A mixed bond and equity fund model for the valuation of variable annuities*

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Abstract

Variable annuity policies are typically issued on mutual funds invested in both fixed income and equity asset classes. However, due to the lack of specialized models to represent the dynamics of fixed income fund returns, the literature has primarily focused on studying long-term investment guarantees on single-asset equity funds. This article develops a mixed bond and equity fund model in which the fund return is linked to movements of the yield curve. Theoretical motivation for our proposed specification is provided through an analogy with a portfolio of rolling horizon bonds. Moreover, basis risk between the portfolio return and its risk drivers is naturally incorporated into our framework. Numerical results show that the fit of our model to Canadian variable annuity data is adequate. Finally, the valuation of variable annuities is illustrated and it is found that the prevailing interest rate environment can have a substantial impact on guarantee costs.

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Keywords: mutual fund model, bond fund, yield curve, investment guarantee, variable annuities, basis risk

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1 Introduction

The modeling of asset returns has evolved significantly since the groundbreaking contribution of Louis Bachelier. There is now a plethora of processes to model a time series of returns, including GARCH, regime-switching and stochastic volatility models. Many extensions to jointly model series of returns have also been entertained. Yet, the emphasis in the literature has been on modeling equity returns, either on a specific stock or on a market index, and returns on fixed income funds have received considerably less attention. Fixed income funds pool investments that pay a stream of interest payments, such as treasury bills, government bonds, investment-grade corporate bonds and high-yield corporate bonds. Since these securities have fixed times to maturity, new investments must be periodically purchased to keep the fund active. Moreover, bond fund managers often do not hold the underlying securities until maturity, and make trades to target a specified duration or to position themselves with respect to a subjective view of the future interest rate environment. Consequently, fixed income fund returns behave very differently from equity returns, and are in particular strongly related to variations in the term structure of interest rates.

The main contribution of this paper is the proposal of a mixed bond and equity fund model that links the return on the fixed income component of the fund to movements of the yield curve. Importantly, we motivate this link theoretically through an analogy with a portfolio of rolling horizon bonds (see Andersson and Lagerås, 2013; Ekeland and Taflin, 2005; Rutkowski, 1999, for an overview). Our fund return model can be thought of as a multifactor asset pricing model in the spirit of arbitrage pricing theory (see Chen et al., 1986; Elton et al., 1995; Ross, 1976). In short, we express the fund return as a linear function of term structure factor variations and returns on equity market indices, plus a heteroskedastic error term. Although we choose to represent term structure dynamics with the discrete-time multifactor Vasicek process, our model specification is valid in greater generality for any yield curve model in the affine class. Moreover, we coherently define our process under real-world and risk-neutral measures, which enables the use of our model for both risk assessment and asset pricing.

It must be stressed that we do not take the approach of directly modeling the fund return with a univariate stochastic model, but rather we model its underlying risk drivers (i.e., yield curve and stock market factors) and relate them to the fund return. A valuable advantage of this specification is that it naturally incorporates basis risk between the fund return and its risk drivers. We remark that for a pure equity fund, our model simplifies to a fund mapping model and is thus consistent with the way equity fund returns are commonly analyzed in the financial industry (Sharpe, 1992).

Our motivation is twofold. First, bond fund returns are usually modeled in the same way as equity returns in the econometric literature (e.g., Guidolin and Timmermann, 2006; Nystrup et al., 2017), and there is therefore a need for more specialized models that reflect the intrinsic relationship between fixed income fund returns and term structure dynamics. In other words, treating a bond fund return as a risky asset return that is correlated with other risky asset returns misses a key feature of this asset class. Our second motivation is more specific to our research interests in actuarial science, and relates to the relevance of our contribution to the literature on variable annuities (VAs). VAs are retirement planning products sold by insurance companies that combine an investment into mutual funds with a life insurance policy and capital guarantees. These products are also known as segregated funds in Canada and unit-linked contracts in the United Kingdom. The VA literature has so far focused on modeling guarantees on single-asset equity funds with stochastic volatility, stochastic interest rates or both (e.g., Augustyniak and Boudreault, 2012, 2017; Donnelly et al., 2014; Hardy, 2003; Kling et al., 2011). However, VAs are typically written on so-called balanced mutual funds with exposure to both fixed income and equity asset classes, and some guarantees are even issued on pure bond funds. Although multi-asset funds are also occasionally considered (e.g., Boudreault and Panneton, 2009; Gan and Valdez, 2017; Ng and Li, 2013), the underlying modeling frameworks used do not distinguish between bond and equity fund returns; a shortcoming that results from the lack of specialized bond fund models (i.e., our first motivation).

Hence, in this paper we emphasize an application of our model to the valuation of VAs with the hope that our process will be used as a benchmark in future studies, and improved upon.

In the first part of our application, we assess the fit of our model to three Canadian funds. Two of these funds are actual underlying assets of VA policies, and one is a bond market index exchange-traded fund (ETF). Fit diagnostics do not find evidence for the presence of significant residual autocorrelations and heteroskedasticity, and confirm the suitability of our model for the data considered. In the second part, we compare the valuation of long-term capital guarantees issued on a bond fund and on a mixed fund. We include realistic features of VAs associated to the periodic fee, mortality rates, dynamic lapses, surrender penalties, and ratchet provisions. Furthermore, we examine the impact of the initial economic environment (i.e., interest rates and volatilities) on valuation. We find that the initial interest rate setting can have a strong influence on valuation, but that it is long-term, rather than current, volatility assumptions that matter in this respect.

This paper is structured as follows. Section 2 introduces the real-world and risk-neutral dynamics of our proposed fund model under a discrete-time multifactor Vasicek term structure. The theoretical justification for modeling the return on the fixed income component of the fund as a linear function of yield curve factor variations is presented in this section. Section 3 describes the VA data used, details our estimation procedure, and analyzes the fit of our model. Section 4 illustrates the application of our model to the valuation of a VA. Section 5 concludes. An online supplementary appendix provides proofs as well as complementary information and results.

2 Modeling framework

Consider an arbitrage-free and frictionless discrete-time market model with monthly time steps denoted by $t = 0, 1, \ldots, T$. The market dynamics are defined on a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ endowed with a filtration $\mathcal{F} := \{\mathcal{F}_t\}_{t=0}^{T}$, where $\mathbb{P}$ represents the real-world probability measure. We assume that money can be invested and borrowed at the risk-free rate, and that the market includes zero-coupon bonds of different maturities as well as a collection of stock indices.

2.1 Interest rate term structure model

2.1.1 Real-world dynamics of the term structure factors

The dynamics of the interest rate term structure are modeled by a discrete-time version of the multifactor Vasicek model (Wüthrich and Merz, 2013). Allowing for multiple factors is important as Brigo and Mercurio (2007) explain that this is needed to reproduce the different shapes of the yield curve observed in practice, and to account for the imperfect correlation between spot rates at future dates.

The annualized continuously compounded risk-free rate applying to the monthly time period $[t, t+1)$ is denoted by r_t and modeled as

$$r_t = \sum_{i=1}^{p} x_t^{(i)}, \tag{2.1}$$

$$x_{t+1}^{(i)} = x_t^{(i)} + \kappa_i \Big(\mu_i - x_t^{(i)} \Big) + \sigma_i z_{t+1}^{(i)}, \quad i = 1, \ldots, p, \tag{2.2}$$

where p is the number of term structure factors, $x_0^{(i)} \in \mathbb{R}$ is a constant, $z^{(i)} := \{z_t^{(i)}\}_{t=1}^{T}$, $i = 1, \ldots, p$, are $\mathcal{F}$-adapted standard Gaussian white noises with contemporaneous correlation $p \times p$ matrix Γ under $\mathbb{P}$, and $(\kappa_i, \mu_i, \sigma_i)$ are model parameters characterizing the $\mathbb{P}$-dynamics of the factor process $x^{(i)} := \{x_t^{(i)}\}_{t=0}^{T}$. The parameter μ_i corresponds to the long-term mean of factor $x^{(i)}$, κ_i represents the rate at which factor $x^{(i)}$ reverts to its mean, and σ_i is a volatility parameter.

2.1.2 Risk-neutral dynamics of the term structure factors

The passage to the risk-neutral probability measure, denoted by $\mathbb{Q}$, is achieved by way of a discrete-time version of the Girsanov theorem. It follows from this theorem that for given interest rate risk premium parameters $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, p$, there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the processes $\tilde{z}^{(i)} := \{\tilde{z}_t^{(i)}\}_{t=1}^{T}$, $i = 1, \ldots, p$, defined by $\tilde{z}_{t+1}^{(i)} := z_{t+1}^{(i)} - \lambda_i x_t^{(i)}$, are $\mathcal{F}$-adapted standard Gaussian white noises with contemporaneous correlation matrix Γ under $\mathbb{Q}$. Substituting $z_{t+1}^{(i)} = \tilde{z}_{t+1}^{(i)} + \lambda_i x_t^{(i)}$ in Eq. (2.2) yields the $\mathbb{Q}$-dynamics of the factor processes:

$$x_{t+1}^{(i)} = x_t^{(i)} + \tilde{\kappa}_i \big(\tilde{\mu}_i - x_t^{(i)} \big) + \sigma_i \tilde{z}_{t+1}^{(i)}, \quad i = 1, \ldots, p, \tag{2.3}$$

where

$$\tilde{\kappa}_i := \kappa_i - \sigma_i \lambda_i, \qquad \tilde{\mu}_i := \frac{\kappa_i \mu_i}{\kappa_i - \sigma_i \lambda_i}.$$

Under this specific change of measure, the factor processes continue to have discrete-time Vasicek dynamics, but with the modified parameters $\left(\tilde{\kappa}_i, \tilde{\mu}_i, \sigma_i\right)$.

2.1.3 Zero-coupon bond price

Let $P_{t,T}$ denote the price at time t of one dollar received with certainty at time T. The arbitrage-free price of this zero-coupon bond under the risk-neutral measure $\mathbb{Q}$ satisfies

$$P_{t,T} = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\Delta \sum_{j=0}^{T-t-1} r_{t+j}\right) \middle| \mathcal{F}_t\right],$$

where $\Delta := 1/12$. The pricing of zero-coupon bonds under the multifactor Vasicek model is studied by Wüthrich and Merz (2013) under a different parametrization. Proposition 2.1 provides the zero-coupon bond price formula in our setting.

Proposition 2.1. *The arbitrage-free price of a zero-coupon bond in the discrete-time multifactor Vasicek model under the risk-neutral measure $\mathbb{Q}$ defined in Section 2.1.2 is given by*

$$P_{t,T} = \exp\left(A_\tau - \Delta \sum_{i=1}^{p} B_\tau^{(i)} x_t^{(i)}\right),$$

where $\tau := T - t$ and

$$A_\tau := \frac{\Delta^2}{2} \mathbb{1}_p^\top v_\tau \mathbb{1}_p - \Delta \sum_{i=1}^{p} \tilde{m}_\tau^{(i)},$$
$$B_\tau^{(i)} := \frac{1 - (1 - \tilde{\kappa}_i)^\tau}{\tilde{\kappa}_i}, \quad i = 1, \ldots, p.$$

Here, $\mathbb{1}_p$ symbolizes the p-dimensional column vector of ones, $\tilde{m}_\tau^{(i)} := \tilde{\mu}_i \left[\tau - B_\tau^{(i)}\right]$, and v_τ is a $p \times p$ matrix with element on row i and column ℓ, denoted by $v_\tau^{(i,\ell)}$, equal to

$$v_\tau^{(i,\ell)} = \frac{\sigma_i \sigma_\ell}{\tilde{\kappa}_i \tilde{\kappa}_\ell} \Gamma_{i,\ell} \left[\tau - B_\tau^{(i)} - B_\tau^{(\ell)} + \frac{1 - (1 - \tilde{\kappa}_i)^\tau (1 - \tilde{\kappa}_\ell)^\tau}{1 - (1 - \tilde{\kappa}_i)(1 - \tilde{\kappa}_\ell)}\right],$$

where $\Gamma_{i,\ell}$ corresponds to the element on row i and column ℓ of the factors' contemporaneous correlation matrix Γ.

Proof. The proof is available in the online appendix. $\square$

Remark 2.1. *The interpretation of r_t and $x_t^{(i)}$ differs in the discrete and continuous-time settings. In the continuous-time setting, r_t and $x_t^{(i)}$ denote values at exact time t, whereas in the discrete-time setting they denote values in the time period $[t, t+1)$ that are assumed to remain constant across the entire interval. Consequently, the continuous-time and discrete-time multifactor Vasicek models are not equivalent. In particular, they give rise to different zero-coupon bond price formulas.*

2.2 Mixed bond and equity fund model

The main contribution of this paper is the proposal of a model for the dynamics of a fund that comprises both equity and fixed income assets. We let $\{F_t\}_{t=0}^{T}$ represent the value process of this fund, and define the corresponding log-return from t to $t+1$ by

$$R_{t+1}^{(F)} := \log\left(\frac{F_{t+1}}{F_t}\right), \quad t = 0, 1, \dots, T-1.$$

The return on the equity component of the fund is generally strongly related to the returns of broad stock market indices, such as the S&P 500. Consequently, we assume that we observe the dynamics of q stock indices, where $\{S_t^{(j)}\}_{t=0}^{T}$ represents the value process of equity index j for $j = 1, \dots, q$. The corresponding log-return from t to $t+1$ is defined by

$$R_{t+1,j}^{(S)} := \log\left(\frac{S_{t+1}^{(j)}}{S_t^{(j)}}\right), \quad t = 0, 1, \dots, T-1.$$

2.2.1 Fund model specification

Our proposed mixed bond and equity fund model corresponds to a regression model that expresses the fund return in excess of the risk-free rate as a linear function of term structure variations and returns on equity index portfolios as follows:

$$R_{t+1}^{(F)} - r_t \Delta = \theta_0 + \sum_{i=1}^{p} \theta_i \left(x_{t+1}^{(i)} - (1 - \tilde{\kappa}_i) x_t^{(i)} \right) + \sum_{j=1}^{q} \theta_j^{(S)} R_{t+1,j}^{(S)} + \sqrt{h_t^{(F)}} z_{t+1}^{(F)}, \tag{2.4}$$

where $(\theta_0, \theta_1, \ldots, \theta_p)$ and $(\theta_1^{(S)}, \ldots, \theta_q^{(S)})$ are regression coefficient parameters, $\{h_t^{(F)}\}_{t=0}^{T-1}$ is an $\mathcal{F}$-adapted conditional variance process (to be defined below), and $z^{(F)} := \{z_t^{(F)}\}_{t=1}^{T}$ is a standard Gaussian white noise under $\mathbb{P}$, independent of $z^{(i)}$, $i = 1, \ldots, p$, and of $R_j^{(S)} := \{R_{t,j}^{(S)}\}_{t=1}^{T}$, $j = 1, \ldots, q$. The independence between $z^{(F)}$ and $z^{(i)}$, and between $z^{(F)}$ and $R_j^{(S)}$, is justified by the assumption that all dependence between the fund returns and term structure shocks or equity index returns is captured by the covariates in the regression relationship. The error term $\sqrt{h_t^{(F)}} z_{t+1}^{(F)}$ can thus be interpreted as a basis risk component that reflects the idiosyncratic risks not captured by these covariates.

Our model expands on the large literature on fund style analysis pioneered by Sharpe (1988, 1992). Fund style analysis is a statistical technique that maps mutual fund returns onto a set of market index returns to determine the portfolio manager's investment style. Note that if $\theta_i = 0$ for $i = 1, \ldots, p$ in Eq. (2.4), the fund's return would be modeled as a linear function of returns on reference portfolios, which is consistent with fund style analysis. VA providers typically use such fund mapping techniques (see for example, Gan and Valdez, 2017; Trottier et al., 2018b), so our specification is compatible with an industry benchmark. The addition of term structure factor variations in the model is our original contribution and allows us to link the return of the fixed income component of the fund to movements in the term structure. The linear form of this link is motivated theoretically in Section 2.2.2. Even though the equity index return variables $R_{t+1,j}^{(S)}$, $j = 1, \ldots, q$, are primarily included to explain the return on the equity component of the fund, these variables can still prove useful when modeling a pure bond fund that includes some corporate debt. This is due to the empirical observation that credit spreads and equity returns are correlated.

To complete our model specification, we propose to use the following EGARCH dynamics (Nelson,

1991) to represent the conditional variance process $\{h_t^{(F)}\}_{t=0}^{T-1}$ in Eq. (2.4):

$$\log h_t^{(F)} = \omega^{(F)} + \alpha^{(F)} z_t^{(F)} + \gamma^{(F)} \left(|z_t^{(F)}| - 2/\sqrt{2\pi} \right) + \beta^{(F)} \log h_{t-1}^{(F)}, \tag{2.5}$$

where $(\omega^{(F)}, \alpha^{(F)}, \gamma^{(F)}, \beta^{(F)})$ are parameters. In the applications presented in Section 3, we found that a heteroskedastic error term in Eq. (2.4) significantly improved the model fit. Although other processes generating heteroskedastic dynamics could have been entertained, we chose the EGARCH model as it combines the strengths of GARCH and autoregressive stochastic volatility models (see Carnero et al., 2004). For instance, Asai and McAleer (2011) explain that the EGARCH model can flexibly accommodate the leverage effect, as well as an asymmetric response of volatility to positive and negative shocks. Moreover, no parameter restrictions are required to guarantee the positivity of the conditional variance in the EGARCH model. Nelson (1991) argued that the constraints that one one needs to impose in (non-exponential) GARCH processes unduly restrict volatility dynamics. Finally, the EGARCH model can generally better match the sample autocorrelation function of squared errors in empirical applications, which leads to a better representation of volatility persistence (Rodríguez and Ruiz, 2012).

2.2.2 Theoretical motivation

This section motivates the linear relationship between the fund return and term structure factor variations considered in our model. We show that this relationship naturally arises in the context of a simplified bond fund process and can thus be used as a basis to model the fund's fixed income component.

A simplified way to think about a bond fund is to consider a trading strategy where bonds are repeatedly sold and bought so that time to maturities of the bonds in the portfolio are fixed. Such a strategy was first studied theoretically by Rutkowski (1999) and then by Ekeland and Taflin (2005). They referred to it as "rolling horizon bonds" or "roll-overs," because the maturities of the bonds are rolled-over from one period to the next. We note that similar strategies were considered by Stefanovits and Wüthrich (2014) as part of a methodology to value and hedge non-tradable long-term zero-coupon bonds with tradable ones under a given risk tolerance.

Consider a bond fund with initial value $V_0^{(\tau)} = 1$ that aims to hold a risk-free zero-coupon bond with a fixed time to maturity τ. At $t = 0$, $1/P_{0,\tau}$ units of the zero-coupon bond with price $P_{0,\tau}$ are purchased. Then, at $t = 1$, this zero-coupon is sold and the proceeds, $V_1^{(\tau)} = P_{1,\tau}/P_{0,\tau}$, are reinvested in a zero-coupon bond with a time to maturity equal to τ and price $P_{1,\tau+1}$. This process is repeated at every time step and the value of this self-financing strategy at time t corresponds to

$$V_t^{(\tau)} = V_{t-1}^{(\tau)} \frac{P_{t,t+\tau-1}}{P_{t-1,t-1+\tau}} = \prod_{n=1}^{t} \frac{P_{n,n+\tau-1}}{P_{n-1,n-1+\tau}}, \quad t = 1, \ldots, T.$$

The associated log-return process, denoted by $\{R_t^{(\tau)}\}_{t=1}^{T}$, under our discrete-time multifactor Vasicek model is given by

$$\begin{aligned}
R_{t+1}^{(\tau)} &:= \log \left(\frac{V_{t+1}^{(\tau)}}{V_t^{(\tau)}} \right) \\
&= \log \left(\frac{P_{t+1,t+\tau}}{P_{t,t+\tau}} \right) \\
&= \log P_{t+1,t+\tau} - \log P_{t,t+\tau} \\
&= A_{\tau-1} - \Delta \sum_{i=1}^{p} B_{\tau-1}^{(i)} x_{t+1}^{(i)} - \left(A_\tau - \Delta \sum_{i=1}^{p} B_\tau^{(i)} x_t^{(i)} \right) \\
&= A_{\tau-1} - A_\tau - \Delta \sum_{i=1}^{p} \left(B_{\tau-1}^{(i)} x_{t+1}^{(i)} - B_\tau^{(i)} x_t^{(i)} \right).
\end{aligned}$$

It then follows that the log-return in excess of the risk-free rate is equal to[1]

$$\begin{aligned}
R_{t+1}^{(\tau)} - r_t \Delta &= A_{\tau-1} - A_\tau - \Delta \sum_{i=1}^{p} \left(B_{\tau-1}^{(i)} x_{t+1}^{(i)} - B_\tau^{(i)} x_t^{(i)} \right) - \Delta \sum_{i=1}^{p} x_t^{(i)} \\
&= A_{\tau-1} - A_\tau - \Delta \sum_{i=1}^{p} \left(B_{\tau-1}^{(i)} x_{t+1}^{(i)} - \left(B_\tau^{(i)} - 1 \right) x_t^{(i)} \right) \\
&= A_{\tau-1} - A_\tau - \Delta \sum_{i=1}^{p} B_{\tau-1}^{(i)} \left(x_{t+1}^{(i)} - \left(\frac{B_\tau^{(i)} - 1}{B_{\tau-1}^{(i)}} \right) x_t^{(i)} \right)
\end{aligned}$$

[1]Note that:

$$\frac{B_\tau^{(i)} - 1}{B_{\tau-1}^{(i)}} = \frac{\frac{1-(1-\tilde{\kappa}_i)^\tau}{\tilde{\kappa}_i} - 1}{\frac{1-(1-\tilde{\kappa}_i)^{\tau-1}}{\tilde{\kappa}_i}} = \frac{1 - \tilde{\kappa}_i - (1 - \tilde{\kappa}_i)^\tau}{1 - (1 - \tilde{\kappa}_i)^{\tau-1}} = 1 - \tilde{\kappa}_i.$$

$$= A_{\tau-1} - A_{\tau} - \Delta \sum_{i=1}^{p} B_{\tau-1}^{(i)} \left(x_{t+1}^{(i)} - (1 - \tilde{\kappa}_i)\, x_t^{(i)} \right).$$

Now consider a portfolio of rolling horizon bonds that aims to hold M zero-coupon bonds with fixed time to maturities $\tau_1, \ldots, \tau_M$ at each time t in the value-weighted proportions $w_1, \ldots, w_M$, where $\sum_{j=1}^{M} w_j = 1$. The log-return on this fund between t and $t+1$, denoted by $R_{t+1}^{(B)}$, in excess of the risk-free rate is approximately equal to

$$\begin{aligned}
R_{t+1}^{(B)} - r_t \Delta &\approx w_1 R_{t+1}^{(\tau_1)} + \cdots + w_M R_{t+1}^{(\tau_M)} - r_t \Delta \\
&= w_1 \left(R_{t+1}^{(\tau_1)} - r_t \Delta \right) + \cdots + w_M \left(R_{t+1}^{(\tau_M)} - r_t \Delta \right) \\
&= \sum_{j=1}^{M} w_j \left(A_{\tau_j - 1} - A_{\tau_j} - \Delta \sum_{i=1}^{p} B_{\tau_j - 1}^{(i)} \left(x_{t+1}^{(i)} - (1 - \tilde{\kappa}_i)\, x_t^{(i)} \right) \right) \\
&= \sum_{j=1}^{M} w_j \left(A_{\tau_j - 1} - A_{\tau_j} \right) - \sum_{i=1}^{p} \left(\Delta \sum_{j=1}^{M} w_j B_{\tau_j - 1}^{(i)} \right) \left(x_{t+1}^{(i)} - (1 - \tilde{\kappa}_i)\, x_t^{(i)} \right).
\end{aligned}$$

Clearly, this log-return specification is a special case of our fund model defined in Eq. (2.4). To recover it, simply set $\theta_j^{(S)} = 0$ for $j = 1, \ldots, q$, $h_t^{(F)} = 0$ for $t = 0, 1, \ldots, T-1$, and

$$\begin{aligned}
\theta_0 &= \sum_{j=1}^{M} w_j \left(A_{\tau_j - 1} - A_{\tau_j} \right), \\
\theta_i &= -\Delta \sum_{j=1}^{M} w_j B_{\tau_j - 1}^{(i)}, \quad i = 1, \ldots, p.
\end{aligned}$$

Of course, in practice portfolio managers do not exactly manage a bond fund by rolling over bond maturities at every time step. However, they typically structure their portfolios to target a specified duration or choose to hold bonds with roughly constant maturities. The portfolio of rolling horizon bonds can therefore be considered as an approximation to bond fund dynamics. Consequently, this analysis offers support for modeling the fund's fixed income component based on a linear relationship with variations in the term structure factors.

Remark 2.2. *Our analysis entails that the excess return on the portfolio of rolling horizon bonds is linearly related to term structure factor variations whenever* $\log P_{t,T}$ *is itself a linear function* *of yield curve factors. Therefore, this relationship is not limited to the multifactor Vasicek process considered here but holds more generally in the affine model class.*

2.2.3 Equity index model

To complete our mixed bond and equity fund model, a model for the equity index return variables $R^{(S)}_{t+1,j}$, $j = 1, \dots, q$, must be specified. We choose to model log-returns on equity index j with EGARCH dynamics as follows:

$$R^{(S)}_{t+1,j} - r_t\Delta = \lambda^{(S)}_j \sqrt{h^{(S)}_{t,j}} - \frac{1}{2} h^{(S)}_{t,j} + \sqrt{h^{(S)}_{t,j}}\, z^{(S)}_{t+1,j}, \tag{2.6}$$

$$\log h^{(S)}_{t,j} = \omega^{(S)}_j + \alpha^{(S)}_j z^{(S)}_{t,j} + \gamma^{(S)}_j \left(|z^{(S)}_{t,j}| - 2/\sqrt{2\pi} \right) + \beta^{(S)}_j \log h^{(S)}_{t-1,j}, \tag{2.7}$$

where $(\omega^{(S)}_j, \alpha^{(S)}_j, \gamma^{(S)}_j, \beta^{(S)}_j)$ are parameters of the $\mathcal{F}$-adapted EGARCH volatility process $\{h^{(S)}_{t,j}\}_{t=0}^{T-1}$, $\lambda^{(S)}_j$ is an equity risk premium parameter, and $z^{(S)}_j := \{z^{(S)}_{t,j}\}_{t=1}^{T}$ is a standard Gaussian white noise under $\mathbb{P}$, independent of $z^{(F)}$ and $z^{(i)}$, $i = 1, \dots, p$.[2] Since equity indices in different regions of the world are generally strongly correlated, we assume that for each t, $(z^{(S)}_{t,1}, \dots, z^{(S)}_{t,q})$ is a centered normal random vector with $\mathrm{Corr}(z^{(S)}_{t,j}, z^{(S)}_{t,k}) = \rho_{jk} \in (-1, 1)$ for $j, k = 1, \dots, q$ and $j \neq k$.

2.3 Risk-neutral dynamics of the mixed bond and equity fund model

The physical dynamics of our mixed fund model are required for a statistical estimation of model parameters, and to generate the actual distribution of fund returns in a risk assessment analysis. However, for the purpose of pricing derivatives or guarantees on the mixed fund, the risk-neutral dynamics must also be derived.

The risk-neutral dynamics of our model are obtained by risk-neutralizing the three underlying risk drivers: $z^{(i)}$, $i = 1, \dots, p$ (interest rate risk), $z^{(S)}_j$, $i = 1, \dots, q$ (equity risk), and $z^{(F)}$ (basis risk). The interest rate risk drivers were already risk-neutralized in Section 2.1.2. Risk-neutralization of

[2]The independence between equity and interest rate innovations $z^{(S)}_j$ and $z^{(i)}$ is assumed due to the difficulty of modeling such dependence with the way interest rates have evolved in the past decades. For instance, interest rates have been steadily declining since the 1990s in most developed countries. Therefore, disentangling the impact of equity market fluctuations from other causes in the decline of interest rates through econometric methods is a challenging endeavor. Moreover, with respect to the specific data sets that we consider in Section 3, the correlation between the S&P/TSX Composite (resp. S&P 500) equity index monthly returns and the monthly changes in the 3-month Canadian interest rate over our data sample is -0.1% (resp. 4.0%). Hence, the independence assumption is empirically supported in our application.

the equity index model is accomplished by invoking Duan (1995)'s local risk-neutral valuation relationship for the EGARCH model (Duan et al., 2006). Under this valuation principle, the shifted innovation process $\tilde{z}_j^{(S)} := \{\tilde{z}_{t,j}^{(S)}\}_{t=1}^T$, defined by $\tilde{z}_{t,j}^{(S)} := z_{t,j}^{(S)} + \lambda_j^{(S)}$, is a standard Gaussian white noise under our risk-neutral measure $\mathbb{Q}$ for each $j = 1, \ldots, q$. Accordingly, the $\mathbb{Q}$-dynamics of the log-returns on equity index j are given by

$$
\begin{aligned}
R_{t+1,j}^{(S)} - r_t\Delta &= -\frac{1}{2}h_{t,j}^{(S)} + \sqrt{h_{t,j}^{(S)}}\tilde{z}_{t+1,j}^{(S)}, \\
\log h_{t,j}^{(S)} &= \omega_j^{(S)} + \alpha_j^{(S)}(\tilde{z}_{t,j}^{(S)} - \lambda_j^{(S)}) + \gamma_j^{(S)}\left(|\tilde{z}_{t,j}^{(S)} - \lambda_j^{(S)}| - 2/\sqrt{2\pi}\right) + \beta_j^{(S)}\log h_{t-1,j}^{(S)}.
\end{aligned}
$$

To complete the risk-neutralization process, first note that Eq. (2.4) can be written as,

$$
R_{t+1}^{(F)} - r_t\Delta = \phi_t + \sum_{i=1}^{p}\theta_i\sigma_i\tilde{z}_{t+1}^{(i)} + \sum_{j=1}^{q}\theta_j^{(S)}\sqrt{h_{t,j}^{(S)}}\tilde{z}_{t+1,j}^{(S)} + \sqrt{h_t^{(F)}}z_{t+1}^{(F)},
$$

where

$$
\phi_t := \theta_0 + \sum_{i=1}^{p}\theta_i\tilde{\kappa}_i\tilde{\mu}_i + \sum_{j=1}^{q}\theta_j^{(S)}\left(r_t\Delta - \frac{1}{2}h_{t,j}^{(S)}\right).
$$

To risk-neutralize the remaining risk driver $z^{(F)}$, we make three assumptions that follow from an extended Girsanov principle: (i) the fund grows at the risk-free rate under the risk-neutral measure $\mathbb{Q}$, (ii) the conditional variance of the basis risk component remains unchanged when moving from $\mathbb{P}$ to $\mathbb{Q}$, and (iii) the conditional distribution of the fund return is still Gaussian under $\mathbb{Q}$. These assumptions imply that the risk-neutral innovation process $\tilde{z}^{(F)} := \{\tilde{z}_t^{(F)}\}_{t=1}^T$ of the fund model is obtained by shifting the real-world innovation process $z^{(F)}$, that is, under $\mathbb{Q}$ we have that $\tilde{z}_{t+1}^{(F)} := z_{t+1}^{(F)} + \lambda_t^{(F)}$ is a standard Gaussian white noise, independent of $\tilde{z}^{(i)}$, $i = 1, \ldots, p$ and $\tilde{z}_j^{(S)}$, $j = 1, \ldots, q$, where

$$
\begin{aligned}
\lambda_t^{(F)} &:= \frac{1}{\sqrt{h_t^{(F)}}}\left[\phi_t + \frac{1}{2}\left(\sigma_t^{(F)}\right)^2\right], \qquad (2.8) \\
\left(\sigma_t^{(F)}\right)^2 &:= \sum_{i=1}^{p}\sum_{\ell=1}^{p}\theta_i\theta_\ell\sigma_i\sigma_\ell\Gamma_{i,\ell} + \sum_{j=1}^{q}\left(\theta_j^{(S)}\right)^2 h_{t,j}^{(S)} + 2\sum_{j=1}^{q-1}\sum_{k=j+1}^{q}\theta_j^{(S)}\theta_k^{(S)}\rho_{jk}\sqrt{h_{t,j}^{(S)}h_{t,k}^{(S)}} + h_t^{(F)}.
\end{aligned}
$$

Consequently, the $\mathbb{Q}$-dynamics of the log-returns on the fund can be expressed as,

$$R_{t+1}^{(F)} - r_t\Delta = -\frac{1}{2}\left(\sigma_t^{(F)}\right)^2 + \sigma_t^{(F)}\tilde{\epsilon}_{t+1}^{(F)},$$

where $\tilde{\epsilon}_{t+1}^{(F)}$, defined by

$$\tilde{\epsilon}_{t+1}^{(F)} := \frac{\sum_{i=1}^{p}\theta_i\sigma_i\tilde{z}_{t+1}^{(i)} + \sum_{j=1}^{q}\theta_j^{(S)}\sqrt{h_{t,j}^{(S)}}\,\tilde{z}_{t+1,j}^{(S)} + \sqrt{h_t^{(F)}}\,\tilde{z}_{t+1}^{(F)}}{\sigma_t^{(F)}},$$

is a standard Gaussian innovation under $\mathbb{Q}$ conditionally on $\mathcal{F}_t$. Finally, we note that the $\mathbb{Q}$-dynamics of the EGARCH fund volatility process $\{h_t^{(F)}\}_{t=0}^{T-1}$ are given by

$$\log h_t^{(F)} = \omega^{(F)} + \alpha^{(F)}(\tilde{z}_t^{(F)} - \lambda_{t-1}^{(F)}) + \gamma^{(F)}\left(|\tilde{z}_t^{(F)} - \lambda_{t-1}^{(F)}| - 2/\sqrt{2\pi}\right) + \beta^{(F)}\log h_{t-1}^{(F)}.$$

3 Model estimation and analysis

We created a freely available R script to estimate our mixed bond and equity fund model introduced in Section 2.[3] Parts of our code make use of functions available in the R packages `DEoptim` (Ardia et al., 2016), `FKF` (Luethi et al., 2018), `Rsolnp` (Ghalanos and Theussl, 2015) and `rugarch` (Ghalanos, 2018).

3.1 Fund data

We analyze our model on monthly return data from the following three Canadian funds.

1. The *iShares Core Canadian Universe Bond Index* ETF. This is a Canadian fixed income ETF comprised of approximately 70% government and 30% corporate securities.[4] We consider monthly return data on this fund from December 2000 to October 2018 for a total of 215 returns (source: Yahoo! Finance).

2. The *RBC Bond GIF Series 1* fund. This is a bond fund sold by RBC Insurance comprised

[3]The code is available on the website https://dms.umontreal.ca/~augusty/. The data sets used for estimation are also available on this website.

[4]According to https://www.blackrock.com/, the effective duration of the index on January 16, 2020 was 8.06 years.

of approximately 55% government and 45% corporate securities. We consider monthly return data on this fund from October 2006 to October 2018 for a total of 145 returns (source: https://lipper.rbcinsurance.com/rbc/).

3. The *Assumption/CI Harbour Growth & Income Fund Series A* fund. This is a mixed bond and equity fund sold by Assumption Life comprised of approximately 35% Canadian equity, 25% U.S. and international equity, and 40% fixed income (65% government and 35% corporate securities). We consider monthly return data on this fund from February 2002 to October 2018 for a total of 201 returns (source: https://assumption.lipperweb.com/assumplife/).

The reason why we consider two fixed income funds along with a mixed fund in this analysis is that the bond funds allow us to test the adequacy of the linear relationship between the fund return and term structure factor variations in a more isolated setting than if a significant equity component was present. Testing this relationship is important as the main novelty of our model is to improve the modeling of the fund's fixed income component. Although the iShares bond ETF is not sold as a VA policy, it acts as a proxy for the return on a passive investment strategy in the Canadian fixed income market. As such, it is well suited to assess the validity of our model specification, and the associated data set also has the advantage of being available for a longer period of time (since December 2000) than VA funds data.

3.2 Estimation procedure and results

Our proposed estimation procedure is based on the maximum likelihood method and is divided in two steps. In the first step, the interest rate model is calibrated to yield curve data. Since the factors $x_t^{(i)}$, $i = 1, \ldots, p$, in the multifactor Vasicek model are latent, we infer their values from the observed data with filtering techniques. Note that this also allows us to infer the risk-free rate as it is not directly observed (recall that $r_t = \sum_{i=1}^{p} x_t^{(i)}$). In the second step, we take the inferred factor values as given and estimate the equity index model and our mixed bond and equity fund model. This two-step approach simplifies the estimation process because on one hand, it allows us to take advantage of the Kalman filter to estimate our interest rate model, and on the other hand, a joint estimation of all model parameters is avoided. Separating the calibration of the interest rate model also leads to other practical advantages. For instance, it allows the use of a common interest rate model to estimate different VAs. In contrast, if a joint estimation was carried out, the parameters of the interest rate model would be influenced by the specific VA fund data used.

3.2.1 Estimation of the discrete-time multifactor Vasicek model

We estimate the discrete-time multifactor Vasicek model to Canadian end-of-month yield curve data from January 1986 to October 2018 (source: `https://www.bankofcanada.ca/rates/interest-rates/bond-yield-curves/`). Refer to Bolder et al. (2004) for the methodology used by the Bank of Canada to construct this yield curve data set. We consider $p = 3$ factors as empirical studies often suggest that the dynamics of the term structure of interest rates are well represented by three factors (see Diebold and Rudebusch, 2013, and references therein). Our calibration procedure takes advantage of the Kalman filter (see e.g., Shumway and Stoffer, 2017, Section 6) and assumes that we observe at each time t a set of M annualized continuously-compounded spot rates with times to maturity $n_1, \ldots, n_M$, denoted by

$$\hat{\boldsymbol{y}}(t) := (\hat{y}(t, t+n_1), \ldots, \hat{y}(t, t+n_M))^\top \in \mathbb{R}^M.$$

We include all integer times to maturities from 1 to 30 years available in the Canadian yield curve data set, as well as short-end maturities of 3, 6 and 9 months. Therefore, we observe a yield curve with $M = 33$ rates on each month from January 1986 to October 2018 (394 months). Refer to the online appendix for a description of the estimation procedure based on the Kalman filter.

Table 1 reports our estimation results, whereas Figure 1 illustrates the factor states and short rate inferred by the model for our data sample. We observe that the first factor trends downward in our data sample and mimics the direction of interest rates since the 1980s. This factor is much more persistent than the two others with a speed of mean reversion parameter $\kappa_1 = 0.00594$, which implies a half-life of almost 10 years.[5] Although a lengthy downward trend of interest

[5]From Eq. (2.2), each factor $x^{(i)}$ follows an independent first-order autoregressive process, and we have that $x_{t+1}^{(i)} - \mu_i = (1-\kappa_i)(x_t^{(i)} - \mu_i) + \sigma_i z_{t+1}^{(i)}$. Consequently, a value of $(1-\kappa_i)$ closer to one leads to a more persistent autoregressive process. The half-life is a quantity used to measure the speed of mean reversion of a process and

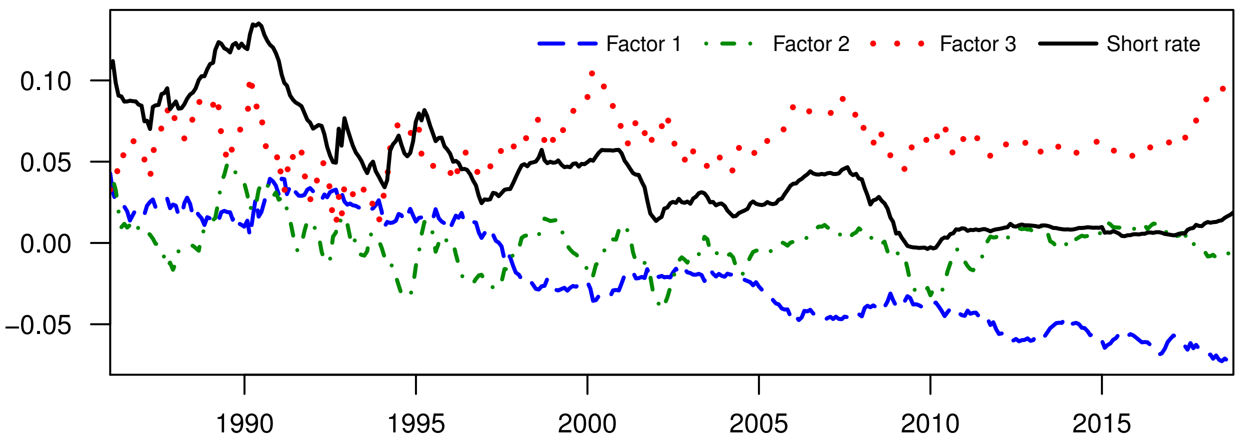
Table 1: Maximum likelihood estimates of the discrete-time multifactor Vasicek model ($p = 3$)

i	κ_i	μ_i	σ_i	λ_i
1	0.00594	0.01176	0.00524	0.918
2	0.04228	−0.00043	0.00482	−5.473
3	0.02049	0.04627	0.00788	1.134

$$\Gamma = \begin{bmatrix} 1 & 0.135 & -0.787 \\ 0.135 & 1 & -0.539 \\ -0.787 & -0.539 & 1 \end{bmatrix}$$

Notes: The discrete-time multifactor Vasicek model is presented in Section 2.1. Model estimation was performed using Canadian end-of-month yield curve data from January 1986 to October 2018.

Figure 1: Model-implied factors and short rate



Notes: Model-implied factors correspond to the smoothed state inferences $\mathbb{E}^{\mathbb{P}}\left[x_t^{(i)} \,\middle|\, \hat{\boldsymbol{y}}(1), \hat{\boldsymbol{y}}(2), \ldots, \hat{\boldsymbol{y}}(T)\right]$, for $i = 1, 2, 3$, whereas the inferred short rate is simply the sum of these estimates.

rates is unlikely to repeat itself in the near future due to rates currently being at very low levels, the high persistence of the first factor is deemed a favorable feature of our model. Indeed, it enables the model to generate interest rates which depart from their long-term averages for extended periods of times. This can lead to a wide variety of interest rate scenarios in Monte Carlo simulations, which is positive for risk management.

Regarding the overall fit of our model to the observed yield curve data, a metric that can be considered is the variance of the error term between observed and model-implied spot rates. This variance was estimated at 3.90×10^{-6}, which corresponds to a standard deviation of 0.2%.[6] This

corresponds to the expected number of periods that the factor needs to halve its distance from its long-term mean and is given by $\log(0.5)/\log|1-\kappa_i|$ here.

[6]The variance of the error term is symbolized by the parameter h in our estimation procedure described in the

low error rate suggests that in general model-implied yields are close to the observed ones. The online appendix presents additional figures comparing model-implied and observed sport rates, which show that our estimated model is flexible enough to match the evolution of observed yields in the data sample and to reproduce different observed term structure shapes reasonably well.

3.2.2 Estimation of the equity index model

We incorporate two country-specific equity indices in our mixed equity and bond fund model ($q = 2$). They are the Canadian S&P/TSX Composite and the U.S. S&P 500 price indices. We estimate the equity index model presented in Section 2.2.3 on monthly return data from February 1986 to October 2018 for a total of 393 returns by index (source: Federal Reserve Economic Database and Yahoo! Finance). We note that the risk-free rate r_t used as an input in this model is inferred from our interest rate model and approximated by its smoothed inferred value, $\sum_{i=1}^{p} \mathbb{E}^{\mathbb{P}} \left[x_t^{(i)} \,\middle|\, \boldsymbol{y}(1), \boldsymbol{y}(2), \ldots, \boldsymbol{y}(T) \right]$ (these expectations are computed by way of the Kalman smoother algorithm presented in the online appendix). Since the equity index model corresponds to a bivariate EGARCH model with a constant correlation parameter, joint maximum likelihood estimation of this model is relatively straightforward. Refer to our R code for more details.

Table 2: Maximum likelihood estimates of the equity index model ($q = 2$)

Stock index	j	$\lambda_j^{(S)}$	$\omega_j^{(S)}$	$\alpha_j^{(S)}$	$\gamma_j^{(S)}$	$\beta_j^{(S)}$	ρ_{12}
S&P/TSX	1	0.08477	−1.0132	−0.01083	0.29438	0.84031	0.76384
		(0.03972)	(0.5561)	(0.06172)	(0.07536)	(0.08539)	(0.02079)
S&P500	2	0.12810	−1.5390	−0.16422	0.28580	0.75939	
		(0.04327)	(0.5620)	(0.07693)	(0.07389)	(0.08689)	

Notes: The equity index model is presented in Section 2.2.3 and corresponds to a bivariate EGARCH model. Model estimation was performed using monthly return data from February 1986 to October 2018 for the S&P/TSX Composite and S&P 500 price indices (393 returns by index). Standard errors of the estimators are displayed in parentheses below the estimates.

Table 2 reports our estimation results. As expected, the two equity indices are highly correlated with ρ_{12} estimated at 0.76384. Moreover, since the values of parameters $\beta_j^{(S)}$ are close to one,

online appendix. The relationship between observed and model-implied rates is given in Eq. (B.2) of this appendix.

significant volatility clustering effects are present. Negative values of $\alpha_j^{(S)}$ imply the existence of a leverage effect, notably for the S&P 500 index, where negative returns generate larger shocks on volatility than positive ones. Finally, the positive values of $\lambda_j^{(S)}$ indicate the presence of an equity risk premium.

3.2.3 Estimation of the fund model

The estimation of our fund model introduced in Section 2.2.1 takes as inputs the smoothed factor states $\mathbb{E}^{\mathbb{P}}\left[x_t^{(i)} \,\middle|\, \boldsymbol{y}(1), \boldsymbol{y}(2), \ldots, \boldsymbol{y}(T)\right]$, $i = 1, \ldots, p$, inferred from our interest rate model. These are used to approximate the latent factors $x_t^{(i)}$ and the risk-free rate r_t in Eq. (2.4). Consequently, with our two-step estimation procedure, our fund model can be interpreted as a linear regression model with heteroskedastic EGARCH errors. As this type of specification is covered in the `rugarch` package, our code for maximum likelihood estimation takes advantage of the functions in this package. We note that the maximization of the log-likelihood is carried out in a single step with respect to both conditional mean and variance parameters. Refer to our R code for more details.

Table 3 reports our estimation results, and the online appendix provides fit diagnostics outputs for all three estimated models. These diagnostics indicate that our fund model fits the data very well. For instance, Ljung-Box tests on residuals and squared residuals do not detect the presence of significant residual autocorrelations or heteroskedasticity. ARCH Lagrange multiplier tests also do not find evidence of residual heteroskedasticity. Moreover, an adjusted Pearson goodness-of-fit test suggests that the assumed standard Gaussian distribution for the errors is not inappropriate. Finally, the interest rate factor variation coefficients θ_1, θ_2 and θ_3 are highly significant in all three estimated fund models, which indicates that the fund model return is affected by all three factors. Note that these coefficients are all highly negative for the iShares bond ETF and the RBC bond fund. This was to be expected as interest rate factor variations are fundamental risk drivers of bond fund returns. Moreover, the negative values of the θ_j parameters reflect the inverse relationship between interest rate variations and bond prices. These results thus offer support for the suitability of our model specification to represent fund returns with a fixed income

Table 3: Maximum likelihood estimates of the fund model

	iShares bond ETF	RBC bond fund	Assumption mixed fund
θ_0	0.00340	0.00258	0.00020
	(0.00024)	(0.00020)	(0.00002)
θ_1	-6.62026	-6.80690	0.39095
	(0.25358)	(0.30873)	(0.02236)
θ_2	-1.31380	-1.50116	-0.32789
	(0.19534)	(0.25166)	(0.00641)
θ_3	-3.82370	-3.86662	-0.99664
	(0.18664)	(0.18263)	(0.01305)
$\theta_1^{(S)}$	0.02801	0.06202	0.49773
	(0.01184)	(0.01889)	(0.00051)
$\theta_2^{(S)}$	0.03378	0.04657	0.05913
	(0.03289)	(0.00758)	(0.00007)
$\omega^{(F)}$	-0.40446	-0.29261	-0.51325
	(0.15432)	(0.02199)	(0.00019)
$\alpha^{(F)}$	-0.09064	-0.25294	-0.06924
	(0.08216)	(0.06051)	(0.00170)
$\gamma^{(F)}$	0.20352	0.19228	-0.20790
	(0.06175)	(0.05292)	(0.00023)
$\beta^{(F)}$	0.96457	0.97454	0.94287
	(0.01287)	(0.00005)	(0.00329)

Notes: The fund model is presented in Section 2.2.1. *iShares Core Canadian Universe Bond Index* ETF: Model estimation was performed using monthly return data from December 2000 to October 2018 (215 returns); *RBC Bond GIF Series 1* fund: Model estimation was performed using monthly total return data from October 2006 to October 2018 (145 returns); *Assumption/CI Harbour Growth & Income Fund Series A* fund: Model estimation was performed using monthly total return data from February 2002 to October 2018 (201 returns). Standard errors of the estimators are displayed in parentheses below the estimates.

component.

4 Application to the valuation of a variable annuity

We consider the valuation of Guaranteed Minimum Maturity Benefit (GMMB)[7] policies issued on two of the funds studied in Section 3, namely the *RBC Bond GIF Series 1* fund and the

[7] A GMMB is a guarantee offered in a VA policy that protects the policyholder's investment in mutual funds against a market downturn. The insurer agrees to pay the policyholder the shortfall, if any, between the guaranteed benefit of the policy and the account value at a pre-determined maturity date, provided that the policyholder is alive at that date. This capital protection is financed via periodic fees collected from the policyholder's account.

Assumption/CI Harbour Growth & Income Fund Series A fund. Indeed, these two funds are actual underlying assets of VA policies issued in Canada, and considering them allows us to illustrate the valuation of long-term guarantees issued on a pure bond fund (RBC) as well as on a mixed fund (Assumption). The *iShares Core Canadian Universe Bond Index* ETF is not considered in this application as it is not a underlying asset in a VA.

4.1 Description of the VA policy cash flows

The mathematical representation of the VA policy cash flows presented in this section is inspired from Trottier et al. (2018a). The policyholder is assumed to invest an amount $A_0 > 0$ at time $t = 0$ in an account that tracks the total return on a fund with value process $\{F_t\}_{t=0}^{T}$. No further contributions are made throughout the life of the product. The policy account offers a T-month maturity benefit guarantee of K_T, which evolves according to the process $\{K_t\}_{t=0}^{T}$. The guaranteed amount can either be a fixed value, or it can vary stochastically if some reset, ratchet or roll-up provisions is included. A constant monthly fee rate ω_{tot} is charged to the policyholder at the end of the period provided that he is active at the beginning of the period to cover guarantee costs, investment management fees, expenses and commissions. This fee is commonly known as the management expense ratio (MER) of the VA fund. The policyholder account value at time t, denoted by A_t, evolves according to

$$A_t = A_{t-1}(1 - \omega_{tot})\frac{F_t}{F_{t-1}}, \qquad t = 1, \ldots, T. \tag{4.1}$$

The policy that is active at time $t = 0$ can become inactive for three possible reasons: (i) the policyholder deceases, (ii) the policyholder lapses his policy, or (iii) the maturity date of the guarantee is attained. We assume that idiosyncratic mortality and policyholder behavior risks are fully diversified so that only the systematic components of these risks are taken into account. Denote by ${}_ta_x$ the proportion of policies still active at time t from a homogeneous pool of policyholders aged x months at contract inception. The inactivity decrements satisfy

$${}_0a_x = 1, \quad {}_ta_x = {}_{t-1}a_x \left(\frac{{}_tp_x}{{}_{t-1}p_x}\right)(1 - \mathcal{L}(m_{t-1})), \quad t = 1, \ldots, T,$$

where ${}_tp_x$ is the survival probability to time t for a policyholder aged x months at time 0, $m_{t-1} := A_{t-1}/K_{t-1}$ is an indicator of the moneyness of the guarantee at time $t-1$, and $\mathcal{L} : (0,\infty) \to [0,1]$ is a function that indicates the proportion of policyholders active at time $t-1$ that will survive to time t and lapse their policies ($\mathcal{L}$ will be explicitly defined in Section 4.3.2). We remark that the dependence of the function $\mathcal{L}$ on m_{t-1} allows us to incorporate dynamic lapsation.

The total fee charge collected by the insurer at time t is given by

$$\text{Total Fee}_t := {}_{t-1}a_x \, \omega_{tot} A_{t-1} \frac{F_t}{F_{t-1}} = {}_{t-1}a_x \frac{\omega_{tot}}{1-\omega_{tot}} A_t, \quad t = 1, \ldots, T.$$

Insurers typically impose penalties for lapses in early years to compensate for underwriting fees incurred at the onset which cannot be fully recovered from regular fees in the case of an early surrender. The lapse penalties collected by the insurer are assumed to take the form

$$\text{Lapse penalty}_t := \underbrace{\left({}_{t-1}a_x \frac{{}_tp_x}{{}_{t-1}p_x} \mathcal{L}(m_{t-1}) \right)}_{\text{conditional lapse proportion}} A_t \mathcal{P}(t), \quad t = 1, \ldots, T,$$

where $\mathcal{P}$ is a deterministic function of time that specifies the proportion of the account value that is retained by the insurer in the event of surrender ($\mathcal{P}$ will be defined explicitly in Section 4.3.2). Note that when a policyholder lapses his policy, he receives the difference between the current account value and lapse penalties charged by the insurer.

Consequently, the insurer's total cash inflow at time t, denoted by CF_t, is given by

$$\text{CF}_t := \text{Total Fee}_t + \text{Lapse penalty}_t, \quad t = 1, \ldots, T.$$

Finally, the GMMB policy ensures that each active policyholder at maturity time T has a minimal amount of K_T in his account. This entails that the insurer might have to pay a maturity benefit corresponding to

$$\text{Benefit}_T := {}_Ta_x \max(0, K_T - A_T),$$

if the policy guarantee is in-the-money at time T.

4.2 Valuation of the VA policy

The valuation of the VA policy is performed through a risk-neutral framework (Bauer et al., 2008). This approach entails that the guarantee cost at time t, denoted by $\Pi_t^{(guar)}$, is given by

$$\Pi_t^{(guar)} := B_t \mathbb{E}^{\mathbb{Q}} \left[\frac{\text{Benefit}_T}{B_T} \right], \quad t = 0, 1, \ldots, T, \tag{4.2}$$

where $B_0 := 1$ and $B_t := \exp\left(\Delta \sum_{j=0}^{t-1} r_t\right)$, $t = 1, \ldots, T$. Moreover, the value of the insurer's future cash inflows at time t, denoted by $\Pi_t^{(in)}$, corresponds to

$$\Pi_t^{(in)} := B_t \mathbb{E}^{\mathbb{Q}} \left[\sum_{j=t+1}^{T} \frac{\text{CF}_j}{B_j} \,\middle|\, \mathcal{F}_t \right], \quad t = 0, 1, \ldots, T. \tag{4.3}$$

The net value of the VA policy at time t, denoted by Π_t, is then defined as $\Pi_t := \Pi_t^{(in)} - \Pi_t^{(guar)}$. The excess over the guarantee cost is generally allocated to pay investment management fees, expenses and commissions, and for profits.

4.3 Assumptions used to value the VA policy

We assume that the GMMB policy is issued to a homogeneous pool of policyholders aged 55. The initial investment in the policy account is set to $A_0 = 100$. Economic scenarios are generated with the model presented in Section 2 and estimated in Section 3. To bring our analysis closer in line with reality, we suppose that the annualized MER, defined as $\omega_{tot}^{(ann)} := 1 - (1 - \omega_{tot})^{12}$, is given by the actual MER charged by RBC and Assumption Life for their baseline guarantees as described in the fund prospectuses, that is, $\omega_{tot}^{(ann)} = 0.0206$ ($\Rightarrow \omega_{tot} = 0.001733$) for the RBC fund and $\omega_{tot}^{(ann)} = 0.0286$ ($\Rightarrow \omega_{tot} = 0.002415$) for the Assumption fund. Assumptions related to mortality, lapsing behavior and associated penalties are chosen to be representative for a Canadian life insurance company.

4.3.1 Mortality assumptions

The mortality rates considered in our application are based on the Canadian Pensioners' Mortality 2014 (CPM2014) male mortality table (CIA, 2014). Projected mortality improvement is applied on entries of this table in conformity with recommendations of the Canadian Institute of Actuaries (see CIA, 2010, Appendix C) so that time $t = 0$ in our study corresponds to the beginning of 2019. Data for both the baseline mortality table and mortality improvement rates can be obtained through a hyperlink on p.8 of CIA (2014). A constant force of mortality assumption between integral ages is considered to obtain monthly rates from the yearly mortality rates.

4.3.2 Lapse assumptions

According to a 2011 survey from the Society of Actuaries (Society of Actuaries, 2011), a majority of life insurers use dynamic lapse assumptions. The impact of the moneyness of the guarantee is intuitive, since policyholders are more likely to surrender their policies if their guarantee is out-of-the-money (i.e., less likely to provide a payment from the insurer) than if it is in-the-money. This is confirmed empirically by different studies (e.g., Knoller et al., 2016; Sun and Mo, 2011). It is common in the literature to incorporate dynamic lapsation by way of a piecewise linear function of the moneyness of the guarantee, see Feng et al. (2017), Ledlie et al. (2008) and Ngai and Sherris (2011), among others.

We model the annualized lapse proportion at time $t + 1$, denoted by $\mathcal{L}^{(ann)}(m_t)$, by the function

$$\mathcal{L}^{(ann)}(m_t) = \begin{cases} \gamma_1, & \text{if } m_t < \delta_1, \\ \gamma_2, & \text{if } m_t > \delta_2, \\ \gamma_1 + (\gamma_2 - \gamma_1)\frac{m_t - \delta_1}{\delta_2 - \delta_1}, & \text{if } \delta_1 \leq m_t \leq \delta_2, \end{cases}$$

with parameters $\gamma_1 = 0.02$, $\gamma_2 = 0.10$, $\delta_1 = 0.4434$ and $\delta_2 = 1.7420$, which are inspired from the assumptions made on p.149 of AMF (2018). This function entails that the annual lapse rate is bounded below and above by γ_1 and γ_2, respectively. When the moneyness m_t is between δ_1 and δ_2, the lapse rate increases linearly between these two values. The monthly lapse proportion is obtained from the relation $\mathcal{L}(m_t) := 1 - \left(1 - \mathcal{L}^{(ann)}(m_t)\right)^{1/12}$.

Finally, surrender penalties are assumed to decrease by 1% every year from 7% in the first year to 0% after the seventh year, that is,

$$\mathcal{P}(t) = \max\left(0, 0.07 - 0.01\lfloor (t-1)/12 \rfloor\right), \quad t = 1, \ldots, T,$$

where $\lfloor \cdot \rfloor$ denotes the integer part function.

4.3.3 Ratchet provision

Some of the GMMB policies considered in our work are assumed to include a ratchet provision[8] based on the mechanism outlined on p.149 of AMF (2018), which is deemed representative of a realistic case encountered in practice. The initial guaranteed amount is set to $K_0 = \xi A_0$, where ξ denotes the guaranteed account ratio (in our case, we assume $\xi = 1$). Under the ratchet provision, the guaranteed amount is increased to a percentage ξ of the account value if the ratio of the latter over the former becomes greater than or equal to 115%, with a maximum of $\bar{\zeta}$ adjustments per year, and no adjustments being performed during the last 10 years before the maturity of the policy. In other words, the ratchet specification entails

$$K_{t+1} = \begin{cases} \xi A_{t+1}, & \text{if } t < T - 120, \ \dfrac{\xi A_{t+1}}{K_t} \geq 1.15 \text{ and } \zeta_{t+1} < \bar{\zeta}, \\ K_t, & \text{otherwise}, \end{cases}$$

where ζ_{t+1} is the number of times the guaranteed amount was increased prior to time $t+1$ and during the current year. A maximum of $\bar{\zeta} = 1$ ratchet adjustment per year is assumed in our application.[9]

4.3.4 Other assumptions

Similarly to CIA (2017), we apply a floor of -0.75% on the risk-free interest rates simulated with our three-factor Vasicek model to avoid highly negative values which could be obtained due to the

[8] In a VA contract, a ratchet provision allows for an increase in the guaranteed benefit amount of the policy according to a pre-determined rule. Depending on the provision, the increase can occur automatically when some specific market conditions are met, or at a time when the policyholder elects the ratchet.

[9] We note that in some setups the resetting of the guaranteed amount also resets the guarantee maturity, see Armstrong (2001). This feature is not considered here.

normality of the term structure factors (i.e., Eq. (2.1) is modified to $r_t = \max(-0.0075, \sum_{i=1}^{p} x_t^{(i)})$). CIA (2017) mentions that the floor assumption of -0.75% is based "on the lowest observed point in German historical 1-year data." Moreover, Alberts (2020) states that -0.75% was the lowest central bank negative interest rate observed, which occurred in Switzerland. Although negative interest rate policies have not been implemented in Canada and the US, an article from the Bank of Canada (Witmer and Yang, 2016) recently estimated the effective lower bound for interest rates in Canada to be between -0.25% and -0.75%. Given that the purpose of our model is to make long-term projections and that guarantee costs are inversely related to interest rates, we consider that an interest rate floor of -0.75% is a justifiable and conservative assumption in our context.

In addition, a floor of 2.0833×10^{-6} is imposed on the fund conditional variance $h_t^{(F)}$ to ensure that the annualized volatility of the fund is at least 0.5% at any given time (0.5% was the minimal EGARCH fund volatility observed in our data sample).

4.4 Valuation results

We value two different GMMB policies on the RBC and Assumption funds: a 20-year 100% capital guarantee (i.e., $T = 240$ and $\forall t : K_t = 100$), and a 20-year 100% capital guarantee with a ratchet that is applied once a year for the first 10 years whenever the account value exceeds 115% of the guaranteed amount (see Section 4.3.3). Such guarantees are representative of maturity benefits offered by life insurers.

The risk-neutral expectations (4.2) and (4.3) are computed by way of Monte Carlo simulations. We simulated three different sets of 1,000,000 economic scenarios using the market and fund models estimated in Section 3. To assess the impact of initial market conditions on the VA valuation, each scenario set is based on different starting values of term structure factors $x_0^{(i)}$, $i = 1, 2, 3$, and volatilities $\sqrt{h_0^{(F)}}$ and $\sqrt{h_{0,j}^{(S)}}$, $j = 1, 2$. Initial risk factor values are inspired from those inferred by our model on specific dates in the data sample used for estimation, and are provided in Table 4.

Scenario set I is our baseline scenario set and assumes that the initial economic environment

Table 4: Initial risk factor values assumed in each simulated scenario set

	Interest rate factors			S&P/TSX	S&P 500	RBC	Assumption
Scenario set	$x_0^{(1)}$	$x_0^{(2)}$	$x_0^{(3)}$	$\sqrt{12h_{0,1}^{(S)}}$	$\sqrt{12h_{0,2}^{(S)}}$	$\sqrt{12h_0^{(F)}}$	$\sqrt{12h_0^{(F)}}$
I	−6.90%	−0.62%	9.40%	14.12%	17.85%	0.90%	4.16%
II	−6.90%	−0.62%	9.40%	22.68%	22.68%	5.40%	5.68%
III	−4.62%	0.96%	7.95%	14.12%	17.85%	0.90%	4.16%

Notes: Initial risk factor values are given on an annualized basis. Starting values in scenario set I are based on those inferred by our model on October 31, 2018. Scenario set II considers the same interest rate factors, but sets volatilities to levels observed on December 31, 2008. Scenario set III combines volatility values on October 31, 2018 with interest rate factors inferred on December 29, 2006.

is given by the one inferred by our model on October 31, 2018, which is the last date of our sample. Scenario set II is based on the same initial interest rate factor values as scenario set I, but volatilities are now set to financial crisis levels, as observed on December 31, 2008. This hypothetical scenario is considered to isolate the impact of a sudden surge in equity and bond asset volatilities. Scenario set III uses the same starting volatility values as scenario set I, but the initial term structure factors are mapped to their inferred values on December 29, 2006. At the end of 2006, the yield curve was approximately flat at a 4% level so this scenario set is representative of an economic environment in which interest rates are at an equilibrium level.

Valuation results are presented in Table 5. First, we observe that at the MER levels charged by RBC and Assumption Life for their baseline guarantees, the cost of the GMMB represents a relatively small portion of the insurer's cash inflows (never more than 25%). Cash inflows are comprised of fee income and lapse penalties, approximately in the proportions 95% and 5%, respectively. Therefore, the majority of the insurer's revenue coming from the MER is used to pay for investment management fees, expenses and commissions, and for profits. If we relate a VA product to an investment in an exchange traded fund which offers exposure to market risk at very low fees, it could be argued that policyholders pay a substantial premium for the guarantees embedded in VA contracts.

Moreover, our results confirm a general expectation that a long-term return-of-capital guarantee

Table 5: Valuation of GMMB policies according to different initial market conditions

	RBC bond fund				Assumption mixed fund			
	Π_0	$\Pi_0^{(in)}$	$\Pi_0^{(guar)}$	$\frac{\Pi_0^{(guar)}}{\Pi_0^{(in)}}$	Π_0	$\Pi_0^{(in)}$	$\Pi_0^{(guar)}$	$\frac{\Pi_0^{(guar)}}{\Pi_0^{(in)}}$
20-year GMMB, no ratchets								
Scenario set I	20.73	21.35	0.62	2.9%	22.53	27.86	5.33	19.1%
Scenario set II	20.71	21.35	0.64	3.0%	22.50	27.85	5.35	19.2%
Scenario set III	20.50	20.56	0.06	0.3%	24.46	26.97	2.51	9.3%
20-year GMMB, with ratchets during the first 10 years								
Scenario set I	19.77	21.95	2.18	9.9%	22.13	29.00	6.87	23.7%
Scenario set II	19.77	21.96	2.19	10.0%	22.07	29.02	6.95	23.9%
Scenario set III	20.65	21.61	0.96	4.4%	24.82	28.53	3.71	13.0%

Notes: $\Pi_0^{(in)}$ represents the value at time 0 of all of the insurer's cash inflows to be received during the life of the policy, whereas $\Pi_0^{(guar)}$ corresponds to the value of the maturity benefit at time 0 (see Section 4.2 for more details). The value of cash inflows in excess of the guarantee cost is denoted by Π_0 (i.e., $\Pi_0 := \Pi_0^{(in)} - \Pi_0^{(guar)}$). These values are computed for each scenario set based on 1,000,000 economic scenarios simulated with the model estimated in Section 3. Initial risk factor values used for each simulated scenario set are provided in Table 4.

on a bond fund is close to being worthless. Indeed, the value of the 20-year GMMB without ratchets on the RBC bond fund is at most 0.64 for all scenarios sets. However, the inclusion of ratchets significantly raises the guarantee cost for the RBC bond fund as it is now more probable that the return of the fund net of fees will be insufficient to match the resetted guarantee level. For example, for scenario set I the ratchet mechanism increases the guarantee cost by a factor of 3.5 for the RBC bond fund (from 0.62 to 2.18). Although the impact of ratchets is much less significant in relative terms for the Assumption mixed fund, this is mainly due to the very low guarantee costs associated to the RBC bond fund in the no ratchet case. In absolute terms, the increase in the value of the maturity benefit due to ratchets is similar for the two funds. Nevertheless, as could be expected, the market protection (with or without ratchets) is more costly on the mixed fund than on the bond fund, due of course to the mixed fund's exposure to equity risk. So even if ratchets lead to a larger relative increase of the guarantee cost in the case of the bond fund, the ratchet guarantee for the mixed fund is still over three times more valuable and is associated to a greater share of the insurer's cash inflows.

Interestingly, we observe that the incremental cost of the guarantee due to ratchets is partly offset by additional fee income received by the insurer. The increase in revenue is due to the reset mechanism lowering the propensity to lapse the policy. Consequently, offering ratchets during an initial temporary period does not necessarily put an excessive strain on the insurer's net financial position, and could actually constitute an attractive product design for policyholders and insurers alike (see MacKay et al., 2017; Bernard and Moenig, 2019, who discuss why a reduced lapse incentive can be beneficial for policyholders and guarantee providers).

Comparing results across scenario sets I and II, we observe that the surge in equity and bond asset volatilities implied by scenario set II has a marginal impact. This therefore suggests that valuation is little affected by starting volatility conditions. In contrast, the initial interest rate environment has a substantial impact on guarantee costs. Indeed, we note a significant drop in the value of the guarantee from scenario set I to III. This decrease is a consequence of the higher level of interest rates implied by scenario set III. ~~On one hand~~ Indeed, a higher short rate in our model specification entails that equity and bond asset returns are subject to a larger drift, which reduces the likelihood of the guarantee ending in-the-money. ~~On the other hand,~~ This effect is further magnified by the fact that higher rates lead to a bigger discounting factor, thus lowering the expected present value of the maturity benefit at time 0.

4.5 Sensitivity to term structure and equity risk factors

To further understand how each risk driver impacts the valuation of a VA policy, we compute sensitivities of the insurer's revenue and costs with respect to variations in the term structure and equity risk factors. This experiment involves shocking each risk factor in isolation to assess its marginal impact on valuation. Such sensitivities are commonly known as *Greeks*.

Let $\Pi_0(z)$ denote the net value of the VA policy at time 0 when a given risk factor is equal to z, and let Δ_z be the size of the shock on this risk factor. The general formula that we use to define

Table 6: Greeks for the 20-year GMMB policy with ratchets during the first 10 years

	$x_0^{(1)}$	$x_0^{(2)}$	$x_0^{(3)}$	$\sqrt{h_0^{(F)}}$	$\sqrt{h_{0,1}^{(S)}}$	$\sqrt{h_{0,2}^{(S)}}$
RBC bond fund						
Greek for $\Pi_0^{(in)}$	−0.068	−0.012	−0.067	0.023	0.000	0.000
Greek for $\Pi_0^{(guar)}$	−0.259	−0.005	−0.123	0.034	0.000	0.000
Greek for Π_0	0.191	−0.007	0.056	−0.011	0.000	0.000
Assumption mixed fund						
Greek for $\Pi_0^{(in)}$	−0.089	−0.016	−0.091	0.022	0.012	0.001
Greek for $\Pi_0^{(guar)}$	−0.637	−0.047	−0.428	0.067	0.034	0.004
Greek for Π_0	0.548	0.031	0.337	−0.045	−0.022	−0.003

Notes: Initial unshocked risk factor values correspond to those considered in our baseline scenario set I (see Table 4). The first-order difference in our Greek formula (4.4) is computed by way of Monte Carlo simulations with 1,000,000 economic scenarios generated for the baseline and shocked risk factor values using a common random seed.

a Greek on $\Pi_0(z)$ is based on the forward first-order difference equation,[10]

$$\text{Greek for } \Pi_0(z) = \frac{\Pi_0(z+\Delta_z) - \Pi_0(z)}{\Delta_z} \times \sigma_z, \tag{4.4}$$

where σ_z is a scaling term associated to the underlying risk factor. Greeks for the value of the insurer's cash inflows at time 0, $\Pi_0^{(in)}$, and for the guarantee cost, $\Pi_0^{(guar)}$, are defined analogously. We set σ_z equal to the standard deviation of the risk factor's monthly variations inferred in our data estimation sample.[11] This allows us to compare Greeks for the different risk factors on a common basis, as each Greek then reflects the impact of one historical monthly standard deviation shock on the risk factor.

Table 6 reports Greeks for the 20-year GMMB policy with ratchets studied in Section 4.4, assuming

[10]The shock magnitude is set to $\Delta_z = 0.001$ for the term structure factors $x_0^{(i)}$, $i = 1, 2, 3$ and for the fund volatility $\sqrt{h_0^{(F)}}$, and to $\Delta_z = 0.005$ for the equity index volatilities $\sqrt{h_{0,i}^{(S)}}$, $i = 1, 2$.

[11]The scaling term σ_z is set to respectively 0.00302, 0.00389 and 0.00541 for yield curve factors $x_0^{(i)}$, $i = 1, 2, 3$, to respectively 0.00555 and 0.00947 for equity index volatilities $\sqrt{h_{0,i}^{(S)}}$, $i = 1, 2$, and to respectively 0.00092 and 0.00083 for RBC and Assumption fund volatilities $\sqrt{h_0^{(F)}}$.

that initial unshocked risk factor values correspond to those in our baseline scenario set I (i.e., initial term structure and volatility factors are set to those inferred on October 31, 2018). This analysis confirms that for both funds investigated, risk factors having the highest impact are those related to the term structure. In fact, we observe that initial fund and equity volatilities have little to no effect on valuation. Intuitively, this is due to the fact that the initial volatility level does not significantly influence the total volatility that will be realized over the 20-year guarantee period. A stronger impact would have surely been felt for a short-term guarantee.

Moreover, we observe that the Greeks for $\Pi_0^{(in)}$ and $\Pi_0^{(guar)}$ pertaining to the term structure factors are all negative; this is expected as an increase in an interest rate factor implies a higher short rate and leads to discounting cash flows at a higher rate, thus decreasing the expected present value of these cash flows. Additionally, the impact of the first factor is much stronger than the other two. Recall that in Section 3, we noted that this factor is very persistent and can be interpreted as governing the yield curve level. The factor's high persistence is the reason why its starting value can have a long-lasting impact over the 20-year guarantee period.

The takeaway from this investigation is that the prevailing interest rate environment is an important input to consider when valuating long-term guarantees. In contrast, it is long-term, rather than current, volatility assumptions that matter in this respect. This can be justified by the fact that interest rate conditions tend to be more persistent than stock volatility over long periods of time.

5 Conclusion

This paper develops a time series model to represent the return dynamics of mutual funds invested in both equity and fixed income asset classes. The model is made up of three building blocks: a discrete-time multifactor Vasicek model for the term structure of interest rates, a multivariate EGARCH model for equity index returns, and finally the fund return model which includes loadings on risk factors associated with the first two components and a heteroskedastic error term. Our model framework is particularly relevant in the context of VAs. Indeed, the literature has so far concentrated on studying investment guarantees on single-asset equity funds, whereas in practice such protections are typically offered on mixed funds. The long-term nature of the financial guarantees offered makes it essential to model the fund's fixed income component in a way that is consistent with movements of the yield curve. Moreover, an important feature of our model is the natural integration of basis risk which, although a critical issue for actuaries in the variable annuity business, has often been neglected in past studies.

Maximum likelihood estimation of the proposed model was performed on historical data for a sample of bond and mixed funds. Goodness-of-fit tests all indicated that the fit of our model was adequate. The valuation of VA policies on those funds was then illustrated to assess the importance of various risk drivers in our model. Realistic characteristics related to the periodic fee, mortality rates, dynamic lapses, surrender penalties, and ratchet provisions were included into our analysis. Our results confirmed that a long-term return-of-capital guarantee on a pure bond fund is close to being worthless. Moreover, we observed that the initial interest rate environment can have a material impact on the value of cash flows to be received by the insurer and on guarantee costs. This finding is important as it implies that yield curve modeling must be given special care in the context of variable annuities.

Finally, we note that although we adopted an EGARCH specification to model volatility persistence of equity and fund returns, other approaches such as regime-switching processes could be entertained. It would also be interesting to evaluate how valuation of VAs is impacted by the choice or calibration of the interest rate model. These questions are left for future research.

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