# New loss reserve models with persistence effects to forecast trapezoidal losses in run-off triangles

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# Appendix

A. To simulate random variables $Y$ from GB2 distribution for reserve forecast, we make use of the fact that GB2 distribution is transformed from a beta distribution such that

$$\text{if } U = \frac{(\frac{Y}{b})^a}{1+(\frac{Y}{b})^a} \sim \text{Beta}(p,q), \;\; Y \sim \text{GB2}(a,b,p,q). \tag{31}$$

Hence the simulation procedures are

$$Y = b\Big(\frac{U}{1-U}\Big)^{1/a} = \frac{\mu\Gamma(p)\Gamma(q)}{\Gamma(p+1/a)\Gamma(q-1/a)}\left(\frac{U}{1-U}\right)^{1/a} \;\; \text{where } U \sim \text{Beta}(p,q) \tag{32}$$

using (17). We note that the inverse method $Y = F^{-1}_{\text{GB2}}(U),\; U \sim \text{Uniform}(0,1)$ fails as the distribution function $F_{\text{GB2}}(\cdot)$ has no closed inverse function.

To implement the Bayesian inference, we use the `Stan` package running under the `R` environment. As GB2 distribution is not a built-in distribution in `Stan`, we define the distribution by defining its log density called `beta2_log` using (18) as below

```
functions{
real beta2_log(real x, real a, real mu, real p, real q) {
return log(fabs(a))+(a*p-1)*log(x)+lgamma(p+q)
 -(a*p)*log((mu*tgamma(p)*tgamma(q))/(tgamma(p+(1/a))*tgamma(q-(1/a))))
 -lgamma(p)-lgamma(q)-(p+q)*log(1+pow((x*tgamma(p+(1/a))*tgamma(q-(1/a)))
  /(mu*tgamma(p)*tgamma(q)),a));
} }
```

where `tgamma()` is a gamma function and `lgamma()` is a log-gamma function. Then the model is specified as `y` $\sim$ `beta2(a,mu,p,q)` where `y` and `mu` are the observation vector and mean vector respectively.

B. The generalized gamma (GG) distribution has two shape parameters with the density:

$$f_{\text{GG}}(y;a,\lambda,p) = \frac{a y^{ap-1} e^{-\left(\frac{y}{\lambda}\right)^a}}{\lambda^{ap}\Gamma(p)} \tag{33}$$

where $\lambda$ is the scale parameter, $a$ and $p$ are the shape parameters ($\boldsymbol{\nu} = (a,p)$), and $\text{E}(Y) = \mu = \frac{\lambda\Gamma(p+1/a)}{\Gamma(p)}$. Expressing the scale parameter

$$\lambda = \frac{\mu\Gamma(p)}{\Gamma(p+1/a)} \tag{34}$$

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in terms of $\mu$, the density in (33) can be written as

$$f_{\mathrm{GG}}(y;a,\mu,p)=\frac{ay^{ap-1}e^{-\left[\frac{y\Gamma(p+1/a)}{\mu\Gamma(p)}\right]^{a}}}{\left[\frac{\mu\Gamma(p)}{\Gamma(p+1/a)}\right]^{ap}\Gamma(p)}. \tag{35}$$

The GG distribution nests Weibull ($p=1$), gamma ($a=1$), and exponential ($a=p=1$) distributions as special cases and lognormal as a limiting distribution ($b=(\sigma^2a^2)^{1/a}$, $p=(a\mu+1)/(\sigma^2a^2)$, $a\to 0$).

To simulate random variables $Y$ from GG distribution for reserve forecast, we make use of the fact that GG distribution is transformed from a gamma distribution such that if $U=(\frac{Y}{\lambda})^a\sim$ Gamma$(p,1)$, $Y\sim$ GG$(p,\lambda,a)$. Hence, the simulation procedures are

$$Y=\lambda U^{1/a}=\frac{\mu\Gamma(p)}{\Gamma(p+1/a)}U^{1/a}\ \text{ where }\ U\sim\text{Gamma}(p,1). \tag{36}$$

Again, GG distribution is not a built-in distribution in `Stan`. We define the distribution directly by defining its log density called `ggamma_log` using (35), similar to the case of GB2 distribution. Then the model is specified as `y ~ ggamma(p,mu,a)`.

C. Weibull distribution has the density and distribution functions given by respectively

$$f_{\mathrm{W}}(y;a,\lambda)=\frac{ay^{a-1}}{\lambda^a}e^{-(y/\lambda)^a}\quad\text{and}\quad F_{\mathrm{W}}(y;a,\lambda)=1-\exp\left[-\left(\frac{y}{\lambda}\right)^a\right] \tag{37}$$

where $\lambda$ is the scale parameter, $a$ is the shape parameter ($\nu=a$), and $\mathrm{E}(Y)=\mu=\lambda\Gamma(1+1/a)$. Expressing the scale parameter $\lambda=\frac{\mu}{\Gamma(1+1/a)}$ in terms of $\mu$, the density in (37) can be written as

$$f_{\mathrm{W}}(y;a,\mu)=\frac{ay^{a-1}\Gamma(1+1/a)^a}{\mu^a}\exp\left\{-\left[\frac{y\Gamma\left(1+\frac{1}{a}\right)}{\mu}\right]^a\right\}. \tag{38}$$

To simulate random variables $Y$ from Weibull distribution for reserve forecast, we can use the inverse method since the distribution function in (37) has close form inverse. Hence, the simulation procedures are

$$U\sim\text{uniform}(0,1),\quad Y=F_{\mathrm{W}}^{-1}(U)=\lambda[-\ln(1-U)]^{1/a}. \tag{39}$$

Since Weibull distribution is a built-in function in `Stan`, the model is specified as `y ~ Weibull(a,lam)` where `lam=mu/tgamma(1+1/a)`.

D. To simulate random variables $Y$ from EE distribution for reserve forecast, we can use the inverse method so that if $U\sim$ uniform$(0,1)$,

$$Y=F_{\mathrm{EE}}^{-1}(U)=\frac{\mu}{e\Gamma\left(\frac{1}{a}+1,1\right)-1}\left\{\left[1-\log(1-U)\right]^{\frac{1}{a}}-1\right\}$$

substituting $\lambda=\mu\left[e\Gamma\left(\frac{1}{a}+1,1\right)-1\right]^{-1}$ in (20). Alternatively, we can use the fact that EE distribution is transformed from a standard exponential distribution with pdf $f_{\mathrm{E}}(u)=e^{-u}$ such that if $U=-\left[1-\left(1+\frac{Y}{\lambda}\right)^a\right]\sim$ Exp$(1)$, $Y\sim$ EE$(a,\mu)$. Hence the simulation procedures are

$$Y=\frac{\mu}{e\Gamma\left(\frac{1}{a}+1,1\right)-1}\left[(U+1)^{1/a}-1\right]\ \text{ where }\ U\sim\text{Exp}(1). \tag{40}$$

Again, EE distribution is not a built-in distribution in `Stan`. We define the distribution directly by defining its log pdf called `eexp_log` using (21). Then the model is specified as `y ~ eexp(a,mu)`.

E. Both GB2 and GG are distributions generated from Gamma distributions. Let $Z_1$ and $Z_2$ be gamma distributed random variables with scale parameter 1 and shape parameter $p$ and $q$ respectively. Then

$$b\left(\frac{Z_1}{Z_2}\right)^{1/a} \sim \text{GB2}(a,b,p,q) \text{ and } b\left(Z_1\right)^{1/a} \sim \text{GG}(a,b,p) \tag{41}$$

Thus, the tail of $Z_1/Z_2$ will be thinner (heavier) relative to the tail of $Z_1$ alone as the parameter $q$ is large (near or less than one).

F. To calculate the log of GB2 density for any $(i,j)$ cell at 90% quantile as an example can be computed using the commands `u90=qbeta(0.90,p,q); z90=ln(u90/(1-u90)); y90=b exp(z90/a); ld90=logf.gb2(y90,a,b,p,q)` applying some built-in R functions where the parameter estimates of $a, p$ and $q$ and $b$ for the $(i,j)$ cell are given by the posterior mean. For GG distribution, we use the commands

```
y90=qggamma(0.90,alpha=a,scale=lam,psi=p);
ld90=dggamma(y90,alpha=a,scale=lam,psi=p,log=T)
```

where `lam` for the $(i,j)$ cell is given by (34).