Scheduling Servers in a Two-stage Queue with Abandonments and Costs: Online Appendix

Gabriel Zayas-Cabán

Industrial and Systems Engineering

BerbeeWalsh Department of Emergency Medicine

University of Wisconsin

Madison, WI

zayascaban@wisc.edu

Amy L. Cochran

Department of Mathematics

Department of Population Health Sciences

University of Wisconsin

Madison, WI

cochran4@wisc.edu

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A Online Appendix

A.1 Existence of a solution to the optimality equations

In this section we show that the optimality equations have a solution for the multiserver model with $N < \infty$ servers when service can be preempted *and* when customers during service can abandon. Whenever possible, we discuss when these results can (or cannot) extended by relaxing either one of these assumptions. Let $\mathbb{A} = \cup_x A(x)$ denote the action space with action $a \in \mathbb{A}$ denoting an allocation decision of idling servers to stations 1 and 2. The set of actions available at state $x \in \mathbb{X}$ is $A(x) \subset \mathbb{A}$. Note that $\mathbb{A}$, and hence $A(x)$ for all $x \in \mathbb{X}$, are Polish spaces with the discrete metric on the set of natural numbers. For states $y \neq x$, let $q(y|x,a)$ be the rate at which a process leaves x and goes to y given that action a is chosen. Moreover, let $-q(x|x,a)$ be the rate at which a Markov process leaves state x under action a. Note for the current study, the transition rate kernel is *conservative* (i.e., $\sum_{y\in\mathbb{X}} q(y|x,a) = 0$ for all $x \in \mathbb{X}, a \in A(x)$) and *stable* (i.e., $q(x) := \sup\{-q(x|x,a) : a \in A(x)\} < \infty$ for all $x \in \mathbb{X}$). The following notation is adopted from [2, 4] and will be used throughout this section.

- For any measurable function $h \geq 1$ on $\mathbb{X}$, we define the h-weighted supremum norm $\|\cdot\|_h$ of a real-valued measurable function f on $\mathbb{X}$ by

$$\|f\|_h := \sup_{x\in\mathbb{X}}\{h(x)^{-1}|f(x)|\}$$

  and the Banach space $\mathcal{B}_h(\mathbb{X}) := \{f : \|f\|_h < \infty\}$.

- Let $K := \{(x,a) \in \mathbb{X} \times \mathbb{A} : a \in A(x)\}$ denote the family of state-action pairs.

- Let Π be the set of all randomized Markov policies, Π_s the set of all randomized Markov stationary policies, and F the set of all stationary, non-idling policies.

Customer abandonments imply that the transition rates are unbounded. As a result, we verify that each process generated by each *Markov* policy yields a transition kernel that (in the one-dimensional case) has row sums equal to one for all time. That is to say, we do not have an infinite number of transitions in finite time. To do so, we verify the following assumption from [1] (see Assumption A in [1]).

Assumption A. There exists a sequence of subsets of $\mathbb{X}_m \subset \mathbb{X}$, a non-decreasing function $h_A \geq 1$ on $\mathbb{X}$, and constants $b_A \geq 0$ and $c_A \in \mathbb{R}$ such that

1. $\mathbb{X}_m \uparrow \mathbb{X}$ and for each $m \geq 1$, $\sup\{q(x)|x \in \mathbb{X}_m\} < \infty$.

2. $\inf\{h_A(x)|x \notin \mathbb{X}_m\} \to \infty$ as $m \to \infty$.

3. $\sum_{y\in\mathbb{X}} q(y|x,a)h_A(y) \le c_A h_A(x) + b_A$ for all $(x,a) \in K$.

Lemma A.1 *Fix $\gamma \in (0,\alpha)$, where $\alpha > 0$ is a fixed, positive discount rate, and take $h_A(x) = h_A\left((x_1,x_2,y_1,y_2)\right) = e^{\epsilon(x_1+x_2)}$ with $\epsilon = \log(\gamma/(\lambda_1+\lambda_2)+1) > 0$. It follows that Assumption A holds with $\mathbb{X}_m = \{(x_1,x_2,y_1,y_2)|x_1+x_2+y_1+y_2 \le m\}$; $h_A(x) = h_A\left((x_1,x_2,y_1,y_2)\right) = e^{\epsilon(x_1+x_2)}$; $c_A = \gamma$; and $b_A = 0$.*

Proof. Only the third statement is nontrivial. For (x_1,x_2,y_1,y_2) and $a \in A((x_1,x_2,y_1,y_2))$, a little algebra yields

$$\begin{aligned}
&\sum_{(x_1',x_2',y_1',y_2')\in\mathbb{X}} q((x_1',x_2',y_1',y_2')|(x_1,x_2,y_1,y_2),a)h_A(x_1',x_2',y_1',y_2') \\
&= e^{\epsilon(x_1+x_2)}\Big[(\lambda_1+\lambda_2)\left[e^{\epsilon}-1\right] + \big((x_1-y_1-a_1)\beta_1 + (x_2-y_2-a_2)\beta_2 + q\min\{x_1,y_1+a_1\}\mu_1 \\
&\quad + \min\{x_2,y_2+a_1\}\mu_2\big)\left[e^{-\epsilon}-1\right]\Big] \\
&\le e^{\epsilon(x_1+x_2)}\left[\lambda_1+\lambda_2\right]\left[e^{\epsilon}-1\right] \\
&= \gamma e^{\epsilon(x_1+x_2)} \\
&= \gamma h_A\left((x_1,x_2,y_1,y_2)\right),
\end{aligned}$$

as desired. ∎

In addition to Assumption **A**, we will need to verify the following assumption from [1] (see Assumption B in [1]).

Assumption B.

With c_A and $h_A(\cdot)$ as in Assumption **A**:

1. either $c_A \le 0$, or $c_A - \alpha < 0$ when $c_A > 0$; and
2. there exist non-negative constants M_1 and M_2 such that

$$|c(x,a)| \le |x_1(h_1+K_1\beta_1) + x_2(h_2+\beta_2K_2)| \le M_1 + M_2h_A(x)$$

for every $x \in \mathbb{X}$.

Lemma A.2 *Assumption **B** holds with $M_1 = 0$ and $M_2 = \frac{\max\{h_1+\beta_1K_1, h_2+\beta_2K_1\}}{\epsilon}$, where ϵ is defined in Lemma A.1 above.*

Proof. Note that we have chosen $c_A = \gamma$ in Assumption **A** so that $0 < c_A < \alpha$, and hence, the first statement trivially holds.

For the second statement, note that

$$\begin{aligned}\frac{c(x,a)}{h_A(x)} \leq \frac{x_1(h_1+\beta_1 K_1)+x_2(h_2+\beta_2 K_2)}{h_A(x)} &= \frac{x_1(h_1+\beta_1 K_1)+x_2(h_2+\beta_2 K_2)}{e^{\epsilon(x_1+x_2)}} \\ &\leq \frac{x_1(h_1+\beta_1 K_1)+x_2(h_2+\beta_2 K_2)}{1+\epsilon(x_1+x_2)} \\ &\leq \frac{\max\{h_1+\beta_1 K_1, h_2+\beta_2 K_1\}(x_1+x_2)}{1+\epsilon(x_2+x_2)} \\ &\leq \frac{\max\{h_1+\beta_1 K_1, h_2+\beta_2 K_2\}}{\epsilon} \\ &= M_2,\end{aligned}$$

as desired. ∎

Assumptions **A** and **B** imply that the function v_α is the unique solution within $\mathcal{B}_{h_A}(\mathbb{X})$ of the discounted-cost optimality equations (see part (b) of Theorem 3.2 in [1]).

The final assumption we need to check is Assumption **C** below (c.f., Assumption C in [1]). That is, Assumptions **A-C** imply that Theorem 3.2 in [1] holds. In particular, there exists an optimal deterministic stationary optimal policy, and this policy attains the minimum in the right hand side of the DCOE (Theorem 4.2).

Assumption C.

1. For each $x \in \mathbb{X}$, the set of available actions in state x, $A(x)$, is compact;
2. The functions $q(x'|x,a)$, $c(x,a)$, and $\sum_{x'\in\mathbb{X}} q(x'|x,a)h_A(x')$ are continuous in $a \in A(x)$ for each fixed $x, x' \in \mathbb{X}$; and
3. Given h_A (the function in Assumption **A**), there exists a non-negative function $h_C : \mathbb{X} \to \mathbb{R}$ and constants $c_C > 0$, $b_C \geq 0$, and $M_C > 0$ such that $q(x)h_A(x) \leq M_C h_C(x)$ for every $x \in \mathbb{X}$

$$\sum_{x'\in\mathbb{X}} q(x'|x,a)h_C(y) \leq c_C h_C(x) + b_C$$

for all $(x,a) \in K$.

Lemma A.3 *Assumption **C** holds with $h_C(x) = e^{2\epsilon(x_1+x_2)}$; $c_C = c_A$; $b_C = 0$; and $M_C > 0$ given in the proof below.*

Proof. The first and second statements follow as a consequence of $A(x)$ being finite for all $x \in \mathbb{X}$.

Next, observe that $h_C(x)$ is a non-negative function. Moreover,

$$q(x)h_A(x)/h_C(x) = q(x)/e^{\epsilon(x_1+x_2)}$$

Since $q(x)$ is bounded above by a linear function of the state, it is dominated by the exponential function in the denominator, and hence, it follows that there exists $M_C > 0$ such that $q(x)h_A(x) \leq M_C h_C(x)$ for every $x \in \mathbb{X}$.

Lastly, fix $x \in \mathbb{X}$ and note that

$$\begin{aligned}\textstyle\sum_{x' \in \mathbb{X}} q(x'|x,a)h_C(x') &\leq (\lambda_1 + \lambda_2)\left[e^{\epsilon} - 1\right] e^{2\epsilon(x_1+x_2)} \\ &= \gamma e^{2\epsilon(x_1+x_2)} \\ &= \gamma h_C(x),\end{aligned}$$

as desired. ■

Our final step is to show that there exists a solution to the average cost optimality equations. To do this, we need to verify Assumption **D** below (c.f., Assumption 3 in [**?**]). That is, Assumptions **A-D** imply that Theorem 3.5 in [**?**], which we re-state below, holds. To do this, we will need the following definitions of the hitting time of a state, expected hitting time of a state, and the total expected cost incurred until the hitting time of state. The following definitions and notation are adopted from [**?**] and will be used throughout this section.
Let $\pi \in \Pi$ and $X_s^{\pi} = (Q_1^{\pi}(s), Q_2^{\pi}(s))$.

Definition A.4 *The* hitting time *of a state $x \in \mathbb{X}$ under policy $f \in F$ is*

$$\tau_x(f) := \inf_{t>0}\{X_t^f = x \textit{ and } \exists s \in (0,t) \textit{ such that } X_s^f \neq x\}.$$

Definition A.5 *The expected hitting time of state x (from state x') under policy $f \in F$ is* $m_{x'x}(f) = \mathbb{E}_{x'}^f \tau_x(f)$

Definition A.6 *The total expected cost incurred until the hitting time of state x under policy $f \in F$ is given by* $c_{x'x}(f) = \mathbb{E}_x^{\pi} \int_0^{\tau_x(f)} c(X_t)dt$

Assumption D

1. There exists a state $x_0 \in \mathbb{X}$ and a policy $f_0 \in F$ such that $m_{xx_0}(f_0), c_{xx_0}(f_0) < \infty$ for all $x \in \mathbb{X}$, except for state x_0, which may be absorbing. Note that this implies that the long-run average cost of policy f is independent of $x \in \mathbb{X}$.

2. There exists $\epsilon > 0$ such that $D = \{x' \in \mathbb{X} | c(x') \leq g(f_0) + \epsilon$ for some $f \in F\}$, is a finite set.
3. For all $x \in D_{\epsilon,\pi_0}$, there exists a policy f_x (depending on x) with $m_{zx}(\pi), c_{zx}(\pi) < \infty$.

If Assumptions **A**-**D** hold, then there exist g and w satisfying the ACOEs (Theorem 4.3) with the property that (1) g is the minimum expected average cost (in F); (2) any deterministic stationary policy f that attains the minimum in the ACOE is average cost optimal; and (3) there exists $x^* \in D$ with $w^*_{x^*} = \inf_x w^*_x$. Moreover, let $x_0 \in \mathbb{X}$ be a fixed state. Any sequence of discount factors $\{\alpha_n\}_n$ with $\lim_{n\to\infty} \alpha_n = 0$ has a subsequence, again denoted by $\{\alpha_n\}_n$, along which the following limits exists:

$$
\begin{aligned}
w'_x &= \lim_{n\to\infty} \{v^{\alpha_n}_x - v^{\alpha}_{x_0}\}, x \in \mathbb{X}, \\
g' &= \lim_{n\to\infty} \alpha_n v^{\alpha_n}_x, x \in \mathbb{X}, \\
f' &= \lim_{n\to\infty} f^{\alpha_n}.
\end{aligned}
$$

Furthermore, the tuple (g', w') is a solution to the ACOE with properties (1), (2), and (3) above, so that $g' = g$. Moreover, f' takes minimizing actions in the ACOE for $g = g'$ and $w = w'$. We next provide conditions which imply that our model satisfies Assumption **D**.

Lemma A.7 *Consider the following mutually exclusive conditions:*

1. $\min\{\beta_1, \beta_2\} > 0$
2. $\beta_1 > 0$*;* $\beta_2 = 0$*, and if* $\frac{\lambda_2}{\mu_2}$ *for the multi-server case with non-preemptive service and no abandonments during service or if* $\lambda_1 \cdot (\frac{1}{\pi_0(\mu_1+\beta_1)} + \frac{p(1-P(Ab)}{\mu_2}) + \frac{\lambda_2}{\mu_2} < 1$ *for the single-server case with preemption and abandonments during service, where* π_0 *is long-run fraction of time that station 2 is empty under the non-idling policy that prioritizes station 2 when* $\beta_2 = 0$ *and* $P(Ab)$ *is the probability a customer/job receiving service at station 1 abandons before completing service, which is given by* $\beta_1/(\mu_1 + \beta_1)$ *;*
3. $\beta_1 = 0$*,* $\beta_2 > 0$ *and* $\frac{\lambda_1}{\mu_1} < 1$

*If Assumption **D** holds.*

Proof. There are several cases to consider.

Case 1 $\min\{\beta_1, \beta_2\} > 0$

In this case, since abandonments from both phases of service can occur, any stationary policy yields a stable system.

Case 2 $\beta_1 > 0$*,* $\beta_2 = 0$*, and either* $\frac{\lambda_2}{\mu_2} < 1$ *for the multi-server case with non-preemptive service and no abandonments during service or if* $\lambda_1 \cdot (\frac{1}{\mu_1+\beta_1} + \frac{1}{\mu_2}) + \frac{\lambda_2}{\mu_2} < 1$ *for the single-server case with preemption and abandonments during service.*

Since $\beta_1 > 0$, station 1 is always stable. The condition $\frac{\lambda_2}{\mu_2} < 1$ implies that the potentially idling policy that prioritizes station 2 with one server yields a stable Markov chain for station 2 since it is akin to an M/M/1 queueing system. As a result, the condition implies the stability of the Markov chain induced by a policy that prioritizes station 2. Next, consider a single-server model with preemption and abandonments during service and suppose $\lambda_1 \cdot (\frac{1}{\mu_1+\beta_1} + \frac{1}{\mu_2}) + \frac{\lambda_2}{\mu_2} < 1$. For this system, consider the stationary nonidling policy that prioritizes station 2. First note that this policy is recurrent. Next, consider the following Lyapunov function $s(x_1, x_2) = \frac{x_1}{\pi_0(\mu_1+\beta_1)} + \frac{p\cdot(1-P(Ab)\cdot x_1+x_2}{\mu_2} + 1$ and $(x_1, x_2) \in \mathbb{X} \setminus \mathbb{X}_m$. For this Lyapunov function, we have

$$\begin{aligned}\sum_{(x_1',x_2')\in\mathbb{X}} q((x_1',x_2')|(x_1,x_2),a)s(x_1',x_2') = \lambda_1\Big[\frac{1}{\pi_0(\mu_1+\beta_1)} + \frac{p\cdot(1-P(Ab)}{\mu_2}\Big] + \lambda_2\Big[\frac{1}{\mu_2}\Big] \\ + \frac{p\mu_1\mathbb{1}(j=0,i>0)}{\mu_2} - (\mu_1\mathbb{1}(x_2=0,x_1>0)+x_1\beta_1)\Big[\frac{p\cdot(1-P(Ab))}{\mu_2}\Big] \\ - (\mu_1\mathbb{1}(x_2=0,x_1>0)+x_1\beta_1)\Big[\frac{1}{\pi_0(\mu_1+\beta_1)}\Big] - \mathbb{1}(x_2>0)\end{aligned}$$

If $x_2 > 0$, then this last expression is bounded above by

$$\lambda_1\Big[\frac{1}{\pi_0(\mu_1+\beta_1)} + \frac{p\cdot(1-P(Ab)}{\mu_2}\Big] + \lambda_2\Big[\frac{1}{\mu_2}\Big] - 1,$$

which is negative as a consequence of our assumption that $\lambda_1 \cdot (\frac{1}{\mu_1+\beta_1} + \frac{1}{\mu_2}) + \frac{\lambda_2}{\mu_2} < 1$. If, however, $x_2 = 0$ so that $x_1 = m > 0$, the first expression above is now bounded above by

$$\begin{aligned}&\lambda_1\Big[\frac{1}{\pi_0(\mu_1+\beta_1)} + \frac{p\cdot(1-P(Ab)}{\mu_2}\Big] + \lambda_2\Big[\frac{1}{\mu_2}\Big] + \frac{p\mu_1}{\mu_2} - (\mu_1+\beta_1)\Big[\frac{p\cdot(1-P(Ab))}{\mu_2}\Big] \\ &\quad - (\mu_1+\beta_1)\Big[\frac{1}{\pi_0(\mu_1+\beta_1)}\Big] \\ &\le \lambda_1\Big[\frac{1}{\pi_0(\mu_1+\beta_1)} + \frac{p\cdot(1-P(Ab)}{\mu_2}\Big] + \lambda_2\Big[\frac{1}{\mu_2}\Big] + \frac{p\mu_1}{\mu_2} - (\mu_1+\beta_1)\Big[\frac{p\cdot(1-P(Ab))}{\mu_2}\Big] \\ &\quad - \frac{1}{\pi_0} \\ &= \lambda_1\Big[\frac{1}{\pi_0(\mu_1+\beta_1)} + \frac{p\cdot(1-P(Ab)}{\mu_2}\Big] + \lambda_2\Big[\frac{1}{\mu_2}\Big] - \frac{1}{\pi_0} < 0,\end{aligned}$$

where the last inequality again follows from our assumption. Applying an analogue to *Foster's criterion* for continuous-time processes (see Meyn and Tweedie [3]; Theorem 4.2 with $f = 1$) yields that the Markov process associated with prioritizing station 2 has all states that communicate with say $(0, 0)$ as positive recurrent. Theorem 4.3(i) of Meyn and Tweedie [3] (again with $f = 1$) implies that under this policy, the Markov process generated starting in any initial state reaches $(0, 0)$ in finite expected time. ∎

Case 3 $\beta_1 = 0$, $\beta_2 > 0$, *and* $\frac{\lambda_1}{\mu_1} < 1$

Since $\beta_2 > 0$, station 2 is always stable. The condition $\frac{\lambda_1}{\mu_1} < 1$ implies that the potentially idling policy that prioritizes station 1 with one server yields a stable Markov chain for station 1 since it is akin to an M/M/1 queueing system. It follows that the output process into station 2 from station 1 is a Poisson process of rate $p\lambda_1$. The queue length process at station 2 is bounded below by the queue length process of an $M/M/\infty$ queue with birth rate equal to $p\lambda_1 + \lambda_2$ and death rate equal to $x_2\beta_2$ when there are x_2 customers in station 2. As a result, the condition implies the stability of the Markov chain induced by a policy that prioritizes station 1. ∎

A.2 Proof of of Theorem 4.5 under the discounted cost criterion

Proof of 1 continued. Recall we are trying to show that

$$\mu_1[pv_\alpha(x_1-1,x_2+1)+qv_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2)]+\mu_2[v_\alpha(x_1,x_2)-v_\alpha(x_1,x_2-1)]\geq 0. \tag{A.1}$$

using a sample path argument. Processes 1-5 started in states (x_1-1,x_2+1), (x_1-1,x_2), (x_1,x_2), (x_1,x_2), and (x_1,x_2-1), respectively. Processes 1, 2, and 4 use stationary optimal policies, which we denotes by π_1, π_2, and π_4, respectively. We are showing how to construct (potentially sub-optimal) policies for Processes 3 and 5 which we denote by π_3 and π_5, so that

$$\mu_1[pv_\alpha^{\pi_1}(x_1-1,x_2+1)+qv_\alpha^{\pi_2}(x_1-1,x_2)-v_\alpha^{\pi_3}(x_1,x_2)]+\mu_2[v_\alpha^{\pi_4}(x_1,x_2)-v_\alpha^{\pi_5}(x_1,x_2-1)]\geq 0. \tag{A.2}$$

Since π_3 and π_5 are potentially sub-optimal, (A.1) follows from (A.2).

Case 2 *Customer service completions*

Suppose that policies $\pi_i (i=1,2,3,4)$ all serve a class 2 customer whereas π_5 serves a class 1 customer. If the first event is a class 1 service completion in Process 5 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), after which all processes follow optimal controls, then the remaining costs in the left side of the inequality in (A.2) (with the denominator of the probability suppressed) are

$$\mu_1\Big[\left(\mu_2h_2-\mu_1[h_1+\beta_1K_1-ph_2]\right)t_1+\mu_1[pv_\alpha(x_1-1,x_2+1)+qv_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2)] \\ +\mu_2[v_\alpha(x_1,x_2)-pv_\alpha(x_1-1,x_2)-qv_\alpha(x_1-1,x_2-1)]\Big]. \tag{A.3}$$

If the first event is a class 2 service completion in Processes 1-4 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), again after which optimal controls are used, then the remaining costs (with the denominator of the probability suppressed) are

$$\mu_2\Big[\left(\mu_2h_2-\mu_1[h_1+\beta_1K_1-ph_2]\right)t_1+\mu_1[pv_\alpha(x_1-1,x_2)+qv_\alpha(x_1-1,x_2-1)-v_\alpha(x_1,x_2-1)]\Big]. \tag{A.4}$$

Adding expressions (A.3) and (A.4) yields

$$(\mu_1+\mu_2)\left(\mu_2 h_2-\mu_1[h_1+\beta_1 K_1-ph_2]\right)t_1$$
$$+\mu_1\Big[\mu_1\big[pv_\alpha(x_1-1,x_2+1)+qv_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2)\big]+\mu_2\big[v_\alpha(x_1,x_2)-v_\alpha(x_1,x_2-1)\big]\Big].$$

The terms inside the brackets in this last expression above are implied by the expression on left side of the inequality in (A.1). That is, we may restart the argument from here.

Suppose that policy π_2 and π_5 serve a class 1 customer, whereas π_1 and π_4 serves a class 2 customer. In this case, let π_3 serve a class 2 customer. If the first event is a class 1 service completion in Processes 2 and 5 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), after which all processes follow optimal controls, then the remaining costs in the left side of the inequality in (A.2) are

$$\begin{aligned}\mu_1\Big[&\left(\mu_2 h_2-\mu_1[h_1+\beta_1 K_1-ph_2]\right)t_1\\&+\mu_1[pv_\alpha(x_1-1,x_2+1)+q(pv_\alpha(x_1-2,x_2+1)+qv_\alpha(x_1-2,x_2))-v_\alpha(x_1,x_2)]\\&\quad+\mu_2[v_\alpha(x_1,x_2)-pv_\alpha(x_1-1,x_2)-qv_\alpha(x_1-1,x_2-1)]\Big].\end{aligned}\tag{A.5}$$

If the first event is a class 2 service completion in Processes 1, 3, and 4 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), again after which optimal controls are used, then the remaining costs are

$$\mu_2\Big[\left(\mu_2 h_2-\mu_1[h_1+\beta_1 K_1-ph_2]\right)t_1+\mu_1[v_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2-1)]\Big].\tag{A.6}$$

Adding (A.5) and (A.6) (and after rearranging terms) we get

$$\begin{aligned}&(\mu_1+\mu_2)\left(\mu_2 h_2-\mu_1[h_1+\beta_1 K_1-ph_2]\right)t_1\\&\quad+\mu_1\Big[\mu_1\big[pv_\alpha(x_1-1,x_2+1)+qv_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2)\big]+\mu_2\big[v_\alpha(x_1,x_2)-v_\alpha(x_1,x_2-1)\big]\Big]\\&\quad+q\mu_1\Big[\mu_1\big[pv_\alpha(x_1-2,x_2+1)+qv_\alpha(x_1-2,x_2)-v_\alpha(x_1-1,x_2)\big]\\&\qquad+\mu_2\big[v_\alpha(x_1-1,x_2)-v_\alpha(x_1-1,x_2-1)\big]\Big].\end{aligned}$$

The expression inside the first and second pair of brackets above are the expression on left side of inequality in (A.2) evaluated at (x_1,x_2) and (x_1-1,x_2), respectively. In both cases, we may relabel the states and continue as though we had started in these states.

Suppose that policies $\pi_1-\pi_3$ and π_5 serve a class 1 customer whereas π_4 serves a class 2 customer. If the first event is a class 1 service completion in Processes 1-3 and 5 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), after which all processes follow an optimal controls, then the

remaining costs in the left side of the inequality in (A.2) are

$$\begin{aligned}
&\mu_1\left(\mu_2 h_2 - \mu_1[h_1 + \beta_1 K_1 - p h_2]\right) t_1 \\
&\quad + \mu_1\Big[\mu_1\big[p(p v_\alpha(x_1-2, x_2+2) + q v_\alpha(x_1-2, x_2+1)) + q(p v_\alpha(x_1-2, x_2+1) + q v_\alpha(x_1-2, x_2)) \\
&\qquad - p v_\alpha(x_1-1, x_2+1) - q v_\alpha(x_1-1, x_2)\big] \\
&\quad + \mu_2\big[v_\alpha(i,j) - p v_\alpha(x_1-1, x_2) - q v_\alpha(x_1-1, x_2-1)\big]\Big] \qquad \text{(A.7)}
\end{aligned}$$

If the first event is a class 2 service completion in Process 4 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), again after which optimal controls are used, then the remaining costs are

$$\mu_2\left(\mu_2 h_2 - \mu_1[h_1 + \beta_1 K_1 - p h_2]\right) t_1 + \mu_2\Big[\mu_1\big[p v_\alpha(x_1-1, x_2+1) + q v_\alpha(x_1-1, x_2) - v_\alpha(x_1, x_2)\big]\Big]. \qquad \text{(A.8)}$$

Adding (A.7) and (A.8) we get

$$\begin{aligned}
&(\mu_1+\mu_2)\left(\mu_2 h_2 - \mu_1[h_1 + \beta_1 K_1 - p h_2]\right) t_1 \\
&\quad + p\mu_1\Big[\mu_1\big[p v_\alpha(x_1-2, x_2+2) + q v_\alpha(x_1-2, x_2+1) - v_\alpha(x_1-1, x_2+1)\big] \\
&\qquad + \mu_2\big[v_\alpha(x_1-1, x_2+1) - v_\alpha(x_1-1, x_2)\big]\Big] \\
&\quad + q\mu_1\Big[\mu_1\big[p v_\alpha(x_1-2, x_2+1) + q v_\alpha(x_1-2, x_2) - v_\alpha(x_1-1, x_2)\big] \\
&\qquad + \mu_2\big[v_\alpha(x_1-1, x_2) - v_\alpha(x_1-1, x_2-1)\big]\Big].
\end{aligned}$$

The expression inside the first pair of brackets above is the expression on the left side of the inequality in (A.2) but evaluated at (x_1-1, x_2+1). Similarly, the expression inside the second pair of brackets in the expression above is the left side of inequality (A.2) evaluated at (x_1-1, x_2). In both cases, we may relabel the states and continue as though we had started in these states.

Suppose that policies π_1, π_3 and π_5 serve a class 1 customer whereas π_2 and π_4 serve a class 2 customer. If the first event is a class 1 service completion in Processes 1,3, and 5 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), after which all processes follow an optimal controls, then the remaining costs in the left side of the inequality in (A.2) are

$$\begin{aligned}
&\mu_1\left(\mu_2 h_2 - \mu_1[h_1 + \beta_1 K_1 - p h_2]\right) t_1 + \mu_1\Big[\mu_1\big[p(p v_\alpha(x_1-2, x_2+2) + q v_\alpha(x_1-2, x_2+1)) + q v_\alpha(x_1-1, x_2) \\
&\qquad - p v_\alpha(x_1-1, x_2+1) - q v_\alpha(x_1-1, x_2)\big] \\
&\quad + \mu_2\big[v_\alpha(x_1, x_2) - p v_\alpha(x_1-1, x_2) - q v_\alpha(x_1-1, x_2-1)\big]\Big]. \qquad \text{(A.9)}
\end{aligned}$$

If the first event is a class 2 service completion in Processes 2 and 4 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), again after which optimal controls are used, then the remaining costs are

$$\begin{aligned}
&\mu_2\left(\mu_2 h_2 - \mu_1[h_1 + \beta_1 K_1 - p h_2]\right) t_1 \\
&\quad + \mu_2\Big[\mu_1\big[p v_\alpha(x_1-1, x_2+1) + q v_\alpha(x_1-1, x_2-1) - v_\alpha(x_1, x_2)\big]\Big]. \qquad \text{(A.10)}
\end{aligned}$$

Adding (A.9) and (A.10) we get

$$
\begin{aligned}
&(\mu_1+\mu_2)\left(\mu_2 h_2-\mu_1[h_1+\beta_1 K_1-ph_2]\right)t_1\\
&+p\mu_1\Big[\mu_1\big[pv_\alpha(x_1-2,x_2+2)+qv_\alpha(x_1-2,x_2+1)-v_\alpha(x_1-1,x_2+1)\big]\\
&\quad+\mu_2\big[v_\alpha(x_1-1,x_2+1)-v_\alpha(x_1-1,x_2)\big]\Big].
\end{aligned}
$$

The expression inside the pair of brackets above is the expression on the left side of the inequality in (A.29) but evaluated at (x_1-1,x_2+1). In this case, we may relabel the states and continue as though we had started in these states.

Suppose that policy π_4 serves a class 1 customer whereas π_1 and π_2 serve a class 2 customer. In this case, let policies π_3 and π_5 have the server work at station 2 and 1, respectively. If the first event is a class 1 service completion in Processes 4 and 5 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), after which all processes follow an optimal controls, then the remaining costs in the left side of the inequality in (A.2) are

$$
\begin{aligned}
&\mu_1\left(\mu_2 h_2-\mu_1[h_1+\beta_1 K_1-ph_2]\right)t_1\\
&\quad+\mu_1\Big[\mu_1\big[pv_\alpha(x_1-1,x_2+1)+qv_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2)\big]\\
&\qquad+\mu_2\big[pv_\alpha(x_1-1,x_2+1)+qv_\alpha(x_1-1,x_2)-pv_\alpha(x_1-1,x_2)-qv_\alpha(x_1-1,x_2-1)\big]\Big].
\end{aligned}
\tag{A.11}
$$

If the first event is a class 2 service completion in Processes 1-3 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), again after which optimal controls are used, then the remaining costs are

$$
\begin{aligned}
&\mu_2\left(\mu_2 h_2-\mu_1[h_1+\beta_1 K_1-ph_2]\right)t_1+\mu_2\Big[\mu_1\big[pv_\alpha(x_1-1,x_2)+qv_\alpha(x_1-1,x_2-1)-v_\alpha(x_1,x_2-1)\big]\\
&\quad+\mu_2\big[v_\alpha(x_1,x_2)-v_\alpha(x_1,x_2-1)\big]\Big].
\end{aligned}
\tag{A.12}
$$

Adding (A.11) and (A.12) we get

$$
\begin{aligned}
&(\mu_1+\mu_2)\left(\mu_2 h_2-\mu_1[h_1+\beta_1 K_1-ph_2]\right)t_1\\
&+(\mu_1+\mu_2)\Big[\mu_1\big[pv_\alpha(x_1-1,x_2+1)+qv_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2)\big]+\mu_2\big[v_\alpha(x_1,x_2)-v_\alpha(x_1,x_2-1)\big]\Big].
\end{aligned}
$$

The expression inside the pair of brackets above is the expression on the left side of the inequality in (A.2). In this case, we may relabel the states and continue as though we had started in these states.

Suppose that policies π_2 and π_4 serve a class 1 customer whereas π_1 serves a class 2 customer. In this case, let policies π_3 and π_5 have the server work at station 1. If the first event is a class 1 service completion in Processes 2-5 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), after which all

processes follow an optimal controls, then the remaining costs in the left side of the inequality in (A.2) are

$$\begin{aligned}
&\mu_1\left(\mu_2 h_2 - \mu_1[h_1 + \beta_1 K_1 - p h_2]\right) t_1 \\
&\quad + \mu_1\Big[\mu_1\big[p v_\alpha(x_1 - 1, x_2 + 1) + q(p v_\alpha(x_1 - 2, x_2 + 1) + q v_\alpha(x_1 - 2, x_2)) \\
&\qquad - p v_\alpha(x_1 - 1, x_2 + 1) - q v_\alpha(x_1 - 1, x_2)\big] \\
&\qquad + \mu_2\big[p v_\alpha(x_1 - 1, x_2 + 1) + q v_\alpha(x_1 - 1, x_2) - p v_\alpha(x_1 - 1, x_2) - q v_\alpha(x_1 - 1, x_2 - 1)\big]\Big].
\end{aligned} \tag{A.13}$$

If the first event is a class 2 service completion in Process 1 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), again after which optimal controls are used, then the remaining costs are

$$\begin{aligned}
&\mu_2\left(\mu_2 h_2 - \mu_1[h_1 + \beta_1 K_1 - p h_2]\right) t_1 \\
&\quad + \mu_2\Big[\mu_1\big[p v_\alpha(x_1 - 1, x_2) + q v_\alpha(x_1 - 1, x_2) - v_\alpha(x_1, x_2)\big] + \mu_2\big[v_\alpha(x_1, x_2) - v_\alpha(x_1, x_2 - 1)\big]\Big].
\end{aligned} \tag{A.14}$$

Adding (A.13) and (A.14) we get

$$\begin{aligned}
&(\mu_1 + \mu_2)\left(\mu_2 h_2 - \mu_1[h_1 + \beta_1 K_1 - p h_2]\right) t_1 \\
&+ q\mu_1\Big[\mu_1\big[p v_\alpha(x_1 - 2, x_2 + 1) + q v_\alpha(x_1 - 2, x_2) - v_\alpha(x_1 - 1, x_2)\big] + \mu_2\big[v_\alpha(x_1 - 1, x_2) - v_\alpha(x_1 - 1, x_2 - 1\big) \\
&+ \mu_2\Big[\mu_1\big[p v_\alpha(x_1 - 1, x_2 + 1) + q v_\alpha(x_1 - 1, x_2) - v_\alpha(x_1, x_2)\big] + \mu_2\big[v_\alpha(x_1, x_2) - v_\alpha(x_1, x_2 - 1)\big]\Big].
\end{aligned}$$

The expression inside the first pair of brackets in the expression above is the left side of the inequality (A.2) evaluated at $(x_1 - 1, x_2)$. The expression inside the second pair of brackets in the expression above is the left side of inequality (A.2). In both cases, we may relabel the states and continue as though we had started in these states.

Suppose that policies π_1 and π_4 serves a class 1 customer whereas π_2 serves a class 2 customer. In this case, let policies π_3 and π_5 have the server work at station 1. If the first event is a class 1 service completion in Processes 1, 3, 4, and 5 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), after which all processes follow an optimal controls, then the remaining costs in the left side of the inequality in (A.2) are

$$\begin{aligned}
&\mu_1\left(\mu_2 h_2 - \mu_1[h_1 + \beta_1 K_1 - p h_2]\right) t_1 \\
&\quad + \mu_1\Big[\mu_1\big[p\left(p v_\alpha(x_1 - 2, x_2 + 2) + q v_\alpha(x_1 - 2, x_2 + 1)\right) + q v_\alpha(x_1 - 1, x_2) - p v_\alpha(x_1 - 1, x_2 + 1) \\
&\qquad - q v_\alpha(x_1 - 1, x_2)\big] \\
&\qquad + \mu_2\big[p v_\alpha(x_1 - 1, x_2 + 1) + q v_\alpha(x_1 - 1, x_2) - p v_\alpha(x_1 - 1, x_2) - q v_\alpha(x_1 - 1, x_2 - 1)\big]\Big].
\end{aligned} \tag{A.15}$$

If the first event is a class 2 service completion in Process 2 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+x_1\beta_1}$), again after which optimal controls are used, then the remaining costs are

$$\mu_2\left(\mu_2 h_2 - \mu_1[h_1+\beta_1 K_1 - p h_2]\right) t_1 \\ + \mu_2\Big[\mu_1\big[p v_\alpha(x_1-1,x_2+1) + q v_\alpha(x_1-1,x_2-1) - v_\alpha(x_1,x_2)\big] + \mu_2\big[v_\alpha(x_1,x_2) - v_\alpha(x_1,x_2-1)\big]\Big]. \tag{A.16}$$

Adding (A.15) and (A.16) we get

$$(\mu_1+\mu_2)\left(\mu_2 h_2 - \mu_1[h_1+\beta_1 K_1 - p h_2]\right) t_1 \\ + p\mu_1\Big[\mu_1\big[p v_\alpha(x_1-2,x_2+2) + q v_\alpha(x_1-2,x_2+1) - v_\alpha(x_1-1,x_2+1)\big] + \mu_2\big[v_\alpha(x_1-1,x_2+1) \\ - v_\alpha(x_1-1,x_2)\big]\Big] \\ + \mu_2\Big[\mu_1\big[v_\alpha(x_1-1,x_2+1) + q v_\alpha(x_1-1,x_2) - v_\alpha(x_1,x_2)\big] + \mu_2\big[v_\alpha(x_1,x_2) - v_\alpha(x_1,x_2-1)\big]\Big].$$

The expression inside the first pair of brackets in the expression above is the left side of the inequality (A.2) evaluated at (x_1-1,x_2+1). The expression inside the second pair of brackets in the expression above is the left side of inequality (A.2). In both cases, we may relabel the states and continue as though we had started in these states.

Proof of 2. The proof is given for the discounted expected cost model. The proof of the long-run average cost case is similar. Suppose $\mu_1 = \mu_2 := \mu$ and that $\beta_1 - \beta_2 - \mu \geq 0$. Note that the optimality equations imply that it is optimal to prioritize a class 2 customer in state (x_1,x_2) with $x_1, x_2 \geq 1$ when

$$p v_\alpha(x_1-1,x_2+1) + q v_\alpha(x_1-1,x_2) - v_\alpha(x_1,x_2-1) \geq 0. \tag{A.17}$$

We show (A.17) via a sample path argument. Fix $x_1, x_2 \geq 1$ and start three processes on the same probability space. Processes 1-3 begin in states (x_1-1,x_2+1), (x_1-1,x_2), and (x_1,x_2-1), respectively. Processes 1-2 us a stationary optimal policies, which we denote by π_1 and π_2, respectively. In what follows, we show how to construct a (potentially sub-optimal) policy for Process 3 which we denote by π_3, so that

$$p v_\alpha^{\pi_1}(x_1-1,x_2+1) + q v_\alpha^{\pi_2}(x_1-1,x_2) - v_\alpha^{\pi_3}(x_1,x_2-1)] \geq 0. \tag{A.18}$$

Since π_2 and π_3 are potentially sub-optimal, (A.17) follows from (A.18). In what follows, discounting is suppressed without any loss of generality.

Observe that starting from (A.18), the immediate costs incurred at the next event are $([h_2+\beta_2 K_2] - [h_1+\beta_1 K_1 - p\,(h_2+\beta_2 K_2)])t_1 \geq 0$, where t_1 is the time of the next event and the inequality is due to the assumption that $h_2+\beta_2 K_2 \geq h_1+\beta_1 K_1 - p\,(h_2-\beta_2 K_2)$. Moreover, if

the relative position (as measured by the current states) of the four processes at the next event remains the same, then we may relabel the initial states and continue from the beginning of the argument. This occurs when any of the uncontrolled events occur that are seen by all three processes. It also occurs when π_1 and π_2 serve a customer class $k \in \{1, 2\}$ by letting π_3 also serve the same customer class k customer provided there is one or more class k customer in Process 3, and the next event service completion. Consider now the other cases.

Case 1 *Customer abandonments*

If the first event is a class 2 abandonment in Process 1 only (with probability $\frac{\beta_2}{\lambda+2\mu+x_1\beta_1+(x_2+1)\beta_2}$), after which all processes follow an optimal control, it follows that the remaining costs in the left side of (A.18) (with the denominator of the probability of this event suppressed) are

$$\beta_2\Big[\left([h_2+\beta_2K_2]-[h_1+\beta_1K_1-p\left(h_2+\beta_2K_2\right)]\right)t_1+\beta_2[v_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2-1)]\Big]. \tag{A.19}$$

If the first event is a class 2 abandonment in Processes 1-2 (with probability $\frac{\beta_2}{\lambda+2\mu+x_1\beta_1+(x_2+1)\beta_2}$), again after which optimal controls are used, then the remaining costs (with the denominator of the probability of this event suppressed) are

$$\begin{aligned}\beta_2\Big[&\left([h_2+\beta_2K_2]-[h_1+\beta_1K_1-p\left(h_2+\beta_2K_2\right)]\right)t_1\\ &+\beta_2[pv_\alpha(x_1-1,x_2)+qv_\alpha(x_1-1,x_2-1)-v_\alpha(x_1,x_2-1)]\Big].\end{aligned} \tag{A.20}$$

If the first event is a class 1 abandonment in Process 3 (with probability $\frac{\beta_1}{\lambda+2\mu+x_1\beta_1+(x_2+1)\beta_2}$), again after which optimal controls are used, then the remaining costs (with the denominator of the probability of this event suppressed) are

$$\begin{aligned}\beta_1\Big[&\left([h_2+\beta_2K_2]-[h_1+\beta_1K_1-p\left(h_2+\beta_2K_2\right)]\right)t_1\\ &+\beta_1[pv_\alpha(x_1-1,x_2+1)+qv_\alpha(x_1-1,x_2)-v_\alpha(x_1-1,x_2-1)]\Big].\end{aligned} \tag{A.21}$$

Adding expressions (A.19)- (A.21) and a little algebra yields

$$\begin{aligned}&\left([h_2+\beta_2K_2]-[h_1+\beta_1K_1-p\left(h_2+\beta_2K_2\right)]\right)t_1+(\beta_1-\beta_2)[pv_\alpha(x_1-1,x_2+1)+qv_\alpha(x_1-1,x_2)\\ &\quad-v_\alpha(x_1-1,x_2-1)]\\ &+\beta_2[pv_\alpha(x_1-1,x_2+1)+qv_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2-1)]\\ &+\beta_2[(1+p)v_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2-1)-pv_\alpha(x_1-1,x_2-1)].\end{aligned}$$

Note that the expression inside the first pair of brackets is nonnegative as a consequence of $\beta_1 \geq \beta_2+\mu$ and Proposition 4.1. The expression inside the second pair of brackets is implied

by (A.17), and so, for this expression, we may simply restart the argument from there. To complete the proof it suffices to consider the remaining costs from

$$(1+p)v_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2-1)-pv_\alpha(x_1-1,x_2-1). \tag{A.22}$$

We continue follow the sample paths of three processes (on the same probability space) Processes 1-3 begin in states (x_1-1,x_2), (x_1,x_2-1), and (x_1-1,x_2-1), respectively. Process 1 uses a stationary optimal policy, which we denote by π_1. In what follows, we show how to construct (potentially sub-optimal) policies for Processes 2 and 3, which we denote by π_2 and π_3, respectively, to evaluate

$$(1+p)v_\alpha^{\pi_1}(x_1-1,x_2)-v_\alpha^{\pi_2}(x_1,x_2-1)-pv_\alpha^{\pi_3}(x_1-1,x_2-1). \tag{A.23}$$

Since π_2 and π_3 are potentially sub-optimal, (A.22) follows from (A.23).

Note that if all three processes see an arrival or they all see a an abandonment, the costs incurred are $[(1+p)(h_2+\beta_2K_2)-(h_1+\beta_1K_1)]t_1 \geq 0$ where t_1 is the time of the next event and the inequality is due to the assumption that $h_2+\beta_2K_2 \geq h_1+\beta_1K_1-p(h_2+\beta_2K_2)$, and the relative position of the new states as measured with respect to the starting states is maintained. We may relabel the states and continue as though we started in these states. Similarly, if π_1 serves station $k \in \{1,2\}$ by letting π_2 and π_3 also serve the same phase of service whenever possible. Consider now the other cases.

Subcase 1.1 *Customer abandonments*

Suppose that the next event is a class 1 abandonment for Processes 2 and 3 (with probability $\frac{\beta_1}{\lambda_1+\lambda_2+2\mu+x_1\beta_1+x_2\beta_2}$), after which all processes follow optimal controls. Suppressing the denominator of the probability, the left hand side of inequality (A.23) becomes

$$[(1+p)(h_2+\beta_2K_2)-(h_1+\beta_1K_1)]t_1+\beta_1\left[(1+p)v_\alpha(x_1-1,x_2)-(1+p)v_\alpha(x_1-1,x_2-1)\right]. \tag{A.24}$$

If, however, the next event is a class 2 abandonment in Process 1 (with probability $\frac{\beta_2}{\lambda_1+\lambda_2+2\mu+x_1\beta_1+x_2\beta_2}$), after which all processes follow an optimal control, then, after suppressing the denominator of the probability, the left hand side of inequality (A.23) becomes

$$[(1+p)(h_2+\beta_2K_2)-(h_1-\beta_1K_1)]t_1+\beta_2[v_\alpha(x_1-1,x_2-1)-v_\alpha(x_1,x_2-1)]. \tag{A.25}$$

Adding (A.24) and (A.25) (with a little algebra) we get

$$\begin{aligned}&[(1+p)(h_2+\beta_2K_2)-(h_1+\beta_1K_1)]t_1+(\beta_1-\beta_2)\left[(1+p)v_\alpha(x_1-1,x_2)-(1+p)v_\alpha(x_1-1,x_2-1)\right] \\ &+\beta_2\left[(1+p)v_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2-1)-pv_\alpha(x_1-1,x_2-1)\right].\end{aligned} \tag{A.26}$$

The expression inside the second pair of brackets is (A.22). In this case, we may relabel the starting states and repeat the argument.

Subcase 1.2 *Service completions*

Suppose policies π_1, π_2, and π_3 assign the server to work at stations 1, 2, and 2, respectively and that the next event is a service completion seen by all processes (with probability $\frac{\mu}{\lambda_1+\lambda_2+2\mu+x_1\beta_1+x_2\beta_2}$), after which all processes follow an optimal control. Suppressing the denominator of the probability, the left hand side of inequality (A.23) becomes

$$[(1+p)(h_2+\beta_2K_2)-(h_1+\beta_1K_1)]t_1+\mu\left[v_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2-1)\right]. \tag{A.27}$$

To complete the proof, we add the remaining running costs from (A.26) and (A.27), to obtain

$$\begin{aligned}&[(1+p)(h_2+\beta_2K_2)-(h_1+\beta_1K_1)]t_1+(\beta_2-\beta_1)\left[v_\alpha(x_1-1,x_2)-v_\alpha(x_1-1,x_2-1)\right]\\&+(\beta_2-\beta_1-\mu)\left[pv_\alpha(x_1-1,x_2)-pv_\alpha(x_1-1,x_2-1)\right]\\&+\mu\left[(1+p)v_\alpha(x_1-1,x_2)-v_\alpha(x_1,x_2-1)-pv_\alpha(x_1-1,x_2-1)\right].\end{aligned}$$

The expression inside the first and second pair of brackets in the expression above are nonnegative while the third one is (A.22) for which we may simply restart the argument.

Case 2 *Service completions*

This case follows exactly the same arguments in the Proof of Case 2 above with μ_2h_2 replaced with $h_2+\beta_2K_2$ and $\mu_1[h_1+\beta_1K_1-p(h_2-\beta_2K_2)]$ replaced with $h_1+\beta_1K_2-p(h_2+\beta_2K_2)$.

A.3 Proof of Theorem 4.6 under the discounted cost criterion

Proof of 1. The proof is given for the discounted expected cost model. The proof of the long-run average cost case is similar. Note that the optimality equations imply that it is optimal to prioritize a class 1 customer in state (x_1,x_2) with $x_1,x_2\geq 1$ when

$$\mu_2[v_\alpha(x_1,x_2-1)-v_\alpha(x_1,x_2)]+\mu_1[v_\alpha(x_1,x_2)-pv_\alpha(x_1-1,x_2+1)-qv_\alpha(x_1-1,x_2)]\geq 0. \tag{A.28}$$

We show (A.28) via a sample path argument. Fix $x_1,x_2\geq 1$ and start four processes on the same probability space. Processes 1-5 begin in states (x_1,x_2-1), (x_1,x_2), (x_1,x_2), (x_1-1,x_2+1), and (x_1-1,x_2), respectively. Processes 1 and 3 use stationary optimal policies, which we

denote by π_1, π_3, respectively. In what follows, we show how to construct (potentially sub-optimal) policies for Processes 2, 4, and 5 which we denote by π_2, π_4, and π_5, so that

$$\mu_2[v_\alpha^{\pi_1}(x_1, x_2 - 1) - v_\alpha^{\pi_2}(x_1, x_2)] + \mu_1[v_\alpha^{\pi_3}(x_1, x_2) - pv_\alpha^{\pi_4}(x_1 - 1, x_2 + 1) - qv_\alpha^{\pi_5}(x_1 - 1, x_2)] \geq 0. \tag{A.29}$$

Since π_2, π_4, π_5 are potentially sub-optimal, (A.28) follows from (A.29). In what follows, discounting is suppressed without any loss of generality.

Observe that starting from (A.29), the immediate costs incurred at the next event are $(\mu_1[h_1 - p\,(h_2 + \beta_2 K_2)] - \mu_2[h_2 + \beta_2 K_2])t_1 \geq 0$, where t_1 is the time of the next event and the inequality is due to the assumption that $\mu_1[h_1 - p\,(h_2 - \beta_2 K_2)] \geq \mu_2[h_2 + \beta_2 K_2]$. Moreover, if the relative position (as measured by the current states) of the four processes at the next event remains the same, then we may relabel the initial states and continue from the beginning of the argument. This occurs when any of the uncontrolled events occur that are seen by all four processes. It also occurs when π_1 and π_3 both serve the same customer class $k \in \{1, 2\}$ by letting π_2 and π_4 also serve the same customer class k customer provided there is one or more customer class k customer in all four Processes, and the next event service completion. Consider now the other cases.

Case 1 *Customer abandonments*

If the first event is an abandonment in Process 4 only (with probability $\frac{\beta_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+(x_2+1)\beta_2}$), after which all processes follow an optimal control, it follows that the remaining costs in the left side of (A.29) (with the probability of this event in the expression suppressed) are

$$\begin{aligned}&(\mu_1\,[h_1 - p\,(h_2 + \beta_2 K_2)] - \mu_2\,[h_2 + \beta_2 K_2])\,t_1 \\ &\qquad + \mu_2[v_\alpha(x_1, x_2 - 1) - v_\alpha(x_1, x_2)] + \mu_1[v_\alpha(x_1, x_2) - v_\alpha(x_1 - 1, x_2)].\end{aligned} \tag{A.30}$$

If the first event is an abandonment in Processes 2-5 (with probability $\frac{\beta_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+(x_2+1)\beta_2}$), again after which optimal controls are used, then the remaining costs (with the probability of this event in the expression suppressed) are

$$\begin{aligned}&(\mu_1\,[h_1 - p\,(h_2 + \beta_2 K_2)] - \mu_2\,[h_2 + \beta_2 K_2])\,t_1 \\ &\qquad + \mu_1[v_\alpha(x_1, x_2 - 1) - pv_\alpha(x_1 - 1, x_2) - qv_\alpha(x_1 - 1, x_2 - 1)].\end{aligned} \tag{A.31}$$

Adding expressions (A.30) and (A.31) yields

$$\begin{aligned}&(\mu_1\,[h_1 - p\,(h_2 + \beta_2 K_2)] - \mu_2\,[h_2 + \beta_2 K_2])\,t_1 \\ &+ \mu_2[v_\alpha(x_1, x_2 - 1) - v_\alpha(x_1, x_2)] + \mu_1[v_\alpha(x_1, x_2) - pv_\alpha(x_1 - 1, x_2 + 1) - qv_\alpha(x_1 - 1, x_2)] \\ &+ \mu_1[pv_\alpha(x_1 - 1, x_2 + 1) + pv_\alpha(x_1, x_2 - 1) - 2pv_\alpha(x_1 - 1, x_2) + qv_\alpha(x_1, x_2 - 1) - qv_\alpha(x_1 - 1, x_2 - 1)].\end{aligned}$$

To complete the proof it suffices to show that

$$v_\alpha(x_1-1,x_2+1)+v_\alpha(x_1-1,x_2-1)-2v_\alpha(x_1-1,x_2)\geq 0. \tag{A.32}$$

We follow the sample paths of three processes (on the same probability space) to show (A.32) via a sample path argument. Processes 1-3 begin in states (x_1-1,x_2+1), (x_1-1,x_2-1), and (x_1-1,x_2), respectively. Processes 1 and 2 use stationary optimal policies, which we denote by π_1 and π_2. In what follows, we show how to construct (potentially sub-optimal) policy for Process 3, which we denote by π_3, so that

$$v_\alpha^{\pi_1}(x_1-1,x_2+1)+v_\alpha^{\pi_2}(x_1-1,x_2-1)-2v_\alpha^{\pi_3}(x_1-1,x_2)\geq 0. \tag{A.33}$$

Since π_3 is potentially sub-optimal, (A.32) follows from (A.33). Note that if all three processes see an arrival or they all see a station 2 abandonment, the immediate costs incurred are 0 and the relative position of the new states as measured with respect to the starting states is maintained. We may relabel the states and continue as though we started in these states. Similarly, if π_1 and π_2 both serve the same customer class $k\in\{1,2\}$ by letting π_3 also serve the same class k customer. Finally, station 2 abandonments that are not seen by all three processes lead to no immediate costs incurred followed by all processes coupling. Because the proof is simple, it is omitted. Consider now the other cases.

Subcase 1.1 *Suppose policies π_1 and π_2 assign the server to work at station 2 and 1, respectively. Assume that π_3 works at station 1.*

Suppose that the next event is a service completion at station 1 for Processes 2 and 3 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+(x_2+1)\beta_2}$), after which all processes follow optimal controls. Suppressing the denominator of the probability, the left hand side of inequality (A.33) becomes

$$\mu_1\left[v_\alpha(x_1-1,x_2+1)+pv_\alpha(x_1-2,x_2)+qv_\alpha(x_1-2,x_2-1)-2pv_\alpha(x_1-2,x_2+1)-2qv_\alpha(x_1-2,x_2)\right].$$

A little algebra yields that this last expression equals

$$\begin{aligned}&\mu_2\left[v_\alpha(x_1-1,x_2)-v_\alpha(x_1-1,x_2+1)\right]+\mu_1[v_\alpha(x_1-1,x_2+1)-pv_\alpha(x_1-2,x_2+2)\\&\quad-qv_\alpha(x_1-2,x_2+1)]-\mu_2\left[v_\alpha(x_1-1,x_2)-v_\alpha(x_1-1,x_2+1)\right]\\&+p\mu_1\left[v_\alpha(x_1-2,x_2+2)+v_\alpha(x_1-2,x_2)-2v_\alpha(x_1-2,x_2+1)\right]\\&+q\mu_1\left[v_\alpha(x_1-2,x_2+1)+v_\alpha(x_1-2,x_2-1)-2v_\alpha(x_1-2,x_2)\right].\end{aligned} \tag{A.34}$$

If, however, the next event is a service completion at station 2 in Process 1 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+(x_2+1)\beta_2}$), after which all processes follow an optimal control, then, after suppressing the denominator of the probability, the left hand side of inequality (A.33) becomes

$$\mu_2[v_\alpha(x_1-1,x_2-1)-v_\alpha(x_1-1,x_2)]. \tag{A.35}$$

Adding (A.34) and (A.35) (with a little algebra) we get

$$\begin{aligned}
&\mu_2\left[v_\alpha(x_1-1,x_2)-v_\alpha(x_1-1,x_2+1)\right]+\mu_1[v_\alpha(x_1-1,x_2+1)-pv_\alpha(x_1-2,x_2+2)\\
&\quad -qv_\alpha(x_1-2,x_2+1)]\\
&+p\mu_1\left[v_\alpha(x_1-2,x_2+2)+v_\alpha(x_1-2,x_2)-2v_\alpha(x_1-2,x_2+1)\right]\\
&+q\mu_1\left[v_\alpha(x_1-2,x_2+1)+v_\alpha(x_1-2,x_2-1)-2v_\alpha(x_1-2,x_2)\right]\\
&+\mu_2\left[v_\alpha(x_1-1,x_2+1)+v_\alpha(x_1-1,x_2-1)-2v_\alpha(x_1-1,x_2)\right].
\end{aligned}$$

The first expression is (A.28) evaluated at (x_1-1,x_2+1). The second, third, and fourth expressions, respectively, are (A.32) evaluated at (x_1-2,x_2+1), (x_1-2,x_2), and (x_1-1,x_2). In each case, we can relabel the starting states and repeat the argument.

Subcase 1.2 *Suppose policies π_1 and π_2 assign the server to work at station 1 and 2, respectively. Assume that π_3 works at station 1.*

Suppose that the next event is a service completion at station 1 for Processes 1 and 3 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+(x_2+1)\beta_2}$), after which all processes follow an optimal control. Suppressing the denominator of the probability, the left hand side of inequality (A.33) becomes

$$\begin{aligned}
&\mu_1[pv_\alpha(x_1-2,x_2+2)+qv_\alpha(x_1-2,x_2+1)+v_\alpha(x_1-1,x_2-1)-2pv_\alpha(x_1-2,x_2+1)\\
&\quad -2qv_\alpha(x_1-2,x_2)].
\end{aligned}$$

A little algebra yields

$$\begin{aligned}
&\mu_1\left[v_\alpha(x_1-1,x_2-1)-pv_\alpha(x_1-2,x_2)-qv_\alpha(x_1-2,x_2-1)\right]\\
&+p\mu_1\left[v_\alpha(x_1-2,x_2+2)+v_\alpha(x_1-2,x_2)-2v_\alpha(x_1-2,x_2+1)\right]\\
&+q\mu_1\left[v_\alpha(x_1-2,x_2+1)+v_\alpha(x_1-2,x_2-1)-2v_\alpha(x_1-2,x_2)\right].
\end{aligned}\tag{A.36}$$

If, however, the next event is a service completion at station 2 in Process 1 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+(x_2+1)\beta_2}$), after which all processes follow an optimal control, then, after suppressing the probability of this event in the expression, the left hand side of inequality (A.33) becomes

$$\mu_2\left[v_\alpha(x_1-1,x_2+1)+v_\alpha(x_1-1,x_2-2)-2v_\alpha(x_1-1,x_2)\right],$$

which equals

$$\begin{aligned}
&\mu_2[v_\alpha(x_1-1,x_2-2)-v_\alpha(x_1-1,x_2-1)]\\
&+\mu_2[v_\alpha(x_1-1,x_2+1)+v_\alpha(x_1-1,x_2-1)-2v_\alpha(x_1-1,x_2)].
\end{aligned}\tag{A.37}$$

Adding (A.36) and (A.37) (with a little algebra) we get

$$\begin{aligned}
&\mu_2[v_\alpha(x_1-1,x_2-2)-v_\alpha(x_1-1,x_2-1)]+\mu_1[v_\alpha(x_1-1,x_2-1)-pv_\alpha(x_1-2,x_2)\\
&\quad -qv_\alpha(x_1-2,x_2+1)]\\
&+\mu_2[v_\alpha(x_1-1,x_2+1)+v_\alpha(x_1-1,x_2-1)-2v_\alpha(x_1-1,x_2)]\\
&+p\mu_1\left[v_\alpha(x_1-2,x_2+2)+v_\alpha(x_1-2,x_2)-2v_\alpha(x_1-2,x_2+1)\right]\\
&+q\mu_1\left[v_\alpha(x_1-2,x_2+1)+v_\alpha(x_1-2,x_2-1)-2v_\alpha(x_1-2,x_2)\right].
\end{aligned}$$

The first expression is (A.28) evaluated at (x_1-1,x_2-1). The second, third, and fourth expressions, respectively, are (A.32) evaluated at (x_1-1,x_2), (x_1-2,x_2+1) and (x_1-2,x_2). In each case, we can relabel the starting states and repeat the argument.

Case 2 *Customer service completions.*

Suppose that policies $\pi_i(i=1,2,3)$ serve a class 1 customer whereas π_4 and π_5 serve a class 2 customer. If the first event is a class 1 service completion in Process 1-3 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+(x_2+1)\beta_2}$), after which all processes follow optimal controls, then the remaining costs in the left side of the inequality in (A.29) (with the denominator of the probability suppressed) are

$$\begin{aligned}
&\left(\mu_1\left[h_1-p\left(h_2+\beta_2K_2\right)\right]-\mu_2\left[h_2+\beta_2K_2\right]\right)t_1\\
&+\mu_1\left[\mu_2\big[pv_\alpha(x_1-1,x_2)+qv_\alpha(x_1-1,x_2-1)-pv_\alpha(x_1-1,x_2+1)-qv_\alpha(x_1-1,x_2)\big]\right.
\end{aligned}\tag{A.38}$$

If the first event is a class 2 service completion in Processes 4 and 5 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+(x_2+1)\beta_2}$), again after which optimal controls are used, then the remaining costs (with the denominator of the probability suppressed) are

$$\begin{aligned}
&\left(\mu_1\left[h_1-p\left(h_2+\beta_2K_2\right)\right]-\mu_2\left[h_2+\beta_2K_2\right]\right)t_1\\
&\quad+\mu_2\left[\mu_2\big[v_\alpha(x_1,x_2-1)-v_\alpha(x_1,x_2)\big]+\mu_1\left[v_\alpha(x_1,x_2)-pv_\alpha(x_1-1,x_2)-qv_\alpha(x_1-1,x_2-1)\right]\right.
\end{aligned}\tag{A.39}$$

Adding expressions (A.38) and (A.39) yields

$$\begin{aligned}
&\left(\mu_1+\mu_2\right)\left(\mu_1\left[h_1-p\left(h_2+\beta_2K_2\right)\right]-\mu_2\left[h_2+\beta_2K_2\right]\right)t_1\\
&+\mu_2\Big[\mu_2\big[v_\alpha(x_1,x_2-1)-v_\alpha(x_1,x_2)\big]+\mu_1\big[v_\alpha(x_1,x_2)-pv_\alpha(x_1-1,x_2+1)-qv_\alpha(x_1-1,x_2)\big]\Big].
\end{aligned}$$

The terms inside the brackets in this last expression above are implied by the expression on left side of the inequality in (A.28). That is, we may restart the argument from here.

Suppose that policy π_1 serves a class 2 customer, whereas π_3 serves a class 1 customer. In this case, let π_2 serve a class 1 customer and $\pi_4 - \pi_5$ a class 2 customer. If the first event is a class 1 service completion in Processes 2 and 3 (with probability $\frac{\mu_1}{\lambda+\mu_1+\mu_2+(x_2+1)\beta_2}$), after which all processes follow optimal controls, then the remaining costs in the left side of the inequality in (A.29) are

$$\begin{aligned}&\mu_1\left(\mu_1\left[h_1 - p\left(h_2+\beta_2 K_2\right)\right] - \mu_2\left[h_2+\beta_2 K_2\right]\right)t_1\\&\quad+\mu_1\Big[\mu_2\big[v_\alpha(x_1,x_2-1) - pv_\alpha(x_1-1,x_2+1) - qv_\alpha(x_1-1,x_2)\big]\Big].\end{aligned} \tag{A.40}$$

If the first event is a class 2 service completion in Processes 1, 4, and 5 (with probability $\frac{\mu_2}{\lambda+\mu_1+\mu_2+(x_2+1)\beta_2}$), again after which optimal controls are used, then the remaining costs are

$$\begin{aligned}&\mu_2\left(\mu_1\left[h_1 - p\left(h_2-\beta_2 K_2\right)\right] - \mu_2\left[h_2+\beta_2 K_2\right]\right)t_1\\&\quad+\mu_2\Big[\mu_2[v_\alpha(x_1,x_2-2) - v_\alpha(x_1,x_2)] + \mu_1[v_\alpha(x_1,x_2) - pv_\alpha(x_1-1,x_2) - qv_\alpha(x_1-1,x_2-1)]\Big].\end{aligned} \tag{A.41}$$

Adding (A.40) and (A.41) we get

$$\begin{aligned}&(\mu_1+\mu_2)\left(\mu_1\left[h_1 - p\left(h_2+\beta_2 K_2\right)\right] - \mu_2\left[h_2+\beta_2 K_2\right]\right)t_1\\&\quad+\mu_2\Big[\mu_2\big[v_\alpha(x_1,x_2-2) - v_\alpha(x_1,x_2-1)\big] + \mu_1\big[v_\alpha(x_1,x_2-1) - pv_\alpha(x_1-1,x_2) - qv_\alpha(x_1-1,x_2-1)\big]\Big]\\&\quad+\mu_2\Big[\mu_2\big[v_\alpha(x_1,x_2-1) - v_\alpha(x_1,x_2)\big] + \mu_1\big[v_\alpha(x_1,x_2) - pv_\alpha(x_1-1,x_2+1) - qv_\alpha(x_1-1,x_2)\big]\Big].\end{aligned}$$

The expression inside the first and second pair of brackets above are the expression on left side of inequality in (A.29) evaluated at (x_1, x_2-1) and (x_1, x_2), respectively. In both cases, we may relabel the states and continue as though we had started in these states.

Suppose that policy π_1 serve a class 1 customer whereas π_3 serves a class 2 customer. In this case, let π_2 serve a class 1 customer and π_4 and π_5 a class 2 customer. If the first event is a class 1 service completion in Processes 1 and 2 (with probability $\frac{\mu_1}{\lambda_1+\lambda_2+\mu_1+\mu_2+(x_2+1)\beta_2}$), after which all processes follow an optimal control, then the remaining costs in the left side of the inequality in (A.29) are

$$\begin{aligned}&\mu_1\left(\mu_1\left[h_1 - p\left(h_2+\beta_2 K_2\right)\right] - \mu_2\left[h_2+\beta_2 K_2\right]\right)t_1\\&\quad+\mu_1\Big[\mu_2\big[pv_\alpha(x_1-1,x_2) + qv_\alpha(x_1-1,x_2-1) - pv_\alpha(x_1-1,x_2+1) - qv_\alpha(x_1-1,x_2)\big]\\&\quad+\mu_1\big[v_\alpha(x_1,x_2) - pv_\alpha(x_1-1,x_2+1) - qv_\alpha(x_1-1,x_2)\big]\Big].\end{aligned} \tag{A.42}$$

If the first event is a class 2 service completion in Processes 3-5 (with probability $\frac{\mu_2}{\lambda_1+\lambda_2+\mu_1+\mu_2+(x_2+1)\beta_2}$),

again after which optimal controls are used, then the remaining costs are

$$\mu_2\left(\mu_1\left[h_1 - p\left(h_2+\beta_2 K_2\right)\right] - \mu_2\left[h_2+\beta_2 K_2\right]\right)t_1 \\ + \mu_2\Big[\mu_2[v_\alpha(x_1,x_2-1)-v_\alpha(x_1,x_2)] + \mu_1\big[v_\alpha(x_1,x_2-1)-pv_\alpha(x_1-1,x_2)-qv_\alpha(x_1-1,x_2-1)\big]\Big]. \tag{A.43}$$

Adding (A.42) and (A.43) we get

$$\left(\mu_1+\mu_2\right)\left(\mu_1\left[h_1 - p\left(h_2+\beta_2 K_2\right)\right] - \mu_2\left[h_2+\beta_2 K_2\right]\right)t_1 \\ +(\mu_1+\mu_2)\Big[\mu_2\big[v_\alpha(x_1,x_2-1)-v_\alpha(x_1,x_2)\big]+\mu_1\big[v_\alpha(x_1,x_2)-pv_\alpha(x_1-1,x_2+1)-qv_\alpha(x_1-1,x_2)\big]\Big].$$

The expression inside the pair of brackets above is the expression on the left side of the inequality in (A.2), and hence, we may relabel the states and continue as though we had started in these states.

In every case save one (i.e., when a class 2 abandonments that are not seen by all three process in (A.32)) we may relabel the states and continue. By doing so we can wait until class 2 abandonments that are not seen by all three process in (A.32) occurs. In particular, the latter case yields the result.

Proof of 2. The proof is given for the discounted expected cost model. The proof of the long-run average cost case is similar. Suppose $\mu_1 = \mu_2$ and that $\beta_2 \geq \beta_1 > 0$. Note that the optimality equations imply that it is optimal to prioritize a class 1 customer in state (x_1, x_2) with $i, j \geq 1$ when

$$v_\alpha(x_1,x_2-1) - pv_\alpha(x_1-1,x_2+1) - qv_\alpha(x_1-1,x_2) \geq 0. \tag{A.44}$$

We show (A.44) via a sample path argument. Fix $x_1, x_2 \geq 1$ and start three processes on the same probability space. Processes 1-3 begin in states (x_1, x_2-1), (x_1-1, x_2+1), and (x_1-1, x_2), respectively. Process 1 uses a stationary optimal policy, which we denote by π_1. In what follows, we show how to construct (potentially sub-optimal) policies for Processes 2 and 3 which we denote by π_2 and π_3, so that

$$v_\alpha^{\pi_1}(x_1,x_2-1) - pv_\alpha^{\pi_2}(x_1-1,x_2+1) - qv_\alpha^{\pi_3}(x_1-1,x_2)] \geq 0. \tag{A.45}$$

Since π_2 and π_3 are potentially sub-optimal, (A.44) follows from (A.45). In what follows, discounting is suppressed without any loss of generality.

Observe that starting from (A.45), the immediate costs incurred at the next event are $([h_1 + \beta_1 K_1 - p\left(h_2+\beta_2 K_2\right)] - [h_2+\beta_2 K_2])t_1 \geq 0$, where t_1 is the time of the next event and the inequality is due to the assumption that $h_1+\beta_1 K_1 - p\left(h_2-\beta_2 K_2\right) \geq h_2+\beta_2 K_2$. Moreover, if the relative position (as measured by the current states) of the four processes at the next event

remains the same, then we may relabel the initial states and continue from the beginning of the argument. This occurs when any of the uncontrolled events occur that are seen by all three processes. It also occurs when π_1 serves a customer class $k \in \{1, 2\}$ by letting π_2 and π_3 also serve the same customer class k customer provided there is one or more customer class k customer Processes 2 and 3, and the next event service completion. Consider now the other cases.

Case 1 *Customer abandonments*

If the first event is an abandonment in Process 2 only (with probability $\frac{\beta_2}{\lambda_1+\lambda_2+2\mu+x_1\beta_1+(x_2+1)\beta_2}$), after which all processes follow an optimal control, it follows that the remaining costs in the left side of (A.45) are

$$\beta_2 \left[\left([h_1 - p\left(h_2 + \beta_2 K_2\right)] - [h_2 + \beta_2 K_2] \right) t_1 + [v_\alpha(x_1, x_2 - 1) - v_\alpha(x_1 - 1, x_2)] \right]. \quad \text{(A.46)}$$

If the first event is an abandonment in Processes 2-3 (with probability $\frac{\beta_2}{\lambda_1+\lambda_2+2\mu+x_1\beta_1+(x_2+1)\beta_2}$), again after which optimal controls are used, then the remaining costs are

$$\beta_2 \Big[\left([h_1 - p\left(h_2 + \beta_2 K_2\right)] - [h_2 + \beta_2 K_2] \right) t_1 \\ + [v_\alpha(x_1, x_2 - 1) - p v_\alpha(x_1 - 1, x_2) - q v_\alpha(x_1 - 1, x_2 - 1)] \Big]. \quad \text{(A.47)}$$

If the first event is an abandonment in Process 1, again after which optimal controls are used, then the remaining costs are

$$\beta_1 \Big[\left([h_1 - p\left(h_2 + \beta_2 K_2\right)] - [h_2 + \beta_2 K_2] \right) t_1 \\ + [v_\alpha(x_1 - 1, x_2 - 1) - p v_\alpha(x_1 - 1, x_2 + 1) - q v_\alpha(x_1 - 1, x_2)] \Big]. \quad \text{(A.48)}$$

Adding expressions (A.46)- (A.48) and a little algebra yields

$$\begin{aligned} &(\beta_1 + \beta_1 + \beta_2) \left([h_1 - p\left(h_2 + \beta_2 K_2\right)] - [h_2 + \beta_2 K_2] \right) t_1 + (\beta_2 - \beta_1)[v_\alpha(x_1, x_2 - 1) - v_\alpha(x_1 - 1, x_2)] \\ &+ (\beta_2 - \beta_1)[v_\alpha(x_1, x_2 - 1) - p v_\alpha(x_1 - 1, x_2) - q v_\alpha(x_1 - 1, x_2 - 1)] \\ &+ 2\beta_1[v_\alpha(x_1, x_2 - 1) - p v_\alpha(x_1 - 1, x_2 + 1) - q v_\alpha(x_1 - 1, x_2)] \\ &+ p\beta_1[v_\alpha(x_1 - 1, x_2 + 1) + v_\alpha(x_1 - 1, x_2 - 1) - 2v_\alpha(x_1 - 1, x_2)]. \end{aligned}$$

To complete the proof it suffices to show that

$$v_\alpha(x_1 - 1, x_2 + 1) + v_\alpha(x_1 - 1, x_2 - 1) - 2v_\alpha(x_1 - 1, x_2) \geq 0. \quad \text{(A.49)}$$

The proof of A.49 is the same as the proof of A.33 (with $\mu_2(h_2+\beta_2 K_2)$ replaced with $h_2+\beta_2 K_2$ and $\mu_1(h_1 - p(h_2+\beta_2 K_2)$ with $h_1 + \beta_1 K_1 - p(h_2+\beta_2 K_2)$) with one exception: when all three processes see a class 1 abandonment. In this latter case, we may simply relabel the states and continue from the beginning of the argument.

Case 2 *Service completions.*

The proof of this case is the same as the proof of Case 2 above with $\mu_2(h_2 + \beta_2 K_2)$ replaced with $h_2 + \beta_2 K_2$ and $\mu_1(h_1 - p(h_2 + \beta_2 K_2)$ with $h_1 + \beta_1 K_1 - p(h_2 + \beta_2 K_2)$.

A.4 Numerical simulation of single-server model

Parameters are summarized in Table 1 below for the simulation. Parameters were chosen to satisfy the four different sets of conditions under which we know prioritizing one phase is optimal. For the single-server model, exponential distributions are assumed for inter-arrival times, service times, and abandonment times. We then systemically vary parameters for a single-server model to capture situations when the optimal policy remains elusive. We repeat these parameters for the simulation of the multi-server model, with minor adjustments to parameters and distributions, which we describe below. It is without any loss of generality that we can fix one cost, and so, the abandonment cost K_2 at phase 2 is fixed at 1. As in the simulation in the main text, parameters are given a time unit of hours, and for each set of parameters examined, we simulated the system over a simulated time horizon of 5 years after a 5 year warm-up period and then performed 50 replications of this simulation. Average costs were averaged of the time horizon and then over the replications.

		Scenario				
Parameters	**Description**	**1**	**2**	**3**	**4**	**General**
λ_1	Arrival rate at 1	[1.5, 4.5]	[1.5, 4.5]	[1.5, 4.5]	[1.5, 4.5]	3
λ_2	Arrival rate at 2	0	0	0	0	[0, 1]
μ_1	Service rate at 1	8	8	10	8	[4, 12]
μ_2	Service rate at 2	10	8	8	8	[4, 12]
β_1	Abandonment rate at 1	0.5	10	0	0.25	[0.1, 3]
β_2	Abandonment rate at 2	0	0.5	0.5	0.5	[0.1, 3]
p	Joining probability	1	1	1	1	[0.25, 1]
h_1	Holding cost rate at 1	1	1.25	2	1.75	[0.1, 3]
h_2	Holding cost rate at 2	1	0.75	0.5	0.5	[0.1, 3]
K_1	Abandonment cost at 1	2	0.1	2	2	[0.1, 3]
K_2	Abandonment cost at 2	1	1	1	1	1

Table 1: Parameters used for the simulation.

Scenario 1 (P2 optimal). Parameters are selected to satisfy $\beta_2 = 0$ and

$$\mu_1[h_1 + \beta_1 K_1 - ph_2] \leq \mu_2 h_2,$$

which from Theorem 4.5, ensures that the policy that prioritizing phase 2 (i.e. P2) is optimal. Arrival rate λ_1 is varied between 1.5 and 4.5 to explore different traffic intensities.

Scenario 2 (P2 optimal). Parameters are selected to satisfy $\mu_1 = \mu_2 := \mu$,

$$\beta_1 - \beta_2 - \mu \geq 0,$$

and

$$h_1 + \beta_1 K_1 - p(h_2 + \beta_2 K_2) \leq h_2 + \beta_2 K_2.$$

which from Theorem 4.5, ensures P2 is optimal. Arrival rate λ_1 is varied between 1.5 and 4.5.

Scenario 3 (P1 optimal). Parameters are selected to satisfy $\beta_1 = 0$ and

$$\mu_2[h_2 + \beta_2 K_2] \leq \mu_1[h_1 - p(h_2 + \beta_2 K_2)],$$

which from Theorem 4.6, ensures that the policy that prioritizing phase 1 (i.e. P1) is optimal. Arrival rate λ_1 is varied between 1.5 and 4.5.

Scenario 4 (P1 optimal). Parameters are selected to satisfy $\mu_1 = \mu_2$, $\beta_2 \geq \beta_1$, and

$$h_2 + \beta_2 K_2 \leq h_1 + \beta_1 K_1 - p(h_2 + \beta_2 K_2)],$$

which from Theorem 4.6, ensures P1 is optimal. Arrival rate λ_1 is varied between 1.5 and 4.

General single-server scenario. Parameters are selected to explore a variety of situations in which the optimal policy is unknown. Given the importance of the classic c-μ inequality and its extended version, parameters were selected and varied to both satisfy and violate these situations. We first explored parameter space using a full factorial design of 6 parameters (λ_2, μ_1, μ_2, β_1, β_2, and p); each parameter had two levels corresponding to the lowest and highest value in the parameter range listed in Table 1. For each of these 64 parameters, we then sampled 10,000 sets of costs (h_1, h_2, K_1) uniformly from the parameter range listed in Table 1. We then systematically varied parameters while keeping fixed (unless otherwise specified) $\lambda_1 = 3$, $\lambda_2 = 0$, $\mu_1 = \mu_2 = 8$, $p = \beta_1 = \beta_2 = h_1 = h_2 = K_2 = 1$, and $K_1 = 2$. Service rates μ_1 and μ_2 were systematically varied, followed by abandonment rates β_1 and β_2, holding cost rates h_1 and h_2, and arrival rate λ_2 and probability of transfer p.

A.4.1 Scenarios 1–4 for the single-server model

For the single-server model, we first benchmarked the heuristic policies in four scenarios when the optimal policy is known (Figure 1). Table 2 shows how far average cost for each policy is away from optimal. In Scenario 1, the optimal policy (i.e. policy P2) performs increasingly better as the traffic intensity increases. For example, policy P1 is only 6% away from optimal when $\lambda_1 = 1.5$, but 222% away when $\lambda_1 = 4.5$. This poor performance is due to no abandonments in phase 2 ($\beta_2 = 0$), causing customers to aggregate at phase 2. A threshold policy can be found to perform within 10% of the optimal policy: average costs for the policy P1(5) is 5% away from optimal when $\lambda_1 = 1.5$ and 9% away from optimal when $\lambda_1 = 4.5$.

Policy P2 is also optimal in Scenario 2, but does not dramatically outperform other policies. In fact, the threshold policy P2(5) is less than 1% away from optimal for all traffic intensities.

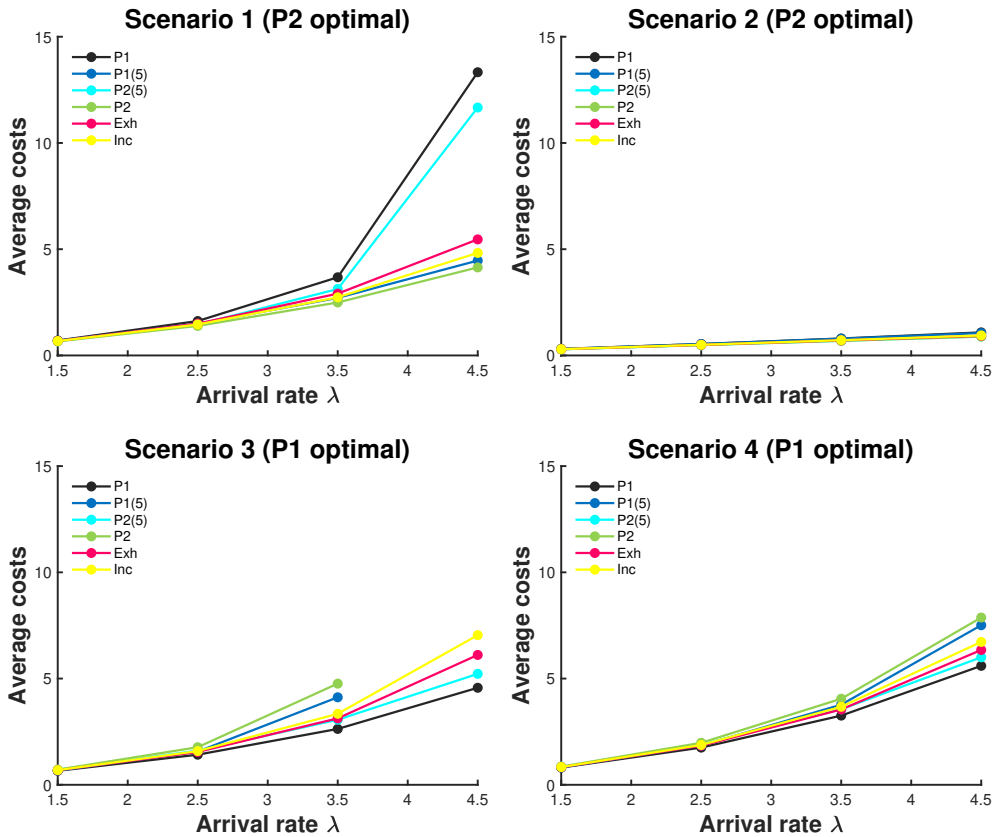

Figure 1: Benchmarking policies for the single-server model when the optimal policy is known with respect to average costs. To help visualization, average costs that exceed 15 are not shown.

Similar performance in Scenario 2 can be attributed to the large abandonment rates at phase 1 ($\beta_1 = 10$), causing few customers to aggregate at phase 1 and leaving the server to work at phase 2 even when they prioritize phase 1.

Many trends reverse when moving from Scenarios 1–2 to Scenarios 3–4. Policy P1 is now optimal, and policy P2 yields poor performance, especially when traffic is high. In Scenario 3, for instance, policy P2 is only 8% away from optimal when $\lambda_1 = 1.5$ but 1478% away when $\lambda_2 = 4.5$ (which is why it was not plotted in Figure 1). Poor performance can again be attributed to no abandonments ($\beta_1 = 0$), causing customers to aggregate at phase 1 and leaving policies that prioritize phase 2 to neglect these patients. In Scenario 4, however, performance is closer: policy P2 is 5% away from optimal when $\lambda_1 = 1.5$ in Scenario 4 and 41% away when $\lambda_2 = 4.5$. Meanwhile, the threshold policy P2(5) is close to optimal in both Scenario 3 and 4: at worst, 16% away from optimal in Scenario 3 and 9% away in Scenario 4.

	Policy					
λ_1	**P1**	**P2**	**P1(5)**	**P2(5)**	**Exh**	**Inc**
		Scenario 1 (P2 optimal)				
1.5	6%	0%	5%	0%	3%	2%
2.5	17%	0%	8%	5%	8%	5%
3.5	48%	0%	9%	25%	17%	10%
4.5	222%	0%	8%	182%	32%	16%
		Scenario 2 (P2 optimal)				
1.5	6%	0%	6%	0%	1%	2%
2.5	11%	0%	10%	0%	2%	3%
3.5	16%	0%	15%	0%	3%	4%
4.5	22%	0%	19%	0%	4%	6%
		Scenario 3 (P1 optimal)				
1.5	0%	8%	1%	7%	4%	5%
2.5	0%	25%	10%	13%	9%	12%
3.5	0%	81%	57%	16%	19%	27%
4.5	0%	1478%	1420%	14%	34%	54%
		Scenario 4 (P1 optimal)				
1.5	0%	5%	1%	5%	2%	3%
2.5	0%	13%	5%	8%	5%	7%
3.5	0%	24%	16%	9%	9%	13%
4.5	0%	41%	34%	7%	13%	20%

Table 2: Percent away from optimal average costs in four simulation scenarios of a single-server model, where the optimal policy is known.

A.4.2 General scenario for single-server model

We varied service rates μ_1 and μ_2 (Figure 2). We find that the extended c-μ inequality provides a good guide for deciding which policy to prioritize. When the inequality is satisfied, policy P2 performs the best out of the policies considered, whereas when the inequality is not satisfied, policy P1 performs best. By contrast, the classic c-μ equality does not enjoy the same insight. That is, even when $\mu_1 h_1 > \mu_2 h_2$, policy P2 is better than policy P1. Again, we find that the threshold policies perform well in that the threshold policy P1(5) performs at worst 4% away from the best priority rule.

When varying holding cost rates h_1 and h_2 (Figure 3), we again find that the extended c-μ inequality provides a good guide for deciding which policy to prioritize, with policy P2 performing well when this inequality is satisfied and policy P1 otherwise. The classic c-μ

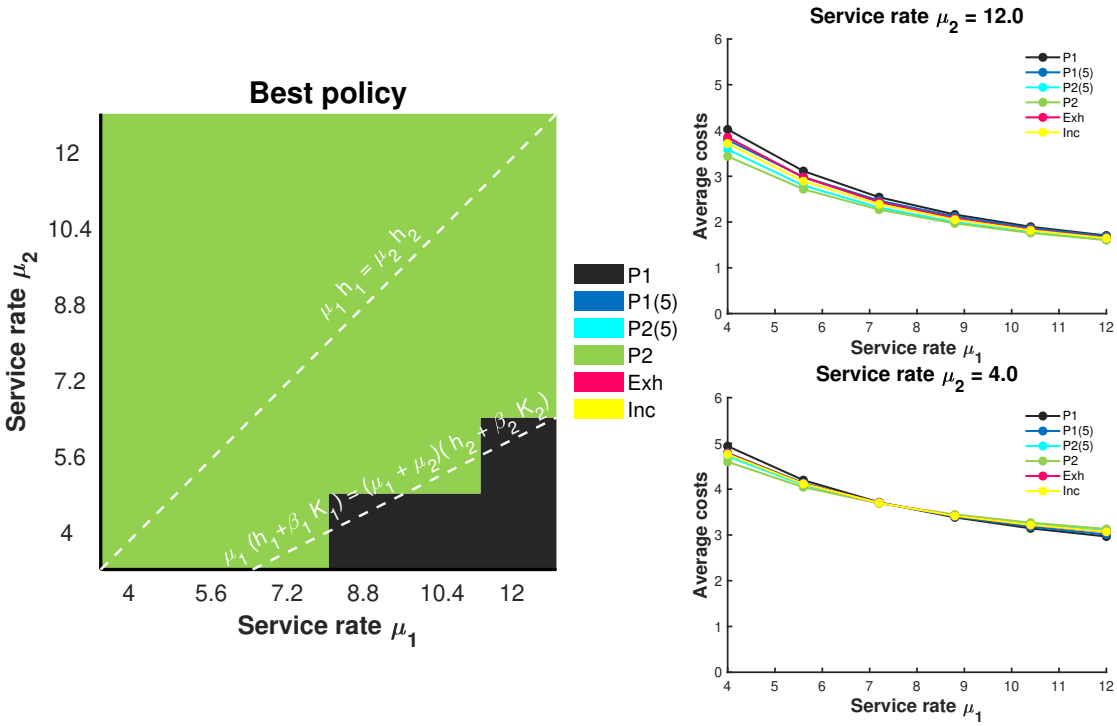

Figure 2: Average cost comparison for single-server model when the optimal policy is unknown and service rates μ_1 and μ_2 are varied.

inequality does not provide similar insight. We do find one case when a threshold policy outperforms both priority rules, but the improvement is negligible ($< 1\%$).

When varying abandonment rates β_1 and β_2 (Figure 4), we find that policy P2 can perform better than P1 even when the extended c-μ inequality is satisfied. This occurs when the abandonment rate β_2 is low, reinforcing what we found in Scenario 1, i.e. that neglecting phase 2 when there are few abandonments at 2 can yield poor performance. In addition, we find several cases when the threshold policy P1(5) performs better than the other heuristic policies, albeit they are all close ($\leq 4\%$ away).

The final parameters varied were the joining probability p and arrival rate λ_2 (Figure 5). Once again, we find that when the extended c-μ inequality is satisfied, policy P2 performs best, whereas when the inequality is not satisfied, policy P1 or the threshold policy P1(5) perform best.

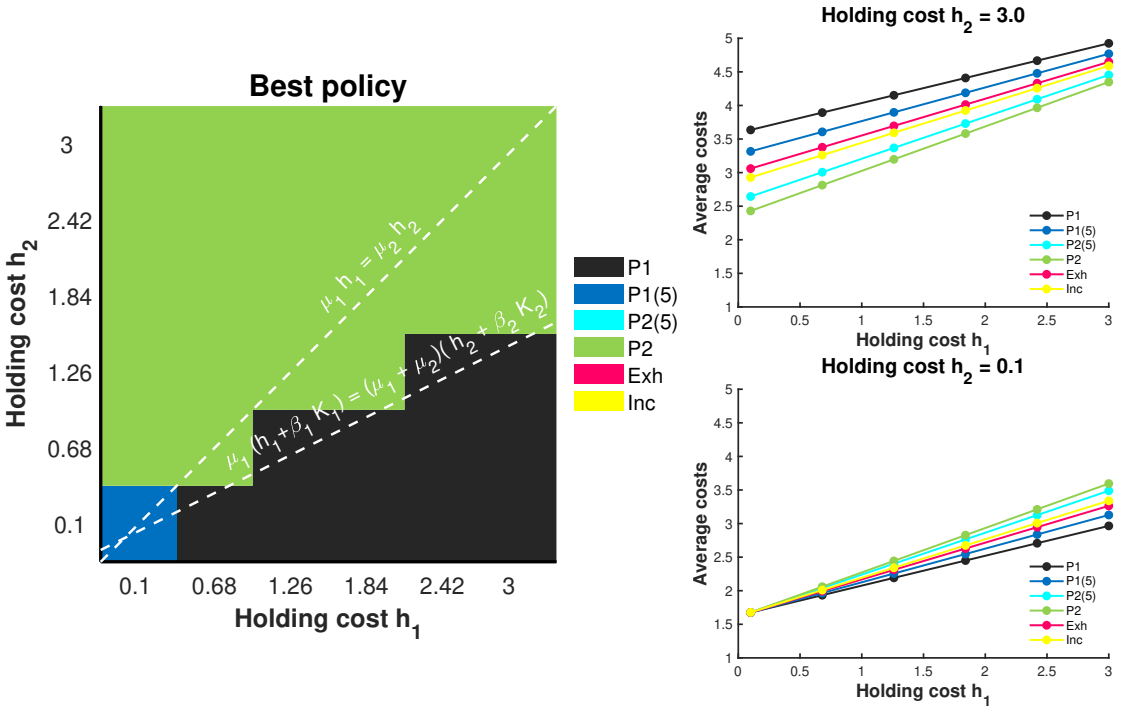

Figure 3: Average cost comparison for single-server model when the optimal policy is unknown and holding cost rates h_1 and h_2 are varied.

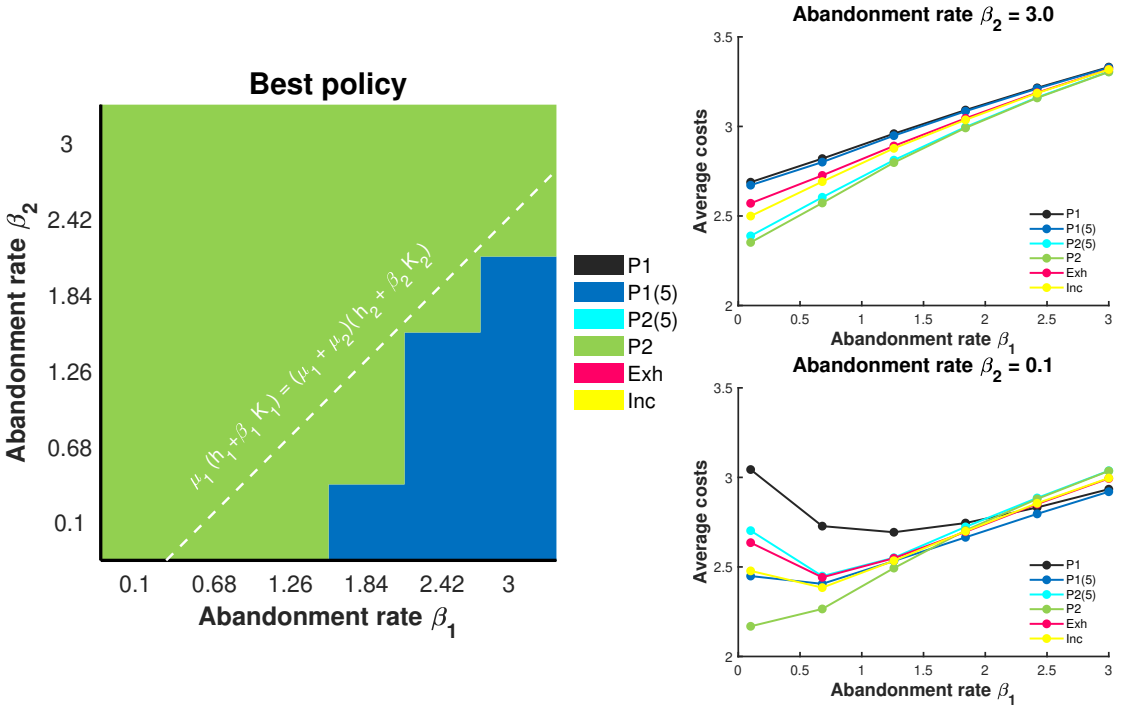

Figure 4: Average cost comparison for single-server model when the optimal policy is unknown and abandonment rates β_1 and β_2 are varied.

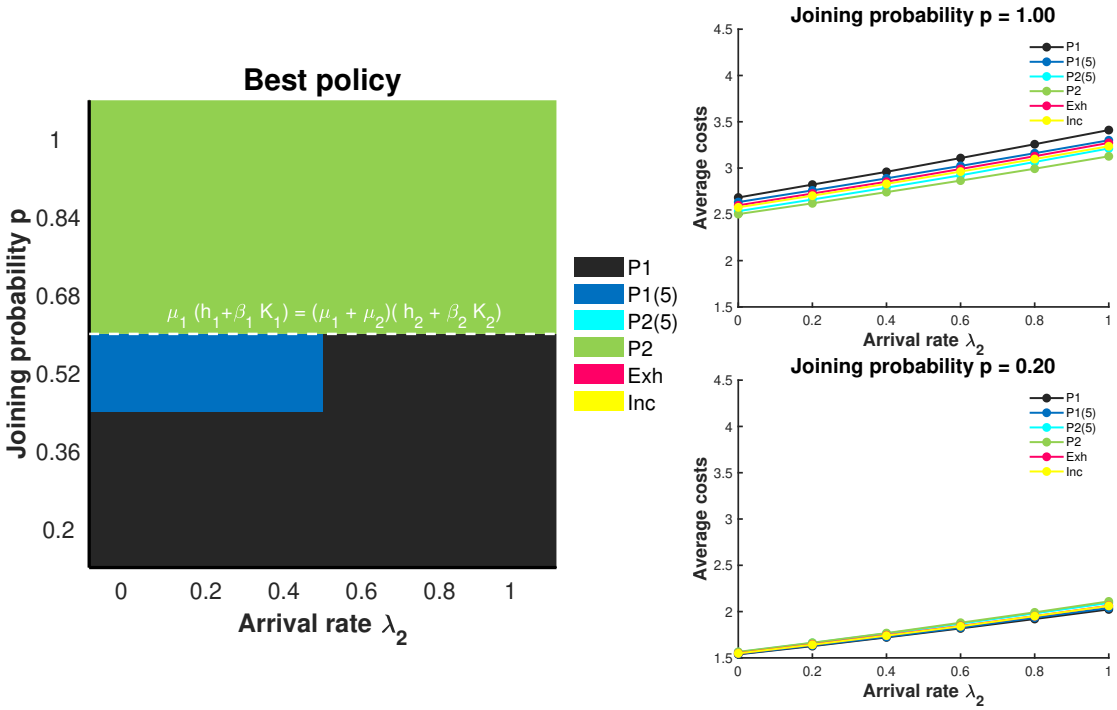

Figure 5: Average cost comparison for single-server model when the optimal policy is unknown and joining probability p and arrival rate λ_2 are varied.

A.4.3 Results from factorial design

Tables below report the results from the 64,000 samples of parameters space. Policy performance is averaged over 10,000 samples of costs (h_1, h_2, and K_1) for each set of the remaining parameters (μ_1, μ_2, p, λ_2, β_1, and β_2).

				Policy							
β_1	β_2	p	λ_2	P1	P2	P1(5)	P2(5)	Exh	Inc	c-μ	Ext. c-μ
0.1	0.1	0.25	0	36.7	62.3	0.3	0	0.7	0	86.4	99
0.1	0.1	0.25	1	37.6	62.4	0	0	0	0	87.3	99.7
0.1	0.1	1	0	20.4	72.4	7.2	0	0	0	70	92.6
0.1	0.1	1	1	17.6	75.4	4.3	2.6	0	0.1	67.3	93
0.1	3	0.25	0	41.3	21.4	0	19.4	0	17.9	55.9	21.4
0.1	3	0.25	1	58.2	14.9	0	20.5	0	6.5	58.1	14.9
0.1	3	1	0	53.3	16.8	0	14.6	0	15.3	57.9	16.8
0.1	3	1	1	62.3	13.6	0	11.6	0	12.5	58.2	13.6
3	0.1	0.25	0	34.1	50.4	15	0	0.5	0	65.3	42.6
3	0.1	0.25	1	17.3	65.2	17.5	0	0	0	59.8	25.8
3	0.1	1	0	5.5	83	11.5	0	0	0	53.7	26.3
3	0.1	1	1	1	89.4	9.6	0	0	0	50.1	21.7
3	3	0.25	0	12.8	84.1	0	0	2.7	0.4	58.8	96.9
3	3	0.25	1	16.1	83.9	0	0	0	0	60.7	98.4
3	3	1	0	0	100	0	0	0	0	49.6	100
3	3	1	1	0	100	0	0	0	0	49.6	100

Table 3: Percent samples of costs that policy yields lowest average costs when $\mu_1 = \mu_2 = 4$.

				Policy							
β_1	β_2	p	λ_2	P1	P2	P1(5)	P2(5)	Exh	Inc	c-μ	Ext. c-μ
0.1	0.1	0.25	0	7.8	90.1	0	0.9	1.3	0	92.9	97.8
0.1	0.1	0.25	1	9.7	90.3	0	0	0	0	94.8	99.2
0.1	0.1	1	0	6.2	93.7	0	0	0.1	0	91.3	99.6
0.1	0.1	1	1	6.2	91.4	2.5	0	0	0	91.2	97.3
0.1	3	0.25	0	0	87.5	0	12.5	0	0	79.8	87.5
0.1	3	0.25	1	0	75.8	0	24.2	0	0	70.4	75.8
0.1	3	1	0	0	75	0	24.8	0	0.2	69.7	75
0.1	3	1	1	0	65.2	0	34.6	0	0.2	61.5	65.2
3	0.1	0.25	0	19.8	71.8	8.4	0	0	0	81.5	64.8
3	0.1	0.25	1	16.9	73.1	10.1	0	0	0	81.8	61.9
3	0.1	1	0	9	81.2	9.8	0	0	0	84.8	65.6
3	0.1	1	1	7.1	83.3	9.7	0	0	0	84.9	63.7
3	3	0.25	0	0	100	0	0	0	0	85.1	100
3	3	0.25	1	0	100	0	0	0	0	85.1	100
3	3	1	0	0	100	0	0	0	0	85.1	100
3	3	1	1	0	100	0	0	0	0	85.1	100

Table 4: Percent samples of costs that policy yields lowest average costs when $\mu_1 = 4$ and $\mu_2 = 12$.

				Policy							
β_1	β_2	p	λ_2	P1	P2	P1(5)	P2(5)	Exh	Inc	c-μ	Ext. c-μ
0.1	0.1	0.25	0	73.7	26.3	0	0	0	0	87.7	99.7
0.1	0.1	0.25	1	73	25.9	0	1.1	0	0	87	98.9
0.1	0.1	1	0	34.2	65.8	0	0	0	0	48.2	99.9
0.1	0.1	1	1	34.1	64.2	1.7	0	0	0	48.1	98.2
0.1	3	0.25	0	37.7	37.3	0	16	0	8.9	51.8	37.3
0.1	3	0.25	1	47.4	30.7	0	18.4	0	3.4	61.5	30.7
0.1	3	1	0	22.7	34.6	0	26.7	0	16	36.8	34.6
0.1	3	1	1	35.8	27.3	0	28.6	0	8.4	49.8	27.3
3	0.1	0.25	0	78.3	12.7	8.6	0	0.1	0.3	74.6	79.4
3	0.1	0.25	1	55.9	19.3	20.8	0	1.4	2.6	57.9	57
3	0.1	1	0	10.2	70.7	17.8	0	1.3	0	23.5	19.8
3	0.1	1	1	3.5	79.7	15.9	0	0.9	0	17.3	13.2
3	3	0.25	0	68.2	31.5	0	0.3	0	0	67.2	97.3
3	3	0.25	1	61.2	30.5	6.2	1.1	0	0.9	61.2	90.3
3	3	1	0	10.3	89.3	0.3	0	0.2	0	24.3	98.8
3	3	1	1	9.3	89.2	1.3	0.2	0	0	23.3	98.5

Table 5: Percent samples of costs that policy yields lowest average costs when $\mu_1 = 12$ and $\mu_2 = 4$.

				Policy							
β_1	β_2	p	λ_2	P1	P2	P1(5)	P2(5)	Exh	Inc	c-μ	Ext. c-μ
0.1	0.1	0.25	0	73.7	26.3	0	0	0	0	87.7	99.7
0.1	0.1	0.25	1	73	25.9	0	1.1	0	0	87	98.9
0.1	0.1	1	0	34.2	65.8	0	0	0	0	48.2	99.9
0.1	0.1	0.25	0	37.2	62.2	0.1	0.6	0	0	86.8	99.4
0.1	0.1	0.25	1	36.7	62.8	0	0	0	0.5	86.3	99.4
0.1	0.1	1	0	20.5	79.5	0	0	0	0	70.1	99.7
0.1	0.1	1	1	20.3	79.6	0	0	0	0	70	99.9
0.1	3	0.25	0	0	100	0	0	0	0	49.6	100
0.1	3	0.25	1	0	99.8	0	0.2	0	0	49.6	99.8
0.1	3	1	0	0	100	0	0	0	0	49.6	100
0.1	3	1	1	0	100	0	0	0	0	49.6	100
3	0.1	0.25	0	71.7	18.1	10.2	0	0	0	60.1	80.2
3	0.1	0.25	1	63.9	20.7	15.4	0	0	0	59	72.3
3	0.1	1	0	32.5	50.3	17.1	0	0	0	64.2	53.3
3	0.1	1	1	28.8	53.8	17.4	0	0	0	63.6	49.5
3	3	0.25	0	7.6	82.7	9.7	0	0	0	54	90.3
3	3	0.25	1	12.8	82	2.8	1.5	0.9	0	57.6	94.8
3	3	1	0	0	100	0	0	0	0	49.6	100
3	3	1	1	0	100	0	0	0	0	49.7	100

Table 6: Percent samples of costs that policy yields lowest average costs when $\mu_1 = 12$ and $\mu_2 = 12$.

A.5 Additional simulations for multi-server model

Scenarios 1–4 capture a situation when the optimal policy is known for the single-server model, *but not the multi-server model*. Parameters are the same as the single-server model with the following exceptions. First, we fixed the number of workers to be 3. Second, abandonment and service times were modeled as Gamma random variables as opposed to exponential random variables. Gamma shape parameters ranged from $1/2$ or 3, yielding random times that have standard deviations larger than their mean and smaller than their mean, complementing exponential random times, which have standard deviations equal to their mean. Coefficient of variation (cv) were respectively 1.4 and 0.6 for the two shape values. Last, parameters μ_1, μ_2, β_1, and β_2 refer to *average* rates, which meant that the rate parameters for the gamma distributions needed to be $\mu_1, \mu_2, \beta_1, \beta_2$ scaled by the corresponding shape parameter.

Figure 6 compares average costs for the various policies when the cv is 0.6. The policy known to be optimal for the single-server model (i.e. P2 for Scenarios 1–2 and P1 for Scenarios 3–4) performs best among the policies. Further for each case, there is at least one threshold policy that is within 2% of the best priority rule for all arrival rates. Last, neglecting any phase of service that has no abandonments can have severe consequences: P1 can be over 200,000% away from the best policy in Scenario 1 and P2 can be over 200,000% away from the best policy in Scenario 3.

Figure 7 compares average costs for the various policies when the cv is 1.4, which captures a situation when the random times have a standard deviation that is greater than their mean. Surprisingly, the policy known to be optimal for the single-server model (i.e. P2 for Scenarios 1–2 and P1 for Scenarios 3–4) performs best among the policies in most, but not all, cases. Yet, it is still always within 1% of the best policy. For each case, there is at least one threshold policies that is within 5% of the best priority rule for all arrival rates. Last, neglecting any phase of service that has no abandonments can have consequences: P1 can be as far as 61% away from the best policy in Scenario 1 and P2 can be as far as 200,000% away from the best policy in Scenario 3.

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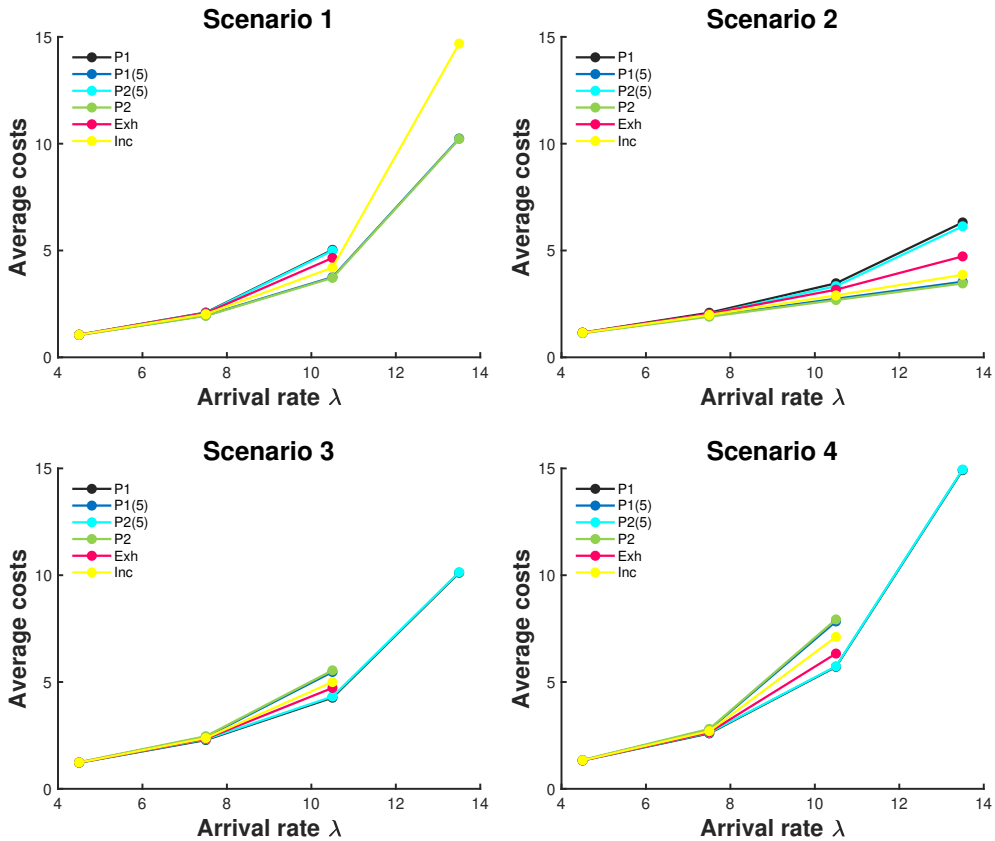

Figure 6: Average costs for policies for the multi-server model and when the cv is 0.6. The optimal policy was known for the corresponding scenarios in the single-server model. To help visualization, average costs that exceed 15 are not shown.

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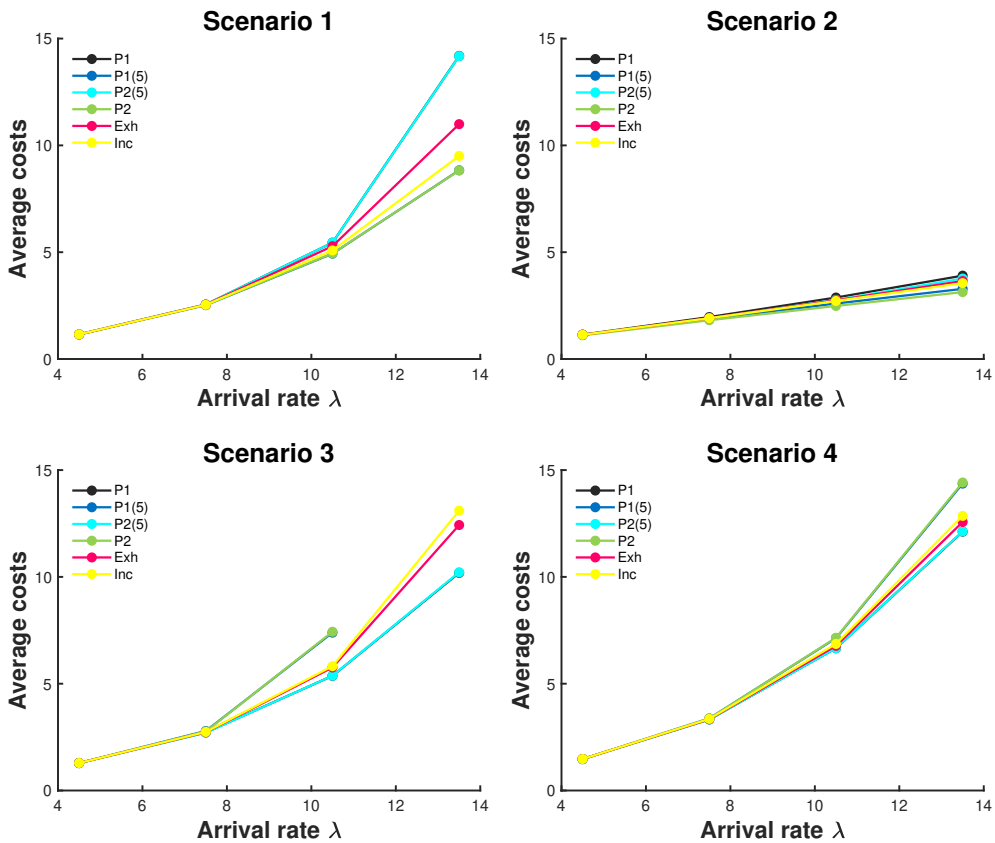

Figure 7: Average costs for policies for the multi-server model and when the cv is 1.4. The optimal policy was known for the corresponding scenarios in the single-server model. To help visualization, average costs that exceed 15 are not shown.