta(\delta(x)) \approx \delta(y)} \\
\multicolumn{2}{l}{s^{1,1}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y}
\end{array}
$$

Graph mappings such as those in (38) satisfy this definition.

2 Bundled to separated: expansion

Γ^{sb} is a transduction defined over a separated representation model signature which translates any bundled model into an equivalent separated model. It is defined over a copy set of size three.

(58)
$$
\begin{array}{lll}
P^1_\sigma(x) \stackrel{\text{def}}{=} P_\sigma(x) & P^2_\sigma(x) \stackrel{\text{def}}{=} \mathrm{F} & P^3_\sigma(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P^1_T(x) \stackrel{\text{def}}{=} P_r(x) & P^2_T(x) \stackrel{\text{def}}{=} \mathrm{F} & P^3_T(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P^1_{+\mathrm{u}}(x) \stackrel{\text{def}}{=} \mathrm{F} & P^2_{+\mathrm{u}}(x) \stackrel{\text{def}}{=} P_r(x) & P^3_{+\mathrm{u}}(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P^1_{-\mathrm{u}}(x) \stackrel{\text{def}}{=} \mathrm{F} & P^2_{-\mathrm{u}}(x) \stackrel{\text{def}}{=} P_r(x) & P^3_{-\mathrm{u}}(x) \stackrel{\text{def}}{=} \mathrm{F} \\
P^1_c(x) \stackrel{\text{def}}{=} \mathrm{F} & P^2_c(x) \stackrel{\text{def}}{=} \mathrm{F} & P^3_c(x) \stackrel{\text{def}}{=} P_r(x) \\
P^1_h(x) \stackrel{\text{def}}{=} \mathrm{F} & P^2_h(x) \stackrel{\text{def}}{=} \mathrm{F} & P^3_h(x) \stackrel{\text{def}}{=} P_t(x) \\
P^1_l(x) \stackrel{\text{def}}{=} \mathrm{F} & P^2_l(x) \stackrel{\text{def}}{=} \mathrm{F} & P^3_l(x) \stackrel{\text{def}}{=} P_t(x) \\
\alpha^{1,1}(x) \approx y \stackrel{\text{def}}{=} \alpha(x) \approx y & \alpha^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \alpha^{1,3}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\alpha^{2,1}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \alpha^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \alpha^{2,3}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\alpha^{3,1}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \alpha^{3,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \alpha^{3,3}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\delta^{1,1}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \delta^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \delta^{1,3}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\delta^{2,1}(x) \approx y \stackrel{\text{def}}{=} P_r(x) \wedge P_r(y) \wedge x \approx y & \delta^{2,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \delta^{2,3}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
\delta^{3,1}(x) \approx y \stackrel{\text{def}}{=} P_r(x) \wedge P_r(y) \wedge x \approx y & \delta^{3,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & \delta^{3,3}(x) \approx y \stackrel{\text{def}}{=} \delta(x) \approx y \\
s^{1,1}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y & s^{1,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & s^{1,3}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
s^{2,1}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & s^{2,2}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y & s^{2,3}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} \\
s^{3,1}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & s^{3,2}(x) \approx y \stackrel{\text{def}}{=} \mathrm{F} & s^{3,3}(x) \approx y \stackrel{\text{def}}{=} s(x) \approx y
\end{array}
$$

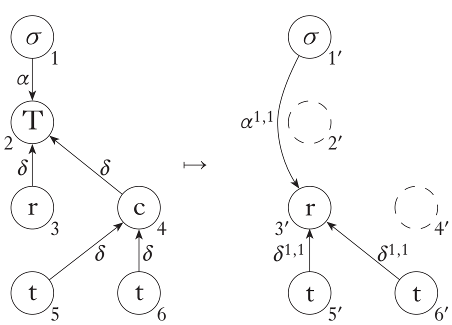
Graph mappings such as those in (43) satisfy this definition.

3 Isomorphism

The second main part of the bi-interpretability definition requires that the composition Γ^{bs} on Γ^{sb} ($\Gamma^{bs} \circ \Gamma^{sb}$) is isomorphic to the identity map on separated models (id_s). Similarly, it requires the composition $\Gamma^{sb} \circ \Gamma^{bs}$ to be isomorphic to the identity map on bundled models (id_b). Thus applying Γ^{sb} to any separated model and then applying Γ^{bs} to its output is the same mapping as a map from the separated model to itself. Additionally, the reverse application over any bundled model is the same mapping as a map from the bundled model to itself.

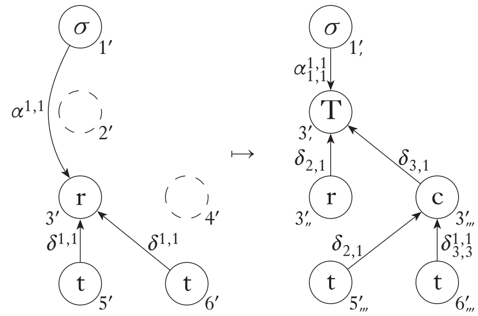
Below, I illustrate this with generalised graph structures, in which register nodes are labelled *r* and binary branching terminals *t*. This shows that this component of bi-interpretability holds for any contour tones representable in either representation. This also generalises to any level tone – by replacing the binary branching graphs below with unary branching ones – and thus holds for the full extent of tonal contrasts formalisable in the two models.

3.1 *Separated model*. First, apply Γ^{sb} to any separated model to generate an equivalent bundled model. In (59), *r* indicates register nodes and *t* indicates terminal tonal nodes, regardless of specification; the transduction preserves register and tonal node features. As before, primes denote output nodes, and superscripts output edges within a single copy set.

(59)


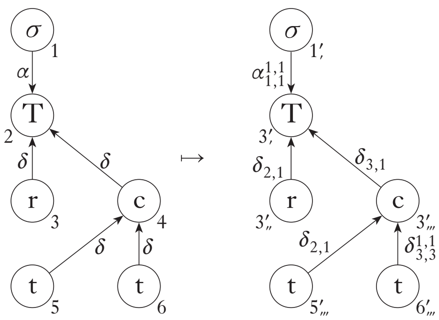
The resulting graph becomes the input structure to which Γ^{bs} is applied, as shown in (60). Here, output copies are denoted with subscripted primes indicating copy set (e.g. $1_{\prime}$ for the first copy, $1_{\prime\prime}$ for the second copy, $1_{\prime\prime\prime}$ for the third), and output edges are denoted with subscripts in the same manner.

(60)



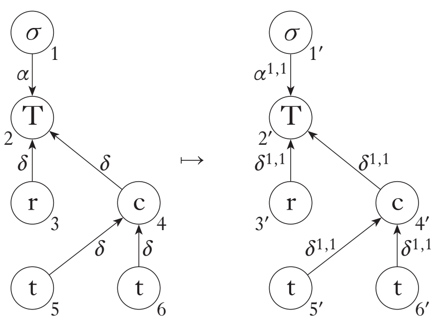
Taken together, the mappings in (59) and (60) illustrate the composition $\Gamma^{bs} \circ \Gamma^{sb}$, shown in (61).

(61)



Now consider the identity map (id_s) which maps every separated structure to itself, as when applied to the generalised separated structure in (62).

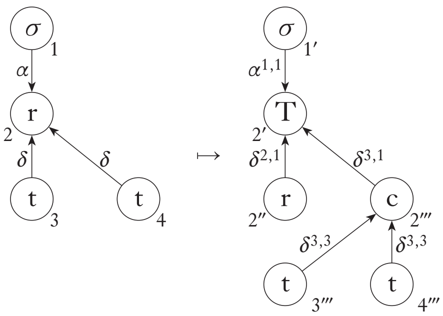
(62)



The composition $\Gamma^{bs} \circ \Gamma^{sb}$ in (61) is isomorphic to id_s in (62); their respective outputs comprise structures with the same set of elements (nodes) and the same relations between those elements (edges). This extends from the generalised graph above to any tonal structure describable in separated representation.

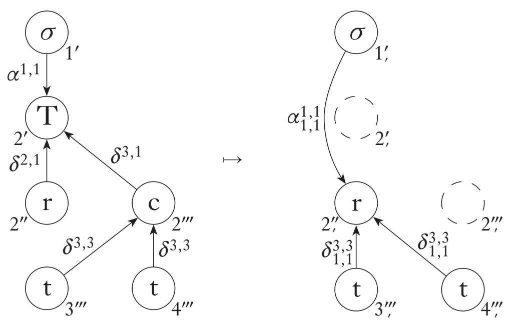
3.2 *Bundled model*. In a similar manner to the above, first apply Γ^{bs} to any bundled model to generate an equivalent separated model, as in (63).

(63)



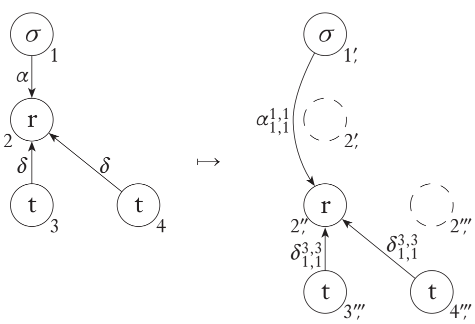
The resulting graph becomes the input structure to which Γ^{sb} is applied, as illustrated in (64). This yields a bundled structure.

(64)



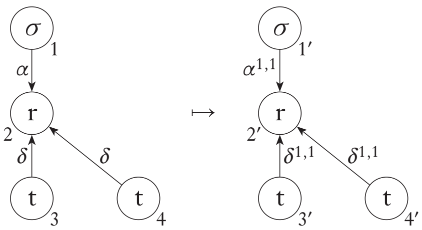
Taken together, the mappings in (63) and (64) illustrate the composition $\Gamma^{sb} \circ \Gamma^{bs}$, shown in (65).

(65)



Now consider the identity map (id_b) which maps every bundled structure to itself, as when applied to the generalised bundled structure in (66).

(66)



The composition $\Gamma^{sb} \circ \Gamma^{bs}$ in (65) is isomorphic to id_b in (66); their respective outputs comprise structures with the same set of elements (nodes) and the same relations between those elements (edges). This extends from the generalised graph above to any tonal structure describable in bundled representation.

ADDITIONAL REFERENCE

Engelfriet, Joost & Hendrik Jan Hoogeboom (2001). MSO definable string transductions and two-way finite transducers. *ACM Transactions on Computational Logic* **2**. 216–254.