# Supplementary online appendix

## A Proofs

### A.1 Proof of Proposition 2.1

Before proving Proposition 2.1, we show the following lemma.

**Lemma A.1.** *For $i = 1, \ldots, p$ and $n = 1, \ldots, T - t$, we have that,*

$$x_{t+n}^{(i)} = x_t^{(i)}(1-\tilde{\kappa}_i)^n + \tilde{\mu}_i\tilde{\kappa}_i \sum_{j=1}^{n}(1-\tilde{\kappa}_i)^{n-j} + \tilde{\sigma}_i \sum_{j=1}^{n}(1-\tilde{\kappa}_i)^{n-j}\tilde{z}_{t+j}^{(i)}.$$

*Proof.* The result is trivially true for $n = 1$. Assuming that it holds for given $n$, the proof is completed by induction as follows:

$$\begin{aligned}
x_{t+n+1}^{(i)} &= x_{t+n}^{(i)} + \tilde{\kappa}_i\big(\tilde{\mu}_i - x_{t+n}^{(i)}\big) + \tilde{\sigma}_i\tilde{z}_{t+n+1}^{(i)} \\
&= x_{t+n}^{(i)}(1-\tilde{\kappa}_i) + \tilde{\kappa}_i\tilde{\mu}_i + \tilde{\sigma}_i\tilde{z}_{t+n+1}^{(i)} \\
&= \left[x_t^{(i)}(1-\tilde{\kappa}_i)^n + \tilde{\kappa}_i\tilde{\mu}_i \sum_{j=1}^{n}(1-\tilde{\kappa}_i)^{n-j} + \tilde{\sigma}_i \sum_{j=1}^{n}(1-\tilde{\kappa}_i)^{n-j}\tilde{z}_{t+j}^{(i)}\right](1-\tilde{\kappa}_i) \\
&\quad + \tilde{\kappa}_i\tilde{\mu}_i + \tilde{\sigma}_i\tilde{z}_{t+n+1}^{(i)} \\
&= x_t^{(i)}(1-\tilde{\kappa}_i)^{n+1} + \tilde{\kappa}_i\tilde{\mu}_i \sum_{j=1}^{n}(1-\tilde{\kappa}_i)^{n+1-j} \\
&\quad + \tilde{\sigma}_i \sum_{j=1}^{n}(1-\tilde{\kappa}_i)^{n+1-j}\tilde{z}_{t+j}^{(i)} + \tilde{\kappa}_i\tilde{\mu}_i + \tilde{\sigma}_i\tilde{z}_{t+n+1}^{(i)} \\
&= x_t^{(i)}(1-\tilde{\kappa}_i)^{n+1} + \tilde{\kappa}_i\tilde{\mu}_i \sum_{j=1}^{n+1}(1-\tilde{\kappa}_i)^{n+1-j} + \tilde{\sigma}_i \sum_{j=1}^{n+1}(1-\tilde{\kappa}_i)^{n+1-j}\tilde{z}_{t+j}^{(i)}.
\end{aligned}$$

□

We now prove Proposition 2.1. By Lemma A.1, we have that:

$$\sum_{\ell=0}^{n-1} x_{t+\ell}^{(i)} = \sum_{\ell=0}^{n-1}\left[x_t^{(i)}(1-\tilde{\kappa}_i)^{\ell} + \tilde{\kappa}_i\tilde{\mu}_i \sum_{j=1}^{\ell}(1-\tilde{\kappa}_i)^{\ell-j} + \sigma_i \sum_{j=1}^{\ell}(1-\tilde{\kappa}_i)^{\ell-j}\tilde{z}_{t+j}^{(i)}\right]$$

$$
\begin{aligned}
&= x_t^{(i)} \sum_{\ell=0}^{n-1} (1-\tilde{\kappa}_i)^{\ell} + \tilde{\kappa}_i \tilde{\mu}_i \sum_{\ell=1}^{n-1} \sum_{j=1}^{\ell} (1-\tilde{\kappa}_i)^{\ell-j} + \sigma_i \sum_{\ell=1}^{n-1} \sum_{j=1}^{\ell} (1-\tilde{\kappa}_i)^{\ell-j} \tilde{z}_{t+j}^{(i)} \\
&= x_t^{(i)} \frac{1-(1-\tilde{\kappa}_i)^n}{\tilde{\kappa}_i} + \tilde{\kappa}_i \tilde{\mu}_i \sum_{\ell=1}^{n-1} \sum_{j=0}^{\ell-1} (1-\tilde{\kappa}_i)^{j} + \sigma_i \sum_{\ell=1}^{n-1} \sum_{j=1}^{n-1} \mathbb{1}_{\{j \le \ell\}} (1-\tilde{\kappa}_i)^{\ell-j} \tilde{z}_{t+j}^{(i)} \\
&= x_t^{(i)} \frac{1-(1-\tilde{\kappa}_i)^n}{\tilde{\kappa}_i} + \tilde{\kappa}_i \tilde{\mu}_i \sum_{\ell=0}^{n-1} \frac{1-(1-\tilde{\kappa}_i)^{\ell}}{\tilde{\kappa}_i} + \sigma_i \sum_{j=1}^{n-1} \sum_{\ell=1}^{n-1} \mathbb{1}_{\{j \le \ell\}} (1-\tilde{\kappa}_i)^{\ell-j} \tilde{z}_{t+j}^{(i)} \\
&= x_t^{(i)} \frac{1-(1-\tilde{\kappa}_i)^n}{\tilde{\kappa}_i} + \tilde{\mu}_i n - \tilde{\mu}_i \sum_{\ell=0}^{n-1} (1-\tilde{\kappa}_i)^{\ell} + \sigma_i \sum_{j=1}^{n-1} \left[ \sum_{\ell=j}^{n-1} (1-\tilde{\kappa}_i)^{\ell-j} \right] \tilde{z}_{t+j}^{(i)} \\
&= \underbrace{\left(x_t^{(i)} - \tilde{\mu}_i\right) \left[\frac{1-(1-\tilde{\kappa}_i)^n}{\tilde{\kappa}_i}\right] + \tilde{\mu}_i n}_{=m_{n,t}^{(i)}} + \sigma_i \sum_{j=1}^{n-1} \frac{1-(1-\tilde{\kappa}_i)^{n-j}}{\tilde{\kappa}_i} \tilde{z}_{t+j}^{(i)}.
\end{aligned}
$$

Define $X_{n,t}^{(i)} := \sum_{\ell=0}^{n-1} x_{t+\ell}^{(i)}$. Thus, under $\mathbb{Q}$ and conditional on $\mathcal{F}_t$, $X_{n,t} := [X_{n,t}^{(1)} \cdots X_{n,t}^{(p)}]^{\top}$ follows a multivariate Gaussian distribution with mean vector $m_{n,t} := \left[m_{n,t}^{(1)} \cdots m_{n,t}^{(p)}\right]^{\top}$ and covariance matrix $v_n := \left[v_n^{(i,\ell)}\right]_{i,\ell=1}^{p}$, where

$$
m_{n,t}^{(i)} = \left(x_t^{(i)} - \tilde{\mu}_i\right) \left[\frac{1-(1-\tilde{\kappa}_i)^n}{\tilde{\kappa}_i}\right] + \tilde{\mu}_i n,
$$

and

$$
\begin{aligned}
v_n^{(i,\ell)} &= \mathrm{Cov}^{\mathbb{Q}} \left[ \sigma_i \sum_{j=1}^{n-1} \frac{1-(1-\tilde{\kappa}_i)^{n-j}}{\tilde{\kappa}_i} \tilde{z}_{t+j}^{(i)},\ \sigma_\ell \sum_{j'=1}^{n-1} \frac{1-(1-\tilde{\kappa}_\ell)^{n-j'}}{\tilde{\kappa}_\ell} \tilde{z}_{t+j'}^{(\ell)} \,\middle|\, \mathcal{F}_t \right] \\
&= \frac{\sigma_i \sigma_\ell}{\tilde{\kappa}_i \tilde{\kappa}_\ell} \Gamma_{i,\ell} \sum_{j=1}^{n-1} \left[1-(1-\tilde{\kappa}_i)^{n-j}\right] \left[1-(1-\tilde{\kappa}_\ell)^{n-j}\right] \\
&= \frac{\sigma_i \sigma_\ell}{\tilde{\kappa}_i \tilde{\kappa}_\ell} \Gamma_{i,\ell} \sum_{j=1}^{n-1} \left[1-(1-\tilde{\kappa}_i)^{j}\right] \left[1-(1-\tilde{\kappa}_\ell)^{j}\right] \\
&= \frac{\sigma_i \sigma_\ell}{\tilde{\kappa}_i \tilde{\kappa}_\ell} \Gamma_{i,\ell} \sum_{j=1}^{n-1} \left[1-(1-\tilde{\kappa}_i)^{j} - (1-\tilde{\kappa}_\ell)^{j} + (1-\tilde{\kappa}_i)^{j}(1-\tilde{\kappa}_\ell)^{j}\right] \\
&= \frac{\sigma_i \sigma_\ell}{\tilde{\kappa}_i \tilde{\kappa}_\ell} \Gamma_{i,\ell} \Bigg[ n - 1 - \frac{1-(1-\tilde{\kappa}_i)^n}{\tilde{\kappa}_i} + 1 - \frac{1-(1-\tilde{\kappa}_\ell)^n}{\tilde{\kappa}_\ell} + 1 \\
&\qquad\qquad + \frac{1-(1-\tilde{\kappa}_i)^n(1-\tilde{\kappa}_\ell)^n}{1-(1-\tilde{\kappa}_i)(1-\tilde{\kappa}_\ell)} - 1 \Bigg]
\end{aligned}
$$

$$= \frac{\sigma_i \sigma_\ell}{\tilde{\kappa}_i \tilde{\kappa}_\ell} \Gamma_{i,\ell} \left[ n - \frac{1-(1-\tilde{\kappa}_i)^n}{\tilde{\kappa}_i} - \frac{1-(1-\tilde{\kappa}_\ell)^n}{\tilde{\kappa}_\ell} + \frac{1-(1-\tilde{\kappa}_i)^n(1-\tilde{\kappa}_\ell)^n}{1-(1-\tilde{\kappa}_i)(1-\tilde{\kappa}_\ell)} \right].$$

Using the expression for the moment generating function of the multivariate normal, we obtain,

$$\begin{aligned} P_{t,T} &= \mathbb{E}^{\mathbb{Q}} \left[ \exp\left( -\Delta \sum_{j=0}^{T-t-1} r_{t+j} \right) \,\middle|\, \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \exp\left( -\Delta \mathbb{1}_p^\top X_{T-t,t} \right) \mid \mathcal{F}_t \right] \\ &= \exp\left\{ -\Delta \mathbb{1}_p^\top m_{T-t,t} + \frac{\Delta^2}{2} \mathbb{1}_p^\top v_{T-t} \mathbb{1}_p \right\}, \end{aligned}$$

from which the result of Proposition 2.1 is deduced.

# B Estimation procedure for the discrete-time multifactor Vasicek model

Let $y(t, t+n)$ be the annualized continuously-compounded model-implied spot rate with time to maturity $n$, that is,

$$\begin{aligned} y(t, t+n) &:= -\frac{1}{n\Delta} \log P_{t,t+n} \\ &= -\frac{A_n}{n\Delta} + \sum_{i=1}^{p} \frac{B_n^{(i)}}{n} x_t^{(i)}, \end{aligned} \tag{B.1}$$

and define,

$$\begin{aligned} \boldsymbol{x}_t &:= \left( x_t^{(1)}, \ldots, x_t^{(p)} \right)^\top \in \mathbb{R}^p, \\ \boldsymbol{y}(t) &:= \left( y(t, t+n_1), \ldots, y(t, t+n_M) \right)^\top \in \mathbb{R}^M, \\ \boldsymbol{a} &:= \left( -\frac{A_{n_1}}{n_1 \Delta}, \ldots, -\frac{A_{n_M}}{n_M \Delta} \right)^\top \in \mathbb{R}^M, \\ \boldsymbol{\beta}_n &:= \left( \frac{B_n^{(1)}}{n}, \ldots, \frac{B_n^{(p)}}{n} \right) \in \mathbb{R}^p, \\ \mathbf{B} &:= \left( \boldsymbol{\beta}_{n_1}, \ldots, \boldsymbol{\beta}_{n_M} \right)^\top \in \mathbb{R}^{M \times p}. \end{aligned}$$

We model the relationship between observed and model-implied spot rates with the following equation:

$$
\begin{aligned}
\hat{\boldsymbol{y}}(t) &= \boldsymbol{y}(t) + \boldsymbol{\eta}_t \\
&= \boldsymbol{a} + \mathbf{B}\boldsymbol{x}_t + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim N(\mathbf{0}, \mathbf{H}),
\end{aligned} \tag{B.2}
$$

where $\{\boldsymbol{\eta}_t\}$ is an independent sequence of Gaussian distributed random vectors with mean $\mathbf{0}$ and diagonal variance-covariance matrix $\mathbf{H}$. All diagonal entries in the matrix $\mathbf{H}$ are assumed to be equal to the parameter $h > 0$.

Eq. (B.2) expresses the observed spot rates as a linear function of the term structure factors plus a Gaussian noise term. In filtering terminology, this equation is called the measurement equation. To complete the state space representation, the transition equation describing the dynamics of the latent factors $\boldsymbol{x}_t$ must be specified. It is given in vector form from Eq. (2.2) by:

$$
\boldsymbol{x}_{t+1} = \boldsymbol{b} + \mathbf{D}\boldsymbol{x}_t + \boldsymbol{\xi}_{t+1}, \quad \boldsymbol{\xi}_{t+1} \sim N(\mathbf{0}, \mathbf{Q}), \tag{B.3}
$$

where

$$
\begin{aligned}
\boldsymbol{b} &:= (\kappa_1\mu_1, \dots, \kappa_p\mu_p)^\top, \\
\mathbf{D} &:= \operatorname{diag}(1-\kappa_1, \dots, 1-\kappa_p), \\
\boldsymbol{\xi}_{t+1} &:= \left(\sigma_1 z_{t+1}^{(1)}, \dots, \sigma_p z_{t+1}^{(p)}\right)^\top, \\
\mathbf{Q} &:= \begin{bmatrix} \sigma_1^2 & \Gamma_{12}\sigma_1\sigma_2 & \cdots & \Gamma_{1p}\sigma_1\sigma_p \\ \Gamma_{21}\sigma_2\sigma_1 & \sigma_2^2 & \cdots & \Gamma_{2p}\sigma_2\sigma_p \\ \vdots & & \ddots & \\ \Gamma_{p1}\sigma_p\sigma_1 & & & \sigma_p^2 \end{bmatrix}.
\end{aligned}
$$

Eqs. (B.2) and (B.3) define a linear Gaussian state space model. This representation allows us to calibrate the discrete-time multifactor Vasicek model to yield curve data using the Kalman filter. This algorithm recursively computes the Gaussian densities $p(\hat{\boldsymbol{y}}(t) \mid \hat{\boldsymbol{y}}(1), \dots, \hat{\boldsymbol{y}}(t-1))$,

$p(\boldsymbol{x}_t \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(t))$ and $p(\boldsymbol{x}_{t+1} \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(t))$ for $t = 1, \ldots, T$. Maximization of the log-likelihood, $\log p(\hat{\boldsymbol{y}}(1), \hat{\boldsymbol{y}}(2), \ldots, \hat{\boldsymbol{y}}(T)) = \sum_{t=1}^{T} \log p(\hat{\boldsymbol{y}}(t) \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(t-1))$, as a function of the model parameters is accomplished using the following optimization strategy to avoid local maxima. In a first step we took advantage of the R package `DEoptim` (Ardia et al., 2016), which implements a differential evolution algorithm for global optimization. Starting from a randomly generated parameter value, this algorithm generates successive parameter iterates that are each more likely to represent the optimum of the objective function. In a second step, we used the best parameter iterate generated in the first step as the initial value in a standard gradient-based optimizer based on Newton's method. We repeated these two steps a large number of times and retained the parameter vector associated with the highest log-likelihood.

### B.1 Kalman recursions for the discrete-time multifactor Vasicek model

The Kalman filter algorithm used to calibrate the discrete-time multifactor Vasicek model performs the following recursions for $t = 1, \ldots, T$ assuming given initial values $\bar{\boldsymbol{x}}_{1|0} \in \mathbb{R}^p$ and $\boldsymbol{P}_{1|0} \in \mathbb{R}^{p \times p}$:

*Step* 1. Calculate $p(\hat{\boldsymbol{y}}(t) \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(t-1))$:

$$\begin{aligned}
p(\hat{\boldsymbol{y}}(t) \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(t-1)) &\sim N(\bar{\boldsymbol{y}}(t), \boldsymbol{\Sigma}_t) \\
\bar{\boldsymbol{y}}(t) &= \boldsymbol{a} + \boldsymbol{B}\bar{\boldsymbol{x}}_{t|t-1} \\
\boldsymbol{\Sigma}_t &= \boldsymbol{B}\boldsymbol{P}_{t|t-1}\boldsymbol{B}^\top + \boldsymbol{H}
\end{aligned}$$

*Step* 2. Calculate $p(\boldsymbol{x}_t \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(t))$:

$$\begin{aligned}
p(\boldsymbol{x}_t \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(t)) &\sim N(\bar{\boldsymbol{x}}_{t|t}, \boldsymbol{P}_{t|t}) \\
\bar{\boldsymbol{x}}_{t|t} &= \bar{\boldsymbol{x}}_{t|t-1} + \boldsymbol{P}_{t|t-1}\boldsymbol{B}^\top \boldsymbol{\Sigma}_t^{-1}(\hat{\boldsymbol{y}}(t) - \bar{\boldsymbol{y}}(t)) \\
\boldsymbol{P}_{t|t} &= \boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1}\boldsymbol{B}^\top \boldsymbol{\Sigma}_t^{-1}\boldsymbol{B}\boldsymbol{P}_{t|t-1}
\end{aligned}$$

*Step* 3. Calculate $p(\boldsymbol{x}_{t+1} \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(t))$:

$$p(\boldsymbol{x}_{t+1} \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(t)) \sim N(\bar{\boldsymbol{x}}_{t+1|t}, \boldsymbol{P}_{t+1|t})$$
$$\bar{\boldsymbol{x}}_{t+1|t} = \boldsymbol{b} + \boldsymbol{D}\bar{\boldsymbol{x}}_{t|t}$$
$$\boldsymbol{P}_{t+1|t} = \boldsymbol{D}\boldsymbol{P}_{t|t}\boldsymbol{D}^\top + \boldsymbol{Q}$$

The density $p(\boldsymbol{x}_t \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(t))$ which infers the factors at time $t$ based on the available observed information up to that time is called the filtered state density. On the other hand, the density $p(\boldsymbol{x}_t \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(T))$ which infers the factors at time $t$ conditionally on all observations is called the smoothed state density. This density can be obtained by way of a backward procedure known as the Kalman smoother (Shumway and Stoffer, 2017, Section 6.2). This algorithm follows the Kalman filter and computes,

$$p(\boldsymbol{x}_t \mid \hat{\boldsymbol{y}}(1), \ldots, \hat{\boldsymbol{y}}(T)) \sim N(\bar{\boldsymbol{x}}_{t|T}, \boldsymbol{P}_{t|T}),$$

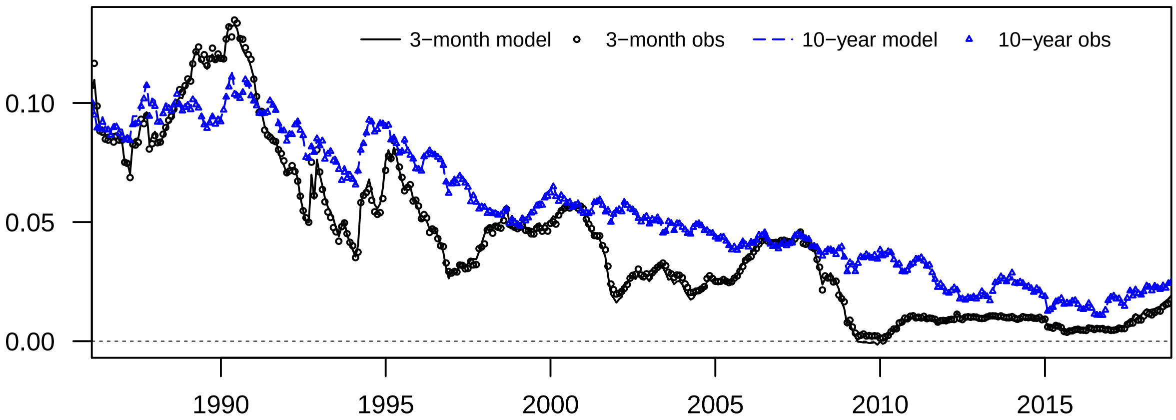
by performing the following recursions for $t = T-1, \ldots, 1$:

$$\boldsymbol{J}_t = \boldsymbol{P}_{t|t}\boldsymbol{D}^\top\boldsymbol{P}_{t+1|t}^{-1},$$
$$\bar{\boldsymbol{x}}_{t|T} = \bar{\boldsymbol{x}}_{t|t} + \boldsymbol{J}_t\left(\bar{\boldsymbol{x}}_{t+1|T} - \bar{\boldsymbol{x}}_{t+1|t}\right),$$
$$\boldsymbol{P}_{t|T} = \boldsymbol{P}_{t|t} + \boldsymbol{J}_t\left(\boldsymbol{P}_{t+1|T} - \boldsymbol{P}_{t+1|t}\right)\boldsymbol{J}_t^\top.$$

### B.2 Additional figures comparing model-implied and observed sport rates

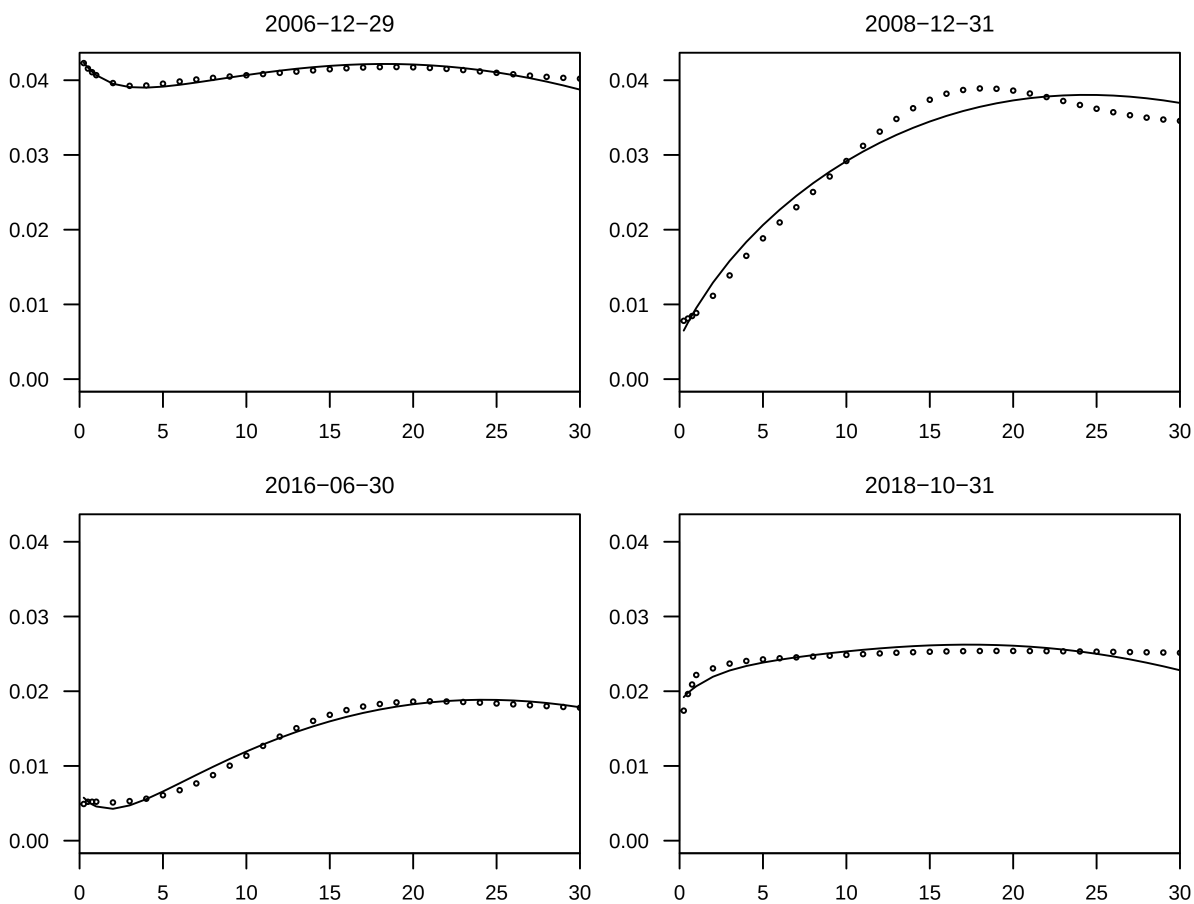
This section presents additional figures comparing model-implied and observed sport rates. Figure B.1 illustrates the evolution of model-implied and observed 3-month and 10-year spot rates in the data sample, whereas Figure B.2 shows model-implied and empirical yield curves at specific dates.

**Figure B.1:** Model-implied versus observed rates




Notes: The model-implied rates are obtained by plugging in the smoothed factor states, $\mathbb{E}^{\mathbb{P}}\left[x_t^{(i)} \;\middle|\; \hat{\boldsymbol{y}}(1), \hat{\boldsymbol{y}}(2), \ldots, \hat{\boldsymbol{y}}(T)\right]$, for $i = 1, 2, 3$, in Eq. (B.1).

**Figure B.2:** Model-implied and observed yield curves




Notes: *Solid line*: model-implied yield curve, *Open dots*: observed yield curve. The date associated with each yield curve is displayed above each graph.

# C Fit diagnostics output for our fund model

This section presents a summary of the fit diagnostics for our fund model introduced in Section 2.2.1 and estimated on the three data sets described in Section 3.1. Note that `mu`, `mxreg1`, `mxreg2`, `mxreg3`, `mxreg4` and `mxreg5` in the outputs below correspond to $\theta_0$, $\theta_1$, $\theta_2$, $\theta_3$, $\theta_1^{(S)}$ and $\theta_2^{(S)}$, respectively.

### iShares bond ETF

```
Optimal Parameters
------------------------------------
        Estimate  Std. Error  t value Pr(>|t|)
mu      0.003399    0.000241  14.0842 0.000000
mxreg1 -6.620257    0.253579 -26.1072 0.000000
mxreg2 -1.313802    0.195340  -6.7257 0.000000
mxreg3 -3.823703    0.186637 -20.4874 0.000000
mxreg4  0.028009    0.011837   2.3661 0.017976
mxreg5  0.033781    0.032885   1.0273 0.304293
omega  -0.404463    0.154322  -2.6209 0.008770
alpha1 -0.090638    0.082155  -1.1033 0.269912
beta1   0.964566    0.012869  74.9544 0.000000
gamma1  0.203524    0.061746   3.2961 0.000980

LogLikelihood : 880.2946

Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic p-value
Lag[1]                    0.06528  0.7983
Lag[2*(p+q)+(p+q)-1][2]   0.07221  0.9396
Lag[4*(p+q)+(p+q)-1][5]   0.58368  0.9440
d.o.f=0
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
                        statistic p-value
Lag[1]                    0.05526  0.8141
Lag[2*(p+q)+(p+q)-1][5]   0.12630  0.9970
Lag[4*(p+q)+(p+q)-1][9]   0.50962  0.9984
d.o.f=2

Weighted ARCH LM Tests
------------------------------------
            Statistic Shape Scale P-Value
ARCH Lag[3]   0.07896 0.500 2.000  0.7787
ARCH Lag[5]   0.11254 1.440 1.667  0.9845
ARCH Lag[7]   0.45528 2.315 1.543  0.9823

Adjusted Pearson Goodness-of-Fit Test:
------------------------------------
```

```
  group statistic p-value(g-1)
1    20     28.44      0.07529
2    30     40.77      0.07215
3    40     40.81      0.39067
4    50     64.77      0.06502
```

## RBC bond fund

```
Optimal Parameters
------------------------------------
        Estimate  Std. Error    t value Pr(>|t|)
mu      0.002580    0.000197    13.1012 0.000000
mxreg1 -6.806903    0.308729   -22.0481 0.000000
mxreg2 -1.501163    0.251663    -5.9650 0.000000
mxreg3 -3.866624    0.182625   -21.1724 0.000000
mxreg4  0.062015    0.018891     3.2827 0.001028
mxreg5  0.046569    0.007581     6.1427 0.000000
omega  -0.292606    0.021987   -13.3084 0.000000
alpha1 -0.252937    0.060505    -4.1805 0.000029
beta1   0.974537    0.000049 19870.4757 0.000000
gamma1  0.192276    0.052918     3.6334 0.000280

LogLikelihood : 607.0695

Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic p-value
Lag[1]                     0.7349  0.3913
Lag[2*(p+q)+(p+q)-1][2]    1.2667  0.4193
Lag[4*(p+q)+(p+q)-1][5]    2.8909  0.4273
d.o.f=0
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
                        statistic p-value
Lag[1]                    0.02661  0.8704
Lag[2*(p+q)+(p+q)-1][5]   0.32127  0.9814
Lag[4*(p+q)+(p+q)-1][9]   1.22617  0.9749
d.o.f=2

Weighted ARCH LM Tests
------------------------------------
            Statistic Shape Scale P-Value
ARCH Lag[3]  0.001018 0.500 2.000  0.9745
ARCH Lag[5]  0.670042 1.440 1.667  0.8326
ARCH Lag[7]  1.332163 2.315 1.543  0.8543

Adjusted Pearson Goodness-of-Fit Test:
------------------------------------
  group statistic p-value(g-1)
1    20     15.69       0.6779
2    30     21.55       0.8383
3    40     37.34       0.5455
4    50     47.07       0.5517
```

### Assumption mixed fund

```
Optimal Parameters
------------------------------------
        Estimate  Std. Error   t value Pr(>|t|)
mu      0.000201    0.000015    13.329        0
mxreg1  0.390952    0.022361    17.483        0
mxreg2 -0.327890    0.006408   -51.167        0
mxreg3 -0.996643    0.013047   -76.388        0
mxreg4  0.497730    0.000513   970.991        0
mxreg5  0.059126    0.000067   888.757        0
omega  -0.513249    0.000188 -2734.566        0
alpha1 -0.069243    0.001701   -40.714        0
beta1   0.942871    0.003293   286.306        0
gamma1 -0.207904    0.000232  -895.978        0

LogLikelihood : 614.8825

Weighted Ljung-Box Test on Standardized Residuals
------------------------------------
                        statistic p-value
Lag[1]                     0.8939 0.34444
Lag[2*(p+q)+(p+q)-1][2]    4.7554 0.04729
Lag[4*(p+q)+(p+q)-1][5]    7.4151 0.04108
d.o.f=0
H0 : No serial correlation

Weighted Ljung-Box Test on Standardized Squared Residuals
------------------------------------
                        statistic p-value
Lag[1]                      1.223  0.2687
Lag[2*(p+q)+(p+q)-1][5]     1.478  0.7454
Lag[4*(p+q)+(p+q)-1][9]     2.825  0.7874
d.o.f=2

Weighted ARCH LM Tests
------------------------------------
            Statistic Shape Scale P-Value
ARCH Lag[3]   0.01537 0.500 2.000  0.9013
ARCH Lag[5]   0.42782 1.440 1.667  0.9047
ARCH Lag[7]   1.83416 2.315 1.543  0.7524

Adjusted Pearson Goodness-of-Fit Test:
------------------------------------
  group statistic p-value(g-1)
1    20     16.61       0.6161
2    30     37.66       0.1302
3    40     35.22       0.6430
4    50     46.01       0.5949
```

# D Additional VA valuation results

This section presents additional results on the valuation of our VA policies to complement those shown in Section 4. Table D.1 repeats the analysis in Table 5 for a GMMB with a lower guarantee (75%) and a shorter maturity (10 years). We note that the basic GMMB offered for the RBC bond fund and the Assumption mixed fund is, respectively, a 10 and 15-year 75% capital guarantee without ratchets. According to our analysis these guarantees have little to no value. When comparing the 100% guarantee for maturities of 10 and 20 years, we notice that the costs increase for the shorter maturity. The greater costs are partly due to the fact that the maturity benefit is less heavily discounted over a 10-year period. In addition, the shorter maturity offers less recovery time in the event of a crash in the years that follow the signature of the contract. Interestingly, for the 75% guarantee on the Assumption mixed fund, the opposite effect is observed; the shorter maturity guarantee is less valuable. We believe that this is related to the very low level of the guarantee. Indeed, for the 75% guarantee to mature in-the-money, a very severe market crash would have to occur, which is more likely to happen over a time horizon of 20 than 10 years.

The second analysis that we performed to complement the results shown in Section 4 relates to a robustness check on our base interest rate floor assumption of $-0.75\%$. Hence, we repeated the experiment in Table 5 with a floor of 0%. Results are illustrated in Table D.2. As expected, guarantee costs decrease when the floor is increased from $-0.75\%$ to 0%. This follows from the one-sided impact of the floor; it either has no influence on interest rates in a given scenario or it increases them. A floor of $-0.75\%$ is thus a conservative assumption. Nevertheless, this assumption does not affect the qualitative analysis of our results.

**Table D.1:** Valuation of GMMB policies according to different guarantee terms and levels

| | RBC bond fund | | | | Assumption mixed fund | | | |
|---|---|---|---|---|---|---|---|---|
| | $\Pi_0$ | $\Pi_0^{(in)}$ | $\Pi_0^{(guar)}$ | $\frac{\Pi_0^{(guar)}}{\Pi_0^{(in)}}$ | $\Pi_0$ | $\Pi_0^{(in)}$ | $\Pi_0^{(guar)}$ | $\frac{\Pi_0^{(guar)}}{\Pi_0^{(in)}}$ |
| *20-year GMMB, no ratchets* | | | | | | | | |
| 100% guarantee | 20.73 | 21.35 | 0.62 | 2.9% | 22.53 | 27.86 | 5.33 | 19.1% |
| 75% guarantee | 18.91 | 18.91 | 0.00 | 0.0% | 24.12 | 24.86 | 0.74 | 3.0% |
| *10-year GMMB, no ratchets* | | | | | | | | |
| 100% guarantee | 13.87 | 15.48 | 1.60 | 10.4% | 13.32 | 20.37 | 7.05 | 34.6% |
| 75% guarantee | 14.62 | 14.62 | 0.00 | 0.0% | 18.90 | 19.17 | 0.27 | 1.4% |

**Table D.2:** Valuation of GMMB policies according to different interest rate floor assumptions

| | RBC bond fund | | | | Assumption mixed fund | | | |
|---|---|---|---|---|---|---|---|---|
| | $\Pi_0$ | $\Pi_0^{(in)}$ | $\Pi_0^{(guar)}$ | $\frac{\Pi_0^{(guar)}}{\Pi_0^{(in)}}$ | $\Pi_0$ | $\Pi_0^{(in)}$ | $\Pi_0^{(guar)}$ | $\frac{\Pi_0^{(guar)}}{\Pi_0^{(in)}}$ |
| *20-year GMMB, no ratchets* | | | | | | | | |
| With floor of −0.75% | 20.73 | 21.35 | 0.62 | 2.9% | 22.53 | 27.86 | 5.33 | 19.1% |
| With floor of 0% | 20.89 | 21.33 | 0.44 | 2.0% | 22.99 | 27.81 | 4.82 | 17.3% |
| *20-year GMMB, with ratchets during the first 10 years* | | | | | | | | |
| With floor of −0.75% | 19.77 | 21.95 | 2.18 | 9.9% | 22.13 | 29.00 | 6.87 | 23.7% |
| With floor of 0% | 20.19 | 21.92 | 1.73 | 7.9% | 22.81 | 28.98 | 6.17 | 21.3% |