

Online Supplementary Material on “Higher-Order Approximation of IV Estimators with Invalid Instruments”

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Abstract This Online Supplementary Material contains the proofs of Propositions in the main text. Supplementary material also includes auxiliary MSE approximation results under different asymptotics and additional simulation results.

S1. PROOFS OF RESULTS

Section S1 contains the proofs of propositions and lemmas in the main paper including higher-order mean square error (MSE) approximation of 2SLS, LIML FULL, B2SLS, JIVE2, HLIM, and HFUL estimators. Section S1.1 and Section S1.2 contain the MSE approximations under $N^{-\gamma}$ locally invalid instrument specifications for $\gamma = 1/2$ and $\gamma > 1/2$, respectively. Section S1.3 includes the asymptotic optimality of instrument selection criteria based on Donald and Newey (2001) under $\gamma > 1/2$ and asymptotic unbiasedness property of invalidity-robust selection criteria proposed in the main paper.

In the following proofs, each IV estimator has a representation $\sqrt{N}(\hat{\delta}(K) - \delta_0) = \hat{H}^{-1}\hat{h} + \hat{H}^{-1}\hat{h}_g$. Define $h = f'v/\sqrt{N}$, $H = f'f/N$, $H_g = f'g/N$, and $\rho_{K,N} = \text{tr}(G + L(K))$. Let $\sum_i = \sum_{i=1}^N$, $\sum_{i \neq j} = \sum_{i=1}^N \sum_{j \neq i}$, and LLN denotes (weak) law of large numbers, and CLT denotes the Lindeberg-Levy central limit theorem. $o_p(\cdot)$ and $O_p(\cdot)$ denote the usual stochastic order symbols, convergence in probability, and bounded in probability, respectively. $\|A\| = \sqrt{\text{tr}(A'A)}$ denotes the Euclidean norm.

S1.1. Proofs of Lemmas and Propositions 3.1-3.5

Lemma S1.1 will be used for MSE derivations of IV estimators discussed in the main text, and this requires modifications of the lemma in Donald and Newey (2001) due to locally invalid instruments.

LEMMA S1.1. *If there is a decomposition $\hat{h} = h + T^h + Z^h$, $\hat{H} = H + T^H + Z^H$, $\hat{h}_g = H_g + T^g + Z^g$, and*

$$\begin{aligned} (h + T^h)(h + T^h)' - hh'H^{-1}T^{H'} - T^HH^{-1}hh' &= \hat{A}_1(K) + Z^{A_1}(K), \\ (H_g + T^g)(H_g + T^g)' - H_gH'_gH^{-1}T^{H'} - T^HH^{-1}H_gH'_g &= \hat{A}_2(K) + Z^{A_2}(K), \\ (h + T^h)(H_g + T^g)' - hH'_gH^{-1}T^{H'} - T^HH^{-1}hH'_g &= \hat{A}_3(K) + Z^{A_3}(K), \end{aligned}$$

such that $T^h = o_p(1)$, $h = O_p(1)$, $H_g = O_p(1)$, $T^g = o_p(1)$, $H = O_p(1)$, $T^H = o_p(1)$, the determinant of H is bounded away from zero with probability 1, $\rho_{K,N} = \text{tr}(G + L(K))$, and $\rho_{K,N} = o_p(1)$,

$$\begin{aligned}\|T^H\|^2 &= o_p(\rho_{K,N}), \|T^h\|\|T^H\| = o_p(\rho_{K,N}), \|Z^h\| = o_p(\rho_{K,N}), \|Z^H\| = o_p(\rho_{K,N}), \\ \|Z^g\| &= o_p(\rho_{K,N}), \|T^g\|\|T^H\| = o_p(\rho_{K,N}), Z^{A_i}(K) = o_p(\rho_{K,N}) \quad \text{for all } i = 1, 2, 3, \\ \mathbb{E}[\hat{A}_1(K) + \hat{A}_2(K) + \hat{A}_3(K) + \hat{A}_3(K)'|X] &= H\Phi H + H(G + L(K))H + o_p(\rho_{K,N}),\end{aligned}$$

then

$$\begin{aligned}N(\hat{\delta}(K) - \delta_0)(\hat{\delta}(K) - \delta_0)' &= \hat{Q}(K) + \hat{r}(K), \\ \mathbb{E}[\hat{Q}(K)|X] &= \Phi + G + L(K) + T(K), \\ [\hat{r}(K) + T(K)]/\text{tr}(G + L(K)) &= o_p(1), \quad \text{as } K \rightarrow \infty, N \rightarrow \infty.\end{aligned}\tag{S1.1}$$

Proof: First, we observe

$$\begin{aligned}\hat{H}^{-1}\hat{h} + \hat{H}^{-1}\hat{h}_g &= H^{-1}(\hat{h} - (\hat{H} - H)H^{-1}h) + \hat{Z} + H^{-1}(\hat{h}_g - (\hat{H} - H)H^{-1}H_g) + \hat{Z}_g, \\ \hat{Z} &= H^{-1}(H - \hat{H})\hat{H}^{-1}(H - \hat{H})H^{-1}h + \hat{H}^{-1}(H - \hat{H})H^{-1}(\hat{h} - h), \\ \hat{Z}_g &= H^{-1}(H - \hat{H})\hat{H}^{-1}(H - \hat{H})H^{-1}H_g + \hat{H}^{-1}(H - \hat{H})H^{-1}(\hat{h}_g - H_g).\end{aligned}$$

Note that, $\hat{H} - H = T^H + Z^H$, $\|T^H\|^2 = o_p(\rho_{K,N})$, $\|Z^H\|^2 = o_p(\rho_{K,N})$, $\|T^g\|\|T^H\| = o_p(\rho_{K,N})$, $\|Z^g\| = o_p(\rho_{K,N})$, and $\hat{h}_g = H_g + T^g + Z^g = O_p(1)$. Thus, $\|\hat{H} - H\|^2 \leq 2(\|T^H\|^2 + \|Z^H\|^2) = o_p(\rho_{K,N})$, and $\|\hat{h}_g - H_g\|\|H - \hat{H}\| \leq \|T^g\|\|T^H\| + \|Z^g\|\|T^H\| + \|T^g\|\|Z^H\| + \|Z^g\|\|Z^H\| = o_p(\rho_{K,N})$. Also, H is nonsingular with probability approaching 1 (w.p.a.1.), so that $H^{-1} = O_p(1)$. Moreover, $\hat{H} = H + o_p(1)$ and $\hat{H}^{-1} = H^{-1} + o_p(1) = O_p(1)$. Thus,

$$\|\hat{Z}_g\| \leq \|H^{-1}\|\|H - \hat{H}\|^2\|\hat{H}^{-1}\|\|H^{-1}H_g\| + \|\hat{h}_g - H_g\|\|H - \hat{H}\|\|H^{-1}\|\|\hat{H}^{-1}\| = o_p(\rho_{K,N}).$$

Similarly, we can show that $\|\hat{Z}\| = o_p(\rho_{K,N})$. Define $\tilde{\tau}_g = H_g + T^g - T^H H^{-1}H_g$ and we obtain

$$\hat{H}^{-1}\hat{h}_g = H^{-1}\tilde{\tau}_g + o_p(\rho_{K,N})$$

by using $\|\hat{Z}_g\| = o_p(\rho_{K,N})$, $\|Z^g\| = o_p(\rho_{K,N})$, $\|Z^H\| = o_p(\rho_{K,N})$. Similarly, for $\hat{h} = h + T^h + o_p(\rho_{K,N}) = O_p(1)$, we obtain $\hat{H}^{-1}\hat{h} = H^{-1}\tilde{\tau} + o_p(\rho_{K,N})$ with $\tilde{\tau} = h + T^h - T^H H^{-1}h$ by $\|\hat{Z}\| = o_p(\rho_{K,N})$, $\|Z^h\| = o_p(\rho_{K,N})$, and $\|Z^H\| = o_p(\rho_{K,N})$.

Then, we have

$$\begin{aligned}\tilde{\tau}\tilde{\tau}' &= \hat{A}_1(K) + Z^{A_1}(K) - T^h h' H^{-1} T^{H'} - T^H H^{-1} h T^{h'} + T^H H^{-1} h h' H^{-1} T^{H'} \\ &= \hat{A}_1(K) + o_p(\rho_{K,N})\end{aligned}$$

by $\|T^h\|\|T^H\| = o_p(\rho_{K,N})$, $\|T^H\|^2 = o_p(\rho_{K,N})$, and $\|Z^{A_1}(K)\| = o_p(\rho_{K,N})$. Also,

$$\begin{aligned}\tilde{\tau}_g\tilde{\tau}'_g &= \hat{A}_2(K) + Z^{A_2}(K) - T^g H'_g H^{-1} T^{H'} - T^H H^{-1} H_g T^{g'} + T^H H^{-1} H_g H'_g H^{-1} T^{H'} \\ &= \hat{A}_2(K) + o_p(\rho_{K,N})\end{aligned}$$

by using $\|T^g\|\|T^H\| = o_p(\rho_{K,N})$, $\|T^H\|^2 = o_p(\rho_{K,N})$, and $\|Z^{A_2}(K)\| = o_p(\rho_{K,N})$.

For the cross term, we obtain

$$\begin{aligned}\tilde{\tau}\tilde{\tau}'_g &= \hat{A}_3(K) + Z^{A_3}(K) - T^h H'_g H^{-1} T^{H'} - T^H H^{-1} h T^{g'} + T^H H^{-1} h H'_g H^{-1} T^{H'} \\ &= \hat{A}_3(K) + o_p(\rho_{K,N})\end{aligned}$$

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by $\|T^h\|\|T^H\| = o_p(\rho_{K,N})$, $\|T^g\|\|T^H\| = o_p(\rho_{K,N})$, $\|T^H\|^2 = o_p(\rho_{K,N})$ and $\|Z^{A_3}(K)\| = o_p(\rho_{K,N})$. Since $\sqrt{N}(\hat{\delta}(K) - \delta_0) = H^{-1}\tilde{\tau} + H^{-1}\tilde{\tau}_g + o_p(\rho_{K,N})$, it follows that

$$N(\hat{\delta}(K) - \delta_0)(\hat{\delta}(K) - \delta_0)' = H^{-1}(\hat{A}_1(K) + \hat{A}_2(K) + \hat{A}_3(K) + \hat{A}_3(K)')H^{-1} + o_p(\rho_{K,N}).$$

Then, the desired conclusion directly follows from the assumption in the lemma. \square

Next we provide useful lemmas for the proofs of Propositions 3.1-3.5. Define $e_f(K) = f'(I - P^K)f/N$, $\Delta(K) = \text{tr}(e_f(K))$, $e_g(K) = g'(I - P^K)g/N$, and $\Delta_g(K) = \text{tr}(e_g(K))$.

LEMMA S1.2. (*Donald and Newey (2001) Lemma A.2, A.3*) If Assumptions 2.1, 2.2 and 2.3 are satisfied, then

- (a) $\text{tr}(P^K) = K$; (b) $\sum_i(P_{ii}^K)^2 = o_p(K)$; (c) $\sum_{i \neq j}P_{ii}^K P_{jj}^K = K^2 + o_p(K)$; (d) $\sum_{i \neq j}P_{ij}^K P_{ij}^K = K + o_p(K)$;
- (e) $h = f'v/\sqrt{N} = O_p(1)$, $H = f'f/N = O_p(1)$;
- (f) $\Delta(K) = o_p(1)$;
- (g) $f'(I - P^K)v/\sqrt{N} = O_p(\Delta(K)^{1/2})$;
- (h) $u'P^Kv = O_p(K)$;
- (i) $\mathbb{E}[u'P^Kvv'P^Ku|X] = \sigma_{uv}\sigma'_{uv}K^2 + (\sigma_v^2\Sigma_u + \sigma_{uv}\sigma'_{uv})K + o_p(K) = \sigma_{uv}\sigma'_{uv}K^2 + o_p(K^2)$;
- (j) $\mathbb{E}[f'vv'P^Ku|X] = \sum_i f_i P_{ii}^K \mathbb{E}[v_i^2 u'_i | x_i] = O_p(K)$;
- (k) $\Delta(K)^{1/2}/\sqrt{N} = o_p(K/N + \Delta(K))$;
- (l) $\mathbb{E}[hh'H^{-1}u'f/N|X] = \sum_i f_i f'_i H^{-1} \mathbb{E}[v_i^2 u_i | x_i] f'_i / N^2 = O_p(1/N)$;
- (m) $\mathbb{E}[f'(I - P^K)vv'P^Ku/N|X] = o_p(\Delta(K)^{1/2}\sqrt{K}/\sqrt{N})$.

The next lemma gives useful calculations that will appear in the MSE approximation due to the invalid instruments.

LEMMA S1.3. If Assumptions 2.1, 2.2 and 2.3 are satisfied, then

- (a) $H_g = f'g/N = O_p(1)$;
- (b) $\Delta_g(K) = o_p(1)$;
- (c) $g'(I - P^K)v/\sqrt{N} = O_p(\Delta_g(K)^{1/2})$, $f'(I - P^K)g/N = O_p((\Delta(K)\Delta_g(K))^{1/2})$;
- (d) $\mathbb{E}[f'vg'g/N|X] = H_g\sigma'_{uv}$;
- (e) $\mathbb{E}[u'P^Kv|X] = K\sigma_{uv}$;
- (f) $\mathbb{E}[(u'f + f'u)/NH^{-1}f'v\sqrt{N}|X] = (\sum_i \sigma_{uv}f'_i H^{-1}f_i + \sum_i f_i\sigma'_{uv}H^{-1}f_i)/N^{3/2} = O_p(1/\sqrt{N})$;
- (g) $\mathbb{E}[f'v(f'g)'H^{-1}(u'f + f'u)|X] = \sum_i f_i (f'g)'H^{-1}\sigma_{uv}f'_i + \sum_i f_i (f'g)'H^{-1}f_i\sigma'_{uv}$.

Proof: (a) holds by LLN. Observe that $(I - P^K)$ is idempotent, and

$$\mathbb{E}[\Delta_g(K)] \leq \mathbb{E}[\text{tr}(g - \Psi\pi_K^g)'(g - \Psi\pi_K^g)]/N = \mathbb{E}[|g(x) - \pi_K^g\psi^K(x)|^2] \rightarrow 0,$$

by Assumption 2.2(b). Thus, $\Delta_g(K) = o_p(1)$ by the Markov inequality. Next, $\mathbb{E}[g'(I - P^K)v/\sqrt{N}|X] = 0$, and

$$\mathbb{E}[g'(I - P^K)vv'(I - P^K)g/N|X] = \sigma_v^2 e_g(K).$$

Therefore, $g'(I - P^K)v/\sqrt{N} = O_p(\Delta_g(K)^{1/2})$ by the Chebyshev inequality. Moreover,

$$\|f'(I - P^K)g/N\| \leq \sqrt{\text{tr}(f'(I - P^K)f/N)} \sqrt{\text{tr}(g'(I - P^K)g/N)} = O_p(\Delta(K)^{1/2}\Delta_g(K)^{1/2}),$$

by the Cauchy-Schwarz inequality, $(I - P^K)$ is idempotent.

Also, $\mathbb{E}[f'vg'u/N|X] = \sum_{i,j} \mathbb{E}[f_iv_ig_ju'_j|X]/N = \sum_i f_ig_i\mathbb{E}[v_iu'_i|x_i]/N = H_g\sigma'_{uv}$, and this gives (d). Moreover, (e) holds by

$$\mathbb{E}[u'P^Kv|X] = \sum_i \mathbb{E}[u_iP_{ii}^Kv_i|X] + \sum_{i,j} \mathbb{E}[u_iP_{ij}^Kv_j|X] = \sum_i P_{ii}^K\mathbb{E}[u_iv_i|X] = K\sigma_{uv}.$$

Next, $\mathbb{E}[u'f/NH^{-1}f'v/\sqrt{N}|X] = \sum_i \mathbb{E}[u_if'_iH^{-1}f_iv_i|x_i]/N^{3/2} = \sigma_{uv}(\sum_i f'_iH^{-1}f_i/N^{3/2})$, and similarly, $\mathbb{E}[f'u/NH^{-1}f'v/\sqrt{N}|X] = (\sum_i f_i\sigma'_{uv}H^{-1}f_i)/N^{3/2}$, and this gives (f). Furthermore, (g) holds by $\mathbb{E}[f'v(f'g)'H^{-1}u'f|X] = \Sigma_i \mathbb{E}[f_iv_i(f'g)'H^{-1}u_if'_i|X] = \Sigma_i f_i(f'g)'H^{-1}\sigma_{uv}f'_i$, and $\mathbb{E}[f'v(f'g)'H^{-1}f'u|X] = \Sigma_i f_i(f'g)'H^{-1}f_i\sigma'_{uv}$. \square

Proof of Proposition 3.1: The 2SLS estimator, $\hat{\delta}_{2SLS}(K) = (W'P^KW)^{-1}(W'P^Ky)$ has the following decomposition with invalid instrument specification in Section 2 of the main paper,

$$\sqrt{N}(\hat{\delta}(K) - \delta_0) = \hat{H}^{-1}\hat{h} + \hat{H}^{-1}\hat{h}_g,$$

where

$$\hat{H} = \frac{W'P^KW}{N}, \hat{h} = \frac{W'P^Kv}{\sqrt{N}}, \hat{h}_g = \frac{W'P^Kg}{N}.$$

Also, \hat{h} , \hat{H} and \hat{h}_g can be decomposed as

$$\begin{aligned} \hat{h} &= h + T_1^h + T_2^h, \\ T_1^h &= -f'(I - P^K)v/\sqrt{N} = O_p(\Delta(K)^{1/2}), \quad T_2^h = u'P^Kv/\sqrt{N} = O_p(K/\sqrt{N}), \\ \hat{H} &= H + T_1^H + T_2^H + Z^H, \\ T_1^H &= -f'(I - P^K)f/N = -e_f(K) = O_p(\Delta(K)), \quad T_2^H = (u'f + f'u)/N = O_p(1/\sqrt{N}), \\ Z^H &= (u'P^Ku - u'(I - P^K)f - f'(I - P^K)u)/N = O_p(K/N + \Delta(K)^{1/2}/\sqrt{N}), \\ \hat{h}_g &= H_g + T_1^g + Z^g, \\ T_1^g &= -f'(I - P^K)g/N = O_p(\Delta(K)^{1/2}\Delta_g(K)^{1/2}), \\ Z^g &= u'g/N - u'(I - P^K)g/N = O_p(1/\sqrt{N} + \Delta_g(K)^{1/2}/\sqrt{N}). \end{aligned}$$

We show that the conditions of Lemma S1.1 are satisfied, and $L(K)$ has the representations given in the proposition. Note that $L(K)$ contains the terms that are proportional to K/\sqrt{N} and K^2/N . To show that a term is $o_p(\rho_{K,N})$, it is enough to show it is $o_p(K/\sqrt{N} + K^2/N + \Delta(K) + \Delta(K)^{1/2}\Delta_g(K)^{1/2})$.

Note that $h = O_p(1)$, $H = O_p(1)$ by Lemma S1.2(e). Also, $T^h = -f'(I - P^K)v/\sqrt{N} + u'P^Kv/\sqrt{N} = O_p(\Delta(K)^{1/2}) + O_p(K/\sqrt{N}) = o_p(1)$ by Lemma S1.2(g), (h), and using $\Delta(K) = o_p(1)$, $K/\sqrt{N} = o(1)$ with $Z^h = 0$. Moreover, $T_1^H = -f'(I - P^K)f/N = O_p(\Delta_K)$ by the definition of Δ_K , and $T_2^H = (u'f + f'u)/N = O_p(1/\sqrt{N})$ by the CLT. Thus $\|T^H\|^2 \leq \|T_1^H\|^2 + \|T_2^H\|^2 + 2\|T_1^H\|\|T_2^H\| = O_p(\Delta(K)^2) + O_p(1/N) + O_p(\Delta(K)/\sqrt{N}) = o_p(\rho_{K,N})$. Also,

$$\begin{aligned} \|T^h\|\|T^H\| &= O_p(\Delta(K)^{3/2}) + O_p(\Delta(K)^{1/2}/\sqrt{N}) + O_p(\Delta(K)K/\sqrt{N}) + O_p(K/N) = o_p(\rho_{K,N}), \\ \text{since } \Delta(K)^{1/2}/\sqrt{N} &= o_p(\rho_{K,N}) \text{ by Lemma S1.2(k). Next, } Z^H = (u'P^Ku - u'(I - P^K)f - f'(I - P^K)u)/N = O_p(K/N) + O_p(\Delta(K)^{1/2}/\sqrt{N}) = o_p(\rho_{K,N}) \text{ by Lemma S1.2(g), (h), (k).} \end{aligned}$$

Next, $H_g = O_p(1)$ by Lemma S1.3(a), and $T_1^g = O_p(\Delta(K)^{1/2}\Delta_g(K)^{1/2}) = o_p(1)$ by

Lemma S1.2(f), S1.3(b),(c). Moreover, $u'g/N = O_p(1/\sqrt{N}) = o_p(\rho_{K,N})$ by CLT and $1/\sqrt{N} = o(K/\sqrt{N})$. Also, $u'(I - P^K)g/N = O_p(\Delta_g(K)^{1/2}/\sqrt{N}) = o_p(\rho_{K,N})$ by Lemma S1.3(c) (replacing v with u) and this gives $\|Z^g\| = o_p(\rho_{K,N})$.

Also,

$$\|T^g\|\|T^H\| = O_p(\Delta(K)^{3/2}\Delta_g(K)^{1/2}) + O_p(\Delta(K)^{1/2}\Delta_g(K)^{1/2}/\sqrt{N}) = o_p(\rho_{K,N}),$$

by $\Delta(K)^{3/2} = o_p(\rho_{K,N})$, $\Delta(K)^{1/2}/\sqrt{N} = o_p(\rho_{K,N})$ using Lemma S1.2(k).

Next, we calculate the expectation of each term $\hat{A}_1(K)$, $\hat{A}_2(K)$, and $\hat{A}_3(K)$ defined in Lemma S1.1. For $Z^{A_1}(K) = 0$, $\hat{A}_1(K) = (h + T_1^h + T_2^h)(h + T_1^h + T_2^h)' - hh'H^{-1}(T_1^H + T_2^H)' - (T_1^H + T_2^H)H^{-1}hh'$, by the proof of Proposition 1 in Donald and Newey (2001), $\mathbb{E}[\hat{A}_1(K)|X] = \sigma_v^2 H + \sigma_v^2 e_f(K) + \sigma_{uv}\sigma'_{uv}K^2/N + o_p(\rho_{K,N})$.

Next, for $Z^{A_2}(K) = 0$, we analyze expectation of $\hat{A}_2(K) = (H_g + T_1^g)(H_g + T_1^g)' - H_g H'_g H^{-1}(T_1^H + T_2^H)' - (T_1^H + T_2^H)H^{-1}H_g H'_g$. First of all, $\mathbb{E}[H_g T_1^g | X] = -H_g g'(I - P^K)f/N$ and $\mathbb{E}[T_1^g H'_g | X] = -f'(I - P^K)g/NH'_g$. Second, $\mathbb{E}[T_1^g T_1^g | X] = O_p(\Delta(K)\Delta_g(K)) = o_p(\rho_{K,N})$ by Lemma S1.3(b), (c). Next,

$$\mathbb{E}[H_g H'_g H^{-1}T_1^H | X] = -H_g H'_g H^{-1}e_f(K).$$

Lastly,

$$\mathbb{E}[H_g H'_g H^{-1}T_2^H | X] = H_g H'_g H^{-1}\mathbb{E}\left[\frac{u'f + f'u}{N} | X\right] = 0.$$

Thus,

$$\begin{aligned} \mathbb{E}[\hat{A}_2(K)|X] &= H_g H'_g + H_g H'_g H^{-1}e_f(K) + e_f(K)H^{-1}H_g H'_g - H_g g'(I - P^K)f/N \\ &\quad - f'(I - P^K)g/NH'_g + o_p(\rho_{K,N}). \end{aligned}$$

For $Z^{A_3}(K) = (\sum_{j=1}^2 T_j^h)T_1^g$, we investigate expectation of $\hat{A}_3(K) = h(H_g + T_1^g)' + (T_1^h + T_2^h)H'_g - hH'_g H^{-1}(T_1^H + T_2^H)' - (T_1^H + T_2^H)H^{-1}hH'_g$. First, observe that $\mathbb{E}[hH'_g | X] = \mathbb{E}[f'v/\sqrt{N}(f'g/N)' | X] = 0$, $\mathbb{E}[hT_1^g | X] = -\mathbb{E}[f'v/\sqrt{N}(f'(I - P^K)g/N)' | X] = 0$, and $\mathbb{E}[T_1^h H'_g | X] = -\mathbb{E}[f'(I - P^K)v/\sqrt{N}H'_g | X] = 0$. Second,

$$\mathbb{E}[T_2^h H'_g | X] = \mathbb{E}[u'P^K v \sqrt{N}H'_g | X] = \frac{K}{\sqrt{N}}\sigma_{uv}H'_g$$

by Lemma S1.3(e). Second, $\mathbb{E}[hH'_g H^{-1}T_1^H | X] = \mathbb{E}[f'v/\sqrt{N}|X]H'_g H^{-1}(f'(I - P)f/N) = 0$, and $\mathbb{E}[T_1^H H^{-1}hH'_g | X] = 0$. Third, by Lemma S1.3(g)

$$\begin{aligned} \mathbb{E}[hH'_g H^{-1}T_2^H | X] &= \mathbb{E}\left[\frac{f'v}{\sqrt{N}}H'_g H^{-1}\frac{u'f + f'u}{N} | X\right] \\ &= (\Sigma_i f_i H'_g H^{-1}\sigma_{uv}f'_i + \Sigma_i f_i H'_g H^{-1}f_i \sigma'_{uv})/N^{3/2} = O_p\left(\frac{1}{\sqrt{N}}\right). \end{aligned}$$

Fourth, by Lemma S1.3(f),

$$\begin{aligned} \mathbb{E}[T_2^H H^{-1}hH'_g | X] &= \mathbb{E}\left[\frac{u'f + f'u}{N}H^{-1}\frac{f'v}{\sqrt{N}}H'_g | X\right] \\ &= \frac{\sum_i \sigma_{uv}f'_i H^{-1}f_i + \sum_i f_i \sigma'_{uv}H^{-1}f_i}{N^{3/2}}H'_g = O_p\left(\frac{1}{\sqrt{N}}\right). \end{aligned}$$

Note that $\|T_1^h\|\|T_1^g\| = O_p(\Delta(K)\Delta_g(K)^{1/2}) = o_p(\rho_{K,N})$ by $\Delta(K) = O_p(\rho_{K,N})$ and

$\|T_2^h\| \|T_1^g\| = O_p(K/\sqrt{N}\Delta(K)^{1/2}\Delta_g(K)^{1/2})$ by $\Delta(K)^{1/2}K/\sqrt{N} \leq K^2/N + \Delta_K$, thus $Z^{A_3}(K) = (T_1^h + T_2^h)T_1^{g'} = o_p(\rho_{K,N})$. Then, we have

$$\mathbb{E}[\hat{A}_3(K)|X] = \frac{K}{\sqrt{N}}\sigma_{uv}H'_g + o_p(\rho_{K,N}),$$

by $1/\sqrt{N} = o(K/\sqrt{N})$.

In sum,

$$\begin{aligned} \mathbb{E}[\hat{A}_1(K) + \hat{A}_2(K) + \hat{A}_3(K) + \hat{A}_3(K)'|X] &= \sigma_v^2 H + \sigma_v^2 e_f(K) + \sigma_{uv}\sigma'_{uv}K^2/N + \\ &+ H_g H'_g + H_g H'_g H^{-1} e_f(K) + e_f(K) H^{-1} H_g H'_g + \frac{K}{\sqrt{N}}(H_g \sigma'_{uv} + \sigma_{uv} H'_g) \\ &- H_g g'(I - P^K)f/N - f'(I - P^K)g/NH'_g + o_p(\rho_{K,N}) \\ &= H\Phi H + H(G + L(K))H + o_p(\rho_{K,N}) \end{aligned}$$

with $\Phi = \sigma_v^2 H^{-1} + H^{-1} H_g H'_g H^{-1}$ and $G = 0$. We have further simplification because $K^2/N = o(K/\sqrt{N})$ and this completes the proof. \square

We also use the following lemma for the proof of Proposition 3.2. Let $\tilde{\sigma}_v^2 = v'v/N$, and $\tilde{\Lambda}(K) = v'P^K v/N\sigma_v^2$.

LEMMA S1.4. Suppose that Assumptions in Proposition 3.2 are satisfied, then

- (a) $\hat{\Lambda}(K) = \tilde{\Lambda}(K) - (\tilde{\sigma}_v^2/\sigma_v^2 - 1)\tilde{\Lambda}(K) - (h + H_g)'H^{-1}(h + H_g)/N\sigma_v^2 + \hat{R}_\Lambda = \tilde{\Lambda}(K) + o_p(K/N)$, $\sqrt{N}\hat{R}_\Lambda = o_p(\rho_{K,N})$;
- (b) $u'P^K u/N - \tilde{\Lambda}(K)\Sigma_u = o_p(K/N)$;
- (c) $\mathbb{E}[h\tilde{\Lambda}(K)v'\eta/\sqrt{N}|X] = (K/N)\Sigma_i f_i \mathbb{E}[v_i^2 \eta'_i | x_i]/N + O_p(K/N^2)$;
- (d) $\mathbb{E}[h(h + H_g)'H^{-1}(h + H_g)/\sqrt{N}|X] = O_p(1/\sqrt{N})$;
- (e) $\mathbb{E}[v'P^K v\eta'v/\sqrt{N}|X] = \sum_i P_{ii}^K \mathbb{E}[v_i^3 \eta_i | x_i]/\sqrt{N} = O(K/\sqrt{N})$.

Proof: Lemma S1.4(a)-(c) follows similarly to the proof of Lemma A.7 and A.8. of Donald and Newey (2001), and it can be shown similarly below in the proof of Lemma S1.5. For (d), we have $\mathbb{E}[hh'H^{-1}h/\sqrt{N}|X] = O_p(1/N)$ and

$$\mathbb{E}[(hh'H^{-1}H_g + hH'_g H^{-1}h)/\sqrt{N}|X] = \frac{\sum_i f_i f'_i H^{-1} H_g + f_i H'_g H^{-1} f_i}{N} \frac{\sigma_v^2}{\sqrt{N}} = O_p(1/\sqrt{N}).$$

For (e), we have

$$\begin{aligned} \mathbb{E}[v'P^K v\eta'v/\sqrt{N}|X] &= \sum_{i,j,k} \mathbb{E}[v_i P_{ij}^K v_j \eta_k v_k | X] / \sqrt{N} \\ &= \sum_i P_{ii}^K \mathbb{E}[v_i^3 \eta_i | x_i] / \sqrt{N} + \sum_i \mathbb{E}[v_i^2 | x_i] P_{ii}^K \mathbb{E}[\eta_i v_i] = O(K/\sqrt{N}) \end{aligned}$$

because $\mathbb{E}[v_i^3 \eta_i | x_i]$ is bounded and $\mathbb{E}[\eta_i v_i] = 0$ by construction. \square

Below, we provide proofs of Propositions 3.2-3.5 which show that LIML (FULL), B2SLS, JIVE2, HLIM (HFUL) estimators satisfy the decomposition (S1.1) (equation (2.10) in the main paper).

It is important to note that different IV estimators have different expressions for G as

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follows, and they can be also found in the proof of Propositions 3.2-3.5 (equations (S1.3), (S1.4), (S1.5), (S1.6));

$$\begin{aligned}
G_{\text{LIML}} &= H^{-1} \left[(H_g \sigma'_{uv} + \sigma_{uv} H'_g) \left(-\frac{2 \sum_i f'_i H^{-1} f_i}{N^{3/2}} - \frac{H'_g H^{-1} H_g}{\sqrt{N} \sigma_v^2} \right) - \sum_i [f_i \sigma'_{uv} H^{-1} (f_i H'_g \right. \\
&\quad \left. + H_g f'_i) + (f_i H'_g + H_g f'_i) H^{-1} \sigma_{uv} f'_i + 2(f_i H'_g H^{-1} f_i \sigma'_{uv} + \sigma_{uv} f'_i H^{-1} H_g f'_i)] / N^{3/2} \right] H^{-1}, \\
G_{\text{B2SLS}} &= H^{-1} \left[(H_g \sigma'_{uv} + \sigma_{uv} H'_g) \left(\frac{d+3}{\sqrt{N}} - \frac{\sum_i f'_i H^{-1} f_i}{N^{3/2}} \right) - \sum_i [f_i \sigma'_{uv} H^{-1} (f_i H'_g + H_g f'_i) \right. \\
&\quad \left. + (f_i H'_g + H_g f'_i) H^{-1} \sigma_{uv} f'_i + (f_i H'_g H^{-1} f_i \sigma'_{uv} + \sigma_{uv} f'_i H^{-1} H_g f'_i)] / N^{3/2} \right] H^{-1}, \quad (\text{S1.2}) \\
G_{\text{JIVE2}} &= H^{-1} \left[(H_g \sigma'_{uv} + \sigma_{uv} H'_g) \left(\frac{1}{\sqrt{N}} - \frac{\sum_i f'_i H^{-1} f_i}{N^{3/2}} \right) - \sum_i [f_i \sigma'_{uv} H^{-1} (f_i H'_g + H_g f'_i) \right. \\
&\quad \left. + (f_i H'_g + H_g f'_i) H^{-1} \sigma_{uv} f'_i + (f_i H'_g H^{-1} f_i \sigma'_{uv} + \sigma_{uv} f'_i H^{-1} H_g f'_i)] / N^{3/2} \right] H^{-1}, \\
G_{\text{HLIM}} &= H^{-1} \left[(H_g \sigma'_{uv} + \sigma_{uv} H'_g) \left(-\frac{2 \sum_i f'_i H^{-1} f_i}{N^{3/2}} - \frac{H'_g H^{-1} H_g}{\sqrt{N} \sigma_v^2} \right) - \sum_i [f_i \sigma'_{uv} H^{-1} (f_i H'_g \right. \\
&\quad \left. + H_g f'_i) + (f_i H'_g + H_g f'_i) H^{-1} \sigma_{uv} f'_i + 2(f_i H'_g H^{-1} f_i \sigma'_{uv} + \sigma_{uv} f'_i H^{-1} H_g f'_i)] / N^{3/2} \right] H^{-1}.
\end{aligned}$$

Proof of Proposition 3.2:

LIML estimator, $\hat{\delta}_{\text{LIML}}(K) = (W' P^K W - \hat{\Lambda}(K) W' W)^{-1} (W' P^K y - \hat{\Lambda}(K) W' y)$ has the following form $\sqrt{N}(\hat{\delta}(K) - \delta_0) = \hat{H}^{-1} \hat{h} + \hat{H}^{-1} \hat{h}_g$ where

$$\hat{H} = \frac{W' P^K W}{N} - \hat{\Lambda}(K) \frac{W' W}{N}, \quad \hat{h} = \frac{W' P^K v}{\sqrt{N}} - \hat{\Lambda}(K) \frac{W' v}{\sqrt{N}}, \quad \hat{h}_g = \frac{W' P^K g}{N} - \hat{\Lambda}(K) \frac{W' g}{N}.$$

We have a following decomposition for \hat{h} and \hat{H}

$$\begin{aligned}
\hat{h} &= h + \sum_{j=1}^5 T_j^h + Z^h, \\
T_1^h &= -f'(I - P^K)v/\sqrt{N} = O_p(\Delta(K)^{1/2}), \quad T_2^h = \eta' P^K v/\sqrt{N} = O_p(\sqrt{K}/\sqrt{N}), \\
T_3^h &= -\tilde{\Lambda}(K)h = O_p(K/N), \quad T_4^h = -\tilde{\Lambda}(K)\eta' v/\sqrt{N} = O_p(K/N), \\
T_5^h &= -(h + H_g)' H^{-1} (h + H_g) \sigma_{uv} / \sqrt{N} \sigma_v^2 = O_p(1/\sqrt{N}), \\
Z^h &= (\tilde{\Lambda}(K) - \hat{\Lambda}(K) + \hat{R}_\Lambda) \sqrt{N} \left(\frac{W' v}{N} - \sigma_{uv} \right) - \hat{R}_\Lambda \frac{W' v}{\sqrt{N}}, \\
\hat{H} &= H + \sum_{j=1}^3 T_j^H + Z^H, \\
T_1^H &= -f'(I - P^K)f/N = -e_f(K) = O_p(\Delta(K)), \quad T_2^H = (u' f + f' u)/N = O_p(1/\sqrt{N}), \\
T_3^H &= -\tilde{\Lambda}(K)H = O_p(K/N), \\
Z^H &= \frac{u' P^K u}{N} - \tilde{\Lambda}(K) \Sigma_u - \hat{\Lambda}(K) \frac{W' W}{N} + \tilde{\Lambda}(K)(H + \Sigma_u) \\
&\quad - u'(I - P^K)f/N - f'(I - P^K)u/N
\end{aligned}$$

where the O_p results follow similarly to the proof of Proposition 2 in Donald and Newey

(2001). We observe that $o_p(\rho_{K,N}) = o_p(1/\sqrt{N} + K/N + \Delta(K) + \Delta(K)^{1/2}\Delta_g(K)^{1/2})$, and $T^h = o_p(1)$, $\|T^H\|^2 = o_p(\rho_{K,N})$, $\|T^h\|\|T^H\| = o_p(\rho_{K,N})$, $\|Z^h\| = o_p(\rho_{K,N})$, and $\|Z^H\| = o_p(\rho_{K,N})$ using Lemma S1.4(a), (b).

Also, \hat{h}_g is decomposed as

$$\begin{aligned}\hat{h}_g &= H_g + \sum_{j=1}^4 T_j^g + Z^g, \\ T_1^g &= -f'(I - P^K)g/N = O_p(\Delta(K)^{1/2}\Delta_g(K)^{1/2}), \\ T_2^g &= u'g/N = O_p(1/\sqrt{N}), \quad T_3^g = -u'(I - P^K)g/N = O_p(\Delta_g(K)^{1/2}/\sqrt{N}), \\ T_4^g &= -\tilde{\Lambda}(K)H_g = O_p(K/N), \\ Z^g &= -\tilde{\Lambda}(K)\frac{W'g}{N} + \tilde{\Lambda}(K)H_g\end{aligned}$$

where the O_p results follow from the proof of Proposition 3.1 and $\tilde{\Lambda}(K) = O_p(K/N)$. Also,

$$\begin{aligned}\hat{\Lambda}(K)\frac{W'g}{N} - \tilde{\Lambda}(K)H_g &= (\hat{\Lambda}(K) - \tilde{\Lambda}(K))\frac{W'g}{N} + \tilde{\Lambda}(K)(\frac{W'g}{N} - H_g) \\ &= o_p(K/N) + O_p(K/N)o_p(1) = o_p(\rho_{K,N})\end{aligned}$$

by Lemma S1.4(a), and by the LLN, $W'g/N = H_g + o_p(1)$, which implies $\|Z^g\| = o_p(\rho_{K,N})$. Moreover, $\|T^g\|\|T_1^H\| = o_p(\rho_{K,N})$, $\|T^g\|\|T_3^H\| = o_p(\rho_{K,N})$,

$\|T^g\|\|T_2^H\| = O_p(\Delta(K)^{1/2}\Delta_g(K)^{1/2}/\sqrt{N}) + O_p(1/N) + O_p(\Delta_g(K)^{1/2}/N) + O_p(K/N^{3/2})$, and $\|T^g\|\|T_2^H\| = o_p(\rho_{K,N})$ by Lemma S1.2(k), $1/N = o(K/N)$, and $K/N^{3/2} = o(K/N)$. Thus, $\|T^g\|\|T_2^H\| = o_p(\rho_{K,N})$ holds by the Cauchy-Schwarz inequality.

Next, we calculate expectation of each term $\hat{A}_1(K)$, $\hat{A}_2(K)$, and $\hat{A}_3(K)$ to apply Lemma S1.1. First, for $\hat{A}_1(K) = hh' + h(\sum_{j=1}^5 T_j^h)' + (\sum_{j=1}^5 T_j^h)h' + (\sum_{j=1}^2 T_j^h)(\sum_{j=1}^2 T_j^h)' - hh'H^{-1}T^{H'} - T^H H^{-1}hh'$,

$$\mathbb{E}[\hat{A}_1(K)|X] = \sigma_v^2 H + \sigma_v^2 e_f(K) + \sigma_v^2 \Sigma_\eta \frac{K}{N} + \hat{\zeta} + \hat{\zeta}' + \hat{\zeta}_\gamma + \hat{\zeta}'_\gamma$$

where

$$\begin{aligned}\hat{\zeta} &= \sum_i f_i P_{ii}^K \mathbb{E}[v_i^2 \eta'_i | X]/N - \frac{K}{N} \sum_i f_i \mathbb{E}[v_i^2 \eta'_i | X]/N, \\ \hat{\zeta}_\gamma &= -\frac{1}{\sqrt{N}} H_g \sigma'_{uv} - \frac{\sum_i f_i H_g' H^{-1} f_i}{N} \frac{\sigma'_{uv}}{\sqrt{N}}\end{aligned}$$

and $\|Z^{A_1}(K)\| = o_p(\rho_{K,N})$ by Lemma S1.4(d) and the proof of Proposition 2 in Donald and Newey (2001).

Next, for $Z^{A_2}(K) = (\sum_{j=2}^4 T_j^g)(\sum_{j=2}^4 T_j^g)'$, we analyze expectation of $\hat{A}_2(K) = (H_g + T_1^g)(H_g + T_1^g)' + (H_g + T_1^g)(\sum_{j=2}^4 T_j^g)' + (\sum_{j=2}^4 T_j^g)(H_g + T_1^g)' - H_g H_g' H^{-1} T^{H'} - T^H H^{-1} H_g H_g'$. Note that $H_g T_4^g - H_g H_g' H^{-1} T^{H'} = 0$, and $\mathbb{E}[H_g T_2^g | X] = \mathbb{E}[H_g g' u/N | X] = H_g \sum_i \mathbb{E}[u'_i | X] g_i / N = 0$. Similarly, we have $\mathbb{E}[H_g T_3^g | X] = 0$, $\mathbb{E}[T_1^g T_2^g | X] = 0$, and $\mathbb{E}[T_1^g T_3^g | X] = 0$. Also,

$$\mathbb{E}[T_1^g T_4^g | X] = \mathbb{E}\left[\frac{f'(I - P^K)g}{N} \tilde{\Lambda}(K) H_g' | X\right] = \frac{K}{N} \frac{f'(I - P^K)g}{N} H_g' = o_p(\rho_{K,N})$$

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by $\mathbb{E}[\tilde{\Lambda}(K)|X] = \mathbb{E}[v'P^K v/N\sigma_v^2|X] = K/N$ and using $K/N = O(\rho_{K,N})$. Lastly, $1/N = o(\rho_{K,N})$, $\Delta_g(K) = o_p(1)$, and $K/N^{3/2} = o(K/N)$ so that $\|T_2^g\|\|T_j^g\|$ for each $j \geq 2$. It also follows similarly that $\|T_3^g\|\|T_3^g\|$, $\|T_3^g\|\|T_4^g\|$ and $\|T_4^g\|\|T_4^g\|$ are $o_p(\rho_{K,N})$. Thus, $Z^{A_2}(K) = o_p(\rho_{K,N})$.

With the same calculations in the proof of Proposition 3.1, we have

$$\begin{aligned}\mathbb{E}[\hat{A}_2(K)|X] &= H_g H'_g + H_g H'_g H^{-1} e_f(K) + e_f(K) H^{-1} H_g H'_g - H_g g'(I - P^K) f / N \\ &\quad - f'(I - P^K) g / NH'_g + o(\rho_{K,N}).\end{aligned}$$

Next, for $Z^{A_3}(K) = (\sum_{j=3}^5 T_j^h)(\sum_{j=1}^4 T_j^g)' + (\sum_{j=1}^2 T_j^h)(\sum_{j=2}^4 T_j^g)'$, we investigate expectation of $\hat{A}_3(K)$. From the proof of Proposition 3.1, we have calculations of $\mathbb{E}[h(H_g + T_1^{g'}) + T_1^h H'_g - hH'_g H^{-1}(T_1^H + T_2^H)' - (T_1^H + T_2^H)H^{-1}hH'_g|X]$. First, note that $hT_4^{g'} - hH'_g H^{-1}T_3^H = 0$ and $T_3^h H'_g - T_3^H H^{-1}hH'_g = 0$, and

$$\mathbb{E}[T_2^h H'_g|X] = \mathbb{E}[\eta' P^K v / \sqrt{N} H'_g|X] = 1/\sqrt{N} \sum_i P_{ii}^K \mathbb{E}[\eta_i v_i] H'_g = 0.$$

Second, by Lemma S1.3(d),

$$\mathbb{E}[hT_2^{g'}|X] = \mathbb{E}[f'v\sqrt{N}g'u/N|X] = \frac{1}{\sqrt{N}} H_g \sigma'_{uv} = O_p(1/\sqrt{N}).$$

Third,

$$\mathbb{E}[hT_3^{g'}|X] = -\mathbb{E}[f'v\sqrt{N}g'(I - P^K)u/N|X] = -\frac{1}{\sqrt{N}} \frac{f'(I - P^K)g}{N} \sigma'_{uv} = o_p(\rho_{K,N}).$$

Fourth, $\mathbb{E}[T_1^h T_1^{g'}|X] = \mathbb{E}[f'(I - P^K)v\sqrt{N}g'(I - P^K)f/N|X] = 0$. Fifth, $\mathbb{E}[T_2^h T_1^{g'}|X] = -\mathbb{E}[\eta' P^K v \sqrt{N}|X] g'(I - P^K)f/N = 0$ as $\mathbb{E}[\eta_i v_i] = 0$. Sixth,

$$\mathbb{E}[T_4^h H'_g|X] = -\mathbb{E}\left[\frac{v' P^K v}{N \sigma_v^2} \eta' v / \sqrt{N} H'_g|X\right] = O_p(\sqrt{K}/N^{3/2}) = o_p(\rho_{K,N})$$

by Lemma S1.4(e). Seventh,

$$\begin{aligned}\mathbb{E}[T_5^h H'_g|X] &= -\mathbb{E}[h' H^{-1} h|X] \frac{\sigma_{uv}}{\sqrt{N} \sigma_v^2} H'_g - H'_g H^{-1} H_g \sigma_{uv} H'_g / \sqrt{N} \sigma_v^2 \\ &= -\frac{\sum_i f'_i H^{-1} f_i}{N} \frac{\sigma_{uv}}{\sqrt{N}} H'_g - H'_g H^{-1} H_g \frac{\sigma_{uv}}{\sqrt{N} \sigma_v^2} H'_g = O_p(1/\sqrt{N}).\end{aligned}$$

Lastly, $K/N = o(\sqrt{K}/\sqrt{N})$, $1/\sqrt{N} = o(\sqrt{K}/\sqrt{N})$, and $o_p(\sqrt{K}/\sqrt{N}(\Delta(K)^{1/2} \Delta_g(K)^{1/2} + 1/\sqrt{N} + \Delta_g(K)^{1/2}/\sqrt{N}) + K/N) = o_p(\rho_{K,N})$, thus $\|T_j^h\|\|T_k^g\| = o_p(\rho_{K,N})$ for $j \geq 3$ and each k . It also follows similarly that $\|T_1^h\|\|T_j^g\| = o_p(\rho_{K,N})$ and $\|T_2^h\|\|T_j^g\| = o_p(\rho_{K,N})$ for $j \geq 2$, and this gives $Z^{A_3}(K) = o_p(\rho_{K,N})$. Thus,

$$\begin{aligned}\mathbb{E}[\hat{A}_3(K)|X] &= \frac{1}{\sqrt{N}} H_g \sigma'_{uv} - \frac{2 \sum_i f'_i H^{-1} f_i}{N^{3/2}} \sigma_{uv} H'_g - H'_g H^{-1} H_g \frac{\sigma_{uv}}{\sqrt{N} \sigma_v^2} H'_g \\ &\quad - \sum_i [f_i H'_g H^{-1} \sigma_{uv} f'_i + f_i H'_g H^{-1} f_i \sigma'_{uv} + f_i \sigma'_{uv} H^{-1} f_i H'_g] / N^{3/2} = O_p(1/\sqrt{N}).\end{aligned}$$

In sum, we have

$$\begin{aligned} \mathbb{E}[\hat{A}_1(K) + \hat{A}_2(K) + \hat{A}_3(K) + \hat{A}_3(K)'|X] &= \sigma_v^2 H + \sigma_v^2 e_f(K) + \sigma_v^2 \Sigma_\eta K/N + \hat{\zeta} + \hat{\zeta}' \\ &+ H_g H_g' + H_g H_g' H^{-1} e_f(K) + e_f(K) H^{-1} H_g H_g' - H_g g'(I - P^K) f/N - f'(I - P^K) g/N H_g' \\ &+ O_p(1/\sqrt{N}) + o_p(\rho_{K,N}). \end{aligned}$$

If we assume $\mathbb{E}[v_i^2 \eta_i | x_i] = 0$, then $\hat{\zeta} = 0$, and we get the desired results with $\Phi = \sigma_v^2 H^{-1} + H^{-1} H_g H_g' H^{-1}$, $L(K)$ provided in Proposition 3.2, and $G = G_{\text{LIML}}$,

$$\begin{aligned} G_{\text{LIML}} &= H^{-1} \left[(H_g \sigma'_{uv} + \sigma_{uv} H_g') \left(-\frac{2 \sum_i f_i' H^{-1} f_i}{N^{3/2}} - \frac{H_g' H^{-1} H_g}{\sqrt{N} \sigma_v^2} \right) - \sum_i [f_i \sigma'_{uv} H^{-1} (f_i H_g' \right. \\ &\quad \left. + H_g f_i') + (f_i H_g' + H_g f_i') H^{-1} \sigma_{uv} f_i' + 2(f_i H_g' H^{-1} f_i \sigma'_{uv} + \sigma_{uv} f_i' H^{-1} H_g f_i')] / N^{3/2} \right] H^{-1}. \end{aligned} \quad (\text{S1.3})$$

For Fuller estimator, $\hat{\delta}_{\text{FULL}}(K) = (W' P^K W - \check{\Lambda}(K) W' W)^{-1} (W' P^K y - \check{\Lambda}(K) W' y)$, observe that

$$\begin{aligned} \check{\Lambda}(K) &= \hat{\Lambda}(K) - \frac{\frac{C}{N-K}(1-\hat{\Lambda}(K))^2}{1-\frac{C}{N-K}(1-\hat{\Lambda}(K))} \\ &= \hat{\Lambda}(K) - \frac{C(1-\hat{\Lambda}(K))^2}{N-K-C(1-\hat{\Lambda}(K))} = \hat{\Lambda}(K) + O_p(1/N) \end{aligned}$$

by $0 \leq 1 - \hat{\Lambda}(K) \leq 1$. Therefore we have

$$\begin{aligned} \frac{W' P^K W}{N} - \check{\Lambda}(K) \frac{W' W}{N} &= \frac{W' P^K W}{N} - \hat{\Lambda}(K) \frac{W' W}{N} + o_p(\rho_{K,N}), \\ \frac{W' P^K v}{\sqrt{N}} - \check{\Lambda}(K) \frac{W' v}{\sqrt{N}} &= \frac{W' P^K v}{\sqrt{N}} - \hat{\Lambda}(K) \frac{W' v}{\sqrt{N}} + o_p(\rho_{K,N}), \\ \frac{W' P^K g}{N} - \check{\Lambda}(K) \frac{W' g}{N} &= \frac{W' P^K g}{N} - \hat{\Lambda}(K) \frac{W' g}{N} + o_p(\rho_{K,N}), \end{aligned}$$

by using $W' W/N = O_p(1)$, $W' v/\sqrt{N} = O_p(1)$, $W' g/N = O_p(1)$ and $1/N = o_p(\rho_{K,N})$. Thus, FULL estimator has the same higher-order MSE decomposition with LIML estimator. \square

Proof of Proposition 3.3:

B2SLS estimator, $\hat{\delta}_{\text{B2SLS}}(K) = (W' P^K W - \bar{\Lambda}(K) W' W)^{-1} (W' P^K y - \bar{\Lambda}(K) W' y)$ with $\bar{\Lambda}(K) = (K-d-2)/N$ has the following decomposition $\sqrt{N}(\hat{\delta}(K) - \delta_0) = \hat{H}^{-1} \hat{h} + \hat{H}^{-1} \hat{h}_g$,

$$\hat{H} = \frac{W' P^K W}{N} - \bar{\Lambda}(K) \frac{W' W}{N}, \quad \hat{h} = \frac{W' P^K v}{\sqrt{N}} - \bar{\Lambda}(K) \frac{W' v}{\sqrt{N}}, \quad \hat{h}_g = \frac{W' P^K g}{N} - \bar{\Lambda}(K) \frac{W' g}{N}.$$

We have following decomposition for \hat{h} , \hat{H} and \hat{h}_g ,

$$\begin{aligned}
 \hat{h} &= h + \sum_{j=1}^4 T_j^h, \quad T_1^h = -f'(I - P^K)v/\sqrt{N} = O_p(\Delta(K)^{1/2}), \\
 T_2^h &= u'P^Kv/\sqrt{N} - \sqrt{N}\bar{\Lambda}(K)\sigma_{uv} = O_P(\sqrt{K}/\sqrt{N}), \\
 T_3^h &= -\bar{\Lambda}(K)h = O_p(K/N), \quad T_4^h = -\bar{\Lambda}(K)\sqrt{N}(u'v/N - \sigma_{uv}) = O_p(K/N), \\
 \hat{H} &= H + \sum_{j=1}^3 T_j^H + Z^H, \\
 T_1^H &= -f'(I - P^K)f/N = -e_f(K) = O_p(\Delta(K)), \quad T_2^H = (u'f + f'u)/N = O_p(1/\sqrt{N}), \\
 T_3^H &= -\bar{\Lambda}(K)H = O_p(K/N), \\
 Z^H &= \frac{u'P^Ku}{N} - \bar{\Lambda}(K)\Sigma_u - \bar{\Lambda}(K)(\frac{W'W}{N} - H - \Sigma_u) - u'(I - P^K)f/N - f'(I - P^K)u/N, \\
 \hat{h}_g &= H_g + \sum_{j=1}^4 T_j^g + Z^g, \\
 T_1^g &= -f'(I - P^K)g/N = O_p(\Delta(K)^{1/2}\Delta_g(K)^{1/2}), \\
 T_2^g &= u'g/N = O_p(1/\sqrt{N}), \quad T_3^g = -u'(I - P^K)g/N = O_p(\Delta_g(K)^{1/2}/\sqrt{N}), \\
 T_4^g &= -\bar{\Lambda}(K)H_g = O_p(K/N), \quad Z^g = -\bar{\Lambda}(K)(\frac{W'g}{N} - H_g),
 \end{aligned}$$

where the O_p results, $T^h = o_p(1)$, $\|T^H\|^2 = o_p(\rho_{K,N})$, $\|T^h\|\|T^H\| = o_p(\rho_{K,N})$, $\|Z^h\| = o_p(\rho_{K,N})$, and $\|Z^H\| = o_p(\rho_{K,N})$ follow immediately from the proof of Proposition 3 in Donald and Newey (2001), and $\|Z^g\| = o_p(\rho_{K,N})$, $\|T^g\|\|T^H\| = o_p(\rho_{K,N})$ similarly to the proof of Proposition 3.2 using $\bar{\Lambda}(K) = O(K/N)$.

Next, we calculate expectation of each term $\hat{A}_1(K)$, $\hat{A}_2(K)$, and $\hat{A}_3(K)$ to apply Lemma S1.1. First, for $\hat{A}_1(K) = hh' + h(\sum_{j=1}^4 T_j^h)' + (\sum_{j=1}^4 T_j^h)h' + (\sum_{j=1}^2 T_j^h)(\sum_{j=1}^2 T_j^h)' - hh'H^{-1}T^{H'} - T^H H^{-1}hh'$

$$\mathbb{E}[\hat{A}_1(K)|X] = \sigma_v^2 H + \sigma_v^2 e_f(K) + (\sigma_v^2 \Sigma_u + \sigma_{uv} \sigma'_{uv}) \frac{K}{N} + \hat{\zeta} + \hat{\zeta}' + o_p(\rho_{K,N})$$

by the proof of Proposition 3 in Donald and Newey (2001), where

$$\hat{\zeta} = \sum_i f_i P_{ii}^K \mathbb{E}[v_i^2 u'_i | X]/N - \bar{\Lambda}(K) \sum_i f_i \mathbb{E}[v_i^2 u'_i | X]/N.$$

Next, for $Z^{A_2}(K) = (\sum_{j=2}^4 T_j^g)(\sum_{j=2}^4 T_j^g)'$, we analyze expectation of $\hat{A}_2(K) = (H_g + T_1^g)(H_g + T_1^g)' + (H_g + T_1^g)(\sum_{j=2}^4 T_j^g)' + (\sum_{j=2}^4 T_j^g)(H_g + T_1^g)' - H_g H_g' H^{-1} T^{H'} - T^H H^{-1} H_g H_g'$. Observe that $H_g T_4^{g'} - H_g H_g' H^{-1} T_3^{H'} = 0$ and $\mathbb{E}[T_1^g T_4^{g'} | X] = \bar{\Lambda}(K) \frac{f'(I - P^K)g}{N} H_g' = o_p(\rho_{K,N})$. Similar to the proof of Proposition 3.2 by replacing $\bar{\Lambda}(K)$ with $\tilde{\Lambda}(K)$, we have

$$\begin{aligned}
 \mathbb{E}[\hat{A}_2(K)|X] &= H_g H_g' + H_g H_g' H^{-1} e_f(K) + e_f(K) H^{-1} H_g H_g' - H_g g'(I - P^K)f/N \\
 &\quad - f'(I - P^K)g/N H_g' + o(\rho_{K,N}),
 \end{aligned}$$

and $Z^{A_2}(K) = o_p(\rho_{K,N})$. For $Z^{A_3}(K) = (\sum_{j=3}^4 T_j^h)(\sum_{j=1}^4 T_j^g)' + (\sum_{j=1}^2 T_j^h)(\sum_{j=2}^4 T_j^g)'$,

we investigate expectation of $\hat{A}_3(K)$. From the proof of Propositions 3.1 and 3.2, we have

$$\begin{aligned} & \mathbb{E}[h(H_g + T_1^g + T_2^g + T_3^g)' + T_1^h(H_g + T_1^g)' - hH_g' H^{-1} T_2^{H'} - T_2^h H^{-1} hH_g' | X] \\ &= \frac{1}{\sqrt{N}} H_g \sigma'_{uv} - \frac{\sum_i f'_i H^{-1} f_i}{N^{3/2}} \sigma_{uv} H_g' \\ &\quad - \sum_i [f_i H_g' H^{-1} \sigma_{uv} f'_i + f_i H_g' H^{-1} f_i \sigma'_{uv} + f_i \sigma'_{uv} H^{-1} f_i H_g'] / N^{3/2} = O_p(1/\sqrt{N}) \end{aligned}$$

and $Z^{A_3}(K) = o_p(\rho_{K,N})$. Also observe that $hT_4^{g'} - hH_g' H^{-1} T_3^{H'} = 0$ and $T_3^h H_g' - T_3^h H^{-1} hH_g' = 0$. Next,

$$\begin{aligned} \mathbb{E}[T_2^h H_g' | X] &= \mathbb{E}[(u' P^K v / \sqrt{N} - \sqrt{N} \bar{\Lambda}(K) \sigma_{uv}) H_g' | X] \\ &= (K / \sqrt{N} - \sqrt{N} \bar{\Lambda}(K)) \sigma_{uv} H_g' = \frac{d+2}{\sqrt{N}} \sigma_{uv} H_g' = O_p(1/\sqrt{N}) \end{aligned}$$

by the definition of $\bar{\Lambda}(K)$. Similarly,

$$\mathbb{E}[T_2^h T_1^{g'} | X] = O_p\left(\frac{1}{\sqrt{N}} f'(I - P^K) g / N\right) = o_p(\rho_{K,N}).$$

Lastly,

$$\mathbb{E}[T_4^h H_g' | X] = -\mathbb{E}[\bar{\Lambda}(K) \sqrt{N} (u' v / N - \sigma_{uv}) H_g' | X] = 0.$$

In sum,

$$\mathbb{E}[\hat{A}_3(K) | X] = O_p(1/\sqrt{N}) + o(\rho_{K,N}).$$

Thus, we have

$$\begin{aligned} \mathbb{E}[\hat{A}_1(K) + \hat{A}_2(K) + \hat{A}_3(K) + \hat{A}_3(K)' | X] &= \sigma_v^2 H + \sigma_v^2 e_f(K) + (\sigma_v^2 \Sigma_u + \sigma_{uv} \sigma'_{uv}) K / N \\ &+ H_g H_g' + H_g H_g' H^{-1} e_f(K) + e_f(K) H^{-1} H_g H_g' - H_g g'(I - P^K) f / N \\ &- f'(I - P^K) g / N H_g' + O_p(1/\sqrt{N}) + \hat{\zeta} + \hat{\zeta}' + o_p(\rho_{K,N}). \end{aligned}$$

Using $\sigma_v^2 \Sigma_u = \sigma_v^2 \Sigma_\eta + \sigma_{uv} \sigma'_{uv}$ and assuming $\mathbb{E}[v_i^2 u_i | X] = 0$, Lemma S1.1 holds with $\Phi = \sigma_v^2 H^{-1} + H^{-1} H_g H_g' H^{-1}$, $L(K)$ provided in Proposition 3.3, and $G = G_{\text{B2SLS}}$,

$$\begin{aligned} G_{\text{B2SLS}} &= H^{-1} \left[(H_g \sigma'_{uv} + \sigma_{uv} H_g') \left(\frac{d+3}{\sqrt{N}} - \frac{\sum_i f'_i H^{-1} f_i}{N^{3/2}} \right) - \sum_i [f_i \sigma'_{uv} H^{-1} (f_i H_g' + H_g f'_i) \right. \\ &\quad \left. + (f_i H_g' + H_g f'_i) H^{-1} \sigma_{uv} f'_i + (f_i H_g' H^{-1} f_i \sigma'_{uv} + \sigma_{uv} f'_i H^{-1} H_g f'_i)] / N^{3/2} \right] H^{-1}. \end{aligned} \tag{S1.4}$$

□

Proof of Proposition 3.4:

JIVE2 estimator, $\hat{\delta}_{\text{JIVE2}}(K) = (W' P^K W - \sum_i P_{ii}^K W_i W_i')^{-1} (W' P^K y - \sum_i P_{ii}^K W_i y_i)$ has the following decomposition $\sqrt{N}(\hat{\delta}(K) - \delta_0) = \hat{H}^{-1} \hat{h} + \hat{H}^{-1} \hat{h}_g$,

$$\hat{H} = \frac{W' P^K W}{N} - \frac{\sum_i P_{ii}^K W_i W_i'}{N}, \hat{h} = \frac{W' P^K v}{\sqrt{N}} - \frac{\sum_i P_{ii}^K W_i v_i}{\sqrt{N}}, \hat{h}_g = \frac{W' P^K g}{N} - \frac{\sum_i P_{ii}^K W_i g_i}{N}.$$

We have a following decomposition for \hat{h} , \hat{H} and \hat{h}_g ,

$$\begin{aligned}
 \hat{h} &= h + \sum_{j=1}^4 T_j^h, \quad T_1^h = -f'(I - P^K)v/\sqrt{N} = O_p(\Delta(K)^{1/2}), \\
 T_2^h &= u'P^Kv/\sqrt{N} - K\sigma_{uv}/\sqrt{N} = O_P(\sqrt{K}/\sqrt{N}), \\
 T_3^h &= -\frac{\sum_i P_{ii}^K f_i v_i}{\sqrt{N}} = O_P(\sqrt{K}/\sqrt{N}), \quad T_4^h = -\left(\frac{\sum_i P_{ii}^K u_i v_i}{\sqrt{N}} - \frac{K}{\sqrt{N}}\sigma_{uv}\right) = O_P(\sqrt{K}/\sqrt{N}), \\
 \hat{H} &= H + \sum_{j=1}^3 T_j^H + Z^H, \\
 T_1^H &= -f'(I - P^K)f/N = -e_f(K) = O_p(\Delta(K)), \quad T_2^H = (u'f + f'u)/N = O_p(1/\sqrt{N}), \\
 T_3^H &= -\frac{\sum_i P_{ii}^K f_i f'_i}{N} = O_p(K/N), \\
 Z^H &= \left(\frac{u'P^Ku}{N} - \frac{K}{N}\Sigma_u\right) + \left(\frac{K}{N}\Sigma_u - \frac{\sum_i P_{ii}^K u_i u'_i}{N}\right) - \frac{\sum_i P_{ii}^K (f_i u'_i + u_i f'_i)}{N} \\
 &\quad - u'(I - P^K)f/N - f'(I - P^K)u/N, \\
 \hat{h}_g &= H_g + \sum_{j=1}^4 T_j^g + Z^g, \\
 T_1^g &= -f'(I - P^K)g/N = O_p(\Delta(K)^{1/2}\Delta_g(K)^{1/2}), \\
 T_2^g &= u'g/N = O_p(1/\sqrt{N}), \quad T_3^g = -u'(I - P^K)g/N = O_p(\Delta_g(K)^{1/2}/\sqrt{N}), \\
 T_4^g &= -\frac{\sum_i P_{ii}^K f_i g_i}{N} = O_p(K/N), \quad Z^g = -\frac{\sum_i P_{ii}^K u_i g_i}{N},
 \end{aligned}$$

where the O_p results, $T^h = o_p(1)$, $\|T^H\|^2 = o_p(\rho_{K,N})$, $\|T^h\|\|T^H\| = o_p(\rho_{K,N})$, $\|Z^h\| = o_p(\rho_{K,N})$, $\|Z^H\| = o_p(\rho_{K,N})$, $\|Z^g\| = o_p(\rho_{K,N})$, and $\|T^g\|\|T^H\| = o_p(\rho_{K,N})$ follow similarly to the proof of Proposition 3.3 and using $\sum_i P_{ii}^K u_i u'_i / N - K/N\Sigma_u = o_p(K/N)$, $\sum_i P_{ii}^K (f_i u'_i + u_i f'_i) / N = o_p(K/N)$ by the Markov Inequality. Note that $o_p(\rho_{K,N}) = o_p(1/\sqrt{N} + K/N + \Delta(K) + \Delta(K)^{1/2}\Delta_g(K)^{1/2})$ because $f'D^K f/N + f'D^K g/N = O_p(K/N)$.

First, for $Z_1^A(K) = 0$, we have $\mathbb{E}[hT_2^{h'}|X] = -\mathbb{E}[hT_4^{h'}|X] = \sum_i f_i P_{ii}^K \mathbb{E}[v_i^2 u'_i | x_i]/N$ by Lemma S1.2(j). Next,

$$\mathbb{E}[hT_3^{h'}|X] = -\mathbb{E}[f'v/\sqrt{N} \sum_i P_{ii}^K f'_i v_i/\sqrt{N}|X] = -\frac{\sum_i P_{ii}^K f_i f'_i}{N} \sigma_v^2,$$

$$\mathbb{E}[hh'H^{-1}T_3^{H'}|X] = -\mathbb{E}[f'vv'f/NH^{-1} \sum_i P_{ii}^K f_i f'_i|X] = -\frac{\sum_i P_{ii}^K f_i f'_i}{N} \sigma_v^2,$$

and $\mathbb{E}[hT_3^{h'} - hh'H^{-1}T_3^{H'}|X] = 0$. Next, $\mathbb{E}[T_1^h T_2^{h'}|X] = \mathbb{E}[T_1^h T_4^{h'}|X] = o_p(\Delta_K^{1/2} \sqrt{K/N}) = o_p(\rho_{K,N})$ by Lemma S1.2(m). We also have

$$|\mathbb{E}[T_1^h T_3^{h'}|X]| = \frac{|\mathbb{E}[\sum_{i,j} f_i Q_{ij}^K v_j \sum_i P_{ii}^K f'_i v_i|X]|}{N} \leq \frac{|f'Q\mu|}{N} = o_p(\Delta_K^{1/2} \sqrt{K/N}) = o_p(\rho_{K,N})$$

where $\mu_i = f'_i P_{ii}^K \sigma_v^2$ by using $\sum_i (P_{ii}^K)^2 = o(K)$ and the Cauchy-Schwarz inequality. Using the similar calculations in the proof of Proposition 3.3, $\mathbb{E}[T_2^h T_2^{h'}|X] = (\sigma_v^2 \Sigma_u +$

$\sigma_{uv}\sigma'_{uv})\frac{K}{N} + o_p(K/N)$. Next,

$$\mathbb{E}[T_2^h T_3^{h'} | X] = -\frac{\sum_i (P_{ii}^K)^2 f'_i \mathbb{E}[u_i v_i^2 | X]}{N} = o_p(K/N),$$

$$\begin{aligned} \mathbb{E}[T_2^h T_4^{h'} | X] &= -\mathbb{E}[(u' P^K v / \sqrt{N} - K / \sqrt{N} \sigma_{uv})(\sum_i P_{ii}^K u_i v_i / \sqrt{N} - K / \sqrt{N} \sigma_{uv})'] \\ &= -\frac{\sum_i (P_{ii}^K)^2 \mathbb{E}[u_i u'_i v_i^2 | X]}{N} - \frac{\sum_{i \neq j} P_{ii}^K P_{jj}^K}{N} \sigma_{uv} \sigma'_{uv} + \frac{K^2}{N} \sigma_{uv} \sigma'_{uv} = o_p(K/N) \end{aligned}$$

by Lemma S1.2(b) and (c). Similarly, we can show $\mathbb{E}[T_3^h T_4^{h'} | X] = -\sum_i (P_{ii}^K)^2 f'_i \mathbb{E}[u'_i v_i^2 | X]/N = o_p(K/N)$ and $\mathbb{E}[T_4^h T_4^{h'} | X] = o_p(K/N)$. Thus, for $\hat{A}_1(K) = hh' + h(\sum_{j=1}^4 T_j^h)' + (\sum_{j=1}^4 T_j^h)h' + (\sum_{j=1}^4 T_j^h)(\sum_{j=1}^4 T_j^h)' - hh'H^{-1}T^{H'} - T^H H^{-1}hh'$, we have

$$\mathbb{E}[\hat{A}_1(K) | X] = \sigma_v^2 H + \sigma_v^2 e_f(K) + (\sigma_v^2 \Sigma_u + \sigma_{uv} \sigma'_{uv}) \frac{K}{N} + o_p(\rho_{K,N}).$$

Next, for $Z^{A_2}(K) = (\sum_{j=2}^4 T_j^g)(\sum_{j=2}^4 T_j^g)'$, we analyze expectation of $\hat{A}_2(K) = (H_g + T_1^g)(H_g + T_1^g)' + (H_g + T_1^g)(\sum_{j=2}^4 T_j^g)' + (\sum_{j=2}^4 T_j^g)(H_g + T_1^g)' - H_g H'_g H^{-1} T^{H'} - T^H H^{-1} H_g H'_g$. Similar to the proof of Proposition 3.3, we have $\mathbb{E}[T_1^g T_4^{g'} | X] = f'(I - P^K)g/N \sum_i P_{ii}^K f_i g_i/N = o_p(\rho_{K,N})$,

$$\mathbb{E}[H_g T_4^{g'} | X] = -H_g \frac{\sum_i P_{ii}^K f'_i g_i}{N},$$

$$\mathbb{E}[H_g H'_g H^{-1} T_3^{H'} | X] = -H_g H'_g H^{-1} \frac{\sum_i P_{ii}^K f_i f'_i}{N},$$

$$\begin{aligned} \mathbb{E}[\hat{A}_2(K) | X] &= H_g H'_g + H_g H'_g H^{-1} e_f(K) + e_f(K) H^{-1} H_g H'_g - H_g g'(I - P^K)f/N \\ &\quad - f'(I - P^K)g/N H'_g - H_g g'D^K f/N - f'D^K g/N H'_g \\ &\quad + H_g H'_g H^{-1} f'D^K f/N + f'D^K f/N H^{-1} H_g H'_g + o_p(\rho_{K,N}) \end{aligned}$$

and $Z^{A_2}(K) = o_p(\rho_{K,N})$.

Lastly, for $Z^{A_3}(K) = (\sum_{j=1}^4 T_j^h)(\sum_{j=1}^4 T_j^g)'$, we investigate expectation of $\hat{A}_3(K)$. We have $\mathbb{E}[T_2^h H'_g | X] = \mathbb{E}[(u' P^K v / \sqrt{N} - K \sigma_{uv} / \sqrt{N}) H'_g | X] = 0$, $\mathbb{E}[T_2^h T_1^{g'} | X] = 0$, $\mathbb{E}[h T_4^{g'} | X] = 0$, $\mathbb{E}[T_3^h H'_g | X] = 0$, $\mathbb{E}[T_3^h T_1^{g'} | X] = 0$ and similarly, $\mathbb{E}[T_4^h H'_g | X] = 0$, $\mathbb{E}[T_4^h T_1^{g'} | X] = 0$. Also observe that $\mathbb{E}[h H'_g H^{-1} T_3^{H'} | X] = 0$, $\mathbb{E}[T_3^H H^{-1} h H'_g | X] = 0$. From the similar calculations as in Propositions 3.1-3.3,

$$\begin{aligned} \mathbb{E}[\hat{A}_3(K) | X] &= \frac{1}{\sqrt{N}} H_g \sigma'_{uv} - \frac{\sum_i f'_i H^{-1} f_i}{N^{3/2}} \sigma_{uv} H'_g \\ &\quad - \sum_i [f_i H'_g H^{-1} \sigma_{uv} f'_i + f_i H'_g H^{-1} f_i \sigma'_{uv} + f_i \sigma'_{uv} H^{-1} f_i H'_g]/N^{3/2} + o_p(\rho_{K,N}), \end{aligned}$$

and $Z^{A_3}(K) = o_p(\rho_{K,N})$. In sum,

$$\begin{aligned} \mathbb{E}[\hat{A}_1(K) + \hat{A}_2(K) + \hat{A}_3(K) + \hat{A}_3(K)'|X] &= \sigma_v^2 H + \sigma_v^2 \frac{f'(I - P^K)f}{N} + H_g H_g' \\ &+ (\sigma_v^2 \Sigma_u + \sigma_{uv} \sigma'_{uv}) K/N + H_g H_g' H^{-1} \frac{f'(I - (P^K - D^K))f}{N} + \frac{f'(I - (P^K - D^K))f}{N} H^{-1} H_g H_g' \\ &- H_g g'(I - (P^K - D^K))f/N - f'(I - (P^K - D^K))g/N H_g' + O_p(1/\sqrt{N}) + o_p(\rho_{K,N}). \end{aligned}$$

Thus, Lemma S1.1 holds with $\Phi = \sigma_v^2 H^{-1} + H^{-1} H_g H_g' H^{-1}$, $L(K)$ provided in Proposition 3.4, and $G = G_{\text{JIVE2}}$,

$$\begin{aligned} G_{\text{JIVE2}} &= H^{-1} [(H_g \sigma'_{uv} + \sigma_{uv} H_g') (\frac{1}{\sqrt{N}} - \frac{\sum_i f'_i H^{-1} f_i}{N^{3/2}}) - \sum_i [f_i \sigma'_{uv} H^{-1} (f_i H_g' + H_g f'_i) \\ &+ (f_i H_g' + H_g f'_i) H^{-1} \sigma_{uv} f'_i + (f_i H_g' H^{-1} f_i \sigma'_{uv} + \sigma_{uv} f'_i H^{-1} H_g f'_i)] / N^{3/2}] H^{-1}. \end{aligned} \quad (\text{S1.5})$$

We use the following lemma for the HLIM/HFUL results. Let $\tilde{\lambda}(K) = \frac{v'(P^K - D^K)v}{N\sigma_v^2}$. \square

LEMMA S1.5. Suppose that Assumptions in Proposition 3.5 are satisfied, then

- (a) $\hat{\lambda}(K) = \tilde{\lambda}(K) - (\tilde{\sigma}_v^2 / \sigma_v^2 - 1) \tilde{\lambda}(K) - (h + H_g)' H^{-1} (h + H_g) / N \sigma_v^2 + \hat{R}_\lambda = \tilde{\lambda}(K) + o_p(K/N)$, $\sqrt{N} \hat{R}_\lambda = o_p(\rho_{K,N})$;
- (b) $\mathbb{E}[h \tilde{\lambda}(K) v' \eta / \sqrt{N} | X] = o_p(K/N^2)$.

Proof: Note that the HLIM estimator is $\hat{\delta}_{\text{HLIM}} = \arg \min_\delta \lambda(\delta)$, $\lambda(\delta) = A(\delta)/B(\delta)$ where $A(\delta) = (y - W\delta)'(P^K - D^K)(y - W\delta)/N$, $B(\delta) = (y - W\delta)'(y - W\delta)/N$, and $\hat{\lambda}(K) = \lambda(\hat{\delta}_{\text{HLIM}})$. By the Taylor expansion, we have

$$\hat{\lambda}(K) = \lambda(\delta_0) - \lambda_\delta(\delta_0)' \lambda_{\delta\delta}(\delta_0)^{-1} \lambda_\delta(\delta_0) / 2 + O_p(1/N^{3/2})$$

where we use $\hat{\delta}_{\text{HLIM}} - \delta_0 = -\lambda_{\delta\delta}(\delta_0)^{-1} \lambda_\delta(\delta_0) + O_p(1/N)$ from the FOC, $\lambda_\delta(\hat{\delta}_{\text{HLIM}}) = 0$ w.p.1. We have straightforward calculations of the gradient and Hessian

$$\lambda_\delta(\delta) = B(\delta)^{-1} [A_\delta(\delta) - \lambda(\delta) B_\delta(\delta)],$$

$$\lambda_{\delta\delta}(\delta) = B(\delta)^{-1} [A_{\delta\delta}(\delta) - \lambda(\delta) B_{\delta\delta}(\delta)] - B(\delta)^{-1} [B_\delta(\delta) \lambda_\delta(\delta)' + \lambda_\delta(\delta) B_\delta(\delta)'].$$

It can be shown that $B(\delta_0) = \tilde{\sigma}_v^2 + g'g/N^2 + (g'v + v'g)/N^{3/2} = \tilde{\sigma}_v^2 + O_p(1/N)$,

$$\begin{aligned} -\tilde{\sigma}_v^2 \sqrt{N} \lambda_\delta(\delta_0) / 2 &= W'(P^K - D^K) \varepsilon / N^{1/2} - \varepsilon'(P^K - D^K) \varepsilon / \sqrt{N} W' \varepsilon / \varepsilon' \varepsilon \\ &= h + H_g + O_p(\Delta(K)^{1/2} + \sqrt{K}/\sqrt{N}) - \frac{v'(P^K - D^K)v}{\sqrt{N}} \left(\frac{W'v}{v'v} - \frac{\sigma_{uv}}{\sigma_v^2} \right) \\ &= h + H_g + o_p(1) \xrightarrow{d} N(H_g, \sigma_v^2 H) \end{aligned}$$

by the similar calculations from Proposition 3.4 and that $\frac{W'v}{v'v} - \frac{\sigma_{uv}}{\sigma_v^2} = O_p(1/\sqrt{N})$.

Therefore, we have $\lambda_\delta(\delta_0) = O_p(1/\sqrt{N})$. Similarly, we have

$$\begin{aligned} \tilde{\sigma}_v^2 \lambda_{\delta\delta}(\delta_0) / 2 &= W'(P^K - D^K) W / N - \lambda(\delta_0) W' W / N + O_p(1/\sqrt{N}) \\ &= H + O_p(\Delta(K) + 1/\sqrt{N} + K/N) \end{aligned}$$

where we use $B_\delta(\delta_0) = O_p(1)$ and the calculations in the proof of Proposition 3.4. Thus,

$$\hat{\lambda}(K) = \lambda(\delta_0) - (h + H_g)' H^{-1} (h + H_g) / N \sigma_v^2 + O_p(\Delta(K)^{1/2}/N + \sqrt{K}/N^{3/2}).$$

From the proof of Lemma A.7 in Donald and Newey (2001), it can be shown similarly that $\lambda(\delta_0) = \tilde{\lambda}(K) - (\tilde{\sigma}_v^2/\sigma_v^2 - 1)\hat{\lambda}(K) + O_p(K/N^2)$. For (b),

$$f' v v' (P^K - D^K) v v' \eta = 2 \sum_{i \neq j} P_{ij}^K f_i v_i^2 v_j^2 \eta'_j - \sum_{i \neq j} f_i P_{ii}^K v_i^3 v_j \eta_j - \sum_{i \neq j} P_{ii}^K v_i^3 \eta_i f_j v_j.$$

Similar to Lemma S1.4(c), we have

$$\mathbb{E}[h\tilde{\lambda}(K)v'\eta/\sqrt{N}|X] = \mathbb{E}[f'vv'(P^K - D^K)vv'\eta/N^2\sigma_v^2|X] = o_p(K/N^2)$$

as $\mathbb{E}[v_j^2\eta'_j|X]$ is bounded and $\mathbb{E}[v_j\eta_j|x_j] = 0$. \square

Proof of Proposition 3.5:

HLIM estimator, $\hat{\delta}_{\text{HLIM}}(K) = (W'(P^K - D^K)W - \hat{\lambda}(K)W'W)^{-1}(W'(P^K - D^K)y - \hat{\lambda}(K)W'y)$ has the following form $\sqrt{N}(\hat{\delta}(K) - \delta_0) = \hat{H}^{-1}\hat{h} + \hat{H}^{-1}\hat{h}_g$ where $\hat{H} = W'(P^K - D^K)W/N - \hat{\lambda}(K)W'W/N$,

$$\hat{h} = \frac{W'(P^K - D^K)v}{\sqrt{N}} - \hat{\lambda}(K) \frac{W'v}{\sqrt{N}}, \quad \hat{h}_g = \frac{W'(P^K - D^K)g}{N} - \hat{\lambda}(K) \frac{W'g}{N}.$$

We have a following decomposition for \hat{h} and \hat{H}

$$\begin{aligned} \hat{h} &= h + \sum_{j=1}^7 T_j^h + Z^h, \\ T_1^h &= -f'(I - P^K)v/\sqrt{N} = O_p(\Delta(K)^{1/2}), \quad T_2^h = \eta' P^K v / \sqrt{N} = O_p(\sqrt{K}/\sqrt{N}), \\ T_3^h &= -\frac{f'D^K v}{\sqrt{N}} = O_p(\sqrt{K}/\sqrt{N}), \quad T_4^h = -\frac{\eta'D^K v}{\sqrt{N}} = O_p(\sqrt{K}/\sqrt{N}), \\ T_5^h &= -\tilde{\lambda}(K)h = O_p(K/N), \quad T_6^h = -\tilde{\lambda}(K)\eta'v/\sqrt{N} = O_p(K/N), \\ T_7^h &= -(h + H_g)' H^{-1} (h + H_g) \sigma_{uv} / \sqrt{N} \sigma_v^2 = O_p(1/\sqrt{N}), \\ Z^h &= (\tilde{\lambda}(K) - \hat{\lambda}(K) + \hat{R}_\lambda) \sqrt{N} \left(\frac{W'v}{N} - \sigma_{uv} \right) - \hat{R}_\lambda \frac{W'v}{\sqrt{N}}, \\ \hat{H} &= H + \sum_{j=1}^4 T_j^H + Z^H, \\ T_1^H &= -f'(I - P^K)f/N = -e_f(K) = O_p(\Delta(K)), \quad T_2^H = (u'f + f'u)/N = O_p(1/\sqrt{N}), \\ T_3^H &= -\tilde{\lambda}(K)H = O_p(K/N), \quad T_4^H = -\frac{\sum_i P_{ii}^K f_i f'_i}{N} = O_p(K/N), \\ Z^H &= \frac{u'(P^K - D^K)u}{N} - \tilde{\lambda}(K)\Sigma_u - \frac{\sum_i P_{ii}^K (f_i u'_i + u_i f'_i)}{N} \\ &\quad - \hat{\lambda}(K) \frac{W'W}{N} + \tilde{\lambda}(K)(H + \Sigma_u) - u'(I - P^K)f/N - f'(I - P^K)u/N. \end{aligned}$$

Also, \hat{h}_g is decomposed as

$$\begin{aligned}\hat{h}_g &= H_g + \sum_{j=1}^5 T_j^g + Z^g, \\ T_1^g &= -f'(I - P^K)g/N = O_p(\Delta(K)^{1/2} \Delta_g(K)^{1/2}), \\ T_2^g &= u'g/N = O_p(1/\sqrt{N}), \quad T_3^g = -u'(I - P^K)g/N = O_p(\Delta_g(K)^{1/2}/\sqrt{N}), \\ T_4^g &= -\tilde{\lambda}(K)H_g = O_p(K/N), \quad T_5^g = -\frac{\sum_i P_{ii}^K f_i g_i}{N} = O_p(K/N), \\ Z^g &= -\hat{\lambda}(K) \frac{W'g}{N} + \tilde{\lambda}(K)H_g - \frac{\sum_i P_{ii}^K u_i g_i}{N},\end{aligned}$$

$T^h = o_p(1)$, $\|T^H\|^2 = o_p(\rho_{K,N})$, $\|T^h\| \|T^H\| = o_p(\rho_{K,N})$, $\|Z^h\| = o_p(\rho_{K,N})$, $\|Z^H\| = o_p(\rho_{K,N})$, $\|Z^g\| = o_p(\rho_{K,N})$ and $\|T^g\| \|T^H\| = o_p(\rho_{K,N})$ and O_p results follow similarly from the proof of Propositions 3.2 and 3.4.

For, $Z^{A_1}(K) = (\sum_{j=5}^7 T_j^h)(\sum_{j=5}^7 T_j^h)' + (\sum_{j=5}^7 T_j^h)(\sum_{j=1}^4 T_j^h)' + (\sum_{j=1}^4 T_j^h)(\sum_{j=5}^7 T_j^h)'$ and $\hat{A}_1(K) = hh' + h(\sum_{j=1}^7 T_j^h)' + (\sum_{j=1}^7 T_j^h)h' + (\sum_{j=1}^4 T_j^h)(\sum_{j=1}^4 T_j^h)' - hh'H^{-1}T^{H'} - T^H H^{-1}hh'$, we have $hT_5^{h'} - hh'H^{-1}T_3^{H'} = 0$, $\mathbb{E}[hT_3^{h'} - hh'H^{-1}T_4^{H'}|X] = 0$, $\mathbb{E}[hT_2^{h'}|X] = -\mathbb{E}[hT_3^{h'}|X]$, $\mathbb{E}[T_j^h T_k^h|X] = o_p(K/N)$ for $k \geq j \geq 2$ as in the proof of Proposition 3.4 by replacing u_i with η_i and using $\sigma_{\eta v} = 0$. Further, $\mathbb{E}[hT_6^{h'}] = o_p(K/N^2) = o_p(\rho_{K,N})$ by Lemma S1.5 (b). Then, we have

$$\mathbb{E}[\hat{A}_1(K)|X] = \sigma_v^2 H + \sigma_v^2 e_f(K) + \sigma_v^2 \Sigma_\eta \frac{K}{N} + \hat{\zeta}_\gamma + \hat{\zeta}'_\gamma + o_p(\rho_{K,N}),$$

where

$$\hat{\zeta}_\gamma = -\frac{1}{\sqrt{N}} H_g \sigma'_{uv} - \frac{\sum_i f_i H'_g H^{-1} f_i}{N} \frac{\sigma'_{uv}}{\sqrt{N}}.$$

Next, for $Z^{A_2}(K) = (\sum_{j=2}^5 T_j^g)(\sum_{j=2}^5 T_j^g)'$, we analyze expectation of $\hat{A}_2(K) = (H_g + T_1^g)(H_g + T_1^g)' + (H_g + T_1^g)(\sum_{j=2}^5 T_j^g)' + (\sum_{j=2}^5 T_j^g)(H_g + T_1^g)' - H_g H'_g H^{-1} T^{H'} - T^H H^{-1} H_g H'_g$. First, note that $H_g T_4^g - H_g H'_g H^{-1} T_3^{H'} = 0$. By combining results in the proof of Propositions 3.2 and 3.4, replacing $\tilde{\Lambda}(K)$ with $\tilde{\lambda}(K)$, we have

$$\begin{aligned}\mathbb{E}[\hat{A}_2(K)|X] &= H_g H'_g + H_g H'_g H^{-1} e_f(K) + e_f(K) H^{-1} H_g H'_g - H_g g'(I - P^K) f / N \\ &\quad - f'(I - P^K) g / N H'_g - H_g g' D^K f / N - f' D^K g / N H'_g \\ &\quad + H_g H'_g H^{-1} f' D^K f / N + f' D^K f / N H^{-1} H_g H'_g + o_p(\rho_{K,N})\end{aligned}$$

and $Z^{A_2}(K) = o_p(\rho_{K,N})$.

Next, for $Z^{A_3}(K) = (\sum_{j=5}^7 T_j^h)(\sum_{j=1}^4 T_j^h)' + (\sum_{j=1}^4 T_j^h)(\sum_{j=2}^5 T_j^g)',$ we investigate expectation of $\hat{A}_3(K)$. First, note that $hT_4^{h'} - hH_g H^{-1} T_3^{H'} = 0$ and $T_5^h H'_g - T_3^H H^{-1} hH'_g = 0$. It follows from S1.4(e) and the proof of Proposition 3.2,

$$\mathbb{E}[T_6^h H'_g | X] = -\mathbb{E}[\tilde{\lambda}(K) \eta' v / \sqrt{N} | X] H'_g = 0,$$

$$\mathbb{E}[T_7^h H'_g | X] = -\frac{\sum_i f'_i H^{-1} f_i}{N} \frac{\sigma_{uv}}{\sqrt{N}} H'_g - H'_g H^{-1} H_g \frac{\sigma_{uv}}{\sqrt{N} \sigma_v^2} H'_g = O_p(1/\sqrt{N}).$$

Then, we have

$$\begin{aligned}\mathbb{E}[\hat{A}_3(K)|X] &= \frac{1}{\sqrt{N}}H_g\sigma'_{uv} - \frac{2\sum_i f'_i H^{-1} f_i}{N^{3/2}}\sigma_{uv}H'_g - H'_g H^{-1} H_g \frac{\sigma_{uv}}{\sqrt{N}\sigma_v^2} H'_g \\ &\quad - \sum_i [f_i H'_g H^{-1} \sigma_{uv} f'_i + f_i H'_g H^{-1} f_i \sigma'_{uv} + f_i \sigma'_{uv} H^{-1} f_i H'_g]/N^{3/2} = O_p(1/\sqrt{N}).\end{aligned}$$

In sum,

$$\begin{aligned}\mathbb{E}[\hat{A}_1(K) + \hat{A}_2(K) + \hat{A}_3(K) + \hat{A}_3(K)'|X] &= \sigma_v^2 H + \sigma_v^2 \frac{f'(I - P^K)f}{N} + H_g H'_g \\ &\quad + (\sigma_v^2 \Sigma_u + \sigma_{uv} \sigma'_{uv})K/N + H_g H'_g H^{-1} \frac{f'(I - (P^K - D^K))f}{N} + \frac{f'(I - (P^K - D^K))f}{N} H^{-1} H_g H'_g \\ &\quad - H_g g'(I - (P^K - D^K))f/N - f'(I - (P^K - D^K))g/N H'_g + O_p(1/\sqrt{N}) + o_p(\rho_{K,N}).\end{aligned}$$

Thus, Lemma S1.1 holds with $\Phi = \sigma_v^2 H^{-1} + H^{-1} H_g H'_g H^{-1}$, $L(K)$ provided in Proposition 3.5, and $G = G_{\text{HLIM}}$,

$$\begin{aligned}G_{\text{HLIM}} &= H^{-1} \left[(H_g \sigma'_{uv} + \sigma_{uv} H'_g) \left(-\frac{2\sum_i f'_i H^{-1} f_i}{N^{3/2}} - \frac{H'_g H^{-1} H_g}{\sqrt{N}\sigma_v^2} \right) - \sum_i [f_i \sigma'_{uv} H^{-1} (f_i H'_g \right. \\ &\quad \left. + H_g f'_i) + (f_i H'_g + H_g f'_i) H^{-1} \sigma_{uv} f'_i + 2(f_i H'_g H^{-1} f_i \sigma'_{uv} + \sigma_{uv} f'_i H^{-1} H_g f'_i)]/N^{3/2} \right] H^{-1}. \tag{S1.6}\end{aligned}$$

HFUL estimator, $\hat{\delta}_{\text{HFUL}}(K) = (W'(P^K - D^K)W - \bar{\lambda}(K)W'W)^{-1}(W'(P^K - D^K)y - \bar{\lambda}(K)W'y)$ has the same higher-order MSE decomposition with HLIM estimator as $\bar{\lambda}(K) = \hat{\lambda}(K) + O_p(1/N)$. This completes the proof. \square

S1.2. Proofs of Propositions 4.1-4.5

For the proofs of Propositions 4.1-4.5, we use the following Lemma S1.6 to handle higher-order terms due to the $N^{-\gamma}$ ($\gamma > 1/2$) locally invalid instruments specification. This is a slight modification of Lemma S1.1 without $O_p(1)$ term in the decomposition of \hat{h}_g , and immediately follows from Lemma S1.1 replacing $H_g = 0$. Define $\rho_{K,N} = \text{tr}(G + L(K))$.

LEMMA S1.6. *If there is a decomposition $\hat{h} = h + T^h + Z^h$, $\hat{H} = H + T^H + Z^H$, $\hat{h}_g = T^g + Z^g$, and*

$$\begin{aligned}(h + T^h)(h + T^h)' - hh'H^{-1}T^H - T^H H^{-1}hh' &= \hat{A}_1(K) + Z^{A_1}(K), \\ T^g T^{g'} &= \hat{A}_2(K) + Z^{A_2}(K), \quad (h + T^h)T^{g'} = \hat{A}_3(K) + Z^{A_3}(K),\end{aligned}$$

such that $T^h = o_p(1)$, $h = O_p(1)$, $T^g = o_p(1)$, $T^H = o_p(1)$, and $H = O_p(1)$, the determinant of H is bounded away from zero with probability 1, $\rho_{K,N} = \text{tr}(G + L(K))$, and $\rho_{K,N} = o_p(1)$,

$$\begin{aligned}\|T^H\|^2 &= o_p(\rho_{K,N}), \|T^h\| \|T^H\| = o_p(\rho_{K,N}), \|Z^h\| = o_p(\rho_{K,N}), \|Z^H\| = o_p(\rho_{K,N}), \\ \|Z^g\| &= o_p(\rho_{K,N}), \|T^g\| \|T^H\| = o_p(\rho_{K,N}), \quad Z^{A_i}(K) = o_p(\rho_{K,N}) \quad \text{for all } i = 1, 2, 3,\end{aligned}$$

$$\mathbb{E}[\hat{A}_1(K) + \hat{A}_2(K) + \hat{A}_3(K) + \hat{A}_3(K)'|X] = H\Phi H + H(G + L(K))H + o_p(\rho_{K,N}),$$

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then

$$\begin{aligned} N(\hat{\delta}(K) - \delta_0)(\hat{\delta}(K) - \delta_0)' &= \hat{Q}(K) + \hat{r}(K), \\ \mathbb{E}[\hat{Q}(K)|X] &= \Phi + G + L(K) + T(K), \\ [\hat{r}(K) + T(K)]/tr(G + L(K)) &= o_p(1), \quad \text{as } K \rightarrow \infty, N \rightarrow \infty. \end{aligned}$$

Proof of Proposition 4.1: The 2SLS estimator, $\hat{\delta}^{2SLS}(K)$ has the following decomposition with $N^{-\gamma}$ locally invalid instruments specification in the main paper,

$$\sqrt{N}(\hat{\delta}(K) - \delta_0) = \hat{H}^{-1}\hat{h} + \hat{H}^{-1}\hat{h}_g,$$

where \hat{h}, \hat{H} is defined and decomposed as in the proof of Proposition 3.1, but \hat{h}_g can be decomposed as follows,

$$\begin{aligned} \hat{h}_g &= \frac{W'P^K g}{N^{1/2+\gamma}} = T_0^g + T_1^g + Z^g, \\ T_0^g &= \frac{H_g}{N^{\gamma-1/2}} = O_p(\frac{1}{N^{\gamma-1/2}}), \quad T_1^g = \frac{-f'(I-P^K)g}{N} \frac{1}{N^{\gamma-1/2}} = O_p(\frac{\Delta(K)^{1/2}\Delta_g(K)^{1/2}}{N^{\gamma-1/2}}), \\ Z^g &= (\frac{u'g}{N} - \frac{u'(I-P^K)g}{N}) \frac{1}{N^{\gamma-1/2}} = O_p(1/N^\gamma + \Delta_g(K)^{1/2}/N^\gamma), \end{aligned}$$

where O_p results immediately follow from the proof of Proposition 3.1. We show that the conditions of Lemma S1.6 are satisfied. Note that $G + L(K)$ in Proposition 4.1 contains the terms of order $1/N^{2\gamma-1}, K/N^\gamma$, and K^2/N . It is important to note that these terms may have same order in MSE approximation. To show a term is $o_p(\rho_{K,N})$, it is enough to show that it is $o_p(1/N^{2\gamma-1} + K/N^\gamma + K^2/N + \Delta(K))$.

By similar arguments as in the proof of Proposition 3.1, $T^g = o_p(1)$, $\|Z^g\| = o_p(\rho_{K,N})$ by $1/N^\gamma = o(K/N^\gamma)$ and $\|T^g\|\|T^H\| = o(K/N^\gamma)$.

Calculation of the expectation of $\hat{A}_1(K)$ and $Z^{A_1} = o_p(\rho_{K,N})$ defined in Lemma S1.6 follows similarly to the proof of Proposition 3.1. For $\hat{A}_2(K) = (\sum_{j=0}^1 T_j^g)(\sum_{j=0}^1 T_j^g)'$ and $Z^{A_2}(K) = 0$, we have

$$\begin{aligned} \mathbb{E}[\hat{A}_2(K)|X] &= \frac{H_g H_g'}{N^{2\gamma-1}} - H_g \frac{g'(I-P^K)f}{N} \frac{1}{N^{2\gamma-1}} - \frac{f'(I-P^K)g}{N} H_g' \frac{1}{N^{2\gamma-1}} + o_p(\frac{1}{N^{2\gamma-1}}) \\ &= \frac{H_g H_g'}{N^{2\gamma-1}} + o_p(\rho_{K,N}) \end{aligned}$$

by Lemma S1.3(c) and $1/N^{2\gamma-1} = O_p(\rho_{K,N})$.

For $Z^{A_3}(K) = (T_1^h + T_2^h)T_1^{g'}$, we can easily show $\mathbb{E}[hT_0^{g'}|X] = 0, \mathbb{E}[hT_1^{g'}|X] = 0, \mathbb{E}[T_1^h T_0^{g'}|X] = 0$, and

$$\mathbb{E}[T_2^h T_0^{g'}|X] = \mathbb{E}[\frac{u'P^K v}{\sqrt{N}} \frac{H_g'}{N^{\gamma-1/2}}|X] = \frac{K}{N^\gamma} \sigma_{uv} H_g'.$$

Moreover, $Z^{A_3}(K) = o_p(\rho_{K,N})$ by inspection. Thus, $\mathbb{E}[\hat{A}_3(K)|X] = \frac{K}{N^\gamma} \sigma_{uv} H_g'$. In sum,

$$\begin{aligned} \mathbb{E}[\hat{A}_1(K) + \hat{A}_2(K) + \hat{A}_3(K) + \hat{A}_3(K)'|X] \\ &= \sigma_v^2 H + \sigma_v^2 e_f(K) + \sigma_{uv} \sigma'_{uv} K^2/N + \frac{H_g H_g'}{N^{2\gamma-1}} + \frac{K}{N^\gamma} (H_g \sigma'_{uv} + \sigma_{uv} H_g') + o_p(\rho_{K,N}) \\ &= H \Phi H + H(G + L(K))H + o_p(\rho_{K,N}) \end{aligned}$$

with $\Phi = \sigma_v^2 H^{-1}$, $G = H^{-1} H_g H'_g H^{-1} / N^{2\gamma-1}$. Second result holds because $O_p(1/N^{2\gamma-1}) = O_p((\frac{N^{1-\gamma}}{K})^2 \frac{K^2}{N}) = o_p(K^2/N)$ under $\frac{K}{N^{1-\gamma}} \rightarrow \infty$, and this completes the proof. \square

Proof of Proposition 4.2: For LIML estimator, we have a similar decomposition as in the proof of Proposition 3.2 with \hat{h}_g ,

$$\begin{aligned}\hat{h}_g &= \sum_{j=0}^2 T_j^g + Z^g, \\ T_0^g &= \frac{H_g}{N^{\gamma-1/2}} = O_p(\frac{1}{N^{\gamma-1/2}}), \quad T_1^g = -\frac{f'(I-P^K)g}{N} \frac{1}{N^{\gamma-1/2}} = O_p(\frac{\Delta(K)^{1/2} \Delta_g(K)^{1/2}}{N^{\gamma-1/2}}), \\ T_2^g &= \frac{u'g}{N} \frac{1}{N^{\gamma-1/2}} = O_p(\frac{1}{N^\gamma}), \\ Z^g &= -\frac{u'(I-P^K)g}{N} \frac{1}{N^{\gamma-1/2}} - \tilde{\Lambda}(K) \frac{H_g}{N^{\gamma-1/2}} - \hat{\Lambda}(K) \frac{W'g}{N^{\gamma-1/2}} + \tilde{\Lambda}(K) \frac{H_g}{N^{\gamma-1/2}}\end{aligned}$$

where the O_p results and $T^g = o_p(1)$, $Z^g = o_p(\rho_{K,N})$, $\|T^g\| \|T^H\| = o_p(\rho_{K,N})$ follow from the proof of Proposition 3.2, and using $1/N^\gamma = o_p(\frac{1}{N^{2\gamma-1}} + \frac{K}{N})$. To see $1/N^\gamma = o_p(\frac{1}{N^{2\gamma-1}} + \frac{K}{N})$, consider the function $(K/a) + a$ which is convex, and has a global minimum at $a = \sqrt{K}$ which gives function value $2\sqrt{K}$. Therefore, for $a = N^{1-\gamma}$, $(1/N^\gamma)/(1/N^{2\gamma-1} + K/N) = 1/(a + K/a) \leq 1/(2\sqrt{K}) \rightarrow 0$. To show a term is $o_p(\rho_{K,N})$ it is enough to show that it is $o_p(K/N^{2\gamma-1} + K/N + \Delta(K))$.

Calculation of the expectation of $\hat{A}_1(K)$ and $Z^{A_1} = o_p(\rho_{K,N})$ follows from the proof of Proposition 3.2. Next, for $Z^{A_2}(K) = 0$, we have $\mathbb{E}[T_0^g T_2^g | X] = 0$, $\mathbb{E}[T_1^g T_2^g | X] = 0$, and

$$\mathbb{E}[T_2^g T_2^g | X] = \frac{1}{N^{2\gamma-1}} \frac{1}{N^2} \mathbb{E}[u'gg'u | X] = \frac{1}{N^{2\gamma-1}} \frac{g'g}{N^2} \Sigma_u = o_p(\rho_{K,N})$$

by $1/N^{2\gamma-1} = O_p(\rho_{K,N})$, $g'g/N = O_p(1)$. Therefore, we have

$$\mathbb{E}[\hat{A}_2(K) | X] = \frac{H_g H'_g}{N^{2\gamma-1}} + o_p(\rho_{K,N})$$

with similar calculations as in the proof of Proposition 4.1.

Next, for $Z^{A_3}(K) = (\sum_{j=3}^5 T_j^h)(\sum_{j=0}^2 T_j^g)' + (T_1^h + T_2^h)T_2^g$, observe that $\mathbb{E}[hT_0^g | X] = 0$, $\mathbb{E}[hT_1^g | X] = 0$, $\mathbb{E}[T_1^h T_0^g | X] = 0$, $\mathbb{E}[T_1^h T_1^g | X] = 0$, $\mathbb{E}[T_1^h T_2^g | X] = 0$, and $\mathbb{E}[hT_2^g | X] = \frac{1}{N^\gamma} H_g \sigma'_{uv}$. Moreover, $\mathbb{E}[T_2^h T_0^g | X] = \mathbb{E}[\eta' P^K v / \sqrt{N} H'_g / N^{\gamma-1/2} | X] = 0$, $\mathbb{E}[T_2^h T_1^g | X] = 0$ by using $\mathbb{E}[\eta_i v_i | x_i] = 0$. Thus,

$$\mathbb{E}[\hat{A}_3(K) | X] = \frac{1}{N^\gamma} H_g \sigma'_{uv} = o_p(\rho_{K,N})$$

by using $O_p(1/N^\gamma) = o_p(\rho_{K,N})$. We can also verify $Z^{A_3}(K) = o_p(\rho_{K,N})$ from the proof of Proposition 3.2 by inspection. In sum,

$$\begin{aligned}\mathbb{E}[\hat{A}_1(K) + \hat{A}_2(K) + \hat{A}_3(K) + \hat{A}_3(K)' | X] \\ &= \sigma_v^2 H + \sigma_v^2 e_f(K) + \sigma_v^2 \Sigma_\eta K/N + \hat{\zeta} + \hat{\zeta}' + \frac{H_g H'_g}{N^{2\gamma-1}} + o_p(\rho_{K,N}) \\ &= H \Phi H + H(G + L(K))H + o_p(\rho_{K,N})\end{aligned}$$

with $\Phi = \sigma_v^2 H^{-1}$, $G = H^{-1} H_g H'_g H^{-1} / N^{2\gamma-1}$. The results for FULL estimator follows

similarly to the proof of Proposition 3.2, and this completes the proof. \square

Proof of Propositions 4.3-4.5: This follows similarly as in the proof of Propositions 4.1-2 with the results in the proof of Propositions 3.3-3.5. \square

S1.3. Proofs of Propositions 5.1 and 5.2

Proof of Proposition 5.1: This immediately follows by Propositions 4.1-4.5 and Proposition 4 of Donald and Newey (2001) under Assumptions 5.1 and 5.2. \square

Proof of Proposition 5.2:

Similar to the proof of Propositions 3.1-3.5, to show a term is $o_p(\bar{\rho}_{K,N})$, it is enough to show that it is $o_p(K/\sqrt{N} + K^2/N + \Delta(K) + \Delta(K)^{1/2}\Delta_g(K)^{1/2})$ for 2SLS and $o_p(K/N + \Delta(K) + \Delta(K)^{1/2}\Delta_g(K)^{1/2})$ for other estimators.

First, we consider the 2SLS estimator. By straightforward algebra, we have

$$2\hat{H}_g\hat{\sigma}_{uv}\frac{K}{\sqrt{N}} + \hat{\sigma}_{uv}^2\frac{K^2}{N} = 2(H_g + \Lambda_g)\sigma_{uv}\frac{K}{\sqrt{N}} + \sigma_{uv}^2\frac{K^2}{N} + o_p(\bar{\rho}_{K,N})$$

by Assumption 5.3.

Because the choice of K is unaffected by subtracting constants from $\hat{L}_{IR}(K)$, we can assume without loss of generality that $\hat{L}_{IR}(K)$ can be constructed using $\tilde{R}(K) = \hat{R}(K) - u'u/N$, $\tilde{G}(K) = \hat{G}(K) + u'u/NH_z^{-1}h_z + u'u/NT_z - u'v/\sqrt{N}$ where $H_z = f'P_zf/N$, $h_z = f'P_zv/\sqrt{N}$. Then we have a following decomposition for the Mallows criterion:

$$\begin{aligned} \tilde{R}(K) - \hat{\sigma}_u^2\frac{K}{N} &= \frac{W'(I - P^K)W}{N} + \hat{\sigma}_u^2\frac{K}{N} - u'u/N \\ &= \frac{f'(I - P^K)f}{N} + \frac{f'(I - P^K)u}{N} + \frac{u'(I - P^K)f}{N} - \left(\frac{u'P^Ku}{N} - \hat{\sigma}_u^2\frac{K}{N}\right) + o_p\left(\frac{K}{N}\right) \\ &= \frac{f'(I - P^K)f}{N} + o_p(\bar{\rho}_{K,N}) \end{aligned}$$

by Lemma S1.2, Assumption 5.3 and the same calculations in the proof of Proposition 3.3. It can be similarly shown for the cross-validation criterion.

Note that $\hat{\varepsilon} = y - W\hat{\delta} = -W(\hat{\delta} - \delta_0) + \frac{g}{\sqrt{N}} + v$, and

$$\begin{aligned} \sqrt{N}(\hat{\delta} - \delta_0) &= \hat{H}_z^{-1}\hat{h}_z = H_z^{-1}h_z + T_z, \\ T_z &= O_p(1/\sqrt{N}), \hat{H}_z = H_z + O_p(1/\sqrt{N}), \hat{h}_z = h_z + O_p(1/\sqrt{N}), \end{aligned}$$

where $H_z = f'P_zf/N$, $h_z = f'P_zv/\sqrt{N}$ by the assumption in the Proposition with valid instruments z . Then, we have following decompositions for \hat{H} , \hat{H}_g and $\tilde{G}(K)$,

$$\begin{aligned} \hat{H} &= \frac{W'P^{\tilde{K}}W}{N} - \hat{\sigma}_u^2\tilde{K}/N = H + T_{1,H} + T_{2,H} + Z_H, \\ T_{1,H} &= -f'(I - P^{\tilde{K}})f/N = -e_f(\tilde{K}) = O_p(\Delta(\tilde{K})), \quad T_{2,H} = (u'f + f'u)/N = O_p(1/\sqrt{N}), \\ Z_H &= u'P^{\tilde{K}}u/N - \hat{\sigma}_u^2\tilde{K}/N - u'(I - P^{\tilde{K}})f/N - f'(I - P^{\tilde{K}})u/N, \end{aligned}$$

$$\begin{aligned}
\hat{H}_g &= \frac{W' P^{\tilde{K}} \hat{\varepsilon}}{\sqrt{N}} - \tilde{K}/\sqrt{N} \hat{\sigma}_{uv} = H_g + \Lambda_g + \sum_{j=1}^7 T_{j,g} + Z_{H,g}, \\
\Lambda_g &= h - HH_z^{-1}h_z = O_p(1), \quad T_{1,g} = -T_{1,H}H_z^{-1}h_z = O_p(\Delta(\tilde{K})), \\
T_{2,g} &= u'P^{\tilde{K}}v/\sqrt{N} - \sigma_{uv}\tilde{K}/\sqrt{N} = O_p(\tilde{K}^{1/2}/\sqrt{N}), \quad T_{3,g} = -u'P^{\tilde{K}}u/NH_z^{-1}h_z = O_p(\tilde{K}/N), \\
T_{4,g} &= -T_{2,H}H_z^{-1}h_z - HT_z + u'g/N = O_p(1/\sqrt{N}), \quad T_{5,g} = -\sqrt{N}(\hat{\sigma}_{uv} - \sigma_{uv})\tilde{K}/N = O_p(\tilde{K}/N), \\
T_{6,g} &= -f'(I - P^{\tilde{K}})g/N = O_p(\Delta^{1/2}(\tilde{K})\Delta_g^{1/2}(\tilde{K})), \quad T_{7,g} = -f'(I - P^{\tilde{K}})v/\sqrt{N} = O_p(\Delta^{1/2}(\tilde{K})), \\
Z_{H,g} &= -u'(I - P^{\tilde{K}})g/N - T_{1,H}T_z - T_{2,H}T_z = O_p(\Delta_g^{1/2}(\tilde{K})/\sqrt{N} + \Delta(\tilde{K})/\sqrt{N} + 1/N),
\end{aligned}$$

$$\begin{aligned}
\tilde{G}(K) &= \frac{W'(I - P^K)\hat{\varepsilon}}{\sqrt{N}} + \frac{K}{\sqrt{N}}\hat{\sigma}_{uv} - u'v/\sqrt{N} + u'u/NH_z^{-1}h_z + u'u/NT_z = \sum_{j=1}^5 T_{j,G} + o_p(\bar{\rho}_{K,N}), \\
T_{1,G} &= -f'(I - P^K)f/NH_z^{-1}h_z = O_p(\Delta(K)), \quad T_{2,G} = u'P^Ku/NH_z^{-1}h_z = O_p(K/N), \\
T_{3,G} &= f'(I - P^K)g/N = O_p(\Delta^{1/2}(K)\Delta_g^{1/2}(K)), \\
T_{4,G} &= f'(I - P^K)v/\sqrt{N} = O_p(\Delta^{1/2}(K)), \\
T_{5,G} &= -(u'P^Kv/\sqrt{N} - K/\sqrt{N}\sigma_{uv}) = O_p(\sqrt{K}/\sqrt{N})
\end{aligned}$$

where we use the similar calculations as in the proof of Propositions 3.1-3.3 using Lemma S1.2 and S1.3. We can show that $Z_H = o_p(\bar{\rho}_{K,N})$, $Z_{H,g} = o_p(\bar{\rho}_{K,N})$ and the above decomposition satisfies the condition of the Assumption 5.3 by assuming $\bar{\rho}_{\tilde{K},N} = o(\bar{\rho}_{K,N})$ as $\tilde{K}, K \rightarrow \infty$.

Next, we observe that

$$\begin{aligned}
\hat{H}_g \tilde{G}(K) &= (H_g + \Lambda_g + \sum_{j=1}^7 T_{j,g} + o_p(\bar{\rho}_{K,N}))(\sum_{j=1}^5 T_{j,G} + o_p(\bar{\rho}_{K,N})) \\
&= (H_g + \Lambda_g) \sum_{j=1}^5 T_{j,G} + o_p(\bar{\rho}_{K,N})
\end{aligned}$$

by Assumption 5.3 and

$$\begin{aligned}
\hat{H}^{-1} \hat{H}_g^2 (\tilde{R}(K) - \hat{\sigma}_u^2 \frac{K}{N}) &= (H^{-1} + o_p(1))((H_g + \Lambda_g)^2 + o_p(1))(\frac{f'(I - P^K)f}{N} + o_p(\bar{\rho}_{K,N})) \\
&= H^{-1} H_g^2 \frac{f'(I - P^K)f}{N} + H^{-1} \Lambda_g^2 \frac{f'(I - P^K)f}{N} \\
&\quad + 2H^{-1} H_g \Lambda_g \frac{f'(I - P^K)f}{N} + o_p(\bar{\rho}_{K,N}).
\end{aligned}$$

In sum,

$$\begin{aligned}
 \hat{L}_{IR}(K) &= 2\hat{H}_g\hat{\sigma}_{uv}\frac{K}{\sqrt{N}} + \hat{\sigma}_{uv}^2\frac{K^2}{N} + (\hat{\sigma}_v^2 + \frac{2\hat{H}_g^2}{\hat{H}})(\tilde{R}(K) - \hat{\sigma}_u^2\frac{K}{N}) - 2\hat{H}_g\tilde{G}(K) + 2\frac{K}{N}\hat{R}_g(K) \\
 &= 2(H_g + \Lambda_g)\sigma_{uv}\frac{K}{\sqrt{N}} + \sigma_{uv}^2\frac{K^2}{N} + (\sigma_v^2 + \frac{2H_g^2}{H})(\frac{f'(I - P^K)f}{N}) + 2H^{-1}\Lambda_g^2\frac{f'(I - P^K)f}{N} \\
 &\quad + 4H^{-1}H_g\Lambda_g\frac{f'(I - P^K)f}{N} - 2(H_g + \Lambda_g)\sum_{j=1}^5 T_{j,G} \\
 &\quad + \frac{2K}{N}\sigma_{v,z}^2\sigma_u^2HH_z^{-1}(I - HH_z^{-1}) - 2HH_z^{-1}\sigma_v^2\frac{f'P_z(I - P^K)f}{N} \\
 &\quad + 2\frac{\sum_i f_i P_{ii,z} f'_i}{N} H_z^{-1}\sigma_v^2\frac{f'(I - P^K)f}{N} + o_p(\bar{\rho}_{K,N}),
 \end{aligned}$$

where $\sigma_{v,z}^2 = \mathbb{E}[v'v_z/N|X] = \mathbb{E}[v'_z v_z/N|X]$, $v_z = P_z v$.

Next, we calculate expectation of each term in $\hat{L}_{IR}(K)$. First,

$$\begin{aligned}
 \mathbb{E}[\Lambda_g^2|X] &= \mathbb{E}[hh' - 2hh'_z H_z^{-1}H + HH_z^{-1}h_z h'_z H_z^{-1}H|X] \\
 &= \sigma_v^2 H - 2\frac{\sum_i f_i P_{ii,z} f'_i}{N} \sigma_v^2 H_z^{-1}H + HH_z^{-1}\frac{\sum_i f_i P_{ii,z} f'_i}{N} \sigma_v^2 H_z^{-1}H.
 \end{aligned}$$

Second, $\mathbb{E}[\Lambda_g K/\sqrt{N}\sigma_{uv}|X] = \mathbb{E}[H^{-1}H_g\Lambda_g\frac{f'(I - P^K)f}{N}|X] = 0$ as $\mathbb{E}[\Lambda_g|X] = 0$. Also note that $\mathbb{E}[H_g T_{1,G}|X] = 0$,

$$\mathbb{E}[H_g T_{2,G}|X] = \mathbb{E}[H_g \frac{u'P^K u}{N} H_z^{-1} \frac{f'P_z v}{\sqrt{N}}] = -H_g \frac{\sum_{i=1}^K P_{ii}^K P_{ii,z} \mathbb{E}[u_i u'_i v_i|X] H_z^{-1} f_i}{N^{3/2}} = o_p(\bar{\rho}_{K,N}).$$

Next, $\mathbb{E}[H_g T_{3,G}] = H_g f'(I - P^K)g/N$,

$$\mathbb{E}[H_g T_{4,G}] = \mathbb{E}[H_g f'(I - P^K)v/\sqrt{N}|X] = 0$$

$$\mathbb{E}[H_g T_{5,G}] = -H_g \mathbb{E}[(u'P^K v/\sqrt{N} - K/\sqrt{N}\sigma_{uv})|X] = 0$$

by Lemma S1.2 (h). Next, we have

$$\begin{aligned}
 \mathbb{E}[\Lambda_g T_{1,G}|X] &= -\mathbb{E}[(h - HH_z^{-1}h_z)\frac{f'(I - P^K)f}{N} H_z^{-1}h_z|X] = -\frac{\sum_i f_i P_{ii,z} f'_i}{N} \sigma_v^2 H_z^{-1} \frac{f'(I - P^K)f}{N} \\
 &\quad + HH_z^{-1} \frac{\sum_i f_i P_{ii,z}^2 f'_i}{N} \sigma_v^2 H_z^{-1} \frac{f'(I - P^K)f}{N}.
 \end{aligned}$$

Note that

$$\begin{aligned}
 \mathbb{E}[f'vv'_z f H_z^{-1}u'P^K u/N^2|X] &= \frac{\sum_i f_i P_{ii}^K f'_i H_z^{-1}}{N^2} \mathbb{E}[v_i v_{z,i} u_i^2|X] \\
 &\quad + \sum_{i \neq j} f_i \mathbb{E}[v_i v_{z,i}|X] f'_i P_{jj}^K H_z^{-1} \mathbb{E}[u_j^2|X]/N^2 + \sum_{i \neq j} f_i f'_j P_{ij}^K H_z^{-1} \mathbb{E}[v_i u_i|X] \mathbb{E}[v_{z,i} u_i|X]/N^2 \\
 &= O_p(K/N^2) + \frac{K}{N} H H_z^{-1} \sigma_{v,z}^2 \sigma_u^2.
 \end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}[\Lambda_g T_{2,G}|X] &= \mathbb{E}[(h - HH_z^{-1}h_z) \frac{u'P^K u}{N} H_z^{-1} h_z | X] \\ &= \frac{K}{N} HH_z^{-1} \sigma_{v,z}^2 \sigma_u^2 - HH_z^{-1} \frac{K}{N} HH_z^{-1} \sigma_{v,z}^2 \sigma_u^2.\end{aligned}$$

Finally, by the zero third-moment condition assumption,

$$\mathbb{E}[\Lambda_g T_{5,G}|X] = -\frac{\sum_i f_i P_{ii}^K \mathbb{E}[v_i^2 u_i | X]}{N} + HH_z^{-1} \frac{\sum_i f_i P_{ii,z} P_{ii}^K \mathbb{E}[v_i^2 u_i | X]}{N} = 0,$$

and

$$\mathbb{E}[\Lambda_g T_{3,G}|X] = \mathbb{E}[(h - HH_z^{-1}h_z)|X] f'(I - P^K) g / N = 0$$

$$\begin{aligned}\mathbb{E}[\Lambda_g T_{4,G}|X] &= \mathbb{E}[(h - HH_z^{-1}h_z) f'(I - P^K) v / \sqrt{N} | X] \\ &= \sigma_v^2 \frac{f'(I - P^K) f}{N} - HH_z^{-1} \frac{f' P_z (I - P^K) f}{N} \sigma_v^2.\end{aligned}$$

Therefore,

$$\mathbb{E}[\hat{L}_{IR}(K)|X] = 2H_g \sigma_{uv} \frac{K}{\sqrt{N}} + \sigma_{uv}^2 \frac{K^2}{N} + (\sigma_v^2 + \frac{2H_g^2}{H}) \left(\frac{f'(I - P^K) f}{N} \right) - 2H_g \frac{f'(I - P^K) g}{N} + o_p(\bar{\rho}_{K,N}).$$

For JIVE2 and HLIM/HFUL estimators, we have following decompositions for $\tilde{R}_D(K), \tilde{G}_D(K)$,

$$\begin{aligned}\tilde{R}_D(K) &= \frac{W'(I - (P^K - D^K))W}{N} - u'u/N \\ &= \frac{f'(I - (P^K - D^K))f}{N} - \left(\frac{u'P^K u}{N} - \sigma_u^2 \frac{K}{N} \right) + \left(\frac{\sum_i u_i u'_i P_{ii}^K}{N} - \sigma_u^2 \frac{K}{N} \right) + o_p(\bar{\rho}_{K,N}) \\ &= \frac{f'(I - (P^K - D^K))f}{N} + o_p(K/N) + o_p(\bar{\rho}_{K,N}),\end{aligned}$$

$$\tilde{G}_D(K) = W'(I - (P^K - D^K))\hat{\varepsilon}/\sqrt{N} - u'v/\sqrt{N} + u'u/NH_z^{-1}h_z + u'u/NT_z = \sum_{j=1}^6 T_{j,G} + o_p(\bar{\rho}_{K,N}),$$

$$T_{1,G} = -f'(I - (P^K - D^K))f/NH_z^{-1}h_z = O_p(\Delta(K) + K/N),$$

$$T_{2,G} = f'(I - (P^K - D^K))g/N = O_p(\Delta^{1/2}(K)\Delta_g^{1/2}(K) + K/N),$$

$$T_{3,G} = f'(I - P^K)v/\sqrt{N} = O_p(\Delta^{1/2}(K)),$$

$$T_{4,G} = -(u'P^K v/\sqrt{N} - \frac{K}{\sqrt{N}}\sigma_{uv}) = O_p(\sqrt{K}/\sqrt{N}),$$

$$T_{5,G} = \left(\frac{\sum_i P_{ii}^K u_i v_i}{\sqrt{N}} - \frac{K}{\sqrt{N}}\sigma_{uv} \right) = O_p(\sqrt{K}/\sqrt{N}), T_{6,G} = \frac{\sum_i P_{ii}^K f_i v_i}{\sqrt{N}} = O_p(\sqrt{K}/\sqrt{N})$$

where we use similar calculations as in the proof of Propositions 3.4-3.5. The results for other estimators follow similarly. The results for the case $\gamma > 1/2$ also follows similarly as in the proof of Propositions 4.1-4.5. This completes the proof. \square

S2. AUXILIARY RESULTS

S2.1. MSE Approximation of 2SLS under $K = O(\sqrt{N})$

This subsection provides MSE approximation of 2SLS under $K = O(\sqrt{N})$ that is faster than those imposed in Proposition 3.1. This rate is considered in many-instruments literature, such as Morimune (1983) and Hahn and Hausman (2005). Note that the assumption in Proposition 3.1 limits the growth rate of the number of instruments, $K = o(\sqrt{N})$, and this guarantees the first-order asymptotic properties of the 2SLS estimator, where the bias from many instruments is of order $O_p(K/\sqrt{N})$. Therefore, we find a different decomposition here rather than the equation (2.10) in the main paper.

Specifically, in the next corollary, we will find the following first-order approximations of the conditional MSE,

$$\begin{aligned} N(\hat{\delta}(K) - \delta_0)(\hat{\delta}(K) - \delta_0)' &= \hat{Q}(K) + o_p(1), \\ E(\hat{Q}(K)|X) &= \sigma_v^2 H^{-1} + H^{-1} H_g H_g' H^{-1} + L(K) + o_p(1), \quad K \rightarrow \infty, N \rightarrow \infty. \end{aligned} \quad (\text{S2.1})$$

COROLLARY S2.1. Suppose that Assumptions 2.1, 2.2, 2.3 are satisfied with $\gamma = 1/2$. If $K/\sqrt{N} \rightarrow \alpha$ ($0 < \alpha < \infty$), $\sigma_{uv} \neq 0$, $H_g \neq 0$, then the approximate MSE for the 2SLS estimator satisfies decomposition (S2.1) with the following terms

$$L(K) = H^{-1} \left[\frac{K}{\sqrt{N}} (H_g \sigma'_{uv} + \sigma_{uv} H_g') + \sigma_{uv} \sigma'_{uv} \frac{K^2}{N} \right] H^{-1}. \quad (\text{S2.2})$$

Proof: Similar to the proof of Proposition 3.1, the 2SLS estimator has the following form with locally invalid instruments specification

$$\sqrt{N}(\hat{\delta}(K) - \delta_0) = \hat{H}^{-1} \hat{h} + \hat{H}^{-1} \hat{h}_g,$$

where $\hat{H} = W' P^K W / N$, $\hat{h} = W' P^K v / \sqrt{N}$, $\hat{h}_g = W' P^K g / N$. Also, \hat{h} , \hat{H} and \hat{h}_g are decomposed as

$$\begin{aligned} \hat{h} &= h + T^h, \\ h &= \frac{f'v}{\sqrt{N}} + u' P^K v / \sqrt{N} = o_p(1), \quad T^h = -f'(I - P^K)v / \sqrt{N} = o_p(1), \\ \hat{H} &= H + T^H, \\ T^H &= -f'(I - P^K)f / N + (u' f + f'u) / N \\ &\quad + (u' P^K u - u'(I - P^K)f - f'(I - P^K)u) / N = o_p(1), \\ \hat{h}_g &= H_g + T^g, \\ T^g &= -f'(I - P^K)g / N + u' g / N - u'(I - P^K)g / N = o_p(1). \end{aligned}$$

It is important to note that h includes term $u' P^K v / \sqrt{N}$ which is $O_p(K/\sqrt{N}) = O_p(1)$ as $K/\sqrt{N} = O(1)$, where $o_p(1)$ results immediately follows from the proof of Proposition 3.1.

By using similar arguments as in the proof of Lemma S1.1, we have

$$\sqrt{N}(\hat{\delta}(K) - \delta_0) = H^{-1} \tilde{\tau} + H^{-1} \tilde{\tau}_g + o_p(1)$$

where $\tilde{\tau} = h + T^h - T^H H^{-1} h = h + o_p(1)$, $\tilde{\tau}_g = H_g + T^g - T^H H^{-1} H_g = H_g + o_p(1)$

by using $T^h = o_p(1)$, $T^H = o_p(1)$, $h = O_p(1)$, $H_g = O_p(1)$, $T^g = o_p(1)$, and $H^{-1} = O_p(1)$, $\hat{H}^{-1} = O_p(1)$.

Since $\sqrt{N}(\hat{\delta}(K) - \delta_0) = H^{-1}h + H^{-1}H_g + o_p(1)$, it follows that

$$N(\hat{\delta}(K) - \delta_0)(\hat{\delta}(K) - \delta_0)' = H^{-1}(hh' + hH_g' + H_gh' + H_gH_g')H^{-1} + o_p(1).$$

First, we have

$$\begin{aligned}\mathbb{E}(hh'|X) &= \mathbb{E}\left(\frac{f'vv'f}{N}\right) + \mathbb{E}\left(\frac{f'vv'P^Ku}{N}\right) + \mathbb{E}\left(\frac{u'P^Kvv'f}{N}\right) + \mathbb{E}\left(\frac{u'P^Kvv'P^Ku}{N}\right) \\ &= \sigma_v^2 H + \sigma_{uv}\sigma'_{uv}\frac{K^2}{N} + O_p\left(\frac{K}{N}\right) + o_p\left(\frac{K^2}{N}\right) \\ &= \sigma_v^2 H + \sigma_{uv}\sigma'_{uv}\frac{K^2}{N} + o_p(1)\end{aligned}$$

by Lemma S1.2 (i), (j), and $K/N = o(K/\sqrt{N})$, $K^2/N = O(1)$. Second,

$$\mathbb{E}(hH_g'|X) = \mathbb{E}\left[\left(\frac{f'v}{\sqrt{N}} + \frac{u'P^Kv}{\sqrt{N}}\right)H_g'|X\right] = \frac{K}{\sqrt{N}}\sigma_{uv}H_g'$$

by Lemma S1.3 (d). Therefore, we have the approximate MSE for the 2SLS estimator as in decomposition (S2.1). \square

S2.2. Rothenberg (1984)'s approximation with invalid instruments

Here, we derive the bias and variance of the 2SLS estimator similar to the approaches in Rothenberg (1984). We consider a model

$$y = W\delta_0 + \frac{\Psi\tau}{\mu} + v, \quad W = \Psi\pi + u,$$

where δ_0 is a scalar, Ψ is a $K \times 1$ (nonrandom) instrument matrix, $\mu^2 = \pi'\Psi'\Psi\pi/\sigma_u^2$ is a concentration parameter, and (v_i, u_i) are bivariate normal with mean zero, variance σ_v^2 , σ_u^2 , and the correlation coefficient ρ . 2SLS estimator can be written as

$$\sqrt{\frac{\pi'\Psi'\Psi\pi}{\sigma_v^2}}(\hat{\delta}_{2SLS} - \delta_0) = \frac{X_1 + (X_2 + \bar{\mu}A)/\mu + \bar{\mu}B/\mu^2}{1 + 2Y_1/\mu + Y_2/\mu^2}$$

where $X_1 = \frac{\pi'\Psi'v}{\sigma_v\sqrt{\pi'\Psi'\Psi\pi}}$, $X_2 = \frac{u'Pv}{\sigma_v\sigma_u}$, $Y_1 = \frac{\pi'\Psi'u}{\sigma_u\sqrt{\pi'\Psi'\Psi\pi}}$, $Y_2 = \frac{u'Pu}{\sigma_u^2}$, $A = \frac{\pi'\Psi'\Psi\tau}{\sqrt{\tau'\Psi'\Psi\tau}\sqrt{\pi'\Psi'\Psi\pi}}$, $B = \frac{\tau'\Psi'u}{\sigma_u\sqrt{\tau'\Psi'\Psi\tau}}$, $\bar{\mu}^2 = \tau'\Psi'\Psi\tau/\sigma_u^2$, and $P = \Psi(\Psi'\Psi)^{-1}\Psi'$. Under conventional asymptotics (μ is large, K is small, and $\bar{\mu}$ is small), we have following expansions similar to Rothenberg (1984),

$$\begin{aligned}&\sqrt{\frac{\pi'\Psi'\Psi\pi}{\sigma_v^2}}(\hat{\delta}_{2SLS} - \delta_0) \\ &= X_1 + \underbrace{\frac{X_2 - 2X_1Y_1 + \bar{\mu}A}{\mu}}_{\equiv \tilde{X}} + \frac{4X_1Y_1^2 - X_1Y_2 - 2Y_1X_2 - 2Y_1\bar{\mu}A + \bar{\mu}B}{\mu^2} + o(1/\mu^2).\end{aligned}$$

Note that (X_1, Y_1) and (X_1, B) are bivariate normal with mean zeros, variances 1, and correlation coefficient ρ , and $\mathbb{E}(X_2) = K\rho$, $Var(X_2) = K(1 + \rho^2)$, $\mathbb{E}(Y_2) = K$, $Var(Y_2) = 2K$, $\mathbb{E}(B) = 0$, $Var(B) = 1$. To the order $o(\mu^{-2})$, we have

$$\mathbb{E}(\tilde{X}) = \frac{(K-2)\rho}{\mu} + \frac{\tilde{\mu}}{\mu^2}, \quad \text{Var}(\tilde{X}) = 1 - \frac{(K-4)(1+3\rho^2) + 4\rho^2 + 2\rho\bar{\mu}}{\mu^2}$$

where $\tilde{\mu} = \pi' \Psi' \Psi \tau / (\sigma_u \sigma_v)$.

S3. GENERAL INSTRUMENT SELECTION CRITERIA

In this section, we provide invalidity-robust (IR) instrument selection criteria for the general vector endogenous variable case. The selection of the instrument K is based on the MSE approximations in Section 3 and 4 of the main paper. Specifically, we choose K to minimize $\hat{L}_\lambda(K)$ which is an estimate of $L_\lambda(K) = \lambda' L(K) \lambda$ with user-specified $\lambda \in \mathbb{R}^p$, where $L(K)$ is a part of the dominating term in the MSE approximations in Propositions 3.1-3.5.

Let $\tilde{\delta}$ be some preliminary estimator, e.g., the IV estimator using all available instruments, or IV estimator where the instruments \tilde{K} are chosen to minimize the first-stage CV or Mallows' criteria. Let $\tilde{\varepsilon}$ as residuals $\tilde{\varepsilon} = y - W\tilde{\delta}$, and let $\hat{H} = W'P^{\tilde{K}}W/N - \hat{\sigma}_{u_\lambda}^2 \tilde{K}/N$ as a preliminary estimator of $H = f'f/N$. Also, let $\tilde{u} = (I - P^{\tilde{K}})W$ as a preliminary residual vector of the first-stage reduced-form regression. Define $\tilde{u}_\lambda = \tilde{u}\hat{H}^{-1}\lambda$ and

$$\hat{\sigma}_v^2 = \tilde{\varepsilon}'\tilde{\varepsilon}/N, \quad \hat{\sigma}_u^2 = \tilde{u}'\tilde{u}/N, \quad \hat{\sigma}_{uv} = \tilde{u}'\tilde{\varepsilon}/N, \quad \hat{\sigma}_{u_\lambda v}^2 = \tilde{u}_\lambda'\tilde{u}_\lambda/N, \quad \hat{\sigma}_{u_\lambda v} = \tilde{u}_\lambda'\tilde{\varepsilon}/N.$$

Let $\hat{\delta}$ as an IV estimator with known valid instruments z_i , and residuals as $\hat{\varepsilon} = y - W\hat{\delta}$. For example, 2SLS estimator $\hat{\delta} = (W'P_z W)^{-1}(W'P_z y)$ where $z = [z_1, \dots, z_N]', P_z = z(z'z)^{-1}z'$. Finally, let $\hat{H}_g = W'P^{\tilde{K}}\hat{\varepsilon}/\sqrt{N} - \tilde{K}/\sqrt{N}\hat{\sigma}_{u_\lambda v}$ as a preliminary estimator of $H_g = f'g/N$, let $\lambda'\hat{H}^{-1}\hat{H}_g = \hat{H}_{g\lambda}$. It is important to note that all of these preliminary estimates remain fixed (do not depend on K) while the criterion is calculated for a different set of instruments. Based on Propositions 3.1-3.5, the invalidity-robust criterion $\hat{L}_{IR}(K)$ is

$$\begin{aligned} \text{2SLS : } & \hat{H}_{g\lambda}\hat{\sigma}_{u_\lambda v} \frac{2K}{\sqrt{N}} + \hat{\sigma}_{u_\lambda v}^2 \frac{K^2}{N} + \hat{\sigma}_v^2(\hat{R}_\lambda(K) - \hat{\sigma}_{u_\lambda}^2 \frac{K}{N}) \\ & + 2\hat{H}_{g\lambda}(\hat{F}_\lambda(K) - \hat{G}_\lambda(K)), \\ \text{LIML : } & \hat{\sigma}_v^2(\hat{R}_\lambda(K) - \frac{\hat{\sigma}_{u_\lambda v}^2 K}{\hat{\sigma}_v^2 N}) + 2\hat{H}_{g\lambda}(\hat{F}_\lambda(K) - \hat{G}_\lambda(K)), \\ \text{B2SLS : } & \hat{\sigma}_v^2(\hat{R}_\lambda(K) + \frac{\hat{\sigma}_{u_\lambda v}^2 K}{\hat{\sigma}_v^2 N}) + 2\hat{H}_{g\lambda}(\hat{F}_\lambda(K) - \hat{G}_\lambda(K)), \\ \text{JIVE2 : } & \hat{\sigma}_v^2(\hat{R}_\lambda(K) + \frac{\hat{\sigma}_{u_\lambda v}^2 K}{\hat{\sigma}_v^2 N}) + 2\hat{H}_{g\lambda}(\hat{F}_{\lambda D}(K) - \hat{G}_{\lambda D}(K)), \\ \text{HLIM/HFUL : } & \hat{\sigma}_v^2(\hat{R}_\lambda(K) - \frac{\hat{\sigma}_{u_\lambda v}^2 K}{\hat{\sigma}_v^2 N}) + 2\hat{H}_{g\lambda}(\hat{F}_{\lambda D}(K) - \hat{G}_{\lambda D}(K)), \end{aligned}$$

where $\hat{u}^K = (I - P^K)W$, $\hat{u}_\lambda^K = \hat{u}^K \hat{H}^{-1}\lambda$ denote residual vectors, $\hat{F}_\lambda(K) = \lambda' \hat{H}^{-1}(\hat{R}(K) - \hat{\sigma}_u^2 \frac{K}{N}) \hat{H}^{-1} \hat{H}_g$, $\hat{F}_{\lambda D}(K) = \lambda' \hat{H}^{-1} \frac{W'(I - (P^K - D^K))W}{N} \hat{H}^{-1} \hat{H}_g$, $\hat{G}_\lambda(K) = \lambda' \hat{H}^{-1} \frac{W'(I - P^K)\hat{\varepsilon}}{\sqrt{N}} + \lambda' \hat{H}^{-1} K / \sqrt{N} \hat{\sigma}_{u_\lambda v}$, and $\hat{G}_{\lambda D}(K) = \lambda' \hat{H}^{-1} \frac{W'(I - (P^K - D^K))\hat{\varepsilon}}{\sqrt{N}}$. For the $\hat{R}(K)$ and $\hat{R}_\lambda(K)$,

we can use the Mallows' criterion

$$\hat{R}_\lambda(K) = \frac{\hat{u}_\lambda^{K'} \hat{u}_\lambda^K}{N} + 2\hat{\sigma}_{u\lambda}^2 \frac{K}{N}, \quad \hat{R}(K) = \frac{\hat{u}^{K'} \hat{u}^K}{N} + 2\hat{\sigma}_u^2 \frac{K}{N}.$$

CV criterion can also be used

$$\hat{R}_\lambda(K) = \frac{1}{N} \sum_{i=1}^N \frac{(\hat{u}_{\lambda i}^K)^2}{(1 - P_{ii}^K)^2}, \quad \hat{R}(K) = \frac{1}{N} \sum_{i=1}^N \frac{(\hat{u}_i^K)^2}{(1 - P_{ii}^K)^2}.$$

Note that when $\hat{H}_{g\lambda} = 0$, $\hat{L}_{IR}(K)$ reduces to the criterion in Donald and Newey (2001).

S4. ADDITIONAL SIMULATIONS

We report additional simulation results in addition to the model considered in the main paper. The main model we consider is

$$\begin{aligned} y_i &= x_i \beta_0 + \frac{\tau' Z_i}{N^\gamma} + v_i, \\ x_i &= \pi' Z_i + u_i, \end{aligned}$$

where we set $\beta_0 = 0.1$, instruments Z_i as $N(0, I_{\bar{K}})$ and the errors (v_i, u_i) as bivariate normal with mean zero, variances 1, and covariances $\sigma_{uv} - \pi'\tau/N^\gamma$ to ensure an endogeneity $Cov(x_i, \varepsilon_i) = \sigma_{uv}$. This is same simulation design as in the main paper without an intercept in the model. We consider $\gamma \in \{\infty, 1, 1/2, 1/3\}$, $(\bar{K}, N) \in \{(20, 100), (30, 1000)\}$ and the first-stage $R^2 \in \{0.1, 0.01\}$. We use the same simulation design of Donald and Newey (2001) allowing for potentially invalid instruments.

Model 1 (Baseline Specification)

In the baseline specification, we set $\pi = (\pi_1, \dots, \pi_{\bar{K}})', \pi_k = c(\bar{K})(1 - k/(\bar{K} + 1))^4, \forall k$ where $c(\bar{K})$ is chosen to set the first-stage R^2 . We also set $\tau = (\tau_1, \dots, \tau_{\bar{K}})'$,

$$\tau_k = 0 \text{ for } k = 1, \quad \tau_k = 0.5 \text{ for } k = 2, \dots, \bar{K}/2, \text{ and } \tau_k = 0 \text{ for } k > \bar{K}/2,$$

Here, we explore Model 1 with different endogeneity parameters $\sigma_{uv} \in \{0.2, 0.5, 0.8\}$. We consider the same specification for $\tau = (\tau_1, \dots, \tau_{\bar{K}})'$ as in the main paper.

Model 2 (equal strength of instruments)

In Model 2, we explore the following specifications for π (equal strength),

$$\pi_k = \sqrt{\frac{R^2}{\bar{K}(1 - R^2)}}$$

for all k . This is the case where the instruments have equal strengths as in Donald and Newey (2001, Model 2). We again use the same specification for $\tau = (\tau_1, \dots, \tau_{\bar{K}})'$ as in the baseline specification with endogeneity $\sigma_{uv} = 0.5$.

Model 3 (heteroskedastic error)

In Model 3, we analyze a heteroskedastic error case. Specifically, we set $\mathbb{E}[v_i^2 | Z_i] = (1 + Z_{1i} + \dots + Z_{\bar{K}i})^2$. We consider the same specification for $\tau = (\tau_1, \dots, \tau_{\bar{K}})'$ as follows;

$$\tau_k = 0 \text{ for } k = 1, \quad \tau_k = 0.5 \text{ for } k = 2, \dots, \bar{K}/2, \text{ and } \tau_k = 0 \text{ for } k > \bar{K}/2.$$

Model 4 (wrong order of instrument strengths) In Model 4, we consider the case where the order of the IV strengths is wrong,

$$\pi_k = c(\bar{K})(1 - (\bar{K} + 1 - k)/(\bar{K} + 1))^4,$$

with the same specification for τ as above in Models 1-3.

Model 5 (different instrument validity)

Using the baseline specification (Model 1) for instrument strength, $\pi = (\pi_1, \dots, \pi_{\bar{K}})', \pi_k = c(\bar{K})(1 - k/(\bar{K} + 1))^4, \forall k$, we also consider the various specifications in terms of instrument invalidity. We vary τ as follows; 1) $\tau \propto (0, 1, 1, 1, 0, \dots, 0)$, 2) $\tau \propto (0, 1, 0, \dots, 0)$, 3) $\tau \propto (0, 0, 0, 0, 1, \dots, 1)$.

Median bias (Bias), interdecile range (IDR), root mean square error (MSE), and root trimmed mean square error (TMSE) of the OLS, 2SLS, averaging GMM estimator (GMM-AVE) in Cheng et al. (2019), LIML, FULL (with $C = 1$), JIVE2, HLIM and HFUL are reported. For models 1 ($\sigma_{uv} \in \{0.2, 0.8\}$), 3, 4 and 5, we only report the case $(N, K) = (100, 20)$ for brevity. The results are calculated using 10,000 simulation replications.

Tables 1-4 contain results for the Model 1 (baseline specification) with $\sigma_{uv} = 0.5$. The $\gamma = \infty$ case in Tables 1-4 replicate the Monte Carlo studies in Tables I-II in Donald and Newey (2001) with all valid instruments.

Tables 5-8 contain results for Model 1 with different σ_{uv} . Tables 9-12 give results for Model 2 and they replicate Tables III-IV in Donald and Newey (2001) when $\gamma = \infty$ (with $\sigma_{uv} = 0.5$). Tables 13-14 and Tables 15-16 give results for Model 3 and Model 4, respectively. Tables 17-22 give results for Model 5 with different specifications of τ (instruments invalidity).

In Model 1 (Tables 1-4) with endogeneity $\sigma_{uv} = 0.5$, the results are qualitatively similar to the results in the main paper (with an intercept in the model), except that the jackknife versions of the k -class estimators using only valid instruments ($K = 1$) perform considerably worse. For Model 2 (Tables 9-12), similar to Donald and Newey (2001), this different specification (equal instrument strengths) tends to increase the median bias/MAD as well as dispersion (IDR/MSE) in most cases compared with the baseline specification (decreasing order of π). Model 1 (Tables 7-8) with high endogeneity σ_{uv} and heteroskedastic setup in Model 3 (Tables 13-14) also tends to increase the bias and the TMSE compared with the baseline specification (Tables 1-2). However, the results are qualitatively similar to the baseline specification; Fuller/HFUL estimators combined with instrument selection perform well with lower IDR/MSE. This suggests that when we have no prior information on which instruments are relatively important or in heteroskedastic setups, instrument selections can still be useful for estimators with finite-sample moments, but less so for “no-moment” estimators.

Next consider Model 4 when the order of the IV strengths is wrong (Tables 15-16). First, we consider the strong identification case ($R^2 = 0.1$). When the degree of invalidity is small (i.e., $\gamma = \infty, \gamma = 1$), DN/IR tend to choose a relatively small number of instruments (which are weak), while K^{MSE} that minimizes the trimmed MSE is 20 (using all instruments) for most cases (2SLS, FULL, HLIM, and HFUL), and this leads to worse performances than the baseline scenario with the correct order of IV strengths. However, when invalidity increases $\gamma = 1/3$, this also makes it costly to include strong, but invalid instruments ($K^{MSE} = 1$ for FULL/HLIM/HFUL), and instrument selection

criteria tend to choose a smaller number of instruments than the baseline specifications by excluding weak invalid instruments (the median \hat{K} for DN/IR is 1 for FULL/HFUL). In the weakly identified cases ($R^2 = 0.01$), all instruments are weak instruments anyway, so the performances are similar to the baseline specification.

Finally, we consider Model 5 with alternative specifications on τ . As we expected, the results are very similar to the baseline specification ($\tau \propto (0, 1, 1, 1, \dots, 1, 0, 0, \dots, 0)$) when the degree of invalidity is small (i.e., $\gamma = \infty, \gamma = 1$), so we focus on the case $\gamma = 1/3$ where the degree of invalidity is relatively large.

Tables 17-18 report the case $\tau \propto (0, 1, 1, 1, 0, \dots, 0)$. In the strong identification case ($R^2 = 0.1$), some relatively strong instruments become valid instruments compared with the baseline specification. Thus, the median bias/MAD and IDR/TMSE of using all instruments are slightly lower than those in the baseline model except MSE of “no-moment” estimators. IR tends to choose slightly more valid instruments than the baseline case on average, however, the differences are relatively small. In the weakly identified case ($R^2 = 0.01$), all valid instruments are weak instruments anyway, so the performances are no better than the baseline scenario as DN/IR tends to choose only a small number of instruments in such cases.

Tables 19-20 consider the case $\tau \propto (0, 1, 0, 0, \dots, 0)$. In the strong identification case, the third, fourth and fifth strong instruments become valid instruments compared with the above case, and we find that the performance of using all instruments and the instrument selections are substantially better than the baseline specification.

Tables 21-22 consider $\tau \propto (0, 0, 0, 0, 1, \dots, 1)$. In this case, many invalid instruments mainly coincide with weak instruments, thus we find that DN/IR perform better by not including weak and invalid instruments in terms of lower bias/MAD and IDR/MSE in most cases compared with the baseline specification.

S4.1. Simulation Design in Hausman et al. (2012)

In this subsection, we explore the following simulation design as in Hausman et al. (2012). Although we can consider the main simulation design with a scalar endogenous regressor as a case where the covariates (including an intercept) have already been partialled out, we note that theoretical results of HLIM/HFUL in Hausman et al. (2012, Assumption 1) include an intercept in the model. The model we consider is

$$y_i = \beta_0 + \beta_1 x_i + \frac{\tau'(D_i - \frac{1}{2})z_i}{N^\gamma} + \varepsilon_i,$$

$$x_i = \pi z_i + u_i,$$

where $z_i \sim N(0, 1)$, $u_i \sim N(0, 1)$, π is chosen so that the concentration parameter $\mu^2 = n\pi^2 = 8, 32$, $D'_i = (D_{i,1}, D_{i,2}, \dots, D_{i,\bar{K}-5})$, $D_{i,k} \in \{0, 1\}$, $Pr(D_{i,k} = 1) = 1/2$, and the full set of available instruments is

$$Z'_i = (1, z_i, z_i^2, z_i^3, z_i^4, z_i D_{i,1}, z_i D_{i,2}, \dots, z_i D_{i,\bar{K}-5}).$$

The structural error term ε is given by

$$\varepsilon = \rho u + \sqrt{\frac{1 - \rho^2}{\phi^2 + (0.86)^4}} (\phi v_1 + 0.86 v_2)$$

where $v_1 \sim N(0, z^2)$, $v_2 \sim N(0, (0.86)^2)$, and v_1, v_2 are independent of u . We set $N = 800$, $\rho = 0.3$, and choose ϕ so that the R -squared for the regression of ε^2 on the instruments is 0. Results are based on 1,000 simulations.

We consider various IV estimators with different numbers of instruments $K \in \mathcal{K} = \{2, 3, \dots, \bar{K} = 30\}$ where we use the first K elements of Z_i . For examples, the instruments are $(1, z_i)$ for $K = 2$, $(1, z_i, z_i^2, z_i^3, z_i^4)$ for $K = 5$, and $(1, z_i, z_i^2, z_i^3, z_i^4, z_i D_{i,1}, z_i D_{i,2}, \dots, z_i D_{i,25})$ for $K = 30$.

Again, γ captures “degree of invalidity” and we vary $\gamma = \infty, 1, 1/2$ and $1/3$, and we consider the following specification for $\tau = (\tau_1, \dots, \tau_{\bar{K}-5})'$:

$$\tau_k = 1 \text{ for } k = 1, \dots, \bar{K}/2 - 3, \text{ and } \tau_k = 0 \text{ for } k > \bar{K}/2 - 3.$$

When $\gamma = \infty$, any instruments from the full set of instruments Z_i are valid, and the simulation design corresponds to the same setup in Hausman et al. (2012). When $\gamma < \infty$, the instruments consisting of interactions with dummy variables are invalid.

Similar to the main paper, we report the median bias (Bias), median absolute deviation (MAD), interdecile range (IDR), root mean squared error (MSE), and root trimmed mean squared error (TMSE) of OLS, 2SLS, LIML, FULL (with $C = 1$), JIVE2, HLIM, and HFUL (with $C = 1$). For the IDR, we report the range between the 0.05 and 0.95 quantiles to be comparable to the results in Hausman et al. (2012). For all IV estimators, we consider four different cases: using all available instruments $K = 30$ (all), using only the valid instrument $(1, z_i)$, $K = 2$ (val), utilizing the instruments chosen by Donald and Newey (2001)’s criterion (DN), and the instruments selected by the invalidity-robust criterion in this paper (IR). The $\gamma = \infty$ case in Table 23 ($\mu^2 = 8$) and Table 24 ($\mu^2 = 32$) with $K = 2$ and $K = 30$ replicate the Monte Carlo studies in Hausman et al. (2012, Tables 1 and 2) without invalid instruments.

Again, Fuller/HFUL estimators perform well with lower IDR/MSE/TMSE and are much less dispersed than LIML/HLIM. Further, instrument selection can help to lower IDR/MSE/TMSE. In Table 23 ($\mu^2 = 8$, the weakly identified case), when the instruments are slightly invalid ($\gamma > 1/2$) or all valid ($\gamma = \infty$), we find that all estimators combined with the DN criterion can lead to a reduction in the IDR and MSE/TMSE compared to those of the estimators using all instrument sets or only valid instruments, except HLIM. For FULL/HFUL, results are similar across DN/IR and using only valid instruments.

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Table 1. Monte Carlo Results: Model 1, $\sigma_{uv} = 0.5$ ($R^2 = 0.1, N = 100$)

	$N = 100$	$\gamma = \infty$				$\gamma = 1$				$\gamma = 1/2$				$\gamma = 1/3$						
		Bias	MAD	IDR	MSE	Bias	MAD	IDR	MSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	
OLS	0.45	0.45	0.21	0.46	0.46	0.45	0.45	0.22	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.23	0.46	0.46	
2SLS-all	0.32	0.32	0.43	0.36	0.36	0.33	0.33	0.43	0.37	0.40	0.40	0.44	0.44	0.44	0.51	0.51	0.48	0.54	0.54	
2SLS-val	0.02	0.35	1.63	97.12	1.54	0.03	0.35	1.61	73.27	1.62	0.03	0.36	1.70	26.30	1.66	0.03	0.37	1.73	28.82	1.67
2SLS-DN	0.19	0.24	0.72	96.80	0.57	0.21	0.26	0.70	18.04	0.58	0.34	0.36	0.70	1.05	0.60	0.50	0.51	0.74	2.76	0.73
2SLS-IR	0.09	0.25	0.96	1.03	0.62	0.10	0.25	0.99	1.42	0.66	0.17	0.32	1.08	1.41	0.70	0.24	0.40	1.26	2.66	0.81
GMM-AVE	0.18	0.25	0.79	0.36	0.36	0.18	0.26	0.81	0.37	0.37	0.21	0.28	0.86	0.40	0.40	0.24	0.31	0.93	0.43	0.43
LIML-all	0.03	0.33	1.62	126.39	1.71	0.07	0.33	1.58	61.07	1.71	0.29	0.47	1.93	63.91	1.90	0.74	0.96	3.81	425.70	2.88
LIML-val	0.02	0.35	1.63	97.12	1.54	0.03	0.35	1.61	73.27	1.62	0.03	0.36	1.70	26.30	1.66	0.03	0.37	1.73	28.82	1.67
LIML-DN	0.10	0.23	0.87	96.82	0.79	0.13	0.24	0.88	9.63	0.81	0.27	0.33	0.97	2.13	0.88	0.45	0.51	1.22	17.18	1.11
LIML-IR	0.03	0.25	1.12	5.81	1.02	0.05	0.26	1.11	57.79	1.03	0.12	0.31	1.25	14.07	0.98	0.20	0.42	1.57	62.23	1.35
FULL-all	0.07	0.29	1.19	0.53	0.53	0.10	0.29	1.18	0.53	0.53	0.31	0.41	1.39	0.66	0.66	0.70	0.78	2.08	1.05	1.05
FULL-val	0.15	0.26	0.88	0.38	0.38	0.15	0.26	0.89	0.38	0.38	0.15	0.27	0.92	0.39	0.39	0.15	0.28	0.93	0.40	0.40
FULL-DN	0.14	0.22	0.72	0.33	0.33	0.17	0.23	0.72	0.34	0.34	0.29	0.31	0.78	0.42	0.42	0.45	0.47	0.99	0.61	0.61
FULL-IR	0.11	0.22	0.83	0.37	0.37	0.13	0.23	0.84	0.37	0.37	0.19	0.28	0.92	0.43	0.43	0.28	0.37	1.20	0.59	0.59
JIVE-all	0.09	0.41	2.35	28.70	2.24	0.12	0.41	2.26	37.88	2.11	0.34	0.51	2.08	52.95	2.06	0.60	0.71	2.23	19.40	2.18
JIVE-val	0.43	0.93	4.96	304.48	3.13	0.44	0.93	5.03	1080.51	3.18	0.42	0.92	5.10	81.63	3.20	0.44	0.98	5.26	501.27	3.23
JIVE-DN	0.12	0.28	1.32	12.13	1.56	0.15	0.29	1.27	31.77	1.55	0.30	0.38	1.21	12.76	1.47	0.50	0.56	1.32	11.66	1.65
JIVE-IR	0.09	0.29	1.21	10.90	1.12	0.11	0.30	1.22	42.08	1.11	0.29	0.37	1.17	30.34	1.06	0.50	0.55	1.29	59.68	1.17
HLIM-all	0.03	0.33	1.62	206.43	1.72	0.06	0.33	1.60	20.68	1.74	0.29	0.47	1.93	59.73	1.90	0.74	0.96	3.83	126.75	2.91
HLIM-val	0.38	0.89	4.73	61.06	3.06	0.38	0.86	4.73	63.17	3.09	0.39	0.91	5.22	376.90	3.21	0.37	0.92	5.29	58.47	3.18
HLIM-DN	0.13	0.25	1.04	9.22	1.31	0.15	0.27	1.01	43.88	1.33	0.30	0.37	1.13	64.15	1.47	0.50	0.57	1.36	12.02	1.62
HLIM-IR	0.09	0.28	1.16	4.39	0.51	0.12	0.28	1.13	6.53	0.51	0.28	0.37	1.15	3.06	0.56	0.50	0.57	1.54	19.25	0.77
HFUL-all	0.08	0.28	1.14	0.50	0.50	0.11	0.29	1.14	0.50	0.50	0.31	0.41	1.32	0.63	0.63	0.69	0.75	1.91	0.99	0.99
HFUL-val	0.39	0.72	2.65	1.05	1.05	0.39	0.70	2.63	1.03	1.03	0.40	0.72	2.69	1.07	1.07	0.38	0.74	2.85	1.10	1.10
HFUL-DN	0.16	0.24	0.89	0.52	0.52	0.18	0.25	0.88	0.52	0.52	0.31	0.36	0.96	0.59	0.59	0.50	0.53	1.15	0.74	0.74
HFUL-IR	0.12	0.27	1.07	0.50	0.50	0.15	0.28	1.06	0.50	0.50	0.29	0.36	1.06	0.55	0.55	0.49	0.54	1.34	0.75	0.75

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 2. Monte Carlo Results: Model 1, $\sigma_{uv} = 0.5$ ($R^2 = 0.01, N = 100$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$					
	$N = 100$	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE			
OLS	0.50	0.50	0.22	0.50	0.50	0.50	0.49	0.49	0.23	0.50	0.50	0.49	0.49	0.24	0.50	0.50	0.49	0.49	0.24	0.50	0.50			
2SLS-all	0.48	0.48	0.50	0.52	0.52	0.48	0.48	0.50	0.52	0.52	0.51	0.51	0.53	0.56	0.56	0.56	0.56	0.56	0.62	0.61	0.61			
2SLS-val	0.34	0.84	4.72	79.74	3.06	0.35	0.85	4.72	41.26	3.03	0.35	0.85	4.72	32.80	3.02	0.34	0.88	5.12	62.82	3.22				
2SLS-DN	0.42	0.52	1.54	49.59	1.86	0.44	0.53	1.55	15.69	1.83	0.51	0.59	1.59	19.35	1.88	0.60	0.70	1.93	20.35	2.04				
2SLS-IR	0.42	0.49	1.40	4.82	1.40	0.43	0.52	1.48	3.01	1.37	0.50	0.57	1.51	7.28	1.44	0.58	0.67	1.72	18.08	1.52				
GMM-AVE	0.44	0.45	0.92	0.57	0.57	0.45	0.46	0.92	0.58	0.58	0.46	0.47	0.96	0.60	0.60	0.49	0.51	1.05	0.64	0.64				
LIML-all	0.38	0.88	4.80	81.71	3.08	0.39	0.89	4.98	251.09	3.13	0.62	1.17	6.04	180.30	3.45	1.34	2.36	12.28	171756.62	4.78				
LIML-val	0.34	0.84	4.72	79.74	3.06	0.35	0.85	4.72	41.26	3.03	0.35	0.85	4.72	32.80	3.02	0.34	0.88	5.12	62.82	3.22				
LIML-DN	0.38	0.61	2.81	41.16	2.39	0.39	0.62	2.82	24.99	2.39	0.49	0.72	2.92	34.67	2.39	0.61	0.89	3.46	29.55	2.65				
LIML-IR	0.37	0.64	3.11	67.58	2.54	0.38	0.66	3.27	251.34	2.59	0.49	0.77	3.49	121.60	2.77	0.68	1.04	5.52	384.91	3.38				
FULL-all	0.40	0.60	2.03	0.88	0.88	0.41	0.61	2.08	0.90	0.90	0.60	0.77	2.31	1.04	1.04	1.05	1.24	3.28	1.47	1.47				
FULL-val	0.45	0.46	0.91	0.57	0.57	0.45	0.46	0.92	0.58	0.58	0.45	0.46	0.96	0.58	0.58	0.45	0.46	0.98	0.59	0.59				
FULL-DN	0.44	0.45	0.92	0.57	0.57	0.45	0.46	0.95	0.58	0.58	0.49	0.50	1.00	0.63	0.63	0.53	0.55	1.24	0.75	0.75				
FULL-IR	0.44	0.46	1.01	0.61	0.61	0.45	0.47	1.03	0.62	0.62	0.49	0.51	1.11	0.68	0.68	0.55	0.59	1.50	0.87	0.87				
JIVE-all	0.48	0.78	3.88	64.84	2.79	0.49	0.77	3.62	30.25	2.68	0.51	0.81	3.83	129222.93	2.83	0.54	0.93	4.90	42.13	3.12				
JIVE-val	0.51	1.01	5.50	38.13	3.31	0.50	1.03	5.38	44.29	3.28	0.49	1.00	5.40	118.61	3.29	0.50	1.05	5.68	288.14	3.35				
JIVE-DN	0.45	0.75	3.68	23.98	2.75	0.44	0.75	3.72	38.67	2.75	0.55	0.82	3.71	28.21	2.75	0.65	0.95	4.11	287.59	2.87				
JIVE-IR	0.44	0.71	3.47	23.26	2.66	0.43	0.70	3.45	25.77	2.68	0.51	0.76	3.41	116.27	2.65	0.62	0.92	3.93	64.31	2.83				
HLIM-all	0.38	0.88	4.85	292.13	3.05	0.40	0.88	4.96	81.73	3.10	0.63	1.18	5.98	129.68	3.44	1.32	2.35	12.08	445.14	4.75				
HLIM-val	0.47	0.96	5.19	45.26	3.26	0.48	0.96	5.22	226.94	3.20	0.49	1.01	5.35	5486.40	3.22	0.49	1.03	5.30	55.83	3.25				
HLIM-DN	0.42	0.67	3.21	30.25	2.58	0.43	0.67	3.15	222.61	2.54	0.55	0.78	3.27	5486.31	2.56	0.68	0.96	3.64	18.06	2.66				
HLIM-IR	0.40	0.59	2.45	22.80	0.83	0.42	0.60	2.43	185.47	0.84	0.53	0.71	2.64	134.60	0.90	0.70	0.99	3.72	33.17	1.03				
HFUL-all	0.40	0.56	1.87	0.83	0.83	0.42	0.57	1.91	0.84	0.84	0.59	0.73	2.12	0.97	0.97	0.99	1.14	2.94	1.35	1.35				
HFUL-val	0.48	0.77	2.75	1.10	1.10	0.48	0.76	2.73	1.10	1.10	0.49	0.79	2.81	1.13	1.13	0.49	0.82	2.90	1.17	1.17				
HFUL-DN	0.43	0.58	2.09	0.92	0.92	0.44	0.58	2.09	0.93	0.93	0.54	0.68	2.17	0.98	0.98	0.64	0.79	2.37	1.08	1.08				
HFUL-IR	0.43	0.53	1.68	0.84	0.84	0.44	0.54	1.67	0.84	0.84	0.52	0.62	1.82	0.90	0.90	0.63	0.76	2.19	1.05	1.05				

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 3. Monte Carlo Results: Model 1, $\sigma_{uv} = 0.5$ ($R^2 = 0.1, N = 1000$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$						
	N = 100	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE
OLS	0.45	0.45	0.07	0.45	0.45	0.45	0.45	0.45	0.07	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45
2SLS-all	0.10	0.10	0.20	0.13	0.13	0.11	0.11	0.21	0.13	0.13	0.20	0.20	0.20	0.20	0.22	0.22	0.41	0.41	0.21	0.42	0.42	0.42	0.42	0.42	0.42
2SLS-val	-0.00	0.13	0.49	0.21	0.21	-0.00	0.13	0.50	0.21	0.21	-0.00	0.13	0.50	0.21	0.21	0.21	0.00	0.13	0.51	0.21	0.21	0.21	0.21	0.21	0.21
2SLS-DN	0.04	0.07	0.24	0.10	0.10	0.05	0.07	0.24	0.10	0.10	0.15	0.15	0.24	0.17	0.17	0.17	0.40	0.40	0.24	0.41	0.41	0.41	0.41	0.41	0.41
2SLS-IR	-0.01	0.08	0.32	0.14	0.14	-0.00	0.08	0.32	0.14	0.14	0.05	0.10	0.35	0.15	0.15	0.15	0.13	0.16	0.39	0.20	0.20	0.20	0.20	0.20	0.20
GMM-AVE	0.05	0.10	0.37	0.16	0.16	0.05	0.10	0.37	0.16	0.16	0.07	0.13	0.42	0.18	0.18	0.18	0.06	0.14	0.49	0.21	0.21	0.21	0.21	0.21	0.21
LIML-all	-0.00	0.07	0.27	0.11	0.11	0.00	0.07	0.27	0.11	0.11	0.12	0.13	0.26	0.16	0.16	0.16	0.39	0.39	0.34	0.41	0.41	0.41	0.41	0.41	0.41
LIML-val	-0.00	0.13	0.49	0.21	0.21	-0.00	0.13	0.50	0.21	0.21	-0.00	0.13	0.50	0.21	0.21	0.21	0.00	0.13	0.51	0.21	0.21	0.21	0.21	0.21	0.21
LIML-DN	0.01	0.07	0.25	0.10	0.10	0.02	0.07	0.25	0.10	0.10	0.12	0.13	0.24	0.16	0.16	0.16	0.37	0.37	0.29	0.39	0.39	0.39	0.39	0.39	0.39
LIML-IR	-0.03	0.08	0.32	0.14	0.14	-0.02	0.08	0.31	0.14	0.14	0.04	0.10	0.35	0.15	0.15	0.15	0.13	0.15	0.39	0.20	0.20	0.20	0.20	0.20	0.20
FULL-all	0.00	0.07	0.27	0.11	0.11	0.01	0.07	0.26	0.10	0.10	0.13	0.13	0.26	0.16	0.16	0.16	0.39	0.39	0.33	0.41	0.41	0.41	0.41	0.41	0.41
FULL-val	0.02	0.12	0.46	0.19	0.19	0.02	0.12	0.47	0.19	0.19	0.02	0.12	0.47	0.19	0.19	0.19	0.02	0.12	0.48	0.19	0.19	0.19	0.19	0.19	0.19
FULL-DN	0.02	0.07	0.25	0.10	0.10	0.02	0.07	0.24	0.10	0.10	0.13	0.13	0.24	0.16	0.16	0.16	0.37	0.37	0.29	0.39	0.39	0.39	0.39	0.39	0.39
FULL-IR	-0.01	0.08	0.31	0.13	0.13	-0.01	0.08	0.31	0.13	0.13	0.04	0.10	0.34	0.15	0.15	0.15	0.13	0.15	0.38	0.20	0.20	0.20	0.20	0.20	0.20
JIVE-all	-0.00	0.07	0.29	0.12	0.12	-0.00	0.07	0.29	0.12	0.12	0.13	0.13	0.27	0.16	0.16	0.16	0.40	0.40	0.27	0.42	0.42	0.42	0.42	0.42	0.42
JIVE-val	0.43	0.95	5.30	106.43	3.24	0.44	0.92	4.97	367.25	3.14	0.43	0.94	5.17	50.61	3.19	0.44	0.95	5.11	38.89	3.13	0.44	0.95	5.11	38.89	3.13
JIVE-DN	0.01	0.07	0.26	0.10	0.10	0.01	0.07	0.25	0.10	0.10	0.13	0.13	0.25	0.16	0.16	0.16	0.38	0.38	0.26	0.39	0.39	0.39	0.39	0.39	0.39
JIVE-IR	-0.01	0.11	1.38	1.09	1.08	-0.01	0.11	1.32	1.04	1.02	0.10	0.17	1.16	0.81	0.81	0.81	0.34	0.36	0.56	0.41	0.41	0.41	0.41	0.41	0.41
HLIM-all	-0.00	0.07	0.27	0.11	0.11	0.00	0.07	0.27	0.11	0.11	0.12	0.13	0.27	0.16	0.16	0.16	0.39	0.39	0.34	0.41	0.41	0.41	0.41	0.41	0.41
HLIM-val	0.39	0.87	4.78	82.69	2.99	0.37	0.90	4.78	61.76	3.06	0.39	0.88	5.00	107.43	3.12	0.38	0.90	4.90	169.58	3.11	0.38	0.90	4.90	169.58	3.11
HLIM-DN	0.01	0.07	0.25	0.10	0.10	0.02	0.07	0.25	0.10	0.10	0.12	0.13	0.25	0.16	0.16	0.16	0.37	0.37	0.29	0.39	0.39	0.39	0.39	0.39	0.39
HLIM-IR	-0.02	0.11	1.47	1.10	0.53	-0.01	0.12	1.43	1.06	0.54	0.09	0.17	1.14	0.80	0.48	0.48	0.31	0.34	0.57	0.41	0.39	0.39	0.39	0.39	0.39
HFUL-all	0.00	0.07	0.27	0.11	0.11	0.01	0.07	0.26	0.10	0.10	0.13	0.13	0.26	0.16	0.16	0.16	0.39	0.39	0.33	0.41	0.41	0.41	0.41	0.41	0.41
HFUL-val	0.39	0.86	4.38	1.99	0.37	0.88	4.34	2.01	2.01	0.39	0.86	4.48	2.05	2.05	2.05	0.38	0.87	4.41	2.07	2.07	2.07	2.07	2.07	2.07	
HFUL-DN	0.02	0.07	0.25	0.10	0.10	0.02	0.07	0.24	0.10	0.10	0.13	0.13	0.24	0.16	0.16	0.16	0.37	0.37	0.29	0.39	0.39	0.39	0.39	0.39	0.39
HFUL-IR	-0.01	0.12	1.72	1.09	1.09	-0.01	0.12	1.70	1.08	1.08	0.09	0.18	1.33	0.88	0.88	0.88	0.31	0.33	0.59	0.42	0.42	0.42	0.42	0.42	0.42

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 4. Monte Carlo Results: Model 1, $\sigma_{uv} = 0.5$ ($R^2 = 0.01, N = 1000$)

$N = 100$	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$							
	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	
OLS	0.49	0.49	0.07	0.50	0.50	0.50	0.50	0.07	0.50	0.50	0.49	0.49	0.07	0.50	0.50	0.49	0.49	0.07	0.50	0.50	0.49	0.49	0.07	0.50	0.50	
2SLS-all	0.37	0.37	0.37	0.40	0.40	0.38	0.38	0.37	0.40	0.40	0.48	0.48	0.36	0.50	0.50	0.70	0.70	0.47	0.73	0.73	0.70	0.70	0.47	0.73	0.73	
2SLS-val	0.05	0.43	2.18	1690.31	2.00	0.06	0.43	2.16	93.12	2.03	0.05	0.43	2.17	23.57	2.05	0.06	0.44	2.15	9.49	2.02	0.06	0.44	2.15	9.49	2.02	
2SLS-DN	0.28	0.30	0.65	1.11	0.57	0.28	0.31	0.65	25.01	0.52	0.45	0.46	0.58	1.44	0.64	0.77	0.78	0.71	2.78	1.03	0.77	0.78	0.71	2.78	1.03	
2SLS-IR	0.19	0.24	0.78	4.61	0.62	0.19	0.24	0.78	9.33	0.60	0.29	0.33	0.85	0.75	0.63	0.49	0.60	1.37	2.97	0.91	0.49	0.60	1.37	2.97	0.91	
GMM-AVE	0.25	0.30	0.79	0.40	0.40	0.25	0.30	0.78	0.40	0.40	0.30	0.34	0.84	0.43	0.43	0.35	0.40	1.02	0.53	0.53	0.35	0.40	1.02	0.53	0.53	
LIML-all	0.04	0.38	1.92	84.31	1.90	0.07	0.37	1.91	25.49	1.89	0.43	0.60	2.43	47.65	2.25	2.29	2.62	8.39	174.27	4.61	2.29	2.62	8.39	174.27	4.61	
LIML-val	0.05	0.43	2.18	1690.34	2.00	0.06	0.43	2.16	93.12	2.03	0.05	0.43	2.17	23.57	2.05	0.06	0.44	2.15	9.49	2.02	0.06	0.44	2.15	9.49	2.02	
LIML-DN	0.13	0.26	0.98	1685.96	1.05	0.14	0.26	0.96	9.97	0.93	0.33	0.40	1.06	18.91	1.09	0.88	0.92	1.79	48.22	1.74	0.88	0.92	1.79	48.22	1.74	
LIML-IR	0.07	0.26	1.15	152.37	1.03	0.08	0.26	1.14	8.80	1.04	0.22	0.32	1.20	2.96	1.10	0.62	0.75	2.50	48.77	1.97	0.62	0.75	2.50	48.77	1.97	
FULL-all	0.08	0.33	1.39	0.60	0.60	0.10	0.33	1.39	0.61	0.61	0.44	0.53	1.64	0.79	0.79	1.91	1.94	3.07	2.09	2.09	1.91	1.94	3.07	2.09	2.09	
FULL-val	0.22	0.30	0.94	0.43	0.43	0.22	0.30	0.93	0.43	0.43	0.22	0.30	0.94	0.43	0.43	0.22	0.30	0.95	0.44	0.44	0.22	0.30	0.95	0.44	0.44	
FULL-DN	0.17	0.24	0.77	0.36	0.36	0.19	0.24	0.75	0.36	0.36	0.35	0.37	0.82	0.48	0.48	0.80	0.80	1.37	1.00	1.00	0.80	0.80	1.37	1.00	1.00	
FULL-IR	0.16	0.24	0.87	0.41	0.41	0.17	0.24	0.87	0.42	0.42	0.28	0.32	0.94	0.51	0.51	0.53	0.60	1.90	1.03	1.03	0.53	0.60	1.90	1.03	1.03	
JIVE-all	0.14	0.47	2.69	20.69	2.34	0.15	0.48	2.67	136.50	2.31	0.45	0.60	2.27	19.11	2.18	1.11	1.28	4.01	68.38	3.11	2.18	2.27	19.11	2.18	2.18	1.11
JIVE-val	0.50	1.01	5.41	81.43	3.29	0.51	0.97	5.06	93.09	3.14	0.50	1.00	5.54	34.34	3.29	0.49	0.99	5.40	135.42	3.28	0.49	0.99	5.40	135.42	3.28	
JIVE-DN	0.15	0.31	1.46	9.82	1.67	0.17	0.32	1.47	83.96	1.61	0.38	0.45	1.34	21.54	1.64	0.83	0.88	1.63	56.88	1.85	0.83	0.88	1.63	56.88	1.85	
JIVE-IR	0.17	0.32	1.24	4.77	1.26	0.18	0.32	1.22	9.11	1.20	0.41	0.46	1.18	11.58	1.28	0.88	0.91	1.77	11.68	1.67	0.88	0.91	1.77	11.68	1.67	
HLIM-all	0.04	0.38	1.93	14.70	1.91	0.07	0.37	1.90	17.19	1.89	0.43	0.61	2.44	49.75	2.24	2.29	2.61	8.31	132.16	4.60	2.29	2.61	8.31	132.16	4.60	
HLIM-val	0.50	0.97	5.10	86.74	3.17	0.47	1.01	5.33	47.88	3.21	0.48	0.99	5.42	123.92	3.24	0.49	1.02	5.42	86.88	3.29	0.49	1.02	5.42	86.88	3.29	
HLIM-DN	0.15	0.28	1.17	19.53	1.41	0.16	0.29	1.17	44.04	1.52	0.38	0.44	1.23	64.29	1.58	0.93	0.99	1.81	55.28	1.98	0.93	0.99	1.81	55.28	1.98	
HLIM-IR	0.17	0.29	1.09	4.85	0.52	0.18	0.30	1.08	25.77	0.52	0.40	0.45	1.13	14.40	0.63	1.18	1.31	2.98	153.64	1.14	1.18	1.31	2.98	153.64	1.14	
HFUL-all	0.08	0.33	1.38	0.59	0.59	0.11	0.33	1.38	0.60	0.60	0.44	0.53	1.62	0.79	0.79	1.90	1.92	3.04	2.07	2.07	1.90	1.92	3.04	2.07	2.07	
HFUL-val	0.50	0.94	4.59	2.12	2.12	0.47	0.97	4.69	2.13	2.13	0.48	0.96	4.73	2.15	2.15	0.49	0.99	4.76	2.20	2.20	0.49	0.99	4.76	2.20	2.20	
HFUL-DN	0.18	0.27	1.04	0.95	0.95	0.19	0.27	1.05	1.01	1.01	0.39	0.43	1.08	1.08	1.08	0.87	0.92	1.51	1.33	1.33	0.87	0.92	1.51	1.33	1.33	
HFUL-IR	0.19	0.29	0.99	0.74	0.74	0.21	0.29	0.99	0.75	0.75	0.41	0.44	1.02	0.85	0.85	1.06	1.13	2.27	1.52	1.52	1.06	1.13	2.27	1.52	1.52	

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 5. Monte Carlo Results: Model 1, $\sigma_{uv} = 0.2$ ($R^2 = 0.1, N = 100$)

	$N = 100$	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$					
		Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE
OLS	0.18	0.18	0.24	0.20	0.20	0.18	0.18	0.24	0.20	0.20	0.18	0.18	0.24	0.20	0.20	0.18	0.18	0.25	0.21	0.21	0.18	0.18	0.25	0.21	0.21
2SLS-all	0.13	0.16	0.47	0.23	0.23	0.14	0.16	0.47	0.23	0.23	0.21	0.22	0.48	0.29	0.29	0.32	0.32	0.53	0.38	0.38	0.32	0.32	0.53	0.38	0.38
2SLS-val	0.01	0.35	1.68	9.99	1.47	0.00	0.36	1.64	50.40	1.50	0.01	0.35	1.64	193.84	1.47	0.01	0.37	1.72	13.09	1.56	0.01	0.37	1.72	13.09	1.56
2SLS-DN	0.08	0.19	0.69	0.81	0.50	0.09	0.19	0.70	0.83	0.54	0.22	0.25	0.70	0.50	0.47	0.38	0.40	0.78	1.42	0.66	0.38	0.40	0.78	1.42	0.66
2SLS-IR	0.03	0.25	1.06	6.38	0.65	0.04	0.25	1.04	1.17	0.68	0.14	0.30	1.10	1.10	0.61	0.29	0.42	1.28	1.97	0.75	0.14	0.29	1.28	1.97	0.75
GMM-AVE	0.07	0.21	0.83	0.35	0.35	0.07	0.21	0.82	0.35	0.35	0.11	0.23	0.85	0.36	0.36	0.16	0.27	0.94	0.41	0.41	0.14	0.16	0.27	0.94	0.41
LIML-all	0.02	0.36	1.74	20.15	1.71	0.04	0.36	1.76	169.39	1.71	0.29	0.48	2.07	94.52	1.94	0.80	0.96	3.27	161.15	2.71	0.80	0.96	3.27	161.15	2.71
LIML-val	0.01	0.35	1.68	9.99	1.47	0.00	0.36	1.64	50.40	1.50	0.01	0.35	1.64	193.84	1.47	0.01	0.37	1.72	13.09	1.56	0.01	0.37	1.72	13.09	1.56
LIML-DN	0.05	0.23	0.92	20.91	0.78	0.05	0.23	0.92	2.35	0.79	0.20	0.29	0.98	3.13	0.76	0.41	0.46	1.21	2.85	0.99	0.20	0.29	0.98	3.13	0.99
LIML-IR	-0.01	0.30	1.24	30.55	1.01	0.01	0.30	1.21	44.35	0.94	0.13	0.34	1.29	18.17	0.95	0.32	0.49	1.67	5.02	1.23	0.13	0.34	1.29	18.17	0.95
FULL-all	0.03	0.31	1.36	0.57	0.57	0.06	0.32	1.37	0.58	0.58	0.28	0.42	1.53	0.70	0.70	0.72	0.81	2.04	1.07	1.07	0.72	0.81	2.04	1.07	1.07
FULL-val	0.06	0.24	0.96	0.39	0.39	0.06	0.25	0.94	0.38	0.38	0.06	0.25	0.95	0.38	0.38	0.06	0.26	0.99	0.40	0.40	0.06	0.26	0.99	0.40	0.40
FULL-DN	0.06	0.20	0.78	0.33	0.33	0.07	0.20	0.77	0.33	0.33	0.20	0.26	0.82	0.39	0.39	0.38	0.41	1.03	0.57	0.57	0.38	0.41	1.03	0.57	0.57
FULL-IR	0.03	0.24	0.94	0.39	0.39	0.04	0.24	0.93	0.39	0.39	0.14	0.28	1.00	0.44	0.44	0.32	0.42	1.29	0.60	0.60	0.32	0.42	1.29	0.60	0.60
JIVE-all	0.04	0.40	2.20	33.35	2.11	0.06	0.41	2.12	555.41	2.02	0.27	0.47	2.19	22.54	2.05	0.54	0.72	2.77	21.33	2.45	0.54	0.72	2.77	21.33	2.45
JIVE-val	0.17	0.94	5.83	316.17	3.35	0.16	0.94	5.60	258.49	3.32	0.18	0.91	5.76	70.89	3.29	0.16	0.98	6.15	308.24	3.43	0.16	0.98	6.15	308.24	3.43
JIVE-DN	0.05	0.28	1.30	308.85	1.67	0.06	0.28	1.29	10.06	1.58	0.22	0.34	1.29	60.37	1.53	0.43	0.52	1.50	17.72	1.68	0.22	0.34	1.29	60.37	1.68
JIVE-IR	0.04	0.28	1.22	308.50	1.13	0.05	0.28	1.24	4.94	1.10	0.22	0.34	1.23	3.68	1.06	0.45	0.51	1.47	18.20	1.29	0.22	0.34	1.23	3.68	1.29
HLIM-all	0.01	0.36	1.75	30.01	1.73	0.05	0.37	1.78	107.98	1.73	0.29	0.48	2.08	8.59	1.96	0.79	0.95	3.32	54.03	2.69	0.79	0.95	3.32	54.03	2.69
HLIM-val	0.16	0.86	5.19	309.12	3.17	0.15	0.88	5.37	58.57	3.25	0.16	0.90	5.37	105.61	3.21	0.16	0.93	5.56	45.36	3.27	0.16	0.93	5.56	45.36	3.27
HLIM-DN	0.06	0.25	1.10	43.80	1.47	0.06	0.25	1.10	45.09	1.43	0.22	0.32	1.17	82.94	1.47	0.45	0.51	1.38	9.57	1.52	0.22	0.33	1.17	82.94	1.52
HLIM-IR	0.05	0.26	1.12	2.03	0.50	0.06	0.26	1.12	41.35	0.50	0.22	0.33	1.17	3.96	0.55	0.48	0.55	1.60	31.19	0.76	0.22	0.33	1.17	3.96	0.76
HFUL-all	0.03	0.31	1.30	0.54	0.54	0.06	0.32	1.32	0.55	0.55	0.28	0.41	1.45	0.66	0.66	0.70	0.78	1.90	1.00	1.00	0.70	0.78	1.90	1.00	1.00
HFUL-val	0.17	0.71	2.91	1.07	1.07	0.16	0.72	2.92	1.07	1.07	0.16	0.74	2.98	1.09	1.09	0.16	0.76	3.06	1.13	1.13	0.16	0.76	3.06	1.13	1.13
HFUL-DN	0.07	0.23	0.96	0.54	0.54	0.08	0.23	0.97	0.54	0.54	0.21	0.30	1.02	0.59	0.59	0.42	0.48	1.19	0.71	0.71	0.21	0.30	1.02	0.71	0.71
HFUL-IR	0.06	0.25	1.05	0.50	0.07	0.25	1.06	0.50	0.50	0.50	0.21	0.31	1.10	0.56	0.56	0.44	0.51	1.41	0.75	0.75	0.21	0.31	1.10	0.56	0.75

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 6. Monte Carlo Results: Model 1, $\sigma_{uv} = 0.2$ ($R^2 = 0.01, N = 100$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$						
$N = 100$	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE
OLS	0.20	0.20	0.25	0.22	0.22	0.20	0.20	0.25	0.22	0.22	0.20	0.20	0.26	0.22	0.22	0.20	0.20	0.27	0.22	0.22	0.20	0.20	0.27	0.22	0.22
2SLS-all	0.19	0.21	0.58	0.30	0.30	0.19	0.21	0.57	0.30	0.30	0.23	0.24	0.59	0.33	0.33	0.28	0.29	0.70	0.39	0.39	0.19	0.19	0.83	3.26	1.21
2SLS-val	0.15	0.83	4.89	31.15	3.05	0.12	0.84	5.00	261.73	3.10	0.12	0.86	5.13	69.36	3.18	0.14	0.90	5.44	131.55	3.25	0.19	0.19	0.70	2.59	1.06
2SLS-DN	0.17	0.35	1.66	13.38	1.80	0.17	0.35	1.69	261.45	1.89	0.25	0.40	1.82	11.95	1.94	0.35	0.53	2.11	87.76	2.05	0.19	0.19	0.86	4.04	1.06
2SLS-IR	0.17	0.38	1.60	9.17	1.37	0.16	0.37	1.59	10.12	1.38	0.25	0.42	1.70	57.69	1.49	0.32	0.52	1.94	8.18	1.56	0.19	0.19	0.86	4.04	1.06
GMM-AVE	0.18	0.29	1.00	0.45	0.45	0.17	0.29	1.01	0.44	0.44	0.20	0.31	1.04	0.46	0.46	0.22	0.34	1.13	0.50	0.50	0.19	0.19	0.86	4.04	1.06
LIML-all	0.16	0.87	5.22	59.64	3.20	0.18	0.87	5.29	1434.06	3.21	0.40	1.16	6.81	135.86	3.59	1.10	2.32	12.60	585.64	4.79	0.19	0.19	0.86	4.04	1.06
LIML-val	0.15	0.83	4.89	31.15	3.05	0.12	0.84	5.00	261.73	3.10	0.12	0.86	5.13	69.36	3.18	0.14	0.90	5.44	131.55	3.25	0.19	0.19	0.86	4.04	1.06
LIML-DN	0.17	0.55	2.87	30.75	2.33	0.15	0.56	3.05	261.52	2.43	0.25	0.62	3.15	27.10	2.55	0.42	0.81	3.65	92.41	2.70	0.19	0.19	0.86	4.04	1.06
LIML-IR	0.15	0.60	3.30	40.69	2.60	0.14	0.60	3.38	57.33	2.64	0.29	0.69	3.87	71.62	2.84	0.46	0.95	5.70	34.66	3.41	0.19	0.19	0.86	4.04	1.06
FULL-all	0.17	0.61	2.27	0.89	0.89	0.18	0.61	2.29	0.89	0.89	0.36	0.75	2.57	1.02	1.02	0.79	1.24	3.56	1.44	1.44	0.19	0.19	0.86	4.04	1.06
FULL-val	0.18	0.28	1.01	0.44	0.44	0.18	0.28	1.01	0.43	0.43	0.18	0.28	1.03	0.45	0.45	0.18	0.29	1.06	0.46	0.46	0.19	0.19	0.86	4.04	1.06
FULL-DN	0.19	0.29	1.02	0.45	0.45	0.18	0.28	1.02	0.45	0.45	0.22	0.31	1.09	0.50	0.50	0.27	0.39	1.33	0.63	0.63	0.19	0.19	0.86	4.04	1.06
FULL-IR	0.19	0.31	1.11	0.51	0.51	0.17	0.30	1.11	0.51	0.51	0.23	0.34	1.23	0.59	0.59	0.29	0.43	1.61	0.76	0.76	0.19	0.19	0.86	4.04	1.06
JIVE-all	0.20	0.74	4.30	212.56	2.91	0.21	0.72	4.24	492.08	2.89	0.19	0.76	4.45	1909.40	2.97	0.25	0.89	5.16	29.86	3.18	0.19	0.19	0.86	4.04	1.06
JIVE-val	0.21	0.99	5.98	578.56	3.35	0.21	0.98	6.09	133.90	3.39	0.22	0.98	6.01	56.41	3.40	0.21	1.05	6.30	421.98	3.48	0.19	0.19	0.86	4.04	1.06
JIVE-DN	0.18	0.69	4.16	53.06	2.82	0.19	0.69	4.16	125.83	2.88	0.28	0.73	4.15	51.15	2.85	0.40	0.87	4.55	421.20	2.97	0.19	0.19	0.86	4.04	1.06
JIVE-IR	0.18	0.64	3.72	576.69	2.71	0.19	0.64	3.72	19.78	2.73	0.28	0.69	3.80	45.29	2.75	0.37	0.84	4.39	422.62	2.91	0.19	0.19	0.86	4.04	1.06
HLIM-all	0.16	0.87	5.33	118.45	3.22	0.18	0.88	5.32	59.71	3.19	0.41	1.17	6.91	180.91	3.61	1.09	2.30	12.60	1675.25	4.78	0.19	0.19	0.86	4.04	1.06
HLIM-val	0.18	0.98	6.05	107.38	3.41	0.19	1.00	6.20	231.84	3.45	0.16	1.04	6.37	70.71	3.46	0.19	1.04	6.36	1068.98	3.50	0.19	0.19	0.86	4.04	1.06
HLIM-DN	0.18	0.60	3.54	24.56	2.70	0.18	0.61	3.64	41.62	2.72	0.28	0.69	3.88	20.30	2.79	0.45	0.86	4.04	65.23	2.85	0.19	0.19	0.86	4.04	1.06
HLIM-IR	0.17	0.52	2.64	27.56	0.81	0.16	0.50	2.66	31.46	0.81	0.27	0.61	3.09	46.82	0.87	0.44	0.90	4.30	28.95	1.00	0.19	0.19	0.86	4.04	1.06
HFUL-all	0.17	0.57	2.11	0.82	0.82	0.18	0.58	2.11	0.82	0.82	0.35	0.70	2.34	0.94	0.94	0.74	1.14	3.22	1.32	1.32	0.19	0.19	0.86	4.04	1.06
HFUL-val	0.18	0.78	3.16	1.15	1.15	0.20	0.79	3.12	1.15	1.15	0.17	0.81	3.22	1.17	1.17	0.19	0.83	3.26	1.21	1.21	0.19	0.19	0.86	4.04	1.06
HFUL-DN	0.18	0.51	2.37	0.94	0.94	0.18	0.51	2.37	0.94	0.94	0.26	0.57	2.52	0.98	0.98	0.39	0.70	2.59	1.06	1.06	0.19	0.19	0.86	4.04	1.06
HFUL-IR	0.17	0.43	1.92	0.83	0.83	0.17	0.43	1.93	0.84	0.84	0.25	0.50	2.08	0.89	0.89	0.36	0.66	2.46	1.01	1.01	0.19	0.19	0.86	4.04	1.06

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 7. Monte Carlo Results: Model 1, $\sigma_{uv} = 0.8$ ($R^2 = 0.1, N = 100$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$					
$N = 100$	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE				
OLS	0.72	0.72	0.16	0.72	0.72	0.72	0.72	0.16	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	0.72	
2SLS-all	0.51	0.51	0.34	0.53	0.53	0.52	0.52	0.33	0.54	0.54	0.60	0.60	0.34	0.61	0.61	0.70	0.70	0.39	0.72	0.72	0.72	0.72	0.72	
2SLS-val	0.04	0.33	1.72	106.39	1.79	0.04	0.33	1.67	123.18	1.84	0.04	0.34	1.71	57.40	1.91	0.03	0.35	1.80	56.22	1.89				
2SLS-DN	0.25	0.29	0.74	106.01	0.74	0.27	0.31	0.72	20.31	0.71	0.42	0.44	0.70	1.30	0.71	0.62	0.63	0.72	2.66	0.80				
2SLS-IR	0.16	0.26	0.83	5.97	0.70	0.16	0.26	0.83	1.85	0.71	0.22	0.31	0.93	8.05	0.76	0.28	0.36	1.11	6.28	0.80				
GMM-AVE	0.25	0.29	0.71	0.39	0.39	0.26	0.29	0.71	0.39	0.39	0.27	0.31	0.77	0.42	0.42	0.28	0.32	0.83	0.45	0.45				
LIML-all	0.01	0.27	1.32	31.09	1.40	0.06	0.28	1.28	36.25	1.40	0.26	0.41	1.58	9.28	1.80	0.56	0.94	4.59	51.17	3.00				
LIML-val	0.04	0.33	1.72	106.39	1.79	0.04	0.33	1.67	123.18	1.84	0.04	0.34	1.71	57.40	1.91	0.03	0.35	1.80	56.22	1.89				
LIML-DN	0.14	0.24	0.83	106.04	0.87	0.16	0.25	0.82	9.49	0.91	0.30	0.35	0.90	39.59	1.05	0.46	0.53	1.33	21.57	1.45				
LIML-IR	0.06	0.26	1.11	105.91	1.13	0.08	0.26	1.08	12.64	1.20	0.15	0.30	1.14	5.70	1.20	0.20	0.36	1.49	29.00	1.54				
FULL-all	0.09	0.23	0.87	0.40	0.40	0.12	0.24	0.86	0.40	0.40	0.31	0.36	1.06	0.56	0.56	0.59	0.66	2.05	0.99	0.99				
FULL-val	0.22	0.26	0.73	0.38	0.38	0.23	0.27	0.73	0.39	0.39	0.23	0.27	0.76	0.40	0.40	0.23	0.28	0.80	0.40	0.40				
FULL-DN	0.20	0.24	0.63	0.33	0.33	0.22	0.25	0.63	0.34	0.34	0.35	0.35	0.67	0.44	0.44	0.50	0.50	0.97	0.65	0.65				
FULL-IR	0.19	0.25	0.74	0.36	0.36	0.20	0.26	0.73	0.36	0.36	0.26	0.29	0.75	0.42	0.42	0.31	0.34	1.03	0.58	0.58				
JIVE-all	0.12	0.42	2.94	36.27	2.54	0.15	0.42	2.72	84.68	2.41	0.38	0.52	2.02	81.93	2.14	0.66	0.73	1.76	20.39	1.95				
JIVE-val	0.70	0.94	3.78	351.26	2.78	0.70	0.94	3.82	842.65	2.85	0.68	0.93	3.86	53.63	2.87	0.69	0.99	4.21	360.28	2.95				
JIVE-DN	0.20	0.32	1.41	10.54	1.50	0.22	0.32	1.35	22.46	1.51	0.37	0.42	1.13	9.77	1.41	0.58	0.61	1.15	7.92	1.50				
JIVE-IR	0.15	0.31	1.26	65.19	1.15	0.17	0.32	1.24	21.09	1.17	0.34	0.41	1.12	4.97	1.06	0.56	0.59	1.13	14.44	1.09				
HLIM-all	0.01	0.27	1.33	6.57	1.39	0.06	0.28	1.30	64.16	1.36	0.26	0.41	1.58	19.49	1.80	0.57	0.94	4.61	38.06	3.01				
HLIM-val	0.61	0.88	3.86	68.06	2.78	0.62	0.87	3.71	56.49	2.74	0.62	0.91	4.12	26.57	2.92	0.61	0.92	4.35	24.69	2.93				
HLIM-DN	0.17	0.26	0.90	57.86	1.09	0.19	0.27	0.89	4.72	1.06	0.34	0.39	0.98	8.57	1.29	0.53	0.59	1.42	10.78	1.66				
HLIM-IR	0.11	0.28	1.17	9.49	0.52	0.14	0.29	1.14	8.31	0.52	0.28	0.37	1.12	4.82	0.55	0.48	0.57	1.69	98.00	0.78				
HFUL-all	0.11	0.23	0.81	0.37	0.37	0.14	0.24	0.80	0.38	0.38	0.32	0.36	0.99	0.54	0.54	0.60	0.65	1.85	0.94	0.94				
HFUL-val	0.62	0.68	2.03	0.99	0.99	0.63	0.68	2.00	0.98	0.98	0.63	0.69	2.09	1.01	1.01	0.62	0.71	2.25	1.04	1.04				
HFUL-DN	0.22	0.25	0.74	0.47	0.47	0.24	0.27	0.73	0.47	0.47	0.38	0.39	0.80	0.56	0.56	0.55	0.56	1.09	0.74	0.74				
HFUL-IR	0.17	0.27	0.91	0.45	0.45	0.19	0.28	0.92	0.45	0.45	0.32	0.37	0.96	0.52	0.52	0.50	0.53	1.30	0.74	0.74				

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 8. Monte Carlo Results: Model 1, $\sigma_{uv} = 0.8$ ($R^2 = 0.01, N = 100$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$							
	$N = 100$	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE
OLS	0.79	0.79	0.15	0.80	0.80	0.79	0.79	0.16	0.79	0.79	0.79	0.79	0.79	0.16	0.79	0.79	0.79	0.79	0.18	0.79	0.79	0.79	0.79	0.79		
2SLS-all	0.76	0.76	0.36	0.78	0.78	0.76	0.76	0.36	0.78	0.78	0.80	0.80	0.39	0.81	0.81	0.84	0.84	0.51	0.87	0.87	0.87	0.87	0.87			
2SLS-val	0.55	0.83	4.02	28.44	2.91	0.54	0.84	3.95	64.11	2.86	0.55	0.84	4.12	172.26	2.89	0.55	0.87	4.18	2578.00	2.94						
2SLS-DN	0.69	0.73	1.27	19.80	1.80	0.70	0.74	1.25	15.76	1.78	0.77	0.81	1.26	9.47	1.78	0.85	0.90	1.64	15.30	2.03						
2SLS-IR	0.68	0.71	1.09	8.90	1.39	0.69	0.71	1.07	6.79	1.35	0.75	0.78	1.16	4.56	1.47	0.82	0.86	1.48	9.82	1.58						
GMM-AVE	0.70	0.70	0.74	0.76	0.76	0.70	0.70	0.74	0.76	0.76	0.72	0.72	0.78	0.79	0.79	0.75	0.75	0.89	0.82	0.82	0.82					
LIML-all	0.58	0.86	3.97	21.87	2.81	0.62	0.86	3.75	63.45	2.79	0.85	1.17	4.88	58.34	3.19	1.77	2.73	13.23	18302.57	4.99						
LIML-val	0.55	0.83	4.02	28.44	2.91	0.54	0.84	3.95	64.11	2.86	0.55	0.84	4.12	172.26	2.89	0.55	0.87	4.18	2577.96	2.94						
LIML-DN	0.60	0.71	2.33	23.68	2.28	0.60	0.73	2.40	58.35	2.30	0.71	0.83	2.39	14.41	2.34	0.86	1.05	3.25	18.48	2.66						
LIML-IR	0.58	0.72	2.78	19.96	2.49	0.59	0.74	2.68	46.56	2.47	0.71	0.86	3.03	174.50	2.63	0.92	1.22	5.55	38.58	3.48						
FULL-all	0.63	0.63	1.56	0.90	0.90	0.65	0.66	1.50	0.90	0.90	0.83	0.83	1.73	1.05	1.05	1.37	1.38	2.89	1.53	1.53						
FULL-val	0.72	0.72	0.73	0.76	0.76	0.72	0.72	0.74	0.76	0.76	0.72	0.72	0.77	0.77	0.77	0.72	0.72	0.81	0.77	0.77						
FULL-DN	0.70	0.70	0.75	0.76	0.76	0.70	0.70	0.74	0.76	0.76	0.75	0.75	0.79	0.81	0.81	0.80	0.81	1.12	0.95	0.95						
FULL-IR	0.70	0.70	0.83	0.77	0.77	0.70	0.70	0.82	0.78	0.78	0.75	0.75	0.90	0.84	0.84	0.82	0.82	1.43	1.04	1.04						
JIVE-all	0.76	0.88	2.68	291.27	2.42	0.77	0.89	2.70	63.00	2.43	0.78	0.91	2.80	129.74	2.47	0.84	1.03	3.78	46.98	2.80						
JIVE-val	0.78	0.99	3.67	55.17	2.78	0.80	1.00	3.79	132.39	2.81	0.77	0.98	3.77	404.99	2.84	0.81	1.05	4.22	47.96	3.03						
JIVE-DN	0.69	0.82	2.61	44.71	2.34	0.71	0.83	2.63	123.19	2.36	0.80	0.90	2.58	82.11	2.39	0.91	1.05	3.05	44.04	2.65						
JIVE-IR	0.68	0.79	2.41	45.08	2.27	0.70	0.80	2.49	28.30	2.34	0.78	0.88	2.47	81.92	2.32	0.88	1.01	2.95	93.33	2.63						
HLIM-all	0.59	0.86	3.96	37.26	2.83	0.62	0.86	3.77	133.67	2.82	0.84	1.16	4.92	117.57	3.19	1.77	2.68	13.14	258.63	4.98						
HLIM-val	0.79	0.99	3.70	19.66	2.81	0.77	0.98	3.71	515.25	2.84	0.79	1.03	4.11	59.90	2.96	0.77	1.03	4.33	288.14	3.00						
HLIM-DN	0.68	0.79	2.33	18.74	2.25	0.70	0.79	2.23	94.59	2.27	0.80	0.90	2.46	26.58	2.38	0.96	1.11	3.17	46.42	2.65						
HLIM-IR	0.66	0.74	1.93	16.43	0.89	0.68	0.76	1.81	134.85	0.89	0.77	0.85	2.02	21.85	0.96	0.95	1.16	3.81	36.49	1.10						
HFUL-all	0.64	0.64	1.41	0.87	0.87	0.66	0.66	1.37	0.87	0.87	0.83	0.83	1.58	1.01	1.01	1.31	1.31	2.60	1.43	1.43						
HFUL-val	0.79	0.80	1.93	1.06	1.06	0.77	0.78	1.93	1.04	1.04	0.79	0.80	2.03	1.08	1.08	0.78	0.80	2.20	1.11	1.11						
HFUL-DN	0.70	0.71	1.50	0.94	0.94	0.72	0.72	1.48	0.93	0.93	0.80	0.80	1.57	1.00	1.00	0.91	0.92	1.83	1.11	1.11						
HFUL-IR	0.70	0.70	1.27	0.89	0.89	0.71	0.71	1.22	0.88	0.88	0.78	0.79	1.32	0.96	0.96	0.89	0.90	1.86	1.11	1.11						

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 9. Monte Carlo Results: Model 2 (equal strength of instruments) ($R^2 = 0.1, N = 100$)

	$N = 100$	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$									
		Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE			
OLS	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.23	0.46	0.46	0.45	0.45	0.23	0.46	0.46	0.46			
2SLS-all	0.31	0.32	0.43	0.36	0.36	0.33	0.33	0.42	0.37	0.37	0.41	0.41	0.43	0.44	0.44	0.52	0.52	0.48	0.55	0.55	0.52	0.52	0.52	0.48	0.55	0.55			
2SLS-val	0.27	0.75	4.39	31.39	2.98	0.29	0.75	4.28	47.45	2.96	0.27	0.75	4.43	50.91	2.94	0.29	0.79	4.57	180.43	3.03	0.29	0.79	4.57	180.43	3.03	0.29	0.79	4.57	
2SLS-DN	0.30	0.34	0.86	8.65	1.35	0.32	0.36	0.85	24.17	1.29	0.44	0.47	0.83	47.08	1.31	0.60	0.63	1.03	16.68	1.52	0.60	0.63	1.03	16.68	1.52	0.60	0.63	1.03	
2SLS-IR	0.29	0.38	1.32	4.12	1.41	0.31	0.39	1.32	8.32	1.40	0.41	0.48	1.32	12.98	1.43	0.58	0.64	1.60	16.82	1.56	0.58	0.64	1.60	16.82	1.56	0.58	0.64	1.60	
GMM-AVE	0.31	0.33	0.85	0.47	0.47	0.32	0.35	0.87	0.48	0.48	0.37	0.39	0.86	0.51	0.51	0.44	0.46	0.94	0.57	0.57	0.44	0.46	0.94	0.57	0.57	0.44	0.46	0.94	0.57
LIML-all	0.03	0.33	1.61	31.53	1.63	0.06	0.34	1.66	46.70	1.72	0.31	0.49	1.98	48.41	1.89	0.76	0.97	3.56	154.58	2.76	0.76	0.97	3.56	154.58	2.76	0.76	0.97	3.56	
LIML-val	0.27	0.75	4.39	31.39	2.98	0.29	0.75	4.28	47.45	2.96	0.27	0.75	4.43	50.91	2.94	0.29	0.79	4.57	180.43	3.03	0.29	0.79	4.57	180.43	3.03	0.29	0.79	4.57	
LIML-DN	0.24	0.39	1.65	11.22	1.88	0.27	0.41	1.63	44.12	1.85	0.43	0.54	1.73	57.70	1.90	0.68	0.79	2.06	19.47	2.04	0.40	0.57	2.20	17.06	2.14	0.67	0.85	3.05	
LIML-IR	0.20	0.44	2.21	137.83	2.14	0.23	0.47	2.27	46.75	2.15	0.40	0.57	2.20	17.06	2.14	0.67	0.85	3.05	26.65	2.48	0.40	0.57	2.20	17.06	2.14	0.67	0.85	3.05	
FULL-all	0.07	0.29	1.21	0.52	0.52	0.09	0.30	1.23	0.54	0.54	0.32	0.43	1.43	0.67	0.67	0.72	0.80	2.07	1.06	1.06	0.72	0.80	2.07	1.06	0.72	0.80	2.07	1.06	
FULL-val	0.39	0.41	0.90	0.52	0.52	0.40	0.42	0.91	0.53	0.53	0.40	0.41	0.90	0.53	0.53	0.40	0.42	0.95	0.54	0.54	0.40	0.42	0.95	0.54	0.40	0.42	0.95	0.54	
FULL-DN	0.31	0.33	0.83	0.46	0.46	0.33	0.35	0.84	0.48	0.48	0.44	0.45	0.85	0.56	0.56	0.59	0.60	1.08	0.74	0.74	0.44	0.45	0.85	0.56	0.44	0.45	0.85	0.56	
FULL-IR	0.30	0.34	0.94	0.49	0.49	0.32	0.36	0.95	0.51	0.51	0.42	0.44	0.95	0.58	0.58	0.58	0.60	1.26	0.79	0.79	0.44	0.46	0.94	0.58	0.44	0.46	0.94	0.58	
JIVE-all	0.08	0.42	2.36	103.08	2.20	0.12	0.42	2.33	111.86	2.26	0.35	0.51	2.03	57.94	2.06	0.61	0.72	2.33	162.60	2.23	0.35	0.51	2.03	57.94	2.06	0.61	0.72	2.33	162.60
JIVE-val	0.46	0.96	5.15	168.98	3.23	0.42	0.94	5.14	135.72	3.18	0.45	0.96	5.25	198.27	3.25	0.43	0.98	5.40	47.58	3.28	0.43	0.98	5.40	47.58	3.28	0.43	0.98	5.40	47.58
JIVE-DN	0.29	0.48	2.47	164.29	2.32	0.30	0.49	2.41	133.21	2.22	0.46	0.60	2.33	45.84	2.30	0.66	0.80	2.64	22.41	2.32	0.46	0.60	2.33	45.84	2.30	0.66	0.80	2.64	22.41
JIVE-IR	0.26	0.46	2.33	163.05	2.24	0.28	0.47	2.28	82.45	2.21	0.44	0.57	2.22	39.93	2.26	0.64	0.78	2.66	31.48	2.36	0.44	0.57	2.22	39.93	2.26	0.64	0.78	2.66	31.48
HLIM-all	0.03	0.33	1.63	13.87	1.62	0.06	0.34	1.66	14.23	1.75	0.31	0.49	1.97	20.42	1.93	0.76	0.97	3.63	90.49	2.75	0.31	0.49	1.97	20.42	1.93	0.76	0.97	3.63	90.49
HLIM-val	0.42	0.93	5.23	26.86	3.20	0.46	0.95	5.13	141.82	3.18	0.43	0.96	5.10	296.00	3.15	0.47	1.01	5.40	70.38	3.25	0.43	0.96	5.40	70.38	3.25	0.43	0.96	5.40	70.38
HLIM-DN	0.27	0.41	1.86	13.64	1.97	0.29	0.43	1.93	117.82	2.06	0.46	0.57	1.96	293.77	2.09	0.72	0.83	2.20	26.20	2.20	0.46	0.57	1.96	293.77	2.09	0.72	0.83	2.20	26.20
HLIM-IR	0.23	0.39	1.56	19.30	0.66	0.26	0.40	1.56	12.31	0.67	0.44	0.52	1.62	17.45	0.75	0.70	0.81	2.21	32.50	0.94	0.44	0.52	1.62	17.45	0.75	0.70	0.81	2.21	32.50
HFUL-all	0.07	0.28	1.15	0.49	0.49	0.11	0.29	1.17	0.51	0.51	0.33	0.42	1.35	0.63	0.63	0.71	0.78	1.92	1.00	1.00	0.71	0.78	1.92	1.00	0.71	0.78	1.92	1.00	
HFUL-val	0.43	0.74	2.66	1.07	1.07	0.46	0.75	2.67	1.08	1.08	0.43	0.76	2.70	1.07	1.07	0.46	0.79	2.84	1.13	1.13	0.46	0.79	2.84	1.13	0.46	0.79	2.84	1.13	
HFUL-DN	0.29	0.39	1.49	0.74	0.74	0.31	0.40	1.50	0.76	0.76	0.46	0.53	1.47	0.81	0.81	0.68	0.75	1.70	0.96	0.96	0.68	0.75	1.70	0.96	0.68	0.75	1.70	0.96	
HFUL-IR	0.27	0.38	1.42	0.71	0.71	0.30	0.39	1.40	0.71	0.71	0.43	0.50	1.43	0.78	0.78	0.64	0.72	1.76	0.96	0.96	0.64	0.72	1.76	0.96	0.64	0.72	1.76	0.96	

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 10. Monte Carlo Results: Model 2 (equal strength of instruments) ($R^2 = 0.01, N = 100$)

	$N = 100$	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$								
		Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE		
OLS	0.50	0.50	0.22	0.50	0.50	0.49	0.49	0.22	0.50	0.50	0.49	0.49	0.23	0.50	0.50	0.49	0.49	0.24	0.49	0.49	0.50	0.50	0.50	0.50	0.50	0.50		
2SLS-all	0.47	0.47	0.50	0.52	0.52	0.48	0.48	0.50	0.52	0.52	0.51	0.51	0.53	0.55	0.55	0.56	0.56	0.63	0.63	0.61	0.61	0.61	0.61	0.61	0.61			
2SLS-val	0.50	0.97	5.13	2074.38	3.15	0.49	0.98	5.54	53.89	3.31	0.47	0.99	5.34	117.49	3.29	0.48	1.02	5.58	219.46	3.29								
2SLS-DN	0.48	0.56	1.72	22.73	1.94	0.47	0.56	1.77	35.94	2.10	0.53	0.62	1.82	42.91	2.06	0.60	0.72	2.23	64.67	2.22								
2SLS-IR	0.48	0.55	1.48	31.22	1.50	0.48	0.56	1.53	12.73	1.53	0.52	0.60	1.58	6.96	1.59	0.60	0.70	1.95	63.09	1.71								
GMM-AVE	0.49	0.49	0.90	0.61	0.61	0.48	0.49	0.90	0.61	0.61	0.50	0.50	0.93	0.63	0.63	0.53	0.54	1.02	0.66	0.66								
LIML-all	0.37	0.88	4.85	40.26	3.06	0.40	0.87	4.85	66.20	3.09	0.62	1.16	6.10	61.96	3.41	1.30	2.40	12.36	379.47	4.79								
LIML-val	0.50	0.97	5.13	2074.40	3.15	0.49	0.98	5.54	53.89	3.31	0.47	0.99	5.34	117.49	3.29	0.48	1.02	5.58	219.46	3.29								
LIML-DN	0.49	0.72	3.27	2073.90	2.54	0.47	0.74	3.44	36.70	2.71	0.55	0.79	3.37	103.40	2.64	0.65	0.98	3.83	67.14	2.80								
LIML-IR	0.46	0.74	3.70	71.15	2.72	0.47	0.76	3.73	53.28	2.79	0.55	0.84	4.10	31.23	2.89	0.70	1.17	6.43	185.61	3.57								
FULL-all	0.40	0.59	2.07	0.89	0.89	0.42	0.59	2.03	0.89	0.89	0.58	0.75	2.32	1.03	1.03	1.02	1.24	3.31	1.46	1.46								
FULL-val	0.50	0.50	0.86	0.61	0.61	0.49	0.50	0.87	0.60	0.60	0.49	0.49	0.89	0.61	0.61	0.49	0.50	0.94	0.61	0.61								
FULL-DN	0.49	0.50	0.90	0.61	0.61	0.49	0.49	0.90	0.61	0.61	0.51	0.52	0.96	0.65	0.65	0.54	0.56	1.25	0.76	0.76								
FULL-IR	0.48	0.50	0.99	0.64	0.64	0.48	0.50	0.98	0.64	0.64	0.52	0.53	1.07	0.70	0.70	0.56	0.59	1.52	0.88	0.88								
JIVE-all	0.47	0.76	3.61	53.60	2.68	0.49	0.78	3.76	77.41	2.81	0.49	0.82	3.90	78.66	2.81	0.56	0.93	4.66	123.97	3.03								
JIVE-val	0.48	0.98	5.19	54.60	3.20	0.50	1.00	5.35	358.65	3.21	0.48	0.97	5.20	95.00	3.25	0.51	1.05	5.76	71.38	3.37								
JIVE-DN	0.47	0.78	3.83	44.21	2.75	0.48	0.80	3.89	343.39	2.79	0.54	0.85	3.85	98.23	2.82	0.62	0.98	4.44	65.57	3.01								
JIVE-IR	0.46	0.75	3.76	71.76	2.77	0.47	0.77	3.80	500.02	2.77	0.54	0.82	3.78	164.12	2.81	0.61	0.94	4.35	82.32	3.02								
HLIM-all	0.38	0.88	4.91	64.88	3.06	0.40	0.87	4.84	45.46	3.07	0.62	1.14	5.97	299.12	3.41	1.30	2.39	12.19	6517.97	4.75								
HLIM-val	0.52	1.00	5.24	328.26	3.21	0.49	0.99	5.26	53.62	3.23	0.52	1.03	5.67	48.70	3.37	0.49	1.04	5.75	242.96	3.33								
HLIM-DN	0.49	0.72	3.40	327.87	2.63	0.49	0.73	3.38	39.38	2.66	0.56	0.81	3.48	46.41	2.73	0.66	0.99	4.04	240.97	2.81								
HLIM-IR	0.47	0.66	2.75	35.84	0.88	0.48	0.66	2.56	40.70	0.87	0.55	0.75	2.98	26.28	0.93	0.68	1.04	4.27	168.70	1.05								
HFUL-all	0.40	0.56	1.89	0.84	0.84	0.42	0.56	1.86	0.83	0.83	0.57	0.70	2.11	0.97	0.97	1.13	2.99	1.35	1.35									
HFUL-val	0.52	0.79	2.77	1.13	1.13	0.49	0.78	2.78	1.11	1.11	0.51	0.81	2.83	1.14	1.14	0.49	0.82	2.99	1.17	1.17								
HFUL-DN	0.49	0.64	2.23	0.98	0.98	0.49	0.64	2.17	0.97	0.97	0.55	0.69	2.27	1.01	1.01	0.61	0.81	2.49	1.09	1.09								
HFUL-IR	0.48	0.59	1.84	0.90	0.90	0.48	0.58	1.79	0.88	0.88	0.54	0.65	1.95	0.94	0.94	0.61	0.78	2.37	1.07	1.07								

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 11. Monte Carlo Results: Model 2 (equal strength of instruments) ($R^2 = 0.1, N = 1000$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$							
$N = 1000$	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	
OLS	0.45	0.45	0.07	0.45	0.45	0.45	0.45	0.07	0.45	0.45	0.45	0.45	0.07	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45	0.45		
2SLS-all	0.10	0.10	0.20	0.13	0.13	0.11	0.11	0.20	0.13	0.13	0.20	0.20	0.20	0.21	0.21	0.40	0.40	0.21	0.41	0.41	0.40	0.40	0.21	0.41	0.41	
2SLS-val	0.04	0.36	1.74	17.53	1.74	0.03	0.36	1.70	39.17	1.72	0.03	0.36	1.80	17.76	1.73	0.02	0.37	1.77	27.86	1.74	0.02	0.37	1.77	27.86	1.74	
2SLS-DN	0.11	0.11	0.24	0.21	0.18	0.11	0.12	0.24	0.24	0.21	0.20	0.20	0.20	0.22	0.22	0.40	0.40	0.22	0.41	0.41	0.40	0.40	0.22	0.41	0.41	
2SLS-IR	0.08	0.12	0.43	1.29	0.49	0.09	0.12	0.43	1.31	0.55	0.20	0.20	0.32	0.47	0.40	0.39	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.60	
GMM-AVE	0.09	0.15	0.65	0.29	0.29	0.08	0.15	0.64	0.29	0.29	0.15	0.20	0.65	0.31	0.31	0.25	0.30	0.73	0.37	0.37	0.25	0.30	0.73	0.37	0.37	
LIML-all	-0.00	0.07	0.27	0.11	0.11	0.00	0.07	0.27	0.11	0.11	0.11	0.12	0.27	0.15	0.15	0.36	0.36	0.36	0.39	0.39	0.36	0.36	0.36	0.39	0.39	
LIML-val	0.04	0.36	1.74	17.53	1.74	0.03	0.36	1.70	39.17	1.72	0.03	0.36	1.80	17.76	1.73	0.02	0.37	1.77	27.86	1.74	0.02	0.37	1.77	27.86	1.74	
LIML-DN	0.01	0.07	0.27	0.11	0.11	0.01	0.07	0.27	0.11	0.11	0.12	0.12	0.27	0.16	0.16	0.37	0.37	0.36	0.40	0.40	0.37	0.37	0.36	0.40	0.40	
LIML-IR	-0.01	0.09	0.40	1.93	0.52	-0.00	0.09	0.40	1.93	0.57	0.12	0.13	0.40	0.37	0.31	0.33	0.35	0.72	8.24	0.66	0.33	0.35	0.72	8.24	0.66	
FULL-all	0.00	0.07	0.27	0.11	0.11	0.01	0.07	0.27	0.11	0.11	0.12	0.12	0.27	0.15	0.15	0.36	0.36	0.35	0.39	0.39	0.36	0.36	0.35	0.39	0.39	
FULL-val	0.16	0.26	0.88	0.39	0.39	0.16	0.26	0.87	0.38	0.38	0.16	0.27	0.89	0.39	0.39	0.16	0.26	0.89	0.39	0.39	0.16	0.26	0.89	0.39	0.39	
FULL-DN	0.01	0.07	0.27	0.10	0.10	0.02	0.07	0.26	0.11	0.11	0.12	0.13	0.27	0.16	0.16	0.37	0.37	0.35	0.40	0.40	0.16	0.16	0.27	0.16	0.40	
FULL-IR	0.01	0.09	0.39	0.23	0.23	0.01	0.09	0.38	0.22	0.22	0.13	0.13	0.37	0.24	0.24	0.35	0.35	0.51	0.43	0.43	0.35	0.35	0.51	0.43	0.43	
JIVE-all	-0.01	0.07	0.29	0.12	0.12	-0.00	0.07	0.29	0.12	0.12	0.12	0.12	0.28	0.16	0.16	0.38	0.38	0.28	0.40	0.40	0.38	0.38	0.28	0.40	0.40	
JIVE-val	0.45	0.95	5.08	61.90	3.18	0.45	0.95	5.11	44.75	3.14	0.45	0.96	5.13	56.64	3.22	0.46	0.95	5.24	67.99	3.21	0.46	0.95	5.24	67.99	3.21	
JIVE-DN	0.00	0.07	0.28	0.11	0.11	0.00	0.07	0.28	0.11	0.11	0.12	0.13	0.28	0.16	0.16	0.39	0.39	0.28	0.41	0.41	0.39	0.39	0.28	0.41	0.41	
JIVE-IR	0.02	0.11	0.52	21.61	0.51	0.02	0.11	0.54	0.62	0.48	0.16	0.17	0.43	1.33	0.39	0.44	0.44	0.54	10.32	0.66	0.39	0.44	0.44	0.54	10.32	0.66
HLIM-all	-0.00	0.07	0.27	0.11	0.11	0.00	0.07	0.27	0.11	0.11	0.11	0.12	0.27	0.15	0.15	0.36	0.36	0.36	0.39	0.39	0.36	0.36	0.35	0.39	0.39	
HLIM-val	0.45	0.94	5.02	48.21	3.14	0.43	0.94	5.23	48.08	3.22	0.43	0.93	5.30	147.31	3.24	0.43	0.97	5.33	35.31	3.29	0.43	0.97	5.33	35.31	3.29	
HLIM-DN	0.01	0.07	0.27	0.11	0.11	0.01	0.07	0.27	0.11	0.11	0.12	0.12	0.27	0.16	0.16	0.37	0.37	0.36	0.40	0.40	0.37	0.37	0.36	0.40	0.40	
HLIM-IR	0.02	0.10	0.49	0.40	0.29	0.03	0.10	0.51	1.16	0.29	0.15	0.17	0.44	0.68	0.26	0.43	0.43	0.59	0.61	0.53	0.43	0.43	0.59	0.61	0.53	
HFUL-all	0.00	0.07	0.27	0.11	0.11	0.01	0.07	0.27	0.11	0.11	0.12	0.12	0.27	0.15	0.15	0.36	0.36	0.35	0.39	0.39	0.36	0.36	0.35	0.39	0.39	
HFUL-val	0.45	0.91	4.46	2.07	2.07	0.43	0.91	4.52	2.08	2.08	0.43	0.90	4.62	2.10	2.10	0.43	0.95	4.67	2.14	2.14	0.43	0.95	4.67	2.14	2.14	
HFUL-DN	0.01	0.07	0.27	0.10	0.10	0.02	0.07	0.26	0.11	0.11	0.12	0.13	0.27	0.16	0.16	0.37	0.37	0.36	0.40	0.40	0.37	0.37	0.36	0.40	0.40	
HFUL-IR	0.03	0.10	0.50	0.41	0.41	0.03	0.10	0.53	0.43	0.43	0.16	0.17	0.44	0.35	0.35	0.43	0.43	0.44	0.58	0.59	0.59	0.43	0.44	0.58	0.59	0.59

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 12. Monte Carlo Results: Model 2 (equal strength of instruments) ($R^2 = 0.01, N = 1000$)

$N = 1000$	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$						
	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE
OLS	0.49	0.49	0.07	0.50	0.50	0.50	0.50	0.07	0.50	0.50	0.49	0.49	0.07	0.50	0.50	0.50	0.50	0.07	0.50	0.50	0.50	0.50	0.50	0.50	
2SLS-all	0.37	0.37	0.36	0.40	0.40	0.37	0.37	0.37	0.40	0.40	0.47	0.47	0.37	0.49	0.49	0.69	0.69	0.47	0.72	0.72	0.72	0.72	0.72	0.72	
2SLS-val	0.36	0.84	4.66	166.05	3.03	0.35	0.87	4.93	97.76	3.12	0.35	0.83	4.66	156.68	3.03	0.35	0.84	4.86	420.49	3.09					
2SLS-DN	0.38	0.39	0.60	147.24	1.09	0.37	0.39	0.64	5.33	1.06	0.50	0.51	0.58	4.58	1.09	0.79	0.83	1.10	418.11	1.58					
2SLS-IR	0.37	0.41	1.06	20.37	1.18	0.37	0.41	1.08	86.91	1.21	0.50	0.53	1.04	4.04	1.20	0.78	0.83	1.61	6.56	1.49					
GMM-AVE	0.38	0.39	0.77	0.50	0.50	0.38	0.39	0.77	0.50	0.50	0.45	0.45	0.78	0.55	0.55	0.59	0.60	0.89	0.66	0.66					
LIML-all	0.06	0.38	1.90	30.21	1.98	0.05	0.38	1.97	27.62	1.91	0.40	0.60	2.46	39.61	2.20	2.35	2.74	9.72	156.79	4.78					
LIML-val	0.36	0.84	4.66	166.05	3.03	0.35	0.87	4.93	97.76	3.12	0.35	0.83	4.66	156.68	3.03	0.35	0.84	4.86	420.49	3.09					
LIML-DN	0.32	0.51	2.19	150.87	2.11	0.31	0.50	2.20	35.92	2.13	0.51	0.65	2.29	156.02	2.15	1.21	1.32	3.02	245.73	2.60					
LIML-IR	0.28	0.47	1.88	30.89	1.91	0.28	0.46	1.84	39.99	1.96	0.50	0.61	1.91	152.20	2.00	1.34	1.53	4.26	115.16	3.24					
FULL-all	0.10	0.33	1.38	0.60	0.60	0.09	0.33	1.41	0.62	0.62	0.41	0.52	1.68	0.79	0.79	1.93	1.96	3.35	2.11	2.11					
FULL-val	0.46	0.47	0.90	0.57	0.57	0.46	0.47	0.91	0.57	0.57	0.46	0.47	0.90	0.57	0.57	0.46	0.47	0.91	0.56	0.56					
FULL-DN	0.40	0.42	0.87	0.53	0.53	0.40	0.41	0.87	0.53	0.53	0.50	0.50	0.88	0.62	0.62	0.79	0.80	1.62	1.10	1.10					
FULL-IR	0.37	0.41	0.96	0.54	0.54	0.37	0.40	0.93	0.54	0.54	0.48	0.49	0.94	0.63	0.63	0.75	0.80	2.03	1.23	1.23					
JIVE-all	0.16	0.48	2.76	31.13	2.41	0.15	0.48	2.77	237.84	2.38	0.44	0.59	2.27	36.36	2.16	1.06	1.22	3.91	39.75	3.02					
JIVE-val	0.51	0.97	5.14	65.18	3.21	0.49	0.97	5.24	520.59	3.19	0.49	0.99	5.38	57.67	3.28	0.49	1.01	5.31	49.82	3.23					
JIVE-DN	0.37	0.59	2.90	50.50	2.43	0.35	0.59	2.92	517.13	2.38	0.54	0.72	2.93	51.82	2.45	0.93	1.13	3.54	29.12	2.68					
JIVE-IR	0.33	0.50	2.25	49.79	2.16	0.32	0.50	2.29	519.21	2.16	0.50	0.63	2.24	30.14	2.23	0.87	1.03	3.13	158.33	2.56					
HLIM-all	0.06	0.38	1.90	13.79	1.97	0.05	0.38	1.94	22.85	1.89	0.40	0.60	2.46	42.17	2.21	2.35	2.73	9.71	247.50	4.78					
HLIM-val	0.48	0.01	5.50	946.71	3.28	0.50	1.02	5.40	410.14	3.29	0.51	1.00	5.43	57.77	3.31	0.50	1.02	5.39	103.01	3.28					
HLIM-DN	0.35	0.54	2.53	82.71	2.31	0.35	0.53	2.44	302.50	2.26	0.55	0.69	2.51	27.06	2.37	1.25	1.38	3.15	18.46	2.76					
HLIM-IR	0.29	0.40	1.49	78.68	0.67	0.29	0.41	1.46	36.16	0.67	0.51	0.57	1.54	14.94	0.77	1.33	1.56	4.27	137.59	1.19					
HFUL-all	0.10	0.33	1.37	0.60	0.60	0.09	0.33	1.40	0.61	0.61	0.41	0.52	1.66	0.78	0.78	1.92	1.95	3.30	2.09	2.09					
HFUL-val	0.48	0.99	4.76	2.16	0.50	0.99	4.75	2.17	0.50	0.97	4.73	2.20	0.50	0.99	4.75	2.21	2.21								
HFUL-DN	0.37	0.52	2.29	1.55	1.55	0.37	0.52	2.28	1.53	1.53	0.55	0.67	2.27	1.62	1.62	1.06	1.18	2.68	1.77	1.77					
HFUL-IR	0.33	0.43	1.64	1.30	1.30	0.33	0.44	1.67	1.31	1.31	0.50	0.58	1.65	1.36	1.36	1.07	1.27	3.25	1.87	1.87					

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 13. Monte Carlo Results: Model 3 (heteroskedastic error) ($R^2 = 0.1, N = 100$)

$N = 100$	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$								
	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE		
OLS	0.44	0.49	1.35	0.69	0.45	0.49	1.35	0.70	0.44	0.48	1.32	0.69	0.44	0.48	1.29	0.68	0.44	0.48	1.29	0.68	0.44	0.48	1.27	0.68	0.68		
2SLS-all	0.30	0.67	2.47	1.04	0.31	0.67	2.45	1.03	0.41	0.68	2.44	1.06	0.49	0.73	2.47	1.12	1.12	0.49	0.73	2.47	1.12	1.12	0.49	0.73	2.47	1.12	1.12
2SLS-val	0.00	1.66	7.59	48.14	3.59	0.02	1.70	7.82	43.74	3.61	-0.04	1.68	7.68	91.56	3.64	0.03	1.65	7.50	320.70	3.59	0.03	1.65	7.50	320.70	3.59		
2SLS-DN	0.19	0.92	3.62	14.62	1.73	0.21	0.89	3.60	8.17	1.69	0.34	0.93	3.53	82.69	1.74	0.49	0.94	3.56	2.60	1.72	0.49	0.94	3.56	2.60	1.72		
2SLS-IR	0.06	1.25	5.06	6.52	2.29	0.08	1.25	5.14	13.28	2.34	0.16	1.28	5.16	83.57	2.32	0.32	1.31	5.20	5.76	2.33	0.32	1.31	5.20	5.76	2.33		
GMM-AVE	0.14	0.88	3.64	1.57	1.57	0.12	0.89	3.79	1.59	1.59	0.14	0.89	3.74	1.60	1.60	0.22	0.87	3.67	1.58	1.58	0.22	0.87	3.67	1.58	1.58		
LIML-all	-0.00	2.00	10.2410552.01	4.33	0.02	1.97	10.06	342.49	4.29	0.32	2.01	10.14	42.20	4.28	0.65	2.12	10.60	104.71	4.43	0.65	2.12	10.60	104.71	4.43			
LIML-val	0.00	1.66	7.59	48.14	3.59	0.02	1.70	7.82	43.74	3.61	-0.04	1.68	7.68	91.56	3.64	0.03	1.65	7.50	320.70	3.59	0.03	1.65	7.50	320.70	3.59		
LIML-DN	0.10	1.17	4.71	23.03	2.26	0.14	1.13	4.66	9.22	2.25	0.25	1.15	4.69	83.25	2.32	0.43	1.17	4.69	366.70	2.33	0.43	1.17	4.69	366.70	2.33		
LIML-IR	-0.02	1.49	5.95	28.17	2.74	-0.03	1.49	5.97	29.30	2.78	0.06	1.50	6.04	88.98	2.82	0.24	1.52	6.03	45.72	2.81	0.24	1.52	6.03	45.72	2.81		
FULL-all	0.04	1.74	7.59	3.14	3.12	0.05	1.72	7.46	3.06	3.04	0.33	1.76	7.53	3.11	3.08	0.63	1.86	7.71	3.26	3.21	0.63	1.86	7.71	3.26	3.21		
FULL-val	0.14	1.14	4.46	1.81	1.81	0.15	1.14	4.54	1.83	1.83	0.10	1.14	4.49	1.83	1.83	0.16	1.12	4.47	1.81	1.81	0.16	1.12	4.47	1.81	1.81		
FULL-DN	0.13	1.03	3.98	1.64	1.64	0.18	1.00	3.94	1.63	1.63	0.27	1.02	3.95	1.64	1.64	0.43	1.03	3.96	1.68	1.68	0.43	1.03	3.96	1.68	1.68		
FULL-IR	0.07	1.21	4.60	1.92	1.92	0.08	1.20	4.59	1.93	1.92	0.15	1.20	4.58	1.92	1.92	0.33	1.24	4.69	1.98	1.97	0.33	1.24	4.69	1.98	1.97		
JIVE-all	0.08	2.00	10.60	2546.24	4.39	0.13	1.96	10.78	272.78	4.39	0.35	2.01	10.75	252.55	4.43	0.57	2.03	10.842409.66	4.46	0.57	2.03	10.842409.66	4.46	0.57	2.03	10.842409.66	4.46
JIVE-val	0.47	4.26	26.31	12439.32	6.26	0.39	4.27	26.33	18007.06	6.29	0.39	4.21	25.824790.68	6.22	0.31	4.27	25.63	709.39	6.25	0.31	4.27	25.63	709.39	6.25			
JIVE-DN	0.14	1.42	6.55	73.75	3.49	0.15	1.35	6.39	85.07	3.41	0.29	1.35	6.35	4735.64	3.43	0.47	1.42	6.21	43.84	3.39	0.47	1.42	6.21	43.84	3.39		
JIVE-IR	0.13	1.41	6.12	180.24	2.99	0.10	1.36	6.04	18.49	2.96	0.26	1.39	6.09	69.95	3.00	0.46	1.42	5.97	14.36	2.99	0.46	1.42	5.97	14.36	2.99		
HLIM-all	0.07	1.73	8.54	115.16	3.92	0.08	1.71	8.45	350.48	3.89	0.35	1.74	8.41	336.79	3.90	0.63	1.87	8.86	161.28	4.01	0.63	1.87	8.86	161.28	4.01		
HLIM-val	0.31	3.91	23.5313581.78	6.08	0.41	3.86	23.83	654.83	6.05	0.28	4.03	25.48	132.86	6.16	0.39	3.85	23.94	327.25	6.07	0.39	3.85	23.94	327.25	6.07			
HLIM-DN	0.11	1.25	5.44	88.48	3.03	0.17	1.19	5.28	66.58	3.02	0.28	1.22	5.43	38.60	3.10	0.46	1.27	5.38	17.36	3.06	0.46	1.27	5.38	17.36	3.06		
HLIM-IR	0.14	1.27	5.44	19.33	1.11	0.17	1.24	5.30	23.61	1.11	0.28	1.28	5.31	25.31	1.12	0.49	1.31	5.40	13.62	1.12	0.49	1.31	5.40	13.62	1.12		
HFUL-all	0.10	1.50	6.40	2.65	2.64	0.11	1.49	6.30	2.60	2.60	0.36	1.52	6.39	2.65	2.64	0.61	1.61	6.51	2.77	2.76	0.61	1.61	6.51	2.77	2.76		
HFUL-val	0.32	3.21	13.22	4.87	4.78	0.41	3.15	13.13	4.88	4.78	0.31	3.30	13.50	4.97	4.88	0.38	3.21	13.22	4.93	4.82	0.38	3.21	13.22	4.93	4.82		
HFUL-DN	0.14	1.13	4.78	2.49	2.46	0.20	1.10	4.71	2.45	2.42	0.30	1.12	4.81	2.57	2.53	0.45	1.16	4.77	2.56	2.52	0.45	1.16	4.77	2.56	2.52		
HFUL-IR	0.15	1.20	5.08	2.39	2.37	0.18	1.18	5.11	2.40	2.38	0.28	1.23	5.07	2.41	2.38	0.50	1.24	5.06	2.45	2.43	0.50	1.24	5.06	2.45	2.43		

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 14. Monte Carlo Results: Model 3 (heteroskedastic error) ($R^2 = 0.01, N = 100$)

	$N = 100$	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$					
		Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE
OLS	0.49	0.53	1.40	0.75	0.75	0.48	0.53	1.39	0.73	0.73	0.48	0.53	1.39	0.74	0.74	0.48	0.52	1.41	0.74	0.74	0.48	0.52	1.41	0.74	0.74
2SLS-all	0.50	0.81	2.86	1.25	1.25	0.45	0.79	2.82	1.23	0.51	0.83	2.84	1.25	1.25	0.53	0.82	2.86	1.28	1.28	0.53	0.82	2.86	1.28	1.28	
2SLS-val	0.42	3.90	24.39	329.93	6.11	0.37	3.93	23.87	855.67	6.07	0.37	4.00	24.72	293139.72	6.15	0.28	3.96	24.44	695.97	6.09	0.28	3.96	24.44	695.97	6.09
2SLS-DN	0.48	1.54	8.06	112.40	3.94	0.39	1.51	8.34	94.26	3.96	0.55	1.59	8.28	68.94	3.98	0.59	1.60	8.37	477.20	4.01	0.59	1.60	8.37	477.20	4.01
2SLS-IR	0.46	1.66	7.56	33.21	3.64	0.37	1.65	7.70	240.99	3.61	0.51	1.69	7.64	74.79	3.67	0.57	1.69	7.84	270.05	3.64	0.57	1.69	7.84	270.05	3.64
GMM-AVE	0.39	1.18	4.79	2.01	2.01	0.34	1.14	4.66	1.97	0.35	1.17	4.78	2.00	2.00	0.37	1.14	4.69	1.97	1.97	0.37	1.14	4.69	1.97	1.97	
LIML-all	0.56	4.82	29.44	2631.70	6.54	0.27	4.91	30.76	3294.81	6.58	0.63	4.94	29.53	470.47	6.57	0.84	5.18	31.80	743.83	6.68	0.84	5.18	31.80	743.83	6.68
LIML-val	0.42	3.90	24.39	329.93	6.11	0.37	3.93	23.87	855.67	6.07	0.37	4.00	24.72	291361.05	6.15	0.28	3.96	24.44	695.97	6.09	0.28	3.96	24.44	695.97	6.09
LIML-DN	0.45	2.61	14.78	275.13	5.08	0.32	2.62	14.49	287.92	5.06	0.55	2.70	14.93	114.03	5.12	0.60	2.72	14.52	485.36	5.09	0.60	2.72	14.52	485.36	5.09
LIML-IR	0.46	2.86	17.84	355.36	5.41	0.29	2.87	17.25	3298.98	5.38	0.51	2.93	17.90	1131.59	5.44	0.66	2.97	17.48	410.06	5.41	0.66	2.97	17.48	410.06	5.41
FULL-all	0.55	3.15	11.57	4.47	4.40	0.32	3.19	11.73	4.48	4.41	0.59	3.25	11.87	4.57	4.49	0.77	3.27	11.97	4.64	4.55	0.77	3.27	11.97	4.64	4.55
FULL-val	0.47	1.20	4.84	2.02	2.02	0.45	1.18	4.79	2.01	2.01	0.44	1.22	4.89	2.03	2.03	0.41	1.19	4.84	1.99	1.99	0.41	1.19	4.84	1.99	1.99
FULL-DN	0.48	1.25	5.07	2.10	2.10	0.42	1.23	5.06	2.09	2.09	0.51	1.29	5.08	2.11	2.11	0.52	1.27	5.13	2.15	2.15	0.52	1.27	5.13	2.15	2.15
FULL-IR	0.47	1.34	5.52	2.43	2.41	0.43	1.30	5.61	2.47	2.46	0.48	1.38	5.59	2.49	2.47	0.55	1.37	5.61	2.51	2.49	0.55	1.37	5.61	2.51	2.49
JIVE-all	0.51	3.30	20.06	691.12	5.71	0.43	3.25	19.54	335.68	5.66	0.46	3.30	20.10	415.17	5.70	0.51	3.32	20.09	456.93	5.72	0.51	3.32	20.09	456.93	5.72
JIVE-val	0.47	4.49	27.82	1470.28	6.40	0.57	4.57	29.01	171.67	6.42	0.60	4.54	27.86	663.40	6.41	0.40	4.47	27.67	484.70	6.39	0.40	4.47	27.67	484.70	6.39
JIVE-DN	0.49	3.11	19.04	1307.30	5.59	0.44	3.20	18.99	134.17	5.60	0.64	3.20	18.97	645.10	5.61	0.68	3.25	18.86	468.44	5.60	0.68	3.25	18.86	468.44	5.60
JIVE-IR	0.50	2.93	17.35	252.87	5.41	0.44	3.01	17.80	223.89	5.46	0.61	2.93	17.53	648.77	5.44	0.60	3.02	17.24	221.06	5.41	0.60	3.02	17.24	221.06	5.41
HLIM-all	0.55	3.88	23.77	370.61	6.07	0.36	3.98	24.60	298.23	6.13	0.56	4.06	24.10	632.08	6.11	0.74	4.24	25.45	297.48	6.23	0.74	4.24	25.45	297.48	6.23
HLIM-val	0.50	4.40	27.71	799.97	6.36	0.46	4.37	26.32	224.21	6.32	0.40	4.49	26.74	3668.27	6.35	0.53	4.51	27.10	399.60	6.36	0.53	4.51	27.10	399.60	6.36
HLIM-DN	0.53	2.71	16.48	775.22	5.27	0.42	2.70	16.00	74.57	5.22	0.56	2.76	15.96	90.06	5.23	0.71	2.83	15.75	239.52	5.25	0.71	2.83	15.75	239.52	5.25
HLIM-IR	0.47	2.28	12.43	771.81	1.24	0.40	2.25	12.32	134.70	1.24	0.51	2.36	12.63	72.44	1.24	0.64	2.38	12.80	210.86	1.24	0.64	2.38	12.80	210.86	1.24
HFUL-all	0.52	2.56	9.70	3.76	3.74	0.38	2.61	9.79	3.77	3.75	0.56	2.69	9.96	3.85	3.85	0.69	2.71	10.13	3.92	3.89	0.69	2.71	10.13	3.92	3.89
HFUL-val	0.52	3.48	14.22	5.24	5.09	0.44	3.52	14.07	5.18	5.05	0.42	3.53	14.21	5.23	5.08	0.53	3.57	14.24	5.24	5.09	0.53	3.57	14.24	5.24	5.09
HFUL-DN	0.53	2.24	10.85	4.31	4.19	0.43	2.26	10.87	4.25	4.15	0.53	2.27	10.65	4.29	4.17	0.65	2.34	10.76	4.31	4.18	0.65	2.34	10.76	4.31	4.18
HFUL-IR	0.49	1.92	8.77	3.82	3.73	0.37	1.90	8.73	3.72	3.64	0.47	1.95	8.94	3.80	3.72	0.61	1.98	8.92	3.82	3.72	0.61	1.98	8.92	3.82	3.72

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 15. Monte Carlo Results: Model 4 (wrong order of instrument strength) ($R^2 = 0.1, N = 100$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$							
$N = 100$	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	
OLS	0.45	0.45	0.21	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.23	0.46	0.46	0.45	0.45	0.23	0.46	0.46	
2SLS-all	0.32	0.32	0.43	0.36	0.36	0.32	0.32	0.43	0.36	0.36	0.33	0.33	0.46	0.37	0.37	0.33	0.33	0.52	0.39	0.39	0.44	0.44	0.78	2.84	1.12	
2SLS-val	0.45	0.92	5.02	472.06	3.11	0.42	0.92	4.88	368.13	3.10	0.43	0.95	5.12	62.72	3.17	0.44	0.99	5.40	100.01	3.25	0.37	0.47	1.84	19.99	1.97	0.37
2SLS-DN	0.38	0.44	1.44	255.56	1.82	0.37	0.43	1.44	9.76	1.79	0.38	0.45	1.61	32.60	1.90	0.37	0.47	1.84	19.99	1.97	0.39	0.50	1.75	6.60	1.60	
2SLS-IR	0.39	0.47	1.43	254.65	1.39	0.39	0.45	1.42	18.02	1.43	0.40	0.47	1.49	17.29	1.47	0.39	0.50	1.75	6.60	1.60	0.36	0.37	0.95	0.54	0.54	
GMM-AVE	0.35	0.36	0.83	0.50	0.50	0.35	0.36	0.84	0.50	0.50	0.36	0.37	0.88	0.52	0.52	0.37	0.38	0.95	0.54	0.54	0.37	0.38	0.95	0.54	0.54	
LIML-all	0.02	0.34	1.64	332.62	1.63	0.04	0.34	1.64	67.85	1.81	-0.03	0.41	2.28	49.72	2.18	-0.40	0.93	5.68	117.67	3.34	0.34	0.34	0.34	0.34	0.34	
LIML-val	0.45	0.92	5.02	472.06	3.11	0.42	0.92	4.88	368.13	3.10	0.43	0.95	5.12	62.72	3.17	0.44	0.99	5.40	100.01	3.25	0.37	0.47	1.84	19.99	1.97	
LIML-DN	0.37	0.60	2.95	260.29	2.45	0.36	0.59	2.87	82.90	2.43	0.38	0.63	2.99	14.36	2.44	0.30	0.74	3.67	100.55	2.67	0.34	0.63	3.41	255.41	2.63	
LIML-IR	0.34	0.63	3.41	255.41	2.63	0.32	0.62	3.52	345.96	2.68	0.34	0.70	4.03	31.15	2.87	0.31	0.99	6.61	100.25	3.53	0.34	0.34	0.34	0.34	0.34	
FULL-all	0.06	0.29	1.21	0.53	0.53	0.08	0.30	1.23	0.53	0.53	0.02	0.35	1.51	0.64	0.64	-0.25	0.62	2.61	1.00	1.00	0.06	0.06	0.06	0.06	0.06	
FULL-val	0.45	0.45	0.83	0.56	0.56	0.44	0.45	0.83	0.56	0.56	0.45	0.45	0.85	0.57	0.57	0.45	0.46	0.90	0.58	0.58	0.45	0.45	0.45	0.45	0.45	
FULL-DN	0.42	0.42	0.84	0.54	0.54	0.41	0.42	0.85	0.53	0.53	0.42	0.43	0.93	0.56	0.56	0.41	0.46	1.25	0.65	0.65	0.42	0.42	0.42	0.42	0.42	
FULL-IR	0.41	0.43	0.94	0.56	0.56	0.40	0.42	0.94	0.56	0.56	0.41	0.44	1.06	0.61	0.61	0.40	0.48	1.51	0.76	0.76	0.41	0.41	0.41	0.41	0.41	
JIVE-all	0.10	0.42	2.40	40.43	2.22	0.11	0.42	2.39	13.41	2.26	0.12	0.44	2.48	21.14	2.26	0.12	0.48	2.69	97.94	2.38	0.10	0.10	0.10	0.10	0.10	
JIVE-val	0.44	0.96	5.21	799.72	3.21	0.46	0.98	5.28	43.63	3.22	0.43	0.96	5.12	68.40	3.20	0.46	1.01	5.50	768.71	3.26	0.39	0.39	0.39	0.39	0.39	
JIVE-DN	0.39	0.68	3.60	47.37	2.68	0.40	0.70	3.66	27.62	2.69	0.40	0.71	3.66	65.65	2.69	0.39	0.79	4.02	189.52	2.79	0.36	0.65	3.51	56.72	2.65	
JIVE-IR	0.36	0.65	3.51	56.72	2.65	0.36	0.68	3.70	30.24	2.76	0.37	0.68	3.65	94.56	2.72	0.37	0.76	4.11	746.02	2.82	0.34	0.34	0.34	0.34	0.34	
HLIM-all	0.03	0.34	1.65	15.37	1.68	0.04	0.34	1.67	11.26	1.80	-0.03	0.41	2.29	23.75	2.20	-0.40	0.92	5.63	94.50	3.36	0.03	0.03	0.03	0.03	0.03	
HLIM-val	0.46	0.94	5.08	67.07	3.18	0.44	0.93	5.07	331.65	3.19	0.46	0.96	5.13	211.58	3.20	0.44	0.98	5.31	269.96	3.26	0.38	0.38	0.38	0.38	0.38	
HLIM-DN	0.38	0.61	3.02	50.36	2.51	0.37	0.59	2.90	86.13	2.49	0.38	0.63	3.02	68.17	2.50	0.31	0.75	3.67	271.07	2.67	0.34	0.34	0.34	0.34	0.34	
HLIM-IR	0.34	0.54	2.53	33.78	0.82	0.34	0.52	2.43	21.62	0.81	0.34	0.58	2.89	77.71	0.86	0.28	0.83	4.46	95.94	0.98	0.34	0.34	0.34	0.34	0.34	
HFUL-all	0.08	0.28	1.16	0.50	0.50	0.09	0.29	1.17	0.50	0.50	0.04	0.33	1.43	0.59	0.59	-0.21	0.57	2.37	0.90	0.90	0.46	0.46	0.46	0.46	0.46	
HFUL-val	0.46	0.74	2.64	1.06	1.06	0.45	0.73	2.65	1.06	1.06	0.46	0.75	2.65	1.08	1.08	0.44	0.78	2.84	1.12	1.12	0.46	0.46	0.46	0.46	0.46	
HFUL-DN	0.40	0.54	2.04	0.90	0.90	0.39	0.52	1.97	0.88	0.88	0.40	0.56	2.06	0.91	0.91	0.37	0.62	2.29	0.98	0.98	0.46	0.46	0.46	0.46	0.46	
HFUL-IR	0.36	0.49	1.80	0.83	0.83	0.36	0.48	1.75	0.83	0.83	0.36	0.51	1.92	0.87	0.87	0.32	0.60	2.30	0.96	0.96	0.46	0.46	0.46	0.46	0.46	

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 16. Monte Carlo Results: Model 4 (wrong order of instrument strength) ($R^2 = 0.01, N = 100$)

	$N = 100$	$\gamma = \infty$				$\gamma = 1$				$\gamma = 1/2$				$\gamma = 1/3$				
		Bias	MAD	IDR	MSE	Bias	MAD	IDR	MSE	Bias	MAD	IDR	MSE	Bias	MAD	IDR	MSE	
OLS	0.50	0.50	0.22	0.50	0.50	0.50	0.50	0.22	0.50	0.50	0.50	0.23	0.50	0.49	0.49	0.24	0.50	
2SLS-all	0.48	0.48	0.51	0.52	0.52	0.47	0.47	0.50	0.51	0.48	0.48	0.54	0.52	0.48	0.48	0.65	0.55	
2SLS-val	0.50	0.99	5.33	66.72	3.22	0.48	0.97	5.18	1306.90	3.21	0.50	1.03	5.43	42.90	3.25	0.47	1.05	5.74
2SLS-DN	0.50	0.58	1.84	51.80	2.04	0.48	0.58	1.93	50.60	2.12	0.49	0.60	2.07	20.95	2.11	0.49	0.65	2.45
2SLS-IR	0.50	0.57	1.55	24.27	1.51	0.48	0.56	1.55	13.88	1.55	0.50	0.58	1.66	18.37	1.56	0.49	0.63	2.01
GMM-AVE	0.48	0.49	0.90	0.60	0.60	0.47	0.48	0.90	0.60	0.49	0.49	0.94	0.62	0.62	0.48	0.49	1.04	0.63
LIML-all	0.40	0.89	4.74	61.30	3.07	0.38	0.88	4.80	115.41	3.10	0.34	1.19	7.10	225.34	3.66	0.10	2.70	14.44
LIML-val	0.50	0.99	5.33	66.72	3.22	0.48	0.97	5.18	1306.90	3.21	0.50	1.03	5.43	42.90	3.25	0.47	1.05	5.74
LIML-DN	0.51	0.75	3.37	57.32	2.64	0.48	0.74	3.44	302.49	2.68	0.50	0.81	3.61	26.50	2.66	0.47	0.95	4.38
LIML-IR	0.49	0.77	3.82	53.69	2.82	0.48	0.78	3.94	105.27	2.87	0.49	0.88	4.67	61.12	3.12	0.48	1.20	7.77
FULL-all	0.42	0.61	2.06	0.89	0.89	0.40	0.60	2.06	0.89	0.89	0.39	0.70	2.42	1.01	1.01	0.30	1.17	3.60
FULL-val	0.49	0.50	0.87	0.60	0.60	0.49	0.50	0.87	0.60	0.60	0.50	0.50	0.89	0.61	0.61	0.49	0.49	0.93
FULL-DN	0.50	0.50	0.91	0.62	0.62	0.49	0.50	0.91	0.61	0.61	0.50	0.51	0.98	0.64	0.64	0.49	0.52	1.29
FULL-IR	0.50	0.51	0.98	0.65	0.65	0.49	0.50	1.00	0.65	0.65	0.50	0.52	1.10	0.69	0.69	0.49	0.55	1.62
JIVE-all	0.49	0.79	3.89	291.88	2.78	0.49	0.79	3.83	167.67	2.78	0.50	0.80	3.94	34.75	2.83	0.49	0.90	4.62
JIVE-val	0.51	1.01	5.33	65.23	3.22	0.49	0.96	5.21	408.10	3.23	0.52	0.98	5.31	36.70	3.28	0.49	1.04	5.62
JIVE-DN	0.50	0.82	4.05	18.09	2.84	0.49	0.81	3.99	208.72	2.84	0.51	0.84	4.12	34.52	2.87	0.48	0.94	4.63
JIVE-IR	0.50	0.80	4.02	20.19	2.83	0.50	0.78	3.79	338.55	2.79	0.51	0.81	4.01	28.52	2.88	0.51	0.92	4.62
HLIM-all	0.39	0.89	4.79	194.23	3.09	0.37	0.87	4.89	187.17	3.12	0.35	1.19	7.08	6129.75	3.68	0.10	2.68	14.45
HLIM-val	0.48	0.98	5.42	576.56	3.29	0.49	1.01	5.55	52.83	3.28	0.46	1.02	5.66	215.95	3.34	0.49	1.05	5.74
HLIM-DN	0.50	0.76	3.52	50.27	2.74	0.49	0.77	3.65	47.68	2.73	0.48	0.81	3.86	194.15	2.81	0.49	0.95	4.36
HLIM-IR	0.50	0.69	2.80	91.79	0.89	0.49	0.68	2.92	48.09	0.90	0.50	0.77	3.48	2985.72	0.95	0.48	1.02	5.22
HFUL-all	0.42	0.58	1.89	0.84	0.84	0.41	0.56	1.88	0.83	0.83	0.40	0.65	2.20	0.94	0.94	0.33	1.04	3.26
HFUL-val	0.48	0.78	2.77	1.11	1.11	0.49	0.80	2.79	1.13	1.13	0.47	0.80	2.86	1.14	1.14	0.49	0.82	2.92
HFUL-DN	0.49	0.65	2.22	0.98	0.98	0.50	0.66	2.28	0.99	0.99	0.48	0.68	2.37	1.01	1.01	0.50	0.75	2.54
HFUL-IR	0.50	0.60	1.84	0.90	0.90	0.49	0.60	1.91	0.92	0.92	0.49	0.62	2.09	0.96	0.96	0.48	0.71	2.48

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 17. Monte Carlo Results: Model 5 ($\tau \propto (0, 1, 1, 0, \dots, 0)$) ($R^2 = 0.1, N = 100$)

	$N = 100$	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$					
		Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE
OLS	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.22	0.46	0.46
2SLS-all	0.32	0.32	0.43	0.36	0.36	0.32	0.32	0.43	0.37	0.37	0.37	0.37	0.37	0.42	0.41	0.41	0.43	0.43	0.44	0.44	0.47	0.47	0.47	0.47	0.47
2SLS-val	0.03	0.34	1.70	12.37	1.58	0.03	0.35	1.64	76.57	1.62	0.02	0.34	1.68	226.64	1.60	0.03	0.35	1.66	26.19	1.69	0.03	0.35	1.66	26.19	1.69
2SLS-DN	0.19	0.25	0.72	0.90	0.59	0.19	0.25	0.72	1.30	0.64	0.29	0.32	0.68	1.01	0.55	0.42	0.43	0.69	1.01	0.66	0.42	0.43	0.69	1.01	0.66
2SLS-IR	0.09	0.25	0.97	2.92	0.68	0.10	0.25	0.98	1.23	0.70	0.16	0.29	1.03	1.00	0.68	0.23	0.36	1.15	24.36	0.75	0.23	0.36	1.15	24.36	0.75
GMM-AVE	0.17	0.25	0.80	0.37	0.37	0.17	0.25	0.80	0.36	0.36	0.20	0.26	0.81	0.38	0.38	0.22	0.29	0.85	0.40	0.40	0.22	0.29	0.85	0.40	0.40
LIML-all	0.04	0.33	1.58	1245.43	1.73	0.05	0.33	1.61	5.13	1.66	0.19	0.38	1.64	40.44	1.71	0.39	0.54	2.12	420.28	2.07	0.39	0.54	2.12	420.28	2.07
LIML-val	0.03	0.34	1.70	12.37	1.58	0.03	0.35	1.64	76.57	1.62	0.02	0.34	1.68	226.64	1.60	0.03	0.35	1.66	26.19	1.69	0.03	0.35	1.66	26.19	1.69
LIML-DN	0.11	0.24	0.89	1.39	0.80	0.12	0.24	0.89	1.95	0.84	0.23	0.30	0.91	3.54	0.83	0.38	0.43	1.09	15.07	1.07	0.38	0.43	1.09	15.07	1.07
LIML-IR	0.03	0.26	1.15	4.88	1.02	0.04	0.25	1.14	55.14	0.99	0.10	0.28	1.18	16.82	1.02	0.17	0.37	1.44	18.98	1.17	0.17	0.37	1.44	18.98	1.17
FULL-all	0.08	0.29	1.20	0.53	0.53	0.09	0.29	1.21	0.54	0.54	0.22	0.34	1.25	0.58	0.58	0.40	0.48	1.48	0.72	0.72	0.40	0.48	1.48	0.72	0.72
FULL-val	0.14	0.26	0.89	0.39	0.39	0.15	0.26	0.89	0.38	0.38	0.14	0.26	0.89	0.38	0.38	0.15	0.26	0.91	0.39	0.39	0.39	0.26	0.91	0.39	0.39
FULL-DN	0.15	0.22	0.73	0.34	0.34	0.15	0.22	0.73	0.34	0.34	0.25	0.28	0.74	0.39	0.39	0.39	0.41	0.89	0.53	0.53	0.42	0.48	1.23	15.56	1.51
FULL-IR	0.11	0.23	0.85	0.38	0.38	0.12	0.23	0.85	0.37	0.37	0.17	0.26	0.88	0.41	0.41	0.24	0.32	1.06	0.50	0.50	0.24	0.32	1.06	0.50	0.50
JIVE-all	0.11	0.42	2.38	42.26	2.24	0.11	0.42	2.26	239.23	2.14	0.23	0.45	2.08	51.15	1.99	0.40	0.54	1.97	23.16	1.99	0.40	0.54	1.97	23.16	1.99
JIVE-val	0.43	0.95	5.29	394.16	3.23	0.43	0.94	5.11	214.97	3.18	0.44	0.92	5.07	62.90	3.17	0.42	0.96	5.39	236.87	3.25	0.42	0.96	5.39	236.87	3.25
JIVE-DN	0.13	0.29	1.30	391.33	1.68	0.13	0.29	1.27	205.44	1.55	0.26	0.35	1.23	53.74	1.50	0.43	0.48	1.25	12.10	1.53	0.43	0.48	1.25	12.10	1.53
JIVE-IR	0.09	0.29	1.21	388.61	1.11	0.10	0.30	1.24	3.53	1.13	0.23	0.34	1.14	11.23	1.06	0.40	0.45	1.18	22.19	1.09	0.40	0.45	1.18	22.19	1.09
HLIM-all	0.04	0.33	1.59	80.84	1.73	0.06	0.34	1.62	13.70	1.68	0.19	0.38	1.66	113.59	1.75	0.39	0.54	2.14	13.32	2.05	0.39	0.54	2.14	13.32	2.05
HLIM-val	0.40	0.87	4.79	370.92	3.08	0.38	0.89	4.99	302.90	3.15	0.38	0.89	4.73	41.20	3.06	0.39	0.89	4.92	315.51	3.15	0.40	0.71	2.67	1.05	1.05
HLIM-DN	0.13	0.26	1.06	367.87	1.35	0.14	0.26	1.03	10.51	1.30	0.26	0.33	1.06	30.86	1.36	0.42	0.48	1.23	15.56	1.51	0.42	0.48	1.23	15.56	1.51
HLIM-IR	0.10	0.28	1.14	8.70	0.51	0.11	0.28	1.11	2.49	0.50	0.23	0.33	1.10	5.46	0.53	0.39	0.45	1.23	3.74	0.64	0.39	0.45	1.23	3.74	0.64
HFUL-all	0.09	0.28	1.14	0.50	0.50	0.10	0.29	1.16	0.51	0.51	0.22	0.33	1.20	0.55	0.55	0.39	0.47	1.42	0.69	0.69	0.40	0.71	2.67	1.05	1.05
HFUL-val	0.41	0.70	2.61	1.04	1.04	0.40	0.71	2.63	1.04	1.04	0.39	0.71	2.65	1.05	1.05	0.40	0.71	2.67	1.05	1.05	0.40	0.71	2.67	1.05	1.05
HFUL-DN	0.16	0.25	0.90	0.52	0.52	0.17	0.25	0.90	0.52	0.52	0.28	0.32	0.92	0.57	0.57	0.43	0.46	1.04	0.67	0.67	0.43	0.46	1.04	0.67	0.67
HFUL-IR	0.13	0.28	1.06	0.49	0.49	0.14	0.27	1.04	0.50	0.50	0.25	0.33	1.02	0.52	0.52	0.39	0.44	1.11	0.62	0.62	0.39	0.44	1.11	0.62	0.62

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 18. Monte Carlo Results: Model 5 ($\tau \propto (0, 1, 1, 0, \dots, 0)$) ($R^2 = 0.01, N = 100$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$								
	$N = 100$	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	
OLS	0.49	0.49	0.22	0.50	0.50	0.49	0.49	0.22	0.50	0.50	0.49	0.49	0.22	0.50	0.50	0.50	0.50	0.50	0.23	0.50	0.50	0.50	0.50	0.50	0.50		
2SLS-all	0.48	0.48	0.51	0.52	0.52	0.48	0.48	0.50	0.52	0.52	0.50	0.50	0.51	0.53	0.53	0.52	0.53	0.53	0.55	0.57	0.57	0.57	0.57	0.57	0.57		
2SLS-val	0.37	0.86	4.65	25.95	3.04	0.35	0.82	4.67	116.09	3.05	0.35	0.84	4.67	133.28	3.08	0.35	0.85	4.73	356.53	3.02							
2SLS-DN	0.44	0.51	1.52	12.31	1.80	0.43	0.52	1.57	31.49	1.85	0.49	0.56	1.58	79.21	1.87	0.56	0.65	1.84	37.46	1.90							
2SLS-IR	0.43	0.51	1.44	13.17	1.46	0.42	0.50	1.45	28.84	1.44	0.48	0.55	1.44	18.26	1.42	0.54	0.63	1.66	5.68	1.52							
GMM-AVE	0.45	0.47	0.92	0.58	0.44	0.45	0.94	0.58	0.58	0.46	0.47	0.92	0.59	0.59	0.47	0.48	0.97	0.61	0.61								
LIML-all	0.40	0.88	4.87	36.29	3.05	0.40	0.89	4.81	87.10	3.07	0.52	0.98	5.08	1011.33	3.17	0.70	1.28	6.60	1033.87	3.59							
LIML-val	0.37	0.86	4.65	25.95	3.04	0.35	0.82	4.67	116.09	3.05	0.35	0.84	4.67	133.28	3.08	0.35	0.85	4.73	356.53	3.02							
LIML-DN	0.39	0.62	2.78	70.54	2.39	0.39	0.62	2.76	33.01	2.39	0.47	0.68	2.90	315.05	2.44	0.57	0.83	3.16	37.77	2.43							
LIML-IR	0.39	0.66	3.14	69.11	2.53	0.37	0.65	3.13	61.34	2.53	0.47	0.72	3.37	316.53	2.67	0.58	0.90	4.09	38.13	2.88							
FULL-all	0.42	0.61	2.06	0.90	0.90	0.42	0.61	2.05	0.89	0.89	0.52	0.67	2.09	0.94	0.94	0.65	0.82	2.42	1.10	1.10							
FULL-val	0.46	0.47	0.93	0.58	0.58	0.45	0.46	0.92	0.57	0.57	0.45	0.46	0.93	0.57	0.57	0.45	0.46	0.95	0.58	0.58							
FULL-DN	0.44	0.46	0.94	0.57	0.57	0.45	0.46	0.93	0.57	0.57	0.48	0.49	0.97	0.61	0.61	0.53	0.54	1.15	0.71	0.71							
FULL-IR	0.45	0.47	1.02	0.61	0.61	0.44	0.47	1.03	0.61	0.61	0.48	0.50	1.03	0.65	0.65	0.53	0.56	1.29	0.76	0.76							
JIVE-all	0.49	0.78	3.76	133.76	2.78	0.49	0.78	3.91	169.08	2.80	0.50	0.79	3.72	167.15	2.80	0.51	0.84	4.29	84.32	2.92							
JIVE-val	0.50	1.01	5.32	47.06	3.22	0.48	1.00	5.41	77.72	3.24	0.50	1.00	5.27	54.84	3.19	0.48	1.00	5.34	37.73	3.23							
JIVE-DN	0.44	0.75	3.63	51.45	2.69	0.45	0.74	3.73	42.74	2.68	0.52	0.80	3.77	45.02	2.72	0.63	0.92	3.86	30.99	2.80							
JIVE-IR	0.42	0.70	3.43	42.83	2.59	0.43	0.69	3.49	41.65	2.64	0.49	0.74	3.32	17.27	2.61	0.59	0.84	3.48	29.61	2.70							
HLIM-all	0.41	0.88	4.87	57.70	3.07	0.40	0.88	4.86	37.89	3.06	0.52	0.99	5.04	85.48	3.18	0.70	1.28	6.56	67.55	3.60							
HLIM-val	0.49	0.96	5.00	1033.45	3.19	0.48	0.98	5.20	81.15	3.27	0.47	0.99	5.26	79.99	3.23	0.50	1.01	5.34	72.26	3.23							
HLIM-DN	0.44	0.67	3.00	1030.07	2.47	0.43	0.67	3.05	63.65	2.55	0.52	0.75	3.20	65.79	2.57	0.63	0.89	3.44	60.58	2.62							
HLIM-IR	0.43	0.60	2.42	1032.28	0.84	0.42	0.59	2.47	87.90	0.84	0.50	0.66	2.36	629.87	0.87	0.61	0.83	3.11	61.17	0.96							
HFUL-all	0.43	0.58	1.89	0.84	0.84	0.42	0.57	1.89	0.84	0.84	0.51	0.64	1.94	0.89	0.89	0.64	0.77	2.22	1.02	1.02							
HFUL-val	0.49	0.77	2.72	1.11	1.11	0.48	0.76	2.73	1.11	1.11	0.47	0.78	2.75	1.11	1.11	0.50	0.81	2.85	1.14	1.14							
HFUL-DN	0.45	0.58	2.10	0.94	0.94	0.44	0.58	2.10	0.93	0.93	0.51	0.65	2.08	0.96	0.96	0.61	0.75	2.21	1.03	1.03							
HFUL-IR	0.44	0.53	1.72	0.85	0.85	0.44	0.53	1.73	0.85	0.85	0.49	0.58	1.70	0.87	0.87	0.57	0.68	1.96	0.95	0.95							

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 19. Monte Carlo Results: Model 5 ($\tau \propto (0, 1, 0, \dots, 0)$) ($R^2 = 0.1, N = 100$)

	N = 100	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$						
		Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE
OLS	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.46	0.45	0.45	0.22	0.46	0.46
2SLS-all	0.32	0.32	0.43	0.36	0.36	0.32	0.32	0.43	0.36	0.36	0.34	0.34	0.42	0.38	0.38	0.37	0.37	0.43	0.40	0.40	0.40	0.37	0.37	0.43	0.40	0.40
2SLS-val	0.02	0.34	1.65	6.55	1.59	0.03	0.35	1.70	137.79	1.70	0.02	0.34	1.63	69.59	1.59	0.03	0.35	1.61	67.22	1.57	0.03	0.35	1.61	67.22	1.57	
2SLS-DN	0.18	0.24	0.72	0.77	0.55	0.19	0.25	0.72	0.73	0.61	0.23	0.27	0.68	1.79	0.55	0.29	0.32	0.68	4.16	0.61	0.23	0.27	0.68	4.16	0.61	
2SLS-IR	0.09	0.25	0.96	4.48	0.67	0.10	0.25	0.98	1.13	0.67	0.13	0.26	1.00	2.96	0.69	0.17	0.30	1.05	1.84	0.68	0.17	0.30	1.05	1.84	0.68	
GMM-AVE	0.17	0.25	0.79	0.36	0.36	0.18	0.26	0.80	0.37	0.37	0.19	0.26	0.80	0.37	0.37	0.19	0.27	0.82	0.38	0.38	0.19	0.27	0.82	0.38	0.38	
LIML-all	0.03	0.33	1.58	654.32	1.67	0.04	0.33	1.70	20.49	1.70	0.11	0.35	1.66	11.00	1.72	0.16	0.40	1.86	58.34	1.85	0.03	0.35	1.61	67.22	1.57	
LIML-val	0.02	0.34	1.65	6.55	1.59	0.03	0.35	1.70	137.79	1.70	0.02	0.34	1.63	69.59	1.59	0.03	0.35	1.61	67.22	1.57	0.03	0.35	1.61	67.22	1.57	
LIML-DN	0.10	0.23	0.90	1.96	0.78	0.11	0.24	0.89	2.40	0.84	0.17	0.26	0.88	4.25	0.84	0.23	0.31	0.96	43.40	0.94	0.23	0.31	0.96	43.40	0.94	
LIML-IR	0.03	0.25	1.12	4.96	1.01	0.04	0.25	1.13	120.54	1.03	0.07	0.27	1.17	25.04	0.98	0.10	0.30	1.27	48.57	1.11	0.10	0.30	1.27	48.57	1.11	
FULL-all	0.07	0.29	1.19	0.53	0.53	0.08	0.29	1.24	0.54	0.54	0.14	0.31	1.24	0.55	0.55	0.19	0.35	1.35	0.60	0.60	0.19	0.35	1.35	0.60	0.60	
FULL-val	0.15	0.26	0.89	0.38	0.38	0.15	0.26	0.89	0.39	0.39	0.15	0.26	0.88	0.38	0.38	0.15	0.26	0.89	0.38	0.38	0.15	0.26	0.89	0.38	0.38	
FULL-DN	0.14	0.22	0.73	0.33	0.33	0.15	0.22	0.72	0.34	0.34	0.20	0.24	0.72	0.36	0.36	0.25	0.29	0.78	0.41	0.41	0.23	0.29	0.78	0.41	0.41	
FULL-IR	0.11	0.23	0.83	0.37	0.37	0.12	0.23	0.84	0.37	0.37	0.15	0.24	0.87	0.39	0.39	0.17	0.27	0.92	0.42	0.42	0.11	0.27	0.92	0.42	0.42	
JIVE-all	0.09	0.42	2.29	57.64	2.20	0.11	0.42	2.36	140.23	2.25	0.17	0.43	2.23	172.10	2.16	0.23	0.45	2.13	29.89	2.06	0.23	0.45	2.13	29.89	2.06	
JIVE-val	0.44	0.95	5.09	49.07	3.19	0.40	0.93	5.02	48.08	3.14	0.44	0.92	5.07	101.04	3.14	0.41	0.93	5.17	86.57	3.20	0.41	0.93	5.17	86.57	3.20	
JIVE-DN	0.12	0.29	1.30	26.18	1.58	0.13	0.29	1.30	6.64	1.54	0.19	0.30	1.23	37.43	1.46	0.27	0.36	1.22	63.18	1.55	0.16	0.31	1.19	7.58	1.11	
JIVE-IR	0.09	0.29	1.23	8.43	1.13	0.10	0.29	1.25	7.98	1.09	0.16	0.31	1.19	7.58	1.08	0.24	0.35	1.19	62.62	1.11	0.24	0.35	1.19	62.62	1.11	
HLIM-all	0.03	0.33	1.60	2312.52	1.70	0.04	0.34	1.71	390.11	1.71	0.11	0.35	1.67	7.85	1.71	0.16	0.40	1.88	23.52	1.84	0.16	0.40	1.88	23.52	1.84	
HLIM-val	0.37	0.87	4.91	70.19	3.11	0.41	0.88	4.86	93.33	3.07	0.36	0.88	4.93	135.74	3.15	0.41	0.90	4.96	482.11	3.15	0.41	0.90	4.96	482.11	3.15	
HLIM-DN	0.12	0.25	1.04	7.32	1.37	0.13	0.26	1.04	9.81	1.34	0.19	0.28	1.04	132.85	1.38	0.26	0.34	1.16	39.28	1.43	0.16	0.29	1.12	3.81	0.51	
HLIM-IR	0.09	0.27	1.13	5.46	0.50	0.10	0.28	1.14	6.90	0.51	0.16	0.29	1.12	3.81	0.51	0.23	0.34	1.14	39.02	0.55	0.23	0.34	1.14	39.02	0.55	
HFUL-all	0.08	0.28	1.14	0.50	0.50	0.09	0.29	1.17	0.51	0.51	0.15	0.31	1.18	0.52	0.52	0.20	0.34	1.28	0.57	0.57	0.20	0.34	1.28	0.57	0.57	
HFUL-val	0.38	0.71	2.64	1.04	1.04	0.41	0.71	2.67	1.05	1.05	0.37	0.70	2.64	1.04	1.04	0.41	0.72	2.65	1.05	1.05	0.41	0.72	2.65	1.05	1.05	
HFUL-DN	0.15	0.24	0.91	0.52	0.52	0.16	0.25	0.90	0.52	0.52	0.21	0.27	0.90	0.54	0.54	0.28	0.33	0.99	0.59	0.59	0.28	0.33	0.99	0.59	0.59	
HFUL-IR	0.13	0.27	1.05	0.50	0.50	0.13	0.27	1.05	0.49	0.49	0.19	0.29	1.04	0.51	0.51	0.26	0.34	1.05	0.54	0.54	0.26	0.34	1.05	0.54	0.54	

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 20. Monte Carlo Results: Model 5 ($\tau \propto (0, 1, 0, \dots, 0)$) ($R^2 = 0.01, N = 100$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$					
	$N = 100$	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE			
OLS	0.50	0.50	0.22	0.50	0.50	0.50	0.50	0.50	0.22	0.50	0.50	0.50	0.50	0.22	0.50	0.50	0.49	0.49	0.23	0.50	0.50			
2SLS-all	0.48	0.48	0.51	0.52	0.52	0.47	0.47	0.50	0.52	0.52	0.49	0.49	0.50	0.53	0.53	0.49	0.49	0.52	0.54	0.54	0.54			
2SLS-val	0.35	0.83	4.63	39.55	3.02	0.36	0.85	4.77	29.30	3.06	0.35	0.85	4.81	87.26	3.06	0.34	0.85	4.84	66.57	3.07	3.07			
2SLS-DN	0.44	0.52	1.55	17.43	1.85	0.44	0.52	1.61	20.41	1.92	0.47	0.54	1.55	30.84	1.86	0.49	0.57	1.74	15.42	1.93	1.93			
2SLS-IR	0.42	0.51	1.46	8.15	1.44	0.43	0.51	1.44	21.07	1.44	0.46	0.53	1.45	22.11	1.48	0.48	0.56	1.57	7.08	1.42	1.42			
GMM-AVE	0.45	0.46	0.91	0.58	0.58	0.44	0.45	0.93	0.58	0.58	0.45	0.46	0.93	0.59	0.59	0.45	0.46	0.96	0.59	0.59	0.59			
LIML-all	0.41	0.88	4.82	56.10	3.10	0.39	0.88	4.86	791.89	3.09	0.44	0.91	5.02	133.51	3.21	0.50	1.02	5.43	191.12	3.26	3.26			
LIML-val	0.35	0.83	4.63	39.55	3.02	0.36	0.85	4.77	29.30	3.06	0.35	0.85	4.81	87.26	3.06	0.34	0.85	4.84	66.57	3.07	3.07			
LIML-DN	0.39	0.63	2.82	19.53	2.40	0.38	0.62	2.84	24.54	2.40	0.42	0.66	2.81	35.19	2.40	0.47	0.73	3.08	25.16	2.48	2.48			
LIML-IR	0.39	0.67	3.11	27.64	2.57	0.38	0.66	3.22	25.06	2.56	0.42	0.70	3.29	38.04	2.63	0.47	0.79	3.67	138.80	2.70	2.70			
FULL-all	0.42	0.61	2.05	0.89	0.89	0.41	0.60	2.07	0.89	0.89	0.45	0.61	2.06	0.91	0.91	0.50	0.67	2.16	0.96	0.96	0.96			
FULL-val	0.46	0.47	0.91	0.57	0.57	0.45	0.46	0.92	0.58	0.58	0.45	0.46	0.92	0.58	0.58	0.45	0.46	0.93	0.57	0.57	0.57			
FULL-DN	0.45	0.46	0.92	0.58	0.58	0.44	0.45	0.95	0.58	0.58	0.46	0.47	0.95	0.60	0.60	0.48	0.49	1.03	0.64	0.64	0.64			
FULL-IR	0.45	0.47	1.02	0.62	0.62	0.44	0.46	1.02	0.62	0.62	0.46	0.48	1.04	0.64	0.64	0.49	0.51	1.14	0.68	0.68	0.68			
JIVE-all	0.50	0.79	3.76	24.36	2.74	0.49	0.79	3.70	52.20	2.80	0.48	0.77	3.77	80.77	2.78	0.50	0.79	3.72	182.75	2.70	2.70			
JIVE-val	0.49	0.99	5.18	1427.23	3.18	0.50	0.99	5.37	28.48	3.24	0.50	1.00	5.50	62.72	3.29	0.49	0.97	5.10	503.14	3.20	3.20			
JIVE-DN	0.44	0.73	3.61	116.43	2.69	0.44	0.74	3.62	105.41	2.69	0.48	0.78	3.68	41.79	2.75	0.55	0.83	3.68	45.15	2.70	2.70			
JIVE-IR	0.43	0.69	3.31	108.87	2.58	0.44	0.69	3.38	110.32	2.61	0.47	0.72	3.34	66.89	2.65	0.50	0.75	3.32	502.81	2.59	2.59			
HLIM-all	0.41	0.89	4.86	186.10	3.07	0.39	0.88	4.88	47.51	3.10	0.44	0.91	5.04	72.00	3.18	0.50	1.02	5.50	72.93	3.27	3.27			
HLIM-val	0.50	0.98	5.29	64.93	3.24	0.48	0.99	5.17	44.98	3.19	0.48	0.99	5.19	44.77	3.20	0.49	1.02	5.29	129.10	3.26	3.26			
HLIM-DN	0.43	0.67	3.16	48.38	2.61	0.43	0.68	3.17	44.21	2.53	0.48	0.70	3.15	27.94	2.54	0.54	0.80	3.37	239.81	2.62	2.62			
HLIM-IR	0.42	0.60	2.37	36.02	0.83	0.43	0.61	2.46	241.18	0.84	0.46	0.62	2.45	15.74	0.85	0.51	0.71	2.80	93.53	0.91	0.91			
HFUL-all	0.43	0.58	1.89	0.84	0.84	0.42	0.57	1.90	0.84	0.84	0.45	0.58	1.90	0.86	0.86	0.50	0.64	1.98	0.90	0.90	0.90			
HFUL-val	0.50	0.78	2.75	1.12	0.48	0.78	2.77	1.12	0.48	0.78	2.76	1.11	0.49	0.79	2.78	1.13	0.49	0.79	2.78	1.13	1.13			
HFUL-DN	0.44	0.58	2.08	0.94	0.94	0.44	0.59	2.13	0.95	0.95	0.48	0.62	2.09	0.95	0.95	0.53	0.68	2.17	0.98	0.98	0.98			
HFUL-IR	0.45	0.54	1.68	0.85	0.85	0.44	0.54	1.74	0.86	0.86	0.46	0.56	1.73	0.87	0.87	0.51	0.61	1.80	0.90	0.90	0.90			

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 21. Monte Carlo Results: Model 5 ($\tau \propto (0, 0, 0, 1, \dots, 1)$) ($R^2 = 0.1, N = 100$)

	$N = 100$	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$										
		Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE				
OLS	0.45	0.45	0.21	0.46	0.46	0.45	0.45	0.21	0.46	0.46	0.45	0.45	0.22	0.46	0.46	0.45	0.45	0.25	0.46	0.46	0.46	0.45	0.45	0.25	0.46	0.46				
2SLS-all	0.32	0.32	0.43	0.36	0.36	0.32	0.32	0.42	0.36	0.36	0.36	0.36	0.46	0.40	0.40	0.41	0.41	0.57	0.47	0.47	0.47	0.40	0.40	0.41	0.41	0.47				
2SLS-val	0.03	0.35	1.67	9.11	1.61	0.04	0.35	1.66	49.95	1.61	0.01	0.35	1.70	114.08	1.71	0.02	0.38	1.78	24.97	1.63	0.02	0.38	1.78	24.97	1.63	0.02	0.38	1.78		
2SLS-DN	0.19	0.25	0.72	5.58	0.58	0.19	0.25	0.73	4.20	0.63	0.23	0.28	0.75	7.18	0.64	0.27	0.32	0.85	1.09	0.62	0.27	0.32	0.85	1.09	0.62	0.27	0.32	0.85		
2SLS-IR	0.09	0.25	0.97	1.19	0.66	0.10	0.25	0.98	3.12	0.73	0.10	0.27	1.01	1.15	0.66	0.11	0.30	1.12	1.35	0.71	0.11	0.30	1.12	1.35	0.71	0.11	0.30	1.12		
GMM-AVE	0.17	0.25	0.79	0.36	0.36	0.19	0.26	0.79	0.37	0.37	0.18	0.26	0.83	0.38	0.38	0.19	0.28	0.94	0.42	0.42	0.42	0.19	0.28	0.94	0.42	0.42	0.19	0.28	0.94	
LIML-all	0.03	0.33	1.61	41.50	1.73	0.05	0.34	1.64	12.48	1.65	0.05	0.50	2.79	80.05	2.31	-0.19	1.90	11.03	272.64	4.47	0.02	0.38	1.78	24.97	1.63	0.02	0.38	1.78		
LIML-val	0.03	0.35	1.67	9.11	1.61	0.04	0.35	1.66	49.95	1.61	0.01	0.35	1.70	114.08	1.71	0.02	0.38	1.78	24.97	1.63	0.02	0.38	1.78	24.97	1.63	0.02	0.38	1.78		
LIML-DN	0.11	0.24	0.89	6.67	0.83	0.11	0.24	0.90	36.91	0.77	0.13	0.26	0.97	10.82	0.87	0.16	0.30	1.15	6.05	1.07	0.16	0.30	1.15	6.05	1.07	0.16	0.30	1.15		
LIML-IR	0.03	0.26	1.11	6.18	1.04	0.04	0.26	1.14	14.26	1.01	0.04	0.27	1.20	8.59	1.17	0.05	0.33	1.49	21.09	1.42	0.05	0.33	1.49	21.09	1.42	0.05	0.33	1.49		
FULL-all	0.07	0.29	1.20	0.53	0.53	0.09	0.29	1.22	0.54	0.54	0.10	0.41	1.75	0.72	0.72	-0.02	1.05	3.64	1.39	1.39	0.15	0.27	0.99	0.42	0.42	0.15	0.27	0.99		
FULL-val	0.15	0.26	0.89	0.38	0.38	0.16	0.26	0.89	0.39	0.39	0.14	0.26	0.90	0.39	0.39	0.19	0.28	0.94	0.46	0.46	0.15	0.27	0.99	0.42	0.42	0.15	0.27	0.99		
FULL-DN	0.15	0.22	0.73	0.33	0.33	0.15	0.22	0.73	0.34	0.34	0.17	0.24	0.78	0.37	0.37	0.19	0.28	0.94	0.46	0.46	0.15	0.27	0.99	0.42	0.42	0.15	0.27	0.99		
FULL-IR	0.12	0.23	0.83	0.37	0.37	0.12	0.23	0.84	0.37	0.37	0.12	0.24	0.90	0.41	0.41	0.13	0.28	1.12	0.54	0.54	0.15	0.27	0.99	0.42	0.42	0.15	0.27	0.99		
JIVE-all	0.10	0.42	2.44	293.53	2.26	0.11	0.41	2.32	31.79	2.18	0.21	0.46	2.28	27.43	2.09	0.34	0.57	2.53	32.09	2.22	0.44	1.01	5.64	121.49	3.33	0.44	1.01	5.64	121.49	3.33
JIVE-val	0.43	0.93	5.30	154.25	3.23	0.42	0.93	5.15	42.28	3.20	0.43	0.94	5.07	55.90	3.18	0.44	1.01	5.64	121.49	3.33	0.44	1.01	5.64	121.49	3.33	0.44	1.01	5.64	121.49	3.33
JIVE-DN	0.13	0.29	1.29	45.39	1.59	0.13	0.29	1.28	28.45	1.56	0.16	0.32	1.35	10.64	1.64	0.19	0.36	1.49	73.02	1.65	0.19	0.36	1.49	73.02	1.65	0.19	0.36	1.49	73.02	1.65
JIVE-IR	0.09	0.29	1.21	42.42	1.13	0.10	0.30	1.25	12.12	1.11	0.14	0.33	1.29	23.85	1.19	0.21	0.39	1.45	34.03	1.22	0.21	0.39	1.45	34.03	1.22	0.21	0.39	1.45	34.03	1.22
HLIM-all	0.03	0.33	1.62	11.64	1.70	0.05	0.34	1.64	28.64	1.64	0.05	0.50	2.77	23.50	2.34	-0.19	1.90	10.99	236.83	4.48	0.02	0.38	1.78	24.97	1.63	0.02	0.38	1.78	24.97	1.63
HLIM-val	0.39	0.87	5.06	59.45	3.15	0.39	0.88	4.83	262.11	3.08	0.38	0.90	5.03	127.00	3.13	0.39	0.97	5.46	57.83	3.29	0.39	0.97	5.46	57.83	3.29	0.39	0.97	5.46	57.83	3.29
HLIM-DN	0.13	0.26	1.02	7.29	1.36	0.14	0.26	1.04	16.97	1.29	0.16	0.29	1.12	11.15	1.35	0.19	0.33	1.37	9.48	1.56	0.19	0.33	1.37	9.48	1.56	0.19	0.33	1.37	9.48	1.56
HLIM-IR	0.09	0.28	1.14	3.24	0.51	0.10	0.28	1.14	19.02	0.51	0.13	0.31	1.24	5.84	0.54	0.15	0.41	1.85	11.87	0.70	0.15	0.41	1.85	11.87	0.70	0.15	0.41	1.85	11.87	0.70
HFUL-all	0.08	0.29	1.15	0.50	0.50	0.10	0.29	1.17	0.51	0.51	0.11	0.40	1.64	0.67	0.67	0.02	0.94	3.26	1.25	1.25	0.02	0.94	3.26	1.25	1.25	0.02	0.94	3.26	1.25	1.25
HFUL-val	0.40	0.69	2.64	1.04	1.04	0.39	0.71	2.65	1.04	1.04	0.39	0.72	2.71	1.07	1.07	0.39	0.79	2.96	1.14	1.14	0.39	0.79	2.96	1.14	1.14	0.39	0.79	2.96	1.14	1.14
HFUL-DN	0.16	0.25	0.89	0.52	0.52	0.17	0.25	0.91	0.52	0.52	0.19	0.27	0.96	0.55	0.55	0.21	0.31	1.17	0.63	0.63	0.21	0.31	1.17	0.63	0.63	0.21	0.31	1.17	0.63	0.63
HFUL-IR	0.12	0.27	1.07	0.50	0.50	0.13	0.27	1.06	0.50	0.50	0.15	0.31	1.14	0.54	0.54	0.18	0.38	1.61	0.70	0.70	0.18	0.38	1.61	0.70	0.70	0.18	0.38	1.61	0.70	0.70

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 22. Monte Carlo Results: Model 5 ($\tau \propto (0, 0, 0, 1, \dots, 1)$) ($R^2 = 0.01, N = 100$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$						
$N = 100$	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE
OLS	0.50	0.50	0.22	0.50	0.50	0.49	0.49	0.22	0.50	0.49	0.49	0.22	0.50	0.50	0.49	0.49	0.25	0.50	0.50	0.49	0.49	0.25	0.50	0.50	
2SLS-all	0.48	0.48	0.51	0.52	0.52	0.48	0.48	0.50	0.52	0.49	0.49	0.55	0.54	0.54	0.51	0.51	0.74	0.59	0.59	0.51	0.51	0.74	0.59	0.59	
2SLS-val	0.37	0.86	4.75	185.70	3.01	0.35	0.82	4.66	52.86	3.09	0.35	0.87	4.91	30307.67	3.09	0.34	0.89	5.17	119.99	3.22	0.34	0.89	5.17	119.99	3.22
2SLS-DN	0.43	0.52	1.55	30.47	1.78	0.43	0.51	1.54	39.14	1.88	0.45	0.54	1.67	12.98	1.87	0.46	0.57	1.71	108.49	1.98	0.46	0.57	1.71	108.49	1.98
2SLS-IR	0.43	0.51	1.45	83.95	1.42	0.43	0.50	1.44	10.31	1.45	0.44	0.52	1.51	9.19	1.44	0.45	0.56	1.63	38.28	1.46	0.45	0.56	1.63	38.28	1.46
GMM-AVE	0.45	0.46	0.93	0.58	0.58	0.44	0.45	0.91	0.57	0.57	0.45	0.47	0.97	0.59	0.59	0.46	0.48	1.13	0.64	0.64	0.46	0.48	1.13	0.64	0.64
LIML-all	0.39	0.89	4.72	140.99	3.05	0.40	0.86	4.92	36.31	3.15	0.47	1.39	8.02	2157.61	3.90	1.32	4.06	22.92	274.22	6.09	1.32	4.06	22.92	274.22	6.09
LIML-val	0.37	0.86	4.75	185.70	3.01	0.35	0.82	4.66	52.86	3.09	0.35	0.87	4.91	30123.77	3.09	0.34	0.89	5.17	119.99	3.22	0.34	0.89	5.17	119.99	3.22
LIML-DN	0.39	0.62	2.84	28.74	2.33	0.38	0.61	2.76	44.64	2.44	0.39	0.65	2.91	46.25	2.42	0.41	0.70	3.11	110.31	2.52	0.41	0.70	3.11	110.31	2.52
LIML-IR	0.38	0.66	3.20	100.63	2.53	0.38	0.65	3.22	46.81	2.63	0.39	0.71	3.72	2157.20	2.79	0.42	0.81	5.47	64.19	3.42	0.42	0.81	5.47	64.19	3.42
FULL-all	0.41	0.60	2.08	0.89	0.89	0.42	0.60	2.05	0.90	0.90	0.47	0.80	2.61	1.10	1.10	0.79	1.59	4.35	1.81	1.81	0.79	1.59	4.35	1.81	1.81
FULL-val	0.46	0.47	0.92	0.58	0.58	0.45	0.46	0.90	0.57	0.57	0.45	0.46	0.94	0.58	0.58	0.45	0.46	1.02	0.60	0.60	0.45	0.46	1.02	0.60	0.60
FULL-DN	0.44	0.46	0.95	0.58	0.58	0.44	0.45	0.91	0.57	0.57	0.45	0.46	0.97	0.59	0.59	0.45	0.47	1.13	0.65	0.65	0.45	0.47	1.13	0.65	0.65
FULL-IR	0.44	0.47	1.03	0.62	0.62	0.44	0.46	1.01	0.61	0.61	0.45	0.48	1.09	0.66	0.66	0.45	0.51	1.41	0.82	0.82	0.45	0.51	1.41	0.82	0.82
JIVE-all	0.47	0.78	3.88	142.87	2.76	0.48	0.76	3.65	100.27	2.70	0.48	0.82	4.16	24.55	2.86	0.48	0.98	5.28	93.43	3.21	0.48	0.98	5.28	93.43	3.21
JIVE-val	0.51	1.02	5.38	148.25	3.23	0.50	0.98	5.27	47.94	3.27	0.50	1.01	5.41	109.90	3.25	0.50	1.08	5.96	71.61	3.40	0.50	1.08	5.96	71.61	3.40
JIVE-DN	0.44	0.75	3.74	118.18	2.74	0.44	0.73	3.54	40.62	2.73	0.45	0.77	3.76	108.37	2.75	0.46	0.81	4.04	59.32	2.82	0.46	0.81	4.04	59.32	2.82
JIVE-IR	0.43	0.70	3.37	38.98	2.62	0.44	0.70	3.31	54.20	2.64	0.44	0.72	3.38	107.60	2.62	0.45	0.78	3.70	85.66	2.72	0.45	0.78	3.70	85.66	2.72
HLIM-all	0.38	0.88	4.86	285.39	3.05	0.40	0.87	4.96	40.90	3.12	0.47	1.39	7.92	234.99	3.90	1.32	4.05	22.39	215.84	6.07	1.32	4.05	22.39	215.84	6.07
HLIM-val	0.48	0.95	5.00	565.52	3.17	0.48	0.99	5.41	158.96	3.26	0.49	1.00	5.22	280.79	3.20	0.50	1.08	5.89	142.62	3.43	0.50	1.08	5.89	142.62	3.43
HLIM-DN	0.44	0.66	2.97	47.10	2.48	0.43	0.67	3.21	131.80	2.60	0.44	0.70	3.19	276.09	2.54	0.48	0.78	3.57	111.54	2.74	0.48	0.78	3.57	111.54	2.74
HLIM-IR	0.41	0.60	2.42	96.11	0.83	0.43	0.60	2.48	64.56	0.84	0.44	0.64	2.60	268.32	0.87	0.48	0.81	3.67	203.77	0.96	0.48	0.81	3.67	203.77	0.96
HFUL-all	0.41	0.57	1.90	0.84	0.84	0.42	0.57	1.88	0.84	0.84	0.48	0.74	2.37	1.02	1.02	0.76	1.41	3.92	1.64	1.64	0.76	1.41	3.92	1.64	1.64
HFUL-val	0.48	0.76	2.70	1.10	1.10	0.48	0.77	2.75	1.11	1.11	0.49	0.80	2.82	1.14	1.14	0.50	0.86	3.07	1.22	1.22	0.50	0.86	3.07	1.22	1.22
HFUL-DN	0.44	0.58	2.02	0.92	0.92	0.44	0.59	2.12	0.94	0.94	0.45	0.60	2.17	0.96	0.96	0.48	0.67	2.34	1.04	1.04	0.48	0.67	2.34	1.04	1.04
HFUL-IR	0.43	0.53	1.67	0.84	0.84	0.44	0.54	1.75	0.86	0.86	0.45	0.56	1.78	0.88	0.88	0.48	0.65	2.28	1.02	1.02	0.48	0.65	2.28	1.02	1.02

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\hat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\hat{L}_{IR}(K)$; (v) GMM-AVE - averaging GMM estimator in Cheng et al. (2019), which combines a GMM with valid instruments and a GMM using all instruments.

Table 23. Monte Carlo Results: Hausman et al. (2012) setup ($N = 800, \mu^2 = 8$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$					
	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE				
OLS	0.30	0.30	0.11	0.30	0.30	0.30	0.30	0.11	0.30	0.30	0.30	0.11	0.30	0.30	0.30	0.30	0.12	0.30	0.30	0.30	0.30	0.30	0.30	
2SLS-all	0.24	0.24	0.55	0.29	0.29	0.23	0.23	0.55	0.28	0.24	0.24	0.58	0.29	0.29	0.24	0.25	0.74	0.33	0.33	0.33	0.33	0.33	0.33	
2SLS-val	-0.00	0.24	1.36	0.70	0.70	0.00	0.24	1.48	1.06	0.74	-0.02	0.24	1.53	0.60	0.60	0.01	0.25	1.48	3.01	0.74	0.74	0.74	0.74	
2SLS-DN	0.15	0.24	1.04	0.50	0.50	0.15	0.24	1.14	0.76	0.49	0.14	0.24	1.15	0.44	0.44	0.12	0.25	1.27	0.44	0.44	0.44	0.44	0.44	
2SLS-IR	0.05	0.21	1.10	0.38	0.38	0.05	0.22	1.14	0.38	0.38	0.03	0.21	1.15	0.38	0.38	0.02	0.23	1.32	0.41	0.41	0.41	0.41	0.41	
LIML-all	0.07	0.47	4.68	12.68	2.08	0.04	0.47	4.85	10.62	2.25	-0.02	0.62	7.03	97.93	2.68	-1.05	3.47	43.14	231.68	5.77				
LIML-val	-0.00	0.24	1.36	0.70	0.70	0.00	0.24	1.48	1.06	0.74	-0.02	0.24	1.53	0.60	0.60	0.01	0.25	1.48	3.01	0.74	0.74	0.74	0.74	
LIML-DN	0.03	0.24	1.32	0.64	0.64	0.04	0.25	1.35	0.90	0.69	0.02	0.24	1.53	0.86	0.63	0.04	0.25	1.56	0.80	0.79	0.79	0.79	0.79	
LIML-IR	0.01	0.25	1.38	0.97	0.97	0.02	0.25	1.55	1.61	1.18	-0.01	0.24	1.63	4.26	1.14	0.03	0.29	3.49	31.93	1.85				
FULL-all	0.10	0.39	2.60	0.74	0.74	0.07	0.41	2.31	0.69	0.69	0.03	0.49	2.96	0.85	0.85	-0.63	1.71	5.66	1.91	1.91				
FULL-val	0.04	0.21	1.00	0.32	0.32	0.04	0.20	1.08	0.33	0.33	0.02	0.21	1.08	0.33	0.33	0.05	0.21	1.07	0.33	0.33				
FULL-DN	0.06	0.21	1.00	0.32	0.32	0.07	0.22	1.08	0.34	0.34	0.06	0.22	1.09	0.35	0.35	0.07	0.23	1.24	0.48	0.48				
FULL-IR	0.05	0.22	1.09	0.37	0.37	0.05	0.22	1.17	0.36	0.36	0.04	0.22	1.19	0.41	0.41	0.06	0.26	2.32	0.72	0.72				
JIVE-all	0.16	0.60	7.91	5.12	2.61	0.14	0.57	6.94	8.00	2.58	0.14	0.63	7.29	9.29	2.77	0.11	0.78	9.50	13.51	3.02				
JIVE-val	-0.02	0.35	2.94	6.42	1.70	-0.02	0.36	3.66	9.66	1.80	-0.06	0.36	3.54	10.23	1.85	-0.03	0.36	3.40	4.64	1.75				
JIVE-DN	0.01	0.33	2.44	4.94	1.51	0.03	0.34	2.96	9.61	1.59	-0.00	0.33	2.75	8.49	1.59	-0.01	0.36	2.83	4.51	1.56				
JIVE-IR	-0.01	0.33	2.37	7.30	1.42	-0.02	0.34	2.53	9.40	1.56	-0.05	0.35	2.85	5.78	1.52	-0.05	0.40	3.29	4.80	1.68				
HLIM-all	0.08	0.48	4.42	20.95	2.05	0.04	0.47	4.84	28.80	2.22	-0.02	0.65	7.73	40.46	2.70	-1.03	3.52	36.71	164.20	5.73				
HLIM-val	-0.01	0.24	1.40	0.70	0.70	0.00	0.24	1.48	1.28	0.75	-0.02	0.24	1.53	0.61	0.61	0.01	0.25	1.47	8.21	0.68				
HLIM-DN	0.03	0.23	1.30	0.86	0.66	0.04	0.25	1.35	4.15	0.79	0.03	0.24	1.54	4.04	0.74	0.04	0.25	1.54	0.84	0.69				
HLIM-IR	0.02	0.24	1.41	1.05	0.46	0.01	0.24	1.42	4.36	0.46	0.00	0.25	1.76	1.62	0.51	0.01	0.32	3.46	31.30	0.64				
HFUL-all	0.11	0.40	2.52	0.72	0.72	0.07	0.40	2.35	0.70	0.70	0.03	0.49	2.80	0.84	0.84	-0.56	1.63	5.41	1.83	1.83				
HFUL-val	0.04	0.21	1.00	0.32	0.32	0.04	0.21	1.08	0.33	0.33	0.02	0.21	1.07	0.33	0.33	0.05	0.21	1.06	0.33	0.33				
HFUL-DN	0.06	0.21	1.00	0.32	0.32	0.07	0.21	1.05	0.33	0.33	0.06	0.21	1.08	0.33	0.33	0.07	0.22	1.16	0.37	0.37				
HFUL-IR	0.05	0.21	1.08	0.35	0.35	0.04	0.20	1.12	0.34	0.34	0.03	0.22	1.28	0.39	0.39	0.05	0.26	1.91	0.66	0.66				

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\widehat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\widehat{L}_{IR}(K)$.

Table 24. Monte Carlo Results: Hausman et al. (2012) setup ($N = 800, \mu^2 = 32$)

	$\gamma = \infty$						$\gamma = 1$						$\gamma = 1/2$						$\gamma = 1/3$					
	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE	Bias	MAD	IDR	MSE	TMSE				
OLS	0.29	0.29	0.10	0.29	0.29	0.29	0.29	0.11	0.29	0.29	0.29	0.29	0.11	0.29	0.29	0.29	0.29	0.11	0.29	0.29	0.29	0.29	0.29	
2SLS-all	0.14	0.14	0.42	0.18	0.18	0.14	0.14	0.41	0.19	0.19	0.15	0.16	0.43	0.20	0.20	0.15	0.16	0.52	0.22	0.22	0.22	0.22	0.22	
2SLS-val	-0.01	0.12	0.62	0.19	0.19	-0.00	0.12	0.61	0.19	0.19	0.00	0.13	0.61	0.19	0.19	0.01	0.13	0.62	0.20	0.20	0.20	0.20	0.20	
2SLS-DN	0.04	0.13	0.61	0.19	0.19	0.04	0.13	0.59	0.19	0.19	0.04	0.14	0.62	0.19	0.19	0.06	0.14	0.68	0.21	0.21	0.21	0.21	0.21	
2SLS-IR	-0.02	0.12	0.62	0.19	0.19	-0.01	0.12	0.61	0.19	0.19	-0.00	0.13	0.63	0.19	0.19	-0.01	0.13	0.69	0.20	0.20	0.20	0.20	0.20	
LIML-all	-0.01	0.17	0.96	0.58	0.56	-0.00	0.17	0.87	0.33	0.33	-0.02	0.20	1.12	0.52	0.52	-0.52	0.79	8.54	36.96	2.98				
LIML-val	-0.01	0.12	0.62	0.19	0.19	-0.00	0.12	0.61	0.19	0.19	0.00	0.13	0.61	0.19	0.19	0.01	0.13	0.62	0.20	0.20				
LIML-DN	0.00	0.12	0.60	0.19	0.19	0.01	0.12	0.58	0.19	0.19	0.01	0.13	0.62	0.19	0.19	0.01	0.13	0.68	0.29	0.29				
LIML-IR	-0.02	0.12	0.64	0.21	0.21	-0.01	0.12	0.62	0.19	0.19	-0.01	0.13	0.63	0.20	0.20	-0.01	0.14	1.01	0.43	0.43				
FULL-all	0.00	0.16	0.91	0.28	0.28	0.01	0.16	0.83	0.28	0.28	-0.01	0.19	1.03	0.34	0.34	-0.44	0.68	3.85	1.13	1.13				
FULL-val	-0.00	0.12	0.59	0.18	0.18	0.01	0.12	0.59	0.18	0.18	0.01	0.13	0.58	0.18	0.18	0.02	0.12	0.60	0.18	0.18				
FULL-DN	0.01	0.11	0.57	0.18	0.18	0.02	0.12	0.56	0.18	0.18	0.02	0.13	0.59	0.18	0.18	0.02	0.12	0.65	0.22	0.22				
FULL-IR	-0.01	0.12	0.61	0.19	0.19	-0.00	0.12	0.59	0.18	0.18	0.00	0.13	0.60	0.18	0.18	0.00	0.14	0.89	0.29	0.29				
JIVE-all	-0.03	0.19	1.25	1.16	0.83	-0.03	0.19	1.17	1.38	0.78	0.01	0.20	1.12	0.50	0.50	0.00	0.22	1.41	1.91	0.96				
JIVE-val	-0.03	0.13	0.69	0.23	0.23	-0.02	0.13	0.67	0.22	0.22	-0.02	0.14	0.67	0.21	0.21	-0.01	0.14	0.69	0.39	0.38				
JIVE-DN	-0.01	0.13	0.68	0.22	0.22	-0.01	0.13	0.65	0.21	0.21	-0.01	0.13	0.68	0.21	0.21	-0.00	0.14	0.72	0.22	0.22				
JIVE-IR	-0.04	0.13	0.69	0.23	0.23	-0.04	0.14	0.69	0.22	0.22	-0.03	0.14	0.70	0.22	0.22	-0.02	0.14	0.75	0.23	0.23				
HLIM-all	-0.01	0.16	0.98	4.20	0.67	-0.01	0.17	0.93	1.02	0.50	-0.01	0.20	1.16	3.45	0.74	-0.50	0.79	9.41	10.33	2.95				
HLIM-val	-0.01	0.12	0.62	0.19	0.19	-0.00	0.12	0.61	0.19	0.19	0.00	0.13	0.61	0.19	0.19	0.01	0.13	0.62	0.20	0.20				
HLIM-DN	0.01	0.12	0.60	0.19	0.19	0.01	0.12	0.58	0.19	0.19	0.01	0.13	0.63	0.19	0.19	0.01	0.13	0.67	0.27	0.27				
HLIM-IR	-0.02	0.12	0.63	0.20	0.20	-0.01	0.12	0.62	0.19	0.19	-0.01	0.13	0.63	0.19	0.19	-0.00	0.13	0.85	5.90	0.30				
HFUL-all	0.01	0.16	0.88	0.29	0.29	0.00	0.17	0.86	0.29	0.29	0.00	0.19	1.06	0.34	0.34	-0.42	0.69	3.67	1.09	1.09				
HFUL-val	-0.00	0.11	0.59	0.18	0.18	0.01	0.12	0.59	0.18	0.18	0.01	0.13	0.58	0.18	0.18	0.02	0.12	0.59	0.18	0.18				
HFUL-DN	0.02	0.11	0.57	0.18	0.18	0.02	0.12	0.56	0.18	0.18	0.02	0.13	0.60	0.18	0.18	0.02	0.12	0.64	0.22	0.22				
HFUL-IR	-0.01	0.11	0.60	0.19	0.19	0.00	0.12	0.59	0.18	0.18	0.01	0.12	0.61	0.18	0.18	0.01	0.13	0.91	0.32	0.32				

Note: (i) all - IV estimators using all instruments; (ii) val - using known valid instruments; (iii) DN - using instruments based on Donald and Newey (2001)'s criterion $\widehat{L}_{DN}(K)$; (iv) IR - based on the invalidity-robust criterion $\widehat{L}_{IR}(K)$.