***In situ* measurement of bulk modulus and yield response of glassy thin films via confined layer compression**

**Supplementary Material: Derivation uniaxial strain deformation constitutive relations for an isotropic elastic-plastic material**

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In this supplement the equations relating applied axial stress , axial strain , and radial stress in the uniaxial strain state are derived from linear elasticity theory. The relation between hydrostatic pressure *P* and Von Mises *Q* equivalent deviatoric shear stress is also arrived at. A Von Mises yield criterion is used to determine the confined yield stress *Yc*, while the post yield axial stress-strain relationship, is derived using the Prandtl-Reuss associated flow equations where *K* is the bulk modulus.

# **1. Elastic Stresses and Strains**

For an isotropic linear elastic material, the principal stresses may be expressed in cylindrical coordinates as1:

Where is the radial stress, is the hoop stress, and is the axial stress. *G* is the elastic shear modulus and is Poisson’s ratio. The uniaxial strain condition mandates that all strains orthogonal to the applied axial stress be zero, ie. . As such, equations 1.a-c reduce to:

Where *K* is the elastic bulk modulus, is the axial strain, and is Lamé’s second parameter. *M* is referred to as the confined elastic modulus and is related to Young’s modulus *E* via:

We note that for all values except , where the two are equal. The radial and hoop stresses in equation 2.a can alternatively be written in terms of the applied axial stress :

As such, direct relations exist between all principal stresses and strains in this deformation mode.

In the elastic domain a scalar equivalent hydrostatic pressure *P* may be defined in terms of the trace of the stress tensor :

Inserting equation 4 this becomes:

A similar scalar may be defined for shear, the Von Mises equivalent shear stress *Q*, which is a function of the second invariant of the deviatoric stress tensor, *J2*:

Making the same substitution as in 6 we find:

Equations 6 and 8 can be readily combined to give a functional relationship between *P* and *Q*:

For comparison, the equivalent relation in uniaxial tension/compression is:

# **2. Plastic yield conditions**

The Von Mises yield criterion is commonly used to determine the onset of permanent plastic deformation in both metals and glassy polymers2,3. The criterion states that plastic yield occurs when the Von Mises equivalent stress *Q* reaches a critical value *Y0*, the yield stress as measured in tensile loading:

With Q defined for uniaxial strain deformation in equation 8, this may be written as:

As such, yield occurs when the applied axial stress reaches a value of:

Where *Yc* is the confined yield stress. *Yc* is greater than *Y0* for all except and rises sharply as .

# **3. Plastic Flow Rule**

Once the yield criterion is met, an elastic-plastic material begins to permanently plastically deform. In this regime, the constitutive response generally cannot be formulated in terms of an explicit relationship between stress and strain, but rather in terms of an incremental flow rule such as the Prandtl-Reuss associated flow equations4:

Where is the relevant deviatoric strain term and the scalar plastic multiplier. The first term characterises elastic response, while the latter describe plasticity. For uniaxial strain equation 14 becomes:

Where equation 4 is assumed valid at the point of first yield and the appropriate substitutions made. At yield , therefore:

In the plastic limit the uniaxial strain boundary condition still holds, therefore all in-plane strains are zero. Further, we assume that all principal stress increments are equal as the yielded system cannot support any additional shear: . Therefore:

Equation 17.a now allows determination of the scalar plastic multiplier, :

Substituting this into 17.b allows determination of the slope of the axial stress-strain curve:

Equations 2.b, 13, and 19 therefore totally describe the axial stress-strain response of a system deformed in uniaxial strain, to which the flat punch – thin film geometry outlined in this work asymptotically approaches with increase punch diameter to film thickness aspect ratio.

# **References**

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