Supplement to
"Adaptation for nonparametric estimators of locally stationary processes"

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Notation supplementary material

During the proofs, $c' > 0$ is a generic constant which may depend on

- M, χ, C (from $g \in \mathcal{H}(M, \chi, C)$),
- D, α, ρ (from Assumption 2.1 and Assumption 4.3),
- $K \in \mathcal{K}$ (the kernel). Especially, let L_K denote the Lipschitz constant of K during the proofs.

Its value may change from line to line. Some technical results are postponed to Section D. Furthermore, for some stationary sequence $\tilde{Z}_t = J(\mathcal{F}_t)$, where J is a measurable function and $\mathcal{F}_t = (\zeta_t, \zeta_{t-1}, ...)$ with i.i.d. ζ_t, we define the projection operator

$$P_k \tilde{Z}_t := \mathbb{E}[\tilde{Z}_t | \mathcal{F}_k] - \mathbb{E}[\tilde{Z}_t | \mathcal{F}_{k-1}].$$

In Wu (2005) it was shown that for any $q \geq 1$, $t \in \mathbb{Z}$,

$$\|P_{t-k} \tilde{Z}_t\|_q \leq \delta_q^{\tilde{Z}}(k),$$

where $\delta_q^{\tilde{Z}}$ is the functional dependence measure defined in (6). This relation is used in the following without further reference. Recall the notation $Y_{t,n} = (X_{t,n}, X_{t-1,n}, ..., X_{1,n}, 0, 0, ...)$ and $\tilde{Y}_t(u) = (\tilde{X}_s(u) : s \leq t)$, where $\tilde{X}_t(u)$ is the stationary approximation from Assumption 2.1.

A Proof of Theorem 2.8

We define

$$\tilde{G}_h(u) := \frac{1}{n} \sum_{t=1}^{n} K_h(t/n - u) \cdot g(\tilde{Y}_t(t/n)).$$

Proof of Theorem 2.8. We first analyze the bias. By Lemma D.6(ii), we have in each component that

$$\sup_{u\in[0,1]} \big\|\hat{G}_h(u) - \tilde{G}_h(u)\big\|_1 \le c'(nh)^{-1}. \quad (1)$$

Since K is Lipschitz-continuous, we have for $u \in [\frac{h}{2}, 1-\frac{h}{2}]$ that

$$\begin{aligned}\mathbb{E}\tilde{G}_h(u) &= \frac{1}{n}\sum_{t=1}^{n} K_h(t/n-u)\mathbb{E}g(\tilde{Y}_t(t/n)) = \frac{1}{n}\sum_{t=1}^{n} K_h(t/n-u)G(t/n)\\ &= \int_0^1 K_h(v-u)G(v)dv + O((nh)^{-1}) = \int K(x)G(u+xh)dx + O((nh)^{-1}).\end{aligned}$$

Since G is twice continuously differentiable and K is symmetric, we conclude with standard arguments that

$$\int K(x)G(u+xh)dx = G(u) + \frac{h^2}{2}\int K(x)^2 dx \partial_u^2 G(u) + o(h^2).$$

We obtain that component-wise,

$$\sup_{u\in[\frac{h}{2},1-\frac{h}{2}]} \big|\mathbb{E}\hat{G}_h(u) - G(u) - \frac{h^2}{2}\mu_K\cdot\partial_u^2 G(u)\big| = o(h^2) + O((nh)^{-1}). \quad (2)$$

We now analyze the mean squared error. We have

$$\mathbb{E}|\hat{G}_h(u) - G(u)|_A^2 = \mathbb{E}|\hat{G}_h(u) - \mathbb{E}\hat{G}_h(u)|_A^2 + |\mathbb{E}\hat{G}_h(u) - G(u)|_A^2. \quad (3)$$

By Lemma D.8, we have uniformly in $u \in [\frac{h}{2}, 1-\frac{h}{2}]$,

$$\begin{aligned}\mathbb{E}|\hat{G}_h(u) - \mathbb{E}\hat{G}_h(u)|_A^2 &= \sum_{j,k=1}^{d} A_{jk}\cdot \mathrm{Cov}(\hat{G}_h(u)_j, \hat{G}_h(u)_k)\\ &= \frac{\sigma_K^2}{nh}\sum_{j,k=1}^{d}\sum_{l\in\mathbb{Z}} \mathrm{Cov}(g(\tilde{Y}_0(u))_j, g(\tilde{Y}_l(u))_k) + o((nh)^{-1})\\ &= \sum_{j,k=1}^{d} A_{jk}\cdot\Sigma_g(u)_{jk} + o((nh)^{-1})\\ &= \mathrm{tr}(A\cdot\Sigma_g(u)) + o((nh)^{-1}). \quad (4)\end{aligned}$$

Insertion of (2) and (4) into (3) yields that uniformly in $u \in [\frac{h}{2}, 1-\frac{h}{2}]$,

$$\mathbb{E}|\hat{G}_h(u) - G(u)|_A^2 = \frac{\sigma_K^2}{nh}\cdot\mathrm{tr}(A\cdot\Sigma_g(u)) + \frac{h^4}{4}\mu_K^2\cdot|\partial_u^2 G(u)|_A^2 + o(h^4 + (nh)^{-1}).$$

This proves (7).

Since F is continuously differentiable, we have

$$\begin{aligned}&F(\hat{G}_h(u)) - F(G(u))\\ &= \partial_G F(G(u))\cdot(\hat{G}_h(u) - G(u) + \big\{\partial_G F\big(\bar{G}(u)\big)\big) - \partial_G F\big(G(u)\big)\big\}\cdot(\hat{G}_h(u) - G(u)), \quad (5)\end{aligned}$$

where $\bar{G}(u) \in \mathbb{R}^d$ is such that $|\bar{G}(u) - G(u)|_2 \le |\hat{G}_h(u) - G(u)|_2$. By (7),

$$|\hat{G}_h(u) - G(u)|_2 = o_p(1).$$

Insertion into (5) shows that

$$F(\hat{G}_h(u)) - F(G(u)) = \{\partial_G F(G(u)) + o_p(1)\} \cdot (\hat{G}_h(u) - G(u)),$$

thus

$$\begin{aligned} |F(\hat{G}_h(u)) - F(G(u))|_2^2 &= |\hat{G}_h(u) - G(u)|^2_{A_F(G(u))} + o_p(\hat{G}_h(u) - G(u)) \\ &= |\hat{G}_h(u) - G(u)|^2_{A_F(G(u))} + o_p(h^4 + (nh)^{-1}). \end{aligned}$$

This shows (8). □

B Proof of Theorem 3.1 and Corollary 3.3

In the following, we abbreviate $\mathbb{E}_0(Z) = Z - \mathbb{E}Z$ for real-valued random variables Z.

To prove Theorem 3.1, we use the proof techniques that were used in Richter and Dahlhaus (2019) to prove their Theorem 3.6.

Proof of Theorem 3.1. Recall that

$$d_{ISE}(h) = \int_0^1 |F(\hat{G}_h(u)) - F(G(u))|_2^2 w(u)du.$$

We now define

$$\begin{aligned} \tilde{d}_{ISE}(h) &:= \int_0^1 \big|G(u) - \hat{G}_h(u)|^2_{A_F(G(u))} w(u)du, \\ \tilde{d}_{MISE}(h) &:= \mathbb{E}[\tilde{d}_{ISE}(h)], \\ \tilde{d}^*_{MISE}(h) &:= \frac{\sigma_K^2}{nh}\int_0^1 \mathrm{tr}(\Sigma_g(u)A_F(G(u)))w(u)du + \int_0^1 |\mathbb{E}\hat{G}_h(u) - G(u)|^2_{A_F(G(u))} w(u)du, \\ \tilde{d}_A(h) &:= \frac{1}{n}\sum_{t=1}^n \big|G(t/n) - \hat{G}_h(\frac{t}{n})|^2_{A_F(G(t/n))} w(t/n), \\ \tilde{d}_{A,-}(h) &:= \frac{1}{n}\sum_{t=1}^n \big|G(t/n) - \hat{G}_h^-(\frac{t}{n})|^2_{A_F(\hat{G}_h(\frac{t}{n}))} w(t/n). \end{aligned}$$

We will show that

$$\sup_{h\in H_n} \frac{|\tilde{d}_{MISE}(h) - \tilde{d}^*_{MISE}(h)|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s., \quad (6)$$

$$\sup_{h\in H_n} \frac{|\tilde{d}_{ISE}(h) - \tilde{d}_{MISE}(h)|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s., \quad (7)$$

$$\sup_{h\in H_n} \frac{|\tilde{d}_A(h) - \tilde{d}_{ISE}(h)|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s., \quad (8)$$

$$\sup_{h\in H_n} \frac{|\tilde{d}_A(h) - \tilde{d}_{A,-}(h)|}{\tilde{d}_A(h)} \to 0 \quad a.s., \quad (9)$$

$$\sup_{h\in H_n} \frac{|[d^{(n)}_{ISE,F}(h) - \frac{1}{n}\sum_{t=1}^n |g(Y_{t,n}) - G(t/n)|^2_{A_F(G(t/n))}] - \tilde{d}_{A,-}(h)|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s. \quad (10)$$

and furthermore,

$$\sup_{h\in H_n} \frac{|d_{ISE}(h) - \tilde{d}_{ISE}(h)|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s. \tag{11}$$

From (6) - (11) it follows by elementary calculations (replacement of d_{ISE} by $\tilde{d}_{ISE}$, then by $\tilde{d}_A$, then by $\tilde{d}_{A,-}$ and then by $d^{(n)}_{ISE,F}(h) - \frac{1}{n}\sum_{t=1}^n |g(Y_{t,n}) - G(t/n)|^2_{A_F(G(t/n))}$) that

$$\sup_{h\in H_n} \frac{|[d^{(n)}_{ISE,F}(h) - \frac{1}{n}\sum_{t=1}^n |g(Y_{t,n}) - G(t/n)|^2_{A_F(G(t/n))}] - d_{ISE}(h)|}{d_{ISE}(h)} \to 0 \quad a.s.$$

Insertion of $h' \in \operatorname{argmin}_{h\in H_n} d_{ISE}(h)$ and $\hat{h} \in \operatorname{argmin}_{h\in H_n} d^{(n)}_{ISE,F}(h)$ yields with the abbreviation $J_n := \frac{1}{n}\sum_{t=1}^n |g(Y_{t,n}) - G(t/n)|^2_{A_F(G(t/n))}$ that

$$\begin{aligned} 0 &\le \frac{d_{ISE}(\hat{h}) - d_{ISE}(h')}{d_{ISE}(\hat{h})} \\ &\le \frac{d_{ISE}(\hat{h}) - [d^{(n)}_{ISE,F}(\hat{h}) - J_n] + [d^{(n)}_{ISE,F}(h') - J_n] - d_{ISE}(h')}{d_{ISE}(\hat{h})} \\ &\le \frac{|d_{ISE}(\hat{h}) - [d^{(n)}_{ISE,F}(\hat{h}) - J_n]|}{d_{ISE}(\hat{h})} + \frac{|[d^{(n)}_{ISE,F}(h') - J_n] - d_{ISE}(h')|}{d_{ISE}(h')} \to 0 \quad a.s., \end{aligned}$$

thus

$$\frac{d_{ISE}(\hat{h})}{\inf_{h\in H_n} d_{ISE}(h)} = \frac{d_{ISE}(\hat{h})}{d_{ISE}(h')} \to 1 \quad a.s. \tag{12}$$

It therefore is enough to show (6) - (11).

Discussion of (6): By the uniform convergence in Theorem 2.8, namely

$$\begin{aligned} \sup_{u\in[\frac{h}{2},1-\frac{h}{2}]} \Big|\mathbb{E}|\hat{G}_h(u) - G(u)|^2_{A_F(G(u))} - \frac{\sigma_K^2}{nh}\operatorname{tr}(\Sigma_g(u)A_F(G(u))) \\ -|\mathbb{E}\hat{G}_h(u) - G(u)|^2_{A_F(G(u))}\Big| = o((nh)^{-1}), \end{aligned}$$

and the fact that $\tilde{d}^*_{MISE}(h) \ge \frac{\sigma_K^2}{nh}\int_0^1 \operatorname{tr}(\Sigma_g(u)A_F(G(u)))w(u)du$ is larger than a constant times $(nh)^{-1}$ by assumption (18), we obtain (6).

Discussion of (7): The proof follows similar as the proof of Lemma 3.10 in Richter and Dahlhaus (2019). We give an overview. We have

$$\begin{aligned} &|\tilde{d}_{ISE}(h) - \tilde{d}_{MISE}(h)| \\ \le\ &\Big|\int_0^1 \Big\{\big|\hat{G}_h(u) - \mathbb{E}\hat{G}_h(u)\big|^2_{A_F(G(u))} - \mathbb{E}\big|\hat{G}_h(u) - \mathbb{E}\hat{G}_h(u)\big|^2_{A_F(G(u))}\Big\}w(u)du\Big| \\ &+\Big|\int_0^1 (\hat{G}_h(u) - \mathbb{E}\hat{G}_h(u))'A_F(G(u))(\mathbb{E}\hat{G}_h(u) - G(u))w(u)du\Big| \\ =:\ &A^{(1)}_{n,h} + A^{(2)}_{n,h}. \end{aligned}$$

In the following, abbreviate $\mathbb{E}_0 Z := Z - \mathbb{E}Z$ for random variables Z. By Lemma D.6(i), we have that for any $q > 2$,

$$\|A^{(1)}_{n,h} - \tilde{A}^{(1)}_{n,h}\|_q = O(n^{-1}),$$

where

$$\begin{aligned}\tilde{A}_{n,h}^{(1)} &= \Big|\int_0^1 \Big\{\big|\tilde{G}_h(u) - \mathbb{E}\tilde{G}_h(u)\big|^2_{A_F(G(u))} - \mathbb{E}\big|\tilde{G}_h(u) - \mathbb{E}\tilde{G}_h(u)\big|^2_{A_F(G(u))}\Big\} w(u)du\Big| \\ &= \Big|\int_0^1 \mathbb{E}_0 \frac{1}{n^2}\sum_{s,t=1}^n K_h(t/n-u)K_h(s/n-u)\mathbb{E}_0 g(\tilde{Y}_t(t/n))' A_F(G(u))\mathbb{E}_0 g(\tilde{Y}_s(s/n))w(u)du\Big| \\ &= \Big|\frac{1}{n^2}\sum_{s,t=1}^n \mathbb{E}_0 \int_{(\frac{t}{n}-1)/h}^{\frac{t}{nh}} K(v)K_h(\frac{s-t}{n} + vh)w(\frac{t}{n} - vh) \\ &\qquad \times \mathbb{E}_0 g(\tilde{Y}_t(t/n))' A_F(G(\frac{t}{n} - vh))\mathbb{E}_0 g(\tilde{Y}_s(s/n))dv\Big|.\end{aligned}$$

By Lemma 8.1(ii) in Richter and Dahlhaus (2019), we have for any $q \geq 2$:

$$\begin{aligned}\|\tilde{A}_{n,h}^{(1)}\|_q &\leq c'|w|_\infty |A_F(G(\cdot))|_\infty \cdot \frac{1}{n^2 h}\Big(\sum_{s,t=1}^n \big(\int_{-1/2}^{1/2} K(v)K(\frac{s-t}{nh} + v)dv\big)^2\Big)^{1/2} \\ &\leq c'|w|_\infty |A_F(G(\cdot))|_\infty |K|_\infty^2 \frac{(n(nh))^{1/2}}{n^2 h} = \frac{h^{1/2}}{nh}.\end{aligned}$$

The last inequality is due to the fact that the integral is 0 if $|s-t| > 2nh$. Thus only $n(nh)$ summands are nonzero.

As for $A_{n,h}^{(1)}$, we obtain that for any $q > 2$,

$$\|A_{n,h}^{(2)} - \tilde{A}_{n,h}^{(2)}\|_q = O(n^{-1}),$$

where

$$\begin{aligned}\tilde{A}_{n,h}^{(2)} &= \Big|\int_0^1 (\tilde{G}_h(u) - \mathbb{E}\tilde{G}_h(u))' A_F(G(u))(\mathbb{E}\hat{G}_h(u) - G(u))w(u)du\Big| \\ &= \Big|\int_0^1 \frac{1}{n}\sum_{t=1}^n K_h(t/n-u)\cdot \mathbb{E}_0 g(\tilde{Y}_t(t/n))' \cdot A_F(G(u))(\mathbb{E}\hat{G}_h(u) - G(u))w(u)du\Big| \\ &= \Big|\frac{1}{n}\sum_{t=1}^n \int_{(\frac{t}{n}-1)/h}^{\frac{t}{nh}} K(v)\cdot w(\frac{t}{n} - vh)\cdot \mathbb{E}_0 g(\tilde{Y}_t(t/n))' A_F(G(\frac{t}{n} - vh)) \\ &\qquad \times (\mathbb{E}\hat{G}_h(\frac{t}{n} - vh) - G(\frac{t}{n} - vh))dv\Big|.\end{aligned}$$

By Lemma 8.1(i) in Richter and Dahlhaus (2019), we have

$$\begin{aligned}&\|\tilde{A}_{n,h}^{(2)}\|_q \\ \leq\ & \frac{c'|K|_\infty}{n}\Big(\sum_{t=1}^n \Big(\int_{-1/2}^{1/2} w(\frac{t}{n} - vh)\Big|A_F(G(\frac{t}{n} - vh))\cdot(\mathbb{E}\hat{G}_h(\frac{t}{n} - vh) - G(\frac{t}{n} - vh))\Big|_2 dv\Big)^2\Big)^{1/2} \\ \leq\ & c'|K|_\infty |A_F(G(\cdot))|_\infty^{1/2} n^{-1}\Big(\int_{-1/2}^{1/2}\sum_{t=1}^n w(\frac{t}{n} - vh)|\mathbb{E}\hat{G}_h(\frac{t}{n} - vh) - G(\frac{t}{n} - vh)|^2_{A_F(G(\frac{t}{n} - vh))}dv\Big)^{1/2} \\ \leq\ & c'|K|_\infty |A_F(G(\cdot))|_\infty^{1/2} n^{-1}\Big(n\int_0^1 |\mathbb{E}\hat{G}_h(u) - G(u)|^2_{A_F(G(u))}w(u)du + O(1)\Big)^{1/2} \\ =\ & O\Big(n^{-1/2}\cdot \int_0^1 |\mathbb{E}\hat{G}_h(u) - G(u)|^2_{A_F(G(u))}w(u)du\Big)^{1/2} + n^{-1}\Big).\end{aligned}$$

Summarizing the results for $A_{n,h}^{(1)}$ and $A_{n,h}^{(2)}$, we have seen that

$$\frac{\|\tilde{d}_{ISE}(h) - \tilde{d}_{MISE}(h)\|_q}{\tilde{d}^*_{MISE}(h)} = O(h^{1/2}).$$

By a chaining argument as presented for $W_{n,h}$ below in (22), we obtain

$$\sup_{h\in H_n} \frac{|\tilde{d}_{ISE}(h) - \tilde{d}_{MISE}(h)|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s.,$$

that is, (7).

Discussion of (8): It holds that

$$\begin{aligned}
&|\tilde{d}_A(h) - \tilde{d}_{ISE}(h)| \\
\leq\ & \sum_{t=1}^n \int_{(t-1)/n}^{t/n} \big||G(t/n) - \hat{G}_h(t/n)|^2_{A_F(G(t/n))} w(t/n) - |G(u) - \hat{G}_h(u)|^2_{A_F(G(u))} w(u)\big| du.
\end{aligned}$$

For $q > 2$,

$$\begin{aligned}
&\|\tilde{d}_A(h) - \tilde{d}_{ISE}(h)\|_q \\
\leq\ & \sum_{t=1}^n \int_{(t-1)/n}^{t/n} \big\||G(t/n) - \hat{G}_h(t/n)|^2_{A_F(G(t/n))} w(t/n) \\
&\qquad - |G(t/n) - \hat{G}_h(t/n)|^2_{A_F(G(u))} w(u)\big\|_q du \\
&\quad + \sum_{t=1}^n \int_{(t-1)/n}^{t/n} \big\{|G(t/n) - G(u)|_2 + \big\||\hat{G}_h(t/n) - \hat{G}_h(u)|_2\big\|_{2q}\big\} \\
&\qquad \times \big\||A_F(G(u))(G(t/n) - \hat{G}_h(t/n))|_2 + |A_F(G(u))(G(u) - \hat{G}_h(u))|_2\big\|_{2q} w(u) du.
\end{aligned} \tag{13}$$

Since $\||G(u) - \hat{G}_h(u)|_2^2\|_q$ is $O(1)$ and $A_F(G(\cdot))$, $w(\cdot)$ have bounded variation, the first summand in (13) is $O(n^{-1})$. In the second summand, of (13), two stochastic terms appear. The first one is

$$\big\||\hat{G}_h(t/n) - \hat{G}_h(u)|_2\big\|_{2q} \leq \frac{1}{n}\sum_{s=1}^n \big|K_h((s-t)/n) - K_h(s/n-u)\big| \cdot \||g(Y_{s,n})|_2\|_{2q} = O((nh)^{-1}). \tag{14}$$

The second one is

$$\big\||\hat{G}_h(u) - G(u)|_2\big\|_{2q} \leq \||\hat{G}_h(u) - \mathbb{E}\hat{G}_h(u)|_2\|_{2q} + \||\mathbb{E}\hat{G}_h(u) - G(u)|_2\|_{2q}. \tag{15}$$

By Lemma B.1,

$$\||\mathbb{E}\hat{G}_h(u) - G(u)|_2\|_{2q} \leq c'(h + (nh)^{-1}).$$

By Lemma D.6(ii) and Lemma 8.1(i) from Richter and Dahlhaus (2019), we have

$$\begin{aligned}
&\||\hat{G}_h(u) - \mathbb{E}\hat{G}_h(u)|_2\|_{2q} \\
\leq\ & \||(\hat{G}_h(u) - \tilde{G}_h(u)) - \mathbb{E}(\hat{G}_h(u) - \tilde{G}_h(u))|_2\|_{2q} + \||\tilde{G}_h(u) - \mathbb{E}\tilde{G}_h(u)|_2\|_{2q} \\
\leq\ & c'((nh)^{-1} + (nh)^{-1/2}).
\end{aligned}$$

Insertion of the last two inequalities into (15) yields

$$\big\| |\hat{G}_h(u) - G(u)|_2 \big\|_{2q} \leq c'(h + (nh)^{-1/2}).$$

Together with (14) we obtain that the second summand of (13) is $O((nh)^{-1}(h+(nh)^{-1/2}))$. We have shown that

$$\frac{\|\tilde{d}_A(h) - \tilde{d}_{ISE}(h)\|_q}{\tilde{d}^*_{MISE}(h)} \leq (nh)\|\tilde{d}_A(h) - \tilde{d}_{ISE}(h)\|_q = O(h + (nh)^{-1/2}).$$

A chaining argument similar to the one for $W_{n,h}$ applied in (22) yields

$$\sup_{h\in H_n} \frac{|\tilde{d}_A(h) - \tilde{d}_{ISE}(h)|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s.,$$

that is, (8).

Discussion of (9): It holds that

$$\begin{aligned}
& |\tilde{d}_A(h) - \tilde{d}_{A,-}(h)| \\
\leq\ & \frac{1}{n}\sum_{t=1}^n \big| |G(t/n) - \hat{G}_h(\tfrac{t}{n})|^2_{A_F(G(t/n)) - A_F(\hat{G}_h(t/n))}\big| w(t/n) \\
& \quad + \frac{1}{n}\sum_{t=1}^n |\hat{G}_h(t/n) - \hat{G}_h^-(t/n)|^2_{A_F(G(t/n))} w(t/n) \\
& \quad + \Big|\frac{2}{n}\sum_{t=1}^n (G(t/n) - \hat{G}_h(t/n))' A_F(G(t/n))\big(\hat{G}_h(t/n) - \hat{G}_h^-(t/n)\big) w(t/n)\Big| \\
\leq\ & W_{n,h}^{(2)} + W_{n,h} + 2W_{n,h}^{1/2}\tilde{d}_A(h)^{1/2},
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
W_{n,h} &= \frac{\sup_{u\in[0,1]}|A_F(G(u))|_\infty}{n^3}\sum_{t=1}^n\Big(\sum_{s=1}^n |g(Y_{s,n})|_1 \big\{K_h - \frac{K_h^{(n)}}{k_{n,h}(t)}\big\}((s-t)/n)\Big)^2 w(t/n)^2, \\
W_{n,h}^{(2)} &:= \frac{1}{n}\sum_{t=1}^n \big| |G(t/n) - \hat{G}_h(\tfrac{t}{n})|^2_{A_F(G(t/n)) - A_F(\hat{G}_h(t/n))}\big| w(t/n),
\end{aligned}$$

and $k_{n,h}(t) := \frac{1}{n}\sum_{s=1}^n K_h^{(n)}((s-t)/n)$.

We now discuss $W_{n,h}^{(2)}$. Since A_F is Lipschitz continuous, there exists some constant $c' > 0$ such that

$$\begin{aligned}
W_{n,h}^{(2)} &\leq \frac{c'}{n}\sum_{t=1}^n \big|\hat{G}_h(t/n) - G(t/n)\big|_2^3 w(t/n) \\
&\leq \frac{2^3 c'}{n}\sum_{t=1}^n \big|\hat{G}_h(t/n) - \mathbb{E}\hat{G}_h(t/n)\big|_2^3 w(t/n) \\
&\qquad + \sup_{u\in[\gamma,1-\gamma]} |\mathbb{E}\hat{G}_h(u) - G(u)|_2 \cdot \frac{2^3 c'}{n}\sum_{t=1}^n \big|\mathbb{E}\hat{G}_h(t/n) - G(t/n)\big|_2^2 w(t/n). \\
&= O\Big(\sup_{u\in[\gamma,1-\gamma]} |\hat{G}_h(u) - \mathbb{E}\hat{G}_h(u)|_2 \cdot \int_0^1 |\hat{G}_h(u) - \mathbb{E}\hat{G}_h(u)|_2^2 w(u)du \\
&\qquad + \sup_{u\in[\gamma,1-\gamma]} |\mathbb{E}\hat{G}_h(u) - G(u)|_2 \cdot \int_0^1 |\mathbb{E}\hat{G}_h(u) - G(u)|_2^2 w(u)du\Big).
\end{aligned}$$

By Lemma 3.6 (equation (61)) in Richter and Dahlhaus (2019) applied to $\ell_\theta(y,u) = \frac{1}{2}(g(y)-\theta)^2$, we obtain for arbitrarily small $r > 0$ and fixed $\gamma > 0$,

$$\sup_{h\in H_n}\sup_{u\in[\gamma,1-\gamma]}(nh)^{\frac{1}{2}-r}|\hat{G}_h^-(u)-\mathbb{E}\hat{G}_h^-(u)|_2 \to 0 \quad a.s., \qquad \sup_{h\in H_n}\sup_{u\in[\gamma,1-\gamma]}|\mathbb{E}\hat{G}_h^-(u)-G(u)|_2 \to 0. \tag{17}$$

and

$$\sup_{h\in H_n}\sup_{u\in[\gamma,1-\gamma]}(nh)^{\frac{1}{2}-r}|\hat{G}_h(u)-\mathbb{E}\hat{G}_h(u)|_2 \to 0 \quad a.s., \qquad \sup_{h\in H_n}\sup_{u\in[\gamma,1-\gamma]}|\mathbb{E}\hat{G}_h(u)-G(u)|_2 \to 0. \tag{18}$$

By (18), we obtain

$$\sup_{h\in H_n}\frac{W_{n,h}^{(2)}}{nh+\int_0^1|\mathbb{E}\hat{G}_h(u)-G(u)|_2^2 w(u)du} \to 0 \quad a.s.$$

By (6), (7) and (18),

$$\sup_{h\in H_n}\frac{|W_{n,h}^{(2)}|}{\tilde{d}_{ISE,F}(h)} \to 0 \quad a.s.$$

We now discuss $W_{n,h}$. Note that $h \mapsto (nh)W_{n,h}$ is Lipschitz-continuous in h in the sense that there exists some polynomial $C(n)$ in n such that for all $h,h' \in H_n$ (thus $h,h' \geq n^{-1}$),

$$|(nh)W_{n,h}-(nh')W_{n,h'}| \leq C(n)\cdot|h-h'|\cdot\frac{1}{n}\sum_{s=1}^n|g(Y_{s,n})|_1^2. \tag{19}$$

Furthermore, we will show below (in equations (24), (25), (26), (27)) that there exists $\tau > 0$ and $C > 0$ independent of n (but C may depend on q) such that for any $q > 2$,

$$\|(nh)W_{n,h}\|_q \leq C\cdot n^{-\tau}. \tag{20}$$

This allows us to apply chaining arguments to $W_{n,h}$ to prove $\sup_{h\in H_n}(nh)|W_{n,h}| \to 0$ a.s. in the following simple manner. For $a \in \mathbb{N}$, the set

$$H_n' := \{in^{-a} : i = 1,...,n^a\}\cap H_n$$

is a discretization of H_n with at most $\#H_n' = n^a$ elements. For $\varsigma > 0$, we obtain from (19) that

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{h\in H_n}|(nh)\cdot W_{n,h}| > \varsigma\Big)\\
\leq\ &\mathbb{P}\Big(\sup_{h\in H_n'}|(nh)\cdot W_{n,h}| > \varsigma\Big)+\mathbb{P}\Big(\sup_{|h-h'|\leq n^{-a}}|(nh)\cdot W_{n,h}-(nh')W_{n,h'}| > \varsigma\Big)\\
\leq\ &(\#H_n')\cdot\frac{\|(nh)W_{n,h}\|_q^q}{\varsigma^q}+C(n)n^{-a}\cdot\frac{\frac{1}{n}\sum_{s=1}^n\||g(Y_{s,n})|_1\|_2^2}{\varsigma}\\
\leq\ &n^a\cdot\frac{\|(nh)W_{n,h}\|_q^q}{\varsigma^q}+C(n)n^{-a}\cdot\frac{\frac{1}{n}\sum_{s=1}^n\||g(Y_{s,n})|_1\|_2^2}{\varsigma}.
\end{aligned} \tag{21}$$

If $a \in \mathbb{N}$ is chosen large enough such that $C(n)n^{-a} \leq n^{-2}$, we obtain that the second summand in (21) is $O(n^{-2})$. By (20), the first summand in (21) is $O(n^{-2})$ for q large enough. We therefore have for any $\varsigma > 0$,

$$\mathbb{P}\Big(\sup_{h\in H_n}|(nh)\cdot W_{n,h}| > \varsigma\Big) = O(n^{-2}).$$

By the Borel-Cantelli lemma, it follows that

$$\sup_{h \in H_n} |(nh) \cdot W_{n,h}| \to 0 \quad a.s. \tag{22}$$

To show (20), we use the decomposition

$$W_{n,h} \le 4 \sup_{u\in[0,1]} |A_F(G(u))|_\infty (W_{n,h,1} + W_{n,h,2} + W_{n,h,3}),$$

where

$$\begin{aligned}
W_{n,h,1} &:= \frac{1}{n^3}\sum_{t=1}^n \Big(\sum_{s=1}^n \mathbb{E}|g(\tilde{Y}_s(s/n))|_1 \{K_h - \frac{K_h^{(n)}}{k_{n,h}(t)}\}((s-t)/n)\Big)^2 w(t/n)^2,\\
W_{n,h,2} &:= \frac{1}{n^3}\sum_{s_1,s_2=1}^n \mathbb{E}_0|g(\tilde{Y}_{s_1}(s_1/n))|_1 \cdot \mathbb{E}_0|g(\tilde{Y}_{s_2}(s_2/n))|_1 \cdot c(s_1,s_2),\\
W_{n,h,3} &:= \frac{1}{n^3}\sum_{t=1}^n \Big(\sum_{s=1}^n |g(Y_{s,n}) - g(\tilde{Y}_s(s/n))|_1 \{K_h - \frac{K_h^{(n)}}{k_{n,h}(t)}\}((s-t)/n)\Big)^2 w(t/n)^2,
\end{aligned}$$

where

$$c(s_1,s_2) := \sum_{t=1}^n \{K_h - \frac{K_h^{(n)}}{k_{n,h}(s_1)}\}((s_1-t)/n) \cdot \{K_h - \frac{K_h^{(n)}}{k_{n,h}(s_2)}\}((s_2-t)/n) w(t/n)^2.$$

First note that for all $q \ge 1$, $h \in H_n$,

$$\begin{aligned}
& (nh)\|W_{n,h,3}\|_q \qquad (23)\\
\le\ & \frac{h}{n^2}\sum_{t=1}^n \Big(\sum_{s=1}^n \||g(Y_{s,n}) - g(\tilde{Y}_s(s/n))|_1\|_{2q} \{K_h - \frac{K_h^{(n)}}{k_{n,h}(t)}\}((s-t)/n)\Big)^2 w(t/n)^2\\
=\ & O(h) \qquad (24)
\end{aligned}$$

due to $\||g(Y_{s,n}) - g(\tilde{Y}_s(s/n))|_1\|_{2q} = O(n^{-1})$.

We now discuss $W_{n,h,2}$. Note that for $t \in \{1,...,n\}$,

$$k_{n,h}(t) - 1 = O((nh)^{-1} + n^{-\alpha}),$$

and if $w(t/n) \neq 0$,

$$\sum_{s=1}^n \big|K_h((s-t)/n) - \frac{K_h^{(n)}((s-t)/n}{k_{n,h}(t)}\big| = O(n \cdot (n^{-\alpha} + (nh)^{-1})).$$

By Lemma D.1, we have $\sup_{s,n} \mathbb{E}|g(\tilde{Y}_s(s/n))|_1 = O(1)$, thus with some constant $c' > 0$,

$$|W_{n,h,1}| \le (c')^2 |K|_\infty^2 |w|_\infty^2 (n^{-2\alpha} + (nh)^{-2}). \tag{25}$$

Using Lemma 8.1(ii) from the Supplementary material of Richter and Dahlhaus (2019), we have with some constants $c', c'' > 0$,

$$\begin{aligned}
|\mathbb{E}W_{n,h,2}| &\le c'\frac{|w|_\infty^2}{n^3}\sup_{k\in\mathbb{Z}} \sum_{1\le t,t+k\le n} |c(t,t+k)| \le c''\frac{|w|_\infty^2 |K|_\infty^2}{n(nh)^2}\cdot n \cdot (n^{-\alpha} + (nh)^{-1})(nh)\\
&= c''|w|_\infty^2 |K|_\infty^2 \cdot (n^{-\alpha} + (nh)^{-1})(nh)^{-1}. \qquad (26)
\end{aligned}$$

Finally, by the same Lemma we obtain for all $q > 2$ with some constant $c' = c'(q) > 0, c'' = c''(q) > 0$:

$$\begin{aligned}
\|W_{n,h,2} - \mathbb{E}W_{n,h,2}\|_q &\leq c' \frac{1}{n^3}\Big(\sum_{s_1,s_2=1}^{n} c(s_1,s_2)^2 \Big)^{1/2} \\
&\leq c''|K|_\infty^2 \frac{(n^{-\alpha} + (nh)^{-1})(nh)}{n(nh)^2} \cdot ((n^{-\alpha} + (nh)^{-1})(nh)\cdot n)^{1/2} \\
&\leq c''|K|_\infty^2 \frac{(n^{-\alpha} + (nh)^{-1})^{3/2} h^{1/2}}{nh}. \qquad (27)
\end{aligned}$$

This completes the proof of (22).

From (6) - (8) and (11), we have

$$\sup_{h\in H_n} \frac{(nh)^{-1}}{\tilde{d}_A(h)} \to 0 \quad a.s.$$

This, (16) and the uniform convergence results on $W_{n,h}$, $W_{n,h}^{(2)}$ yield

$$\sup_{h\in H_n} \frac{|\tilde{d}_A(h) - \tilde{d}_{A,-}(h)|}{\tilde{d}_A(h)} \leq \sup_{h\in H_n} \frac{W_{n,h}^{(2)}}{\tilde{d}_A(h)} + \sup_{h\in H_n} \frac{W_{n,h}}{\tilde{d}_A(h)} + 2 \sup_{h\in H_n} \Big(\frac{W_{n,h}}{\tilde{d}_A(h)}\Big)^{1/2} \to 0 \quad a.s.$$

that is, (9).

Discussion of (10): Since F is twice continuously differentiable, we have

$$\begin{aligned}
&[d_{ISE,F}^{(n)}(h) - \frac{1}{n}\sum_{t=1}^{n} |g(Y_{t,n}) - G(t/n)|^2_{A_F(G(t/n))}] - \tilde{d}_{A,-}(h) \\
= &-\frac{2}{n}\sum_{t=1}^{n}(g(Y_{t,n}) - G(t/n))' A_F(\hat{G}_h(t/n)) \cdot (\hat{G}_h^-(t/n) - G(t/n)) \\
= &S_{n,h}^{(1)} + S_{n,h}^{(2)} + S_{n,h}^{(3)},
\end{aligned}$$

where

$$\begin{aligned}
S_{n,h}^{(1)} &:= -\frac{2}{n}\sum_{t=1}^{n}(g(Y_{t,n}) - G(t/n))' A_F(G(t/n)) \cdot (\hat{G}_h^-(t/n) - G(t/n)) w(t/n), \\
S_{n,h}^{(2)} &:= -\frac{2}{n}\sum_{t=1}^{n}(g(Y_{t,n}) - G(t/n))' [\partial_F A_F(G(t/n)) \cdot (\hat{G}_h^-(t/n) - G(t/n))] \\
&\quad \times (\hat{G}_h^-(t/n) - G(t/n)) w(t/n), \\
S_{n,h}^{(3)} &:= -\frac{2}{n}\sum_{t=1}^{n}(g(Y_{t,n}) - G(t/n))' [(\partial_F A(\hat{G}_h(t/n)) - \partial_F A_F(G(t/n))) \cdot (\hat{G}_h^-(t/n) - G(t/n))] \\
&\quad \times (\hat{G}_h^-(t/n) - G(t/n)) w(t/n).
\end{aligned}$$

We will show in the following that for $i = 1, 2, 3$,

$$\sup_{h\in H_n} \frac{|S_{n,h}^{(i)}|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s.$$

Then (10) follows.

We first discuss $S_{n,h}^{(1)}$. Define $\tilde{G}_h^-(u) := \frac{1}{n}\sum_{t=1}^n K_h^{(n)}(t/n-u)\cdot g(\tilde{Y}_t(t/n))$. With Lemma D.6(i) and a chaining argument, we obtain that

$$\tilde{S}_{n,h}^{(1)} := -\frac{2}{n}\sum_{t=1}^n (g(\tilde{Y}_t(t/n)) - G(t/n))' A_F(G(t/n))\cdot(\tilde{G}_h^-(t/n) - G(t/n))w(t/n)$$

fulfills $\sup_{h\in H_n}\frac{|S_{n,h}^{(1)}-\tilde{S}_{n,h}^{(1)}|}{\tilde{d}_{MISE}^*(h)} \to 0$ a.s. It remains to show $\sup_{h\in H_n}\frac{|\tilde{S}_{n,h}^{(1)}|}{\tilde{d}_{MISE}^*(h)} \to 0$ a.s.

We now derive an upper bound of $\mathbb{E}\tilde{S}_{n,h}^{(1)}$: By assumption and Lemma D.7,

$$\sup_{u\in[0,1]} \delta_2^{g(\tilde{Y}(u))}(k) \le c' k^{-\kappa}$$

with some $c' > 0$.

In the case $s > t \ge 1$, the Cauchy Schwarz inequality implies

$$\begin{aligned}
&|\mathrm{Cov}(g(\tilde{Y}_t(t/n)), g(\tilde{Y}_s(s/n)))| \\
\le\ & \sum_{k,l=0}^{\infty} |\mathbb{E}[P_{t-k}g(\tilde{Y}_t(t/n))\cdot P_{s-l}g(\tilde{Y}_s(s/n))]| \\
\le\ & \sum_{k=0}^{\infty} |\mathbb{E}[P_{t-k}g(\tilde{Y}_t(t/n))\cdot P_{t-k}g(\tilde{Y}_s(s/n))]| \\
\le\ & \sum_{k=0}^{\infty} \|P_{t-k}g(\tilde{Y}_t(t/n))\|_2\cdot \|P_{t-k}g(\tilde{Y}_s(s/n))\|_2 \\
\le\ & \sum_{k=0}^{\infty} \sup_{u\in[0,1]}\delta_2^{g(\tilde{Y}(u))}(k)\cdot \sup_{u\in[0,1]}\delta_2^{g(\tilde{Y}(u))}(s-t+k) \\
\le\ & (c')^2\cdot\sum_{k=0}^{\infty} k^{-\kappa}\cdot(s-t)^{-\kappa}.
\end{aligned}$$

A similar result holds for $s < t$. With $c'' := (c')^2\cdot\sum_{k=0}^\infty k^{-\kappa}$, we obtain

$$\begin{aligned}
|\mathbb{E}\tilde{S}_{n,h}^{(1)}| &\le \frac{2|\partial_F A_F(G(\cdot))|_\infty}{n(nh)}\sum_{s,t=1}^n |K^{(n)}((s-t)/(nh)|\cdot|\mathrm{Cov}(g(\tilde{Y}_t(t/n)), g(\tilde{Y}_s(s/n)))| \\
&\le \frac{2|\partial_F A_F(G(\cdot))|_\infty}{n(nh)}\sum_{s,t=1}^n |K^{(n)}((s-t)/(nh)|\cdot c''|s-t|^{-\kappa} \\
&\le \frac{4c''|K|_\infty|\partial_F A_F(G(\cdot))|_\infty}{n(nh)}\sum_{t=1}^n\ \sum_{k=\lfloor(1-\varepsilon)n^{-\alpha}(nh)\rfloor}^{\infty} k^{-\kappa} \\
&= O(\frac{(n^{-\alpha}(nh))^{-\kappa+1}}{nh}).
\end{aligned}$$

By assumption, $h\in H_n$ implies $n^{-\alpha}(nh)\to 0$. Thus

$$\sup_{h\in H_n}\frac{|\mathbb{E}\tilde{S}_{n,h}^{(1)}|}{\tilde{d}_{MISE}^*(h)} \to 0.$$

It remains to show that $\sup_{h\in H_n}\frac{|\tilde{S}_{n,h}^{(1)}-\mathbb{E}\tilde{S}_{n,h}^{(1)}|}{\tilde{d}_{MISE}^*(h)} \to 0$ a.s. We decompose

$$\tilde{S}_{n,h}^{(1)} - \mathbb{E}\tilde{S}_{n,h}^{(1)} = Z_{n,h}^{(1)} + Z_{n,h}^{(2)},$$

where

$$Z_{n,h}^{(1)} := -\mathbb{E}_0\Big[\frac{2}{n}\sum_{t=1}^{n}(g(\tilde{Y}_t(t/n)) - G(t/n))' A_F(G(t/n)) \cdot (\tilde{G}_h^-(t/n) - \mathbb{E}\tilde{G}_h^-(t/n))w(t/n)\Big],$$

$$Z_{n,h}^{(2)} := -\frac{2}{n}\sum_{t=1}^{n}(g(\tilde{Y}_t(t/n)) - G(t/n))' A_F(G(t/n)) \cdot (\mathbb{E}\tilde{G}_h^-(t/n) - G(t/n))w(t/n).$$

We again use a chaining argument to show that $\sup_{h\in H_n}\frac{|Z_{n,h}^{(i)}|}{\tilde{d}^*_{MISE}(h)} \to 0$ a.s., $i = 1, 2$. By Lemma 8.1(ii) in Richter and Dahlhaus (2019), it holds that

$$\|Z_{n,h}^{(1)}\|_q \le \frac{c'|A_F(G(\cdot))|_\infty}{n^2 h}\Big(\sum_{s,t=1}^{n} K^{(n)}(\frac{s-t}{nh})^2\Big)^{1/2} = O((n(nh))^{1/2}(nh)^{-1}) = O(h^{1/2}(nh)^{-1}),$$

that is, $\frac{\|Z_{n,h}^{(1)}\|_q}{\tilde{d}^*_{MISE}(h)} = O(h^{1/2})$ and thus $\sup_{h\in H_n}\frac{|Z_{n,h}^{(1)}|}{\tilde{d}^*_{MISE}(h)} \to 0$.

By Lemma 8.1(i) in Richter and Dahlhaus (2019), it holds that

$$\begin{aligned}
\|Z_{n,h}^{(2)}\|_q &\le \frac{c'}{n}\Big(\sum_{t=1}^{n}\big|A_F(G(t/n)) \cdot (\mathbb{E}\hat{G}_h^-(t/n) - G(t/n))\big|_2^2 w(t/n)^2\Big)^{1/2}\\
&\le \frac{c'|w|_\infty^{1/2}|A_F(G(\cdot))|_\infty^{1/2}}{n}\Big(n\cdot\int_0^1\big|\mathbb{E}\hat{G}_h(u) - G(u)\big|_{A_F(G(u))}^2 w(u)du + O(1)\Big)^{1/2}\\
&= O\big(n^{-1/2}\int_0^1\big|\mathbb{E}\hat{G}_h(u) - G(u)\big|_{A_F(G(u))}^2 w(u)du + n^{-1}\big),
\end{aligned}$$

which shows that $\frac{\|Z_{n,h}^{(2)}\|_q}{\tilde{d}^*_{MISE}(h)} = O(h^{1/2})$ and thus $\sup_{h\in H_n}\frac{|Z_{n,h}^{(1)}|}{\tilde{d}^*_{MISE}(h)} \to 0$. This completes the discussion of $S_{n,h}^{(1)}$.

We now discuss $S_{n,h}^{(2)}$. With Lemma D.6(i) and a chaining argument, we obtain that

$$\begin{aligned}
\tilde{S}_{n,h}^{(2)} := -\frac{2}{n}\sum_{t=1}^{n}&\langle g(\tilde{Y}_t(t/n)) - G(t/n), [\partial_F A_F(G(t/n)) \cdot (\tilde{G}_h^-(t/n) - G(t/n))]\\
&\times(\tilde{G}_h^-(t/n) - G(t/n))\rangle w(t/n)
\end{aligned}$$

fulfills $\sup_{h\in H_n}\frac{|S_{n,h}^{(2)} - \tilde{S}_{n,h}^{(2)}|}{\tilde{d}_{ISE}(h)} \to 0$ a.s. Let

$$\begin{aligned}
\bar{S}_{n,h}^{(2)} := -\frac{2}{n}\sum_{t=1}^{n}&(g(\tilde{Y}_t(t/n)) - G(t/n))'[\partial_F A_F(G(t/n)) \cdot (\tilde{G}_h^-(t/n) - \mathbb{E}\tilde{G}_h^-(t/n))]\\
&\times(\tilde{G}_h^-(t/n) - \mathbb{E}\tilde{G}_h^-(t/n))w(t/n).
\end{aligned}$$

$\bar{S}_{n,h}^{(2)}$ is obtained from $\tilde{S}_{n,h}^{(2)}$ by replacing $G(t/n)$ by $\mathbb{E}\tilde{G}_h^-(t/n)$ in the last two factors. By Lemma 8.1(i),(ii) in Richter and Dahlhaus (2019), we have for any $q > 2$ with some constant $c' > 0$ that

$$\begin{aligned}
&\big\|\tilde{S}^{(2)}_{n,h} - \bar{S}^{(2)}_{n,h}\big\|_q \\
\leq\ & c'\Big[\sup_u |\partial_F A_F(G(u))|_\infty \frac{1}{\sqrt{n}}\Big(\frac{1}{n}\sum_{t=1}^{n} |\mathbb{E}\tilde{G}_h^-(t/n) - G(t/n)|_2^4 w(t/n)^2\Big)^{1/2} \\
& + \frac{2}{n^2 h}\sup_u |\partial_F A_F(G(u))|_\infty \Big(\sum_{s,t=1}^{n} |K^{(n)}(\frac{s-t}{nh})|^2 \cdot |\mathbb{E}\tilde{G}_h^-(t/n) - G(t/n)|_2^2 \cdot w(t/n)^2\Big)^{1/2} \\
=\ & O\Big(\frac{1}{\sqrt{n}}\int_0^1 |\mathbb{E}\hat{G}_h(u) - G(u)|_2^2 w(u)du + (nh)^{-1}h^{1/2}\Big).
\end{aligned}$$

Using a similar chaining argument as above, we obtain with (18) that $\sup_{h\in H_n} \frac{|\tilde{S}_{n,h} - \bar{S}^{(2)}_{n,h}|}{\tilde{d}^*_{MISE}(h)} \to 0$ a.s.

We therefore now analyze $\bar{S}^{(2)}_{n,h}$. By Lemma 8.1(iii) in Richter and Dahlhaus (2019), we have with some constant $c' > 0$,

$$|\mathbb{E}\bar{S}^{(2)}_{n,h}| \leq \frac{c'}{n^3 h^2}\sup_{k,l\in\mathbb{Z}} \sum_{1\leq t,t+k,t+l\leq n} \Big|K^{(n)}(\frac{k}{nh})K^{(n)}(\frac{l}{nh})\Big| = O((nh)^{-2}),$$

that is, $\sup_{h\in H_n} \frac{|\mathbb{E}\bar{S}^{(2)}_{n,h}|}{\tilde{d}^*_{MISE}(h)} \to 0$.

Furthermore, by Lemma 8.1(ii) in Richter and Dahlhaus (2019), the terms

$$\begin{aligned}
R^{(2,1)}_{n,h} &:= -\sum_{j,k,l=1}^{d} \frac{2}{n}\sum_{t=1}^{n} [\partial_F A_F(G(t/n))]_{jkl}[g(\tilde{Y}_t(t/n)) - G(t/n)]_j \cdot \mathbb{E}\big[[\tilde{G}_h^-(t/n) - \mathbb{E}\tilde{G}_h^-(t/n)]_k \\
&\qquad \times [\tilde{G}_h^-(t/n) - \mathbb{E}\tilde{G}_h^-(t/n)]_l\big]\rangle w(t/n), \\
R^{(2,2)}_{n,h} &:= -\sum_{j,k,l=1}^{d} \frac{2}{n}\sum_{t=1}^{n} [\partial_F A_F(G(t/n))]_{jkl}\mathbb{E}\big[[g(\tilde{Y}_t(t/n)) - G(t/n)]_j \cdot [\tilde{G}_h^-(t/n) - \mathbb{E}\tilde{G}_h^-(t/n)]_k\big] \\
&\qquad \times [\tilde{G}_h^-(t/n) - \mathbb{E}\tilde{G}_h^-(t/n)]_l\rangle w(t/n), \\
R^{(2,3)}_{n,h} &:= -\sum_{j,k,l=1}^{d} \frac{2}{n}\sum_{t=1}^{n} [\partial_F A_F(G(t/n))]_{jkl}\mathbb{E}\big[[g(\tilde{Y}_t(t/n)) - G(t/n)]_j \cdot [\tilde{G}_h^-(t/n) - \mathbb{E}\tilde{G}_h^-(t/n)]_l\big] \\
&\qquad \times [\tilde{G}_h^-(t/n) - \mathbb{E}\tilde{G}_h^-(t/n)]_k\rangle w(t/n)
\end{aligned}$$

satisfy for any $q > 2$ with some constant $c' > 0$,

$$\begin{aligned}
\big\|R^{(2,1)}_{n,h}\big\|_q &\leq \frac{c'}{\sqrt{n}}\Big(\sum_{t=1}^{n}\big(|\hat{G}_h^-(t/n) - \mathbb{E}\hat{G}_h^-(t/n)|_2^2\big)^2 w(t/n)^2\Big)^{1/2} = O(n^{-1/2}(nh)^{-1}), \\
\big\|R^{(2,2)}_{n,h}\big\|_q &\leq \frac{c'}{n^2 h}\Big(\sum_{s=1}^{n}\Big[\sum_{t=1}^{n} K^{(n)}(\frac{t-s}{nh})^2 \mathbb{E}[|\tilde{G}_h^-(t/n) - \mathbb{E}\tilde{G}_h^-(t/n)|_2^2]^{1/2}\Big]^2\Big)^{1/2} \\
&= O((n^2h)^{-1}(n\cdot nh)^{1/2}) = O((nh)^{-1}h^{1/2}),
\end{aligned}$$

and a similar result as for $\|R^{(2,2)}_{n,h}\|_q$ holds for $\|R^{(2,3)}_{n,h}\|_q$. Again a chaining argument implies that $\sup_{h\in H_n} \frac{|R^{(2,i)}_{n,h}|}{\tilde{d}^*_{MISE}(h)} \to 0$ a.s. $(i = 1, 2, 3)$.

Finally, by Lemma 8.1(iii) in Richter and Dahlhaus (2019), we have for $q > 2$ and some constant $c' > 0$,

$$\begin{aligned}
&\big\|\bar{S}_{n,h}^{(2)} - \mathbb{E}\bar{S}_{n,h}^{(2)} - \sum_{i=1}^{3} R_{n,h}^{(2,i)}\big\|_q \\
\leq\; & \frac{c'}{n^3h^2}\Big(\sum_{s_1,s_2,t=1}^{n} K^{(n)}\big(\frac{s_1-t}{nh}\big)^2 K^{(n)}\big(\frac{s_2-t}{nh}\big)^2\Big)^{1/2} \\
=\; & O((n^3h^2)^{-1}(n(nh)^2)^{1/2}) = O((nh)^{-1}n^{-1/2}).
\end{aligned}$$

Again a chaining argument implies that $\sup_{h\in H_n} \frac{|\bar{S}_{n,h}^{(2)} - \mathbb{E}\bar{S}_{n,h}^{(2)} - \sum_{i=1}^{3} R_{n,h}^{(2,i)}|}{\tilde{d}^*_{MISE}(h)} \to 0$ a.s.

The results above show that

$$\sup_{h\in H_n} \frac{|\bar{S}_{n,h}^{(2)}|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s.,$$

and then $\sup_{h\in H_n} \frac{|S_{n,h}^{(2)}|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s.$.

We now discuss $S_{n,h}^{(3)}$. Note that G is continuous and therefore, $\{G(u) : u \in [0,1]\}$ is a compact set. Together with the fact that F is twice continuously differentiable with Lipschitz continuous second derivative, the same holds for A_F. We conclude that there exists some constant $c' > 0$ such that

$$\begin{aligned}
|S_{n,h}^{(3)}| \;&\leq\; \frac{c'}{n}\sum_{t=1}^{n} \big|\hat{G}_h^-(t/n) - G(t/n)\big|_2^2 \big|\hat{G}_h(t/n) - G(t/n)\big|_2 w(t/n) \\
&\leq\; c' \cdot \sup_{u\in[\gamma,1-\gamma]} |\hat{G}_h(u) - G(u)|_2 \cdot \frac{1}{n}\sum_{t=1}^{n} |\hat{G}_h^-(t/n) - G(t/n)|_2^2 w(t/n). \qquad (28)
\end{aligned}$$

Mimicking the proofs of (6), (7), (8) and (9) (with $A_F(G(u)) = I_{d\times d}$), the terms

$$\begin{aligned}
\tilde{d}_{A,-,norm}(h) \;&:=\; \frac{1}{n}\sum_{t=1}^{n} |\hat{G}_h^-(t/n) - G(t/n)|_2^2 w(t/n), \\
\tilde{d}^*_{MISE,norm}(h) \;&:=\; \frac{\sigma_K^2}{nh}\int_0^1 \mathrm{tr}(\Sigma_g(u))w(u)du + \int_0^1 \big|\mathbb{E}\hat{G}_h(u) - G(u)\big|_2^2 w(u)du
\end{aligned}$$

fulfill

$$\sup_{h\in H_n} \frac{|\tilde{d}_{A,-,norm}(h) - \tilde{d}^*_{MISE,norm}(h)|}{\tilde{d}^*_{MISE,norm}(h)} \to 0 \quad a.s.$$

Together with (18), we therefore have

$$\sup_{h\in H_n} \frac{|S_{n,h}^{(3)}|}{\tilde{d}^*_{MISE,norm}(h)} \leq c' \cdot \sup_{u\in[\gamma,1-\gamma]} |\hat{G}_h(u) - G(u)|_2 \cdot \Big(\frac{\tilde{d}_{A,-,norm}(h) - \tilde{d}^*_{MISE,norm}(h)}{\tilde{d}^*_{MISE,norm}(h)} + 1\Big) \to 0 \quad a.s. \qquad (29)$$

By (18), we have that

$$\sup_{h\in H_n} \frac{\tilde{d}^*_{MISE}(h)}{\tilde{d}^*_{MISE,norm}(h)}$$

is uniformly bounded in n. Thus (29) implies $\sup_{h\in H_n} \frac{|S_{n,h}^{(3)}|}{\tilde{d}^*_{MISE,norm}(h)}$, which concludes the proof of (10).

Discussion of (11): By a Taylor expansion,

$$\begin{aligned} & F(\hat{G}_h(u)) - F(G(u)) \\ = \; & \partial_G F(G(u)) \cdot (\hat{G}_h(u) - G(u)) + \{\partial_G F(\bar{G}_h(u)) - \partial_G F(G(u))\} \cdot (\hat{G}_h(u) - G(u)), \end{aligned}$$

with $\bar{G}_h(u) \in \mathbb{R}^d$ such that $|\bar{G}_h(u) - G(u)|_2 \leq |\hat{G}_h(u) - G(u)|_2$. Since $G \mapsto \partial_G F(G)$ is continuously differentiable with Lipschitz continuous first derivative, we obtain that for some constant $C > 0$,

$$\begin{aligned} & |d_{ISE}(h) - \tilde{d}_{ISE}(h)| \\ = \; & \Big| \int_0^1 \big\{ \big|F(\hat{G}_h(u)) - F(G(u))\big|_2^2 - \big|\hat{G}_h(u) - G(u)\big|^2_{A_F(G(u))} \big\} w(u) du \Big| \\ \leq \; & 2 \int_0^1 \big| \langle \partial_G F(G(u))(\hat{G}_h(u) - G(u)), \\ & \qquad (\partial_G F(\bar{G}_h(u)) - \partial_G F(G(u)))(\hat{G}_h(u) - G(u)) \rangle \big| w(u) du \\ & \quad + \int_0^1 \big| (\partial_G F(\bar{G}_h(u)) - \partial_G F(G(u)))(\hat{G}_h(u) - G(u)) \big|_2^2 w(u) du \\ \leq \; & C \Big[\sup_{u\in \mathrm{supp}(w)} |\hat{G}_h(u) - G(u)|_2 + \sup_{u \in \mathrm{supp}(w)} |\hat{G}_h(u) - G(u)|_2^2 \Big] \\ & \qquad \times \int_0^1 \big|\hat{G}_h(u) - G(u)\big|_2^2 w(u) du. \end{aligned} \tag{30}$$

We now have exactly the same structure as $S_{n,h}^{(3)}$ in (28). We conclude that as above that

$$\sup_{h\in H_n} \frac{|d_{ISE}(h) - \tilde{d}_{ISE}(h)|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s.$$

□

Proof of Corollary 3.3. By (6) - (11), we have with the abbreviation $J_n := \frac{1}{n}\sum_{t=1}^n |g(Y_{t,n}) - G(t/n)|^2_{A_F(G(t/n))}$ that

$$\sup_{h\in H_n} \frac{|[d_{ISE,F}^{(n)}(h) - J_n] - \tilde{d}^*_{MISE}(h)|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s.$$

By assumption, $h_{opt} \in H_n$ for n large enough and thus $h_{opt} \in \mathrm{argmin}_{h\in H_n} \tilde{d}^*_{MISE}(h)$. As in (12), we obtain

$$\frac{\tilde{d}^*_{MISE}(\hat{h})}{\tilde{d}^*_{MISE}(h_{opt})} = \frac{\tilde{d}^*_{MISE}(\hat{h})}{\inf_{h\in H_n} \tilde{d}^*_{MISE}(h)} \to 1 \quad a.s. \tag{31}$$

By the structure of $h \mapsto \tilde{d}^*_{MISE}(h)$, we obtain

$$\frac{\hat{h}}{h_{opt}} \to 1 \quad a.s.$$

Second, (6) - (11) imply

$$\sup_{h\in H_n}\Big|1-\frac{d_{ISE}(h)}{\tilde{d}^*_{MISE}(h)}\Big| = \sup_{h\in H_n}\frac{|d_{ISE}(h)-\tilde{d}^*_{MISE}(h)|}{\tilde{d}^*_{MISE}(h)} \to 0 \quad a.s.$$

Since $h_{opt}\in H_n$, we obtain

$$\frac{d_{ISE}(h_{opt})}{\tilde{d}^*_{MISE}(h_{opt})} \to 1 \quad a.s., \qquad \frac{d_{ISE}(\hat{h})}{\tilde{d}^*_{MISE}(\hat{h})} \to 1 \quad a.s.,$$

Combination with (31) yields

$$\frac{d_{ISE}(\hat{h})}{d_{ISE}(h_{opt})} \to 1.$$

□

The following lemma is needed to bound a bias term in the proof of Theorem 3.1.

Lemma B.1 (Bias upper bound). *Let $g\in\mathcal{H}(M,\chi,C)$ and let Assumption 2.1 hold. Then component-wise,*

$$\sup_{u\in[0,1]}|\mathbb{E}\hat{G}_h(u)-G(u)| = O((nh)^{-1}+h).$$

Proof of Lemma B.1. By Lemma D.6(ii), we have in each component that

$$\sup_{u\in[0,1]}\|\hat{G}_h(u)-\tilde{G}_h(u)\|_1 = O((nh)^{-1}).$$

Now, we have

$$\begin{aligned}\mathbb{E}\tilde{G}_h(u)-\mathbb{E}g(\tilde{Y}_0(u)) &= \frac{1}{n}\sum_{t=1}^{n}K_h(t/n-u)\big\{\mathbb{E}g(\tilde{Y}_0(t/n))-\mathbb{E}g(\tilde{Y}_0(u))\big\}\\ &\quad +\Big(\frac{1}{n}\sum_{t=1}^{n}K_h(t/n-u)-1\Big)\cdot\mathbb{E}g(\tilde{Y}_0(u)). \end{aligned} \tag{32}$$

By Hölder's inequality,

$$\begin{aligned}|\mathbb{E}g(\tilde{Y}_0(t/n))-\mathbb{E}g(\tilde{Y}_0(u))| &\le C(1+(D|\chi|_1)^{M-1})\cdot\sum_{i=1}^{\infty}\chi_i\|\tilde{X}_0(t/n)-\tilde{X}_0(u)\|_1\\ &\le CD|\chi|_1(1+(D|\chi|_1)^{M-1})\cdot|t/n-u|.\end{aligned}$$

Insertion into (32) and using that $K\in\mathcal{K}$ yields

$$|\mathbb{E}\tilde{G}_h(u)-\mathbb{E}g(\tilde{Y}_0(u))| \le O(h+(nh)^{-1}).$$

□

C Proof of Theorem 4.6

Recall the notation from Section 4:

$$\hat{G}_h(u) := \frac{1}{nw_{n,h}(u)} \sum_{t=1}^{n} K_h(t/n - u) g(Y_{t,n}),$$
$$\tilde{G}_h(u) := \frac{1}{nw_{n,h}(u)} \sum_{t=1}^{n} K_h(t/n - u) g(\tilde{Y}_t(t/n)),$$

where $w_{n,h}(u) = \frac{1}{n}\sum_{t=1}^{n} K_h(t/n - u)$.

Define $G_h(u) := \mathbb{E}\tilde{G}_h(u)$, and

$$h_0(u) := \sup\{h \in H_n : |G_{h'}(u) - G(u)|_2 \le \frac{C^{\#}}{8} v(h,u)\lambda(h) \quad \text{for all } h' \in H_n, h' \le h\}.$$

The main theoretical work for the proof of Theorem 4.6 is devoted to Proposition C.1 below.

Proof of Theorem 4.6. Define $\Delta_h(u) := \sup_{h' \le h, h' \in H_n} |G_{h'}(u) - G(u)|_2$. We first derive an upper bound for $\Delta_h(u)$ with $h \in H_n$. It holds that

$$G_h(u) - G(u) = \frac{1}{nw_{n,h}(u)} \sum_{t=1}^{n} K_h(t/n - u)\{G(t/n) - G(u)\}.$$

For c_H large enough and C_H small enough, $w_{n,h}(u) \ge \frac{1}{2}$. With some constant $c' > 0$ only depending on $|K|_\infty$, $|G|_\infty$ and the corresponding Lipschitz constants of K, G, we have

$$\Big|\frac{1}{n}\sum_{t=1}^{n} K_{h'}(t/n - u)\{G(t/n) - G(u)\} - \int_0^1 K_{h'}(v - u)\{G(v) - G(u)\}\, \mathrm{d}v\Big|_\infty \le c' n^{-1}.$$

Since G is twice continuously differentiable,

$$\int_0^1 K_{h'}(v - u)\{G(v) - G(u)\}\, \mathrm{d}v = \frac{h^2}{2}\mu_K \cdot \partial_u^2 G(u) + o(h^2).$$

Since $|\partial_u^2 G(u)|_2 > 0$, $\int K(x)x^2 dx > 0$, we obtain for C_H small enough that

$$|G_h(u) - G(u)| \le c' n^{-1} + 4 \cdot \frac{h^2}{2}\mu_K \cdot |\partial_u^2 G(u)|_2.$$

This expression is increasing in h, thus

$$\Delta_h(u) \le c' n^{-1} + 4 \cdot \frac{h^2}{2}\mu_K \cdot |\partial_u^2 G(u)|_2. \tag{33}$$

We now derive an upper bound for $\mathbb{E}|\hat{G}^{\circ}_{\hat{h}(u)}(u) - G(u)|_2^2$. By Proposition C.1, there exists some universal constant $c > 0$ and some constant c' not depending on n such that

$$\begin{aligned} \mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - G(u)|_2^2 &\le cv^2(h_0(u), u)\lambda(h_0(u))^2 + c' \cdot \log(n)^2 n^{-1} \\ &\le cv^2(C_H, u)\lambda(C_H)^2 + c\sum_{h \in H_n} v^2(h_0(u), u)\lambda(h_0(u))^2 \mathbb{1}_{\{h_0(u) = ah\}} \\ &\quad + c' \cdot \log(n)^2 n^{-1}. \end{aligned} \tag{34}$$

If $h_0(u) = ah$, then by definition of $h_0(u)$,

$$\Delta_h(u) \geq \frac{C^\#}{8} v(ah,u)\lambda(ah).$$

Thus

$$\begin{aligned} v^2(h_0(u),u)\lambda(h_0(u))^2 \mathbb{1}_{\{h_0(u)=a\cdot h\}} &\leq \min\{v^2(ah,u)\lambda(ah)^2, (\frac{8}{C^\#})^2 \Delta_h(u)^2\} \\ &\leq (\frac{8}{C^\#})^2 \min\{v^2(ah,u)\lambda(ah)^2, \Delta_h(u)^2\}. \end{aligned}$$

Let $h^* \in H_n$ be arbitrary. We find that

$$\begin{aligned} &\sum_{h\in H_n} v^2(h_0(u),u)\lambda(h_0(u))^2 \mathbb{1}_{\{h_0(u)=a\cdot h\}} \\ \leq\ & \sum_{h\in H_n} \min\{v^2(ah,u)\lambda(ah)^2, \Delta_h(u)^2\} \\ \leq\ & \sum_{h\in H_n, h\leq h^*} \Delta_h(u)^2 + \sum_{h\in H_n, h>h^*} v^2(ah,u)\lambda(ah)^2 \\ \leq\ & \frac{4}{(1-a^4)}(h^*)^4 \mu_K^2 |\partial_u^2 G(u)|_2^2 + \frac{1}{1-a}\sigma_K^2 \mathrm{tr}(\Sigma(u))\frac{\log(n)}{nh^*} + c' \log(n) n^{-1}. \end{aligned}$$

Insertion into (34) and using the fact that $v^2(C_H,u)\lambda(C_H)^2 = O(n^{-1})$ yields with some constant $c' > 0$ independent of n that

$$\mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - G(u)|_2^2 \leq c' \log(n)^2 n^{-1} + \frac{4c}{1-a}\Big\{\frac{(h^*)^4}{4}\mu_K^2|\partial_u^2 G(u)|_2^2 + \sigma_K^2 \mathrm{tr}(\Sigma(u))\frac{\log(n)}{nh^*}\Big\}.$$

□

Proposition C.1. *Suppose that the assumptions of Theorem 4.6 hold. Then there exists some universal constant $c > 0$ and some constant $c' > 0$ independent of n such that*

$$\mathbb{E}|\hat{G}^\circ_{\hat{h}(u)}(u) - G(u)|_2^2 \leq c \cdot v^2(h_0(u),u)\lambda(h_0(u))^2 + c' \cdot \log(n)^2 n^{-1}.$$

Proof of Proposition C.1. We follow the proof strategy of Lepski et al. (1997). During the proof, we use $c' > 0$ for a constant not depending on n, h, h'. Put

$$\begin{aligned} v^2(h,u) &:= \frac{1}{nh}\int K(x)^2 \, \mathrm{d}x \cdot \mathrm{tr}(\Sigma(u)), \\ v^2(h,h',u) &:= \frac{1}{n}\int \{K_h(x) - K_{h'}(x)\}^2 \, \mathrm{d}x \cdot \mathrm{tr}(\Sigma(u)). \end{aligned}$$

Define

$$S(u) := \{\hat{h}(u) \geq h_0(u)\}.$$

Then

$$\mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - G(u)|_2^2 = \mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - G(u)|_2^2 \mathbb{1}_{S(u)} + \mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - G(u)|_2^2 \mathbb{1}_{S(u)^c}. \qquad (35)$$

Discussion of the first summand in (35): It holds that

$$\begin{aligned} \mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - G(u)|_2^2 \mathbb{1}_{S(u)} \leq\ & 2\mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - \hat{G}_{h_0(u)}(u)|_2^2 \mathbb{1}_{S(u)} + 2\mathbb{E}|\hat{G}_{h_0(u)}(u) - G_{h_0(u)}(u)|_2^2 \\ & +2|G_{h_0(u)}(u) - G(u)|_2^2. \qquad (36) \end{aligned}$$

Note that for $u \in (0,1)$,

$$\frac{1}{n}\sum_{t=1}^{n} K_h(t/n-u) = \int_0^1 K_h(v-u)dv + O((nh)^{-1}) = \int_{-u/h}^{(1-u)/h} K(x)dx + O((nh)^{-1}).$$

By Lemma D.6 and Lemma D.10 and since $\frac{1}{n}\sum_{t=1}^{n} K_h(t/n-u) \geq \frac{1}{2}$ for n large enough, we have

$$\begin{aligned}\mathbb{E}|\hat{G}_{h_0(u)}(u) - G_{h_0(u)}(u)|_2^2 &\leq 8\mathbb{E}|\hat{G}_{h_0(u)}(u) - \tilde{G}_{h_0(u)}(u)|_2^2 + 8\mathbb{E}|\tilde{G}_{h_0(u)}(u) - G_{h_0(u)}(u)|_2^2 \\ &\leq 8v^2(h_0(u)) + c \cdot n^{-1}. \qquad (37)\end{aligned}$$

By definition of $\hat{h}(u)$ and monotonicity of $\lambda(\cdot)$,

$$\mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - \hat{G}_{h_0(u)}(u)|_2^2 \mathbb{1}_{S(u)} \leq (C^{\#})^2 \mathbb{E}[\hat{v}^2(h_0(u),u)]\lambda(h_0(u))^2. \qquad (38)$$

We now discuss $\mathbb{E}[\hat{v}^2(h_0(u),u)]$. It holds that

$$\mathbb{E}[\hat{v}^2(h_0(u),u)\mathbb{1}_{\{\mathrm{tr}(\hat{\Sigma}_n(u))>2\mathrm{tr}(\Sigma(u))\}}] \leq \|\hat{v}^2(h_0(u),u)\|_2 \cdot \mathbb{P}\big(|\mathrm{tr}(\hat{\Sigma}_n(u)) - \mathrm{tr}(\Sigma(u))| > \mathrm{tr}(\Sigma(u))\big)^{1/2}. \qquad (39)$$

By assumption, $\mathbb{P}\big(|\mathrm{tr}(\hat{\Sigma}_n(u)) - \mathrm{tr}(\Sigma(u))| > \mathrm{tr}(\Sigma(u))\big) \leq c' n^{-2}$ with some constant $c' > 0$. Furthermore, $\|\mathrm{tr}(\hat{\Sigma}_n(u))\|_2 \leq d \||\hat{\Sigma}_n|_\infty\|_2 \leq d c_\Sigma$. Thus $\hat{v}^2(h,u) \leq c_H^{-1} d c_\Sigma$. Insertion into (39) yields that for some $c' > 0$,

$$\mathbb{E}[\hat{v}^2(h_0(u),u)\mathbb{1}_{\{\mathrm{tr}(\hat{\Sigma}_n(u))>2\mathrm{tr}(\Sigma(u))\}}] \leq c' n^{-1}.$$

We therefore have

$$\mathbb{E}\hat{v}^2(h_0(u),u) \leq 2v^2(h_0(u),u) + \mathbb{E}[\hat{v}^2(h_0(u),u)\mathbb{1}_{\{\mathrm{tr}(\hat{\Sigma}_n(u))>2\mathrm{tr}(\Sigma(u))\}}] \leq 2v^2(h_0(u),u) + c' n^{-1}. \qquad (40)$$

Using (38) and (40), we obtain

$$\mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - \hat{G}_{h_0(u)}(u)|_2^2 \mathbb{1}_{S(u)} \leq 3(C^{\#})^2 v^2(h_0(u),u)\lambda(h_0(u))^2. \qquad (41)$$

By definition of $h_0(u)$, we have

$$|G_{h_0(u)}(u) - G(u)|_2 \leq \frac{C^{\#}}{8} v(h_0(u),u)\lambda(h_0(u)). \qquad (42)$$

Inserting (37), (41) and (42) into (36), we obtain

$$\mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - G(u)|_2^2 \mathbb{1}_{S(u)} \leq \big[8 + (\frac{C^{\#}}{8})^2 + 3(C^{\#})^2\big] v^2(h_0(u),u)\lambda(h_0(u))^2 + c' \cdot n^{-1}. \quad (43)$$

Discussion of the second summand in (35): Let $H_n(h) := \{h' \in H_n : h' < h\}$. By definition of $h_0(u)$ and by monotonicity of $v(\cdot)$, $\lambda(\cdot)$, we obtain for $h' \leq h \leq h_0(u)$:

$$|G_{h'}(u) - G(u)|_2 \leq \frac{C^{\#}}{8} v(h_0(u),u)\lambda(h_0(u)) \leq \frac{C^{\#}}{8} v(h,u)\lambda(h). \qquad (44)$$

Decompose

$$S(u)^c = \bigcup_{h \in H_n(a \cdot h_0(u))} \bigcup_{h' \in H_n(h)} E(h,h',u), \qquad E(h,h',u) := \{|\hat{G}_h(u) - \hat{G}_{h'}(u)|_2 > C^{\#}\hat{v}(h',u)\lambda(h')\}.$$

Let

$$A_1 := \{\mathrm{tr}(\hat{\Sigma}_n(u)) \geq \frac{1}{2}\mathrm{tr}(\Sigma(u))\}.$$

Note that by Assumption 4.5,

$$\mathbb{P}(A_1^c) \leq \mathbb{P}\big(|\mathrm{tr}(\hat{\Sigma}_n(u)) - \mathrm{tr}(\Sigma(u))| \geq \frac{1}{2}\mathrm{tr}(\Sigma(u))\big) \leq n^{-2}. \tag{45}$$

Define

$$\begin{aligned}
N(h,h',u) &:= (\hat{G}_h(u) - G_h(u)) - (\hat{G}_{h'}(u) - G_{h'}(u)),\\
\tilde{N}(h,h',u) &:= (\tilde{G}_h(u) - G_h(u) - (\tilde{G}_{h'}(u) - G_{h'}(u)),\\
N(h,u) &:= \hat{G}_h(u) - G_h(u),\\
\tilde{N}(h,u) &:= \tilde{G}_h(u) - G_h(u).
\end{aligned}$$

We have

$$\begin{aligned}
E(h,h',u)\cap A_1 &\subset \{|\hat{G}_h(u) - \hat{G}_{h'}(u)|_2 > \frac{C^{\#}}{2}v(h',u)\lambda(h')\}\\
&\subset \{\frac{2C^{\#}}{8}v(h',u)\lambda(h') + |N(h,h',u)|_2 > \frac{C^{\#}}{2}v(h',u)\lambda(h')\}\\
&\subset \{|N(h,h',u)|_2 > C^{\#}(\frac{1}{2} - \frac{2}{8})v(h',u)\lambda(h')\} =: E_0(h,h',u). \quad (46)
\end{aligned}$$

By (45) and the Cauchy-Schwarz inequality, we obtain that there exists some constant $c' > 0$ such that

$$\begin{aligned}
\mathbb{E}|\hat{G}_h(u) - G(u)|_2^2\mathbb{1}_{E(h,h',u)} &\leq \mathbb{E}|\hat{G}_h(u) - G(u)|_2^2\mathbb{1}_{E_0(h,h',u)} + \mathbb{E}[|\hat{G}_h(u) - G(u)|_2^4]^{1/2}\mathbb{P}(A_1^c)^{1/2}\\
&\leq \mathbb{E}|\hat{G}_h(u) - G(u)|_2^2\mathbb{1}_{E_0(h,h',u)} + c'n^{-1}. \quad (47)
\end{aligned}$$

We conclude with (46), (47) and (44) that:

$$\begin{aligned}
&\mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - G(u)|_2^2\mathbb{1}_{S(u)^c}\\
\leq\ & \sum_{h\in H_n(ah_0(u))}\sum_{h'\in H_n(h)} \mathbb{E}|\hat{G}_h(u) - G(u)|_2^2\mathbb{1}_{E(h,h',u)}\\
\leq\ & \sum_{h\in H_n(ah_0(u))}\sum_{h'\in H_n(h)} \mathbb{E}|\hat{G}_h(u) - G(u)|_2^2\mathbb{1}_{E_0(h,h',u)} + c'\log(n)^2n^{-1}\\
\leq\ & \sum_{h\in H_n(ah_0(u))}\sum_{h'\in H_n(h)} \mathbb{E}\big[\big(|N(h,u)|_2 + \frac{C^{\#}}{8}v(h,u)\lambda(h_0(u))\big)^2\big]\mathbb{1}_{E_0(h,h',u)} + c'\log(n)^2n^{-1}.
\end{aligned}$$

With Lemma D.6, we can replace $|N(h,u)|_2$ with $|\tilde{N}(h,u)|_2$ with error $O(\log(n)^2n^{-1})$ due to

$$\mathbb{E}\big[(|N(h,u)|_2 - |\tilde{N}(h,u)|_2)^2\big] \leq |\ \|N(h,u) - \tilde{N}(h,u)\|_2\ |_2^2 \leq c'n^{-2}.$$

Similarly, the set $A(h,h') := \{|\ |N(h,h',u)|_2 - |\tilde{N}(h,h',u)|_2| \leq \frac{C^{\#}}{2}(\frac{1}{2}-\frac{2}{8})v(h_0(u),u)\lambda(h_0(u))\}$ has the property

$$\mathbb{P}(A(h,h')) \leq \frac{c'n^{-4}}{(v(h_0(u),u)\lambda(h_0(u)))^4} = O(n^{-2}),$$

allowing to replace $|N(h,h',u)|_2$ by $|\tilde N(h,h',u)|_2$ in $E_0(h,h',u)$ with replacement error $O(\log(n)^2 n^{-1})$. Together with $v(h,h',u)\le v(h',u)$ for $h'\le h$, we have shown that

$$\tilde E_0(h,h',u) := \{|\tilde N(h,h',u)|_2 > \frac{C^{\#}}{2}(\frac{1}{2}-\frac{2}{8})v(h,h',u)\lambda(h')\},$$

fulfills

$$\mathbb{E}|\hat G_{\hat h(u)}(u)-G(u)|_2^2\mathbb{1}_{S(u)^c} \le \sum_{h\in H_n(ah_0(u))}\sum_{h'\in H_n(h)}\mathbb{E}\big[|\tilde N(h,u)|_2+\frac{C^{\#}}{8}v(h,u)\lambda(h_0(u))\big]^2\mathbb{1}_{\tilde E_0(h,h',u)} + c'\log(n)^2 n^{-1}. \quad (48)$$

Put $x := \frac{C^{\#}}{8}\lambda(h_0(u))$ and $D^{\#} := \frac{C^{\#}}{2}(\frac12-\frac28)$. We now discuss the summands in (48). It holds that

$$\begin{aligned}
&\mathbb{E}\big[x+v(h,u)^{-1}|\tilde N(h,u)|_2\big]^2\mathbb{1}_{\{|\tilde N(h,h',u)|_2>D^{\#}v(h,h',u)\lambda(h')\}}\\
=\;& 2\int_0^\infty z\cdot\mathbb{P}(v(h,u)^{-1}|\tilde N(h,u)|_2>z-x, |\tilde N(h,h',u)|_2>D^{\#}v(h,h',u)\lambda(h'))\,\mathrm{d}z\\
\le\;& 2\int_0^{x+\sqrt{2^7\log(n)}} z\cdot\mathbb{P}(|\tilde N(h,h',u)|_2>D^{\#}v(h,h',u)\lambda(h'))\,\mathrm{d}z\\
&\quad +2\int_{x+\sqrt{2^7\log(n)}}^\infty z\cdot\mathbb{P}(v(h)^{-1}|\tilde N(h,u)|_2>z-x)\,\mathrm{d}z\\
=\;& (x+\sqrt{2^7\log(n)})^2\mathbb{P}(|\tilde N(h,h',u)|_2>D^{\#}v(h,h',u)\lambda(h'))\\
&\quad +2\int_{\sqrt{2^7\log(n)}}^\infty (x+y)\mathbb{P}(|\tilde N(h,u)|_2>v(h,u)y)\,\mathrm{d}y. \qquad (49)
\end{aligned}$$

Discussion of the first summand in (49). Put $v_j^2(h,u) := \frac{1}{nh}\int K(x)^2\,\mathrm{d}x\cdot\Sigma(u)_{jj}$ and $v_j^2(h,h',u) := \frac1n\int\{K_h(x)-K_{h'}(x)\}^2\,\mathrm{d}x\cdot\Sigma(u)_{jj}$. Then, with Lemma D.5 and $a_n = \log(n)^{1/\tau_2} = \log(n)^{\alpha M}$,

$$\begin{aligned}
&\mathbb{P}(|\tilde N(h,h',u)|_2>D^{\#}v(h,h',u)\lambda(h'))\\
\le\;& \mathbb{P}\Big(\sum_{j=1}^d \tilde N(h,h',u)_j^2 > (D^{\#})^2\sum_{j=1}^d v_j^2(h,h',u)\lambda(h')^2\Big)\\
\le\;& \sum_{j=1}^d \mathbb{P}\big((nh')|\tilde N(h,h',u)_j| > D^{\#}(nh')v_j(h,h',u)\lambda(h')\big)\\
\le\;& d\cdot\sup_{j=1,\dots,d}\Big\{2\exp\Big(-\frac{(D^{\#})^2\lambda(h')^2}{32+c_4(\frac{a_n}{(nh')v_j(h,h',u)})^{1/3}(D^{\#}\lambda(h'))^{5/3}}\Big)+c_5\Big(\frac{n^{-1}}{D^{\#}(nh')v_j(h,h',u)\lambda(h')}\Big)^2\Big\}.
\end{aligned}$$

By Lemma D.9, $(nh')v_j(h,h',u)$ is lower bounded by $cL(nh')^{1/2}$ with some $c_L>0$ if $h,h'\in H_n$. Thus,

$$c_4(\frac{a_n}{(nh')v_j(h,h',u)})^{1/3}(D^{\#}\lambda(h'))^{5/3}\le 32$$

is satisfied for

$$h'\ge c_L^2\Big(\frac{32}{c_4(D^{\#})^{5/3}}\Big)^6\frac{a_n^2}{n}\lambda(h')^{10},$$

that is, for $h'\ge c'\log(n)^{2\alpha M+5}$ with some $c'>0$ large enough which is true due to $h'\in H_n$.

We obtain that for $h \in H_n$, $h' \in H_n(h)$ with $D^{\#} = 8$:

$$\mathbb{P}(|\tilde{N}(h,h',u)|_2 > D^{\#} v(h,h',u)\lambda(h')) \leq 2d \cdot h' + c_5\Big(\frac{n^{-1}}{D^{\#} c_L (nh')^{1/2}\lambda(h')}\Big)^2. \tag{50}$$

Note that the last summand in (50) is upper bounded by a constant times n^{-1}.

Discussion of the second summand in (49). We have by Lemma D.4 with $a_n = \log(n)^{1/\tau_2} = \log(n)^{\alpha M}$ that

$$\begin{aligned} &\mathbb{P}(|\tilde{N}(h,u)|_2 > v(h,u)y) \\ \leq\ & \sum_{j=1}^{d} \mathbb{P}((nh)|\tilde{N}(h,u)_j| > (nh)v_j(h,u)y) \\ \leq\ & d \sup_{j=1,\dots,d} \Big\{ 2\exp\Big(-\frac{y^2}{32 + c_4(\frac{a_n}{(nh)v_j(h,u)})^{1/3} y^{5/3}}\Big) + c_5\Big(\frac{n^{-1}}{(nh)v_j(h,u)y}\Big)^2\Big\}. \end{aligned}$$

We have with $c_n := \frac{a_n}{(nh)v_j(h,u)}$:

$$\exp\Big(-\frac{y^2}{32 + c_4(\frac{a_n}{(nh)v_j(h,u)})^{1/3} y^{5/3}}\Big) \leq \begin{cases} \exp(-\frac{y^2}{64}), & y \leq (\frac{32}{c_4})^{3/5} c_n^{-1/5} =: d_n \\ \exp\big(-\frac{1}{2c_4}(\frac{y}{c_n})^{1/3}\big), & y > d_n \end{cases}.$$

Thus

$$\begin{aligned} &\int_{\sqrt{2^7 \log(n)}}^{\infty} (x+y)\mathbb{P}(|\tilde{N}(h,u)|_2 > v(h,u)y)\ \mathrm{d}y \\ \leq\ & 4d \sup_j \Big[\int_{\sqrt{2^7\log(n)}}^{d_n} (x+y)\exp(-\frac{y^2}{2^6})\ \mathrm{d}y + \int_{d_n}^{\infty} (x+y)\exp\Big(-\frac{1}{2c_4}(\frac{y}{c_n})^{1/3}\Big)\ \mathrm{d}y \\ &\qquad + \frac{c_5}{\sigma_K^2 nh \cdot \inf_{j=1,\dots,d}\Sigma(u)_{jj}} n^{-2}\Big]. \end{aligned} \tag{51}$$

We now discuss the first two summands in (51). We have

$$\int_{\sqrt{2^7\log(n)}}^{d_n} (x+y)\exp(-\frac{y^2}{2^6})\ \mathrm{d}y \leq [x + d_n]n^{-2},$$

and, with some constant $\tilde{c}_4 > 0$ only depending on c_4,

$$\begin{aligned} &\int_{d_n}^{\infty} (x+y)\exp(-\frac{1}{2c_4}(\frac{y}{c_n})^{1/3})\ \mathrm{d}y \\ =\ & c_n \int_{d_n/c_n}^{\infty} (x + c_n z)\exp(-\frac{1}{2c_4} z^{1/3})\ \mathrm{d}z \\ \leq\ & \tilde{c}_4 \cdot c_n \cdot \exp\Big(-\frac{1}{2c_4}\Big(\frac{d_n}{c_n}\Big)^{1/3}\Big) \cdot \Big[x \cdot ((\frac{d_n}{c_n})^{2/3} + 1)) + c_n\big((\frac{d_n}{c_n})^{5/3} + 1\big)\Big]. \end{aligned}$$

Here,

$$\frac{1}{2c_4}(\frac{d_n}{c_n})^{1/3} = \frac{1}{2}(\frac{16}{c_4^6})^{1/5} c_n^{-2/5} = \frac{1}{2}\Big[\frac{16\int K(x)^2\ \mathrm{d}x \cdot \Sigma(u)_{jj}}{c_4^6} \cdot \frac{nh}{a_n^2}\Big]^{1/5}. \tag{52}$$

We conclude that (52) is $\geq 2\log(n)$ if $h \geq c' \cdot \log(n)^{5+\frac{2}{\tau_2}} \cdot n^{-1}$ for $c' > 0$ large enough (which is fulfilled due to $h \in H_n$). Clearly, (52) is $\leq O(n^{1/5})$. Summarizing these results into (51), we obtain for all $h \in H_n$ that

$$\int_{\sqrt{2^7\log(n)}}^{\infty} (x+y)\mathbb{P}(|\tilde{N}(h,u)|_2 > v(h,u)y)\ \mathrm{d}y \leq c' n^{-1} \tag{53}$$

with some constant $c' > 0$.

Inserting (50) and (53) into (49) and (48), we obtain with $\sum_{h' \in H_n(h)} h' \le \frac{h}{1-a}$:

$$\begin{aligned}
&\mathbb{E}|\hat{G}_{\hat{h}(u)}(u) - G(u)|_2^2 \mathbb{1}_{S(u)^c} \\
\le\ & \frac{2d}{C_H} \cdot \sum_{h \in H_n(ah_0(u))} \sum_{h' \in H_n(h)} v^2(h,u) \cdot h' + c' \log(n)^2 n^{-1} \\
\le\ & \frac{\int K^2(x)\,\mathrm{d}x \cdot \mathrm{tr}(\Sigma(u))}{1-a} \cdot \frac{2d}{C_H} \cdot \sum_{h \in H_n(ah_0(u))} n^{-1} + c' \log(n)^2 n^{-1} \\
\le\ & c' \cdot \log(n)^2 n^{-1}. \qquad (54)
\end{aligned}$$

By (43) and (54), the result follows. □

Proof of Lemma 4.10. Define $C(u,k) := \mathrm{Cov}(g(\tilde{Y}_0(u)), g(\tilde{Y}_k(u)))$. We have

$$|\hat{\Sigma}_n(u) - \Sigma(u)|_\infty \le \sum_{k=-r_n}^{r_n} |\hat{c}_\eta^g(u,k) - C(u,k)|_\infty + \sum_{|k|>r_n} |C(u,k)|_\infty. \qquad (55)$$

We start with investigating the second summand of (55). Let $i,j \in \{1,...,d\}$. Then by Lemma D.7, with some $\tilde{\rho} \in (0,1)$, $c' > 0$,

$$\begin{aligned}
|C(u,k)_{ij}| &\le |\mathrm{Cov}(g_i(\tilde{Y}_0(u)), g_j(\tilde{Y}_k(u)))| \\
&\le \sum_{l_1,l_2=0}^{\infty} |\mathrm{Cov}(P_{-l_1} g_i(\tilde{Y}_0(u)), P_{k-l_2} g_j(\tilde{Y}_k(u)))| \\
&= \sum_{l=0}^{\infty} |\mathrm{Cov}(P_{-l} g_i(\tilde{Y}_0(u)), P_{-l} g_j(\tilde{Y}_k(u)))| \\
&\le \sum_{l=0}^{\infty} \|P_{-l} g_i(\tilde{Y}_0(u))\|_2 \|P_{-l} g_j(\tilde{Y}_k(u))\|_2 \\
&\le \sum_{l=0}^{\infty} \delta_2^{g_i(\tilde{Y}(u))}(l) \cdot \delta_2^{g_j(\tilde{Y}(u))}(k+l) \\
&\le c' \sum_{l=0}^{\infty} \tilde{\rho}^l \cdot \tilde{\rho}^{k+l} \le \frac{c'}{1-\tilde{\rho}^2} \cdot \tilde{\rho}^k.
\end{aligned}$$

For $r_n \ge -\frac{\log(n)}{\log(\tilde{\rho})}$, we therefore have

$$\sum_{|k|>r_n} |C(u,k)|_\infty \le \frac{2c'}{1-\tilde{\rho}^2} \cdot \sum_{k=r_n+1}^{\infty} \tilde{\rho}^k = \frac{2c'}{(1-\tilde{\rho}^2)(1-\tilde{\rho})} \cdot \tilde{\rho}^{r_n} \le \frac{2c'}{(1-\tilde{\rho}^2)(1-\tilde{\rho})} \cdot n^{-1}. \qquad (56)$$

We now investigate the first summand of (55). Abbreviate $w_{n,\eta}(u) = \frac{1}{n}\sum_{t=1}^n K_\eta(t/n - u)$. Note that with $p(y) = g(y_1, y_2, ...) g(y_{1-k}, y_{2-k}, ...)'$,

$$\hat{c}_\eta^g(u,k) = \frac{\frac{1}{n}\sum_{t=1}^n K_\eta(t/n-u) p(Y_{t,n})}{w_{n,\eta}(u)} - \hat{G}_\eta(u)\hat{G}_\eta(u)',$$

thus

$$\begin{aligned}
|\hat{c}_\eta^g(u,k) - C(u,k)|_\infty \le\ & |\frac{\frac{1}{n}\sum_{t=1}^n K_\eta(t/n-u) p(Y_{t,n})}{w_{n,\eta}(u)} - \mathbb{E}p(\tilde{Y}_0(u))|_\infty \\
& + |\hat{G}_\eta(u)\hat{G}_\eta(u)' - G(u)G(u)'|_\infty. \qquad (57)
\end{aligned}$$

In the following, we restrict ourselves to the analysis of $\hat{G}_\eta(u) - G(u)$. Due to the similar structure, the results for $\frac{\frac{1}{n}\sum_{t=1}^n K_\eta(t/n-u)p(Y_{t,n})}{w_{n,\eta}(u)}$ are completely similar. By Lemma D.6, we have for any $r \geq 2$ that

$$\|\hat{G}_\eta(u) - \tilde{G}_\eta(u)\|_r \leq c' n^{-1}. \tag{58}$$

As in the proof of Theorem 4.6, eq. (33), we conclude that with some constant $c' > 0$ and C_H small enough that

$$|\mathbb{E}\tilde{G}_\eta(u) - G(u)|_\infty \leq c' n^{-1} + 2\eta^2 \mu_K |\partial_u^2 G(u)|_\infty. \tag{59}$$

Let $\phi(v) = \frac{K_\eta(v-u)}{w_{n,\eta}(u)}$. Then with some constant $c' > 0$,

$$(\frac{1}{n}\sum_{t=1}^n \phi(t/n)^2)^{1/2} \leq c' \eta^{-1/2}.$$

By Lemma D.2(i) applied with the above ϕ, for any $r \geq 2$ there exists a constant $c' = c'(r) > 0$ such that for any $j \in \{1, ..., d\}$,

$$\|\tilde{G}_\eta(u)_j - \mathbb{E}\tilde{G}_\eta(u)_j\|_r \leq c'(r)(n\eta)^{-1/2}. \tag{60}$$

From (58), (59) and (60) we conclude with Markov's inequality that for any $r \geq 2$ and some constant $c' = c'(r) > 0$ that

$$\|\hat{G}_\eta(u) - G(u)\|_r \;\leq\; c'(r)\Big(n^{-1} + (n\eta)^{-1/2} + \eta^2\Big).$$

Insertion into (57) (and application of the same theory to p instead of g) yields that for any $r \geq 2$, there exists a constant $c'(r) > 0$ such that

$$\|\,|\hat{c}_\eta^g(u,k) - C(u,k)|_\infty\|_r \leq d^2 \cdot \sup_{i,j=1,\ldots,d} \|\hat{c}_\eta^g(u,k)_{ij} - C(u,k)_{ij}\|_r \leq c'(r)\big(n^{-1} + (n\eta)^{-1/2} + \eta^2\big).$$

We obtain

$$\Big\| \sum_{k=-r_n}^{r_n} |\hat{c}_\eta^g(u,k) - C(u,k)|_\infty \Big\|_r \leq 2c'(r)\big(n^{-1} + (n\eta)^{-1/2} + \eta^2\big).$$

Combination with (55) and (56) yields

$$\|\,|\hat{\Sigma}_n(u) - \Sigma_g(u)|_\infty\|_r \leq 2c'(r)\big(n^{-1} + (n\eta)^{-1/2} + \eta^2\big) + \frac{2c'}{(1-\tilde{\rho}^2)(1-\tilde{\rho})} \cdot n^{-1}.$$

Under the given upper bound on r_n and the conditions on η, we now can choose r large enough such that $\|\,|\hat{\Sigma}_n(u) - \Sigma_g(u)|_\infty\|_r^r \leq n^{-2}$. This proves Assumption 4.5(i). Using the above inequality for $r = 2$ proves Assumption 4.5(ii). □

D Technical lemmata

For the next sections, we define for $p \in \mathcal{H}(M, \chi, C)$, $K \in \mathcal{K}$ and $u \in [0,1]$ the quantities

$$\begin{aligned}
\mathbb{G}_{n,h,u}(p) &:= \frac{1}{n}\sum_{t=1}^n K_h(t/n-u) \cdot \big\{p(Y_{t,n}) - \mathbb{E}p(Y_{t,n})\big\}, \\
\tilde{\mathbb{G}}_{n,h,u}(p) &:= \frac{1}{n}\sum_{t=1}^n K_h(t/n-u) \cdot \big\{p(\tilde{Y}_t(t/n)) - \mathbb{E}p(\tilde{Y}_t(t/n))\big\}, \\
\tilde{\mathbb{G}}^\circ_{n,h,u}(p) &:= \frac{1}{n}\sum_{t=1}^n K_h(t/n-u) \cdot \big\{p(\tilde{Y}_t(u)) - \mathbb{E}p(\tilde{Y}_t(u))\big\}.
\end{aligned}$$

Furthermore, let

$$\sigma_p(u) := \sum_{k\in\mathbb{Z}} \mathrm{Cov}(p(\tilde{Y}_0(u)), p(\tilde{Y}_k(u))).$$

D.1 A Bernstein-type inequality and Proof of Theorem 4.8

In Lemma D.3 below, we derive a Bernstein-type inequality for locally stationary processes of the form

$$W_n = \sum_{t=1}^{n} \phi(t/n)\{g(\tilde{Y}_t(t/n)) - \mathbb{E}g(\tilde{Y}_t(t/n))\}, \tag{61}$$

where $\phi : [0,1] \to \mathbb{R}$ is a measurable function and $g \in \mathcal{H}(M,\chi,C)$. To do so, we first use Assumption 4.3 to derive bounds exponential moments of $g(\tilde{Y}_t(u))$ in Lemma D.1. These results are used to exclude events where $g(\tilde{Y}_t(u))$ is large during the proof of the Bernstein-type inequality in Lemma D.3.

We then use Lemma D.3 to derive Bernstein-type inequalities for $\tilde{\mathbb{G}}_{n,h,u}(p)$ and $\tilde{\mathbb{G}}_{n,h,u}(p) - \tilde{\mathbb{G}}_{n,h',u}(p)$ in Lemma D.4 and Lemma D.5. Note that Theorem 4.8 is a direct implication of Lemma D.4 below.

Lemma D.1. *Assume that $g \in \mathcal{H}(M,\chi,C)$. Suppose that Assumption 4.3 holds. Put $\tau_2 = (\alpha M)^{-1}$. Then there exist constants $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3 > 0$ only depending on M,χ,C,D such that for $q \geq 2$, $u \in (0,1)$:*

$$\begin{aligned} \big\|g(\tilde{Y}_t(u))\big\|_q &\leq \tilde{c}_1 N_\alpha(qM)^M, \\ \mathbb{E}\exp\Big[\frac{1}{2}\Big(\frac{|g(\tilde{Y}_t(u))|}{\tilde{c}_1}\Big)^{\tau_2}\Big] &\leq \tilde{c}_2. \end{aligned}$$

If additionally, $\chi_j = O(\rho^j)$ with ρ from Assumption 4.3, then there exists $\tilde{\rho} \in (0,1)$ such that

$$\delta_q^{g(\tilde{Y}(u))} \leq \tilde{c}_3 \cdot \tilde{\rho}^t \cdot N_\alpha(qM)^M.$$

Proof of Lemma D.1: (i) It holds that

$$\begin{aligned} \|g(\tilde{Y}_t(u)) - g(0)\|_q &\leq C\sum_{j=1}^{\infty} \chi_j \|\tilde{X}_{t-j+1}(u)\|_{qM} \cdot \big(1 + |\chi|_1^{M-1}\|\tilde{X}_t(u)\|_{qM}^{M-1}\big) \\ &\leq C|\chi|_1 D N_\alpha(qM) \cdot \big(1 + |\chi|_1^{M-1} D^{M-1} N_\alpha(qM)^{M-1}\big). \end{aligned}$$

Since $|g(0)| \leq C$, we obtain $\|g(\tilde{Y}_t(u))\|_q \leq \tilde{c}_1 N_\alpha(qM)^M$ with some $\tilde{c}_1$ only depending on M,χ,C,D.

(ii) Define $\lambda = (2\tilde{c}_1^{\tau_2})^{-1}$. By a series expansion of exp, we have

$$\mathbb{E}\exp\big(\lambda|g(\tilde{Y}_t(u))|^{\tau_2}\big) = \sum_{q=0}^{\infty} \frac{\lambda^q \|g(\tilde{Y}_t(u))\|_{\tau_2 q}^{\tau_2 q}}{q!}.$$

If $\tau_2 q \geq 2$, we have

$$\|g(\tilde{Y}_t(u))\|_{\tau_2 q}^{\tau_2 q} \leq \tilde{c}_1^{\tau_2 q} \cdot \Gamma(\alpha q \tau_2 M + 2) = \tilde{c}_1^{\tau_2 q}\Gamma(q+2).$$

This shows $\sum_{\tau_2 q \geq 2} \frac{\lambda^q \|g(\tilde{Y}_t(u))\|_{\tau_2 q}^{\tau_2 q}}{q!} \leq \sum_{\tau_2 q \geq 0}^{\infty} (\lambda \tilde{c}_1^{\tau_2})^q \cdot \frac{\Gamma(q+2)}{\Gamma(q+1)} = \sum_{\tau_2 q \geq 2} \frac{q+1}{2^q} \leq 4$.
In the case $\tau_2 q < 2$, we have

$$\|g(\tilde{Y}_t(u))\|_{\tau_2 q}^{\tau_2 q} \leq \|g(\tilde{Y}_t(u))\|_2^{\tau_2 q} \leq \tilde{c}_1^{\tau_2 q}\Gamma(2\alpha M + 2)^{\tau_2 q/2} \leq \tilde{c}_1^{\tau_2 q}\Gamma(2\alpha M + 2).$$

This shows $\sum_{\tau_2 q<2}\frac{\lambda^q\|g(\tilde{Y}_t(u))\|_{\tau_2 q}^{\tau_2 q}}{q!} \le \Gamma(2\alpha M+2)\sum_{q=0}^{\infty}\frac{2^{-q}}{q!} = \exp(2^{-1})\Gamma(2\alpha M+2)$. The result is obtained with $\tilde{c}_2 := 4+\exp(2^{-1})\Gamma(2\alpha M+2)$.

The proof of the statement about the functional dependence measure is similar to the proof of Lemma D.7. By Hölder's inequality and Assumption 4.3, we have

$$\begin{aligned}
\delta_q^{g(\tilde{Y}(u))}(t) &= \big\|g(\tilde{Y}_t(u)) - g(\tilde{Y}_t^*(u))\big\|_q \\
&\le C\big(1+2|\chi|_1^{M-1}D^{M-1}\cdot N_\alpha(qM)^{M-1}\big)\textstyle\sum_{j=1}^t \chi_j\|\tilde{X}_{t-j+1}(u)-\tilde{X}_{t-j+1}^*(u)\|_{qM} \\
&\le C\big(1+2|\chi|_1^{M-1}D^{M-1}\cdot N_\alpha(qM)^{M-1}\big)\textstyle\sum_{j=1}^t \chi_j\delta_{qM}^{\tilde{X}(u)}(t-j+1).
\end{aligned}$$

Since $\chi_j = O(\rho^j)$ and $\delta_{qM}^{\tilde{X}(u)}(t-j+1) \le DN_\alpha(qM)\cdot\rho^{t-j+1}$, we obtain that there exists some constant $\tilde{c}'>0$ depending on M,χ,C,D such that

$$\delta_q^{g(\tilde{Y}(u))}(t) \le \tilde{c}'\cdot N_\alpha(qM)^M\cdot\sum_{j=1}^t \rho^j\cdot\rho^{t-j+1} = \tilde{c}'\cdot N_\alpha(qM)^M\cdot t\rho^{t+1},$$

which yields the assertion. □

The following exponential inequality is used in the proof of Lemma 4.10.

Lemma D.2 (Exponential inequality). *Assume that $\phi:[0,1]\to\mathbb{R}$ is some measurable function, and $g\in\mathcal{H}(M,\chi,C)$. Suppose that Assumption 4.3 holds. Define*

$$F_n(\phi,g) := \frac{1}{n}\sum_{t=1}^n \phi(t/n)\cdot\{g(\tilde{Y}_t(t/n)) - \mathbb{E}g(\tilde{Y}_t(t/n))\}.$$

Put $\tau=\tau(\alpha,M) := (\frac{1}{2}+\alpha M)^{-1}$. Then there exist constants $c_1,c_2>0$ only depending on M,χ,C,D such that

(i)

$$\|F_n(\phi,g)\|_q \le c_1(q-1)^{1/2}n^{-1/2}\Big(\frac{1}{n}\sum_{t=1}^n\phi(t/n)^2\Big)^{1/2}\cdot N_\alpha(qM)^M,$$

(ii)

$$\mathbb{P}(|F_n(\phi,g)|>\gamma) \le c_2\exp\Big[-\frac{1}{4e}\Big(\frac{\sqrt{n}\cdot\gamma}{c_1(\frac{1}{n}\sum_{t=1}^n\phi(t/n)^2)^{1/2}}\Big)^{\tau}\Big].$$

Proof of Lemma D.2. (i) Let $\delta(k) := D\rho^k$. By Hölder's inequality, we have with some constant $\tilde{c}$ only dependent on M,χ,C,D:

$$\begin{aligned}
&\big\|g(\tilde{Y}_t(u)) - g(\tilde{Y}_t^*(u))\big\|_q \\
\le\ & C\textstyle\sum_{j=1}^t\chi_j\|\tilde{X}_{t-j+1}(u)-\tilde{X}_{t-j+1}^*(u)\|_{qM}\cdot\big(1+2|\chi|_1^{M-1}D^{M-1}\cdot N_\alpha(qM)^{M-1}\big) \\
\le\ & \tilde{c}\textstyle\sum_{j=1}^t\chi_j\delta(t-j+1)\cdot N_\alpha(qM)^M. \qquad (62)
\end{aligned}$$

Let $\xi(t) := \sum_{j=1}^t\chi_j\cdot\delta(t-j+1)$. Obviously, $\sum_{t=1}^\infty\xi(t) = \sum_{j=1}^\infty\chi_j\sum_{t=j}^\infty\delta(t-j+1) < |\chi|_1|\delta(\cdot)|_1$. We have shown that the dependence measure fulfills $\delta_q^{g(\tilde{Y}(u))}(k)\le\tilde{c}\cdot\xi(k)\cdot N_\alpha(qM)^M$ and is absolutely summable.

By Theorem 2.1 from Rio (2009) for $q > 2$ (and for $q = 2$ directly by calculating the variance of the following term), we have

$$\begin{aligned}
\|F_n(\phi,g)\|_q &\leq \Big\|\frac{1}{n}\sum_{t=1}^{n}\phi(t/n)\{g(\tilde{Y}_t(t/n)) - \mathbb{E}g(\tilde{Y}_t(t/n))\}\Big\|_q \\
&\leq \frac{1}{n}\sum_{k=0}^{\infty}\Big\|\sum_{t=1}^{n}\phi(t/n)P_{t-k}g(\tilde{Y}_t(t/n))\Big\|_q \\
&\leq \frac{1}{n}\sum_{k=0}^{\infty}(q-1)^{1/2}\Big|\Big(\sum_{t=1}^{n}\phi(t/n)^2\|P_{t-k}g(\tilde{Y}_t(t/n))\|_q^2\Big)^{1/2}\Big|_2 \\
&\leq (q-1)^{1/2}n^{-1/2}\Big(\frac{1}{n}\sum_{t=1}^{n}\phi(t/n)^2\Big)^{1/2}\cdot\tilde{c}\sum_{k=0}^{\infty}\xi(k)\cdot N_\alpha(qM)^M.
\end{aligned}$$

(ii) Define $Z_n := \tilde{c}n^{-1/2}\big(\frac{1}{n}\sum_{t=1}^{n}\phi(t/n)^2\big)^{1/2}\cdot\sum_{k=0}^{\infty}\xi(k)$. By Stirling's formula, we have for all $x \geq 1$:

$$\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x} \leq \Gamma(x) \leq e^{1/12}\cdot\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}.$$

By Markov's inequality, we have for $\gamma, \lambda > 0$:

$$\mathbb{P}(|F_n(\phi,g)| \geq \gamma) \leq e^{-\lambda\gamma^\tau}\mathbb{E}[e^{\lambda|F_n(\phi,g)|^\tau}] = e^{-\lambda\gamma^\tau}\sum_{q=0}^{\infty}\frac{\lambda^q\||F_n(\phi,g)|_2\|_{\tau q}^{\tau q}}{q!}.$$

In the case $\tau q \geq 2$, we have

$$\frac{\lambda^q\|F_n(\phi,g)\|_{\tau q}^{\tau q}}{q!} \leq \frac{\lambda^q}{\Gamma(q+1)}(\tau q)^{\frac{\tau q}{2}}D(u)^{\tau q}\cdot\Gamma(\alpha M\tau q+2).$$

Note that $\alpha M\tau \leq 1$ and $\tau(\alpha M + \frac{1}{2}) = 1$, thus

$$\begin{aligned}
q^{\frac{\tau q}{2}}\frac{\Gamma(\alpha M\tau q+2)}{\Gamma(q+1)} &\leq (q+2)^{\frac{\tau q}{2}}\cdot\frac{(\alpha M\tau q+2)^{\alpha M\tau q+\frac{3}{2}}e^{-(\alpha M\tau q+2)}e^{1/12}}{(q+1)^{q+\frac{1}{2}}e^{-(q+1)}} \\
&= e^{1/12}(q+2)\cdot\Big(\frac{q+2}{q+1}\Big)^{q+\frac{1}{2}}e^{-1}e^{q(1-\alpha M\tau)} \\
&\leq e^{1/12}(q+2)e^q.
\end{aligned}$$

Define $\lambda := (4e)^{-1}Z_n^{-\tau}$. Since $\tau \leq 2$, it holds that $\tau^{\tau/2} \leq 2$. Thus

$$\sum_{q\geq 2/\tau}\frac{\lambda^q\|F_n(\phi,g)\|_{\tau q}^{\tau q}}{q!} \leq e^{1/12}\cdot\sum_{q\geq 2/\tau}(q+2)(\lambda\cdot 2eZ_n^\tau)^q \leq e^{1/12}\sum_{q\geq 2/\tau}\frac{q+2}{2^q} \leq 4e^{1/12}.$$

In the case $\tau q < 2$, we have

$$\begin{aligned}
\frac{\lambda^q\|F_n(\phi,g)\|_{\tau q}^{\tau q}}{q!} &\leq \frac{\lambda^q\||E_{n,b}(g,u)|_2\|_2^{\tau q}}{q!} \leq \frac{\lambda^q}{q!}Z_n^{\tau q}\cdot\Gamma(2\alpha M+2)^{\frac{\tau q}{2}} \\
&\leq \frac{(4e)^{-q}}{q!}\cdot\Gamma(2\alpha M+2),
\end{aligned}$$

thus $\sum_{q<2/\tau}\frac{\lambda^q\|F_n(\phi,g)\|_{\tau q}^{\tau q}}{q!} \leq \exp((4e)^{-1})\Gamma(2\alpha M+2)$. So the result is obtained with $c_2 := 4e^{1/2} + \exp((4e)^{-1})\Gamma(2\alpha M+2)$ and $c_1 = \tilde{c}\sum_{k=0}^{\infty}\xi(k) = \tilde{c}|\chi|_1|\delta(\cdot)|_1$. □

Lemma D.3 (Bernstein inequality). *Assume that $g \in \mathcal{H}(M,\chi,C)$ and that Assumption 4.3 holds. Suppose that $\chi_j = O(\rho^j)$ with ρ from Assumption 4.3. For W_n from (61), assume that $s_n := \#\{t \in \{1,...,n\} : \phi(t/n) \neq 0\}$ fulfills with some constant $C_1 > 0$:*

$$Var(W_n) \geq C_1 \cdot s_n. \tag{63}$$

Then there exist some constants $c_4, c_5, c_6 > 0$ only dependent on $M, \chi, C, D, \rho, |\phi|_\infty, C_1$ such that for $s_n \geq c_6$,

$$\mathbb{P}\Big(\Big|\sum_{t=1}^{n}\phi(t/n)\{g(\tilde{Y}_t(t/n)) - \mathbb{E}g(\tilde{Y}_t(t/n))\}\Big| > \gamma\Big) \leq 2\exp\Big(-\frac{\gamma^2}{16\,Var(W_n) + c_4 a_n^{1/3}\gamma^{5/3}}\Big) + c_5\frac{n^{-2}}{\gamma^2},$$

with $a_n := \tilde{c}_1(8\log(n))^{1/\tau_2}$ ($\tilde{c}_1, \tau_2$ from Lemma D.1).

Proof. (i) *Step 1: Truncation.* Define $W_n^\circ := \sum_{t=1}^n Z_t^\circ$, where $Z_t^\circ = Z_t^\circ(t/n)$, $\Psi(x) = x\mathbb{1}_{\{|x|\leq a_n\}} + a_n\mathbb{1}_{\{|x|>a_n\}}$,

$$Z_t^\circ(u) := \phi(u)\cdot[\Psi(g(\tilde{Y}_t(u))) - \mathbb{E}\Psi(g(\tilde{Y}_t(u)))].$$

We have

$$\begin{aligned}\|W_n - W_n^\circ\|_q &\leq 2\sum_{t=1}^n |\phi(t/n)|\cdot\|g(\tilde{Y}_t(t/n))\mathbb{1}_{\{|g(\tilde{Y}_t(t/n))|>a_n\}}\|_q \\ &\leq 2|\phi|_\infty n\cdot\sup_{u\in[0,1]}\|g(\tilde{Y}_0(u))\|_{2q}\cdot\sup_{u\in[0,1]}\mathbb{P}(|g(\tilde{Y}_0(u))| > a_n)^{1/2}. \end{aligned} \tag{64}$$

By Lemma D.1, we have

$$\mathbb{P}(|g(\tilde{Y}_0(u))| > a_n) \leq \tilde{c}_2\cdot\exp\Big(-\frac{1}{2}\Big(\frac{a_n}{\tilde{c}_1}\Big)^{\tau_2}\Big) \leq \tilde{c}_2 n^{-4}.$$

Inserting this into (64) and using Lemma D.1 to bound $\sup_{u\in[0,1]}\|g(\tilde{Y}_0(u))\|_{2q} \leq \tilde{c}_1 N_\alpha(2qM)^M$, we obtain

$$\|W_n - W_n^\circ\|_q \leq 2|\phi|_\infty\tilde{c}_1 N_\alpha(2qM)^M\sqrt{\tilde{c}_2}\cdot n^{-1} =: \tilde{c}'(q)\cdot n^{-1}. \tag{65}$$

With (65) and Markov's inequality, we obtain

$$\begin{aligned}\mathbb{P}(|W_n| > \gamma) &\leq \mathbb{P}(|W_n - W_n^\circ| > \frac{\gamma}{2}) + \mathbb{P}(|W_n^\circ| > \frac{\gamma}{2}) \\ &\leq \tilde{c}'(2)^2\cdot\frac{n^{-2}}{\gamma^2} + \mathbb{P}(|W_n^\circ| > \frac{\gamma}{2}). \end{aligned} \tag{66}$$

Step 2: Apply a Bernstein inequality from Doukhan and Neumann (2007).

For $s_1,...,s_u,t_1,...,t_v \in \mathbb{N}$, it holds that

$$\begin{aligned}&|\mathrm{Cov}(Z_{s_1}^\circ\ldots Z_{s_u}^\circ, Z_{t_1}^\circ\ldots Z_{t_v}^\circ)| \\ \leq& \sum_{k=0}^\infty |P_{s_u-k}(Z_{s_1}^\circ\ldots Z_{s_u}^\circ)\cdot P_{s_u-k}(Z_{t_1}^\circ\ldots Z_{t_v}^\circ)| \\ \leq& \sum_{k=0}^\infty \|P_{s_u-k}(Z_{s_1}^\circ\ldots Z_{s_u}^\circ)\|_2\cdot\|P_{s_u-k}(Z_{t_1}^\circ\ldots Z_{t_v}^\circ)\|_2.\end{aligned}$$

We have

$$\|P_{s_u-k}(Z_{s_1}^\circ\ldots Z_{s_u}^\circ)\|_2 \leq 2\|Z_{s_1}^\circ\ldots Z_{s_u}^\circ\|_2 \leq 2(2|\phi|_\infty a_n)^u.$$

By Lemma D.1, $L := |\phi|_\infty \sup_{u\in[0,1]} \|g(\tilde{Y}_0(u))\|_2 \le |\phi|_\infty \cdot \tilde{c}_1 N_\alpha(2M)^M$. We obtain with Lemma D.1 and Lipschitz continuity of Ψ that

$$\begin{aligned}\|P_{s_u-k}(Z_{t_1}^\circ \dots Z_{t_v}^\circ)\|_2 &\le \|Z_{t_1}^\circ \dots Z_{t_v}^\circ - (Z_{t_1}^\circ)^* \dots (Z_{t_v}^\circ)^*\|_2\\ &\le L|\phi|_\infty(2|\phi|_\infty a_n)^{u-2}\sum_{k=1}^{v}\delta_2^{g(\tilde{Y}(u))}(t_1-s_u+k)\\ &\le \tilde{c}'\cdot L|\phi|_\infty(2|\phi|_\infty a_n)^{u-2}\cdot v\cdot \tilde{\rho}^{t_1-s_u}.\end{aligned}$$

Furthermore,

$$\sum_{s=0}^{\infty}(s+1)^k\tilde{\rho}^s \le k!(\frac{1}{1-\tilde{\rho}})^{k+1}, \qquad \mathbb{E}|Z_t|^k \le (2|K|_\infty a_n)^k$$

(cf. also Doukhan and Neumann (2007), Proposition 8 for the upper bound of the sum). By Theorem 1 in Doukhan and Neumann (2007) (with $\mu = 1$, $\nu = 0$ therein),

$$\mathbb{P}(W_n^\circ > \frac{\gamma}{2}) \le \mathbb{P}(\sum_{t=1}^{n} Z_t^\circ \ge \frac{\gamma}{2}) \le \exp\Big(-\frac{(\gamma/2)^2/2}{A_n + B_n^{1/3}(\gamma/2)^{5/3}}\Big), \tag{67}$$

where $A_n := \mathrm{Var}(W_n^\circ)$ and with $s_n := \#\{t \in \{1,...,n\} : \phi(t/n) \ne 0\}$,

$$B_n = 2(\sqrt{\tilde{c}L|\phi|_\infty} \vee (2|\phi|_\infty a_n))\cdot\frac{1}{1-\tilde{\rho}}\cdot\Big(\Big(\frac{2^5 s_n \frac{\tilde{c}'L|\phi|_\infty}{1-\tilde{\rho}}}{A_n}\Big)\vee 1\Big)$$

(we use s_n instead of n which is possible due to a change in the upper bound in their equation (43)). Here, we have with (65) that

$$|\mathrm{Var}(W_n^\circ) - \mathrm{Var}(W_n)| \le \|W_n^\circ - W_n\|_2 \le \tilde{c}'(2)n^{-1}.$$

If $s_n \ge \frac{2\tilde{c}'(2)}{C_1}$, then

$$\mathrm{Var}(W_n) \ge 2\tilde{c}'(2).$$

Thus, for $s_n \ge \frac{2\tilde{c}'(2)}{C_1}$ we have

$$\frac{3}{2}\mathrm{Var}(W_n) \ge \mathrm{Var}(W_n) + \tilde{c}'(2) \ge A_n = \mathrm{Var}(W_n^\circ) \ge \mathrm{Var}(W_n) - \tilde{c}'(2) \ge \frac{1}{2}\mathrm{Var}(W_n). \tag{68}$$

We obtain that $A_n =\ge \frac{C_1}{2}s_n$, thus

$$B_n \le \mathrm{const.}(M,\chi,C,D,\rho,\phi,C_1)\cdot a_n. \tag{69}$$

The result now follows from (66), (67), (69) and (68). □

As a direct corollary of Lemma D.3, we obtain Bernstein inequalities for $\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)}$ and $\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)} - \frac{\tilde{\mathbb{G}}_{n,h',u}(p)}{w_{n,h'}(u)}$, where $w_{n,h}(u) := \frac{1}{n}\sum_{t=1}^{n} K_h(t/n-u)$.

Lemma D.4. *Fix $u \in [0,1]$. Assume that $g \in \mathcal{H}(M,\chi,C)$ and Assumption 4.3 holds. Suppose that $\sigma_p^2(u) > 0$. Suppose that $K \in \mathcal{K}'$. Then there exist some constants $c_4, c_5, c_H, C_H > 0$ independent of n, h such that for $c_H n^{-1} \le h \le C_H$,*

$$\begin{aligned}&\mathbb{P}\Big((nh)\Big|\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)}\Big| > \gamma\Big)\\ &\le 2\exp\Big(-\frac{\gamma^2}{32(nh)^2 v_p^2(h,u) + c_4 a_n^{1/3}\gamma^{5/3}}\Big) + c_5\frac{n^{-2}}{\gamma^2},\end{aligned}$$

where $a_n := \log(n)^{1/\tau_2}$ (τ_2 is from Lemma D.1) and $v_p^2(h,u) := \frac{1}{nh}\int K(x)^2 dx \cdot \sigma_p^2(u)$.

Proof of Lemma D.4. We apply Lemma D.3 with $\phi(v) := \frac{hK_h(v-u)}{w_{n,h}(u)}$. Then $W_n = (nh)\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)}$. Since $K \in \mathcal{K}'$ is bounded, has support $[-\frac{1}{2}, \frac{1}{2}]$ and $\int K(x)dx = 1$, it follows that

$$s_n = \#\{t \in \{1, ..., n\} : \phi(t/n) \neq 0\} \in [\bar{c}' \cdot nh, \bar{c}'' \cdot nh]$$

with some $\bar{c}'' > \bar{c}' > 0$ independent of n, h.

By Lemma D.10(i), there exist $c', c'' > 0$ such that for $c'n^{-1} \leq h \leq c''$

$$\frac{3}{2} \cdot nh \int K(x)^2 dx \cdot \sigma_p^2(u) \geq \mathrm{Var}(W_n) \geq \frac{1}{2} \cdot nh \int K(x)^2 dx \cdot \sigma_p^2(u) \geq \frac{\int K(x)^2 dx \cdot \sigma_p^2(u)}{2\bar{c}''} \cdot s_n.$$

Choosing c' large enough, we can furthermore ensure that $s_n \geq \bar{c}'nh \geq \bar{c}'c' \geq c_6$, where c_6 is from Lemma D.3. The result now follows from Lemma D.3. □

Lemma D.5 (Bernstein inequality for $\tilde{\mathbb{G}}_{n,h,u}(p) - \tilde{\mathbb{G}}_{n,h',u}(p)$). *Fix $u \in [0, 1]$ and $a \in (0, 1)$. Assume that $g \in \mathcal{H}(M, \chi, C)$ and Assumption 4.3 holds. Suppose that $K \in \mathcal{K}'$ and $\inf_{u \in [0,1]} \sigma_p^2(u) > 0$. Then there exist some constants $c_4, c_5, c_H, C_H > 0$ independent of n, h, h' such that for $c_H n^{-1} \leq h, h' \leq C_H$ and $h' \leq a \cdot h$,*

$$\begin{aligned} & \mathbb{P}\Big((nh')\Big|\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)} - \frac{\tilde{\mathbb{G}}_{n,h',u}(p)}{w_{n,h'}(u)}\Big| > \gamma\Big) \\ \leq \; & 2\exp\Big(-\frac{\gamma^2}{32(nh')^2 v_p^2(h, h', u) + c_4 a_n^{1/3}\gamma^{5/3}}\Big) + c_5\frac{n^{-2}}{\gamma^2}, \end{aligned}$$

with $a_n := (\log(n))^{1/\tau_2}$ (τ_2 from Lemma D.1) and $v_p^2(h, h', u) := \frac{1}{n}\int\{K_h(x) - K_{h'}(x)\}^2 dx \cdot \sigma_p^2(u)$.

Proof of Lemma D.5. We apply Lemma D.3 with $\phi(v) := h'(\frac{K_h(v-u)}{w_{n,h}(u)} - \frac{K_{h'}(v-u)}{w_{n,h'}(u)})$. Then $W_n = (nh')\{\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)} - \frac{\tilde{\mathbb{G}}_{n,h',u}(p)}{w_{n,h'}(u)}\}$. Since K is bounded, has support $[-\frac{1}{2}, \frac{1}{2}]$ (thus $\frac{K_h(v-u)}{w_{n,h}(u)} - \frac{K_{h'}(v-u)}{w_{n,h'}(u)} > 0$ for $v \in [u-h, u-h'] \cup [u+h', u+h]$) and $h' \leq a \cdot h$, it follows that

$$s_n = \#\{t \in \{1, ..., n\} : \phi(t/n) \neq 0\} \in [\bar{c}' \cdot nh, \bar{c}'' \cdot nh]$$

with some $\bar{c}'' > \bar{c}' > 0$ independent of n, h.

By Lemma D.10(ii), there exist $c', c'' > 0$ such that for $c'n^{-1} \leq h \leq c''$

$$\begin{aligned} & \frac{3}{2} \cdot nh' \int \{K_h(x) - K_{h'}(x)\}^2 dx \cdot \sigma_p^2(u) \\ \geq \; & \mathrm{Var}(W_n) \geq \frac{1}{2} \cdot nh' \int \{K_h(x) - K_{h'}(x)\}^2 dx \cdot \sigma_p^2(u) \\ \geq \; & \frac{\int K(x)^2 dx \cdot \sigma_p^2(u)}{2\bar{c}''} \cdot s_n. \end{aligned}$$

Choosing c' large enough, we can furthermore ensure that $s_n \geq \bar{c}'nh \geq \bar{c}'c' \geq c_6$, where c_6 is from Lemma D.3. The result now follows from Lemma D.3. □

D.2 Stationary approximation results, dependence measure and variance expansions

Lemma D.6 (Replacement by sum over stationary sequences). *Let $p \in \mathcal{H}(M, \chi, C)$ and let $r \geq 1$. Suppose that Assumption 2.1 holds for some $q \geq rM$. Then there exists some constant c' only depending on C, D, χ, r, M such that* *(i)* $\|\mathbb{G}_{n,h,u}(p) - \tilde{\mathbb{G}}_{n,h,u}(p)\|_r \le c'n^{-1}$.

(ii) *uniformly in* $u \in [0,1]$, $\|\mathbb{G}_{n,h,u}(p) - \tilde{\mathbb{G}}_{n,h,u}(p)\|_r \le c'(nh)^{-1}$.

(iii) *if* $r \ge 2$ *and Assumption 2.7 holds, then uniformly in* $u \in [0,1]$,

$$\|\tilde{\mathbb{G}}_{n,h,u}(p) - \tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)\|_r \le c'(nh)^{-1/2}\sum_{k=0}^{\infty}\min\{h, \delta_r^{p(\tilde{Y}(u))}(k)\}.$$

Proof of Lemma D.6. (i) By Hölder's inequality,

$$\begin{aligned}
&\|p(Y_{t,n}) - p(\tilde{Y}_t(t/n))\|_r\\
\le\ & C(1+(D|\chi|_1)^{M-1})\cdot\Big[\sum_{i=1}^{t-1}\chi_i\|X_{t-i,n} - \tilde{X}_{t-i}(t/n)\|_{rM} + \sum_{i=t}^{\infty}\chi_i\|\tilde{X}_{t-i}(t/n)\|_{rM}\Big]\\
\le\ & CD(1+(D|\chi|_1)^{M-1})\cdot\Big[n^{-1}\sum_{i=1}^{\infty}\chi_i(i+1) + \sum_{i=t}^{\infty}\chi_i\Big]. \qquad (70)
\end{aligned}$$

Put $c'' = CD(1+2(D|\chi|_1)^{M-1})$. Then

$$\|\mathbb{G}_{n,h,u}(p) - \tilde{\mathbb{G}}_{n,h,u}(p)\|_q \le |K|_\infty c'' n^{-1}\sum_{i=1}^{\infty}\chi_i(i+1) + \frac{c''}{nh}\sum_{t=1}^{n}K\Big(\frac{t/n-u}{h}\Big)\cdot\sum_{i=t}^{\infty}\chi_i.$$

If $h \le u$, then we sum over $\frac{t}{n} \ge u - \frac{h}{2} \ge \frac{u}{2}$, thus $t \ge \frac{u}{2}\cdot n$. Since with some constant $c > 0$, $\sum_{i=t}^{\infty}\chi_i \le ct^{-1} \le \frac{2c}{u}\cdot\frac{1}{n}$, we obtain the result.
If $h \ge u$, then $|\frac{1}{nh}\sum_{t=1}^{n}K\big(\frac{t/n-u}{h}\big)\cdot\sum_{i=t}^{\infty}\chi_i| \le \frac{|K|_\infty}{nh}\sum_{t=1}^{\infty}\sum_{i=t}^{\infty}\chi_i$ and $nh \ge nu$, we obtain the result.

(ii) is immediate from (70) and

$$\|\mathbb{G}_{n,h,u}(p) - \tilde{\mathbb{G}}_{n,h,u}(p)\|_r \le \frac{|K|_\infty}{nh}\sum_{t=1}^{n}\|p(Y_{t,n}) - p(\tilde{Y}_t(t/n))\|_r.$$

(iii) We have

$$\begin{aligned}
&\big\|\tilde{G}_{n,h,u}(p) - \tilde{G}^{\circ}_{n,h,u}(p)\big\|_2\\
\le\ & \sum_{k=0}^{\infty}\frac{1}{nh}\Big(\sum_{t=1}^{n}K\big(\frac{t/n-u}{h}\big)^2\cdot\|P_{t-k}(p(\tilde{Y}_t(t/n)) - p(\tilde{Y}_t(u)))\|_2^2\Big)^{1/2}\\
\le\ & (nh)^{-1/2}|K|_\infty\cdot\sum_{k=0}^{\infty}\sup_{t:|t/n-u|\le h}\|P_{t-k}(p(\tilde{Y}_t(t/n)) - p(\tilde{Y}_t(u)))\|_2. \qquad (71)
\end{aligned}$$

We can bound the summands by two different values: First,

$$\|P_{t-k}(p(\tilde{Y}_t(t/n)) - p(\tilde{Y}_t(u)))\|_2 \le 2\sup_{u\in[0,1]}\|P_{t-k}(p(\tilde{Y}_t(u))\|_2 \le 2\sup_{u\in[0,1]}\delta_2^{p(\tilde{Y}(u))}(k),$$

second, due to the projection property of the conditional expectation and a similar calculation as in (70), for $|t/n-u| \le h$,

$$\|P_{t-k}(p(\tilde{Y}_t(t/n)) - p(\tilde{Y}_t(u)))\|_2 \le \|p(\tilde{Y}_t(t/n)) - p(\tilde{Y}_t(u))\|_2 \le c''\cdot h|\chi|_1.$$

Insertion of these two bounds into (71) yields the assertion.

□

Lemma D.7 (Dependence measure of functions of $\tilde{X}_t(u)$)**.** *Let $p \in \mathcal{H}(M, \chi, C)$ and $r > 0$ with $rM \geq 1$. Suppose that Assumption 2.1 and Assumption 2.7 hold for some $q \geq rM$. Then for all $k \in \mathbb{N}_0$,*

$$\sup_{u\in[0,1]} \delta_r^{p(\tilde{Y}(u))}(k) \leq \tilde{C} \cdot \sum_{j=1}^{k} \chi_j \sup_{u\in[0,1]} \delta_{rM}^{\tilde{X}(u)}(k-j+1),$$

where $\tilde{C} = C \cdot (1 + 2(D|\chi|_1)^{M-1})$. Especially,

$$\sum_{k=0}^{\infty} \sup_{u\in[0,1]} \delta_r^{p(\tilde{Y}(u))}(k) < \infty.$$

Proof of Lemma D.7. By Hölder's inequality, we have with some constant $\tilde{c}$ only dependent on M, χ, C, D:

$$\begin{aligned}
\delta_r^{p(\tilde{Y}(u))}(k) &= \big\|p(\tilde{Y}_k(u)) - p(\tilde{Y}_k^*(u))\big\|_r \\
&\leq C\big\||\tilde{Y}_k(u) - \tilde{Y}_k^*(u)|_\chi\|_{rM}\big(1 + 2\||\tilde{Y}_k(u)\|_\chi\|_{rM}^{M-1}\big) \\
&\leq C\sum_{j=1}^{k} \chi_j \|\tilde{X}_{k-j+1}(u) - \tilde{X}_{k-j+1}^*(u)\|_{rM} \cdot \big(1 + 2|\chi|_1^{M-1}\|\tilde{X}_0(u)\|_{rM}^{M-1}\big) \\
&\leq C\big(1 + 2|\chi|_1^{M-1}\|\tilde{X}_0(u)\|_{rM}^{M-1}\big) \cdot \sum_{j=1}^{k} \chi_j \delta_{rM}^{\tilde{X}(u)}(k-j+1).
\end{aligned}$$

The absolute summability of $\sup_{u\in[0,1]} \delta_r^{p(\tilde{Y}(u))}(k)$ follows from the convolution theorem and the fact that χ is absolutely summable and $\sup_{u\in[0,1]} \delta_{rM}^{\tilde{X}(u)}(k)$ is absolutely summable by Assumption 2.7. □

Lemma D.8 (Asymptotic representation of variance and covariance)**.** *Let $p_1, p_2 \in \mathcal{H}(M, \chi, C)$. Let Assumption 2.1 and Assumption 2.7 hold with some $q \geq 2M$. Additionally, suppose that $\sup_{u\in[0,1]} \delta_q^{\tilde{X}(u)}(k) = O(k^{-2-\varepsilon})$ with some $\varepsilon > 0$. Then for $n \to \infty$, $h \to 0$ and $nh \to \infty$,*

$$\mathrm{Cov}(\mathbb{G}_{n,h,u}(p_1), \mathbb{G}_{n,h,u}(p_2)) = \frac{1}{nh}\int K(x)^2 dx \cdot \sigma_{p_1,p_2}(u) + R_{n,h}(u), \tag{72}$$

where

$$\sigma_{p_1,p_2}(u) := \sum_{k,l=0}^{\infty} \mathbb{E}[P_0 p_1(\tilde{Y}_k(u)) \cdot P_0 p_2(\tilde{Y}_l(u))] = \sum_{j\in\mathbb{Z}} \mathrm{Cov}(p_1(\tilde{Y}_0(u)), p_2(\tilde{Y}_j(u)))$$

and $\sup_{u\in[0,1]} |R_{n,h}(u)| = o((nh)^{-1})$.

Proof of Lemma D.8. We have

$$\mathrm{Cov}(\mathbb{G}_{n,h,u}(p_1), \mathbb{G}_{n,h,u}(p_2)) = \mathbb{E}[\mathbb{G}_{n,h,u}(p_1)\mathbb{G}_{n,h,u}(p_2)].$$

Thus

$$\begin{aligned}
&\big|\mathrm{Cov}(\mathbb{G}_{n,h,u}(p_1), \mathbb{G}_{n,h,u}(p_2)) - \mathrm{Cov}(\tilde{\mathbb{G}}_{n,h,u}^{\circ}(p_1), \tilde{\mathbb{G}}_{n,h,u}^{\circ}(p_2))\big| \\
\leq\ &\big\|\mathbb{G}_{n,h}(p_1) - \tilde{\mathbb{G}}_{n,h,u}^{\circ}(p_1)\big\|_2 \big\|\mathbb{G}_{n,h,u}(p_2) - \tilde{\mathbb{G}}_{n,h,u}^{\circ}(p_2)\big\|_2 \\
&+ \big\|\mathbb{G}_{n,h,u}(p_1) - \tilde{\mathbb{G}}_{n,h,u}^{\circ}(p_1)\big\|_2 \big\|\tilde{\mathbb{G}}_{n,h,u}^{\circ}(p_2)\big\|_2 \\
&+ \big\|\mathbb{G}_{n,h,u}(p_2) - \tilde{\mathbb{G}}_{n,h,u}^{\circ}(p_2)\big\|_2 \big\|\tilde{\mathbb{G}}_{n,h,u}^{\circ}(p_1)\big\|_2.
\end{aligned}$$

Below in (75) (by taking $p_1 = p_2$) we see that $\sup_u \|\tilde{G}^{\circ}_{n,h,u}(p_i)\|_2 = O((nh)^{-1/2})$ $(i = 1, 2)$. By Lemma D.6(ii),(iii),

$$\sup_u \left\|\mathbb{G}_{n,h,u}(p_1) - \tilde{\mathbb{G}}^{\circ}_{n,h,u}(p_1)\right\|_2 = o((nh)^{-1/2}), \quad i = 1, 2.$$

We conclude that

$$\sup_u \left|\mathrm{Cov}(\mathbb{G}_{n,h,u}(p_1), \mathbb{G}_{n,h,u}(p_2)) - \mathrm{Cov}(\tilde{\mathbb{G}}^{\circ}_{n,h,u}(p_1), \tilde{\mathbb{G}}^{\circ}_{n,h,u}(p_2))\right| = o((nh)^{-1}). \tag{73}$$

In the following, we abbreviate $p_{1t} := p_1(\tilde{Y}_t(u))$ and $p_{2t} := p_2(\tilde{Y}_t(u))$. Since $(P_{t-k}p_{1t})_t$ and $(P_{t-l}p_{2t})_t$ are martingale differences with respect to $\mathcal{F}_t$, we obtain

$$\begin{aligned}
&\mathrm{Cov}(\tilde{\mathbb{G}}^{\circ}_{n,h,u}(p_1), \tilde{\mathbb{G}}^{\circ}_{n,h,u}(p_2)) \\
= \; & \frac{1}{n^2}\sum_{k,l=0}^{\infty}\sum_{s,t=1}^{n} K_h(t/n-u)K_h(s/n-u)\mathbb{E}\big[P_{t-k}p_{1t}\cdot P_{s-l}p_{2t}\big] \\
= \; & \frac{1}{n^2}\sum_{k,l=0}^{\infty}\sum_{t=1,\dots,n:1\le t-k+l\le n} K_h(t/n-u)K_h((t-k+l)/n-u)\mathbb{E}\big[P_{t-k}p_{1t}\cdot P_{t-k}p_{2(t-k+l)}\big] \\
= \; & \frac{1}{n^2}\sum_{k,l=0}^{\infty}\mathbb{E}\big[P_0 p_{1k}\cdot P_0 p_{2l}\big]\sum_{t=1,\dots,n:1\le t-k+l\le n} K_h(t/n-u)K_h((t-k+l)/n-u).
\end{aligned} \tag{74}$$

By Lemma D.7, the convolution theorem and $\sup_{u\in[0,1]}\delta_{rM}^{\tilde{X}(u)}(k) = O(k^{-2-\varepsilon})$, we have $\sup_{u\in[0,1]}\delta_2^{p(\tilde{Y}(u))}(k) = O(k^{-2-\varepsilon})$. Therefore, $K_h((t-k+l)/n-u)$ can be replaced by $K_h(t/n-u)$ due to Lipschitz-continuity of K with replacement error bounded by

$$\begin{aligned}
\le \; & \frac{L_K}{(nh)^2}\sum_{k,l=0}^{\infty}(k+l)\cdot|\mathbb{E}[P_0p_{1k}\cdot P_0p_{2l}]|\cdot\frac{1}{n}\sum_{t=1}^{n}|K_h(t/n-u)| \\
\le \; & \frac{|K|_\infty L_K}{(nh)^2}\sum_{k,l=0}^{\infty}(k+l)\cdot\sup_{u\in[0,1]}\delta_2^{p_1(\tilde{Y}(u))}(k)\cdot\sup_{u\in[0,1]}\delta_2^{p_2(\tilde{Y}(u))}(l) = O((nh)^{-2}).
\end{aligned}$$

After the replacement, (74) reads

$$\begin{aligned}
& \frac{1}{(nh)^2}\sum_{t=1}^{n}K((t/n-u)/h)^2\cdot\sum_{k,l=0}^{\infty}\mathbb{E}[P_0p_{1k}\cdot P_0p_{2l}] \\
= \; & \frac{1}{(nh)^2}\sum_{t=1}^{n}K((t/n-u)/h)^2\cdot\sigma_{p_1,p_2}(u).
\end{aligned}$$

Since K is Lipschitz continuous, this can be replaced by $\frac{1}{nh}\int K(x)^2\,\mathrm{d}x\cdot\sigma^2_{p_1,p_2}(u)$ with replacement error

$$\begin{aligned}
& L_K|K|_\infty(nh)^{-2}|\sigma_{p_1,p_2}(u)| \\
\le \; & L_K|K|_\infty(nh)^{-2}\sum_{k,l=0}^{\infty}\sup_{u\in[0,1]}\delta_2^{p_1(\tilde{Y}(u))}(k)\cdot\sup_{u\in[0,1]}\delta_2^{p_2(\tilde{Y}(u))}(l).
\end{aligned}$$

We therefore have shown that

$$\mathrm{Cov}(\tilde{\mathbb{G}}_{n,h}(p_1), \tilde{\mathbb{G}}_{n,h}(p_2)) = \frac{1}{nh}\int K(x)^2dx\cdot\sigma^2_{p_1,p_2}(u) + R_{n,h}(u), \tag{75}$$

where

$$\sup_{u\in[0,1]} |R_{n,h}(u)| = O((nh)^{-2}).$$

The result follows from (73) and (75). □

Lemma D.9 (Lower bound on the variance). *Suppose that $\sigma^2_{p,min} := \inf_{u\in[0,1]} \sigma_p^2(u) > 0$. Suppose that $K \in \mathcal{K}'$. Let $a \in (0,1)$. Then for all $h, h' \in (0,1)$ with $h' \leq a \cdot h$,*

$$\frac{1}{n}\int \{K_h(x) - K_{h'}(x)\}^2 dx \cdot \sigma_p^2(u) \geq (1-a)^2\sigma^2_{p,min} f_{min} \cdot (nh')^{-1},$$

where $f_{min} > 0$ is a constant which only depends on the kernel K.

Proof of Lemma D.9. First note that we have with $Q := \frac{h'}{h} \in (0, a]$:

$$\frac{1}{n}\int \{K_h(x) - K_{h'}(x)\}^2 \,\mathrm{d}x = \frac{1}{nh'}\int \{QK(Qy) - K(y)\}^2 \,\mathrm{d}y$$

Let $f(Q) := \frac{1}{(Q-1)^2}\int \{QK(Qy) - K(y)\}^2 \,\mathrm{d}y$. Then

$$\begin{aligned} \lim_{Q\to 1} f(Q) &= \lim_{Q\to 1}\int \{Qy\frac{K(Qy)-K(y)}{Qy-y} + K(y)\}^2 \,\mathrm{d}y \\ &= \int \{yK'(y) + K(y)\}^2 \,\mathrm{d}y > 0 \end{aligned}$$

by assumption, and

$$\lim_{Q\to 0} f(Q) = \int K(y)^2 \,\mathrm{d}y > 0.$$

Note that $Q \mapsto f(Q)$ is a continuous function and $f(Q) > 0$ for all $Q \in (0,1)$ by the property $K(x) = 0 \iff x \in [-\frac{1}{2}, \frac{1}{2}]$. Thus $f_{min} := \inf_{Q\in[0,1]} f(Q) > 0$. We conclude that

$$\frac{1}{n}\int \{K_h(x) - K_{h'}(x)\}^2 \,\mathrm{d}x \cdot \sigma_p^2(u) \geq \frac{(1-\frac{h'}{h})^2}{nh'}\sigma_p^2(u) f(\frac{h'}{h}) > 0 \geq \frac{(1-a)^2}{nh'}\sigma^2_{min} \cdot f_{min}.$$

□

For the following lemma, define $w_{n,h}(u) := \frac{1}{n}\sum_{t=1}^n K_h(t/n - u)$.

Lemma D.10 (Detailed calculation of the variance). *Let $p \in \mathcal{H}(M, \chi, C)$. Fix $a \in (0,1)$ and $u \in (0,1)$. Let Assumption 4.3 hold. Assume that $\sigma_p^2(u) > 0$. Then there exist constants $c_1, c_2 > 0$ independent of n, h, h' such that for all $c_1 n^{-1} \leq h' \leq h \leq c_2$:*

(i)

$$Var(\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)}) = \frac{1}{nh}\int K(x)^2 \,\mathrm{d}x \cdot \sigma_p^2(u) + R_{n,h},$$

where $R_{n,h}$ satisfies $|R_{n,h}| \leq \frac{1}{2}(nh)^{-1}\int K(x)^2 dx \cdot \sigma_p(u)^2$.

(ii) Let additionally the assumptions of Lemma D.9 hold. Then

$$Var(\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)} - \frac{\tilde{\mathbb{G}}_{n,h',u}(p)}{w_{n,h'}(u)}) = \frac{1}{n}\int \{K_h(x) - K_{h'}(x)\}^2 \,\mathrm{d}x \cdot \sigma_p^2(u) + R_{n,h,h'},$$

where $R_{n,h,h'}$ satisfies $|R_{n,h,h'}| \leq \frac{1}{2n}\int \{K_h(x) - K_{h'}(x)\}^2 \,\mathrm{d}x \cdot \sigma_p^2(u)$.

Proof of Lemma D.10. (i) It holds that

$$\left|w_{n,h}(u) - \int_0^1 K_h(v-u)dv\right| \leq L_K(nh)^{-1}, \tag{76}$$

and for $h \leq u$,

$$\int_0^1 K_h(v-u)dv = \int K(x)dx = 1. \tag{77}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& |\mathrm{Var}(\tilde{\mathbb{G}}_{n,h,u}(p)) - \mathrm{Var}(\tilde{\mathbb{G}}^{\circ}_{n,h,u}(p))| \\
\leq\ & \|\tilde{\mathbb{G}}_{n,h,u}(p) - \tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)\|_2 \Big(\|\tilde{\mathbb{G}}_{n,h,u}(p) - \tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)\|_2 + 2\|\tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)\|_2 \Big).
\end{aligned} \tag{78}$$

By Lemma D.6(iii), there exists some constant $c' > 0$ such that

$$\|\tilde{\mathbb{G}}_{n,h,u}(p) - \tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)\|_2 \leq c'(nh)^{-1/2} \sum_{k=0}^{\infty} \min\{\tilde{\rho}^k, h\}. \tag{79}$$

We have with some constant $\bar{c}$ only dependent on $\tilde{\rho}$ that

$$\sum_{k=0}^{\infty} \min\{\tilde{\rho}^k, h\} \leq \sum_{k=0}^{\lfloor \log(h)/\log(\tilde{\rho}) \rfloor} h + \sum_{k=\lceil \log(h)/\log(\tilde{\rho}) \rceil}^{\infty} \tilde{\rho}^k \leq \bar{c} \cdot h \log(h^{-1}). \tag{80}$$

We obtain with (79) that with some constant $c' > 0$,

$$\|\tilde{\mathbb{G}}_{n,h,u}(p) - \tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)\|_2 \leq c'(nh)^{-1/2} \min\{h \log(h^{-1}), \frac{1}{1-\tilde{\rho}}\}. \tag{81}$$

Abbreviate $p_t := p(\tilde{Y}_t(u))$. Then

$$\begin{aligned}
& \|\tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)\|_2^2 \\
=\ & \frac{1}{n^2} \Big\| \sum_{k=0}^{\infty} \sum_{t=1}^{n} K_h(t/n - u) \cdot P_{t-k} p_t \Big\|_2^2 \\
=\ & \frac{1}{n^2} \sum_{k,l=0}^{\infty} \mathbb{E}\Big[\sum_{t=1}^{n} K_h(t/n - u) \cdot P_{t-k} p_t \cdot \sum_{s=1}^{n} K_h(s/n - u) \cdot P_{s-l} p_s \Big] \\
=\ & \frac{1}{(nh)^2} \sum_{k,l=0}^{\infty} \mathbb{E}[P_0 p_k \cdot P_0 p_l] \\
& \qquad \times \sum_{t: 1 \leq t \leq n, 1 \leq t-k+l \leq n} K((t/n-u)/h) K(((t-k+l)/n - u)/h).
\end{aligned} \tag{82}$$

$K(((t-k+l)/n-u)/h)$ can be replaced by $K((t/n-u)/h)$ due to Lipschitz-continuity (Lipschitz constant L_K) of K with replacement error

$$\begin{aligned}
\leq\ & \frac{L_K}{(nh)^3} \sum_{k,l=0}^{\infty} (k+l) \cdot |\mathbb{E}[P_0 p_k \cdot P_0 p_l]| \sum_{t=1}^{n} |K((t/n-u)/h)| \\
\leq\ & \frac{|K|_\infty L_K}{(nh)^2} \sum_{k,l=0}^{\infty} (k+l) \cdot \sup_{u \in [0,1]} \delta_2^{p(\tilde{Y}(u))}(k) \cdot \sup_{u \in [0,1]} \delta_2^{p(\tilde{Y}(u))}(l) \leq \tilde{C} \cdot |K|_\infty L_K (nh)^{-2}
\end{aligned}$$

due to Lemma D.7 with some $\tilde{C}$ only depending on M, χ, C, D, ρ. After the replacement, (82) reads

$$\begin{aligned} &\frac{1}{(nh)^2}\sum_{t=1}^{n} K((t/n-u)/h)^2 \cdot \sum_{k,l=0}^{\infty} \mathbb{E}[P_0 p_k \cdot P_0 g_l] \\ = \ &\frac{1}{(nh)^2}\sum_{t=1}^{n} K((t/n-u)/h)^2 \cdot \sigma_p^2(u). \end{aligned}$$

Since K is Lipschitz continuous, this can be replaced by $\frac{1}{nh}\int K(x)^2 \, \mathrm{d}x \cdot \sigma_p^2(u)$ with replacement error $L_K|K|_\infty (nh)^{-2}\sigma_p^2(u)$. Together with (76), we have shown that

$$\big|\|\frac{\tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)}{w_{n,h}(u)}\|_2^2 - \frac{1}{nh}\int K(x)^2 dx \cdot \sigma_p^2(u)\big| \le c'(nh)^{-2}.$$

Combination with (78) and (81) yields that with some constant $C' > 0$,

$$\big|\mathrm{Var}(\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)}) - \frac{1}{nh}\int K(x)^2 dx \cdot \sigma_p^2(u)\big| \le C'\big((nh)^{-2} + (nh)^{-1} h\log(h^{-1})\big).$$

Since $\int K(x)^2 dx > 0$, $\sigma_p^2(u) > 0$, we obtain that there exist constants $c', c'' > 0$ such that $c' n^{-1} \le h \le c''$ implies

$$C'((nh)^{-2} + (nh)^{-1} h\log(h^{-1})) \le \frac{1}{2}(nh)^{-1}\int K(x)^2 dx \cdot \sigma_p^2(u).$$

This shows the bound on $R_{n,h}$.

(ii) First note that as before in (81),

$$\begin{aligned} &\|(\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)} - \frac{\tilde{\mathbb{G}}_{n,h',u}(p)}{w_{n,h'}(u)}) - (\frac{\tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)}{w_{n,h}(u)} - \frac{\tilde{\mathbb{G}}^{\circ}_{n,h',u}(p)}{w_{n,h'}(u)})\|_2 \\ \le \ &\frac{1}{w_{n,h}(u)}\|\tilde{\mathbb{G}}_{n,h,u}(p) - \tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)\|_2 + \frac{1}{w_{n,h'}(u)}\|\tilde{\mathbb{G}}_{n,h',u}(p) - \tilde{\mathbb{G}}^{\circ}_{n,h',u}(p)\|_2 \\ \le \ &c'\big((nh)^{-1/2}\min\{h\log(h^{-1}), \frac{1}{1-\tilde{\rho}}\} + (nh')^{-1/2}\min\{h'\log((h')^{-1}), \frac{1}{1-\tilde{\rho}}\}\big). \quad (83) \end{aligned}$$

As in (82), we obtain

$$\begin{aligned} &\|\frac{\tilde{\mathbb{G}}^{\circ}_{n,h,u}(p)}{w_{n,h}(u)} - \frac{\tilde{\mathbb{G}}^{\circ}_{n,h',u}(p)}{w_{n,h'}(u)}\|_2^2 \\ = \ &\frac{1}{n^2}\sum_{k,l=0}^{\infty} \mathbb{E}[P_0 p_k \cdot P_0 p_l] \\ &\quad \times \sum_{t:1\le t\le n, 1\le t-k+l\le n} \{\frac{K_h(t/n-u)}{w_{n,h}(u)} - \frac{K_{h'}(t/n-u)}{w_{n,h'}(u)}\} \\ &\quad\quad \cdot \{\frac{K_h((t-k+l)/n-u)}{w_{n,h}(u)} - \frac{K_{h'}((t-k+l)/n-u)}{w_{n,h'}(u)}\}. \quad (84) \end{aligned}$$

As before, $\{\frac{K_h((t-k+l)/n-u)}{w_{n,h}(u)} - \frac{K_{h'}((t-k+l)/n-u)}{w_{n,h'}(u)}\}$ can be replaced by $\{\frac{K_h(t/n-u)}{w_{n,h}(u)} - \frac{K_{h'}(t/n-u)}{w_{n,h'}(u)}\}$ due to Lipschitz-continuity of K and $w_{n,h}(u), w_{n,h'}(u) \ge \frac{1}{2}$ for $c'n^{-1} \le h, h' \le c''$ (c' large enough, c'' small enough) with replacement error

$$\leq \frac{2L_K}{n^2}\big((nh^2)^{-1}+(n(h')^2)^{-1}\big)\sum_{k,l=0}^{\infty}(k+l)\big|\mathbb{E}[P_0p_k\cdot P_0p_l]\big|$$
$$\times\sum_{t=1}^{n}\Big(\Big|\frac{K_h(t/n-u)}{w_{n,h}(u)}\Big|+\Big|\frac{K_{h'}(t/n-u)}{w_{n,h'}(u)}\Big|\Big)$$
$$\leq \frac{4\tilde{C}L_K|K|_\infty}{n^2}\big((nh^2)^{-1}+(n(h')^2)^{-1}\big)\cdot 2n$$
$$\leq 8\tilde{C}L_K|K|_\infty\cdot\big((nh)^{-2}+(nh')^{-2}\big)=O((nh')^{-2}).$$

After the replacement, (84) reads

$$\frac{1}{n^2}\sum_{t=1}^{n}\{\frac{K_h(t/n-u)}{w_{n,h}(u)}-\frac{K_{h'}(t/n-u)}{w_{n,h'}(u)}\}^2\cdot\sigma_p^2(u).$$

Again, by Lipschitz continuity of K and in view of (76),(77), we obtain that this can be replaced by

$$\frac{1}{n}\int\{K_h(x)-K_{h'}(x)\}^2dx\cdot\sigma_p^2(u)$$

with replacement error $O((nh')^{-2})$. Combination with (83) yields

$$\Big|\mathrm{Var}(\frac{\tilde{\mathbb{G}}_{n,h,u}(p)}{w_{n,h}(u)}-\frac{\tilde{\mathbb{G}}_{n,h',u}(p)}{w_{n,h'}(u)})-\frac{1}{n}\int\{K_h(x)-K_{h'}(x)\}^2dx\cdot\sigma_p^2(u)\Big|$$
$$\leq c'\big((nh')^{-2}+(nh')^{-1}h'\log((h')^{-1})\big). \tag{85}$$

By Lemma D.9,

$$\frac{1}{n}\int\{K_h(x)-K_{h'}(x)\}^2dx\cdot\sigma_p^2(u)\geq(1-a)^2\sigma_{p,min}^2f_{min}(nh')^{-1}.$$

There exist constants $c',c''>0$ such that for $c'n^{-1}\leq h'\leq h\leq c''$, the right hand side of (85) is bounded by $\frac{1}{2}(1-a)^2\sigma_{p,min}^2f_{min}(nh')^{-1}$, which shows the assertion on $R_{n,h,h'}$.

□

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