Online Appendix "Central Limit Theory for Combined Cross-Section and Time Series with an Application to Aggregate Productivity Shocks"

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I Details for Section 2

We consider a simplified version of Olley and Pakes' (1996) model without any aggregate shock. Assume that $\omega_{j,t}$ follows an AR(1) process[1]

$$\omega_{j,t} = \alpha\omega_{j,t-1} + e_{j,t},$$

where we assume that the intercept term is zero for notational simplicity, and that our parameters of interests are (β_k, α). This means that we can write the conditional expectation of $\mathfrak{y}^*_{j,t+1}$ given

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[1] Olley and Pakes (1996) adopted a non-parametric specification for the dynamics of $\omega_{j,t}$, but Ackerberg, Caves and Frazer (2015) adopted a parametric specification. The parametric specification makes it easier to recognize the source of complication in the presence of aggregate shocks.

the information $I_{j,t}$ available at time t as

$$
\begin{aligned}
E\left[\left.\mathfrak{y}^*_{j,t+1}\right| I_{j,t}\right] &= \beta_k k_{j,t+1} + \alpha\omega_{j,t} \\
&= \beta_k k_{j,t+1} + \alpha\left(\phi_t\left(i_{j,t}, k_{j,t}\right) - \beta_k k_{j,t}\right) \\
&= \beta_k k_{j,t+1} + \alpha\left(\phi_t\left(i_{j,t}, k_{j,t}\right) - \beta_k k_{j,t}\right),
\end{aligned}
$$

and that (β_k, α) can be identified by the conditional moment restriction

$$
0 = E\left[\left.\mathfrak{y}^*_{j,t+1} - \left(\beta_k k_{j,t+1} + \alpha\left(\phi_t\left(i_{j,t}, k_{j,t}\right) - \beta_k k_{j,t}\right)\right)\right| I_{j,t}\right]
$$

using cross-sectional variation. This gives the basic intuition underlying (7).

From

$$
\mathfrak{y}^*_{j,t} - \beta_k k_{j,t} = \omega_{j,t} + \eta_{j,t} = \nu_t + \varepsilon_{j,t} + \eta_{j,t},
$$

we can infer the following limit sequences

$$
\begin{aligned}
&\operatorname*{plim}_{n\to\infty} n^{-1}\sum_{j=1}^{n}\left(\mathfrak{y}^*_{j,t} - \nu_t - \beta_k k_{j,t}\right)\left(\mathfrak{y}^*_{j,t-1} - \nu_{t-1} - \beta_k k_{j,t-1}\right) = \alpha^{(C)}\sigma^2_\varepsilon \qquad \text{(S.1)} \\
&\operatorname*{plim}_{n\to\infty} n^{-1}\sum_{j=1}^{n}\left(\mathfrak{y}^*_{j,t} - \nu_t - \beta_k k_{j,t}\right)\left(\mathfrak{y}^*_{j,t-2} - \nu_{t-2} - \beta_k k_{j,t-2}\right) = \left(\alpha^{(C)}\right)^2\sigma^2_\varepsilon \\
&\vdots
\end{aligned}
$$

If the panel consists of T observations, we can identify $T+3$ parameters, including $\nu_1, \ldots, \nu_T$, β_k, $\alpha^{(C)}$, σ^2_ε, using the $T\left(T-1\right)/2$ moments based on all available pairs of time periods.[2]

While the moment conditions in (S.1) demonstrate identification in the cross-section of certain parameters, these moments are not suitable for our limit theory which requires estimating functions to have a martingale difference sequence property. To address this issue we propose the following moment conditions, which is similar to Ackerberg, Caves and Frazer's (2015) parametric rendition

[2] It may be tempting to use the Yule-Walker equation

$$
\begin{aligned}
E\left[\omega_{j,t}\omega_{j,t}\right] &= \sigma^2_\nu + \sigma^2_\varepsilon \\
E\left[\omega_{j,t}\omega_{j,t-1}\right] &= \rho^{(A)}\sigma^2_\nu + \rho^{(C)}\sigma^2_\varepsilon
\end{aligned}
$$

but the expectation operator on the LHS refers to the joint distribution involving both time series and cross-sectional variations, and as such, is not implementable in cross-sectional data.

of Olley and Pakes' (1996) moment condition. Recall our assumption that β_l and $\phi_t\left(i_{j,t}, k_{j,t}\right)$ are known. From

$$\mathfrak{y}_{j,t} = \beta_l l_{j,t} + \phi_t\left(i_{j,t}, k_{j,t}\right) + \eta_{j,t},$$

we obtain

$$\mathfrak{y}^*_{j,t} = \mathfrak{y}_{j,t} - \beta_l l_{j,t} = \phi_t\left(i_{j,t}, k_{j,t}\right) + \eta_{j,t}$$

from which we further obtain

$$\eta_{j,t} = \mathfrak{y}^*_{j,t} - \phi_t\left(i_{j,t}, k_{j,t}\right). \tag{S.2}$$

We also have

$$\mathfrak{y}^*_{j,t} - \beta_k k_{j,t} = \nu_t + \varepsilon_{j,t} + \eta_{j,t}$$

so

$$\varepsilon_{j,t} = \mathfrak{y}^*_{j,t} - \beta_k k_{j,t} - \nu_t - \eta_{j,t}. \tag{S.3}$$

Combining (S.2) and (S.3), we obtain

$$\begin{aligned}\varepsilon_{j,t} &= \mathfrak{y}^*_{j,t} - \beta_k k_{j,t} - \nu_t - \left(\mathfrak{y}^*_{j,t} - \phi_t\left(i_{j,t}, k_{j,t}\right)\right)\\ &= \phi_t\left(i_{j,t}, k_{j,t}\right) - \nu_t - \beta_k k_{j,t},\end{aligned}$$

which can be combined with

$$\begin{aligned}\mathfrak{y}^*_{j,t+1} - \beta_k k_{j,t+1} &= \nu_{t+1} + \varepsilon_{j,t+1} + \eta_{j,t+1}\\ &= \nu_{t+1} + \left(\alpha^{(C)}\varepsilon_{j,t} + e^{(C)}_{j,t+1}\right) + \eta_{j,t+1}\end{aligned}$$

to yield

$$\mathfrak{y}^*_{j,t+1} - \beta_k k_{j,t+1} = \nu_{t+1} + \alpha^{(C)}\left(\phi_t\left(i_{j,t}, k_{j,t}\right) - \nu_t - \beta_k k_{j,t}\right) + e^{(C)}_{j,t+1} + \eta_{j,t+1}.$$

After some straightforward algebra, we obtain

$$\mathfrak{y}^*_{j,t+1} = \beta^*_{0,t+1} + \beta_k k_{j,t+1} + \alpha^{(C)}\left(\phi_t\left(i_{j,t}, k_{j,t}\right) - \beta_k k_{j,t}\right) + \left(e^{(C)}_{j,t+1} + \eta_{j,t+1}\right),$$

where

$$\beta^*_{0,t+1} \equiv \nu_{t+1} - \alpha^{(C)}\nu_t$$

and the error $e^{(C)}_{j,t+1} + \eta_{j,t+1}$ is orthogonal to the past variables such as $k_{j,t}, \mathfrak{y}^*_{j,t}$. Therefore, if the $z_{j,t}$ is an instrument uncorrelated with the error $e^{(C)}_{j,t+1} + \eta_{j,t+1}$, we can use the moment (7) as a basis of estimating the parameters $\left(\beta^*_{0,t+1}, \beta_k, \alpha^{(C)}\right)$.

II Detailed Calculations for Example 1

First note that it follows that

$$E\left[z_s u_{s+1} | \mathcal{G}_{\tau n,(s-\min(1,\tau_0))n+i}\right] = z_s E\left[u_{s+1}|\mathcal{C}\right],$$

where the equality uses the fact that z_s is measurable with respect to $\mathcal{G}_{\tau n,(s-\min(1,\tau_0))n+i}$ and that u_{s+1} is independent of $\sigma\left(\{z_s, z_{s-1}, \ldots, z_{\tau_0}\}\right)$. To evaluate $E\left[u_{s+1}|\mathcal{C}\right]$ consider the joint distribution of u_{s+1} for $s<0$ and $\nu_1 = z_1$, which is Gaussian,

$$N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho^{|s|} \\ \rho^{|s|} & \frac{1}{1-\rho^2} \end{bmatrix}\right).$$

This implies that $E\left[u_{s+1}|\mathcal{C}\right] = \rho^{|s|}\left(1-\rho^2\right) z_1$. Evaluating the L_2 norm of Condition 1(vii) leads to

$$\begin{aligned}
&\left\| E\left[\psi^{\nu}_{\tau,s}\right| \mathcal{G}_{\tau n,(s-\min(1,\tau_0)-1)n+i}\right] \Big\|_2^2 \\
&\leq |\rho|^{|s|}\left(1-\rho^2\right)\left(E\left[\left(z_s z_1 - E\left[z_s z_1\right]\right)^2\right] + \left(E\left[z_s z_1\right]\right)^2\right) \\
&= |\rho|^{|s|}\left(1-\rho^2\right) E\left[\left(z_s z_1 - E\left[z_s z_1\right]\right)^2\right] + |\rho|^{|s|}\left(1-\rho^2\right)\left(\frac{\rho^{1-s}}{1-\rho^2}\right)^2 \\
&= |\rho|^{|s|}\left(1-\rho^2\right)\frac{\rho^{2-2s}+1}{\left(1-\rho^2\right)^2} + |\rho|^{|s|}\frac{\rho^{2-2s}}{1-\rho^2} \\
&= |\rho|^{|s|}\frac{\rho^{2+2|s|}+1}{1-\rho^2} + |\rho|^{|s|}\frac{\rho^{2+2|s|}}{1-\rho^2} \\
&= O\left(|\rho|^{|s|}\right) = o\left(|s|^{-(1+\delta)}\right)
\end{aligned}$$

such that

$$\left\| E\left[\psi^{\nu}_{\tau,s}\right| \mathcal{G}_{\tau n,(s-\min(1,\tau_0)-1)n+i}\right] \Big\|_2 = O\left(|\rho|^{|s|/2}\right) = o\left(|s|^{-(1+\delta)/2}\right).$$

III Details for the Proof of Theorem 1

III.1 Proof of (41)

In this section we show that $\sum_{q=1}^{k_n} E\left[\left|\ddot{\psi}_q\right|^{2+\delta}\right] \to 0$ where $\ddot{\psi}_q$ is defined in (36). Note that, for any fixed n and given q, and thus for a corresponding unique vector (t,i), there exists a unique

$j \in \{1,\ldots,k\}$ such that $\tau_0 + [\tau r_{j-1}] < t \leq \tau_0 + [\tau r_j]$. Then,

$$\begin{aligned}
\ddot{\psi}_{q(i,t)} &= \sum_{l=1}^{k} \lambda_l' \left(\Delta\tilde{\psi}_{it}(r_l) - E\left[\Delta\tilde{\psi}_{it}(r_l) \middle| \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right) \\
&= \lambda_{1,y}' \left(\tilde{\psi}_{it}^{y} - E\left[\tilde{\psi}_{it}^{y} \middle| \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right) 1\{j = 1\} \\
&+ \lambda_{j,\nu}' \left(\tilde{\psi}_{\tau,t}^{\nu}(r_j) - E\left[\tilde{\psi}_{\tau,t}^{\nu}(r_j) \middle| \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right) 1\{[\tau r_{j-1}] < t - \tau_0 \leq [\tau r_j]\} 1\{\ i = 1\},
\end{aligned}$$

where all remaining terms in the sum are zero because of by (36), (38) and (39). For the subsequent inequalities, fix $q \in \{1, ..., k_n\}$ (and the corresponding (t,i) and j) arbitrarily. Introduce the shorthand notation $1_j = 1\{j = 1\}$ and $1_{ij} = 1\{[\tau r_{j-1}] < t - \tau_0 \leq [\tau r_j]\} 1\{\ i = 1\}$.

First, note that for $\delta \geq 0$, and by Jensen's inequality applied to the empirical measure $\frac{1}{4}\sum_{i=1}^{4} x_i$ we have that

$$\begin{aligned}
&\left| \ddot{\psi}_q \right|^{2+\delta} \\
&= 4^{2+\delta} \left| \frac{1}{4} \lambda_{1,y}' \left(\tilde{\psi}_{it}^{y} - E\left[\tilde{\psi}_{it}^{y} | \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right) 1_j + \frac{1}{4} \lambda_{j,\nu}' \left(\tilde{\psi}_{\tau,t}^{\nu}(r_j) - E\left[\tilde{\psi}_{\tau,t}^{\nu}(r_j) | \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right) 1_{ij} \right|^{2+\delta} \\
&\leq 4^{2+\delta} \left(\frac{1}{4} \|\lambda_{1,y}\|^{2+\delta} \left\| \tilde{\psi}_{it}^{y} \right\|^{2+\delta} + \frac{1}{4} \|\lambda_{1,y}\|^{2+\delta} \left\| E\left[\tilde{\psi}_{it}^{y} | \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right\|^{2+\delta} \right) 1_j \\
&+ 4^{2+\delta} \left(\frac{1}{4} \|\lambda_{j,\nu}\|^{2+\delta} \left\| \tilde{\psi}_{\tau,t}^{\nu}(r_j) \right\|^{2+\delta} + \frac{1}{4} \|\lambda_{j,\nu}\|^{2+\delta} \left\| E\left[\tilde{\psi}_{\tau,t}^{\nu}(r_j) | \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right\|^{2+\delta} \right) 1_{ij} \\
&= 2^{2+2\delta} \left(\|\lambda_{1,y}\|^{2+\delta} \left\| \tilde{\psi}_{it}^{y} \right\|^{2+\delta} + \|\lambda_{1,y}\|^{2+\delta} \left\| E\left[\tilde{\psi}_{it}^{y} | \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right\|^{2+\delta} \right) 1_j \\
&+ 2^{2+2\delta} \left(\|\lambda_{j,\nu}\|^{2+\delta} \left\| \tilde{\psi}_{\tau,t}^{\nu}(r_j) \right\|^{2+\delta} + \|\lambda_{j,\nu}\|^{2+\delta} \left\| E\left[\tilde{\psi}_{\tau,t}^{\nu}(r_j) | \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right\|^{2+\delta} \right) 1_{ij}.
\end{aligned}$$

We further use the definitions in (15) such that by Jensen's inequality and for $i = 1$ and $t \in [\tau_0 + 1, \tau_0 + \tau]$

$$\begin{aligned}
&\left\| \tilde{\psi}_{\tau,t}^{\nu}(r_j) \right\|^{2+\delta} + \left\| E\left[\tilde{\psi}_{\tau,t}^{\nu}(r_j) | \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right\|^{2+\delta} \\
&\leq \frac{1}{\tau^{1+\delta/2}} \left(\left\| \psi_{\tau,t}^{\nu} \right\|^{2+\delta} + \left(E\left[\left\| \psi_{\tau,t}^{\nu} \right\| | \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right)^{2+\delta} \right) \\
&\leq \frac{1}{\tau^{1+\delta/2}} \left(\left\| \psi_{\tau,t}^{\nu} \right\|^{2+\delta} + E\left[\left\| \psi_{\tau,t}^{\nu} \right\|^{2+\delta} | \mathcal{G}_{\tau n, t^* n + i - 1} \right] \right)
\end{aligned}$$

while for $i > 1$ or $t \notin [\tau_0 + 1, \tau_0 + \tau]$,

$$\left\| \tilde{\psi}_{it}^{\nu} \right\| = 0.$$

Similarly, for $t \in [1, ..., T]$

$$\begin{aligned}&\left\|\tilde{\psi}_{it}^{y}\right\|^{2+\delta} + \left\|E\left[\tilde{\psi}_{it}^{y}|\mathcal{G}_{\tau n,t^*n+i-1}\right]\right\|^{2+\delta} \\ &\leq \frac{1}{n^{1+\delta/2}}\left(\|\psi_{it}^{y}\|^{2+\delta} + E\left[\|\psi_{it}^{y}\|^{2+\delta}|\mathcal{G}_{\tau n,t^*n+i-1}\right]\right)\end{aligned}$$

while for $t \notin [1, ..., T]$

$$\left\|\tilde{\psi}_{it}^{y}\right\| = 0.$$

Noting that $\|\lambda_{j,y}\| \leq 1$ and $\|\lambda_{j,\nu}\| < 1$,

$$\begin{aligned}E\left[\left|\ddot{\psi}_q\right|^{2+\delta}\middle|\mathcal{G}_{\tau n,q-1}\right] \leq &\frac{2^{3+2\delta}1\{i=1, t\in[\tau_0+1, \tau_0+\tau]\}}{\tau^{1+\delta/2}}E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^{2+\delta}\middle|\mathcal{G}_{\tau n,t^*n+i-1}\right] \\ &+ \frac{2^{3+2\delta}1\{t\in[1,...,T]\}}{n^{1+\delta/2}}E\left[\|\psi_{it}^{y}\|^{2+\delta}\middle|\mathcal{G}_{\tau n,t^*n+i-1}\right], \end{aligned} \tag{S.4}$$

where the inequality in (S.4) holds for $\delta \geq 0$.

To show that (41) holds note that from (S.4), the law of iterated expectations and Condition 1 it follows that for some constant $C < \infty$,

$$\begin{aligned}\sum_{q=1}^{k_n} E\left[\left|\ddot{\psi}_q\right|^{2+\delta}\right] &= \sum_{q=1}^{k_n} E\left[E\left[\left|\ddot{\psi}_q\right|^{2+\delta}|\mathcal{G}_{\tau n,q-1}\right]\right] \\ &\leq \frac{2^{3+2\delta}}{\tau^{1+\delta/2}}\sum_{t=\tau_0+1}^{\tau_0+\tau} E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^{2+\delta}\right] \\ &+ \frac{2^{3+2\delta}}{n^{1+\delta/2}}\sum_{t=1}^{T}\sum_{i=1}^{n} E\left[\|\psi_{it}^{y}\|^{2+\delta}\right] \\ &\leq \frac{2^{3+2\delta}\tau C}{\tau^{1+\delta/2}} + \frac{2^{3+2\delta}nTC}{n^{1+\delta/2}} = \frac{2^{3+2\delta}C}{\tau^{\delta/2}} + \frac{2^{3+2\delta}TC}{n^{\delta/2}} \to 0\end{aligned}$$

because $2^{3+2\delta}C$ and T are fixed as $\tau, n \to \infty$.

III.2 Proof of (42)

Consider the probability limit of $\sum_{q=1}^{k_n} \ddot{\psi}_q^2$. We have

$$
\begin{aligned}
&\sum_{q=1}^{k_n} \ddot{\psi}_q^2 \\
&= \frac{1}{\tau} \sum_{j=1}^{k} \sum_{t=\tau_0+1}^{\tau_0+\tau} \left(\lambda_{j,\nu}' \left(\psi_{\tau,t}^{\nu} - E\left[\psi_{\tau,t}^{\nu} | \mathcal{G}_{\tau n, t^* n}\right]\right)\right)^2 1_{ij} && \text{(S.5)} \\
&+ \frac{2}{\sqrt{\tau n}} \sum_{\substack{t \in (\tau_0+1,\ldots,\tau_0+\tau) \\ \cap \{1,..,T\}}} \sum_{j=1}^{k} \lambda_{j,\nu}' \left(\psi_{\tau,t}^{\nu} - E\left[\psi_{\tau,t}^{\nu} | \mathcal{G}_{\tau n, s^* n}\right]\right) \left(\psi_{1t}^{y} - E\left[\psi_{1t}^{y} | \mathcal{G}_{\tau n, t^* n}\right]\right)' \lambda_{j,y} 1_{1j} 1_j && \text{(S.6)} \\
&+ \frac{1}{n} \left(\sum_{t \in \{1,..,T\}} \sum_{i=1}^{n} \left(\lambda_{1,y}' \left(\psi_{n,it}^{y} - E\left[\psi_{n,it}^{y} | \mathcal{G}_{\tau n, t^* n+i-1}\right]\right)\right) \right)^2 . && \text{(S.7)}
\end{aligned}
$$

Note that

$$
\begin{aligned}
\left(\lambda_{j,\nu}' \left(\psi_{\tau,t}^{\nu} - E\left[\psi_{\tau,t}^{\nu} | \mathcal{G}_{\tau n, t^* n}\right]\right)\right)^2 \leq \left(\lambda_{j,\nu}' \psi_{\tau,t}^{\nu}\right)^2 &+ 2 \left\|\psi_{\tau,t}^{\nu}\right\| \left\|\lambda_{j,\nu}\right\|^2 \left\|E\left[\psi_{\tau,t}^{\nu} | \mathcal{G}_{\tau n, t^* n}\right]\right\| \\
&+ \left\|\lambda_{j,\nu}\right\|^2 \left\|E\left[\psi_{\tau,t}^{\nu} | \mathcal{G}_{\tau n, t^* n}\right]\right\| .
\end{aligned}
$$

Also note that $E\left[\psi_{\tau,t}^{\nu} | \mathcal{G}_{\tau n, t^* n}\right] = 0$ when $t > T$ implies that

$$
\begin{aligned}
&\frac{1}{\tau} \sum_{j=1}^{k} \sum_{t=\tau_0+1}^{\tau_0+\tau} \left| \left(\left(\lambda_{j,\nu}' \left(\psi_{\tau,t}^{\nu} - E\left[\psi_{\tau,t}^{\nu} | \mathcal{G}_{\tau n, t^* n}\right]\right)\right)^2 - \left(\lambda_{j,\nu}' \psi_{\tau,t}^{\nu}\right)^2 \right) 1_{ij} \right| && \text{(S.8)} \\
&\leq \frac{1}{\tau} \sum_{j=1}^{k} \sum_{t \in \{\tau_0+1,\ldots,\tau_0+\tau\} \cap \{1,..,T\}} \left(2 \left\|\psi_{\tau,t}^{\nu}\right\| \left\|\lambda_{j,\nu}\right\|^2 \left\|E\left[\psi_{\tau,t}^{\nu} | \mathcal{G}_{\tau n, t^* n}\right]\right\| + \left\|\lambda_{j,\nu}\right\|^2 \left\|E\left[\psi_{\tau,t}^{\nu} | \mathcal{G}_{\tau n, t^* n}\right]\right\|^2 \right) .
\end{aligned}
$$

Because Condition 2 implies that

$$
\frac{1}{\tau} \sum_{j=1}^{k} \sum_{t=t=\tau_0+1}^{\tau_0+\tau} \left(\lambda_{j,\nu}' \psi_{\tau,t}^{\nu}\right)^2 1\left\{\tau_0 + [\tau r_{j-1}] < t \leq \tau_0 + [\tau r_j]\right\} \xrightarrow{p} \sum_{j=1}^{k} \lambda_{j,\nu}' \left(\Omega_\nu(r_j) - \Omega_\nu(r_{j-1})\right) \lambda_{j,\nu},
$$

the term (S.5) is equal to

$$
\sum_{j=1}^{k} \lambda_{j,\nu}' \left(\Omega_\nu(r_j) - \Omega_\nu(r_{j-1})\right) \lambda_{j,\nu} + o_p(1)
$$

if the RHS of (S.8) is $o_p(1)$. To show that it is indeed the case, note that by the Markov inequality and the Cauchy-Schwarz inequality it is enough to show that

$$\frac{1}{\tau}\sum_{j=1}^{k}\sum_{\substack{t\in\{\tau_0+1,...,\tau_0+\tau\}\\ \cap\{1,..,T\}}}\|\lambda_{j,\nu}\|^2\left(2\sqrt{E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2\right]E\left[\left\|E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,t^*n}\right]\right\|^2\right]}+E\left[\left\|E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,t^*n}\right]\right\|^2\right]\right) \tag{S.9}$$

$$\begin{aligned}
\leq&\frac{1}{\tau}\sum_{j=1}^{k}\|\lambda_{j,\nu}\|^2\left(\sum_{t=0}^{T}3E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2\right]+\sum_{t=\tau_0}^{-1}\left(2\sqrt{E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2\right]}\vartheta_t+\vartheta_t^2\right)\right)\\
\leq&O\left(\tau^{-1}\right)+\frac{C}{\tau}\sum_{t=\tau_0}^{-1}\left(2\sqrt{E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2\right]}\left(|t|^{1+\delta}\right)^{-1/2}+C\left(|t|^{1+\delta}\right)^{-1}\right)\\
\leq&\frac{2C\sup_t\sqrt{E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2\right]}}{\tau}\sum_{t=\tau_0}^{-1}\left(|t|^{1+\delta}\right)^{-1/2}\\
&+o\left(\tau^{-1}\sum_{t=\tau_0}^{-1}\left(|t|^{1+\delta}\right)^{-1/2}\right)+O\left(\tau^{-1}\right)\to 0,
\end{aligned}$$

where the first inequality follows from Condition 1(vii). The second inequality follows from the fact that T is fixed and bounded and Condition 1(vii). The final result uses that $\sup_t E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^{2+\delta}\right]\leq C<\infty$ by Condition 1(iv) and the fact that $|\tau_0|\leq\tau$, $t/\tau\leq 1$ for $t\in[1,...,\tau]$, such that

$$\begin{aligned}
\tau^{-1}\sum_{t=\tau_0}^{-1}\left(|t|^{1+\delta}\right)^{-1/2}&=\tau^{-1}\sum_{t=1}^{|\tau_0|}\left(t^{1+\delta}\right)^{-1/2}\\
&\leq\tau^{-1}\sum_{t=1}^{|\tau_0|}\left(\frac{t}{\tau}\right)^{-1/2}t^{-(1/2+\delta/2)}\\
&\leq\tau^{-1/2}\sum_{t=1}^{\infty}t^{-(1+\delta/2)}=O\left(\tau^{-1/2}\right).
\end{aligned}$$

The last equality above uses the fact $\sum_{t=1}^{\infty}t^{-(1+\delta/2)}<\infty$ for any $\delta>0$.

Next we show that (S.6) is $o_p(1)$. For this purpose, we note

$$E\left[\left\|\frac{2}{\sqrt{\tau n}}\sum_{\substack{t\in(\tau_0+1,\ldots,\tau_0+\tau)\\ \cap\{1,..,T\}}}\sum_{j=1}^{k}\lambda'_{j,\nu}\left(\psi^{\nu}_{\tau,t}-E\left[\psi^{\nu}_{\tau,t}|\mathcal{G}_{\tau n,s^*n}\right]\right)\left(\psi^{y}_{1t}-E\left[\psi^{y}_{1t}|\mathcal{G}_{\tau n,t^*n}\right]\right)'\lambda_{1,y}1_{1j}1_{1}\right\|\right]$$

$$\leq\frac{2}{\sqrt{\tau n}}\sum_{\substack{t\in\{\tau_0+1,\ldots,\tau_0+\tau\}\\ \cap\{1,..,T\}}}\left\{\left(E\left[\left|\sum_{j=1}^{k}\lambda'_{j\nu}\left(\psi^{\nu}_{\tau,t}-E\left[\psi^{\nu}_{\tau,t}|\mathcal{G}_{\tau n,t^*n}\right]\right)\right|^{2}\right]\right)^{1/2}\right.$$

$$\left.\times\left(E\left[\left|\left(\psi^{y}_{1t}-E\left[\psi^{y}_{1t}|\mathcal{G}_{\tau n,t^*n}\right]\right)'\lambda_{1y}\right|^{2}\right]\right)^{1/2}\right\} \tag{S.10}$$

$$\leq\frac{2^2\sqrt{k}}{\sqrt{\tau n}}\sup_t\left(E\left[\left\|\psi^{\nu}_{\tau,t}\right\|\right]\right)^{1/2}\sum_{\substack{t\in\{\tau_0,\ldots,\tau_0+\tau\}\\ \cap\{1,..,T\}}}\left(E\left[\left|\left(\psi^{y}_{1t}-E\left[\psi^{y}_{1t}|\mathcal{G}_{\tau n,t^*n}\right]\right)'\lambda_{1,y}\right|^{2}\right]\right)^{1/2} \tag{S.11}$$

$$\leq\frac{2^3\sqrt{k}T}{\sqrt{\tau n}}\sup_t\left(E\left[\left\|\psi^{\nu}_{\tau,t}\right\|\right]\right)^{1/2}\left(\sup_{i,t}E\left[\left\|\psi^{y}_{n,it}\right\|^{2}\right]\right)^{1/2}\to 0 \tag{S.12}$$

where the first inequality in (S.10) follows from the Cauchy-Schwarz inequalities. Then we have in (S.11), by Condition 1(iii) and the Hölder inequality that

$$E\left[\left|\left(\psi^{y}_{1t}-E\left[\psi^{y}_{1t}|\mathcal{G}_{\tau n,t^*n}\right]\right)'\lambda_{y}\right|^{2}\right]\leq 2E\left[\left\|\psi^{y}_{1t}\right\|^{2}\right]$$

such that (S.12) follows. We note that (S.12) goes to zero because of Condition 1(iv) as long as $T/\sqrt{\tau n}\to 0$. Clearly, this condition holds as long as T is held fixed, but holds under weaker conditions as well.

Next the limit of (S.7) is, by Condition 1(v) and Condition 3,

$$\frac{1}{n}\sum_{t\in\{1,..,T\}}\sum_{i=1}^{n}\left(\lambda'_{1,y}\left(\psi^{y}_{n,it}-E\left[\psi^{y}_{n,it}|\mathcal{G}_{\tau n,t^*n+i-1}\right]\right)\right)^{2}\xrightarrow{p}\sum_{t\in\{1,..,T\}}\lambda'_{1,y}\Omega_{ty}\lambda_{1,y}.$$

This verifies (42).

III.3 Proof of (43)

For (43) we check that

$$\sup_n E\left[\left(\sum_{q=1}^{k_n}E\left[\left|\ddot{\psi}_q\right|^{2}\middle|\mathcal{G}_{\tau n,q-1}\right]\right)^{1+\delta/2}\right]<\infty. \tag{S.13}$$

First, use (S.4) with $\delta = 0$ to obtain

$$\sum_{q=1}^{k_n} E\left[\left|\ddot{\psi}_q\right|^2 \middle| \mathcal{G}_{\tau n,q-1}\right] \leq \frac{2^3}{\tau}\sum_{t=\tau_0}^{\tau_0+\tau} E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2 \middle| \mathcal{G}_{\tau n,t^* n}\right] + \frac{2^3}{n}\sum_{t\in\{1,..,T\}}\sum_{i=1}^{n} E\left[\left\|\psi_{n,it}^{y}\right\|^2 \middle| \mathcal{G}_{\tau n,t^* n+i-1}\right]. \tag{S.14}$$

Applying (S.14) to (S.13) and using the Hölder inequality implies

$$\begin{aligned} &E\left[\left(\sum_{q=1}^{k_n} E\left[\left|\ddot{\psi}_q\right|^2 \middle| \mathcal{G}_{\tau n,q-1}\right]\right)^{1+\delta/2}\right] \\ &\leq 2^{\delta/2} E\left[\left(\frac{2^3}{\tau}\sum_{t=\tau_0+1}^{\tau_0+\tau} E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2 \middle| \mathcal{G}_{\tau n,t^* n+i-1}\right]\right)^{1+\delta/2}\right] \\ &+ 2^{\delta/2} E\left[\left(\frac{2^3}{n}\sum_{t\in\{\tau_0,\ldots,\tau_0+\tau\}\cap\{1,..,T\}}\sum_{i=1}^{n} E\left[\left\|\psi_{n,it}^{y}\right\|^2 \middle| \mathcal{G}_{\tau n,t^* n+i-1}\right]\right)^{1+\delta/2}\right]. \end{aligned}$$

By Jensen's inequality, we have

$$\left(\frac{1}{\tau}\sum_{t=\tau_0+1}^{\tau_0+\tau} E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2 \middle| \mathcal{G}_{\tau n,t^* n+i-1}\right]\right)^{1+\delta/2} \leq \frac{1}{\tau}\sum_{t=\tau_0+1}^{\tau_0+\tau} E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2 \middle| \mathcal{G}_{\tau n,t^* n+i-1}\right]^{1+\delta/2}$$

and

$$E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2 \middle| \mathcal{G}_{\tau n,t^* n+i-1}\right]^{1+\delta/2} \leq E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^{2+\delta} \middle| \mathcal{G}_{\tau n,t^* n+i-1}\right]$$

so that

$$\begin{aligned} E\left[\left(\frac{2^3}{\tau}\sum_{t=\tau_0+1}^{\tau_0+\tau} E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^2 \middle| \mathcal{G}_{\tau n,t^* n+i-1}\right]\right)^{1+\delta/2}\right] &\leq \frac{2^{3+3\delta/2}}{\tau}\sum_{t=\tau_0+1}^{\tau_0+\tau} E\left[E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^{2+\delta} \middle| \mathcal{G}_{\tau n,t^* n+i-1}\right]\right] \\ &\leq 2^{3+3\delta/2}\sup_t E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^{2+\delta}\right] < \infty \end{aligned} \tag{S.15}$$

and similarly,

$$\begin{aligned} &E\left[\left(\frac{2^3}{n}\sum_{t\in\{1,..,T\}}\sum_{i=1}^{n} E\left[\left\|\psi_{n,it}^{y}\right\|^2 |\mathcal{G}_{\tau n,t^* n+i-1}\right]\right)^{1+\delta/2}\right] \\ &\leq \frac{2^{3+3\delta/2}(Tn)^{\delta/2}}{n^{1+\delta/2}}\sum_{t=1}^{T}\sum_{i=1}^{n} E\left[\left\|\psi_{n,it}^{y}\right\|^{2+\delta}\right] \\ &\leq 2^{3+3\delta/2}T^{1+\delta/2}\sup_{i,t} E\left[\left\|\psi_{n,it}^{y}\right\|^{2+\delta}\right] < \infty. \end{aligned} \tag{S.16}$$

By combining (S.15) and (S.16) we obtain the following bound for (S.13),

$$
\begin{aligned}
&E\left[\left(\sum_{q=1}^{k_n} E\left[\left|\ddot{\psi}_q\right|^{2+\delta} |\mathcal{G}_{\tau n,q-1}\right]\right)^{1+\delta/2}\right] \\
&\leq 2^{3+3\delta/2} \sup_t E\left[\left\|\psi^{\nu}_{\tau,t}\right\|^{2+\delta}\right] + 2^{3+3\delta/2} T^{1+\delta/2} \sup_{i,t} E\left[\left\|\psi^{y}_{n,it}\right\|^{2+\delta}\right] < \infty.
\end{aligned}
$$

III.4 Second Term in (40)

Consider

$$
\begin{aligned}
&\sum_{t=\min(1,\tau_0+1)}^{\max(T,\tau_0+\tau)} \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda_j' E\left[\Delta\tilde{\psi}_{it}\left(r_j\right)|\mathcal{G}_{\tau n,t^* n+i-1}\right] \\
&= \tau^{-1/2} \sum_{t=\tau_0+1}^{\tau_0+\tau} \sum_{j=1}^{k} \lambda_{j,\nu}' E\left[\psi^{\nu}_{\tau,t}|\mathcal{G}_{\tau n,t^* n+i-1}\right] 1\left\{\tau_0 + [\tau r_{j-1}] < t \leq \tau_0 + [\tau r_j]\right\} \\
&+ n^{-1/2} \sum_{t=1}^{T} \sum_{i=1}^{n} \lambda_{1,y}' E\left[\psi^{y}_{n,it}|\mathcal{G}_{\tau n,t^* n+i-1}\right],
\end{aligned}
$$

where we defined $r_0 = 0$. Note that

$$
E\left[\psi^{\nu}_{\tau,t}|\mathcal{G}_{\tau n,t^* n+i-1}\right] = 0 \text{ for } t > T
$$

and

$$
E\left[\psi^{y}_{n,it}|\mathcal{G}_{\tau n,t^* n+i-1}\right] = 0.
$$

This implies, using the convention that a term is zero if it is a sum over indices from a to b with $a > b$, as well as the fact that $T \leq \tau_0 + \tau$, that

$$
\begin{aligned}
&\tau^{-1/2} \sum_{t=\tau_0+1}^{\tau_0+\tau} \lambda_{\nu}' E\left[\psi^{\nu}_{\tau,t}|\mathcal{G}_{\tau n,t^* n+i-1}\right] \\
&= \tau^{-1/2} \sum_{t=\tau_0+1}^{T} \sum_{j=1}^{k} \lambda_{j,\nu}' E\left[\psi^{\nu}_{\tau,t}|\mathcal{G}_{\tau n,t^* n+i-1}\right] 1\left\{\tau_0 + [\tau r_{j-1}] < t \leq \tau_0 + [\tau r_j]\right\}.
\end{aligned}
$$

By a similar argument used to show that (S.9) vanishes, and noting that T is fixed while $\tau \to \infty$, it follows that, as long as $\tau_0 \le T$,

$$\begin{aligned}
&E\left[\left|\tau^{-1/2}\sum_{t=\tau_0+1}^{T}\lambda_{j,\nu}' E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,t^*n}\right]\right|^{1+\delta/2}\right] \\
&\le \left(\frac{1}{\tau}\right)^{1/2+\delta/4}(T-\tau_0)^{\delta/2} E\left[\sum_{t=\tau_0+1}^{T}\left|\lambda_{j,\nu}' E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,t^*n}\right]\right|^{1+\delta/2}\right] \\
&\le \left(\frac{1}{\tau}\right)^{1/2+\delta/4}(T-\tau_0)^{\delta/2}\sum_{t=\tau_0+1}^{T} E\left[\left\|E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,t^*n}\right]\right\|^{1+\delta/2}\right] \\
&\le \left(\frac{1}{\tau}\right)^{1/2+\delta/4}(T-\tau_0)^{\delta/2}\sum_{t=\tau_0+1}^{T}\left(E\left[\left\|E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,t^*n}\right]\right\|^{2}\right]\right)^{1/2+\delta/4},
\end{aligned}$$

where the first inequality uses the triangular inequality and the second and third inequalities are based on versions of Jensen's inequality. We continue to use the convention that a term is zero if it is a sum over indices from a to b with $a > b$. Now consider two cases. When $\tau_0 \to -\infty$ use Condition 1(vii) and the fact that $(T-\tau_0)/\tau \le 1$ as well as $(T-\tau_0)^{-\delta/2} \le |\tau_0|^{-\delta/2}$

$$\begin{aligned}
&\left(\frac{1}{\tau}\right)^{1/2+\delta/4}(T-\tau_0)^{\delta/2}\sum_{t=\tau_0}^{T}\left(E\left[\left\|E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,t^*n}\right]\right\|^{2}\right]\right)^{1/2+\delta/4} \\
&\le \tau^{-(1/2+\delta/4)}\sum_{t=0}^{T}\left(E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^{2}\right]\right)^{1/2+\delta/4} + \tau^{-(1/2+\delta/4)} C\sum_{t=\tau_0}^{-1}|t|^{-\left(\frac{1}{2}+\frac{3\delta+\delta^2}{4}\right)} \\
&\le O\left(\tau^{-(1/2+\delta/4)}\right) + \tau^{-(1/2+\delta/4)}|\tau_0|^{1/2-\left(\delta+\delta^2\right)/4} C\sum_{t=\tau_0}^{-1}|t|^{-(1+\delta/2)} \\
&\le O\left(\tau^{-(1/2+\delta/4)}\right) + \tau^{-\delta/2}\left(\frac{|\tau_0|}{\tau}\right)^{1/2-\delta/4} C\sum_{t=1}^{\infty}t^{-(1+\delta/2)} \\
&= O\left(\tau^{-\delta/2}\right) \to 0
\end{aligned}$$

since $\frac{|\tau_0|}{\tau} \to \upsilon$ as $\tau \to \infty$ and $\sum_{t=1}^{\infty} t^{-1}(\log(t+1))^{-(1+\delta)} < \infty$. The second case arises when τ_0 is fixed. Then,

$$\begin{aligned}
&\left(\frac{1}{\tau}\right)^{1/2+\delta/4}(T-\tau_0)^{\delta/2}\sum_{t=\tau_0+1}^{T}\left(E\left[\left\|E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,t^*n}\right]\right\|^{2}\right]\right)^{1/2+\delta/4} \\
&\le \tau^{-1/2-\delta/4}\sup_t\left(E\left[\left\|\psi_{\tau,t}^{\nu}\right\|^{2}\right]\right)^{1/2+\delta/4}(T+|\tau_0|)^{1+\delta/2} \to 0
\end{aligned}$$

as $\tau \to \infty$. In both cases the Markov inequality then implies that

$$\tau^{-1/2} \sum_{t=\tau_0+1}^{\tau_0+\tau} \sum_{i=1}^{n} \sum_{j=1}^{k} \lambda_j' E\left[\Delta\tilde{\psi}_{it}(r_j)\,|\mathcal{G}_{\tau n,t^* n+i-1}\right] = o_p(1),$$

and the conclusion follows consequently.

III.5 Proof of (48)

Define

$$\ddot{\psi}_{\tau,t}^{\nu} = \frac{1}{\sqrt{\tau}}\left(\psi_{\tau,t}^{\nu} - E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right)$$

and

$$\ddot{X}_{n\tau,\nu}(r) = \sum_{t=\tau_0+1}^{\tau_0+[\tau r]} \ddot{\psi}_{\tau,t}^{\nu}$$

such that

$$X_{n\tau,\nu}(r) = \ddot{X}_{n\tau,\nu}(r) + \frac{1}{\sqrt{\tau}} \sum_{t=\tau_0+1}^{\tau_0+[\tau r]} E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right].$$

Let $\ddot{S}_{\tau,s} = \sum_{t=\tau_0+1}^{\tau_0+s} \lambda_\nu' \ddot{\psi}_{\tau,t}^{\nu}$ and

$$S_{\tau,s} = \sum_{t=\tau_0+1}^{\tau_0+s} \lambda_\nu' \left(\ddot{\psi}_{\tau,t}^{\nu} + \frac{1}{\sqrt{\tau}} E\left[\psi_{\tau,t}^{\nu}\middle|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right)$$

as before. Since

$$\begin{aligned}
&P\left(\max_{s\le\tau}|S_{\tau,k+s} - S_{\tau,k}| > c\right)\\
&\le P\left(\max_{s\le\tau}\left|\ddot{S}_{\tau,k+s} - \ddot{S}_{\tau,k}\right| + \max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=\tau_0+k+1}^{\tau_0+s}\lambda_\nu' E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right| > c\right)\\
&\le P\left(\max_{s\le\tau}\left|\ddot{S}_{\tau,k+s} - \ddot{S}_{\tau,k}\right| > \frac{c}{2}\right) \qquad \text{(S.17)}\\
&+ P\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=\tau_0+k+1}^{\tau_0+s}\lambda_\nu' E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right| > \frac{c}{2}\right), \qquad \text{(S.18)}
\end{aligned}$$

note that for each k and τ fixed, $M_s = \ddot{S}_{\tau,s+k} - \ddot{S}_{\tau,k}$ and $\mathcal{F}_{\tau,s} = \sigma(z_{\tau_0},...,z_{\tau_0+s+k})$, $\{M_{\tau,s},\mathcal{F}_{\tau,s}\}$ is a martingale. Note that the filtration $\mathcal{F}_{\tau,s}$ does not depend on τ when τ_0 is held fixed. We prove an extension of Hall and Heyde (1980, Theorems 2.1 and 2.2) to triangular martingale arrays in Lemma 1 and Corollary 2 in Online Appendix IV.[3]

[3]There is only a limited literature on laws of large numbers for triangular arrays of martingales. Andrews (1988) or Kanaya (2017) prove weak laws, de Jong (1996) proves a strong law but without proving a maximal inequality.

We first consider the term (S.18) and show that

$$\lim_{c\to\infty}\limsup_{\tau\to\infty} c^2 P\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=\tau_0+k+1}^{\tau_0+s}\lambda_\nu' E\left[\psi_{\tau,t}^\nu|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right|>\frac{c}{2}\right)=0. \tag{S.19}$$

Using the convention that a term is zero if it is a sum over indices from a to b with $a>b$, note that (S.18) is bounded by Markov inequality by

$$\begin{aligned}
&P\left(\max_{s\le\tau}\left|\sum_{t=\tau_0+k+1}^{\tau_0+s}\lambda_\nu' E\left[\psi_{\tau,t}^\nu|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right|>\frac{c}{2}\sqrt{\tau}\right)\\
&\le\frac{4}{c^2\tau}E\left[\max_{s\le\tau}\left|\sum_{t=\tau_0+k+1}^{\tau_0+s}\lambda_\nu' E\left[\psi_{\tau,t}^\nu|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right|^2\right]\\
&\le\frac{4}{c^2\tau}\sum_{t=\tau_0+k+1}^{\tau_0+\tau}E\left[\left\|E\left[\psi_{\tau,t}^\nu|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right\|^2\right],
\end{aligned}$$

so

$$\begin{aligned}
&c^2P\left(\max_{s\le\tau}\left|\sum_{t=\tau_0+k+1}^{\tau_0+s}\lambda_\nu' E\left[\psi_{\tau,t}^\nu|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right|>\frac{c}{2}\sqrt{\tau}\right)\\
&\le\frac{4}{\tau}\sum_{t=T+1}^{\tau_0+\tau}E\left[\left\|E\left[\psi_{\tau,t}^\nu|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right\|^2\right]\\
&+\frac{4}{\tau}\sum_{t=0}^{T}E\left[\left\|\psi_{\tau,t}^\nu\right\|^2\right]+\frac{4}{\tau}\sum_{t=\tau_0+k+1}^{-1}\vartheta_t.
\end{aligned} \tag{S.20}$$

For the first term in (S.20), we have

$$\frac{4}{\tau}\sum_{t=T+1}^{\tau_0+\tau}E\left[\left\|E\left[\psi_{\tau,t}^\nu|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right\|^2\right]=0$$

because $E\left[\psi_{\tau,t}^\nu|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]=0$ for $t>T$. The second term $\frac{4}{\varepsilon\tau}\sum_{t=0}^{T}E\left[\left\|\psi_{\tau,t}^\nu\right\|^2\right]\to 0$ as $\tau\to\infty$ because T is finite. For the third term in (S.20) note that

$$\begin{aligned}
\frac{4}{\sqrt{\tau}}\sum_{t=\tau_0+k+1}^{-1}\vartheta_t &\le \frac{4K}{\sqrt{\tau}}\sum_{t=\tau_0+k+1}^{-1}\left(|t|^{1+\delta}\right)^{-1/2}\\
&\le\frac{4K\tau^{1/2-\delta/4}}{\sqrt{\tau}}\sum_{t=1}^{\infty}t^{-(1+\delta/4)}=O\left(\tau^{-\delta/4}\right)\to 0.
\end{aligned}$$

These considerations imply the desired result (S.19).

With (S.19), in order to establish (48), it suffices to consider (S.17) and show that

$$\lim_{c\to\infty}\limsup_{\tau\to\infty} c^2P\left(\max_{s\le\tau}\left|\ddot{S}_{\tau,k+s}-\ddot{S}_{\tau,k}\right|>\frac{c}{2}\right)=0.$$

Atchade (2009) and Hill (2010) allow for trinagular arrays but only with respect to a fixed filteration that does not depend on samples size.

Because

$$P\left(\max_{s\le\tau}\left|\ddot{S}_{\tau,k+s}-\ddot{S}_{\tau,k}\right|>\frac{c}{2}\right)\le\frac{4}{c^2\varepsilon}E\left[\max_{s\le\tau}\left|\ddot{S}_{\tau,k+s}-\ddot{S}_{\tau,k}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\ddot{S}_{\tau,k+s}-\ddot{S}_{\tau,k}\right|\ge\frac{c}{2}\right)\right]$$
$$=\frac{4}{c^2\varepsilon}E\left[\max_{s\le\tau}\left|\sum_{t=k+1}^{k+s}\lambda_\nu'\ddot{\psi}_{\tau,t}^{\nu}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\sum_{t=k+1}^{k+s}\lambda_\nu'\ddot{\psi}_{\tau,t}^{\nu}\right|\ge\frac{c}{2}\right)\right]$$

it suffices to prove that

$$\lim_{c\to\infty}\limsup_{\tau\to\infty}E\left[\max_{s\le\tau}\left|\sum_{t=k+1}^{k+s}\lambda_\nu'\ddot{\psi}_{\tau,t}^{\nu}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\sum_{t=k+1}^{k+s}\lambda_\nu'\ddot{\psi}_{\tau,t}^{\nu}\right|\ge c\right)\right]=0. \tag{S.21}$$

We show in Online Appendix III.6 that (S.21) holds as long as $\sup_t E\left[\left|\sqrt{\tau}\lambda_\nu'\ddot{\psi}_{\tau,t}^{\nu}\right|^{2+\delta}\right]<\infty$. The latter is satisfied by Condition 1(iv).

III.6 Proof of (S.21)

We will follow Billingsley (1968, p.208). Let

$$\xi_{\tau,t}=\lambda_\nu'\left(\psi_{\tau,t}^{\nu}-E\left[\psi_{\tau,t}^{\nu}|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right)=\sqrt{\tau}\lambda_\nu'\ddot{\psi}_{\tau,t}^{\nu},$$
$$\xi_{\tau,t}^{u}=\xi_{\tau,t}1\left(|\xi_{\tau,t}|\le u\right),$$
$$\eta_{\tau,t}^{u}=\xi_{\tau,t}^{u}-E\left[\xi_{\tau,t}^{u}\middle|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right],$$
$$\delta_{\tau,t}^{u}=\xi_{\tau,t}-\eta_{\tau,t}^{u}=\xi_{\tau,t}-\xi_{\tau,t}^{u}-E\left[\xi_{\tau,t}-\xi_{\tau,t}^{u}\middle|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]$$

and note that the expectation in (S.21) can be written as

$$E\left[\max_{s\le\tau}\left|\sum_{t=k+1}^{k+s}\lambda_\nu'\ddot{\psi}_{\tau,t}^{\nu}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\sum_{t=k+1}^{k+s}\lambda_\nu'\ddot{\psi}_{\tau,t}^{\nu}\right|\ge c\right)\right]$$
$$=E\left[\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=k+1}^{k+s}\xi_{\tau,t}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=k+1}^{k+s}\xi_{\tau,t}\right|\ge c\right)\right]. \tag{S.22}$$

We will use the fact that

$$\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=k+1}^{k+s}\xi_{\tau,t}\right|^2\le 2\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=k+1}^{k+s}\eta_{\tau,t}^{u}\right|^2+2\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=k+1}^{k+s}\delta_{\tau,t}^{u}\right|^2, \tag{S.23}$$

which also implies that

$$
\begin{aligned}
&1\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\xi_{\tau,t}\right|\ge c\right)\\
&=1\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\xi_{\tau,t}\right|^2\ge c^2\right)\\
&\le 1\left(2\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta^u_{\tau,t}\right|^2\ge \frac{c^2}{2}\right)+1\left(2\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta^u_{\tau,t}\right|^2\ge \frac{c^2}{2}\right)\\
&=1\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta^u_{\tau,t}\right|\ge \frac{c}{2}\right)+1\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta^u_{\tau,t}\right|\ge \frac{c}{2}\right). \qquad \text{(S.24)}
\end{aligned}
$$

Combining (S.22), (S.23) and (S.24), we obtain

$$
\begin{aligned}
&E\left[\max_{s\le\tau}\left|\sum\nolimits_{t=k+1}^{k+s}\lambda'_\nu\ddot{\psi}^\nu_{\tau,t}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\sum\nolimits_{t=k+1}^{k+s}\lambda'_\nu\ddot{\psi}^\nu_{\tau,t}\right|\ge c\right)\right]\\
&\le 2E\left[\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta^u_{\tau,t}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta^u_{\tau,t}\right|\ge \frac{c}{2}\right)\right] \qquad \text{(S.25)}\\
&+2E\left[\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta^u_{\tau,t}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta^u_{\tau,t}\right|\ge \frac{c}{2}\right)\right] \qquad \text{(S.26)}\\
&+2E\left[\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta^u_{\tau,t}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta^u_{\tau,t}\right|\ge \frac{c}{2}\right)\right] \qquad \text{(S.27)}\\
&+2E\left[\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta^u_{\tau,t}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta^u_{\tau,t}\right|\ge \frac{c}{2}\right)\right]. \qquad \text{(S.28)}
\end{aligned}
$$

We will bound each term (S.25) - (S.28). First, we have

$$
\begin{aligned}
(S.25)&=2E\left[\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta^u_{\tau,t}\right|^2\cdot 1\left(\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta^u_{\tau,t}\right|\ge \frac{c}{2}\right)\right]\\
&\le 2\cdot\frac{4}{c^2}E\left[\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta^u_{\tau,t}\right|^4\right].\\
&\le 2\cdot\frac{4}{c^2}\left(\frac{4}{3}\right)^4\frac{1}{\tau^2}E\left[\left|\sum\nolimits_{t=k+1}^{k+\tau}\eta^u_{\tau,t}\right|^4\right]. \qquad \text{(S.29)}
\end{aligned}
$$

We can note that

$$
E\left[\max_{s\le\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta^u_{\tau,t}\right|^4\right]\le\left(\frac{4}{3}\right)^4\frac{1}{\tau^2}E\left[\left|\sum\nolimits_{t=k+1}^{k+\tau}\eta^u_{\tau,t}\right|^4\right]
$$

by using Corollary 1. We can then note the fact that by construction, (i) $\eta^u_{\tau,t}$ is a martingale difference; and (ii) it is bounded by $2u$, which allows us to follow Billingsley's (1968, p.207) argument, leading to $\frac{1}{\tau^2}E\left[\left|\sum_{t=k+1}^{k+s}\eta_{\tau,t}^u\right|^4\right] \leq 6\left(2u\right)^4$. Therefore, we have

$$E\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=k+1}^{k+s}\eta_{\tau,t}^u\right|^4\right] \leq \left(\frac{4}{3}\right)^4\cdot 6\left(2u\right)^4. \tag{S.30}$$

Combining (S.29) and (S.30), we obtain

$$(S.25) \leq 2\frac{4}{c^2}\left(\frac{4}{3}\right)^4 6\left(2u\right)^4 = \frac{Cu^4}{c^2}, \tag{S.31}$$

where C denotes a generic finite constant.

Second, we have

$$\begin{aligned}(S.26) &= 2E\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=k+1}^{k+s}\delta_{\tau,t}^u\right|^2\cdot 1\left(\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=k+1}^{k+s}\delta_{\tau,t}^u\right|\geq\frac{c}{2}\right)\right]\\ &\leq 2E\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=k+1}^{k+s}\delta_{\tau,t}^u\right|^2\right].\end{aligned} \tag{S.32}$$

Using Lemma 1 in Online Appendix IV, and the fact that by construction, $\delta_{\tau,t}^u$ is a martingale difference, we can conclude that

$$\begin{aligned}E\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum_{t=k+1}^{k+s}\delta_{\tau,t}^u\right|^2\right] &= \frac{1}{\tau}E\left[\max_{s\leq\tau}\left|\sum_{t=k+1}^{k+s}\delta_{\tau,t}^u\right|^2\right]\\ &\leq \frac{1}{\tau}\cdot 4E\left[\left(\sum_{t=k+1}^{k+\tau}\delta_{\tau,t}^u\right)^2\right]\\ &= \frac{4}{\tau}\sum_{t=k+1}^{k+\tau}E\left[\left(\delta_{\tau,t}^u\right)^2\right].\end{aligned}$$

Now, note that

$$\begin{aligned}E\left[\left(\delta_{\tau,t}^u\right)^2\right] &= E\left[\left(\xi_{\tau,t}-\xi_{\tau,t}^u-E\left[\xi_{\tau,t}-\xi_{\tau,t}^u\middle|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)n+1)}\right]\right)^2\right]\\ &\leq E\left[\left(\xi_{\tau,t}-\xi_{\tau,t}^u-E\left[\xi_{\tau,t}-\xi_{\tau,t}^u\right]\right)^2\right]\\ &\leq E\left[\left(\xi_{\tau,t}-\xi_{\tau,t}^u\right)^2\right]\\ &= E\left[\left(\xi_{\tau,t}-\xi_{\tau,t}1\left(\left|\xi_{\tau,t}\right|\leq u\right)\right)^2\right]\\ &= E\left[\left(\xi_{\tau,t}1\left(\left|\xi_{\tau,t}\right|>u\right)\right)^2\right]\\ &= E\left[\xi_{\tau,t}^2 1\left(\left|\xi_{\tau,t}\right|>u\right)\right]\\ &\leq \frac{1}{u^\delta}E\left[\xi_{\tau,t}^{2+\delta}\right]\leq\frac{1}{u^\delta}\sup_t E\left[\xi_{\tau,t}^{2+\delta}\right],\end{aligned}$$

where the first inequality is by Billingsley's (1968, p. 184) Lemma 1. It follows that

$$E\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta_{\tau,t}^{u}\right|^{2}\right]\leq\frac{C}{u^{\delta}}\sup_{t}E\left[\xi_{\tau,t}^{2+\delta}\right]. \tag{S.33}$$

Combining (S.32) and (S.33), we obtain

$$(S.26)\leq\frac{C}{u^{\delta}}\cdot\sup_{t}E\left[\xi_{\tau,t}^{2+\delta}\right]. \tag{S.34}$$

Third, we have

$$\begin{aligned}(S.27)&=2E\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta_{\tau,t}^{u}\right|^{2}\cdot 1\left(\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta_{\tau,t}^{u}\right|\geq\frac{c}{2}\right)\right]\\&\leq 2\left(E\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta_{\tau,t}^{u}\right|^{4}\right]\right)^{\frac{1}{2}}\left(E\left[1\left(\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta_{\tau,t}^{u}\right|\geq\frac{c}{2}\right)^{2}\right]\right)^{\frac{1}{2}}\\&=2\cdot\left(\left(\frac{4}{3}\right)^{4}\cdot 6\left(2u\right)^{4}\right)^{\frac{1}{2}}\cdot P\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta_{\tau,t}^{u}\right|\geq\frac{c}{2}\right],\end{aligned} \tag{S.35}$$

where the first inequality is by Cauchy-Schwarz and the last equality is by (S.30). We further have

$$\begin{aligned}P\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta_{\tau,t}^{u}\right|\geq\frac{c}{2}\right]&\leq\frac{4}{c^{2}}E\left[\left(\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta_{\tau,t}^{u}\right|\right)^{2}\right]\\&\leq\frac{4}{c^{2}}\frac{C}{u^{\delta}}\sup_{t}E\left[\xi_{\tau,t}^{2+\delta}\right]\\&=\frac{C}{c^{2}u^{\delta}}\sup_{t}E\left[\xi_{\tau,t}^{2+\delta}\right],\end{aligned} \tag{S.36}$$

where the first inequality is by Markov, and the second inequality is by (S.33). Combining (S.35) and (S.36), we obtain

$$(S.27)\leq\frac{Cu^{2-\delta}}{c^{2}}\sup_{t}E\left[\xi_{\tau,t}^{2+\delta}\right]. \tag{S.37}$$

Fourth, we have

$$\begin{aligned}(S.28)&=2E\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta_{\tau,t}^{u}\right|^{2}\cdot 1\left(\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\eta_{\tau,t}^{u}\right|\geq\frac{c}{2}\right)\right]\\&\leq 2E\left[\max_{s\leq\tau}\left|\frac{1}{\sqrt{\tau}}\sum\nolimits_{t=k+1}^{k+s}\delta_{\tau,t}^{u}\right|^{2}\right]\\&\leq\frac{C}{u^{\delta}}\sup_{t}E\left[\xi_{\tau,t}^{2+\delta}\right],\end{aligned} \tag{S.38}$$

where the second inequality is by (S.33).

Combining (S.31), (S.34), (S.37), (S.28), we obtain that

$$
\begin{aligned}
& E\left[\max_{s\leq\tau}\left|\sum_{t=k+1}^{k+s}\lambda_\nu'\ddot{\psi}_{\tau,t}^{\nu}\right|^2\cdot 1\left(\max_{s\leq\tau}\left|\sum_{t=k+1}^{k+s}\lambda_\nu'\ddot{\psi}_{\tau,t}^{\nu}\right|\geq c\right)\right] \\
& \leq \frac{Cu^4}{c^2}+\frac{C}{u^\delta}.\sup_t E\left[\xi_{\tau,t}^{2+\delta}\right]+\frac{Cu^{2-\delta}}{c^2}\sup_t E\left[\xi_{\tau,t}^{2+\delta}\right]+\frac{C}{u^\delta}\sup_t E\left[\xi_{\tau,t}^{2+\delta}\right].
\end{aligned}
$$

In view of (S.21), it suffices to prove that we can choose $u\to\infty$ as a function of c such that the terms above all converge to zero as $c\to\infty$. This we can do by choosing $u=c^{1/3}$, for example.

IV A Maximal Inequality for Triangular Arrays

In this section we extend Hall and Heyde (1980) Theorem 2.1 and Corollary 2.1 to the case of triangular arrays of martingales. Let $\mathcal{F}_{\tau,s}$ be an increasing filtration such that for each τ, $\mathcal{F}_{\tau,s}\subset\mathcal{F}_{\tau,s+1}$. Let $S_{\tau,s}$ be adapted to $\mathcal{F}_{\tau,s}$ and assume that for all τ and $k>0$, $E\left[S_{\tau,s+k}|\mathcal{F}_{\tau,s}\right]=S_{\tau,s}$. Since for $p\geq 1$, $|.|^p$ is convex, it follows by Jensen's inequality for conditional expectations that $E\left[|S_{\tau,s+k}|^p|\mathcal{F}_{\tau,s}\right]\geq\left|E\left[S_{\tau,s+k}|\mathcal{F}_{\tau,s}\right]\right|^p=|S_{\tau,s}|^p$. Thus, for each τ, $\left\{|S_{\tau,s}|^p,\mathcal{F}_{\tau,s}\right\}$ is a submartingale. We say that $\left\{|S_{\tau,s}|^p,\mathcal{F}_{\tau,s}\right\}$ is a triangular array of submartingales. If $\left\{S_{\tau,s},\mathcal{F}_{\tau,s}\right\}$ is a triangular array of submartingales then the same holds for $\left\{|S_{\tau,s}|^p,\mathcal{F}_{\tau,s}\right\}$. The following Lemma extends Theorem 2.1 of Hall and Heyde (1980) to triangular arrays of submartingales.

Lemma 1 *For each τ, let $\{S_{\tau,s},\mathcal{F}_{\tau,s}\}$ be a submartingale $S_{\tau,s}$ with respect to an increasing filtration $\mathcal{F}_{\tau,s}$. Then for each real λ, and each τ it follows that*

$$
\lambda P\left(\max_{s\leq\tau}S_{\tau,s}>\lambda\right)\leq E\left[S_{\tau,\tau}1\left\{\max_{s\leq\tau}S_{\tau,s}>\lambda\right\}\right].
$$

Proof. The proof closely follows Hall and Heyde (1980, p.14), with the necessary modifications. Define the event

$$
E_\tau=\left\{\max_{s\leq\tau}S_{\tau,s}>\lambda\right\}=\cup_{i=1}^{\tau}\left\{S_{\tau,i}>\lambda;\max_{1\leq j<i}S_{\tau,j}\leq\lambda\right\}=\cup_{i=1}^{\tau}E_{\tau,i}.
$$

These events are $\mathcal{F}_{\tau,i}$ measurable and disjoint. Then,

$$\begin{aligned}
\lambda P\left(E_{\tau}\right) &\leq \sum_{i=1}^{\tau} E\left[S_{\tau,i} 1\left\{E_{\tau,i}\right\}\right] \\
&\leq \sum_{i=1}^{\tau} E\left[E\left[S_{\tau,\tau} | \mathcal{F}_{\tau,i}\right] 1\left\{E_{\tau,i}\right\}\right] \\
&= \sum_{i=1}^{\tau} E\left[E\left[S_{\tau,\tau} 1\left\{E_{\tau,i}\right\} | \mathcal{F}_{\tau,i}\right]\right] \\
&= \sum_{i=1}^{\tau} E\left[S_{\tau,\tau} 1\left\{E_{\tau,i}\right\}\right] = E\left[S_{\tau,\tau} 1\left\{E_{\tau}\right\}\right].
\end{aligned}$$

■

Corollary 2.1 in Hall and Heyde (1980) now follows in the same way. If $\{S_{\tau,s}, \mathcal{F}_{\tau,s}\}$ is a martingale triangular array then $\{|S_{\tau,s}|^p, \mathcal{F}_{\tau,s}\}$ is submartingale for $p \geq 1$.

Corollary 1 *For each τ, let $\{S_{\tau,s}, \mathcal{F}_{\tau,s}\}$ be a triangular array of a martingale $S_{\tau,s}$ with respect to an increasing filtration $\mathcal{F}_{\tau,s}$. Then for each real λ and for each $p \geq 1$ and each τ it follows that*

$$\lambda^p P\left(\max_{s\leq\tau} |S_{\tau,s}| > \lambda\right) \leq E\left[|S_{\tau,\tau}|^p\right].$$

Proof. Note that $P\left(\max_{s\leq\tau} |S_{\tau,s}| > \lambda\right) = P\left(\max_{s\leq\tau} |S_{\tau,s}|^p > \lambda^p\right)$. Then, using the fact that $\{|S_{\tau,s}|^p, \mathcal{F}_{\tau,s}\}$ is submartingale for $p \geq 1$, apply Lemma 1. ■

Below is a triangular array counterpart of Hall and Heyde's (1980) Theorem 2.2.

Corollary 2 *For each τ, let $\{S_{\tau,s}, \mathcal{F}_{\tau,s}\}$ be a triangular array of a martingale $S_{\tau,s}$ with respect to an increasing filtration $\mathcal{F}_{\tau,s}$. Then for each real λ and for each $p > 1$ and each τ it follows that*

$$E\left[\max_{s\leq\tau} |S_{\tau,s}|^p\right] \leq \left(\frac{p}{p-1}\right)^p E\left[|S_{\tau,\tau}|^p\right].$$

Proof. This proof is based on Hall and Heyde (1980, proof of Theorem 2.2). Note that by the layer-cake representation of an integral, we have

$$E\left[\max_{s\leq\tau} |S_{\tau,s}|^p\right] = \int_0^\infty P\left(\max_{s\leq\tau} |S_{\tau,s}|^p > t\right) dt = \int_0^\infty P\left(\max_{s\leq\tau} |S_{\tau,s}| > t^{\frac{1}{p}}\right) dt.$$

With the change of variable $x = t^{\frac{1}{p}}$, we get

$$E\left[\max_{s\leq\tau} |S_{\tau,s}|^p\right] = p\int_0^\infty x^{p-1} P\left(\max_{s\leq\tau} |S_{\tau,s}| > x\right) dt.$$

Because $|S_{\tau,s}|$ is a submartingale, we can apply Lemma 1 and obtain

$$
\begin{aligned}
E\left[\max_{s\leq\tau}|S_{\tau,s}|^p\right] &\leq p\int_0^\infty x^{p-2}E\left[|S_{\tau,\tau}|\,1\left\{\max_{s\leq\tau}|S_{\tau,s}|>x\right\}\right]dx\\
&= pE\left[|S_{\tau,\tau}|\int_0^\infty x^{p-2}1\left\{\max_{s\leq\tau}|S_{\tau,s}|>x\right\}dx\right]\\
&= pE\left[|S_{\tau,\tau}|\int_0^{\max_{s\leq\tau}|S_{\tau,s}|}x^{p-2}dx\right]\\
&= \frac{p}{p-1}E\left[|S_{\tau,\tau}|\max_{s\leq\tau}|S_{\tau,s}|^{p-1}\right]\\
&\leq \frac{p}{p-1}\left(E\left[|S_{\tau,\tau}|^p\right]\right)^{\frac{1}{p}}\left(E\left[\left(\max_{s\leq\tau}|S_{\tau,s}|^{p-1}\right)^q\right]\right)^{\frac{1}{q}},
\end{aligned}
$$

where the last inequality is an application of Hölder's inequality for $q=\frac{1}{1-\frac{1}{p}}=\frac{p}{p-1}$. Dividing both sides by $\left(E\left[\left(\max_{s\leq\tau}|S_{\tau,s}|^{p-1}\right)^q\right]\right)^{\frac{1}{q}}=\left(E\left[\max_{s\leq\tau}|S_{\tau,s}|^p\right]\right)^{\frac{1}{q}}$, we get

$$
\left(E\left[\max_{s\leq\tau}|S_{\tau,s}|^p\right]\right)^{1-\frac{1}{q}}\leq\frac{p}{p-1}\left(E\left[|S_{\tau,\tau}|^p\right]\right)^{\frac{1}{p}}
$$

or

$$
E\left[\max_{s\leq\tau}|S_{\tau,s}|^p\right]\leq\left(\frac{p}{p-1}\right)^pE\left[|S_{\tau,\tau}|^p\right].
$$

■

V Proof of (56)

Lemma 2 *Assume that Conditions 7, 8 and 9 hold. For $r,s\in[0,1]$ fixed and as $\tau\to\infty$ it follows that*

$$
\left|\tau^{-1}\sum_{t=\tau_0+[\tau s]+1}^{\tau_0+[\tau r]}\left((\psi_{\tau,s})^2-e^{(-2\gamma t/\tau)}E\left[\eta_t^2|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n}\right]\right)\right|\xrightarrow{p}0.
$$

Proof. By Hall and Heyde (1980, Theorem 2.23) we need to show that for all $\varepsilon>0$

$$
\tau^{-1}\sum_{t=\tau_0+[\tau s]+1}^{\tau_0+[\tau r]}e^{-2\gamma t/\tau}E\left[\eta_t^21\left\{\left|\tau^{-1/2}e^{-\gamma t/\tau}\eta_t\right|>\varepsilon\right\}\middle|\,\mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n}\right]\xrightarrow{p}0. \tag{S.39}
$$

By Condition 7iv) it follows that for some $\delta > 0$

$$E\left[\tau^{-1}\sum_{t=\tau_0+[\tau s]+1}^{\tau_0+[\tau r]} e^{-2\gamma t/\tau} E\left[\eta_t^2 1\left\{\left|\tau^{-1/2}e^{-\gamma t/\tau}\eta_t\right| > \varepsilon\right\}|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n}\right]\right]$$

$$\leq \tau^{-(1+\delta/2)}\sum_{t=\tau_0+[\tau s]+1}^{\tau_0+[\tau r]} \frac{\left(e^{-\gamma t/\tau}\right)^{2+\delta}}{\varepsilon^\delta} E\left[|\eta_t|^{2+\delta}\right]$$

$$\leq \sup_t E\left[|\eta_t|^{2+\delta}\right]\frac{[\tau r]-[\tau s]}{\tau^{1+\delta/2}\varepsilon^\delta}e^{(2+\delta)|\gamma|} \to 0.$$

This establishes (S.39) by the Markov inequality. Since $\tau^{-1}\sum_{t=\tau_0+[\tau s]+1}^{\tau_0+[\tau r]} e^{(-2\gamma t/\tau)}E\left[\eta_t^2|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n}\right]$ is uniformly integrable by (55) and (59) it follows from Hall and Heyde (1980, Theorem 2.23, Eq 2.28) that

$$E\left[\left|\tau^{-1}\sum_{t=\tau_0+[\tau s]+1}^{\tau_0+[\tau r]}\left((\psi_{\tau,s})^2 - e^{(-2\gamma t/\tau)}E\left[\eta_t^2|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n}\right]\right)\right|\right] \to 0.$$

The result now follows from the Markov inequality. ■

Lemma 3 *Assume that Conditions 7, 8 and 9 hold. For $r, s \in [0,1]$ fixed and as $\tau \to \infty$ it follows that*

$$\tau^{-1}\sum_{t=\tau_0+[\tau s]+1}^{\tau_0+[\tau r]} e^{(-2\gamma t/\tau)}E\left[\eta_t^2|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n}\right] \to^p \sigma^2\int_s^r \exp(-2\gamma t)\,dt.$$

Proof. The proof closely follows Chan and Wei (1987, p. 1060-1062) with a few necessary adjustments. Fix $\delta > 0$ and choose $s = t_0 \leq t_1 \leq ... \leq t_k = r$ such that

$$\max_{i\leq k}\left|e^{-2\gamma t_i} - e^{-2\gamma t_{i-1}}\right| < \delta.$$

This implies

$$\left|\int_s^r e^{-2\gamma t}dt - \sum_{i=1}^k e^{-2\gamma t_i}(t_i - t_{i-1})\right| \leq \sum_{i=1}^k\int_{t_{i-1}}^{t_i}\left|e^{-2\gamma t} - e^{-2\gamma t_i}\right|dt \leq \delta. \quad \text{(S.40)}$$

Let $I_i = \{l : [\tau t_{i-1}] < l \leq [\tau t_i]\}$. Then,

$$\begin{aligned}
&\tau^{-1} \sum_{t=\tau_0+[\tau s]+1}^{\tau_0+[\tau r]} e^{-2\gamma t/\tau} E\left[\eta_t^2 | \mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n}\right] - \sigma^2 \int_s^r e^{-2\gamma t} dt \\
&= \tau^{-1} \sum_{i=1}^{k} \sum_{l \in I_i} e^{-2\gamma l/\tau} E\left[\eta_l^2 | \mathcal{G}_{\tau n,(l-\min(1,\tau_0)-1)n}\right] - \sigma^2 \int_s^r e^{-2\gamma t} dt \\
&= \tau^{-1} \sum_{i=1}^{k} \sum_{l \in I_i} \left(e^{-2\gamma l/\tau} - e^{-2\gamma[\tau t_{i-1}]/\tau}\right) E\left[\eta_l^2 | \mathcal{G}_{\tau n,(l-\min(1,\tau_0)-1)n}\right] \\
&+ \sum_{i=1}^{k} e^{-2\gamma[\tau t_{i-1}]/\tau} \left(\tau^{-1} \sum_{l \in I_i} E\left[\eta_l^2 | \mathcal{G}_{\tau n,(l-\min(1,\tau_0)-1)n}\right] - \sigma^2 (t_i - t_{i-1})\right) \\
&+ \sum_{i=1}^{k} e^{-2\gamma[\tau t_{i-1}]/\tau} \sigma^2 (t_i - t_{i-1}) - \sigma^2 \int_s^r e^{-2\gamma t} dt \\
&= I_n + II_n + III_n.
\end{aligned}$$

For III_n we have that $e^{-2\gamma[\tau t_{i-1}]/\tau} \to e^{-2\gamma t_{i-1}}$ as $\tau \to \infty$. In other words, there exists a τ' such that for all $\tau \geq \tau'$, $\left|e^{-2\gamma[\tau t_{i-1}]/\tau} - e^{-2\gamma t_{i-1}}\right| \leq \delta$ and by (S.40)

$$|III_n| \leq 2\delta.$$

We also have by Condition 7vii) that

$$\tau^{-1} \sum_{l \in I_i} E\left[\eta_l^2 | \mathcal{G}_{\tau n,(l-\min(1,\tau_0)-1)n}\right] \to \sigma^2 (t_i - t_{i-1})$$

as $\tau \to \infty$ such that by $\max_{i \leq k} \left|e^{2\gamma[\tau t_{i-1}]/\tau}\right| \leq e^{2|\gamma|}$

$$|II_n| \leq e^{2|\gamma|} \left|\tau^{-1} \sum_{l \in I_i} E\left[\eta_l^2 | \mathcal{G}_{\tau n,(l-\min(1,\tau_0)-1)n}\right] - \sigma^2 (t_i - t_{i-1})\right| = o_p(1).$$

Finally, there exists a τ' such that for all $\tau \geq \tau'$ it follows that

$$\begin{aligned}
\max_{i \leq k} \max_{l \in I_i} \left|e^{-2\gamma l/\tau} - e^{-2\gamma[\tau t_{i-1}]/\tau}\right| &\leq \max_{i \leq k} \left|e^{-2\gamma[\tau t_i]/\tau} - e^{-2\gamma[\tau t_{i-1}]/\tau}\right| \\
&\leq 2 \max_{i \leq k} \left|e^{-2\gamma[\tau t_i]/\tau} - e^{-2\gamma t_i}\right| \\
&+ \max_{i \leq k} \left|e^{-2\gamma t_i} - e^{-2\gamma t_{i-1}}\right| \\
&\leq 2\delta + \delta = 3\delta.
\end{aligned}$$

We conclude that

$$|I_n| \leq 3\delta \left| \tau^{-1} \sum_{i=1}^{k} \sum_{l \in I_i} E\left[\eta_l^2 | \mathcal{G}_{\tau n,(l-\min(1,\tau_0)-1)n}\right] \right| = 3\delta\sigma^2 \left(1 + o_p(1)\right).$$

The remainder of the proof is identical to Chan and Wei (1987, p. 1062). ■

VI Standard Error for Section 2

We state precise sufficient conditions for our example and establish that they imply the regularity conditions of our general results in Section 4.

Condition 1 (EX-1) *Assume that*

i) $f(y_{j,t}|\theta)$ *is measurable with respect to* $\mathcal{G}_{\tau n,(t-\min(1,\tau_0))n+i}$ *and* $E\left[f(y_{j,t}|\theta_0)|\, \mathcal{G}_{\tau n,(t-\min(1,\tau_0))n+j-1}\right] = 0.$

ii) $g(\nu_t(\beta)|\nu_{t-1}(\beta), \beta, \rho)$ *is measurable with respect to* $\mathcal{G}_{\tau n,(t-\min(1,\tau_0))n+i}$ *for all* $i = 1, ..., n$ *and* $E\left[g(\nu_t|\nu_{t-1}, \beta_0, \rho_0)|\, \mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n+i}\right] = 0$ *for* $t > T$ *and all* $i = 1, ..., n$.

iii) Let $y_{j,t}^k$ *be the k-th element of* $y_{j,t}$*. Then, for some* $\delta > 0$ *and* $C < \infty$, $\sup_{it} E\left[\left|y_{j,t}^k\right|^{2+\delta}\right] \leq C$ *for all* $n \geq 1$.

iv) For some $\delta > 0$ *and* $C < \infty$, $\sup_{t \leq \tau_0+\tau} E\left[|Y_t^*|^{2+\delta}\right] \leq C$ *and* $\sup_{t \leq \tau_0+\tau} E\left[|K_t^*|^{2+\delta}\right] \leq C$ *and for all* $\tau \geq 1$.

v) $\left\|E\left[g(\nu_t|\nu_{t-1}, \beta_0, \rho_0)|\, \mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n+i}\right]\right\|_2 \leq \vartheta_t$ *for* $t < 0$ *and all* $i = 1, ..., n$ *where*

$$\vartheta_t \leq C\left(|t|^{1+\delta}\right)^{-1/2}.$$

Conditions EX-1(i) and (ii) impose that the estimating functions are martingale differences relative to the filtrations defined in (14). These filtrations accumulate information about the time series and cross-section samples up to a common point in time t, as well as information about common shocks of the cross-section sample. The conditions can be interpreted as imposing correct specification of the time series and cross-section models in terms of the conditional mean. Note that for the time series moments $E\left[g(\nu_t|\nu_{t-1}, \beta_0, \rho_0)|\, \mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n+i}\right] \neq 0$ for $t \leq T$ because of possible mean dependence of $g(.)$ with the aggregate shocks generating $\mathcal{C}$. The violation of the moment conditions for $t \leq T$ leads to possible estimator bias that is being controlled by imposing

Condition (v). It is important to stress that we are not assuming that the cross-section and time series samples are independent of each other or the common shocks, or that conditioning on the common shocks leads to conditionally independent samples. Nor do we assume that the cross-section is sampled randomly. Such additional assumptions can be invoked to ensure that laws of large numbers for sample averages hold, but are likely much stronger than needed. In our theory we impose these laws of large numbers as high level regularity conditions. Conditions EX-1(iii) and (iv) impose mild regularity conditions in terms of moments of the marginal distributions of all variables in the cross-section and time series samples. Finally, Condition (v) imposes a mixingale condition on the common shock process. We show in Example 1 that it holds for a stationary Gaussian AR(1) model for ν_t, although the condition is expected to hold for much more general processes. Condition EX-1 parallels Footnote 32 of HKM20 for a different example where the focus is on the cross-sectional parameters, while here we use an example that focuses on the time series parameters as the main object of interest.

Under Condition EX-1 it follows that for $u_{j,t} = e^{(C)}_{j,t+1} + \eta_{j,t+1}$ and $f\left(y_{j,t}|\theta_0\right) = u_{j,t}z_{j,t}$ the cross-sectional moment vector satisfies a martingale difference property such that

$$E\left[u_{j,t+1}z_{j,t}|\mathcal{G}_{\tau n,(t-\min(1,\tau_0))n+j-1}\right] = 0.$$

In our example, $z_{j,t} = (1, k_{j,t-1}, i_{j,t-1})'$ consists of lagged values that are measurable with respect to $\mathcal{G}_{\tau n,(t-\min(1,\tau_0))n+j-1}$. The condition then is equivalent to imposing the martingale difference assumption on the cross-sectional innovation $u_{j,t}$. The construction of $\mathcal{G}_{\tau n,(t-\min(1,\tau_0))n+j-1}$ in (14) guarantees that $u_{j,t}$ are uncorrelated both cross-sectionally and temporally. This implies that when evaluated at the true parameter θ_0,

$$\text{Var}\left(\frac{1}{\sqrt{n}}\sum_{t=1}^{T}\sum_{j=1}^{n} f\left(y_{j,t}\middle|\,\theta_0\right)\right) = \frac{1}{n}\sum_{t=1}^{T}\sum_{j=1}^{n} E\left[u^2_{j,t}z_{j,t}z'_{j,t}\right].$$

The next condition postulates that a law of large numbers holds.

Condition 2 (EX-2) *There exist non-singular constant matrices Ω_f and Ω_g such that*

$$\Omega_f = \operatorname*{plim}_{n\to\infty}\frac{1}{n}\sum_{t=1}^{T}\sum_{j=1}^{n} u^2_{j,t}z_{j,t}z'_{j,t}, \tag{S.41}$$

$$\Omega_g = \operatorname*{plim}_{\tau\to\infty}\frac{1}{\tau}\sum_{s=\tau_0+1}^{\tau_0+\tau}\left(e^{(A)}_s\right)^2\nu^2_{s-1}. \tag{S.42}$$

Also, let $\tilde{\Omega}_f = \frac{1}{n}\sum_{t=1}^{T}\sum_{j=1}^{n}\tilde{u}_{j,t}^2 z_{j,t}z_{j,t}'$ with

$$\tilde{u}_{j,t} = \mathfrak{y}_{j,t}^* - \left(\tilde{\beta}_{0,t}^* + \tilde{\beta}_k k_{j,t} + \tilde{\alpha}^{(C)}\left(\phi_t\left(i_{j,t-1}, k_{j,t-1}\right) - \tilde{\beta}_k k_{j,t-1}\right)\right).$$

and set $W_n^C = \tilde{\Omega}_f^{-1}$, $W^C = \Omega_f^{-1}$ and $F_n(\beta,\nu) = -h_n(\beta,\nu)' W_n^C h_n(\beta,\nu)$.

At the true parameter values β_0 and ρ_0 it follows that $g\left(\nu_s(\beta)\middle|\,\nu_{s-1}(\beta),\beta,\rho\right) = g\left(\nu_s\middle|\,\nu_{s-1},\beta,\rho\right) = e_s^{(A)}\nu_{s-1}$. As for the cross-sectional error, Condition EX-1 implies that $E\left[e_s^{(A)}\nu_{s-1}|\mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n+j}\right] = 0$ for all $j = 1,...,n$. By the same logic as before, ν_{s-1} is measurable with respect to $\mathcal{G}_{\tau n,(t-\min(1,\tau_0)-1)n+j}$ such that a martingale difference assumption may be directly imposed on the aggregate time series shock $e_s^{(A)}$. The martingale difference sequences (mds) property then implies that

$$\operatorname{Var}\left(\frac{1}{\sqrt{\tau}}\sum_{s=\tau_0+1}^{\tau_0+\tau} g\left(z_s\middle|\,\beta_0,\rho_0\right)\right) = \frac{1}{\tau}\sum_{s=\tau_0+1}^{\tau_0+\tau} E\left[\left(e_s^{(A)}\right)^2\nu_{s-1}^2\right].$$

Let $\tilde{\Omega}_g = \frac{1}{\tau}\sum_{s=\tau_0+1}^{\tau_0+\tau}\left(\tilde{e}_s^{(A)}\right)^2\tilde{\nu}_{s-1}^2$ where $\tilde{\nu}_s = Y_s^* - \tilde{\beta}_k K_s^*$ and $\tilde{e}_s^{(A)} = \tilde{\nu}_s - \hat{\alpha}^{(A)}\tilde{\nu}_{s-1}$. Then, set $W_\tau^\tau = \tilde{\Omega}_g^{-1}$, $W^\tau = \Omega_g^{-1}$ and let $G_\tau(\beta,\rho) = -k_\tau(\beta,\rho)' W_\tau^\tau k_\tau(\beta,\rho)$.

We now demonstrate how to obtain the distributional approximations analogous to (23) and (24). Similar arguments for a more complicated and cross-sectionally oriented example can also be found in Section 6, Eq (35) of HKM20. To obtain explicit formulas we turn to the derivatives of the criterion functions. We have

$$\frac{\partial f\left(y_{j,t}|\theta\right)'}{\partial\theta} = \begin{bmatrix} 1 \\ -\left(k_{j,t} - \alpha^{(C)}k_{j,t-1}\right) \\ -\left(\phi_t\left(i_{j,t-1},k_{j,t-1}\right) - \beta_k k_{j,t-1}\right)\end{bmatrix} z_{j,t}'$$

and let $h(\theta) = \operatorname{plim}_{n\to\infty} h_n(\beta,\nu)$ and $k(\beta,\rho) = \operatorname{plim}_{\tau\to\infty} k_\tau(\beta,\rho)$. It follows from standard GMM large sample theory that

$$\varphi_{j,t} = \left(\frac{\partial h(\theta)}{\partial\theta'}W^C\frac{\partial h(\theta)}{\partial\theta}\right)^{-1}\frac{\partial h(\theta)}{\partial\theta'}W^C f\left(y_{j,t}\middle|\,\theta_0\right)$$

such that the asymptotic variance covariance matrix of the cross-sectional GMM estimator is $\left(\frac{\partial h(\theta_0)}{\partial\theta'}\Omega_f^{-1}\frac{\partial h(\theta_0)}{\partial\theta}\right)^{-1}$. Note that $\partial h(\theta)/\partial\theta = \operatorname{plim}_{n\to\infty} n^{-1}\sum_{t=1}^{T}\sum_{i=1}^{n}\partial f\left(y_{j,t}|\theta\right)/\partial\theta'$. Similarly, considering the time series estimator one obtains

$$\begin{bmatrix} \frac{\partial g(z_s|\beta,\rho)}{\partial \theta} \\ \frac{\partial g(z_s|\beta,\rho)}{\partial \rho} \end{bmatrix} = \begin{bmatrix} 0 \\ -\left(K_s^* - \alpha^{(A)} K_{s-1}^*\right) \nu_{s-1}(\beta) - \left(\nu_s(\beta) - \alpha^{(A)} \nu_{s-1}(\beta)\right) K_{s-1}^* \\ 0 \\ -\alpha^{(A)} \left(\nu_{s-1}(\beta)\right)^2 \end{bmatrix}.$$

It then follows again from standard theory that hypothetical estimates for the parameter ρ obtained from the time series data and using the moment function $g(.)$ at the true parameter value for β_k have an asymptotic variance covariance matrix equal to $\left(\frac{\partial k(\beta_0,\rho_0)}{\partial \rho'} \Omega_g^{-1} \frac{\partial k(\beta_0,\rho_0)}{\partial \rho}\right)^{-1}$. Our theory formally allows to handle the case where β_k is estimated from the cross-section sample. While the expressions for the asymptotic distribution of ρ are similar to standard formulas for two step estimators, a rigorous derivation of the asymptotic approximations is much more involved because there are two generally dependent samples involved in the estimation.

With these expressions it is now possible to obtain standard errors, mimicking the procedure laid out in Section 6 of Hahn, Kuersteiner and Mazzocco (2020). For this, we sketch how to obtain the joint limiting distribution of the vector $\phi = (\theta', \rho')'$.

The joint (with $\mathcal{C}$ measurable random variables) limiting distribution of $D_{n\tau}^{-1} J_{n\tau}(\phi_0)$ is established in the following Lemma.

Lemma 4 *Assume that Conditions EX-1 and EX-2 hold, and that (S.41) and (S.42) are well defined. Then,*

$$D_{n\tau}^{-1} J_{n\tau}(\phi_0) \rightarrow_d N(0, \Omega) \;\; \mathcal{C}\text{-stably}$$

where $\Omega = \operatorname{diag}(\Omega_y, \Omega_\nu)$ is the asymptotic variance covariance matrix of the moment functions defined in (29).

Lemma 4 is a direct consequence of Corollary 1 in Section 4 and the fact that Condition EX-1 combined with (S.41) and (S.42) imply that Conditions 1, 2 and 3 hold for $r = 1$, where r is defined in Condition 2.