Appendix

A Earthquake economic losses: the competing regression models

Table 11: Probability density distribution functions and regression components used.

Distribution	Density function	Regression component
Lomax	$\frac{\beta\sigma_i{}^{\beta}}{(y_i+\sigma_i)^{\beta+1}}$	$\sigma_i = \exp(\boldsymbol{x}_i^T\boldsymbol{\beta})$
Lognormal	$\frac{1}{\sqrt{2\pi}y_i}\exp\left[-\frac{(\log y_i-\mu_i)^2}{2\sigma^2}\right]$	$\mu_i = \boldsymbol{x}_i^T\boldsymbol{\beta}$
Burr	$\frac{\lambda\theta^{\lambda}\tau_i y_i{}^{\tau_i-1}}{(\theta+y_i{}^{\tau_i})^{\lambda+1}}$	$\tau_i = \exp(\boldsymbol{x}_i^T\boldsymbol{\beta})$
GlogM	$\frac{\sqrt{\theta}}{\sqrt{2\pi}\sigma_i}\left(\frac{1}{y_i}\right)^{\frac{1}{2\sigma_i}+1}\exp\left[-\frac{\theta}{2}\left(\frac{1}{y_i}\right)^{1/\sigma_i}\right]$	$\sigma_i = \exp(\boldsymbol{x}_i^T\boldsymbol{\beta})$
Exponentiated Fréchet	$\alpha\lambda_i\tau\left\{1-\exp\left[-\left(\frac{\tau}{y_i{}^{\lambda_i}}\right)\right]\right\}^{\alpha-1} y_i^{-(1+\lambda_i)}\exp\left[-\left(\frac{\tau}{y_i{}^{\lambda_i}}\right)\right]$	$\lambda_i = \exp(\boldsymbol{x}_i^T\boldsymbol{\beta})$
Gamma-generalized inverse Gaussian	$\frac{(y_i t_i\mu)^a}{y_i\Gamma(a)}\left(\frac{\psi}{\psi+2\mu^2 t_i y_i}\right)^{\frac{a+\lambda}{2}}\frac{K_{a+\lambda}(\psi/\mu_1^*)}{K_{\lambda}(\psi/\mu)}$	$t_i = \exp(\boldsymbol{x}_i^T\boldsymbol{\beta})$
Exponential-inverse Gaussian	$\frac{\delta}{t_i}\frac{\exp\left[-\delta\phi(y_i,t_i,\delta)-\delta^2\right]}{\phi(y_i,t_i,\delta)^3}\left[\delta\phi(y_i,t_i,\delta)+1\right]$	$t_i = \exp(\boldsymbol{x}_i^T\boldsymbol{\beta})$
GB2	$\frac{\lvert\sigma\rvert}{y_i B(\nu,\tau)}\frac{(y_i/\mu_i)^{pv}}{[1+(y_i/\mu_i)^p]^{\tau+\nu}}$	$\mu_i = \exp(\boldsymbol{x}_i^T\boldsymbol{\beta})$
Gamma	$\frac{1}{\beta_i^{\alpha}}y_i^{\alpha-1}\exp\left(-\frac{y_i}{\beta_i}\right)$	$\beta_i = \exp(\boldsymbol{x}_i^T\boldsymbol{\beta})$
Inverse Gaussian	$\left(\frac{\lambda}{2\pi y_i^3}\right)^{1/2}\exp\left[-\frac{\lambda(y_i-\mu_i)^2}{2\mu_i^2 y_i}\right]$	$\mu_i = \exp(\boldsymbol{x}_i^T\boldsymbol{\beta})$
Generalized Pareto	$\frac{1}{\sigma}\left[1+\frac{\xi_i(y_i-\mu)}{\sigma}\right]^{\left(-\frac{1}{\xi_i}-1\right)}$	$\xi_i = \boldsymbol{x}_i^T\boldsymbol{\beta}$ $\mu = 0$
DPLN	$\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\frac{1}{y_i}\left[\exp\left(\frac{1}{2}\tau^2\lambda_1^2-\lambda_1(\log y_i-\nu_i)\right)\Phi\left(\frac{\log y_i-\tau^2\lambda_1-\nu_i}{\tau}\right)\right.$ $\left.+\exp\left(\frac{1}{2}\tau^2\lambda_2^2-\lambda_2(\nu_i-\log y_i)\right)\Phi^c\left(\frac{\log y_i+\tau^2\lambda_2-\nu_i}{\tau}\right)\right]$	$\nu_i = \boldsymbol{x}_i^T\boldsymbol{\beta}$

Note: In the gamma-generalized inverse Gaussian regression model, $\mu_1^* = \mu\sqrt{\frac{\psi}{\Psi+2\mu^2 t_i y_i}}$, $K_m(\cdot)$ is the modified Bessel function of the third kind with order m. In the exponential-inverse Gaussian regression model, $\phi(y_i,\boldsymbol{\beta},\delta) = (\delta^2+2y_i/t_i)^{1/2}$. In the DPLN regression model, $\Phi(\cdot)$ and $\Phi^c(\cdot)$ represent the cumulative distribution function and survival function of the standard normal distribution respectively.

B Proofs

Proof of Proposition 2.1.

Note that when $Y \sim \text{GLMGA}(\sigma, a, b)$, we derive the marginal distribution $F(y)$ as follows. By letting $z = t^{-\frac{1}{\sigma}}$ and $v = \frac{z}{z+2b}$, we have

$$
\begin{aligned}
F(y) &= \int_0^y \frac{b^a}{\sqrt{2}\sigma B\left(a, \frac{1}{2}\right)} \frac{t^{-\left(\frac{1}{2\sigma}+1\right)}}{\left(\frac{1}{2}t^{-\frac{1}{\sigma}} + b\right)^{a+\frac{1}{2}}} dt \\
&= 1 - \int_0^{y^{-\frac{1}{\sigma}}} \frac{1}{zB(a, \frac{1}{2})} \frac{z^{\frac{1}{2}}(2b)^a}{(z+2b)^{a+\frac{1}{2}}} dz \\
&= 1 - \int_0^{\frac{z}{z+2b}} \frac{1}{B(a, \frac{1}{2})} v^{-\frac{1}{2}}(1-v)^{a-1} dv \\
&= 1 - I_{\frac{1}{2},a}\left[\frac{z}{z+2b}\right].
\end{aligned} \tag{B.1}
$$

The q^{th} quantile of $F^{-1}(q)$ of $\text{GLMGA}(\sigma, a, b)$ can be obtained by inverting the cdf (B.1).

Proof of (2.8). Let $b = \frac{a}{2}\phi^{\frac{1}{\sigma}}$ in $\text{GLMGA}(\sigma, a, b)$. Substituting this in (2.3) yields

$$
\begin{aligned}
f(y; \sigma, a, b, \phi) &= \frac{b^a}{\sqrt{2}\sigma B\left(a, \frac{1}{2}\right)} \frac{y^{-\left(\frac{1}{2\sigma}+1\right)}}{\left(\frac{1}{2}y^{-\frac{1}{\sigma}} + b\right)^{a+\frac{1}{2}}} \\
&= f(y; \sigma, a, \phi) \\
&= \frac{y^{-\frac{1}{2\sigma}-1}}{\sigma a^{\frac{1}{2}} \phi^{\frac{1}{2\sigma}} B(\frac{1}{2}, a)} \left[\frac{1}{1 + \frac{y^{-1/\sigma}}{2b}}\right]^{a+\frac{1}{2}} \\
&= \frac{y^{-\frac{1}{2\sigma}-1}}{\sigma \sqrt{\pi} \phi^{\frac{1}{2\sigma}}} \left[\frac{1}{1 + \frac{y^{-1/\sigma}}{2b}}\right]^{a+\frac{1}{2}} \frac{\Gamma(a+\frac{1}{2})}{\Gamma(a) a^{\frac{1}{2}}}.
\end{aligned}
$$

For large values of a, the gamma function can be approximated by Stirling's formula thus

$$
\lim_{a \to \infty} \frac{\Gamma(a+\frac{1}{2})}{\Gamma(a) a^{\frac{1}{2}}} = \frac{e^{-\frac{1}{2}-a}(a+\frac{1}{2})^{a+\frac{1}{2}-\frac{1}{2}\sqrt{2\pi}}}{e^{-a} a^{a-\frac{1}{2}\sqrt{2\pi}} a^{\frac{1}{2}}} = 1.
$$

For $a \to \infty$, we have

$$\begin{aligned}
\lim_{a\to\infty} f(y) &= \lim_{a\to\infty} \frac{y^{-\frac{1}{2\sigma}-1}}{\sigma\sqrt{\pi}\phi^{\frac{1}{2\sigma}}} \left[\frac{1}{1+\frac{y^{-1/\sigma}}{2b}}\right]^{a+\frac{1}{2}} \frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)a^{\frac{1}{2}}} \\
&= \lim_{a\to\infty} \frac{1}{\sigma\sqrt{\pi}\phi^{\frac{1}{2\sigma}}} y^{-\frac{1}{2\sigma}-1} \left[\frac{a}{a+(\phi y)^{-1/\sigma}}\right]^{a+\frac{1}{2}} \\
&= \frac{1}{\sigma\sqrt{\pi}\phi^{\frac{1}{2\sigma}}} y^{-\frac{1}{2\sigma}-1} \exp\left[-(\phi y)^{-\frac{1}{\sigma}}\right],
\end{aligned}$$

which is the density of generalized inverse gamma distribution with shape parameters $\frac{1}{2}$ and $\frac{1}{\sigma}$, and scale parameter ϕ.
For $\sigma = \frac{1}{2}$ and $a \to \infty$, we have

$$\lim_{a\to\infty} f(y) = \frac{2}{\phi\sqrt{\pi}y^2} \exp\left(-\frac{1}{\phi^2 y^2}\right),$$

which is the pdf of inverse half-normal distribution.

Proof of (2.10) and (2.11). By letting $z = y^{-\frac{1}{\sigma}}$ and $v = \frac{z}{z+2b}$, we have

$$\begin{aligned}
\int_0^u y^k f(y)dy &= \int_0^u y^h \frac{b^a}{\sqrt{2}\sigma B\left(a,\frac{1}{2}\right)} \frac{y^{-\left(\frac{1}{2\sigma}+1\right)}}{\left(\frac{1}{2}y^{-\frac{1}{\sigma}}+b\right)^{a+\frac{1}{2}}} dy \\
&= \int_{u^{-\frac{1}{\sigma}}}^{+\infty} z^{-h\sigma-\frac{1}{2}} \frac{1}{B(a,\frac{1}{2})} \frac{(2b)^a}{(z+2b)^{a+\frac{1}{2}}} dz \\
&= (2b)^{-h\sigma} \frac{B(\frac{1}{2}-h\sigma, a+h\sigma)}{B(a,\frac{1}{2})} \left[1 - \int_0^{\frac{u^{-1/\sigma}}{u^{-1/\sigma}+2b}} \frac{v^{\frac{1}{2}-h\sigma-1}(1-v)^{a+h\sigma-1}}{B(\frac{1}{2}-h\sigma, a+h\sigma)} dv\right] \\
&= \mathbb{E}(Y^h)\left\{1 - I_{\frac{1}{2}-h\sigma, a+h\sigma}\left[\frac{u^{-1/\sigma}}{u^{-1/\sigma}+2b}\right]\right\}.
\end{aligned}$$

Similarly,

$$\int_u^{+\infty} y^k f(y)dy = \mathbb{E}(Y^h) I_{\frac{1}{2}-h\sigma, a+h\sigma}\left[\frac{u^{-1/\sigma}}{u^{-1/\sigma}+2b}\right].$$

C Simulations: extra figures

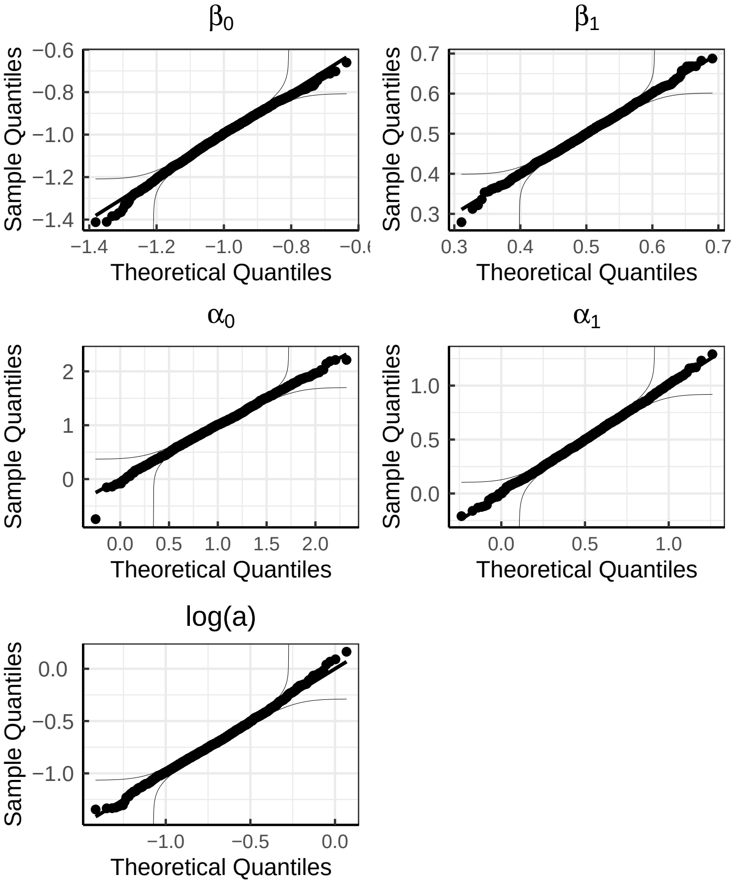

Figure 12: Normal QQ-plots of ML parameter estimators from GLMGA regression simulations with sample size $n = 200$ and $(\beta_0, \beta_1, \alpha_0, \alpha_1, a) = (-1, 0.5, 1, 0.5, 0.5)$.

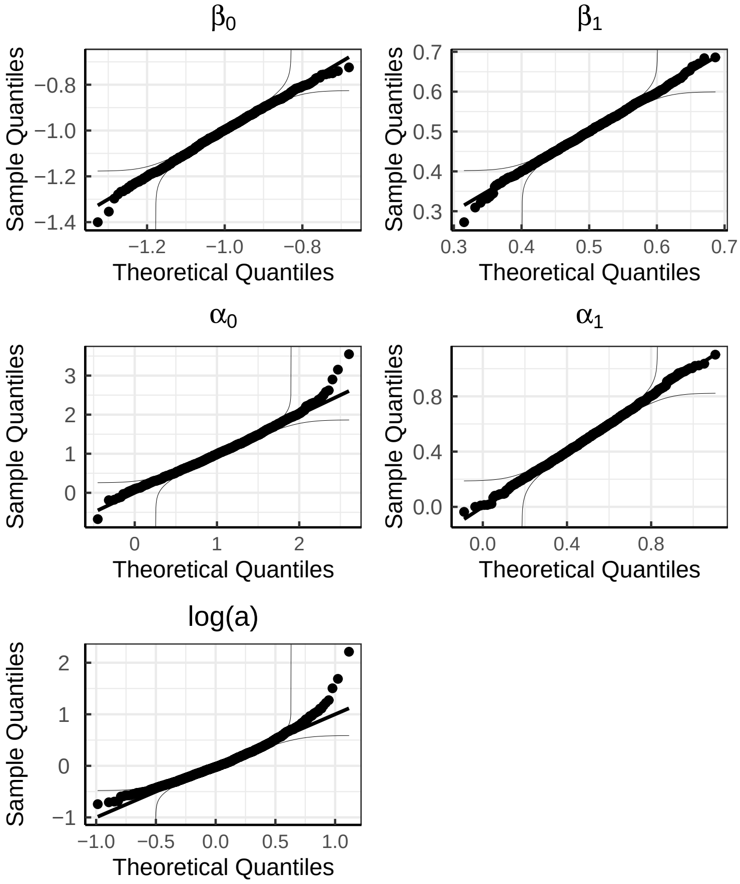

Figure 13: Normal QQ-plots of ML parameter estimators from GLMGA regression simulations with sample size $n = 200$ and $(\beta_0, \beta_1, \alpha_0, \alpha_1, a) = (-1, 0.5, 1, 0.5, 1)$.

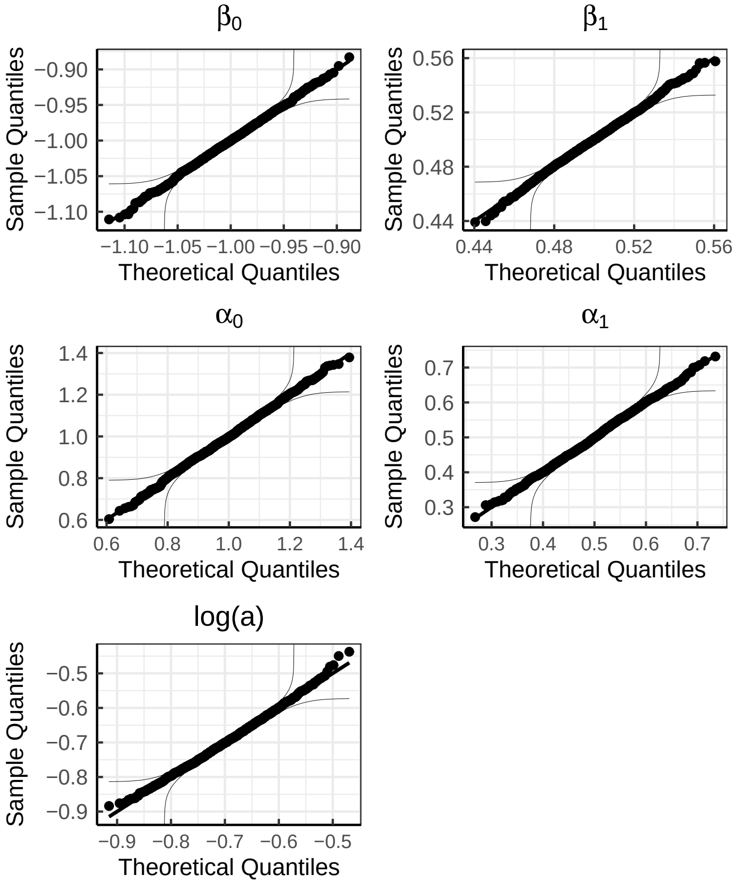

Figure 14: Normal QQ-plots of ML parameter estimators from GLMGA regression simulations with sample size $n = 2000$ and $(\beta_0, \beta_1, \alpha_0, \alpha_1, a) = (-1, 0.5, 1, 0.5, 0.5)$.

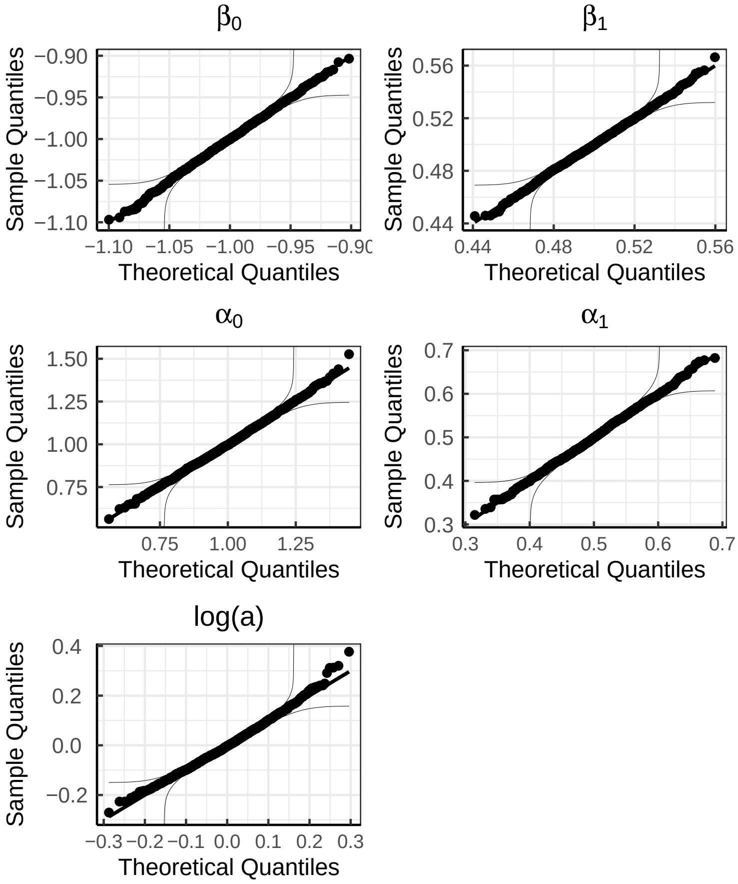

Figure 15: Normal QQ-plots of ML parameter estimators from GLMGA regression simulations with sample size $n = 2000$ and $(\beta_0, \beta_1, \alpha_0, \alpha_1, a) = (-1, 0.5, 1, 0.5, 1)$.

D Norwegian fire data 1991 and 1992

Table 12: Norwegian fire loss data (1991, 1992): model selection measures.

Year	Distribution	Estimates		#Par.	LL	AIC	BIC
1992	GlogM	$\hat{\sigma}$	1.360 (0.035)	2	-5130.5	10264.9	10273.7
		$\hat{\mu}$	182.167 (16.032)				
	GB2	$\hat{\mu}$	794.830 (124.772)	4	**-4915.5**	9838.9	9856.5
		$\hat{\sigma}$	1.708 (0.457)				
		$\hat{\nu}$	0.569 (0.192)				
		$\hat{\tau}$	0.755 (0.280)				
	GLMGA	$\hat{\sigma}$	0.433 (0.028)	3	-4916.0	**9838.1**	**9851.3**
		$\hat{b}$	1×10^{-7} (1×10^{-7})				
		$\hat{a}$	0.398 (0.046)				
	Lomax	$\hat{\beta}$	1.801 (0.193)	2	-4919.0	9842.0	9850.8
		$\hat{\sigma}$	1292.905 (199.178)				
	Log-gamma	$\hat{\alpha}$	1.360 (0.035)	2	-4992.1	9988.1	9996.9
		$\hat{\beta}$	182.167 (16.032)				
	Fréchet	$\hat{a}$	0.675 (0.019)	2	-4972.4	9948.7	9957.5
		$\hat{b}$	1165.882 (74.599)				
	DPLN	$\hat{\lambda}_1$	1.262 (0.156)	4	-4915.4	9838.7	9856.4
		$\hat{\lambda}_2$	0.951 (0.084)				
		$\hat{\nu}$	0.705 (0.126)				
		$\hat{\tau}$	6.596 (0.112)				
1991	GlogM	$\hat{\sigma}$	1.144 (0.031)	2	-5126.1	10256.2	10265.0
		$\hat{\mu}$	204.581 (14.935)				
	GB2	$\hat{\mu}$	937.991 (183.894)	4	**-4943.8**	**9895.6**	9913.3
		$\hat{\sigma}$	1.452 (0.408)				
		$\hat{\nu}$	0.747 (0.281)				
		$\hat{\tau}$	1.147 (0.495)				
	GLMGA	$\hat{\sigma}$	0.383 (0.026)	3	-4945.9	9897.9	9911.1
		$\hat{b}$	1×10^{-8} (2×10^{-8})				
		$\hat{a}$	0.366 (0.044)				
	Lomax	$\hat{\beta}$	2.460 (0.317)	2	-4948.4	9900.8	**9909.7**
		$\hat{\sigma}$	1882.034 (323.675)				
	Log-gamma	$\hat{\alpha}$	1.144 (0.031)	2	-4994.9	9993.7	10002.6
		$\hat{\beta}$	204.581 (14.935)				
	Fréchet	$\hat{a}$	0.763 (0.021)	2	-4986.7	9977.5	9986.3
		$\hat{b}$	1090.447 (61.044)				

DPLN	$\hat{\lambda}_1$ 1.593 (0.258) $\hat{\lambda}_2$ 1.043 (0.103) $\hat{\nu}$ 0.742 (0.121) $\hat{\tau}$ 6.669 (0.113)	4	-4944.0	9895.9	9913.6

Table 13: Norwegian fire loss data (1991-192): goodness-of-fit measures.

Year	Distribution	R	Kolmogorov-Smirnov		Anderson-Darling		Cramer-von Mises	
			Statistic	P-value	Statistic	P-value	Statistic	P-value
1992	GlogM	0.924	0.178	0.000	41.062	0.000	6.887	0.000
	GB2	0.999	0.023	0.226	0.161	0.787	0.028	0.553
	GLMGA	0.999	0.027	0.182	0.217	0.642	0.040	0.378
	Lomax	0.997	0.034	0.058	0.990	0.017	0.152	0.016
	Log-gamma	0.972	0.099	0.000	9.884	0.000	1.658	0.000
	Fréchet	0.985	0.077	0.000	9.784	0.000	1.531	0.000
	DPLN	0.999	0.023	0.221	0.151	0.839	0.026	0.617
1991	GlogM	0.925	0.163	0.000	33.990	0.000	5.581	0.000
	GB2	0.999	0.023	0.257	0.178	0.704	0.023	0.772
	GLMGA	0.999	0.030	0.067	0.386	0.150	0.063	0.119
	Lomax	0.998	0.037	0.033	1.282	0.003	0.175	0.009
	Log-gamma	0.979	0.080	0.000	7.598	0.000	1.260	0.000
	Fréchet	0.988	0.070	0.000	6.927	0.000	1.023	0.000
	DPLN	0.999	0.024	0.226	0.182	0.670	0.023	0.716

*The bootstrap P-values are computed using parametric bootstrap with 1000 simulation runs.

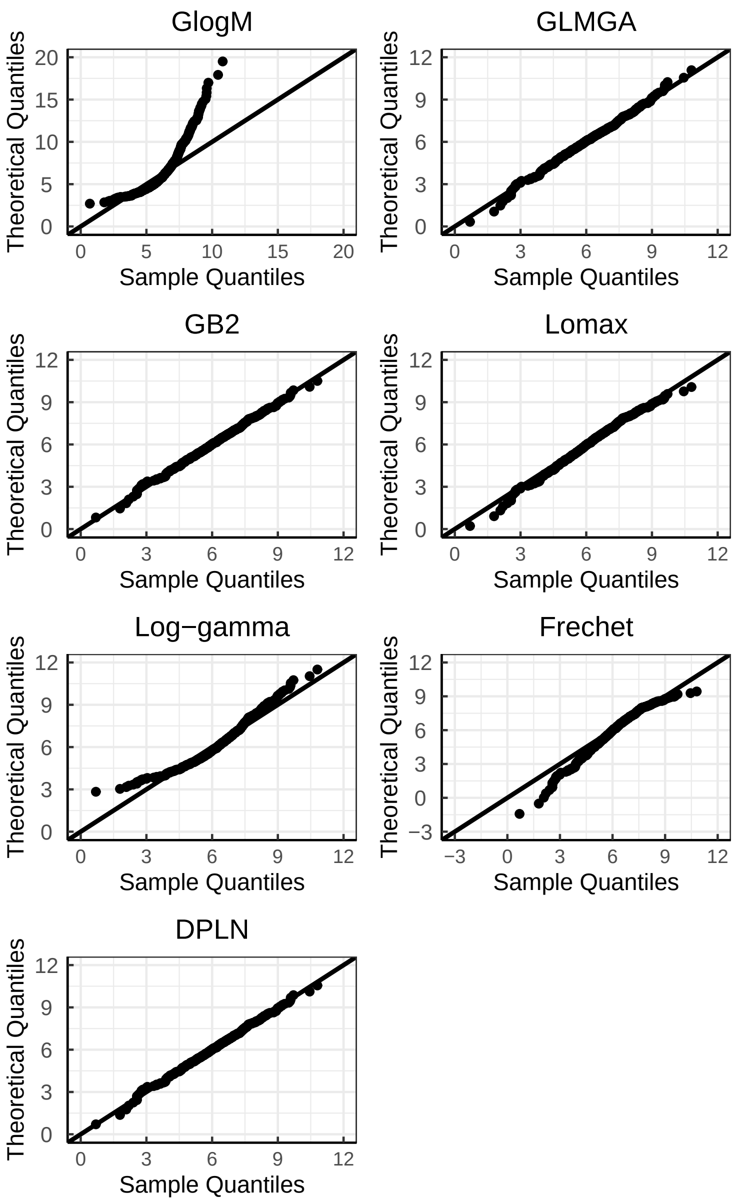

Figure 16: Norwegian fire loss data set (1991): QQ-plots of the log-transformed empirical quantiles against the log-transformed estimated model quantiles.

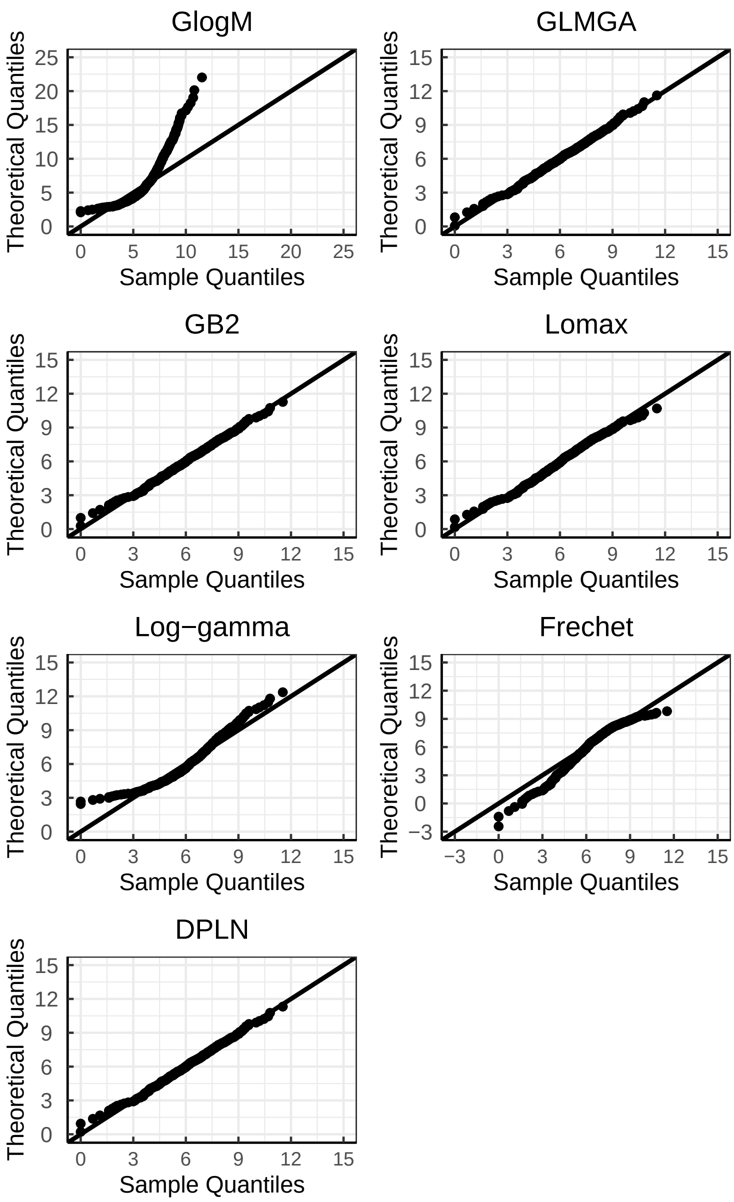

Figure 17: Norwegian fire loss data set (1992): QQ-plots of the log-transformed empirical quantiles against the log-transformed estimated model quantiles.

Table 14: Norwegian fire loss data (1991-1992): model estimates of $\mathrm{VaR}_{0.995}$ and $\mathrm{VaR}_{0.998}$ and relative difference (in percentage) with respect to the empirical VaR.

Year	Model	99.5%	Diff. %	99.8%	Diff. %
	Empirical	34588.54	-	45150.78	
	GlogM	-	5187.76	-	44946.46
	GB2	33350.00	-0.04	49572.40	0.10
1992	GLMGA	42525.09	0.23	66152.20	0.47
	Lomax	23194.11	-0.33	31222.61	-0.31
	Log-gamma	93878.20	1.71	143771.28	2.18
	Fréchet	13811.00	-0.60	15831.32	-0.65
	DPLN	34186.55	-0.01	51250.32	0.14
	Empirical	14585.56	-	30464.67	
	GlogM	-	1545.05	-	6027.16
	GB2	18526.20	0.27	32255.61	0.06
1991	GLMGA	27451.87	0.88	55414.78	0.82
	Lomax	14340.75	-0.02	21663.59	-0.29
	Log-gamma	45555.41	2.12	85873.96	1.82
	Fréchet	9700.66	-0.34	11956.60	-0.61
	DPLN	18974.05	0.30	33723.96	0.11