1 Online supplementary material

In this material, we present additional numerical results and an empirical illustration with financial data.

1.1 Numerical example

In our numerical evaluation, we present two illustrations. In the first experiment, we use the range based risk measures for calculating the risk premium in an insurance setup. Table 1 describes the parameterizations in each scenario evaluated in this numerical example. In the article document, we report the results from the described scenarios.

Insert Table 1

In our second experiment, we perform an extensive numerical risk prediction study. For this experiment, we use the AR(p)-GARCH(q,s) model as a data generating process (DGP). Our chosen parameterizations are described in detail in Table 2. In the article document, we present the descriptive statistics of forecast obtained from the VaR, ES, Expectile, and SDR, and the range based risk measure generated from these functionals.

Insert Table 2

In addition to the descriptive part of the risk predictions, we evaluate the risk forecasts through absolute bias (Bias), relative bias (R. Bias), root-mean-square error (RMSE)[1], and model risk measures. Similar evaluation criteria are also used in the studies of Ozun et al. (2010), Telmoudi et al. (2016), and Syuhada et al. (2021). Y represents a vector of risk forecasts obtained by risk or range based risk measures. To quantify model risk, we consider the following worst-case model risk measures:

$$\begin{aligned}
\mathrm{MR1}(Y) &= \frac{1}{R}\sum_{t=1}^{R}\left[\frac{1}{I}\sum_{i=1}^{I}|y_{i,t+1}-\bar{y}_{I,t+1}|\right],\\
\mathrm{MR2}(Y) &= \frac{1}{R}\sum_{t=1}^{R}\left[\frac{1}{I}\sum_{i=1}^{I}|(y_{i,t+1}-\bar{y}_{I,t+1})^{-}|\right],\\
\mathrm{LR1}(Y) &= \frac{1}{R}\sum_{t=1}^{R}\left[\frac{\frac{1}{I}\sum_{i=1}^{I}|y_{i,t+1}-\bar{y}_{I,t+1}|}{\bar{y}_{I,t+1}}\right],\\
\mathrm{LR2}(Y) &= \frac{1}{R}\sum_{t=1}^{R}\left[\frac{\frac{1}{I}\sum_{i=1}^{I}|(y_{i,t+1}-\bar{y}_{I,t+1})^{-}|}{\bar{y}_{I,t+1}}\right],
\end{aligned} \tag{1}$$

where I are the models used to estimate the distribution function of X, i.e., AR(1)-GARCH(1,1) model with different specifications for the distribution of z_t; $y_{i,t+1}$ is the risk forecast obtained by model $i \in I$ in each Monte Carlo replication; and $\bar{y}_{I,t+1} = \frac{1}{I}\sum_{i=1}^{I} y_{i,t+1}$ is average risk

[1] In the numerical example, we focus on traditional metrics (Bias, R. Bias, and RMSE) instead of scores functions to assess risk forecasts because, in the simulation environment, we know the real risk value. Score functions are suitable for evaluating risk predictions obtained with real data, where we do not know the real risk value.

forecast in each Monte Carlo replication. MR1 is proposed by Krajcovicova et al. (2019). This measure gives an average relative distance of each forecast $y_{i,t+1}$ from $\bar{y}_{I,t+1}$. MR2, which is proposed by Müller and Righi (2020) and explored by Berkhouch et al. (2022), quantifies the downside model risk. For both measures, lower values imply a lower model risk. The third measure, LR1, is defined by Kellner and Rösch (2016). The $\bar{y}_{I,t+1}$ in the denominator is used for the purpose of scaling in order to allow comparisons between the results of the risk and range measures. The higher the value of this measure, the more dispersed are the risk forecasts quantified by I. Following the idea of Kellner and Rösch (2016), we present LR2, which allows the comparison of the downside model risk of the different measures. The last two measures can assume negative values, being desired results closer to zero for both criteria.

Insert Table 3.

In Table 3, we describe the bias, relative bias, and RMSE of risk forecasts. We verify that the models with criteria closer to zero in some cases coincide for R_ρ and ρ. For instance, in Scenario 1 (Table 3), for SDR^α, SDR^β, and R_{SDR^α} the distribution with best results is the sged. In cases where the model with the best value for both criteria does not match for ρ^α and ρ^β, the performance of R_ρ generally is in line with the result obtained for ρ at one of two significance levels. Moreover, in most scenarios, we observed that the sign of the relative and absolute bias of the range based risk measures correspond with the tail measure used to generate it.

We identify that for data generated with $\eta = 8$ (Student-t distribution), the best result for three criteria does not correspond with the data generating process. This result holds for both ρ and R_ρ. For RMSE, the GARCH with ged (Scenario 1 and 3) and std (Scenario 2) have the best values, while for Bias and R.Bias, we verify ged (Scenario 1 and 2) and sged (Scenario 3) are the distributions with both criteria closer to zero. Similar results are found for scenario with $n = 250$. Based on a simulation study, Telmoudi et al. (2016) found that for GARCH models, the best fitting model is not necessarily the true model of the data generating process. According to the authors, this is because the choice of the z_t distribution results in an estimation risk[2]. On the other hand, for $\eta = 800$, we perceive that the data generating process coincides with the model with the best result for absolute and relative bias, and RMSE. For these scenarios, the GARCH with normal distribution has the best results for both criteria. For data with $\eta = 800$, we have a sample with less extreme values than those observed with $\eta = 8$. For this reason, the normal distribution is better suited to model the data distribution.

For the different measures considered in the study, the absolute and relative bias of the predictions obtained by the GARCH with norm and snorm distributions are negative. This result indicates that, on average, the risk forecasts from both models tend to underestimate the risk. Thus, the capital requirement computed from these models can be insufficient to absorb unexpected losses, especially in periods of crisis. The literature shows that GARCH models, in general, are negatively biased, which explains the negative bias of risk predictions. See, for example, the results of Hwang and Valls Pereira (2006). In the multivariate sense, Müller and Righi (2018) identify negative Bias and R. Bias for the VaR, ES, and Expectile forecasts quan-

[2]The estimation risk occurs when pointwise estimate may not correspond to the true risk measure value. In some studies, this risk is named a model risk. See Farkas et al. (2020).

tified by DCC-GARCH (Dynamic Conditional Correlation-Generalized Autoregressive Conditional Heteroskedastic). On the other hand, we realized that for the other specifications of z_t, the absolute and relative bias tend to be positive, i.e., the risk is overestimated. Although the impact of overestimating risk is less than underestimating, the high level of capital requirements limits the leverage of financial institutions, which can compromise financial intermediation in the market.

Relative bias is helpful because it allows us to perform bias analysis concerning the true risk forecast value. Due to this statistic standardizing the bias makes it possible to carry out a comparative analysis between the results of the different risk and range measures. We perceive that VaR and Expectile, in general, have the relative bias closest to zero among the tail measures. This result is valid mainly in scenarios generated with $\eta = 800$, i.e., normal distribution. For scenarios generated with $\eta = 8$, we verified that for GARCH, considering normal and skewed normal distribution, VaR and Expectile display the R. Bias close to zero. In contrast, for GARCH with ged and sged, we see that the SDR and Expectile with best values for this criterion. On the other hand, we observed the worst result for ES in most cases. Concerning range measures, we identify results similar to the tail measures used to generate them. So, we have to $\mathrm{R}_{\mathrm{VaR}}$ (consequently the RVaR also) and $\mathrm{R}_{\mathrm{Expectile}}$ with best results regarding R. Bias, while R_{ES} shows the worst results. Among the justifications that could be given for this, it refers to the fact that the R_ρ estimation is also subject to the estimation errors of the tail measure (ρ) used to generate it.

Insert Table 4.

Finally, we expose in Table 4 model risk estimates. This analysis intends to see how the risk forecasts changes as different probability distributions are used. Higher values for the model risk measures imply a more significant difference. LR1 and LR2 for being standardized allow us to compare the values obtained for the different measures. It can be observed that in almost all cases, the estimates for ES^α seem to vary to a higher degree among different probability distributions in comparison to other risk measures. This result corroborates the findings of Kellner and Rösch (2016), which show that ES about VaR presents a greater probability of regulatory arbitrage[3]. Among the reasons given by the authors for this result, it refers to the fact that the ES estimation considers all tail beyond the α-quantile and not just a single quantile value (VaR case). We also visualize that lower significance levels imply higher values for LR1 and LR2. This result is possibly in line with the fact that lower significance levels quantify the risk of more extreme and less likely events[4], which can be modeled less accurately by some probability distributions. Moreover, we observe that the results from $\mathrm{Expectile}^\beta$ and $\mathrm{R}_{\mathrm{VaR}^\alpha}$ (and RVaR) are competitive with VaR. Thus, we can conclude that these measures are less affected by the change in the probability distribution, which implies fewer changes in the risk forecasts and, by consequence, the capital determination. Besides that, we conclude that

[3]In a sense used, regulatory arbitrage refers to two institutions with the same portfolio and using different internal models, approved by the regulator, and so quantify different amounts of capital requirement. As they keep the same portfolio, they must hold the same or at least almost the same amount of regulatory capital.

[4]This is valid for scenarios generated with $\eta = 8$. For $\eta = 800$ there is a lower probability of extreme values. This implies, as observed results, that distributions that consider asymmetry and heavy tails do not adjust well to this type of distribution.

the greater relative dispersion of the Expectile forecasts does not imply a greater dispersion of the capital determination obtained by the measure when considering different probability distributions for z_t.

2 Capital determination

We consider the VaR, ES, SDR, and Expectile, the range based measures generated from these measures, i.e., $\mathrm{R}_{\mathrm{VaR}}$, R_{ES}, $\mathrm{R}_{\mathrm{SDR}}$, and $\mathrm{R}_{\mathrm{Expectile}}$, and RVaR. We use procedures similar to numerical analysis to predict the risk. Our financial position X refers to log-returns of S&P 500 market index[5], i.e., $x_t = (\ln P_t - \ln P_{t-1})$, where P refers to the closing price at time t and $t-1$. The analyzed period comprises data from January 4, 2010, to December 31, 2019, totaling 2526 observations. We consider this period because it includes periods of easing and turbulence, such as the Eurozone Crisis.

We note that the average of log-returns is closer to zero (0.041%). The skewness is negative (-0.508), i.e., the tail in the left side of the distribution is longer. The excess kurtosis (4.592) is higher than 0. Thus, the distribution is leptokurtic, i.e., it is taller and more concentrated than the normal distribution. The Jarque Bera test (Jarque and Bera, 1980) rejects at a significance level of 1% the null hypothesis that the series follows a normal distribution. We also notice the presence of volatility clusters (see returns evolution in Figure 1). These characteristics, that is, asymmetry, heavy tails, non-normality, and volatility clusters, are common in financial returns (Cont, 2001).

We quantify the risk forecast of S&P 500 log-returns using an AR(1)-GARCH(1,1) model because this model order is competitive according to the Akaike information criterion (AIC) (Akaike, 1974). So, as in the numerical part, we consider seven different probability distributions for z_t and a rolling window estimation of 250 and 1000 observations. The out-of-sample period comprehends December 12, 2013, to December 31, 2019. To quantify capital determination, we select a portfolio value equal to 1. In this way, capital determination value coincides with the risk forecasts.

Consider X a vector with returns for the out-of-sample period (T) and Y a vector with risk forecasts for T. For evaluation of point forecasts, in addition to descriptive statistics, we calculate, for each risk and range measure, realized loss and realized cost. The realized loss is obtained as:

$$\mathcal{L}^{\rho} = \frac{1}{T}\sum_{t=1}^{T} S^{\rho}(X,Y),$$
$$\mathcal{L}^{R_{\rho}} = \frac{1}{T}\sum_{t=1}^{T} S^{R_{\rho}}(X,Y), \tag{2}$$

where $\mathcal{L}^{\rho}$ is the realized loss of tail measures and $\mathcal{L}^{R_\rho}$ from range based measures. S^{ρ} is the score function of VaR, Expectile, ES, RVaR, and SDR. The VaR, Expectile, ES, and RVaR scores are defined in the article document. For SDR we consider an approximation of $ES^{\alpha}(Y)$

[5]We select S&P 500 because is a important index market and is frequently employed in literature; see Righi and Ceretta (2015)

score function, where $Y = X - SD^{\alpha}(X)$. S^{R_ρ} is the range based score. Realized loss values closer to zero indicate better performance of the risk forecast model.

We quantify the realized cost in the following way:

$$\text{Cost}(X, Y) := \text{Cost}_{G,L}(X, Y) = \frac{1}{T}\sum_{t=1}^{T}\left[(x_{t+1} - y_{t+1})^{+} g_{t+1} + (x_{t+1} - y_{t+1})^{-} l_{t+1}\right]$$

where $x_{t+1} \in X$ and $y_{t+1} \in Y$. Besides, $g_{t+1} \in G$ represents costs from risk overestimation and $l_{t+1} \in L$ costs from risk underestimation. G and L are positive random variables. The Cost is based on the robust risk measurement approach proposed by Righi et al. (2020) and explored by Müller and Righi (2020) to assess risk estimates. It allows us to identify the model with the best trade-off between the costs from risk overestimation and underestimation. We use as overestimation and underestimation cost the yield rates of the U.S. Treasury Bill with a maturity of three months (T-bill) and the Overnight London Interbank Offered Rate (LIBOR), based on U.S. Dollar, respectively[6]. T-bill is a risk-free investment with high liquidity where the surplus on the capital need could be applied safely. The LIBOR represents a rate for providential loans when the capital need is not sufficient. We convert both rates to a daily frequency. We also quantify model risk using the measures defined in (1). We describe these results in the following subsection.

We now present the results regarding capital determination obtained by our approach[7]. Due to the similarity of the results considering the rolling estimation windows of 250 and 1000 observations and brevity, we present the results considering a rolling estimation window of 1000 observations. Results with 250 observations as a rolling estimation window are available under request. In the first moment, we assess the descriptive statistics of forecasts, and then we analyze the realized loss, realized cost, and model risk.

Insert Table 5.

In general, we can see that the descriptive statistics are in line with those observed in the numerical analysis. According to Table 5, we note that the average values of the range based risk measures are between the average capital determination obtained for α and β by tail measures, i.e., the inequality $\rho^{\alpha}(X) \geq R_{\rho}(X) \geq \rho^{\beta}(X)$, $\alpha \leq \beta \leq 1$, is maintained using financial data. We verify that the inequality also remains for the standard deviation in most cases. As expected, the average results of capital determination computed by $\mathrm{R_{VaR}}$ are lower than those of $\mathrm{R_{ES}}$. The $\mathrm{R_{ES}}$ is based on ES, which quantifies the expected loss beyond VaR losses, so its value is greater than VaR for the same significance level. $\mathrm{R_{SDR}}$ displays higher values compared to $\mathrm{R_{VaR}}$ and $\mathrm{R_{ES}}$, which is natural because SDR is a combination of ES and Shortfall Deviation. For $\mathrm{R_{Expectile}}$ is observed lower values since Expectile is comparable to ES and VaR in more extreme levels than those used in this study (Bellini et al., 2018). On the other hand, we realize that its forecasts with financial data, like that of Expectile$^{\alpha}$ and Expectile$^{\beta}$, also have the largest relative standard deviation. This result is connected to the fact that the Expectile is more

[6]We downloaded the database from Federal Reserve Economic Data (`https://fred.stlouisfed.org/`).

[7]In the empirical analysis, we keep the results of $\mathrm{R_{VaR^{\alpha}}}$ and RVaR, because here we are considering the loss functions to evaluate the forecasts, which, because they are different for both measures, result in different values.

sensitive to the magnitude of extreme losses of the distribution than the quantile measures, such as VaR (Xie et al., 2014; Yao et al., 2021). This feature makes it interesting to use this measure, especially in catastrophic moments that are usually of concern to practitioners and policymakers.

Insert Figure 1.

Figure 1 presents the historical evolution of forecasts of risk and range based risk measures considering an AR(1)-GARCH(1,1) with skewed generalized error and $\alpha = 1.0\%$ and $\beta = 2.5\%$[8]. Illustrations for the other distributions and significance levels have been omitted for brevity and are available under request. We verify that the evolution of range measures is similar to that of the measure used to generate it, which corroborates with the descriptive statistics of empirical and numerical analysis. Also, risk based measures follow the evolution of losses S&P 500 and capture periods of greater variability.

Insert Table 6.

As Table 6, the models with the lower realized loss for range based risk measures also tend to perform better for the measure used to generate it. For the cases where the model with the lowest value does not match for ρ^α and ρ^β, R^ρ follows the results of one of these measures. This result corroborates the results observed for Bias, R. Bias, and RMSE obtained in the numerical experiment. In addition, through this result, we can conclude that the performance of a given model changes according to the significance level. Financial data tend to have more extreme observations than expected for normal distributions. As lower significance levels are associated with more extreme observations, different distributions can more accurately accommodate the characteristics of the financial data for each level. By a illustration, see $\mathrm{ES}^{2.5\%}$ and $\mathrm{ES}^{5.0\%}$, for which the sged and ged, respectively, have the lower realized loss, while for $\mathrm{R}_{\mathrm{ES}^\alpha}$ the lower value is given by GARCH with ged distribution.

We identify that the model with the lowest realized loss coincides for R_{ES} and $\mathrm{R}_{\mathrm{VaR}}$. For instance, we have GARCH sged and ged with lower realized loss for $\alpha = 1.0\%$ and $\beta = 2.5\%$, and $\alpha = 2.5\%$ and $\beta = 5.0\%$, respectively. The similarity between these two measures can be explained by the direct connection between VaR and ES. For the same significance levels, VaR and ES also present (in general) the same model as the best candidate according to the realized loss. The similarity in the performance of models to predict VaR and ES is also documented in previous research, for example, Meng and Taylor (2020). For $\mathrm{R}_{\mathrm{SDR}}$, we verify lower values for the model with sged and jsu distribution for z_t. Concerning the $\mathrm{R}_{\mathrm{Expectile}}$, we see only for $\alpha = 2.5\%$ and $\beta = 5.0\%$ similar results to R_{ES} and $\mathrm{R}_{\mathrm{VaR}}$ for both rolling estimation window. Thus, as per our results, the superiority of one model does not hold to predict VaR, ES, and Expectile. These results do not corroborate with the findings of Müller and Righi (2018).

We also verify in Table 6 that the models with the lowest realized cost do not coincide with the models with the lowest realized loss. We perceive that GARCH with normal distribution had a better realized cost in many scenarios. We can give one possible explanation for this

[8]We chose to illustrate the predictions obtained considering the skewed generalized error distribution because it performed well according to the loss realized for $\alpha = 1.0\%$ and $\beta = 2.5\%$.

difference by the fact that the realized loss computed from elicitable functions, especially for VaR and Expectile, penalizes more heavily the observations for which we note returns showing risk estimates exceedance. Moreover, unlike Cost, elicitable loss functions only consider forecasting errors rather than the costs associated with such errors. Realized cost becomes interesting in choosing the model used for capital determination once more expensive capital costs may increase expenditures. These results allow us to conclude that the model that presents the best relationship between costs from risk overestimation and underestimation does not correspond with the model recommended by traditional tests used to select risk models, i.e., elicitable loss functions.

Insert Table 7.

In Table 7, we describe model risk estimates of capital determination. We focused on analyzing LR1 and LR2 because it allows for comparing different risk and range measures. VaR^{β} presents lower values for both LR1 and LR2. Based on LR1, we conclude that VaR^{β} results in a less dispersed amount of capital determination when considering different probability distributions. LR2 points out that this dispersion is also lower when we look only at the capital determination obtained below the average capital value quantified by the VaR. The results of $\mathrm{R}_{\mathrm{VaR}}$ also tend to be competitive with VaR. Thus, the bank that uses these measures for capital determination may have an advantage based on regulatory arbitrage of banks that employed the ES with the worst results as a regulatory risk forecast measure. A reason for the worst result of ES is that it considers the entire left tail to quantify the risk. Each probability distribution models differently (albeit with a small difference) the most extreme events, common in financial series. This implies a greater sensitivity of the ES results to different models or probability distributions (Kellner and Rösch, 2016). Although we consider losses beyond a threshold in the case of range risk measures, we do not model the most extreme losses, usually associated with difficulties in estimating ES. For this reason, the range measures, in general, have lower model risk values than ES[9]. This result, in line with the good results of the $\mathrm{R}_{\mathrm{VaR}}$ in the numerical example, suggests that our range measures are good candidates for determining capital and contribute to a balance in the regulatory environment among banking institutions.

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[9]This result is not valid only for R_{ES} in relation to ES^{β}

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Table 1: Parameters of the Weibull distribution, window estimation (n), and confidence intervals (α and β), which we use in the Monte Carlo Simulation to determine risk premium in an insurance setup.

Scenarios	Scale	Shape	α	β	n
1	1	0.5	97.50%	99.00%	1,000
2	1	0.5	95.00%	97.50%	1,000
3	1	0.5	95.00%	99.00%	1,000
4	1	1.5	97.50%	99.00%	1,000
5	1	1.5	95.00%	97.50%	1,000
6	1	1.5	95.00%	99.00%	1,000
7	1	3	97.50%	99.00%	1,000
8	1	3	95.00%	97.50%	1,000
9	1	3	95.00%	99.00%	1,000
10	1	0.5	97.50%	99.00%	250
11	1	0.5	95.00%	97.50%	250
12	1	0.5	95.00%	99.00%	250
13	1	1.5	97.50%	99.00%	250
14	1	1.5	95.00%	97.50%	250
15	1	1.5	95.00%	99.00%	250
16	1	3	97.50%	99.00%	250
17	1	3	95.00%	97.50%	250
18	1	3	95.00%	99.00%	250

Table 2: Parameters of the AR(1)-GARCH(1,1) model, window estimation (n), and significance levels (α and β), which we use in the Monte Carlo Simulation to quantify risk forecasts.

Scenarios	ϕ_1	a_0	a_1	b_1	η	α	β	n
1	0.50	$4.00E-06$	0.10	0.85	8.00	1.00%	2.50%	1,000
2	0.50	$4.00E-06$	0.10	0.85	8.00	2.50%	5.00%	1,000
3	0.50	$4.00E-06$	0.10	0.85	8.00	1.00%	5.00%	1,000
4	0.50	$4.00E-06$	0.10	0.85	800.00*	1.00%	2.50%	1,000
5	0.50	$4.00E-06$	0.10	0.85	800.00	2.50%	5.00%	1,000
6	0.50	$4.00E-06$	0.10	0.85	800.00	1.00%	5.00%	1,000
7	0.50	$4.00E-06$	0.10	0.85	8.00	1.00%	2.50%	250
8	0.50	$4.00E-06$	0.10	0.85	8.00	2.50%	5.00%	250
9	0.50	$4.00E-06$	0.10	0.85	8.00	1.00%	5.00%	250
10	0.50	$4.00E-06$	0.10	0.85	800.00	1.00%	2.50%	250
11	0.50	$4.00E-06$	0.10	0.85	800.00	2.50%	5.00%	250
12	0.50	$4.00E-06$	0.10	0.85	800.00	1.00%	5.00%	250

Note: *As performed by Christoffersen and Gonçalves (2005) we consider a wider η to represent the normal distribution. ϕ_1, a_0, a_1, and b_1, are parameters of AR(1)-GARCH(1,1) model for $p = q = s = 1$, and η represents the parameter of Student′s t-distribution. α and β represent the significance levels, and n the estimation window.

Table 3: Monte Carlo results considering Scenario 1 to Scenario 6 of Table 2. The values are multiplied by 100.

GARCH	norm	snorm	std	sstd	ged	sged	jsu	norm	snorm	std	sstd	ged	sged	jsu	norm	snorm	std	sstd	ged	sged	jsu
	Scenario 1							Scenario 2							Scenario 3						
	Bias																				
VaR^{α}	-0.170	-0.173	0.120	0.116	0.087	0.084	0.118	-0.033	-0.038	-0.029	-0.031	0.036	0.033	-0.012	-0.166	-0.167	0.123	0.126	0.078	0.080	0.128
VaR^{β}	-0.046	-0.048	-0.029	-0.032	0.034	0.032	-0.013	0.028	0.024	-0.085	-0.086	-0.011	-0.013	-0.072	0.020	0.019	-0.087	-0.085	-0.019	-0.018	-0.071
$\text{R}_{\text{VaR}^{\alpha}}$	-0.095	-0.097	0.027	0.024	0.058	0.055	0.038	0.003	-0.001	-0.063	-0.065	0.010	0.008	-0.048	-0.037	-0.039	-0.029	-0.027	0.020	0.021	-0.011
ES^{α}	-0.393	-0.397	0.502	0.495	0.094	0.090	0.357	-0.198	-0.204	0.207	0.202	0.072	0.068	0.159	-0.388	-0.390	0.509	0.512	0.081	0.083	0.372
ES^{β}	-0.214	-0.217	0.217	0.212	0.072	0.069	0.166	-0.097	-0.103	0.072	0.069	0.041	0.038	0.056	-0.108	-0.109	0.078	0.081	0.032	0.034	0.066
$\text{R}_{\text{ES}^{\alpha}}$	-0.286	-0.290	0.328	0.322	0.084	0.080	0.243	-0.140	-0.145	0.127	0.123	0.056	0.052	0.099	-0.200	-0.201	0.210	0.213	0.056	0.058	0.166
SDR^{α}	-0.387	-0.391	0.469	0.463	0.062	0.058	0.324	-0.191	-0.197	0.176	0.171	0.042	0.038	0.128	-0.382	-0.383	0.475	0.479	0.049	0.051	0.339
SDR^{β}	-0.208	-0.211	0.186	0.181	0.041	0.038	0.134	-0.090	-0.096	0.042	0.038	0.011	0.008	0.025	-0.102	-0.103	0.048	0.050	0.001	0.003	0.035
$\text{R}_{\text{SDR}^{\alpha}}$	-0.280	-0.284	0.296	0.291	0.052	0.049	0.210	-0.132	-0.138	0.097	0.093	0.025	0.022	0.068	-0.193	-0.195	0.179	0.182	0.025	0.027	0.134
$\text{Expectile}^{\alpha}$	-0.125	-0.127	0.137	0.134	0.041	0.039	0.099	-0.038	-0.042	0.034	0.033	0.021	0.019	0.025	-0.121	-0.122	0.140	0.142	0.035	0.036	0.107
Expectile^{β}	-0.048	-0.050	0.036	0.035	0.019	0.018	0.026	-0.007	-0.009	-0.008	-0.009	0.002	-0.001	-0.011	-0.013	-0.013	-0.007	-0.006	-0.004	-0.003	-0.008
$\text{R}_{\text{Expectile}^{\alpha}}$	-0.076	-0.078	0.076	0.073	0.030	0.029	0.055	-0.019	-0.023	0.008	0.007	0.009	0.007	0.003	-0.043	-0.043	0.037	0.039	0.013	0.014	0.029
	R. Bias																				
VaR^{α}	-8.088	-8.239	5.700	5.511	4.137	3.986	5.627	-1.929	-2.212	-1.689	-1.830	2.122	1.956	-0.701	-7.909	-7.969	5.839	5.983	3.705	3.792	6.110
VaR^{β}	-2.723	-2.876	-1.746	-1.890	2.015	1.882	-0.768	1.998	1.719	-6.154	-6.253	-0.817	-0.966	-5.187	1.479	1.397	-6.466	-6.351	-1.382	-1.326	-5.258
$\text{R}_{\text{VaR}^{\alpha}}$	-5.096	-5.248	1.442	1.279	3.119	2.977	2.066	0.204	-0.078	-4.164	-4.282	0.655	0.497	-3.138	-2.297	-2.369	-1.815	-1.687	1.248	1.319	-0.671
ES^{α}	-15.075	-15.217	19.241	18.982	3.588	3.429	13.682	-9.081	-9.355	9.502	9.278	3.303	3.126	7.308	-14.917	-14.966	19.539	19.691	3.100	3.196	14.301
ES^{β}	-9.924	-10.071	10.054	9.844	3.346	3.196	7.686	-5.264	-5.541	3.884	3.704	2.214	2.045	3.014	-5.905	-5.970	4.290	4.424	1.778	1.856	3.609
$\text{R}_{\text{ES}^{\alpha}}$	-12.187	-12.333	13.963	13.733	3.564	3.410	10.329	-6.995	-7.271	6.374	6.175	2.788	2.615	4.964	-9.471	-9.529	9.963	10.106	2.677	2.763	7.863
SDR^{α}	-13.226	-13.362	16.034	15.807	2.120	1.991	11.072	-7.641	-7.895	7.058	6.867	1.667	1.519	5.126	-13.084	-13.136	16.289	16.428	1.662	1.753	11.618
SDR^{β}	-8.402	-8.541	7.502	7.324	1.661	1.545	5.415	-4.167	-4.420	1.923	1.773	0.502	0.366	1.153	-4.748	-4.815	2.230	2.349	0.047	0.120	1.618
$\text{R}_{\text{SDR}^{\alpha}}$	-10.502	-10.640	11.105	10.906	1.966	1.844	7.888	-5.727	-5.981	4.182	4.014	1.100	0.958	2.944	-7.979	-8.040	7.370	7.498	1.022	1.103	5.525
$\text{Expectile}^{\alpha}$	-8.037	-8.180	8.820	8.605	2.648	2.495	6.361	-3.071	-3.350	2.778	2.624	1.666	1.518	2.003	-7.833	-7.893	9.026	9.168	2.231	2.321	6.885
Expectile^{β}	-3.976	-4.115	3.010	2.871	1.573	1.466	2.113	-0.678	-0.951	-0.785	-0.932	0.109	-0.089	-1.075	-1.309	-1.374	-0.732	-0.582	-0.400	-0.379	-0.818
$\text{R}_{\text{Expectile}^{\alpha}}$	-5.650	-5.788	5.595	5.417	2.228	2.117	4.086	-1.763	-2.067	0.757	0.604	0.809	0.640	0.274	-3.639	-3.695	3.146	3.302	1.086	1.183	2.461
	RMSE																				
VaR^{α}	0.276	0.293	0.238	0.255	0.222	0.238	0.259	0.239	0.245	0.144	0.157	0.185	0.194	0.160	0.294	0.292	0.246	0.258	0.250	0.264	0.262
VaR^{β}	0.190	0.204	0.125	0.141	0.148	0.162	0.144	0.208	0.209	0.139	0.146	0.137	0.145	0.140	0.169	0.167	0.135	0.136	0.125	0.131	0.127
$\text{R}_{\text{VaR}^{\alpha}}$	0.220	0.235	0.154	0.172	0.179	0.193	0.181	0.220	0.223	0.137	0.148	0.156	0.165	0.145	0.197	0.195	0.131	0.137	0.163	0.173	0.138
ES^{α}	0.472	0.488	0.666	0.678	0.286	0.307	0.528	0.339	0.353	0.330	0.338	0.262	0.273	0.304	0.495	0.491	0.685	0.700	0.336	0.353	0.545
ES^{β}	0.307	0.323	0.337	0.351	0.221	0.238	0.306	0.264	0.273	0.194	0.204	0.207	0.216	0.200	0.238	0.236	0.194	0.203	0.196	0.207	0.193
$\text{R}_{\text{ES}^{\alpha}}$	0.371	0.387	0.463	0.476	0.249	0.267	0.393	0.293	0.305	0.247	0.256	0.231	0.241	0.241	0.315	0.312	0.334	0.345	0.247	0.260	0.299
SDR^{α}	0.484	0.500	0.641	0.653	0.286	0.307	0.507	0.366	0.378	0.317	0.327	0.268	0.280	0.297	0.512	0.506	0.656	0.670	0.348	0.363	0.519
SDR^{β}	0.327	0.342	0.322	0.337	0.224	0.241	0.296	0.298	0.304	0.197	0.210	0.218	0.229	0.208	0.267	0.261	0.196	0.203	0.215	0.224	0.194
$\text{R}_{\text{SDR}^{\alpha}}$	0.388	0.403	0.443	0.457	0.251	0.269	0.378	0.325	0.334	0.241	0.253	0.240	0.251	0.242	0.339	0.333	0.319	0.329	0.262	0.273	0.286
$\text{Expectile}^{\alpha}$	0.205	0.218	0.230	0.242	0.153	0.167	0.208	0.178	0.183	0.126	0.135	0.138	0.145	0.132	0.218	0.217	0.239	0.248	0.177	0.187	0.212
Expectile^{β}	0.143	0.153	0.119	0.131	0.110	0.122	0.125	0.148	0.151	0.090	0.098	0.107	0.113	0.100	0.122	0.121	0.085	0.090	0.097	0.103	0.089
$\text{R}_{\text{Expectile}^{\alpha}}$	0.165	0.177	0.160	0.171	0.128	0.141	0.157	0.161	0.164	0.103	0.111	0.120	0.126	0.112	0.148	0.148	0.121	0.128	0.123	0.131	0.120

GARCH	norm	snorm	std	sstd	ged	sged	jsu	norm	snorm	std	sstd	ged	sged	jsu	norm	snorm	std	sstd	ged	sged	jsu
	Scenario 4							Scenario 5							Scenario 6						
	Bias																				
VaR^{α}	-0.014	-0.016	0.264	0.263	0.237	0.236	0.267	-0.019	-0.022	0.000	-0.002	0.062	0.059	0.017	-0.016	-0.022	0.259	0.260	0.226	0.228	0.264
VaR^{β}	-0.012	-0.013	-0.002	-0.001	0.064	0.064	0.018	-0.016	-0.018	-0.118	-0.118	-0.045	-0.047	-0.104	-0.010	-0.015	-0.111	-0.110	-0.042	-0.041	-0.096
$\text{R}_{\text{VaR}^{\alpha}}$	-0.013	-0.014	0.100	0.100	0.135	0.134	0.115	-0.017	-0.020	-0.072	-0.073	0.000	-0.002	-0.055	-0.013	-0.017	-0.004	-0.003	0.049	0.051	0.013
ES^{α}	-0.017	-0.019	0.856	0.854	0.461	0.460	0.719	-0.024	-0.028	0.408	0.406	0.263	0.260	0.359	-0.020	-0.026	0.831	0.831	0.440	0.443	0.702
ES^{β}	-0.014	-0.016	0.402	0.401	0.265	0.265	0.357	-0.021	-0.024	0.168	0.167	0.132	0.129	0.152	-0.014	-0.019	0.163	0.164	0.128	0.129	0.151
$\text{R}_{\text{ES}^{\alpha}}$	-0.016	-0.017	0.581	0.580	0.347	0.346	0.502	-0.022	-0.025	0.268	0.265	0.189	0.185	0.239	-0.016	-0.022	0.373	0.374	0.238	0.240	0.331
SDR^{α}	-0.025	-0.027	0.807	0.806	0.415	0.414	0.671	-0.033	-0.037	0.363	0.360	0.218	0.214	0.313	-0.026	-0.033	0.786	0.787	0.398	0.400	0.658
SDR^{β}	-0.022	-0.024	0.357	0.356	0.220	0.220	0.311	-0.030	-0.033	0.124	0.122	0.087	0.084	0.107	-0.021	-0.026	0.122	0.123	0.086	0.088	0.109
$\text{R}_{\text{SDR}^{\alpha}}$	-0.023	-0.025	0.534	0.533	0.301	0.301	0.455	-0.031	-0.034	0.223	0.220	0.144	0.140	0.193	-0.023	-0.029	0.331	0.332	0.196	0.198	0.288
$\text{Expectile}^{\alpha}$	-0.011	-0.012	0.241	0.241	0.151	0.150	0.207	-0.014	-0.016	0.073	0.072	0.055	0.053	0.063	-0.012	-0.016	0.235	0.236	0.144	0.145	0.204
Expectile^{β}	-0.009	-0.010	0.070	0.070	0.056	0.056	0.062	-0.011	-0.013	-0.002	-0.003	0.005	0.003	-0.005	-0.008	-0.011	-0.002	-0.001	0.004	0.005	-0.003
$\text{R}_{\text{Expectile}^{\alpha}}$	-0.010	-0.011	0.135	0.135	0.093	0.094	0.118	-0.012	-0.014	0.027	0.026	0.025	0.023	0.022	-0.009	-0.012	0.066	0.067	0.048	0.050	0.059
	R. Bias																				
VaR^{α}	-0.723	-0.809	13.461	13.440	12.080	12.051	13.613	-1.167	-1.324	-0.029	-0.108	3.753	3.590	1.042	-0.855	-1.154	13.868	13.926	12.109	12.211	14.106
VaR^{β}	-0.698	-0.789	-0.100	-0.096	3.866	3.840	1.067	-1.137	-1.278	-8.455	-8.508	-3.236	-3.374	-7.450	-0.797	-1.105	-8.456	-8.387	-3.199	-3.105	-7.301
$\text{R}_{\text{VaR}^{\alpha}}$	-0.708	-0.797	5.595	5.589	7.551	7.524	6.441	-1.150	-1.300	-4.735	-4.800	0.023	-0.127	-3.650	-0.817	-1.122	-0.264	-0.198	3.220	3.317	0.853
ES^{α}	-0.769	-0.849	38.130	38.055	20.533	20.504	32.037	-1.225	-1.395	20.711	20.565	13.352	13.162	18.192	-0.909	-1.204	38.699	38.725	20.517	20.623	32.700
ES^{β}	-0.736	-0.821	20.419	20.381	13.466	13.438	18.104	-1.193	-1.354	9.670	9.559	7.569	7.396	8.715	-0.841	-1.143	9.838	9.893	7.704	7.804	9.110
$\text{R}_{\text{ES}^{\alpha}}$	-0.750	-0.833	27.801	27.748	16.592	16.564	24.032	-1.207	-1.373	14.502	14.376	10.221	10.040	12.922	-0.866	-1.165	20.207	20.254	12.891	12.993	17.924
SDR^{α}	-0.959	-1.039	31.407	31.350	16.126	16.110	26.096	-1.441	-1.590	15.765	15.643	9.467	9.315	13.585	-1.065	-1.353	31.897	31.927	16.126	16.234	26.676
SDR^{β}	-0.942	-1.027	15.514	15.491	9.583	9.569	13.515	-1.435	-1.574	5.981	5.892	4.197	4.064	5.146	-1.039	-1.332	6.153	6.208	4.357	4.460	5.516
$\text{R}_{\text{SDR}^{\alpha}}$	-0.948	-1.031	22.099	22.063	12.460	12.445	18.836	-1.436	-1.580	10.237	10.134	6.598	6.456	8.869	-1.044	-1.335	15.274	15.322	9.053	9.158	13.311
$\text{Expectile}^{\alpha}$	-0.750	-0.836	16.618	16.592	10.381	10.358	14.273	-1.177	-1.328	6.107	6.044	4.623	4.477	5.279	-0.860	-1.165	17.092	17.139	10.430	10.526	14.819
Expectile^{β}	-0.759	-0.846	5.925	5.942	4.709	4.718	5.238	-1.165	-1.297	-0.233	-0.314	0.484	0.294	-0.530	-0.856	-1.159	-0.244	-0.164	0.453	0.506	-0.342
$\text{R}_{\text{Expectile}^{\alpha}}$	-0.769	-0.851	10.392	10.386	7.183	7.197	9.070	-1.146	-1.316	2.557	2.469	2.352	2.189	2.038	-0.852	-1.146	6.103	6.175	4.443	4.570	5.395
	RMSE																				
VaR^{α}	0.239	0.268	0.374	0.387	0.354	0.372	0.393	0.219	0.225	0.142	0.154	0.190	0.201	0.164	0.193	0.210	0.323	0.335	0.299	0.316	0.342
VaR^{β}	0.202	0.222	0.145	0.158	0.184	0.201	0.167	0.185	0.188	0.161	0.165	0.136	0.144	0.157	0.139	0.147	0.139	0.143	0.104	0.114	0.134
$\text{R}_{\text{VaR}^{\alpha}}$	0.218	0.242	0.213	0.226	0.249	0.266	0.241	0.200	0.204	0.140	0.148	0.149	0.159	0.145	0.160	0.171	0.105	0.118	0.135	0.151	0.125
ES^{α}	0.274	0.311	1.030	1.042	0.601	0.619	0.888	0.260	0.269	0.512	0.521	0.386	0.396	0.474	0.220	0.242	0.931	0.942	0.523	0.540	0.802
ES^{β}	0.240	0.269	0.522	0.533	0.384	0.401	0.486	0.230	0.237	0.261	0.271	0.249	0.260	0.261	0.172	0.185	0.225	0.236	0.200	0.216	0.226
$\text{R}_{\text{ES}^{\alpha}}$	0.255	0.287	0.721	0.732	0.473	0.491	0.645	0.243	0.252	0.362	0.372	0.307	0.317	0.347	0.190	0.207	0.441	0.451	0.309	0.326	0.408
SDR^{α}	0.310	0.344	0.984	0.996	0.565	0.585	0.844	0.297	0.303	0.476	0.487	0.362	0.373	0.441	0.249	0.269	0.891	0.902	0.490	0.509	0.763
SDR^{β}	0.277	0.304	0.487	0.499	0.357	0.376	0.452	0.268	0.271	0.242	0.253	0.240	0.251	0.245	0.202	0.213	0.204	0.216	0.186	0.204	0.208
$\text{R}_{\text{SDR}^{\alpha}}$	0.292	0.321	0.680	0.692	0.442	0.461	0.607	0.281	0.286	0.333	0.344	0.290	0.301	0.322	0.220	0.234	0.408	0.419	0.285	0.303	0.377
$\text{Expectile}^{\alpha}$	0.177	0.198	0.332	0.342	0.242	0.256	0.306	0.158	0.164	0.147	0.157	0.146	0.155	0.151	0.145	0.158	0.290	0.299	0.201	0.215	0.266
Expectile^{β}	0.145	0.161	0.150	0.160	0.142	0.155	0.154	0.131	0.135	0.089	0.097	0.101	0.109	0.098	0.101	0.108	0.070	0.078	0.075	0.086	0.079
$\text{R}_{\text{Expectile}^{\alpha}}$	0.159	0.177	0.216	0.226	0.180	0.194	0.210	0.143	0.148	0.108	0.117	0.119	0.127	0.116	0.117	0.127	0.117	0.127	0.108	0.120	0.122

Note: This table describes bias (Bias), relative bias (R. Bias), root-mean-square error (RMSE) of risk, and range based forecasts. The values in bold refer to the models that presented values closer to zero in the metric, that is, better performance according to the criterion. The results are based on 1,000 Monte Carlo replications. For risk estimation we employ an AR(1)-GARCH(1,1) model, where z_t follows normal (norm), skewed normal (snorm), Student-t (std), skewed Student-t (sstd), generalized error (ged), skewed generalized error (sged), or Johnson SU (jsu) distributions.

Table 4: Model risk estimates of risk forecasts quantified in Monte Carlo simulations, considering Scenarios 1 to 6 from Table 2. To quantify model risk, we consider MR1, MR2, LR1 and LR2. The values are multiplied by 100.

	Scenario 1				Scenario 2				Scenario 3			
	MR1	MR2	LR1	LR2	MR1	MR2	LR1	LR2	MR1	MR2	LR1	LR2
VaR^{α}	0.122	0.061	5.952	2.976	0.052	0.026	3.222	1.611	0.124	0.062	5.967	2.983
VaR^{β}	0.051	0.026	3.295	1.647	0.051	0.025	4.185	2.093	0.050	0.025	7.324	3.662
$\text{R}_{\text{VaR}^{\alpha}}$	0.073	0.036	4.142	2.071	0.047	0.024	3.323	1.662	0.050	0.025	3.242	1.621
ES^{α}	0.311	0.155	11.552	5.776	0.152	0.076	7.146	3.573	0.315	0.157	11.627	5.813
ES^{β}	0.158	0.079	7.421	3.711	0.079	0.040	4.494	2.247	0.083	0.042	4.723	2.361
$\text{R}_{\text{ES}^{\alpha}}$	0.217	0.109	9.183	4.591	0.109	0.054	5.655	2.828	0.153	0.077	7.282	3.641
SDR1	0.299	0.150	9.986	4.993	0.142	0.071	5.806	2.903	0.303	0.152	10.052	5.026
SDR^{β}	0.147	0.074	6.010	3.005	0.072	0.036	3.442	1.721	0.076	0.038	3.596	1.798
$\text{R}_{\text{SDR}^{\alpha}}$	0.206	0.103	7.683	3.841	0.100	0.050	4.435	2.217	0.142	0.071	5.869	2.934
$\text{Expectile}^{\alpha}$	0.098	0.049	6.793	3.396	0.043	0.022	4.204	2.102	0.100	0.050	6.828	3.414
Expectile^{β}	0.044	0.022	3.990	1.995	0.027	0.013	2.020	1.010	0.027	0.014	2.390	1.195
$\text{R}_{\text{Expectile}^{\alpha}}$	0.064	0.032	6.652	3.326	0.032	0.016	3.769	1.884	0.044	0.022	3.934	1.967
	Scenario 4				Scenario 5				Scenario 6			
	MR1	MR2	LR1	LR2	MR1	MR2	LR1	LR2	MR1	MR2	LR1	LR2
VaR^{α}	0.119	0.060	5.696	2.848	0.051	0.026	3.196	1.598	0.116	0.058	5.901	2.950
VaR^{β}	0.051	0.025	3.194	1.597	0.049	0.024	4.221	2.111	0.045	0.023	4.445	2.223
$\text{R}_{\text{VaR}^{\alpha}}$	0.071	0.036	3.961	1.980	0.045	0.023	3.260	1.630	0.044	0.022	3.084	1.542
ES^{α}	0.303	0.152	11.215	5.607	0.158	0.079	7.354	3.677	0.296	0.148	11.495	5.748
ES^{β}	0.155	0.077	7.131	3.565	0.081	0.041	4.601	2.301	0.077	0.039	4.641	2.320
$\text{R}_{\text{ES}^{\alpha}}$	0.212	0.106	8.866	4.433	0.112	0.056	5.805	2.903	0.144	0.072	7.198	3.599
SDR1	0.292	0.146	9.680	4.840	0.147	0.074	5.975	2.988	0.285	0.142	9.935	4.967
SDR^{β}	0.143	0.072	5.771	2.886	0.074	0.037	3.517	1.758	0.069	0.035	3.498	1.749
$\text{R}_{\text{SDR}^{\alpha}}$	0.201	0.100	7.413	3.706	0.103	0.051	4.555	2.278	0.133	0.067	5.789	2.894
$\text{Expectile}^{\alpha}$	0.096	0.048	6.440	3.220	0.044	0.022	4.212	2.106	0.093	0.047	6.708	3.354
Expectile^{β}	0.044	0.022	7.149	3.575	0.026	0.013	3.930	1.965	0.024	0.012	5.682	2.841
$\text{R}_{\text{Expectile}^{\alpha}}$	0.063	0.031	6.735	3.368	0.032	0.016	3.443	1.722	0.040	0.020	4.550	2.275

Note: This table shows the numerical results of model risk estimates. The results are based on 1,000 Monte Carlo replications considering. Data generation process of returns corresponds to AR(1)-GARCH(1,1), considering Normal and Student′s t-distribution. As significance levels, we use 1%, 2.5%, and 5%. Table 2 presents the considered scenarios. For risk estimation we consider an AR(1)-GARCH(1,1) model, where z_t follows normal (norm), skewed normal (snorm), Student-t (std), skewed Student-t (sstd), generalized error (ged), skewed generalized error (sged), or Johnson SU (jsu) distributions. Values in bold indicate risk and range measures with the lowest model risk, according to LR1 and LR2.

Table 5: Average and standard values of the risk forecasts of S&P 500 log-returns. These results are multiplied by 100. The sample period refers to January 4, 2010, to December 31, 2019. In this analysis, the rolling window is 1000 observations.

	Mean							Standard Deviation						
$\alpha = 1\%$ and $\beta = 2.5\%$														
GARCH	norm	snorm	std	sstd	ged	sged	jsu	norm	snorm	std	sstd	ged	sged	jsu
RVaR^{α}	1.586	1.693	1.711	1.807	1.748	1.828	1.854	0.660	0.710	0.773	0.805	0.765	0.789	0.820
VaR^{α}	1.748	1.875	1.992	2.102	1.994	2.085	2.158	0.724	0.783	0.899	0.935	0.872	0.900	0.953
VaR^{β}	1.461	1.552	1.511	1.595	1.564	1.636	1.634	0.610	0.653	0.684	0.712	0.684	0.706	0.725
$\mathrm{R}_{\mathrm{VaR}^{\alpha}}$	1.586	1.693	1.711	1.807	1.748	1.828	1.854	0.660	0.710	0.773	0.805	0.765	0.789	0.820
ES^{α}	2.014	2.175	2.643	2.771	2.439	2.547	2.797	0.830	0.905	1.205	1.245	1.071	1.103	1.239
ES^{β}	1.757	1.886	2.084	2.193	2.025	2.116	2.231	0.728	0.788	0.945	0.979	0.887	0.914	0.987
$\mathrm{R}_{\mathrm{ES}^{\alpha}}$	1.869	2.011	2.315	2.433	2.202	2.300	2.469	0.772	0.839	1.052	1.089	0.965	0.995	1.092
SDR^{α}	2.317	2.476	2.926	3.054	2.720	2.827	3.079	0.946	1.021	1.317	1.358	1.179	1.212	1.352
SDR^{β}	2.062	2.187	2.369	2.478	2.307	2.398	2.515	0.844	0.905	1.060	1.096	0.997	1.025	1.102
$\mathrm{R}_{\mathrm{SDR}^{\alpha}}$	2.172	2.312	2.600	2.717	2.484	2.582	2.751	0.888	0.955	1.166	1.204	1.075	1.105	1.206
$\mathrm{Expectile}^{\alpha}$	1.272	1.361	1.460	1.542	1.433	1.501	1.571	0.534	0.576	0.672	0.697	0.635	0.654	0.702
$\mathrm{Expectile}^{\beta}$	1.022	1.086	1.093	1.157	1.101	1.156	1.181	0.435	0.465	0.508	0.527	0.492	0.507	0.532
$\mathrm{R}_{\mathrm{Expectile}^{\alpha}}$	1.130	1.204	1.246	1.318	1.241	1.303	1.344	0.477	0.512	0.576	0.597	0.552	0.569	0.603
$\alpha = 2.5\%$ and $\beta = 5\%$														
	Mean							Standard Deviation						
GARCH	norm	snorm	std	sstd	ged	sged	jsu	norm	snorm	std	sstd	ged	sged	jsu
RVaR^{α}	1.327	1.401	1.320	1.392	1.376	1.439	1.422	0.556	0.592	0.601	0.624	0.603	0.622	0.633
VaR^{α}	1.461	1.552	1.511	1.595	1.564	1.636	1.634	0.610	0.653	0.684	0.712	0.684	0.706	0.725
VaR^{β}	1.215	1.276	1.171	1.233	1.225	1.280	1.255	0.512	0.542	0.536	0.556	0.539	0.555	0.562
$\mathrm{R}_{\mathrm{VaR}^{\alpha}}$	1.327	1.401	1.320	1.392	1.376	1.439	1.422	0.556	0.592	0.601	0.624	0.603	0.622	0.633
ES^{α}	1.757	1.886	2.084	2.193	2.025	2.116	2.231	0.728	0.788	0.945	0.979	0.887	0.914	0.987
ES^{β}	1.542	1.643	1.702	1.792	1.701	1.777	1.827	0.642	0.690	0.772	0.801	0.745	0.768	0.810
$\mathrm{R}_{\mathrm{ES}^{\alpha}}$	1.639	1.753	1.869	1.967	1.845	1.928	2.004	0.681	0.734	0.847	0.878	0.808	0.833	0.887
SDR^{α}	2.062	2.187	2.369	2.478	2.307	2.398	2.515	0.844	0.905	1.060	1.096	0.997	1.025	1.102
SDR^{β}	1.847	1.946	1.989	2.079	1.984	2.061	2.112	0.760	0.808	0.890	0.919	0.857	0.881	0.927
$\mathrm{R}_{\mathrm{SDR}^{\alpha}}$	1.944	2.055	2.155	2.254	2.128	2.211	2.289	0.798	0.851	0.964	0.996	0.919	0.945	1.004
$\mathrm{Expectile}^{\alpha}$	1.022	1.086	1.093	1.157	2.066	1.156	1.181	0.435	0.465	0.508	0.527	37.708	0.506	0.532
$\mathrm{Expectile}^{\beta}$	0.820	0.866	0.836	0.886	0.854	0.898	0.903	0.354	0.377	0.395	0.410	0.387	0.399	0.413
$\mathrm{R}_{\mathrm{Expectile}^{\alpha}}$	0.910	0.965	0.948	1.005	0.963	1.012	1.025	0.390	0.416	0.444	0.461	0.433	0.446	0.465
$\alpha = 1\%$ and $\beta = 5\%$														
	Mean							Standard Deviation						
GARCH	norm	snorm	std	sstd	ged	sged	jsu	norm	snorm	std	sstd	ged	sged	jsu
RVaR^{α}	1.424	1.510	1.467	1.548	1.516	1.585	1.584	0.595	0.636	0.666	0.692	0.664	0.685	0.703
VaR^{α}	1.748	1.875	1.992	2.102	1.994	2.085	2.158	0.724	0.783	0.899	0.935	0.872	0.900	0.953
VaR^{β}	1.215	1.276	1.171	1.233	1.225	1.280	1.255	0.512	0.542	0.536	0.556	0.539	0.555	0.562
$\mathrm{R}_{\mathrm{VaR}^{\alpha}}$	1.424	1.510	1.467	1.548	1.516	1.585	1.584	0.595	0.636	0.666	0.692	0.664	0.685	0.703
ES^{α}	2.014	2.175	2.643	2.771	2.439	2.547	2.797	0.830	0.905	1.205	1.245	1.071	1.103	1.239
ES^{β}	1.542	1.643	1.702	1.792	1.701	1.777	1.827	0.642	0.690	0.772	0.801	0.745	0.768	0.810
$\mathrm{R}_{\mathrm{ES}^{\alpha}}$	1.725	1.850	2.036	2.142	1.979	2.068	2.178	0.715	0.773	0.924	0.957	0.867	0.893	0.964
SDR^{α}	2.317	2.476	2.926	3.054	2.720	2.827	3.079	0.946	1.021	1.317	1.358	1.179	1.212	1.352
SDR^{β}	1.847	1.946	1.989	2.079	1.984	2.061	2.112	0.760	0.808	0.890	0.919	0.857	0.881	0.927
$\mathrm{R}_{\mathrm{SDR}^{\alpha}}$	2.030	2.151	2.322	2.427	2.261	2.350	2.463	0.832	0.890	1.039	1.074	0.977	1.005	1.080
$\mathrm{Expectile}^{\alpha}$	1.272	1.361	1.373	1.542	1.433	1.501	1.571	0.534	0.576	0.679	0.695	0.635	0.654	0.702
$\mathrm{Expectile}^{\beta}$	0.820	0.866	0.799	0.886	0.854	0.897	0.903	0.354	0.377	0.393	0.409	0.387	0.399	0.413
$\mathrm{R}_{\mathrm{Expectile}^{\alpha}}$	0.993	1.054	1.007	1.122	1.067	1.121	1.144	0.423	0.452	0.494	0.511	0.477	0.492	0.517

Note: This table describes mean and standard deviation values of RVaR^{α}, VaR^{α}, VaR^{β}, $\mathrm{R}_{\mathrm{VaR}^{\alpha}}$, ES^{α}, ES^{β}, $\mathrm{R}_{\mathrm{ES}^{\alpha}}$, SDR^{α}, SDR^{β}, $\mathrm{R}_{\mathrm{SDR}^{\alpha}}$, $\mathrm{Expectile}^{\alpha}$, $\mathrm{Expectile}^{\beta}$ and $\mathrm{R}_{\mathrm{Expectile}^{\alpha}}$ forecasts. The results are based in a rolling window of 1000 observations. For risk estimation we consider an AR(1)-GARCH(1,1) model, where z_t follows normal (norm), skewed normal (snorm), Student-t (std), skewed Student-t (sstd), generalized error (ged), skewed generalized error (sged), or Johnson SU (jsu) distributions.

Table 6: Realized loss and realized cost values of the risk forecasts of S&P 500 log-returns. These values are multiplied by 10000. The sample period refers to January 4, 2010, to December 31, 2019. In this analysis, the rolling estimation window is 1000 observations.

	Realized loss							Realized cost						
$\alpha = 1\%$ and $\beta = 2.5\%$														
GARCH	norm	snorm	std	sstd	ged	sged	jsu	norm	snorm	std	sstd	ged	sged	jsu
$RVaR^{\alpha}$	10109.606	10109.427	10109.333	10109.309	10109.285	10109.275	10109.276	0.326	0.358	0.369	0.390	0.381	0.397	0.404
VaR^{α}	3.196	3.066	2.969	2.970	2.964	2.966	2.956	0.372	0.409	0.449	0.471	0.451	0.467	0.486
VaR^{β}	6.028	5.978	5.976	5.950	5.934	5.921	5.930	0.289	0.317	0.309	0.329	0.326	0.342	0.342
$R_{VaR^{\alpha}}$	4.704	4.606	4.560	4.539	4.541	4.523	4.530	0.326	0.358	0.369	0.390	0.381	0.397	0.404
ES^{α}	415.092	402.072	392.352	392.344	391.853	391.978	390.937	0.443	0.487	0.627	0.650	0.571	0.589	0.652
ES^{β}	493.715	491.623	491.491	490.470	489.808	489.296	489.655	0.374	0.412	0.476	0.498	0.459	0.476	0.507
$R_{ES^{\alpha}}$	450.201	443.403	440.381	438.460	438.973	437.353	437.577	0.405	0.445	0.539	0.562	0.507	0.525	0.569
SDR^{α}	486.842	461.779	430.738	429.084	429.884	428.411	423.392	0.520	0.563	0.693	0.716	0.638	0.656	0.718
SDR^{β}	538.030	532.043	531.078	527.863	527.717	522.735	522.487	0.455	0.489	0.548	0.569	0.531	0.546	0.577
$R_{SDR^{\alpha}}$	507.741	492.178	482.685	475.726	480.981	473.086	471.835	0.484	0.521	0.609	0.631	0.577	0.594	0.637
$Expectile^{\alpha}$	0.063	0.061	0.058	0.059	0.059	0.059	0.059	0.229	0.259	0.300	0.319	0.289	0.305	0.327
$Expectile^{\beta}$	0.106	0.105	0.103	0.104	0.102	0.103	0.104	0.143	0.166	0.177	0.194	0.178	0.193	0.203
$R_{Expectile^{\alpha}}$	0.086	0.085	0.083	0.084	0.082	0.084	0.084	0.181	0.207	0.230	0.248	0.227	0.242	0.257
$\alpha = 2.5\%$ and $\beta = 5\%$														
	Realized loss							Realized cost						
GARCH	norm	snorm	std	sstd	ged	sged	jsu	norm	snorm	std	sstd	ged	sged	jsu
$RVaR^{\alpha}$	10268.655	10268.606	10268.620	10268.597	10268.549	10268.553	10268.565	0.444	0.464	0.439	0.463	0.454	0.476	0.470
VaR^{α}	6.028	5.978	5.976	5.950	5.934	5.921	5.930	0.483	0.508	0.493	0.521	0.507	0.532	0.532
VaR^{β}	9.716	9.716	9.730	9.731	9.700	9.715	9.718	0.412	0.429	0.398	0.418	0.412	0.431	0.423
$R_{VaR^{\alpha}}$	7.955	7.947	7.954	7.951	7.926	7.943	7.950	0.444	0.464	0.439	0.463	0.454	0.476	0.470
ES^{α}	493.715	491.623	491.491	490.470	489.808	489.296	489.655	0.571	0.608	0.658	0.696	0.641	0.674	0.708
ES^{β}	711.151	711.115	711.403	711.414	710.782	711.090	711.156	0.506	0.535	0.546	0.577	0.546	0.573	0.587
$R_{ES^{\alpha}}$	599.427	598.873	599.371	598.985	598.338	598.573	598.828	0.535	0.568	0.595	0.629	0.588	0.618	0.640
SDR^{α}	538.030	532.043	531.078	527.863	527.717	522.735	522.487	0.664	0.700	0.746	0.784	0.728	0.761	0.796
SDR^{β}	750.543	745.647	748.953	746.061	745.615	742.754	743.896	0.598	0.627	0.634	0.665	0.633	0.660	0.675
$R_{SDR^{\alpha}}$	641.902	633.835	638.292	630.919	631.500	628.018	630.307	0.628	0.660	0.683	0.717	0.675	0.705	0.728
$Expectile^{\alpha}$	0.106	0.105	0.103	0.104	35.688	0.102	0.104	0.359	0.376	0.373	0.393	0.408	0.393	0.399
$Expectile^{\beta}$	0.155	0.154	0.154	0.154	0.153	0.154	0.154	0.307	0.318	0.307	0.322	0.311	0.325	0.326
$R_{Expectile^{\alpha}}$	0.132	0.132	0.131	0.131	0.130	0.131	0.131	0.330	0.343	0.335	0.353	0.339	0.355	0.357
$\alpha = 1\%$ and $\beta = 5\%$														
	Realized loss							Realized cost						
GARCH	norm	snorm	std	sstd	ged	sged	jsu	norm	snorm	std	sstd	ged	sged	jsu
$RVaR^{\alpha}$	10113.231	10113.107	10113.036	10113.044	10113.000	10113.022	10113.021	0.277	0.304	0.295	0.315	0.311	0.326	0.327
VaR^{α}	3.196	3.066	2.969	2.970	2.964	2.966	2.956	0.372	0.409	0.449	0.471	0.451	0.467	0.486
VaR^{β}	9.716	9.716	9.730	9.731	9.700	9.715	9.718	0.210	0.231	0.199	0.216	0.217	0.231	0.224
$R_{VaR^{\alpha}}$	6.806	6.789	6.773	6.779	6.755	6.767	6.771	0.277	0.304	0.295	0.315	0.311	0.326	0.327
ES^{α}	415.092	402.072	392.352	392.344	391.853	391.978	390.937	0.443	0.487	0.627	0.650	0.571	0.589	0.652
ES^{β}	711.151	711.115	711.403	711.414	710.782	711.090	711.156	0.312	0.344	0.368	0.388	0.368	0.383	0.398
$R_{ES^{\alpha}}$	549.024	546.499	546.700	545.288	544.995	544.047	544.237	0.365	0.402	0.464	0.485	0.447	0.463	0.493
SDR^{α}	486.842	461.779	430.738	429.084	429.884	428.411	423.392	0.520	0.563	0.693	0.716	0.638	0.656	0.718
SDR^{β}	750.543	745.647	748.953	746.061	745.615	742.754	743.896	0.399	0.427	0.446	0.464	0.444	0.458	0.472
$R_{SDR^{\alpha}}$	597.177	588.604	588.848	582.947	584.362	581.276	581.695	0.447	0.480	0.536	0.556	0.519	0.534	0.564
$Expectile^{\alpha}$	0.063	0.061	0.061	0.059	0.059	0.058	0.059	0.229	0.259	0.288	0.319	0.289	0.305	0.327
$Expectile^{\beta}$	0.155	0.154	0.155	0.154	0.153	0.154	0.154	0.066	0.085	0.072	0.095	0.085	0.098	0.101
$R_{Expectile^{\alpha}}$	0.116	0.116	0.116	0.115	0.114	0.115	0.115	0.132	0.155	0.156	0.182	0.166	0.181	0.190

Note: This table describes realized loss and realized cost values of $RVaR^{\alpha}$, VaR^{α}, VaR^{β}, $R_{VaR^{\alpha}}$, ES^{α}, ES^{β}, $R_{ES^{\alpha}}$, SDR^{α}, SDR^{β}, $R_{SDR^{\alpha}}$, $Expectile^{\alpha}$, $Expectile^{\beta}$ and $R_{Expectile^{\alpha}}$ forecasts. The results are based in a rolling window of 1000 observations. For risk estimation we consider an AR(1)-GARCH(1,1) model, where z_t follows normal (norm), skewed normal (snorm), Student-t (std), skewed Student-t (sstd), generalized error (ged), skewed generalized error (sged), or Johnson SU (jsu) distributions. The values in bold correspond to the model with the best result, according to the criterion considered.

Table 7: Model risk estimates of capital determination obtained considering a rolling window estimation of 1000 observations. To quantify model risk, we consider MR1, MR2, LR1 and LR2. The values are multiplied by 100.

	$\alpha = 1\%$ and $\beta = 2.5\%$				$\alpha = 2.5\%$ and $\beta = 5\%$				$\alpha = 1\%$ and $\beta = 5\%$			
	MR1	MR2	LR1	LR2	MR1	MR2	LR1	LR2	MR1	MR2	LR1	LR2
RVaR^{α}	0.070	0.035	4.082	2.041	0.040	0.020	3.024	1.512	0.049	0.024	3.302	1.651
VaR^{α}	0.099	0.049	5.089	2.544	0.052	0.026	3.398	1.699	0.099	0.049	5.089	2.544
VaR^{β}	0.052	0.026	3.398	1.699	0.034	0.017	3.015	1.507	0.034	0.017	3.015	1.507
$\mathrm{R}_{\mathrm{VaR}^{\alpha}}$	0.070	0.035	4.082	2.041	0.040	0.020	3.024	1.512	0.049	0.024	3.302	1.651
ES^{α}	0.197	0.099	8.328	4.164	0.113	0.057	5.729	2.865	0.197	0.099	8.328	4.164
ES^{β}	0.113	0.057	5.729	2.865	0.071	0.035	4.223	2.111	0.071	0.035	4.223	2.111
$\mathrm{R}_{\mathrm{ES}^{\alpha}}$	0.146	0.073	6.805	3.403	0.088	0.044	4.860	2.430	0.109	0.054	5.601	2.800
SDR^{α}	0.191	0.096	7.174	3.587	0.108	0.054	4.740	2.370	0.191	0.096	7.174	3.587
SDR^{β}	0.108	0.054	4.740	2.370	0.067	0.033	3.380	1.690	0.067	0.033	3.380	1.690
$\mathrm{R}_{\mathrm{SDR}^{\alpha}}$	0.140	0.070	5.736	2.868	0.084	0.042	3.943	1.971	0.104	0.052	4.611	2.306
$\mathrm{Expectile}^{\alpha}$	0.072	0.036	5.101	2.551	0.042	0.021	3.886	1.943	0.092	0.046	6.458	3.229
$\mathrm{Expectile}^{\beta}$	0.042	0.021	3.899	1.950	0.027	0.014	3.314	1.657	0.040	0.020	4.583	2.292
$\mathrm{R}_{\mathrm{Expectile}^{\alpha}}$	0.053	0.027	4.375	2.188	0.033	0.017	3.531	1.765	0.055	0.028	5.122	2.561

Note: This table shows model risk estimates of capital determination. For estimation, we consider a rolling window estimation of 1000 observations. As significance levels, we use 1%, 2.5%, and 5%. For risk estimation, we considered an AR(1)-GARCH(1,1) model, where z_t follows normal (norm), skewed normal (snorm), Student-t (std), skewed Student-t (sstd), generalized error (ged), skewed generalized error (sged), or Johnson SU (jsu) distributions. Values in bold indicate the risk and range measures with the lowest model risk, according to LR1 and LR2.

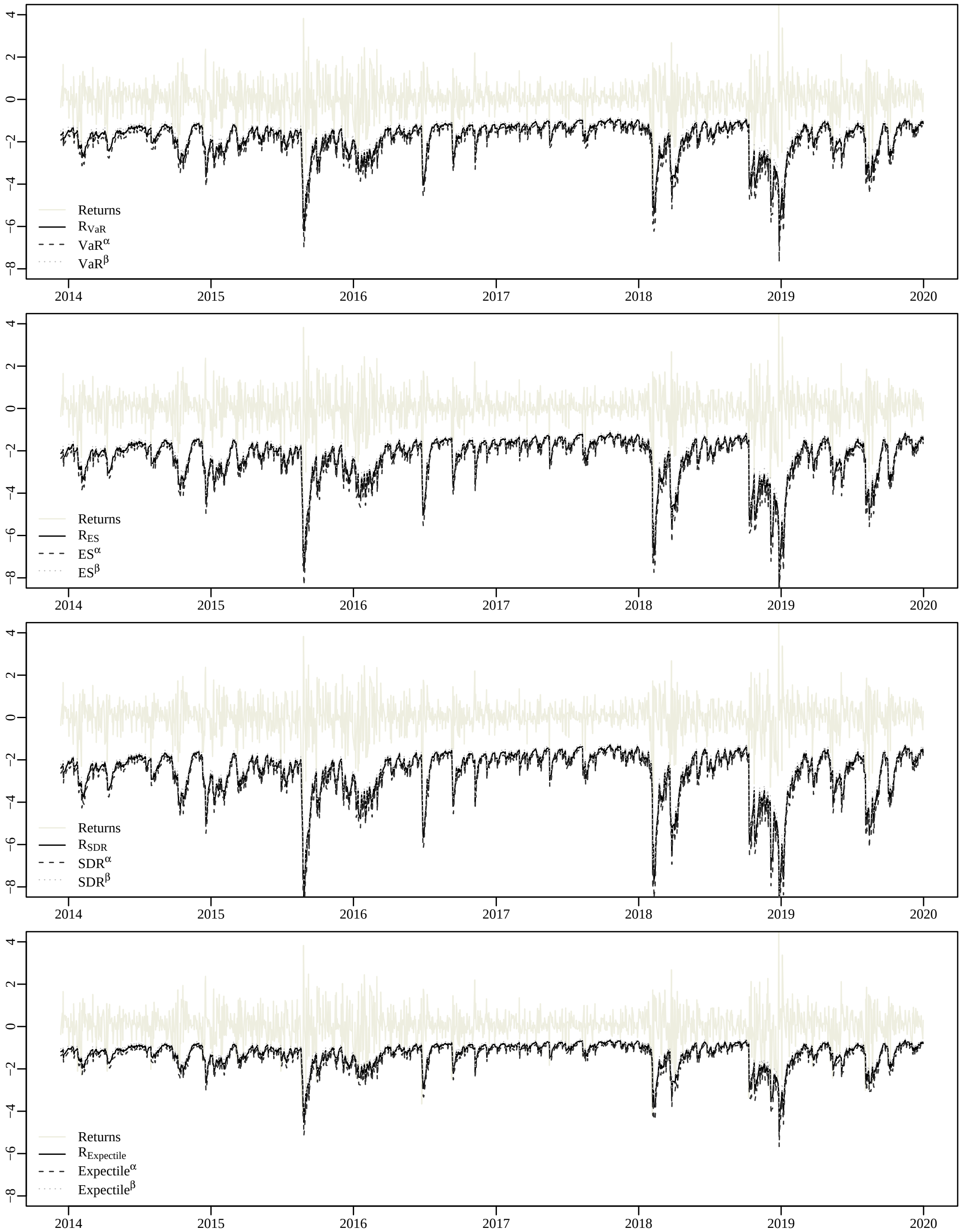

Figure 1: Evolution of risk and range based risk forecasts for S&P 500 log-returns from December 12, 2013, to December 31, 2019 (out-of-sample period). The estimates are obtained using AR(1)-GARCH(1,1) with skewed generalized error and $\alpha = 1.0\%$ and $\beta = 2.5\%$ for a rolling window estimation of 1000 observations. The risk values are with the sign adjusted. Returns and risk estimates are multiplied by 100.