MORTALITY CREDITS WITHIN LARGE SURVIVOR FUNDS

Supplementary online material

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1 Network structure

In this section, we introduce a risk transfer network structure that allows participants to restrict risk sharing to a community of individuals with whom they are connected. Sharing mortality credits among all participants to the pool allows for maximum risk diversification. However, for members of a small community, when one participant dies, it can be preferred that his or her contribution remains in the community rather than being scattered within the entire pool.

We assume that participants' accumulated assets are identical across the pool, that is, $a_i = a$ for some $a > 0$.

1.1 The mortality risk transfer network framework

We introduce a mortality risk transfer network to account for the links existing between some of the participants within the pool and explain how mortality credits can be shared among the sub-pool of participants who are connected with him or her.

The participants' undirected graph is denoted by $G = (V, E)$ where the set of nodes is $V = \{1, ..., n\}$ and the set of edges between participants is $E \subseteq \mathcal{P}_2(V)$ where $\mathcal{P}_2(V)$ is the unordered subsets of V of size 2. The subset of participants who are directly connected with participant i is given by $\mathcal{C}(i) = \{j \in V : (i, j) \in E\}$. The cardinal of $\mathcal{C}(i)$ is referred to as the degree of node i on the graph $G = (V, E)$ and is denoted by $\deg(i)$. This is the number of participants who are connected with participant i. We associate weights $\{\alpha_{ii}, \alpha_{ij} : j \in \mathcal{C}(i)\}$ to participant i and to his or her connected peers such that

$$\alpha_{ii} > 0, \;\; \alpha_{ij} > 0 \text{ for } j \in \mathcal{C}(i) \text{ and } \alpha_{ii} + \sum_{j \in \mathcal{C}(i)} \alpha_{ij} = 1.$$

Participant i's initial contribution to the fund c is then split into $1+\deg(i)$ parts of respective amounts $\alpha_{ii}c$ and $\alpha_{ji}c$ for $j \in \mathcal{C}(i)$.

The mortality risk transfer network is composed of several overlapping sub-pools: one for each participant. Participant i's sub-pool gathers participants belonging to $\mathcal{C}(i)$. In this sub-pool, participant i brings capital $\alpha_{ii}c$ while participant j, for $j \in \mathcal{C}(i)$, brings $\alpha_{ji}c$. With this construction, the mortality credit $\alpha_{ij}X_i$, for $j \in \mathcal{C}(i)$, is attached to participant j's sub-pool while the mortality credit $\alpha_{ii}X_i$ is attached to participant i's sub-pool. The aggregate mortality credit for participant i's sub-pool is then given by

$$S^{(i)} = \alpha_{ii}X_i + \sum_{j \in \mathcal{C}(i)} \alpha_{ji}X_j.$$

The amount received ex post by participant i from his or her sub-pool and the amount received ex post by participant j, $j \in \mathcal{C}(i)$, from participant i's sub-pool are respectively given by

$$h_{ii}(S^{(i)}) = \mathrm{E}\left[\alpha_{ii} X_i \,\middle|\, S^{(i)}\right] \quad \text{and} \quad h_{ij}(S^{(i)}) = \mathrm{E}\left[\alpha_{ji} X_j \,\middle|\, S^{(i)}\right].$$

The ex-post aggregate amount received by participant i is therefore given by

$$h_i(\boldsymbol{X}) = h_{ii}(S^{(i)}) + \sum_{j \in \mathcal{C}(i)} h_{ji}(S^{(j)}),$$

where $\boldsymbol{X} = (X_1, \ldots, X_n)$.

The amounts $h_1(\boldsymbol{X}), \ldots, h_n(\boldsymbol{X})$ satisfy $\sum_{i=1}^{n} h_i(\boldsymbol{X}) = \sum_{j=1}^{n} X_j$, meaning that the entire resources are pooled within the whole group (see Appendix A for a proof). Compared to the survivor fund mechanism described in Section 2 of the paper, the network structure limits the benefits of mortality credits to a reduced number of well-identified participants. Moreover, the mortality risk transfer network is still fair and does not transfer money from some participants to other ones on average (ex ante) since

$$\begin{aligned}
\mathrm{E}[h_i(\boldsymbol{X})] &= \mathrm{E}\left[h_{ii}(S^{(i)}) + \sum_{j \in \mathcal{C}(i)} h_{ji}(S^{(j)})\right] \\
&= \alpha_{ii}\mathrm{E}\left[X_i\right] + \sum_{j \in \mathcal{C}(i)} \alpha_{ij}\mathrm{E}\left[X_i\right] \\
&= \mathrm{E}\left[X_i\right].
\end{aligned}$$

Remark 1.1. *Notice that the structure of the pool presented in Section 2 of the paper is a particular case of the network structure proposed here: it corresponds to the case where the graph is complete/fully-connected (i.e. participants are connected to all other participants by a path of length* 1*), and weights are uniform, i.e.* $\alpha_{ij} = 1/n$ *for all* $i = 1, \ldots, n$ *and* $j = 1, \ldots, n$.

1.2 Large-pool approximations for Erdős–Rényi graphs

In order to assess the diversification benefits, let us consider the case of the Erdős–Rényi models where each edge has a fixed probability p of being present or absent, independently of the other edges. Let $G(n, p)$ be an Erdős–Rényi random graph with n nodes/participants and probability p. Because the existence of each edge is a Bernoulli random variable with parameter p, the degree of a given node is a Binomial random variable with parameters $n-1$ and p. The average degree is thus $\mathrm{E}[\deg(i)] = (n-1)p$. Analysis of probabilistic properties of large Erdős–Rényi graphs is performed as n approaches ∞ when the probability p is a function of n, henceforth denoted as p_n.

We are interested in the connectivity property, that is, whether it is possible to connect all participants (possibly with a path of length larger than 1). It turns out that there exists a threshold function for p_n above which an Erdős–Rényi graph is connected and below which it is not connected with high probability. Precisely, the following result can be found in Chapter 3 of Bollobas (2001). Let $\lambda > 0$ and $p_n = \lambda \log n / n$. Then $G(n, p)$ is connected with high probability for $\lambda > 1$ and disconnected with high probability for $\lambda < 1$. Hence $t_n = \log n / n$ is a threshold function for the connectivity property. Note that if $p_n = \lambda \log n / n$, then the average degree for participant i satisfies $\mathrm{E}\left[\deg(i)\right] \sim \lambda \ln n$ for large n. It tends to ∞ but at a slow rate. If

$$p_n = \frac{1}{n}\left(\log n + k \log\log n + y\right),$$

where $k \in \{2, 3, \ldots\}$ and $y \in \mathbb{R}$, then the minimal degree over the graph is almost surely bounded below, since

$$\lim_{n\to\infty} \mathrm{P}\left[\min_{i=1,\ldots n} \deg(i) = k\right] = 1 - \mathrm{e}^{-\mathrm{e}^{-y}/k!} \text{ and } \lim_{n\to\infty} \mathrm{P}\left[\min_{i=1,\ldots n} \deg(i) = k-1\right] = \mathrm{e}^{-\mathrm{e}^{-y}/k!}.$$

See Theorem 3.5 in Bollobas (2001).

Let us now assume that $p_n = \lambda \log n / n$ with $\lambda > 0$ and that the weights within each sub-pool are uniform, i.e. for $i = 1, \ldots, n$,

$$\alpha_{ii} = \alpha_{ij} = \frac{1}{1 + \deg(i)}, \qquad j \in \mathcal{C}(i).$$

For $i = 1, \ldots, n$, define

$$\bar{\sigma}_{n,i}^2 = \frac{a^2}{1 + \deg(i)} \left(q_{x_i}(1 - q_{x_i}) + \sum_{j \in \mathcal{C}(i)} q_{x_j}\left(1 - q_{x_j}\right) \right).$$

We can then state the following result.

Proposition 1.2. *Assume that the sequence (q_{x_l}) associated to participant i's sub-pool and to sub-pools of all participants belonging to $\mathcal{C}(i)$ is contained in a closed subinterval of $(0, 1)$. Assume moreover that $\lim_{n\to\infty} \bar{\sigma}_{n,i}^2 = \sigma^2 > 0$, and that, for $j \in \mathcal{C}(i)$, $\lim_{n\to\infty} \bar{\sigma}_{n,j}^2 = \sigma^2$. Then, as $n \to \infty$,*

$$\sqrt{\lambda \log n} \frac{\sigma}{q_{x_i}(1 - q_{x_i}) a^2} \left(h_i(\boldsymbol{X}) - q_{x_i} a\right) \xrightarrow{\mathcal{L}} \mathrm{Normal}(0, 1).$$

The proof of Proposition 1.2 is given in Appendix B. Proposition 1.2 shows that the risk diversification intensity is proportional to the square root of the average size of the sub-pool for each participant.

References

- Bollobas, B. (2001). Random Graphs. Cambridge University Press, Second Edition.

- Strasser, H. (2012). Asymptotic expansions for conditional moments of Bernoulli trials. Statistics & Risk Modeling 29, 327-343.

Appendix

A Full allocation for the network structure

In order to establish that the identity $\sum_{i=1}^{n} h_i(\boldsymbol{X}) = \sum_{j=1}^{n} X_j$ holds true, let us write

$$
\begin{aligned}
\sum_{i=1}^{n} h_i(\boldsymbol{X}) &= \sum_{i=1}^{n} h_{ii}(S^{(i)}) + \sum_{i=1}^{n} \sum_{j\in\mathcal{C}(i)} h_{ji}(S^{(j)}) \\
&= \sum_{i=1}^{n} \mathrm{E}\left[\alpha_{ii}X_i \,\middle|\, S^{(i)}\right] + \sum_{i=1}^{n} \sum_{j\in\mathcal{C}(i)} \mathrm{E}\left[\alpha_{ij}X_i \,\middle|\, S^{(j)}\right] \\
&= \sum_{i=1}^{n} \mathrm{E}\left[\alpha_{ii}X_i \,\middle|\, S^{(i)}\right] + \sum_{(i,j)\in E} \mathrm{E}\left[\alpha_{ij}X_i \,\middle|\, S^{(j)}\right] \\
&= \sum_{i=1}^{n} \mathrm{E}\left[\alpha_{ii}X_i \,\middle|\, S^{(i)}\right] + \sum_{(i,j)\in E} \mathrm{E}\left[\alpha_{ji}X_j \,\middle|\, S^{(i)}\right] \\
&= \sum_{i=1}^{n} \mathrm{E}\left[\alpha_{ii}X_i \,\middle|\, S^{(i)}\right] + \sum_{i=1}^{n} \sum_{j\in\mathcal{C}(i)} \mathrm{E}\left[\alpha_{ji}X_j \,\middle|\, S^{(i)}\right] \\
&= \sum_{i=1}^{n} \mathrm{E}\left[\alpha_{ii}X_i + \sum_{j\in\mathcal{C}(i)} \alpha_{ji}X_j \,\middle|\, S^{(i)}\right] \\
&= \sum_{i=1}^{n} S^{(i)} = \sum_{i=1}^{n} \left(\alpha_{ii}X_i + \sum_{j\in\mathcal{C}(i)} \alpha_{ji}X_j\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \alpha_{ii} X_i + \sum_{i=1}^{n} \sum_{j \in \mathcal{C}(i)} \alpha_{ji} X_j = \sum_{i=1}^{n} \alpha_{ii} X_i + \sum_{i=1}^{n} \sum_{j \in \mathcal{C}(i)} \alpha_{ij} X_i \\
&= \sum_{i=1}^{n} X_i \left(\alpha_{ii} + \sum_{j \in \mathcal{C}(i)} \alpha_{ij} \right) = \sum_{j=1}^{n} X_j .
\end{aligned}
$$

B Proof of Proposition 1.2

Defining

$$
\begin{aligned}
a'_{i,n} &= \frac{\lambda \log n}{1 + \deg(i)} a \\
X'_{i,n} &= (1 - I_i) a'_{i,n} = \frac{\lambda \log n}{1 + \deg(i)} X_i \\
S^{(i)\prime} &= X'_{i,n} + \sum_{j \in \mathcal{C}(i)} X'_{j,n} = \lambda \log n S^{(i)}
\end{aligned}
$$

we have that

$$
h_{ii}(S^{(i)}) = \frac{1}{\lambda \log n} \mathrm{E}\left[X'_i \middle| S^{(i)\prime} \right]
$$

and it follows from the proof of Theorem 2.1 in Strasser (2012) that, for large n, and $j \in \mathcal{C}(i)$,

$$
h_{ij}(S^{(i)}) = \frac{1}{\lambda \log n} \left(q_{x_j} a'_{j,n} + \frac{1}{\sqrt{1 + \deg(i)}} \frac{q_{x_j}(1 - q_{x_j}) a'^2_{j,n}}{\overline{\sigma}_n^{(i)}} Z_n^{(i)} + O_{\mathrm{P}}\left((\log n)^{-1} \right) \right)
$$

where

$$
\begin{aligned}
Z_n^{(i)} &= \sqrt{(1 + \deg(i))} \left(\frac{S^{(i)\prime} - \overline{p}_n^{(i)}}{\overline{\sigma}_n^{(i)}} \right) \\
\overline{p}_n^{(i)} &= \frac{1}{1 + \deg(i)} \left(q_{x_i} a'_{i,n} + \sum_{j \in \mathcal{C}(i)} q_{x_j} a'_{j,n} \right) \\
\left(\overline{\sigma}_n^{(i)} \right)^2 &= \frac{1}{1 + \deg(i)} \left(q_{x_i} \left(1 - q_{x_i}\right) \left(a'_{i,n}\right)^2 + \sum_{j \in \mathcal{C}(i)} q_{x_j} \left(1 - q_{x_j}\right) \left(a'_{j,n}\right)^2 \right).
\end{aligned}
$$

Therefore participant i's contribution to the global pool satisfies

$$
\begin{aligned}
& h_i(\boldsymbol{X}) \\
= \; & h_{ii}(S^{(i)}) + \sum_{j \in \mathcal{C}(i)} h_{ji}(S^{(j)})
\end{aligned}
$$

$$= \quad q_{x_i} a_{i,n} + q_{x_i} \left(1 - q_{x_i}\right) \left(a'_{i,n}\right)^2 \left(\frac{Z_n^{(i)}}{\sqrt{1 + \deg(i)} \overline{\sigma}_n^{(i)}} + \sum_{j \in \mathcal{C}(i)} \frac{Z_n^{(j)}}{\sqrt{1 + \deg(j)} \overline{\sigma}_n^{(j)}} \right) + O_{\mathrm{P}}\left((\log n)^{-1} \right).$$

We deduce that

$$\frac{\sqrt{\lambda \log n}}{q_{x_i} \left(1 - q_{x_i}\right) a^2} \left(h_i(\boldsymbol{X}) - q_{x_i} a\right)$$

$$= \quad \left(\frac{\lambda \log n}{1 + \deg(i)} \right)^2 \sqrt{\frac{1 + \deg(i)}{\lambda \log n}} \frac{1}{\sqrt{1 + \deg(i)}} \left(\sqrt{\frac{\lambda \log n}{1 + \deg(i)}} \frac{Z_n^{(i)}}{\overline{\sigma}_n^{(i)}} + \sum_{j \in \mathcal{C}(i)} \sqrt{\frac{\lambda \log n}{1 + \deg(j)}} \frac{Z_n^{(j)}}{\overline{\sigma}_n^{(j)}} \right)$$

$$+ O_{\mathrm{P}}\left((\log n)^{-1/2} \right).$$

The random variables $Z_n^{(i)} / \overline{\sigma}_n^{(i)}$ may asymptotically be considered as independent because, for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathrm{P}\left[Card \left(\bigcup_{j,l \in \mathcal{C}(i), j \neq l} \left(\mathcal{C}(j) \cap \mathcal{C}(l)\right) \right) / \log n > \varepsilon \right] = 0.$$

Since

$$\frac{1 + \deg(i)}{\lambda \log n} \xrightarrow{\mathrm{P}} 1, \qquad \overline{\sigma}_n^{(i)} \xrightarrow{\mathrm{P}} \sigma, \qquad Z_n^{(i)} \xrightarrow{\mathcal{L}} \mathrm{Normal}(0, 1),$$

we deduce the result from the Lindeberg-Feller central-limit theorem.