Online Appendix to "Two-Step Estimation of Quantile Panel Data Models with Interactive Fixed Effects"

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A Proof of the Main Results

Definitions and Notations

For any random variable W_i or W_t, $W_i = \bar{O}_P(1)$ means that $\max_{1\leq i\leq N}\|W_i\| = O_P(1)$, and $W_t = \bar{O}_P(1)$ means that $\max_{1\leq t\leq T}\|W_t\| = O_P(1)$. $\bar{o}_P(1)$ is defined similarly. For notational simplicity, we suppress the dependence of $\lambda_{0i}(\tau)$, $\hat{\beta}(\tau)$ and $\beta_0(\tau)$ on τ. Let $M > 0$ denote a generic bounded constant that does not depend on N or T.

Define:

$$\mathbb{S}_{NT}(\beta,\Lambda,F) = (NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} l_{it}(\beta,\lambda_i,f_t) \quad \mathbb{S}^*_{NT}(\beta,\Lambda,F) = (NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\rho_{it}(\beta,\lambda_i,f_t)$$

$$\mathbb{S}_{i,T}(\beta,\lambda_i,F) = \frac{1}{T}\sum_{t=1}^{T} l_{it}(\beta,\lambda_i,f_t) \quad \mathbb{S}^*_{i,T}(\beta,\lambda_i,F) = \frac{1}{T}\sum_{t=1}^{T}\rho_{it}(\beta,\lambda_i,f_t)$$

where $l(u) = [\tau - K(u/h)]u$, $\rho_\tau(u)$ is the check function, and

$$l_{it}(\beta,\lambda_i,f_t) = l(Y_{it} - \beta' X_{it} - \lambda_i' f_t), \quad \rho_{it}(\beta,\lambda_i,f_t) = \rho_\tau(Y_{it} - \beta' X_{it} - \lambda_i' f_t).$$

For any random function $L(\beta,\Lambda,F)$ and fixed (β,Λ,F), define $\bar{L}(\beta,\Lambda,F) = \mathbb{E}[L(\beta,\Lambda,F)]$ and $\tilde{L}(\beta,\Lambda,F) = L(\beta,\Lambda,F) - \bar{L}(\beta,\Lambda,F)$. Let $l^{(j)}(u)$ denote the jth order derivative of l, i.e.,

$$l^{(1)}(u) = \tau - K(u/h) + k(u/h)u/h, \quad l^{(2)}(u) = 2k(u/h)1/h + k^{(1)}(u/h)u/h^2,$$

$$l^{(3)}(u) = 3k^{(1)}(u/h)1/h^2 + k^{(2)}(u/h)u/h^3, \quad l^{(4)}(u) = 4k^{(2)}(u/h)1/h^3 + k^{(3)}(u/h)u/h^4.$$

Let $l_{it}^{(j)}(\beta, \lambda_i, f_t) = l^{(j)}(Y_{it} - \beta' X_{it} - \lambda_i' f_t)$ for $j = 1, \ldots, 4$, and their arguments are dropped when evaluated at $(\beta_0, \lambda_{0i}, f_{0t})$.

Finally, define $\tilde{\lambda}_{0i} = (\mathbf{H}_0')^{-1}\lambda_{0i}$, $\tilde{f}_{0t} = \mathbf{H}_0 f_{0t}$, $\tilde{\Lambda}_0 = [\tilde{\lambda}_{01}, \ldots, \tilde{\lambda}_{0N}]'$, and $\tilde{F}_0 = [\tilde{f}_{01}, \ldots, \tilde{f}_{0T}]'$.

A.1 Proof of Proposition 1:

Proof. Note that

$$\begin{aligned}
&\|\hat{\Sigma}_{\bar{x}} - \Gamma_0 \Sigma_{f_0} \Gamma_0'\| \\
=& \left\| \bar{\Gamma}\hat{\Sigma}_{f_0}\bar{\Gamma}' - \Gamma_0\Sigma_{f_0}\Gamma_0' + \bar{\Gamma}\cdot \frac{1}{T}\sum_{t=1}^{T} f_{0t}\bar{e}_t' + \frac{1}{T}\sum_{t=1}^{T}\bar{e}_t f_{0t}' \cdot \bar{\Gamma}' + \frac{1}{T}\sum_{t=1}^{T}\bar{e}_t\bar{e}_t' \right\| \\
\leq& \left\| \bar{\Gamma}\hat{\Sigma}_{f_0}\bar{\Gamma}' - \Gamma_0\Sigma_{f_0}\Gamma_0' \right\| + 2\left\| \bar{\Gamma}\cdot\frac{1}{T}\sum_{t=1}^{T} f_{0t}\bar{e}_t' \right\| + \left\| \frac{1}{T}\sum_{t=1}^{T}\bar{e}_t\bar{e}_t' \right\|.
\end{aligned}$$

First, by Assumption 1(ii), $\left\| \bar{\Gamma}\hat{\Sigma}_{f_0}\bar{\Gamma}' - \Gamma_0\Sigma_{f_0}\Gamma_0' \right\| = O(N^{-1/2} + T^{-1/2})$. Second, by Assumption 1(iii) and the Cauchy-Schwarz inequality, we have

$$\left\| \frac{1}{T}\sum_{t=1}^{T} f_{0t}\bar{e}_t' \right\| \leq \frac{1}{\sqrt{N}} \sqrt{\frac{1}{T}\sum_{t=1}^{T}\|f_{0t}\|^2} \sqrt{\frac{1}{T}\sum_{t=1}^{T}\|\sqrt{N}\bar{e}_t\|^2} = O_P(N^{-1/2})$$

and

$$\left\| \frac{1}{T}\sum_{t=1}^{T}\bar{e}_t\bar{e}_t' \right\| \leq \frac{1}{N}\cdot\frac{1}{T}\sum_{t=1}^{T}\|\sqrt{N}\bar{e}_t\|^2 = O_P(N^{-1}).$$

It then follows that $\|\hat{\Sigma}_{\bar{x}} - \Gamma_0\Sigma_{f_0}\Gamma_0'\| = O_P(N^{-1/2} + T^{-1/2})$. Third, by matrix perturbation theory (Hoffman-Wielandt inequality) and the fact that $\Gamma_0\Sigma_{f_0}\Gamma_0'$ is a matrix with rank r (Assumption 1(ii)), it can be concluded that $\hat{\rho}_1, \ldots, \hat{\rho}_r$ converge in probability to some positive constants, while $\hat{\rho}_{r+1}, \ldots, \hat{\rho}_k$ are all $O_P(N^{-1/2} + T^{-1/2})$. Thus, it follows that

$$P[\hat{r} \neq r] \leq P[\hat{r} < r] + P[\hat{r} > r] \leq P[\hat{\rho}_r < \mathbb{P}_{NT}] + P[\hat{\rho}_{r+1} \geq \mathbb{P}_{NT}] \to 0,$$

and the desired result follows. □

A.2 Proof of Theorem 1

Lemma 1. *Define* $\hat{\mathbf{H}} = \hat{\Psi}'\bar{\Gamma}$. *Under Assumptions 1 and 2, (i)* $\hat{f}_t = \hat{\mathbf{H}} f_{0t} + \hat{\Psi}'\bar{e}_t$*; (ii)* $\hat{\Psi} \xrightarrow{p} \Psi_0$ *and* $\hat{\mathbf{H}} \xrightarrow{p} \mathbf{H}_0 = \Psi_0'\Gamma_0$*; (iii)* $\hat{\mathbf{H}}$ *is invertible with probability approaching 1.*

Proof. Result (i) follows directly from $\bar{X}_t = \bar{\Gamma} f_{0t} + \bar{e}_t$ and $\hat{f}_t = \hat{\Psi}'\bar{X}_t$. Given Assumption 2(i), it follows from the Bauer-Fike Theorem and $\|\hat{\Sigma}_{\bar{x}} - \Gamma_0\Sigma_{f_0}\Gamma_0'\| = o_P(1)$ (see the proof of Proposition 1) that $\|\hat{\Psi} - \Psi_0\| = o_P(1)$. Thus, result (ii) follows since $\|\bar{\Gamma} - \Gamma_0\| = o(1)$ by Assumption 1(ii). Finally, suppose that $\text{rank}(\mathbf{H}_0) < r$, and let $\mathcal{D}$ be the diagonal matrix with the eigenvalues of $\Gamma_0\Sigma_{f_0}\Gamma_0'$ as the diagonal elements, then $\text{rank}(\mathcal{D}) = \text{rank}(\Psi_0'\Gamma_0\Sigma_{f_0}\Gamma_0'\Psi_0) = \text{rank}(\mathbf{H}_0\Sigma_{f_0}\mathbf{H}_0') \leq \text{rank}(\mathbf{H}_0) < r$, which contradicts with Assumption 1(ii). Thus, we have $\text{rank}(\mathbf{H}_0) = r$ and result (iii) follows from $\hat{\mathbf{H}} = \mathbf{H}_0 + o_P(1)$. $\square$

Proof of Theorem 1

Proof. **Step 1:** By Lemma 1, we have

$$\frac{1}{T}\sum_{t=1}^{T}\|\hat{f}_t - \tilde{f}_{0t}\|^2 \leq \frac{1}{T}\sum_{t=1}^{T}\|\hat{f}_t - \hat{\mathbf{H}}f_{0t}\|^2 + M\|\mathbf{H}_0 - \hat{\mathbf{H}}\|^2 \leq \frac{r}{NT}\sum_{t=1}^{T}\|\sqrt{N}\bar{e}_t\|^2 + o_P(1) = O_P\left(\frac{1}{N}\right) + o_P(1).$$

Step 2: Adding and subtracting terms, we can write

$$\mathbb{S}_{NT}(\beta,\Lambda,F) = (\mathbb{S}_{NT}(\beta,\Lambda,F) - \mathbb{S}^*_{NT}(\beta,\Lambda,F)) + (\mathbb{S}^*_{NT}(\beta,\Lambda,F) - \mathbb{S}^*_{NT}(\beta,\Lambda,\tilde{F}_0)) + \mathbb{S}^*_{NT}(\beta,\Lambda,\tilde{F}_0).$$

By the definition of the estimators, $\mathbb{S}_{NT}(\hat{\beta},\hat{\Lambda},\hat{F}) \leq \mathbb{S}_{NT}(\beta_0,\tilde{\Lambda}_0,\hat{F})$. Thus, we have

$$\begin{aligned}\mathbb{S}^*_{NT}(\hat{\beta},\hat{\Lambda},\tilde{F}_0) - \mathbb{S}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\tilde{F}_0) \leq{}& [\mathbb{S}_{NT}(\beta_0,\tilde{\Lambda}_0,\hat{F}) - \mathbb{S}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\hat{F})] + [\mathbb{S}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\hat{F}) - \mathbb{S}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\tilde{F}_0)] \\ &- [\mathbb{S}_{NT}(\hat{\beta},\hat{\Lambda},\hat{F}) - \mathbb{S}^*_{NT}(\hat{\beta},\hat{\Lambda},\hat{F})] - [\mathbb{S}^*_{NT}(\hat{\beta},\hat{\Lambda},\hat{F}) - \mathbb{S}^*_{NT}(\hat{\beta},\hat{\Lambda},\tilde{F}_0)]. \quad \text{(A.1)}\end{aligned}$$

Step 3: Let δ be a positive number close to 0. Define $B_{\delta,i} = \{\beta \in \mathcal{B}, \lambda_i \in \mathcal{A} : \|\beta - \beta_0\|_1 + \|\lambda_i - \tilde{\lambda}_{0i}\|_1 \leq \delta\}$. Consider any $(\beta,\lambda_i) \in B^C_{\delta,i}$. Let $m = \|\beta-\beta_0\|_1 + \|\lambda_i - \tilde{\lambda}_{0i}\|_1 > \delta$, then $(\bar{\beta},\bar{\lambda}_i) = (\beta,\lambda_i)\delta/m + (\beta_0,\tilde{\lambda}_{0i})(1-\delta/m)$ is on the boundary of $B_{\delta,i}$. Note that given X_{it} and f_t, the check function ρ_{it} is convex in (β,λ_i). Thus,

$$\delta/m \cdot \rho_{it}(\beta,\lambda_i,\tilde{f}_{0t}) + (1-\delta/m)\cdot\rho_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t}) \geq \rho_{it}(\bar{\beta},\bar{\lambda}_i,\tilde{f}_{0t}),$$

and it follows that

$$\rho_{it}(\beta,\lambda_i,\tilde{f}_{0t}) - \rho_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t}) \geq m/\delta \cdot \left[\rho_{it}(\bar{\beta},\bar{\lambda}_i,\tilde{f}_{0t}) - \rho_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right]$$

and

$$\mathbb{S}^*_{NT}(\beta,\Lambda,\tilde{F}_0) - \mathbb{S}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\tilde{F}_0) \gtrsim \mathbb{S}^*_{NT}(\bar{\beta},\bar{\Lambda},\tilde{F}_0) - \mathbb{S}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\tilde{F}_0)$$

if $(\beta,\lambda_i) \in B^C_{\delta,i}$ for all i.

Write

$$\mathbb{S}^*_{NT}(\bar{\beta},\bar{\Lambda},\tilde{F}_0) - \mathbb{S}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\tilde{F}_0) = \bar{\mathbb{S}}^*_{NT}(\bar{\beta},\bar{\Lambda},\tilde{F}_0) - \bar{\mathbb{S}}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\tilde{F}_0) + \tilde{\mathbb{S}}^*_{NT}(\bar{\beta},\bar{\Lambda},\tilde{F}_0) - \tilde{\mathbb{S}}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\tilde{F}_0).$$

First, by Taylor expansion $\bar{\rho}_{it}(\beta,\lambda_i,\tilde{f}_{0t})$ around $(\beta_0,\tilde{\lambda}_{0i})$ of we have

$$\bar{\mathbb{S}}^*_{NT}(\bar{\beta},\bar{\Lambda},\tilde{F}_0)-\bar{\mathbb{S}}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\tilde{F}_0)=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{\rho}_{it}(\bar{\beta},\bar{\lambda}_i,\tilde{f}_{0t})-\bar{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})$$
$$=\frac{1}{N}\sum_{i=1}^{N}[(\bar{\beta}-\beta_0)',(\bar{\lambda}_i-\lambda_{0i})]\cdot\left(\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\mathrm{f}_{it}(0|X_{it})V_{it}V_{it}']\right)\cdot[(\bar{\beta}-\beta_0)',(\bar{\lambda}_i-\lambda_{0i})]'+o(\delta^2)$$
$$\geq\delta^2\cdot\frac{1}{N}\sum_{i=1}^{N}\varrho_{i,T}+o(\delta^2)\geq\underline{\varrho}\delta^2 \quad \text{(A.2)}$$

by Assumption 2(iii). Second, we have

$$\left|\tilde{\mathbb{S}}^*_{NT}(\bar{\beta},\bar{\Lambda},\tilde{F}_0)-\tilde{\mathbb{S}}^*_{NT}(\beta_0,\tilde{\Lambda}_0,\tilde{F}_0)\right|\leq\max_{1\leq i\leq N}\sup_{(\beta,\lambda_i)\in B_{\delta,i}}\left|\frac{1}{T}\sum_{t=1}^{T}\left[\tilde{\rho}_{it}(\beta,\lambda_i,\tilde{f}_{0t})-\tilde{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right]\right|. \quad \text{(A.3)}$$

Step 4: $\|\hat{\beta}-\beta_0\|_1>\delta$ implies that $(\hat{\beta},\hat{\lambda}_i)\in B^C_{\delta,i}$ for all i. It then follows from (A.1), (A.2) and (A.3) that for small $\delta>0$, there exists an $\epsilon>0$ (depending on δ) such that

$$P[\|\hat{\beta}-\beta_0\|_1>\delta]\leq P\left[\sup_{\beta,\Lambda,F}\left|\mathbb{S}_{NT}(\beta,\Lambda,F)-\mathbb{S}^*_{NT}(\beta,\Lambda,F)\right|>1/3\epsilon\right]$$
$$+P\left[\sup_{\beta,\lambda_i\in\mathcal{A}}\left|\mathbb{S}^*_{NT}(\beta,\Lambda,\hat{F})-\mathbb{S}^*_{NT}(\beta,\Lambda,\tilde{F}_0)\right|>1/3\epsilon\right]$$
$$+P\left[\max_{1\leq i\leq N}\sup_{(\beta,\lambda_i)\in B_{\delta,i}}\left|\frac{1}{T}\sum_{t=1}^{T}\left[\tilde{\rho}_{it}(\beta,\lambda_i,\tilde{f}_{0t})-\tilde{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right]\right|>1/3\epsilon\right]. \quad \text{(A.4)}$$

The first term on the right-hand side of (A.4) is $o(1)$ because it is easy to show that[1]:

$$\sup_{\beta,\Lambda,F}\left|\mathbb{S}_{NT}(\beta,\Lambda,F)-\mathbb{S}^*_{NT}(\beta,\Lambda,F)\right|\lesssim h$$

and $h\to 0$ as $N,T\to\infty$. The second term on the right-hand side of (A.4) is $o(1)$ since by the result of Step 1 and Assumption 2(ii),

$$\sup_{\beta,\lambda_i\in\mathcal{A}}\left|\mathbb{S}^*_{NT}(\beta,\Lambda,\hat{F})-\mathbb{S}^*_{NT}(\beta,\Lambda,F_0)\right|\leq\max_{1\leq i\leq N}\sup_{\beta,\lambda_i\in\mathcal{A}}\left|\frac{1}{T}\sum_{t=1}^{T}\rho_{it}(\beta,\lambda_i,\hat{f}_t)-\frac{1}{T}\sum_{t=1}^{T}\rho_{it}(\beta,\lambda_i,\tilde{f}_{0t})\right|$$
$$\lesssim\sqrt{\frac{1}{T}\sum_{t=1}^{T}\|\hat{f}_t-\tilde{f}_{0t}\|^2}=o_P(1).$$

Finally, for the consistency of $\hat{\beta}$, it remains to show that the third term on the right-hand side of (A.4)

[1] Note that $|l(u)-\rho_\tau(u)|=|(\tau-\mathbf{1}\{u\leq 0\})u-(\tau-K(u/h))u|\leq|u|\cdot|\mathbf{1}\{u\leq 0\}-K(u/h)|\lesssim|u|\cdot\mathbf{1}\{|u|\leq h\}\lesssim h$.

is $o(1)$. By the union bound, it suffices to show that for all i

$$P\left[\sup_{(\beta,\lambda_i)\in B_{\delta,i}}\left|\frac{1}{T}\sum_{t=1}^{T}\left[\tilde{\rho}_{it}(\beta,\lambda_i,\tilde{f}_{0t})-\tilde{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right]\right|>1/3\epsilon\right]=o(N^{-1}). \tag{A.5}$$

Step 5: Write $\theta_i=(\beta',\lambda_i')'$, and $\theta_{0i}=(\beta_0',\tilde{\lambda}_{0i}')'$. Define $\Delta_{it}(\theta_i)=\rho_{it}(\beta,\lambda_i,\tilde{f}_{0t})-\rho_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})$. Note that there exists $C_1,C_2>0$ such that $|\Delta_{it}(\theta_i^a)-\Delta_{it}(\theta_i^b)|\leq C_1\cdot\|\beta^a-\beta^b\|\cdot\|X_{it}\|+C_2\cdot\|\lambda_i^a-\lambda_i^b\|$. Suppose that there exists $M_X>0$ such that $\mathbb{E}\|X_{it}\|\leq M_X$ for all i,t (see Assumption 2(v)).

Since $B_{\delta,i}$ is compact, for any $\eta>0$, there exists a positive integer L and a maximal set of points $\theta_i^{(1)},\ldots,\theta_i^{(L)}$ in $B_{\delta,i}$ such that $\|\theta_i^{(k)}-\theta_i^{(j)}\|\geq\eta$ for any $k\neq j$. For any $\theta_i\in B_{\delta,i}$, let $\theta_i^*=\{\theta_i^{(j)}:1\leq j\leq L,\|\theta_i-\theta_i^{(j)}\|\leq\eta\}$. Then,

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\left[\tilde{\rho}_{it}(\beta,\lambda_i,\tilde{f}_{0t})-\tilde{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right]&=\frac{1}{T}\sum_{t=1}^{T}\left[\Delta_{it}(\theta_i)-\mathbb{E}(\Delta_{it}(\theta_i))\right]\\&=\frac{1}{T}\sum_{t=1}^{T}\left[\Delta_{it}(\theta_i^*)-\mathbb{E}(\Delta_{it}(\theta_i^*))\right]+\frac{1}{T}\sum_{t=1}^{T}\left[\Delta_{it}(\theta_i)-\Delta_{it}(\theta_i^*)-\mathbb{E}(\Delta_{it}(\theta_i)-\Delta_{it}(\theta_i^*))\right],\end{aligned}$$

and

$$\begin{aligned}\sup_{(\beta,\lambda_i)\in B_{\delta,i}}\left|\frac{1}{T}\sum_{t=1}^{T}\left[\tilde{\rho}_{it}(\beta,\lambda_i,\tilde{f}_{0t})-\tilde{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right]\right|&\leq\max_{1\leq j\leq L}\left|\frac{1}{T}\sum_{t=1}^{T}\left[\Delta_{it}(\theta_i^{(j)})-\mathbb{E}(\Delta_{it}(\theta_i^{(j)}))\right]\right|\\&+\sup_{\|\theta^a-\theta^b\|\leq\eta}\left|\frac{1}{T}\sum_{t=1}^{T}\left[\Delta_{it}(\theta_i^a)-\Delta_{it}(\theta_i^b)-\mathbb{E}(\Delta_{it}(\theta_i^a)-\Delta_{it}(\theta_i^b))\right]\right|.\end{aligned}$$

Note that

$$\begin{aligned}&\sup_{\|\theta^a-\theta^b\|\leq\eta}\left|\frac{1}{T}\sum_{t=1}^{T}\left[\Delta_{it}(\theta_i^a)-\Delta_{it}(\theta_i^b)-\mathbb{E}(\Delta_{it}(\theta_i^a)-\Delta_{it}(\theta_i^b))\right]\right|\\&\qquad\leq C_1\eta\left(\frac{1}{T}\sum_{t=1}^{T}(\|X_{it}\|-\mathbb{E}\|X_{it}\|)\right)+2(C_2+C_1M_X)\eta,\end{aligned}$$

it then follows from the previous two inequalities that

$$\begin{aligned}&P\left[\sup_{(\beta,\lambda_i)\in B_{\delta,i}}\left|\frac{1}{T}\sum_{t=1}^{T}\left[\tilde{\rho}_{it}(\beta,\lambda_i,\tilde{f}_{0t})-\tilde{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right]\right|>1/3\epsilon\right]\\&\qquad\leq\sum_{j=1}^{L}P\left[\left|\frac{1}{T}\sum_{t=1}^{T}\left[\Delta_{it}(\theta_i^{(j)})-\mathbb{E}(\Delta_{it}(\theta_i^{(j)}))\right]\right|\geq1/9\epsilon\right]\\&\qquad+P\left[C_1\eta\left|\frac{1}{T}\sum_{t=1}^{T}(\|X_{it}\|-\mathbb{E}\|X_{it}\|)\right|\geq1/9\epsilon\right]+P\left[2(C_2+C_1M_X)\eta\geq1/9\epsilon\right].\end{aligned}\tag{A.6}$$

First, choosing $\eta < \epsilon/(18(C_2 + C_1 M_X))$, the last term on the right-hand side of (A.6) is 0.

Second, for any $\theta_i \in B_{\delta,i}$, $\mathbb{E}|\Delta_{it}(\theta_i)|^{2m+\gamma} \leq M \cdot \mathbb{E}\|X_{it}\|^{2m+\gamma} + O(1) < \infty$ by Assumption 2(v). Thus, by Assumption 2(iv) and Theorem 3 of Yoshihara (1978) we have

$$\mathbb{E}\left|T^{-1/2}\sum_{t=1}^{T}\left[\Delta_{it}(\theta_i^{(j)}) - \mathbb{E}(\Delta_{it}(\theta_i^{(j)}))\right]\right|^{2m} \leq M,$$

and by Markov's inequality,

$$P\left[\left|\frac{1}{T}\sum_{t=1}^{T}\left[\Delta_{it}(\theta_i^{(j)}) - \mathbb{E}(\Delta_{it}(\theta_i^{(j)}))\right]\right| \geq 1/9\epsilon\right] = O(T^{-m}).$$

Finally, we can show that the second term on the right-hand side of (A.6) is $O(T^{-m})$ in a similar way. Thus, (A.5) follows since $N/T^m \to 0$ by Assumption 2(vi). This completes the proof. □

A.3 Proof of Theorem 2

To simplify the notations, write $\check{f}_{0t} = \hat{\mathbf{H}} f_{0t}$ and $\check{\lambda}_{0i} = (\hat{\mathbf{H}}')^{-1}\lambda_{0i}$.

Lemma 2. *Under Assumptions 1 to 4,*
(i) $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \check{f}_{0t}\| = O_P(N^{-1/2})$, $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \check{f}_{0t}\|^2 = O_P(N^{-1})$, $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \check{f}_{0t}\|^3 = O_P(N^{-3/2})$, $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \check{f}_{0t}\|^4 = O_P(N^{-2})$.
(ii) $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \tilde{f}_{0t}\| = O_P(N^{-1/2})$, $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \tilde{f}_{0t}\|^2 = O_P(N^{-1})$, $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \tilde{f}_{0t}\|^3 = O_P(N^{-3/2})$, $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \tilde{f}_{0t}\|^4 = O_P(N^{-2})$.
(iii) $\max_{1\leq t\leq T}\|\hat{f}_t - \check{f}_{0t}\| = O_P(T^{1/2m}/\sqrt{N})$ *and* $\max_{1\leq t\leq T}\|\hat{f}_t - \tilde{f}_{0t}\| = O_P(T^{1/2m}/\sqrt{N})$.

Proof. By the properties of L_p norms in the Euclidean space, it suffices to prove that $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \check{f}_{0t}\|^4 = O_P(N^{-2})$ and $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \tilde{f}_{0t}\|^4 = O_P(N^{-2})$. Note that Assumption 3(iv) implies that $\{e_{1t},\ldots,e_{Nt}\}$ is independent across i and $\mathbb{E}\|e_{it}\|^4 < M$ for all i,t. Thus,

$$\mathbb{E}\|\sqrt{N}\bar{e}_t\|^4 = \mathbb{E}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right\|^4 \leq \left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\|e_{it}\|^2\right)^2 + \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\|e_{it}\|^4 = O(1),$$

and by Lemma 1

$$\frac{1}{T}\sum_{t=1}^{T}\|\hat{f}_t - \check{f}_{0t}\|^4 \leq \|\hat{\Psi}\|^4 \cdot \frac{1}{N^2 T}\sum_{t=1}^{T}\|\sqrt{N}\bar{e}_t\|^4 = O_P(N^{-2}).$$

Moreover, $T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \tilde{f}_{0t}\|^4 \leq T^{-1}\sum_{t=1}^{T}\|\hat{f}_t - \check{f}_{0t}\|^4 + C\|\hat{\mathbf{H}} - \mathbf{H}_0\|^4$. Then result (ii) follows if we can show that $\|\hat{\mathbf{H}} - \mathbf{H}_0\| = O_P(N^{-1/2})$.

By definition, $\|\hat{\mathbf{H}} - \mathbf{H}_0\| \leq O_P(\|\hat{\Psi} - \Psi_0\|) + O_P(\|\bar{\Gamma} - \Gamma_0\|)$. By the proof of Proposition 1 and Lemma 1 we have $\|\hat{\Psi} - \Psi_0\| \lesssim \|\hat{\Sigma}_{\bar{x}} - \Gamma_0\Sigma_{f_0}\Gamma_0'\| = O_P(N^{-1/2} + T^{-1/2})$. Then the result (ii) follows from Assumption 1(ii) and the fact that $N \asymp T$.

Finally, note that $\max_{1\leq t\leq T}\|\hat{f}_t - \check{f}_{0t}\| \leq O_P(1)\cdot\max_{1\leq t\leq T}\|\bar{e}_t\|$. For $1 \leq h \leq r$, it is easy to show that $\mathbb{E}|\sqrt{N}\bar{e}_{th}|^{2m} \leq M$ for all t. Thus, it follows from Lemma 2.2.1 and Lemma 2.2.2 of van der Vaart and Wellner (1996) that $\mathbb{E}[\max_{1\leq t\leq T}\sqrt{N}|\bar{e}_{th}|] = O(T^{1/2m})$. Thus, result (iii) follows since $\|\hat{\mathbf{H}} - \mathbf{H}_0\| = o_P(1)$. □

Lemma 3. *Under Assumptions 1 to 4,* $\max_{1\leq i\leq N}\|\hat{\lambda}_i - \check{\lambda}_{0i}\| = o_P(1)$.

Proof. Recall that $\tilde{\lambda}_{0i} = (\mathbf{H}_0')^{-1}\lambda_{0i}$, $\tilde{f}_{0t} = \mathbf{H}_0 f_{0t}$. By the definition of the estimators we have $\mathbb{S}_{i,T}(\hat{\beta},\hat{\lambda}_i,\hat{F}) \leq \mathbb{S}_{i,T}(\hat{\beta},\tilde{\lambda}_{0i},\hat{F})$ for each i. Note that

$$\mathbb{S}_{i,T}(\beta,\lambda_i,F) = \mathbb{S}^*_{i,T}(\beta_0,\lambda_i,\tilde{F}_0) + (\mathbb{S}_{i,T}(\beta,\lambda_i,F) - \mathbb{S}^*_{i,T}(\beta,\lambda_i,F)) + (\mathbb{S}^*_{i,T}(\beta,\lambda_i,F) - \mathbb{S}^*_{i,T}(\beta_0,\lambda_i,\tilde{F}_0)).$$

Thus, $\mathbb{S}_{i,T}(\hat{\beta},\hat{\lambda}_i,\hat{F}) \leq \mathbb{S}_{i,T}(\hat{\beta},\tilde{\lambda}_{0i},\hat{F})$ implies that

$$\begin{aligned}\mathbb{S}^*_{i,T}(\beta_0,\hat{\lambda}_i,\tilde{F}_0) - \mathbb{S}^*_{i,T}(\beta_0,\tilde{\lambda}_{0i},\tilde{F}_0) \leq (\mathbb{S}_{i,T}(\hat{\beta},\tilde{\lambda}_{0i},\hat{F}) - \mathbb{S}^*_{i,T}(\hat{\beta},\tilde{\lambda}_{0i},\hat{F})) + (\mathbb{S}^*_{i,T}(\hat{\beta},\tilde{\lambda}_{0i},\hat{F}) - \mathbb{S}^*_{i,T}(\beta_0,\tilde{\lambda}_{0i},\tilde{F}_0)) \\ - (\mathbb{S}_{i,T}(\hat{\beta},\hat{\lambda}_{0i},\hat{F}) - \mathbb{S}^*_{i,T}(\hat{\beta},\hat{\lambda}_{0i},\hat{F})) - (\mathbb{S}^*_{i,T}(\hat{\beta},\hat{\lambda}_{0i},\hat{F}) - \mathbb{S}^*_{i,T}(\beta_0,\hat{\lambda}_{0i},\tilde{F}_0)).\end{aligned} \quad \text{(A.7)}$$

Similar to the proof of Theorem 1, for small $\delta > 0$, define $B_{\delta,i} = \{\lambda_i \in \mathcal{A} : \|\lambda_i - \tilde{\lambda}_{0i}\| \leq \delta\}$. For any $\lambda_i \in B^C_{\delta,i}$, let $m = \|\lambda_i - \tilde{\lambda}_{0i}\| > \delta$. Then $\bar{\lambda}_i = \lambda_i \cdot \delta/m + \tilde{\lambda}_{0i}\cdot(1-\delta/m)$ is on the boundary of $B_{\delta,i}$, i.e., $\|\bar{\lambda}_i - \tilde{\lambda}_{0i}\| = \delta$. Given β_0 and $\tilde{f}_{0t}$, the check function is convex in λ_i, thus

$$\delta/m\cdot\rho_{it}(\beta_0,\lambda_i,\tilde{f}_{0t}) + (1-\delta/m)\cdot\rho_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t}) \geq \rho_{it}(\beta_0,\bar{\lambda}_i,\tilde{f}_{0t}),$$

and it follows that

$$\rho_{it}(\beta_0,\lambda_i,\tilde{f}_{0t}) - \rho_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t}) \geq m/\delta\cdot\left(\rho_{it}(\beta_0,\bar{\lambda}_i,\tilde{f}_{0t}) - \rho_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right).$$

Note that $\rho_{it}(\beta_0,\bar{\lambda}_i,\tilde{f}_{0t}) - \rho_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t}) = \bar{\rho}_{it}(\beta_0,\bar{\lambda}_i,\tilde{f}_{0t}) - \bar{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t}) + \tilde{\rho}_{it}(\beta_0,\bar{\lambda}_i,\tilde{f}_{0t}) - \tilde{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})$, and

$$\bar{\rho}_{it}(\beta_0,\bar{\lambda}_i,\tilde{f}_{0t}) - \bar{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t}) \geq (\bar{\lambda}_i - \tilde{\lambda}_{0i})'\left(\mathsf{f}_{it}(0)\tilde{f}_{0t}\tilde{f}_{0t}'\right)(\bar{\lambda}_i - \tilde{\lambda}_{0i}) + o(\delta^2).$$

Thus, if $\|\hat{\lambda}_i - \tilde{\lambda}_{0i}\| > \delta$, by Assumption 3(ii)

$$\begin{aligned}\mathbb{S}^*_{i,T}(\beta_0,\hat{\lambda}_i,\tilde{F}_0) - \mathbb{S}^*_{i,T}(\beta_0,\tilde{\lambda}_{0i},\tilde{F}_0) &= \frac{1}{T}\sum_{t=1}^{T}\left[\rho_{it}(\beta_0,\hat{\lambda}_i,\tilde{f}_{0t}) - \rho_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right] \\ \geq (\hat{\bar{\lambda}}_i - \tilde{\lambda}_{0i})'\left(\frac{1}{T}\sum_{t=1}^{T}\mathsf{f}_{it}(0)\tilde{f}_{0t}\tilde{f}_{0t}'\right)(\hat{\bar{\lambda}}_i - \tilde{\lambda}_{0i}) &+ o(\delta^2) + m/\delta\cdot\frac{1}{T}\sum_{t=1}^{T}\left[\tilde{\rho}_{it}(\beta_0,\hat{\bar{\lambda}}_i,\tilde{f}_{0t}) - \tilde{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right] \\ &\geq C\delta^2 + m/\delta\cdot\frac{1}{T}\sum_{t=1}^{T}\left[\tilde{\rho}_{it}(\beta_0,\hat{\bar{\lambda}}_i,\tilde{f}_{0t}) - \tilde{\rho}_{it}(\beta_0,\tilde{\lambda}_{0i},\tilde{f}_{0t})\right],\end{aligned}$$

where $\hat{\bar{\lambda}}_i$ is between $\tilde{\lambda}_{0i}$ and $\hat{\lambda}_i$ and is on the boundary of $B_{\delta,i}$. Thus, it follows from (A.7) that there exists some $\epsilon > 0$ (depending on δ) such that

$$P\left[\max_{1\le i\le N}\|\hat{\lambda}_i-\tilde{\lambda}_{0i}\|>\delta\right]\le P\left[\max_{1\le i\le N}\sup_{\beta,\lambda_i,F}\left|\mathbb{S}_{i,T}(\beta,\lambda_i,F)-\mathbb{S}^*_{i,T}(\beta,\lambda_i,F)\right|>\epsilon\right]$$
$$+P\left[\max_{1\le i\le N}\sup_{\lambda_i\in\mathcal{A}}\left|\mathbb{S}^*_{i,T}(\hat{\beta},\lambda_i,\hat{F})-\mathbb{S}^*_{i,T}(\beta_0,\lambda_i,\tilde{F}_0)\right|>\epsilon\right]$$
$$+P\left[\max_{1\le i\le N}\sup_{\lambda_i\in B_{\delta,i}}\left|\tilde{\mathbb{S}}^*_{i,T}(\beta_0,\lambda_i,\tilde{F}_0)-\tilde{\mathbb{S}}^*_{i,T}(\beta_0,\lambda_i,\tilde{F}_0)\right|>\epsilon\right]. \quad (A.8)$$

Similar to the proof of Theorem 1, it can be shown that the first and last term on the right-hand side of (A.8) are both $o(1)$. It remains to show that the second term is $o(1)$.

By the property of the check function, we have

$$\begin{aligned}
&\max_{1\le i\le N}\sup_{\lambda_i\in\mathcal{A}}\left|\mathbb{S}^*_{i,T}(\hat{\beta},\lambda_i,\hat{F})-\mathbb{S}^*_{i,T}(\beta_0,\lambda_i,\tilde{F}_0)\right|\\
\le\ &\max_{1\le i\le N}\sup_{\lambda_i\in\mathcal{A}}\left|\mathbb{S}^*_{i,T}(\hat{\beta},\lambda_i,\hat{F})-\mathbb{S}^*_{i,T}(\hat{\beta},\lambda_i,\tilde{F}_0)\right|+\max_{1\le i\le N}\sup_{\lambda_i\in\mathcal{A}}\left|\mathbb{S}^*_{i,T}(\hat{\beta},\lambda_i,\tilde{F}_0)-\mathbb{S}^*_{i,T}(\beta_0,\lambda_i,\tilde{F}_0)\right|\\
\lesssim\ &\frac{1}{T}\sum_{t=1}^{T}\|\hat{f}_t-\tilde{f}_{0t}\|+\max_{1\le i\le N}\sup_{\lambda_i\in\mathcal{A}}\left|\frac{1}{T}\sum_{t=1}^{T}\Delta_{it}(\hat{\beta},\lambda_i)-\bar{\Delta}_{it}(\hat{\beta},\lambda_i)\right|+\max_{1\le i\le N}\sup_{\lambda_i\in\mathcal{A}}\left|\frac{1}{T}\sum_{t=1}^{T}\bar{\Delta}_{it}(\hat{\beta},\lambda_i)\right|\\
\le\ &o_P(1)+\max_{1\le i\le N}\sup_{\lambda_i\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\left|\frac{1}{T}\sum_{t=1}^{T}\Delta_{it}(\beta,\lambda_i)-\bar{\Delta}_{it}(\beta,\lambda_i)\right|+O_P\left(\|\hat{\beta}-\beta_0\|\right)
\end{aligned}$$

where $\Delta_{it}(\beta,\lambda_i)=\rho_{it}(\beta,\lambda_i,\tilde{f}_{0t})-\rho_{it}(\beta_0,\lambda_i,\tilde{f}_{0t})$ and $\bar{\Delta}_{it}(\beta,\lambda_i)=\mathbb{E}\Delta_{it}(\beta,\lambda_i)$. Similar to the proof of (A.2) in Kato et al. (2012) it can be shown that

$$\max_{1\le i\le N}\sup_{\lambda_i\in\mathcal{A}}\sup_{\beta\in\mathcal{B}}\left|\frac{1}{T}\sum_{t=1}^{T}\Delta_{it}(\beta,\lambda_i)-\bar{\Delta}_{it}(\beta,\lambda_i)\right|=o_P(1).$$

It then follows from $\|\hat{\beta}-\beta_0\|=o_P(1)$ and Lemma 2(ii) that

$$\max_{1\le i\le N}\sup_{\lambda_i\in\mathcal{A}}\left|\mathbb{S}^*_{i,T}(\hat{\beta},\lambda_i,\hat{F})-\mathbb{S}^*_{i,T}(\beta_0,\lambda_i,\tilde{F}_0)\right|=o_P(1).$$

This implies that the second term on the right-hand side of (A.8) is $o(1)$. The desired result follows by noting that $\max_i\|\tilde{\lambda}_{0i}-\check{\lambda}_{0i}\|\le O_P(1)\cdot\|\hat{\mathbf{H}}-\mathbf{H}_0\|=o_P(1)$. □

Lemma 4. *Under Assumptions 1 to 4,* $\|\hat{\beta}-\beta_0\|=o_P(\max_i\|\hat{\lambda}_i-\check{\lambda}_{0i}\|)+o_P(1/\sqrt{T})$.

Proof. **Step 1:** Define the following notations:

$$\underbrace{S^{\beta}(\beta,\Lambda,F)}_{p\times 1}=\partial\mathbb{S}_{NT}(\beta,\Lambda,F)/\partial\beta,\quad \underbrace{S^{\lambda}(\beta,\Lambda,F)}_{Nr\times 1}=\partial\mathbb{S}_{NT}(\beta,\Lambda,F)/\partial\Lambda,$$

$$\underbrace{S^{\beta\beta'}(\beta,\Lambda,F)}_{p\times p} = \partial^2\mathbb{S}_{NT}(\beta,\Lambda,F)/\partial\beta\partial\beta', \quad \underbrace{S^{\beta\lambda'}(\beta,\Lambda,F)}_{p\times Nr} = \partial^2\mathbb{S}_{NT}(\beta,\Lambda,F)/\partial\beta\partial\Lambda'.$$

The other functions such as $S^{\beta f'}(\beta,\Lambda,F)$, $S^{\lambda\lambda'}(\beta,\Lambda,F)$, $S^{\lambda f'}(\beta,\Lambda,F)$, $S^{\beta\beta' f_{th}}(\beta,\Lambda,F)$ are defined in a similar fashion. The arguments of these functions are dropped when they are evaluated at $(\beta,\Lambda,F) = (\beta_0,\check{\Lambda}_0,\check{F}_0)$, where $\check{\Lambda}_0 = (\check{\lambda}_{01},\ldots,\check{\lambda}_{0N})'$, $\check{F}_0 = (\check{f}_{01},\ldots,\check{f}_{0T})'$ (recall that $\check{f}_{0t} = \hat{\mathbf{H}}f_{0t}$ and $\check{\lambda}_{0i} = (\hat{\mathbf{H}}')^{-1}\lambda_{0i}$).

Expanding $S^{\beta}_{NT}(\hat\beta,\hat\Lambda,\hat F)$ and $S^{\lambda}_{NT}(\hat\beta,\hat\Lambda,\hat F)$ around $(\beta_0,\check{\Lambda}_0,\check{F}_0)$ up to the third order gives:

$$0 = S^{\beta}(\hat\beta,\hat\Lambda,\hat F) = S^{\beta} + S^{\beta\beta'}(\hat\beta-\beta_0) + S^{\beta\lambda'}(\hat\Lambda-\check\Lambda_0) + S^{\beta f'}(\hat F-\check F_0) + 1/2R^{\beta}(\beta^*,\Lambda^*,F^*), \tag{A.9}$$

$$0 = S^{\lambda}(\hat\beta,\hat\Lambda,\hat F) = S^{\lambda} + S^{\lambda\beta'}(\hat\beta-\beta_0) + S^{\lambda\lambda'}(\hat\Lambda-\check\Lambda_0) + S^{\lambda f'}(\hat F-\check F_0) + 1/2R^{\lambda}(\beta^*,\Lambda^*,F^*), \tag{A.10}$$

where (β^*,Λ^*,F^*) is between $(\beta_0,\check\Lambda_0,\check F_0)$ and $(\hat\beta,\hat\Lambda,\hat F)$, and

$$\begin{aligned}
&R^{\beta}(\beta^*,\Lambda^*,F^*) = \\
&\quad\sum_{k=1}^{p} S_*^{\beta\beta'\beta_k}(\hat\beta_k-\beta_{0k})(\hat\beta-\beta_0) + \sum_{k=1}^{p} S_*^{\beta\lambda'\beta_k}(\hat\beta_k-\beta_{0k})(\hat\Lambda-\check\Lambda_0) + \sum_{k=1}^{p} S_*^{\beta f'\beta_k}(\hat\beta_k-\beta_{0k})(\hat F-\check F_0) \\
&+\sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\beta\beta'\lambda_{ih}}(\hat\lambda_{ih}-\check\lambda_{0i,h})(\hat\beta-\beta_0) + \sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\beta\lambda'\lambda_{ih}}(\hat\lambda_{ih}-\check\lambda_{0i,h})(\hat\Lambda-\check\Lambda_0) + \sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\beta f'\lambda_{ih}}(\hat\lambda_{ih}-\check\lambda_{0i,h})(\hat F-\check F_0) \\
&+\sum_{t=1}^{T}\sum_{h=1}^{r} S_*^{\beta\beta' f_{th}}(\hat f_{th}-\check f_{0t,h})(\hat\beta-\beta_0) + \sum_{t=1}^{T}\sum_{h=1}^{r} S_*^{\beta\lambda' f_{th}}(\hat f_{th}-\check f_{0t,h})(\hat\Lambda-\check\Lambda_0) + \sum_{t=1}^{T}\sum_{h=1}^{r} S_*^{\beta f' f_{th}}(\hat f_{th}-\check f_{0t,h})(\hat F-\check F_0),
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
&R^{\lambda}(\beta^*,\Lambda^*,F^*) = \\
&\quad\sum_{k=1}^{p} S_*^{\lambda\beta'\beta_k}(\hat\beta_k-\beta_{0k})(\hat\beta-\beta_0) + \sum_{k=1}^{p} S_*^{\lambda\lambda'\beta_k}(\hat\beta_k-\beta_{0k})(\hat\Lambda-\check\Lambda_0) + \sum_{k=1}^{p} S_*^{\lambda f'\beta_k}(\hat\beta_k-\beta_{0k})(\hat F-\check F_0) \\
&+\sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\lambda\beta'\lambda_{ih}}(\hat\lambda_{ih}-\check\lambda_{0i,h})(\hat\beta-\beta_0) + \sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\lambda\lambda'\lambda_{ih}}(\hat\lambda_{ih}-\check\lambda_{0i,h})(\hat\Lambda-\check\Lambda_0) + \sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\lambda f'\lambda_{ih}}(\hat\lambda_{ih}-\check\lambda_{0i,h})(\hat F-\check F_0) \\
&+\sum_{t=1}^{T}\sum_{h=1}^{r} S_*^{\lambda\beta' f_{th}}(\hat f_{th}-\check f_{0t,h})(\hat\beta-\beta_0) + \sum_{t=1}^{T}\sum_{h=1}^{r} S_*^{\lambda\lambda' f_{th}}(\hat f_{th}-\check f_{0t,h})(\hat\Lambda-\check\Lambda_0) + \sum_{t=1}^{T}\sum_{h=1}^{r} S_*^{\lambda f' f_{th}}(\hat f_{th}-\check f_{0t,h})(\hat F-\check F_0),
\end{aligned} \tag{A.12}$$

where the asterisk in the subscript of the functions means that these functions are evaluated at (β^*,Λ^*,F^*).

Define $\tilde S^{\lambda\lambda'} = S^{\lambda\lambda'} - \bar S^{\lambda\lambda'}$, $\tilde S^{\beta\lambda'} = S^{\beta\lambda'} - \bar S^{\beta\lambda'}$, where $\bar S^{\beta\lambda'} = N^{-1}(\tilde\Xi_1,\ldots,\tilde\Xi_N)$, $\bar S^{\lambda\lambda'} = N^{-1}\operatorname{diag}(\tilde\Omega_1,\ldots,\tilde\Omega_N)$, and $\tilde\Xi_i = \Xi_i\hat{\mathbf{H}}'$, $\tilde\Omega_i = \hat{\mathbf{H}}\Omega_i\hat{\mathbf{H}}'$. Recall that

$$\Xi_i = \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\mathsf{f}_{it}(0|X_{it})X_{it}]f_{0t}', \quad \Omega_i = \frac{1}{T}\sum_{t=1}^{T}\mathsf{f}_{it}(0)f_{0t}f_{0t}'.$$

Then (A.9) can be written as

$$0 = S^{\beta} + S^{\beta\beta'}(\hat{\beta} - \beta_0) + \bar{S}^{\beta\lambda'}(\hat{\Lambda} - \check{\Lambda}_0) + \tilde{S}^{\beta\lambda'}(\hat{\Lambda} - \check{\Lambda}_0) + S^{\beta f'}(\hat{F} - \check{F}_0) + 1/2R^{\beta}(\beta^*, \Lambda^*, F^*), \quad (A.13)$$

and (A.10) can be written as

$$0 = S^{\lambda} + S^{\lambda\beta'}(\hat{\beta} - \beta_0) + \bar{S}^{\lambda\lambda'}(\hat{\Lambda} - \check{\Lambda}_0) + \tilde{S}^{\lambda\lambda'}(\hat{\Lambda} - \check{\Lambda}_0) + S^{\lambda f'}(\hat{F} - \check{F}_0) + 1/2R^{\lambda}(\beta^*, \Lambda^*, F^*). \quad (A.14)$$

Plugging (A.14) into (A.13) gives

$$\begin{aligned}&[S^{\beta\beta'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda\beta'}](\hat{\beta} - \beta_0) = \\ &- \left[S^{\beta} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda}\right] - \left[S^{\beta f'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda f'}\right](\hat{F} - \check{F}_0) - \left[\tilde{S}^{\beta\lambda'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\lambda'}\right](\hat{\Lambda} - \check{\Lambda}_0) \\ &- 1/2\left[R^{\beta}(\beta^*, \Lambda^*, F^*) - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}R^{\lambda}(\beta^*, \Lambda^*, F^*)\right]. \quad (A.15)\end{aligned}$$

Step 2: The term $S^{\beta\beta'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda\beta'}$ can be written as

$$\bar{S}^{\beta\beta'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\bar{S}^{\lambda\beta'} + \tilde{S}^{\beta\beta'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\beta'},$$

where $\bar{S}^{\beta\beta'} = (NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[\mathsf{f}_{it}(0|X_{it})X_{it}X_{it}']$, $\bar{S}^{\lambda\beta'} = (\bar{S}^{\beta\lambda'})'$. Note that

$$\bar{S}^{\beta\beta'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\bar{S}^{\lambda\beta'} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[\mathbb{E}[\mathsf{f}_{it}(0|X_{it})X_{it}X_{it}' - \Xi_i\Omega_i^{-1}\Xi_i'] = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}Z_{it}'].$$

Next, we show that $\tilde{S}^{\beta\beta'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\beta'} = o_P(1)$. Write

$$\tilde{S}^{\beta\beta'} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(l_{it}^{(2)}X_{it}X_{it}' - \mathbb{E}[l_{it}^{(2)}X_{it}X_{it}']\right) + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\mathbb{E}[l_{it}^{(2)}X_{it}X_{it}'] - \mathbb{E}[\mathsf{f}_{it}(0|X_{it})X_{it}X_{it}']\right)$$

where the second term on the right-side of the above equation is $O(h^q) = o(1)$ by Lemma S1, and for the first term, by Assumption 3(iv) we have

$$\mathbb{E}\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(l_{it}^{(2)}X_{it}X_{it}' - \mathbb{E}[l_{it}^{(2)}X_{it}X_{it}']\right)\right\|^2 \le \frac{1}{Th^2}\cdot\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}h\left(l_{it}^{(2)}X_{it}X_{it}' - \mathbb{E}[l_{it}^{(2)}X_{it}X_{it}']\right)\right\|^2.$$

By Lemma S1 and Assumption 3(iii), $\mathbb{E}\|h \cdot l_{it}^{(2)}X_{it}X_{it}'\|^{2+\gamma/2} \le \mathbb{E}\|X_{it}\|^{4+\gamma} \le M$. Thus, it follows from the mixing property (Assumption 2(iv)), the fact that $Th^2 \to \infty$ (Assumption 3(vii)) and Theorem 3 of Yoshihara (1978) that the right-hand side of the above inequality is $o(1)$. Thus, we have $\tilde{S}^{\beta\beta'} = o_P(1)$. Similarly, we can show that $\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\beta'} = o_P(1)$. Therefore, it follows that

$$S^{\beta\beta'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda\beta'} = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}Z_{it}'] + o_P(1) = \Delta + o_P(1). \quad (A.16)$$

Step 3: $S^{\beta} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda}$ can be written as

$$-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} l_{it}^{(1)} Z_{it} = -\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(l_{it}^{(1)}Z_{it} - \mathbb{E}[l_{it}^{(1)}Z_{it}]\right) - \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[l_{it}^{(1)}Z_{it}].$$

By Lemma S1 and Assumption 3(vii), $(NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[l_{it}^{(1)}Z_{it}] = o(h^q) = o(T^{-1/2})$. Similar to the proof of (A.53) below, the first term on the right-hand side of the above equation is $O_P(1/\sqrt{NT}) = o_P(T^{-1/2})$. Thus, we have

$$S^{\beta} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda} = o_P(1/\sqrt{T}). \tag{A.17}$$

Step 4: By Lemma 8 below

$$[S^{\beta f'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda f'}](\hat{F} - \check{F}_0) = o_P(1/\sqrt{T}). \tag{A.18}$$

Step 5: Now consider the term: $\left[\tilde{S}^{\beta\lambda'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\lambda'}\right](\hat{\Lambda} - \check{\Lambda}_0)$. Write

$$\tilde{S}^{\beta\lambda'}(\hat{\Lambda} - \check{\Lambda}_0) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} X_{it} f_{0t}' - \Xi_i\right)\hat{\mathbf{H}}(\hat{\lambda}_i - \check{\lambda}_{0i}).$$

By the Cauchy-Schwarz inequality, we have

$$\left\|\tilde{S}^{\beta\lambda'}(\hat{\Lambda} - \check{\Lambda}_0)\right\| \lesssim \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} X_{it} f_{0t}' - \Xi_i\right\|^2} \cdot \|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N}.$$

Note that by Lemma S1,

$$\begin{aligned}\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} X_{it} f_{0t}' - \Xi_i\right\|^2 &= \mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T}\left(l_{it}^{(2)}X_{it} - \mathbb{E}[l_{it}^{(2)}X_{it}]\right) f_{0t}' + \frac{1}{T}\sum_{t=1}^{T}\left(\mathbb{E}[l_{it}^{(2)}X_{it}] - \mathbb{E}[\mathsf{f}_{it}(0|X_{it})X_{it}]\right) f_{0t}'\right\|^2 \\ &\leq \frac{1}{h^2}\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T} h\left(l_{it}^{(2)}X_{it} - \mathbb{E}[l_{it}^{(2)}X_{it}]\right) f_{0t}'\right\|^2 + o(1)\end{aligned}$$

and by the mixing property and Theorem 3 of Yoshihara (1978) the first term on the right-hand side of the above inequality is $O(1/(Th^2)) = o(1)$. Thus, it follows that

$$\left\|\tilde{S}^{\beta\lambda'}(\hat{\Lambda} - \check{\Lambda}_0)\right\| = o_P(1) \cdot \|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N}.$$

Similarly, we can show that $\|\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\lambda'}(\hat{\Lambda} - \check{\Lambda}_0)\| = o_P(1) \cdot \|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N}$, and conclude that

$$\left\|\left(\tilde{S}^{\beta\lambda'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\lambda'}\right)(\hat{\Lambda} - \check{\Lambda}_0)\right\| = o_P(1) \cdot \|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N}. \tag{A.19}$$

Step 6: We will show that:

$$R^{\beta}(\beta^*, \Lambda^*, F^*) = o_P(\|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N}) + o_P(\|\hat{\beta} - \beta_0\|) + o_P(1/\sqrt{T}), \tag{A.20}$$

$$\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1} R^{\lambda}(\beta^*, \Lambda^*, F^*) = o_P(\|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N}) + o_P(\|\hat{\beta} - \beta_0\|) + o_P(1/\sqrt{T}). \tag{A.21}$$

To save space, we focus on (A.20), which follows from Results 1 to 9 below. The proof of (A.21) is similar.

Result 1: $S_*^{\beta\beta'\beta_k}(\hat{\beta}_k - \beta_{0k})(\hat{\beta} - \beta_0) = o_P(\|\hat{\beta} - \beta_0\|)$.

Observe that:

$$S_*^{\beta\beta'\beta_k}(\hat{\beta}_k - \beta_{0k})(\hat{\beta} - \beta_0) = -\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} X_{it}' X_{it,k} (\hat{\beta}_k - \beta_{0k})(\hat{\beta} - \beta_0),$$

so

$$\|S_*^{\beta\beta'\beta_k}(\hat{\beta}_k - \beta_{0k})(\hat{\beta} - \beta_0)\| \leq \|\hat{\beta} - \beta_0\|^2 \cdot \left\| \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} X_{it}' X_{it,k} \right\|.$$

Expanding $l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*)$ around $(\beta^*, \lambda_i^*, \tilde{f}_{0t})$ gives

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} X_{it}' X_{it,k} \\
&= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} l_{it}^{(4)}(\beta^*, \lambda_i^*, f_t^{**}) X_{it} X_{it}' X_{it,k} (\lambda_i^*)'(f_t^* - \tilde{f}_{0t}) + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \mathbb{E}[l_{it}^{(3)}(\beta, \lambda_i, \tilde{f}_{0t}) X_{it} X_{it}' X_{it,k}]|_{\beta=\beta^*, \lambda_i=\lambda_i^*} \\
&+ \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \left[l_{it}^{(3)}(\beta^*, \lambda_i^*, \tilde{f}_{0t}) X_{it} X_{it}' X_{it,k} - \mathbb{E}[l_{it}^{(3)}(\beta, \lambda_i, \tilde{f}_{0t}) X_{it} X_{it}' X_{it,k}]|_{\beta=\beta^*, \lambda_i=\lambda_i^*} \right],
\end{aligned} \tag{A.22}$$

where f_t^{**} is between f_t^* and $\check{f}_{0t}$. By Lemma S1, the second term on the right-hand side of (A.22) is $(NT)^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T} \mathbb{E}[\mathsf{f}_{it}^{(1)}(\cdot|X_{it}) X_{it} X_{it}' X_{it,k}] + \bar{O}(h^{q-1}) = O(1)$, and the first term is bounded by

$$\sqrt{\frac{1}{T}\sum_{t=1}^{T} \|f_t^* - \tilde{f}_{0t}\|^2 \cdot \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} [l_{it}^{(4)}(\beta^*, \lambda_i^*, f_t^{**})]^2 \cdot \|X_{it}\|^6 \cdot \|\lambda_i^*\|^2} = O_P\left(1/\sqrt{Th^6}\right) = o_P(1),$$

since we have $|l_{it}^{(4)}(\cdot)| \lesssim 1/h^3$ by Lemma S1, $\mathbb{E}\|X_{it}\|^6 < M$ by Assumption 3(iii), and $T^{-1}\sum_{t=1}^{T} \|f_t^* - \tilde{f}_{0t}\|^2 \leq T^{-1}\sum_{t=1}^{T} \|\hat{f}_t - \tilde{f}_{0t}\|^2 = O_P(N^{-1}) = O_P(T^{-1})$ by Lemma 2 and Assumption 3(vii). Finally, with probability approaching 1, the last term on the right-hand side of (A.22) is bounded by $N^{-1}\sum_{i=1}^{N} \mathcal{Z}_i$, where

$$\mathcal{Z}_i = \sup_{(\beta, \lambda_i) \in B_{\delta,i}} \left\| \frac{1}{T}\sum_{t=1}^{T} \left[l_{it}^{(3)}(\beta, \lambda_i, \tilde{f}_{0t}) X_{it} X_{it}' X_{it,k} - \mathbb{E}[l_{it}^{(3)}(\beta, \lambda_i, \tilde{f}_{0t}) X_{it} X_{it}' X_{it,k}] \right] \right\|$$

and $B_{\delta,i}$ is a neighbourhood of $(\beta_0, \tilde{\lambda}_{0i})$. Then Result 1 follows if we can show that $\max_{1 \leq i \leq N} \mathbb{E}[\mathcal{Z}_i] < \infty$.

For any $\epsilon > 0$, let $\theta_i^{(1)}, \ldots, \theta_i^{(L)}$ be a maximal set of points in $B_{\delta,i}$ such that $\|\theta_i^{(j)} - \theta_i^{(l)}\| \geq \epsilon$ for any $j \neq l$. It is well known that L, the packing number of a Euclidean ball, is bounded (up to a positive constant that only depends on $p+r$) by $(1/\epsilon)^{p+r}$. Thus, we have

$$\mathbb{E}\mathcal{Z}_i \le \frac{1}{\sqrt{Th^4}}\cdot\mathbb{E}\left[\max_{1\le j\le L}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}h^2\left[l_{it}^{(3)}(\beta^{(j)},\lambda_i^{(j)},\tilde{f}_{0t})X_{it}X_{it}'X_{it,k}-\mathbb{E}[l_{it}^{(3)}(\beta^{(j)},\lambda_i^{(j)},\tilde{f}_{0t})X_{it}X_{it}'X_{it,k}]\right]\right\|\right]+O(\epsilon/h^3),$$

where we have used the fact that $|l_{it}^{(3)}(\beta^a,\lambda_i^a,\tilde{f}_{0t})-l_{it}^{(3)}(\beta^b,\lambda_i^b,\tilde{f}_{0t})|\lesssim(\|\beta^a-\beta^b\|+\|\lambda_i^a-\lambda_i^b\|)/h^3$.

By Assumption 3 (iii) and (vii), $2m>12(p+r)>3(p+r)/2c$. Choose $2m/3>L>(p+r)/2c$, then $\mathbb{E}\|h^2 l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})X_{it}X_{it}'X_{it,k}\|^{L+\gamma/3}\lesssim\mathbb{E}\|X_{it}\|^{3L+\gamma}\le\mathbb{E}\|X_{it}\|^{2m+\gamma}<M$. Thus, by Lemma 2.2.2 of van der Vaart and Wellner (1996) and Theorem 3 of Yoshihara (1978), the first term on right-hand side of the above inequality is bounded (up to a positive constant) by $L^{1/J}/\sqrt{Th^4}$. Choosing $\epsilon=1/\sqrt{T}$, we have

$$\mathbb{E}\mathcal{Z}_i\le C\left(\frac{T^{\frac{p+r}{2J}}}{\sqrt{Th^4}}+\frac{1}{\sqrt{Th^6}}\right)$$

for some positive constant C. Since $J>(p+r)/(2c)$, $T^{\frac{p+r}{2J}}/\sqrt{Th^4}=o(\sqrt{Th^6})$. Then from Assumption 3(vii) we have $\max_{1\le i\le N}\mathbb{E}[\mathcal{Z}_i]<\infty$ and the desired result follows.

Result 2: $S_*^{\beta\lambda'\beta_k}(\hat{\beta}_k-\beta_{0k})(\hat{\Lambda}-\check{\Lambda}_0)=o_P(\|\hat{\beta}-\beta_0\|)$.

We have

$$\|S_*^{\beta\lambda'\beta_k}(\hat{\beta}_k-\beta_{0k})(\hat{\Lambda}-\check{\Lambda}_0)\|=\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^*)X_{it}X_{it,k}f_t^{*'}\right)(\hat{\lambda}_i-\check{\lambda}_{0i})(\hat{\beta}-\beta_0)\right\|$$
$$\le\|\hat{\beta}-\beta_0\|\cdot\max_{i\le N}\|\hat{\lambda}_i-\check{\lambda}_{0i}\|\cdot\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^*)X_{it}X_{it,k}f_t^{*'}\right\|.$$

Note that

$$\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^*)X_{it}X_{it,k}f_t^{*'}=\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\beta^*,\lambda_i^*,\tilde{f}_{0t})X_{it}X_{it,k}\tilde{f}_{0t}'+$$
$$+\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^{**})X_{it}X_{it,k}(f_t^*-\tilde{f}_{0t})'+\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(4)}(\beta^*,\lambda_i^*,f_t^{**})X_{it}X_{it,k}\tilde{f}_{0t}'(\lambda_i^*)'(f_t^*-\tilde{f}_{0t}),$$

where f_t^{**} is between f_t^* and $\tilde{f}_{0t}$. Using Lemma S1, Lemma 2, Assumption 3(iii) and the Cauchy-Schwarz inequality, we can show that the last two terms on the right-hand side of the above inequality is $\bar{O}_P(1/\sqrt{Nh^6})=\bar{O}_P(1/\sqrt{Th^6})=\bar{o}_P(1)$. For the first term on the right-hand side of the above inequality, with probability approaching 1, we have

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\beta^*,\lambda_i^*,\tilde{f}_{0t})X_{it}X_{it,k}\tilde{f}_{0t}'\right\|\le\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|\mathbb{E}\left[l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})X_{it}X_{it,k}\tilde{f}_{0t}'\right]|_{\beta=\beta^*,\lambda_i=\lambda_i^*}\right\|$$
$$+\frac{1}{N}\sum_{i=1}^{N}\sup_{\theta_i\in B_{\delta,i}}\left\|\frac{1}{T}\sum_{t=1}^{T}\left(l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})X_{it}X_{it,k}\tilde{f}_{0t}'-\mathbb{E}\left[l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})X_{it}X_{it,k}\tilde{f}_{0t}'\right]\right)\right\|$$

The first term on the right-hand side of the above inequality is $O(1)$ by Lemma S1. Similar to the proof of Result 1, The second term on the right-hand side of the above inequality can be shown to be $o_P(1)$. Thus, we have

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^*)X_{it}X_{it,k}f_t^{*'}\right\| = O_P(1),$$

and the result follows from uniform consistency of $\hat{\lambda}_i$.

Result 3: $S_*^{\beta f'\beta_k}(\hat{\beta}_k-\beta_{0k})(\hat{F}-\check{F}_0) = o_P(\|\hat{\beta}-\beta_0\|)$.

Note that by Lemma 2,

$$\begin{aligned}
& \|S_*^{\beta f'\beta_k}(\hat{\beta}_k-\beta_{0k})(\hat{F}-\check{F}_0)\| \\
\leq\ & \|\hat{\beta}-\beta_0\|\cdot\left\|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^*)X_{it}X_{it,k}\lambda_i^{*'}\right)(\hat{f}_t-\check{f}_{0t})\right\| \\
\leq\ & \|\hat{\beta}-\beta_0\|\cdot\sqrt{\frac{1}{T}\sum_{t=1}^{T}\|\hat{f}_t-\check{f}_{0t}\|^2\cdot\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^*)]^2\cdot\|X_{it}\|^4\cdot\|\lambda_i^*\|^2} \\
=\ & \|\hat{\beta}-\beta_0\|\cdot O_P\left(1/\sqrt{Nh^4}\right) = o_P(\|\hat{\beta}-\beta_0\|),
\end{aligned}$$

because by Lemma S1, $h^2 l_{it}^{(3)}(\beta,\lambda_i,f_t)$ is uniformly bounded, and $Nh^4\to\infty$ by Assumption 3(vii).

Result 4: $\sum_{i=1}^{N}\sum_{h=1}^{r}S_*^{\beta\beta'\lambda_{ih}}(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{\beta}-\beta_0) = o_P(\|\hat{\beta}-\beta_0\|)$.

Observe that for each $h\leq r$,

$$\sum_{i=1}^{N}S_*^{\beta\beta'\lambda_{ih}}(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{\beta}-\beta_0) = -\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})\left(\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^*)X_{it}X_{it}'f_{th}^*\right)(\hat{\beta}-\beta_0),$$

so

$$\left\|\sum_{i=1}^{N}S_*^{\beta\beta'\lambda_{ih}}(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{\beta}-\beta_0)\right\| \leq \|\hat{\beta}-\beta_0\|\cdot\max_{i\leq N}\|\hat{\lambda}_i-\check{\lambda}_{0i}\|\cdot\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^*)X_{it}X_{it}'f_{th}^*\right\|,$$

which can be shown to be $o_P(\|\hat{\beta}-\beta_0\|)$, similar to the proof of Result 2.

Result 5: $\sum_{t=1}^{T}\sum_{h=1}^{r}S_*^{\beta\beta' f_{th}}(\hat{f}_{th}-\check{f}_{0t,h})(\hat{\beta}-\beta_0) = o_P(\|\hat{\beta}-\beta_0\|)$.

The proof is similar to Result 3. For each $h \leq r$, we have

$$\sum_{t=1}^{T} S_*^{\beta\beta' f_{th}}(\hat{f}_{th} - \check{f}_{0t,h}) = -\frac{1}{T}\sum_{t=1}^{T}(\hat{f}_{th} - \check{f}_{0t,h})\left(\frac{1}{N}\sum_{i=1}^{N} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*)X_{it}X_{it}'\lambda_{ih}^*\right),$$

so

$$\left\|\sum_{t=1}^{T} S_*^{\beta\beta' f_{th}}(\hat{f}_{th} - \check{f}_{0t,h})\right\| \leq \sqrt{\frac{1}{T}\sum_{t=1}^{T}\|\hat{f}_t - \check{f}_{0t}\|^2 \cdot \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*)]^2 \cdot \|X_{it}\|^4 \cdot \|\lambda_i^*\|^2},$$

which is $O_P(1/\sqrt{Th^4})$. So Result 5 follows.

Result 6: $\sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\beta f' \lambda_{ih}}(\hat{\lambda}_{ih} - \check{\lambda}_{0,ih})(\hat{F} - \check{F}_0) = o_P(\|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N})$.

Write:

$$\sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\beta f' \lambda_{ih}}(\hat{\lambda}_{ih} - \check{\lambda}_{0i,h})(\hat{F} - \check{F}_0) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[l_{it}^{(2)}(\beta^*, \lambda_i^*, f_t^*)X_{it} - l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*)X_{it}(\lambda_i^*)'f_t^*](\hat{\lambda}_i - \check{\lambda}_{0i})'(\hat{f}_t - \check{f}_{0t}).$$

Thus, by the Cauchy-Schwarz inequality, Lemma 2 and Lemma S1

$$\begin{aligned}\left\|\sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\beta f' \lambda_{ih}}(\hat{\lambda}_{ih} - \check{\lambda}_{0i,h})(\hat{F} - \check{F}_0)\right\| &\leq \|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N} \cdot \\ &\sqrt{\frac{1}{T}\sum_{t=1}^{T}\|\hat{f}_t - \check{f}_{0t}\|^2 \cdot \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|[l_{it}^{(2)}(\beta^*, \lambda_i^*, f_t^*)X_{it} - l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*)X_{it}(\lambda_i^*)'f_t^*]\right\|^2} \\ &= \|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N} \cdot O_P(1/\sqrt{Nh^4}),\end{aligned}$$

so the result follows by Assumption 3(vii).

Result 7: $\sum_{t=T}^{N}\sum_{h=1}^{r} S_*^{\beta\lambda' f_{th}}(\hat{f}_{th} - \check{f}_{0t,h})(\hat{\Lambda} - \check{\Lambda}_0) = o_P(\|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N})$.

The proof is similar to the proof of Result 6.

Result 8: $\sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\beta\lambda' \lambda_{ih}}(\hat{\lambda}_{ih} - \check{\lambda}_{0,ih})(\hat{\Lambda} - \check{\Lambda}_0) = o_P(\|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N})$.

Note that for each $h \leq r$, we have

$$\sum_{i=1}^{N} S_*^{\beta\lambda' \lambda_{ih}}(\hat{\lambda}_{ih} - \check{\lambda}_{0i,h})(\hat{\Lambda} - \check{\Lambda}_0) = -\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*)X_{it}(f_t^*)'f_{th}^*(\hat{\lambda}_{ih} - \check{\lambda}_{0i,h})(\hat{\lambda}_i - \check{\lambda}_{0i}),$$

by Lemma 3

$$\left\| \sum_{i=1}^{N} S_*^{\beta\lambda'\lambda_{ih}} (\hat{\lambda}_{ih} - \check{\lambda}_{0i,h})(\hat{\Lambda} - \check{\Lambda}_0) \right\| \leq \max_{1 \leq N} \|\hat{\lambda}_i - \check{\lambda}_{0i}\| \cdot \frac{1}{N} \sum_{i=1}^{N} \left(\|\hat{\lambda}_i - \check{\lambda}_{0i}\| \cdot \left\| \frac{1}{T} \sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} f_t^* f_{th}^* \right\| \right)$$

$$\leq o_P(1) \cdot \|\hat{\Lambda} - \check{\Lambda}_0\| / \sqrt{N} \cdot \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T} \sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} (f_t^*)' f_{th}^* \right\|^2}.$$

Thus, it remains to show that

$$\frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T} \sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} (f_t^*)' f_{th}^* \right\|^2 = O_P(1). \tag{A.23}$$

First, write

$$\begin{aligned} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} (f_t^*)' f_{th}^* &= l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} (\tilde{f}_{0t})' \tilde{f}_{0,th} + l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} \left((f_t^*)' f_{th}^* - (\tilde{f}_{0t})' \tilde{f}_{0,th} \right) \\ &= l_{it}^{(3)}(\beta^*, \lambda_i^*, \tilde{f}_{0t}) X_{it} (\tilde{f}_{0t})' \tilde{f}_{0,th} - l_{it}^{(4)}(\beta^*, \lambda_i^*, f_t^{**}) (\lambda_i^*)' (f_t^* - \tilde{f}_{0t}) X_{it} (\tilde{f}_{0t})' \tilde{f}_{0,th} \\ &\quad + l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} \left((f_t^*)' f_{th}^* - (\tilde{f}_{0t})' \tilde{f}_{0,th} \right), \end{aligned}$$

thus, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T} \sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} (f_t^*)' f_{th}^* \right\|^2 &\leq \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T} \sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, \tilde{f}_{0t}) X_{it} (\tilde{f}_{0t})' \tilde{f}_{0,th} \right\|^2 \\ &\quad + \frac{1}{T} \sum_{t=1}^{T} \|(f_t^*)' f_{th}^* - (\tilde{f}_{0t})' \tilde{f}_{0,th}\|^2 \cdot \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \|l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it}\|^2 \\ &\quad + \frac{1}{T} \sum_{t=1}^{T} \|f_t^* - \tilde{f}_{0t}\|^2 \cdot \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \|l_{it}^{(4)}(\beta^*, \lambda_i^*, f_t^{**}) (\lambda_i^*) X_{it} (\tilde{f}_{0t})' \tilde{f}_{0,th}\|^2. \end{aligned}$$

The last two terms on the right-hand side of the above inequality are both $o_P(1)$ by Lemma 2 and Lemma S1. For the first term on the right-hand side of the above inequality, by Assumption 3(iii), with probability approaching 1,

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{1}{T} \sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, \tilde{f}_{0t}) X_{it} (\tilde{f}_{0t})' \tilde{f}_{0,th} \right\|^2 &\leq \max_{i,t} \left\| \mathbb{E}[g_{it}(\beta, \lambda_i)|_{\beta=\beta^*, \lambda_i=\lambda_i^*}] \right\|^2 + \\ &\quad \frac{1}{N} \sum_{i=1}^{N} \sup_{\theta_i \in B_{\delta,i}} \left\| \frac{1}{T} \sum_{t=1}^{T} (g_{it}(\beta, \lambda_i) - \mathbb{E}[g_{it}(\beta, \lambda_i)]) \right\|^2, \end{aligned}$$

where $g_{it}(\beta, \lambda_i) = l_{it}^{(3)}(\beta, \lambda_i, \tilde{f}_{0t}) X_{it} (\tilde{f}_{0t})' \tilde{f}_{0,th}$. The first term on the right-hand side of the above inequality is $O(1)$ by Lemma S1. Thus, to prove (A.23), it suffices to show that

$$\max_{1 \leq i \leq N} \mathbb{E} \left[\sup_{\theta_i \in B_{\delta,i}} \left\| \frac{1}{T} \sum_{t=1}^{T} (g_{it}(\beta, \lambda_i) - \mathbb{E}[g_{it}(\beta, \lambda_i)]) \right\|^2 \right] = O(1).$$

Similar to the proof of Result 1, for any $\epsilon > 0$, let $\theta_i^{(1)}, \ldots, \theta_i^{(L)}$ be a maximal set of points in $B_{\delta,i}$ such that $\|\theta_i^{(j)} - \theta_i^{(l)}\| \geq \epsilon$ for any $j \neq l$. Thus, for some constants $C_1, C_2 > 0$, we have

$$\mathbb{E}\left[\sup_{\theta_i \in B_{\delta,i}} \left\| \frac{1}{T}\sum_{t=1}^{T} (g_{it}(\beta, \lambda_i) - \mathbb{E}[g_{it}(\beta, \lambda_i)]) \right\|^2\right] \leq C_1 \frac{1}{Th^4} \mathbb{E}\left[\max_{1 \leq j \leq L} \left\| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} h^2 \left(g_{it}(\beta^{(j)}, \lambda_i^{(j)}) - \mathbb{E}[g_{it}(\beta^{(j)}, \lambda_i^{(j)})]\right) \right\|^2\right] + C_2\epsilon^2/h^6.$$

By Assumption 3 (iii) and (vii), $m > (p+r)/(4c)$. Choose J such that $m > J > (p+r)/(4c)$. Since $\mathbb{E}\|h^2 g_{it}(\beta, \lambda_i)\|^{2J+\gamma} \lesssim \mathbb{E}\|X_{it}\|^{2J+\gamma} \leq \mathbb{E}\|X_{it}\|^{2m+\gamma} < M$. It follows that the right-hand side of the above inequality is

$$O\left(\frac{L^{1/J}}{Th^4} + \epsilon^2/h^6\right) = O\left(\frac{T^{(p+r)/(2J)}}{Th^4}\right) + O\left(\frac{1}{Th^6}\right) = o\left(\frac{T^{2c}}{Th^4}\right) + o(1) = o(1).$$

This completes the proof of Result 8.

Result 9: $\sum_{t=1}^{T}\sum_{h=1}^{r} S_*^{\beta f' f_{th}} (\hat{f}_{th} - \check{f}_{0t,h})(\hat{F} - \check{F}_0) = o_P(1/\sqrt{T})$.

For each $h \leq r$, we have

$$\sum_{t=1}^{T} S_*^{\beta f' f_{th}} (\hat{f}_{th} - \check{f}_{0t,h})(\hat{F} - \check{F}_0) = -\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} (\lambda_i^*)' \lambda_{ih}^* (\hat{f}_{th} - \check{f}_{0t,h})(\hat{f}_t - \check{f}_{0t}),$$

so by Assumption 3(iii), Lemma 2 and Lemma S1,

$$\left\| \sum_{t=1}^{T} S_*^{\beta f' f_{th}} (\hat{f}_{th} - \check{f}_{0t,h})(\hat{F} - \check{F}_0) \right\| \leq \frac{1}{T}\sum_{t=1}^{T} \left(\|\hat{f}_t - \check{f}_{0t}\|^2 \cdot \left\| \frac{1}{N}\sum_{i=1}^{N} l_{it}^{(3)}(\beta^*, \lambda_i^*, f_t^*) X_{it} (\lambda_i^*)' \lambda_{ih}^* \right\| \right)$$
$$\lesssim O_P\left(\frac{1}{h^2}\right) \cdot \frac{1}{T}\sum_{t=1}^{T} \|\hat{f}_t - \check{f}_{0t}\|^2 = O_P\Big(1/Nh^2\Big).$$

Then the result follows since $O_P\left(1/Nh^2\right) = O_P\left(\frac{1}{\sqrt{N}\cdot\sqrt{N}h^2}\right)$ and Assumption 3(vii) implies that $\sqrt{N}h^2 \to \infty$.

Step 7: It follows from (A.15) to (A.21) that:

$$\Delta(\hat{\beta} - \beta_0) = o_P(\|\hat{\beta} - \beta_0\|) + o_P(\|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N}) + o_P(1/\sqrt{T}),$$

then the desired result follows from the assumption that Δ is positive definite and the fact that $\|\hat{\Lambda} - \check{\Lambda}_0\|/\sqrt{N} \leq \max_i \|\hat{\lambda}_i - \check{\lambda}_{0i}\|$. □

Lemma 5. *Under Assumptions 1 to 4, there exists $0 < \nu < 1/6 - c$ (where c is defined in Assumption 3(vii)) such that* $\max_{1\le i\le N} \|\hat{\lambda}_i - \check{\lambda}_{0i}\| = O_P(1/T^{1/2-1/(2m)})$.

Proof. Expanding the first order condition: $T^{-1}\sum_{t=1}^T l_{it}^{(1)}(\hat{\beta}, \hat{\lambda}_i, \hat{f}_t)\hat{f}_t = 0$ gives:

$$\tilde{\Omega}_i(\hat{\lambda}_i - \check{\lambda}_{0i}) = \frac{1}{T}\sum_{t=1}^T l_{it}^{(1)}\check{f}_{0t} + \frac{1}{T}\sum_{t=1}^T l_{it}^{(1)}(\hat{f}_t - \check{f}_{0t}) - \frac{1}{T}\sum_{t=1}^T l_{it}^{(2)}\hat{f}_t[X_{it}'(\hat{\beta}-\beta_0)] - \left(\tilde{\Omega}_i - \frac{1}{T}\sum_{t=1}^T l_{it}^{(2)}\check{f}_{0t}\check{f}_{0t}'\right)(\hat{\lambda}_i - \check{\lambda}_{0i})$$
$$-\left(\frac{1}{T}\sum_{t=1}^T l_{it}^{(2)}(\hat{f}_t - \check{f}_{0t})\check{f}_{0t}'\right)(\hat{\lambda}_i - \check{\lambda}_{0i}) - \left(\frac{1}{T}\sum_{t=1}^T l_{it}^{(2)}\hat{f}_t(\hat{f}_t - \check{f}_{0t})'\right)\check{\lambda}_{0i} + 0.5\frac{1}{T}\sum_{t=1}^T l_{it}^{(3)}(*)\hat{f}_t\cdot[X_{it}'(\hat{\beta}-\beta_0)]^2$$
$$+\frac{1}{T}\sum_{t=1}^T l_{it}^{(3)}(*)\hat{f}_t\cdot[X_{it}'(\hat{\beta}-\beta_0)]\cdot[(f_t^*)'(\hat{\lambda}_i - \check{\lambda}_{0i})] + \frac{1}{T}\sum_{t=1}^T l_{it}^{(3)}(*)\hat{f}_t\cdot[X_{it}'(\hat{\beta}-\beta_0)]\cdot[(\lambda_i^*)'(\hat{f}_t - \check{f}_{0t})]$$
$$+\frac{1}{T}\sum_{t=1}^T l_{it}^{(3)}(*)\hat{f}_t\cdot[(\lambda_i^*)'(\hat{f}_t - \check{f}_{0t})]\cdot[(f_t^*)'(\hat{\lambda}_i - \check{\lambda}_{0i})] + 0.5\frac{1}{T}\sum_{t=1}^T l_{it}^{(3)}(*)\hat{f}_t\cdot[(\lambda_i^*)'(\hat{f}_t - \check{f}_{0t})]^2 + 0.5\frac{1}{T}\sum_{t=1}^T l_{it}^{(3)}(*)\hat{f}_t\cdot[(f_t^*)'(\hat{\lambda}_i - \check{\lambda}_{0i})]^2.$$

Step 1: Let M be a generic bounded constant. By Lemma S1,

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^T l_{it}^{(1)}\check{f}_{0t}\right\| \le \max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^T \tilde{l}_{it}^{(1)} f_{0t}\right\|\cdot\|\hat{\mathbf{H}}\| + O(h^q).$$

Since, $\{l_{it}^{(1)} f_{0t}\}$ is uniformly bounded, by the mixing property of u_{it} and Theorem 3 of Yoshihara (1978), for any $J \ge 2$ we have $\mathbb{E}\left\|T^{-1/2}\sum_{t=1}^T \tilde{l}_{it}^{(1)} f_{0t}\right\|^J < M$ and it follows from Lemma 2.2.2 of van der Vaart and Wellner (1996) that

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^T \tilde{l}_{it}^{(1)} f_{0t}\right\| = O_P(N^{1/J}/\sqrt{T}) = O_P(1/T^{1/2-1/J}).$$

Choosing $J = 2m$, we have

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^T l_{it}^{(1)}\check{f}_{0t}\right\| = O_P(T^{1/(2m)}/T^{1/2}) \tag{A.24}$$

since $O(h^q) = o(T^{-1})$ by Assumption 3(vii).

Step 2: By Lemma S1, $l_{it}^{(1)}$ is uniformly bounded, so it follows from Lemma 2 that

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^T l_{it}^{(1)}(\hat{f}_t - \check{f}_{0t})\right\| \le O_P(1)\cdot\frac{1}{T}\sum_{t=1}^T\|\hat{f}_t - \check{f}_{0t}\| = O_P(N^{-1/2}) = O_P(T^{-1/2}). \tag{A.25}$$

Step 3: Note that:

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)}\hat{f}_t[X_{it}'(\hat{\beta}-\beta_0)]\right\| \le \|\hat{\beta}-\beta_0\|\cdot\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)}\hat{f}_t X_{it}'\right\|$$
$$\le \|\hat{\beta}-\beta_0\|\cdot\|\hat{\mathbf{H}}\|\cdot\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} f_{0t} X_{it}'\right\| + \|\hat{\beta}-\beta_0\|\cdot\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)}(\hat{f}_t-\check{f}_{0t}) X_{it}'\right\|.$$

First, by Assumption 3(iii), Lemma 2 and Lemma S1, we have

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)}(\hat{f}_t-\check{f}_{0t}) X_{it}'\right\| \le O_P(h^{-1})\cdot\frac{1}{T}\sum_{t=1}^{T}\|\hat{f}_t-\check{f}_{0t}\| = O_P(1/(\sqrt{N}h)) = o_P(1).$$

Second, by Lemma S1,

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} f_{0t} X_{it}'\right\| \le \frac{1}{\sqrt{Th^2}}\cdot\max_{1\le i\le N}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h\left(l_{it}^{(2)} f_{0t}X_{it}' - \mathbb{E}[l_{it}^{(2)} f_{0t}X_{it}']\right)\right\| + O(1).$$

Since $\|hl_{it}^{(2)} f_{0t}X_{it}'\|$ is uniformly bounded by Assumption 3(iii) and Lemma S1, it follows from Theorem 3 of Yoshihara (1978) that

$$\mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h\left(l_{it}^{(2)} f_{0t}X_{it}' - \mathbb{E}[l_{it}^{(2)} f_{0t}X_{it}']\right)\right\|^4 < M,$$

and then it follows from Lemma 2.2.2 of van der Vaart and Wellner (1996) that

$$\frac{1}{\sqrt{Th^2}}\cdot\max_{1\le i\le N}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h\left(l_{it}^{(2)} f_{0t}X_{it}' - \mathbb{E}[l_{it}^{(2)} f_{0t}X_{it}']\right)\right\| = O_P(N^{1/4}/\sqrt{Th^2}) = o_P(1).$$

Thus, it can be concluded that

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)}\hat{f}_t[X_{it}'(\hat{\beta}-\beta_0)]\right\| = O_P(\|\hat{\beta}-\beta_0\|). \tag{A.26}$$

Step 4: By the definition of $\tilde{\Omega}_i$, we have

$$\max_{1\le i\le N}\left\|\left(\tilde{\Omega}_i - \frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)}\check{f}_{0t}\check{f}_{0t}'\right)(\hat{\lambda}_i-\check{\lambda}_{0i})\right\| \le \|\hat{\mathbf{H}}\|^2\cdot\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}(l_{it}^{(2)}-\mathsf{f}_{it}(0))f_{0t}f_{0t}'\right\|\cdot\max_{1\le i\le N}\|\hat{\lambda}_i-\check{\lambda}_{0i}\|.$$

Similar to the proof of Step 3, we can show that

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}(l_{it}^{(2)}-\mathsf{f}_{it}(0))f_{0t}f_{0t}'\right\| = o_P(1),$$

thus it follows that

$$\max_{1\le i\le N}\left\|\left(\tilde{\Omega}_i-\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}\check{f}_{0t}\check{f}_{0t}'\right)(\hat{\lambda}_i-\check{\lambda}_{0i})\right\|=o_P\left(\max_i\|\hat{\lambda}_i-\check{\lambda}_{0i}\|\right). \tag{A.27}$$

Step 5: By Lemma 2, Lemma S1 and Assumption 3(vii) we have

$$\max_{1\le i\le N}\left\|\left(\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}(\hat{f}_t-\check{f}_{0t})\check{f}_{0t}'\right)(\hat{\lambda}_i-\check{\lambda}_{0i})\right\|\le\|\hat{\mathbf{H}}\|\cdot O_P(1/\sqrt{Nh^2})\cdot\max_{1\le i\le N}\|\hat{\lambda}_i-\check{\lambda}_{0i}\|=o_P\left(\max_i\|\hat{\lambda}_i-\check{\lambda}_{0i}\|\right). \tag{A.28}$$

Step 6: Note that

$$\max_{1\le i\le N}\left\|\left(\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}\hat{f}_t(\hat{f}_t-\check{f}_{0t})'\right)\check{\lambda}_{0i}\right\|\le O_P(1)\cdot\|\hat{\mathbf{H}}\|\cdot\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}f_{0t}(\hat{f}_t-\check{f}_{0t})'\right\|$$
$$+O_P(1)\cdot\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}(\hat{f}_t-\check{f}_{0t})(\hat{f}_t-\check{f}_{0t})'\right\|.$$

The second term on the right-hand side of the above inequality is $O_P(1/(Th))$ by Lemma S1 and Lemma 2. Next, we have

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}f_{0t}(\hat{f}_t-\check{f}_{0t})'\right\|\lesssim\max_{1\le t\le T}\|\hat{f}_t-\check{f}_{0t}\|\cdot\max_{1\le i\le N}\frac{1}{T}\sum_{t=1}^{T}\left|l_{it}^{(2)}\right|,$$

and

$$\max_{1\le i\le N}\frac{1}{T}\sum_{t=1}^{T}\left|l_{it}^{(2)}\right|\lesssim 1/h\cdot\max_{1\le i\le N}\frac{1}{T}\sum_{t=1}^{T}\mathbf{1}\{|u_{it}|\le h\}$$
$$\le\max_{i,t}P[|u_{it}|\le h]/h+h^{-1}\cdot\max_{i\le N}\left|\frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{1}\{|u_{it}|\le h\}-P(|u_{it}|\le h)\right]\right|.$$

It is easy to see that $\max_{i,t}P[|u_{it}|\le h]=O(h)$. Moreover, similar to the proof of Step 3, we can show that

$$h^{-1}\cdot\max_{i\le N}\left|\frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{1}\{|u_{it}|\le h\}-P(|u_{it}|\le h)\right]\right|=o_P(1).$$

Therefore,

$$\max_{1\le i\le N}\frac{1}{T}\sum_{t=1}^{T}\left|l_{it}^{(2)}\right|=O_P(1),$$

and by Lemma 2

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}f_{0t}(\hat{f}_t-\check{f}_{0t})'\right\|=O_P(T^{1/(2m)}/\sqrt{N}).$$

Thus, we can conclude that

$$\max_{1\le i\le N}\left\|\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)}\hat{f}_t(\hat{f}_t-\check{f}_{0t})'\right)\check{\lambda}_{0i}\right\| = O_P(T^{1/(2m)}/\sqrt{N}) + O_P(1/(Th)) = O_P(T^{1/(2m)}/T^{1/2}). \quad \text{(A.29)}$$

Step 7: By the consistency of $\hat{\beta}$,

$$\begin{aligned}\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)\hat{f}_t\cdot[X_{it}'(\hat{\beta}-\beta_0)]^2\right\| &\le \max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)\hat{f}_t X_{it}'\right\|\cdot\|\hat{\beta}-\beta_0\|^2\\ &= o_P(\|\hat{\beta}-\beta_0\|)\cdot\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)\hat{f}_t X_{it}'\right\|.\end{aligned}$$

Write

$$\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)\hat{f}_t X_{it}' = \frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(\beta^*,\lambda_i^*,\tilde{f}_{0t})\tilde{f}_{0t}X_{it}' + \frac{1}{T}\sum_{t=1}^{T} l_{it}^{(4)}(\beta^*,\lambda_i^*,f_t^{**})\tilde{f}_{0t}X_{it}'\cdot[(f_t^*-\tilde{f}_{0t})\lambda_i^*] + \frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)(\hat{f}_t-\tilde{f}_{0t})X_{it}'.$$

It then follows from Lemma 2 and Lemma S1 that

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)\hat{f}_t X_{it}'\right\| \lesssim \max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(\beta^*,\lambda_i^*,\tilde{f}_{0t})f_{0t}X_{it}'\right\| + O_P(1/\sqrt{Th^6}).$$

Let $B_{\delta,i}$ be a neighborhood of $(\beta_0,\tilde{\lambda}_{0i})$, then by the uniform consistency of $\hat{\lambda}_i$, with probability approaching 1, we have

$$\begin{aligned}\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(\beta^*,\lambda_i^*,\tilde{f}_{0t})f_{0t}X_{it}'\right\| &\le \max_{i,t}\sup_{\theta_i\in B_{\delta,i}}\left\|\mathbb{E}[l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})f_{0t}X_{it}']\right\|\\ &+ \max_{1\le i\le N}\sup_{\theta_i\in B_{\delta,i}}\left\|\frac{1}{T}\sum_{t=1}^{T}\left(l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})f_{0t}X_{it}' - \mathbb{E}[l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})f_{0t}X_{it}']\right)\right\|.\end{aligned}$$

The first term on the right-hand side of the above inequality is $O(1)$ by Lemma S1. Next, for each i, let $\theta_i^{(1)},\ldots,\theta_i^{(L_i)}$ be a maximal set of points in $B_{\delta,i}$ such that $\max_i\|\theta_i^{(j)}-\theta_i^{(k)}\|\le\epsilon$ for some small $\epsilon>0$. It follows that

$$\begin{aligned}&\max_{1\le i\le N}\sup_{\theta_i\in B_{\delta,i}}\left\|\frac{1}{T}\sum_{t=1}^{T}\left(l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})f_{0t}X_{it}' - \mathbb{E}[l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})f_{0t}X_{it}']\right)\right\| \le\\ &\frac{1}{\sqrt{T}h^2}\cdot\max_{1\le i\le N}\max_{1\le j\le L_i}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}h^2\left(l_{it}^{(3)}(\beta^{(j)},\lambda_i^{(j)},\tilde{f}_{0t})f_{0t}X_{it}' - \mathbb{E}[l_{it}^{(3)}(\beta^{(j)},\lambda_i^{(j)},\tilde{f}_{0t})f_{0t}X_{it}']\right)\right\| + O(\epsilon/h^3).\end{aligned}$$

Note $\max_i L_i \lesssim \bar{L} = (1/\epsilon)^{p+r}$. By Assumption 3(iii), $2m > 3(p+r)/(2c) > (p+r+2)/(2c)$. Choose $2m > L > (p+r+2)/(2c)$. Then $\mathbb{E}\|h^2 l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})f_{0t}X_{it}\|^{J+\gamma} \lesssim \mathbb{E}\|X_{it}\|^{J+\gamma} \le \mathbb{E}\|X_{it}\|^{2m+\gamma} < M$. By

Theorem 3 of Yoshihara (1978),

$$\mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}h^2\left(l_{it}^{(3)}(\beta^{(j)},\lambda_i^{(j)},\tilde{f}_{0t})f_{0t}X_{it}'-\mathbb{E}[l_{it}^{(3)}(\beta^{(j)},\lambda_i^{(j)},\tilde{f}_{0t})f_{0t}X_{it}']\right)\right\|^J < M.$$

Thus, by Lemma 2.2.2 of van der Vaart and Wellner (1996), we have

$$\max_{1\le i\le N}\sup_{\theta_i\in B_{\delta,i}}\left\|\frac{1}{T}\sum_{t=1}^{T}\left(l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})f_{0t}X_{it}'-\mathbb{E}[l_{it}^{(3)}(\beta,\lambda_i,\tilde{f}_{0t})f_{0t}X_{it}']\right)\right\| = O_P\left(\frac{(N\bar{L})^{1/J}}{\sqrt{Th^4}}\right)+O(\epsilon/h^3).$$

Choosing $\epsilon = 1/\sqrt{T}$, we have

$$O_P\left(\frac{(N\bar{L})^{1/J}}{\sqrt{Th^4}}\right)+O(\epsilon/h^3) = O_P(1/\sqrt{Th^6}) = o_P(1).$$

Therefore, we can conclude that

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\beta^*,\lambda_i^*,\tilde{f}_{0t})f_{0t}X_{it}'\right\| = O_P(1)$$

and that

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(*)\hat{f}_t\cdot[X_{it}'(\hat{\beta}-\beta_0)]^2\right\| = o_P(\|\hat{\beta}-\beta_0\|). \tag{A.30}$$

Step 8: Similar to the proof of Step 7, we can show that

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(*)\hat{f}_t\cdot[X_{it}'(\hat{\beta}-\beta_0)]\cdot[(f_t^*)'(\hat{\lambda}_i-\check{\lambda}_{0i})]\right\| = o_P(\|\hat{\beta}-\beta_0\|), \tag{A.31}$$

and

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(*)\hat{f}_t\cdot[(f_t^*)'(\hat{\lambda}_i-\check{\lambda}_{0i})]^2\right\| = O_P\left(\max_i\|\hat{\lambda}_i-\lambda_{0i}\|^2\right). \tag{A.32}$$

Step 9: From Lemma 2 and Lemma S1 it follows easily that:

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(*)\hat{f}_t\cdot[X_{it}'(\hat{\beta}-\beta_0)]\cdot[(\lambda_i^*)'(\hat{f}_t-\check{f}_{0t})]\right\| = o_P(\|\hat{\beta}-\beta_0\|), \tag{A.33}$$

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(*)\hat{f}_t\cdot[(\lambda_i^*)'(\hat{f}_t-\check{f}_{0t})]\cdot[(f_t^*)'(\hat{\lambda}_i-\check{\lambda}_{0i})]\right\| = O_P(1/\sqrt{Th^4})\cdot O_P\left(\max_i\|\hat{\lambda}_i-\lambda_{0i}\|\right), \tag{A.34}$$

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(*)\hat{f}_t\cdot[(\lambda_i^*)'(\hat{f}_t-\check{f}_{0t})]^2\right\| = O_P(1/(Th^2)). \tag{A.35}$$

Finally, since $\max_i\|\tilde{\Omega}_i^{-1}\| \le \|\hat{\mathbf{H}}^{-1}\|^2\cdot\max_i\|\Omega_i^{-1}\| = O_P(1)$ by Assumption 3 (ii), it follows from

(A.24) to (A.35) that:

$$\max_{1\leq i\leq N}\|\hat{\lambda}_i-\lambda_{0i}\|=O_P(\|\hat{\beta}-\beta_0\|)+o_P\left(\max_i\|\hat{\lambda}_i-\lambda_{0i}\|\right)+O_P(T^{1/(2m)}/T^{1/2}), \tag{A.36}$$

then the desired result follows from (A.36) and Lemma 4. □

Lemma 6. *Under Assumptions 1 to 4,*

$$\tilde{\Omega}_i(\hat{\lambda}_i-\check{\lambda}_{0i})=\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(1)}\check{f}_{0t}+\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(1)}(\hat{f}_t-\check{f}_{0t})-\left(\tilde{\Omega}_i-\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}\check{f}_{0t}\check{f}_{0t}'\right)(\hat{\lambda}_i-\check{\lambda}_{0i})$$
$$-\left(\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}\check{f}_{0t}(\hat{f}_t-\check{f}_{0t})'\right)\check{\lambda}_{0i}+\bar{O}_P(\|\hat{\beta}-\beta_0\|)+\bar{O}_P(T^{1/(2m)}/(Th^2)). \tag{A.37}$$

Proof. The result follows immediately from the proof of Lemma 5. □

From (A.15) and the proof of Lemma 4 we have:

$$\begin{aligned}\Delta(\hat{\beta}-\beta_0)&=o_P(\|\hat{\beta}-\beta_0\|)+\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}l_{it}^{(1)}Z_{it}\\&-\left[S^{\beta f'}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda f'}\right](\hat{F}-\check{F}_0)-\left[\tilde{S}^{\beta\lambda'}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\lambda'}\right](\hat{\Lambda}-\check{\Lambda}_0)\\&-1/2\sum_{i=1}^{N}\sum_{h=1}^{r}(S_*^{\beta\lambda'\lambda_{ih}}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S_*^{\lambda\lambda'\lambda_{ih}})(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{\Lambda}-\check{\Lambda}_0)\\&-1/2\sum_{i=1}^{N}\sum_{h=1}^{r}(S_*^{\beta f'\lambda_{ih}}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S_*^{\lambda f'\lambda_{ih}})(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{F}-\check{F}_0)\\&-1/2\sum_{t=1}^{T}\sum_{h=1}^{r}(S_*^{\beta\lambda' f_{th}}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S_*^{\lambda\lambda' f_{th}})(\hat{f}_{th}-\check{f}_{0t,h})(\hat{\Lambda}-\check{\Lambda}_0)\\&-1/2\sum_{t=1}^{T}\sum_{h=1}^{r}(S_*^{\beta f' f_{th}}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S_*^{\lambda f' f_{th}})(\hat{f}_{th}-\check{f}_{0t,h})(\hat{F}-\check{F}_0).\end{aligned} \tag{A.38}$$

In the next 5 lemmas, we analyze each term on the right-hand side of (A.38).

Lemma 7. *Under Assumptions 1 to 4,*

$$[\tilde{S}^{\beta\lambda'}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\lambda'}](\hat{\Lambda}-\check{\Lambda}_0)=-\frac{b_1}{T}+o_P(T^{-1})+o_P(\|\hat{\beta}-\beta_0\|).$$

Proof. **Step 1**: Note that we can write:

$$[\tilde{S}^{\beta\lambda'}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\lambda'}](\hat{\Lambda}-\tilde{\Lambda}_0)=\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}Z_{it}f_{0t}'\right)\hat{\mathbf{H}}(\hat{\lambda}_i-\check{\lambda}_{0i}).$$

Plugging in the result of Lemma 6, we have

$$
\begin{aligned}
&[\tilde{S}^{\beta\lambda'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\tilde{S}^{\lambda\lambda'}](\hat{\Lambda} - \check{\Lambda}_0) \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)} f_{0t}\right) + \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}(\hat{\mathbf{H}}')^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)}(\hat{f}_t - \check{f}_{0t})\right) \\
&\qquad - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} f_{0t}(\hat{f}_t - \check{f}_{0t})'\right)\tilde{\lambda}_{0i} \\
&\qquad - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}\left(\Omega_i - \frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} f_{0t} f_{0t}'\right)(\hat{\mathbf{H}}')^{-1}(\hat{\lambda}_i - \check{\lambda}_{0i}) \\
&\qquad + \left[O_P(\|\hat{\beta} - \beta_0\|) + O_P(T^{1/(2m)}/(Th^2))\right] \cdot O_P\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right\|\right). \qquad \text{(A.39)}
\end{aligned}
$$

Step 2: By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}(\hat{\mathbf{H}}')^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)}(\hat{f}_t - \check{f}_{0t})\right)\right\| \\
&\qquad \lesssim \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right\|^2} \cdot \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)}(\hat{f}_t - \check{f}_{0t})\right\|^2}.
\end{aligned}
$$

First, by Theorem 3 of Yoshihara (1978),

$$
\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right\|^2 = \frac{1}{Th^2} \cdot \mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h \cdot l_{it}^{(2)} Z_{it} f_{0t}'\right\|^2 = O(1/(Th^2)).
$$

Second, by Lemma 1, we have

$$
\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)}(\hat{f}_t - \check{f}_{0t})\right\|^2 &\leq \frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{NT}\sum_{j=1}^{N}\sum_{t=1}^{T} l_{it}^{(1)} e_{jt}\right\|^2 \cdot \left\|\hat{\Psi}\right\|^2 \\
&= O_P(1) \cdot \frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{NT}\sum_{j\neq i}^{N}\sum_{t=1}^{T} l_{it}^{(1)} e_{jt}\right\|^2 + O_P(1/N^2) \cdot \frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)} e_{it}\right\|^2
\end{aligned}
$$

For simplicity, consider the case where $p = 1$. Then by the mixing property we have

$$
\begin{aligned}
\mathbb{E}\left\|\frac{1}{NT}\sum_{j\neq i}^{N}\sum_{t=1}^{T} l_{it}^{(1)} e_{jt}\right\|^2 &= \frac{1}{N^2T^2}\sum_{j\neq i}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}[l_{it}^{(1)} l_{is}^{(1)}] \cdot \mathbb{E}[e_{jt}e_{js}] \\
&= \frac{1}{NT} \cdot \frac{1}{N}\sum_{j\neq i}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}[l_{it}^{(1)} l_{is}^{(1)}] \cdot \mathbb{E}[e_{jt}e_{js}]\right) = O((NT)^{-1}) = O(1/T^2).
\end{aligned}
$$

Moreover, it is easy to show that

$$\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)} e_{it}\right\|^2 = O(1).$$

There, it follows that

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}(\hat{\mathbf{H}}')^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)}(\hat{f}_t - \check{f}_{0t})\right)\right\| = O_P\left(\frac{1}{T}\cdot\frac{1}{\sqrt{Th^2}}\right) = o_P(T^{-1}). \quad \text{(A.40)}$$

Similarly, it can be shown that

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} f_{0t}(\hat{f}_t - \check{f}_{0t})'\right)\tilde{\lambda}_{0i}\right\| = O_P\left(\frac{1}{T}\cdot\frac{1}{\sqrt{Th^4}}\right) = o_P(T^{-1}). \quad \text{(A.41)}$$

Step 3: By the Cauchy-Schwarz inequality and Lemma 5,

$$\begin{aligned}
&\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}\left(\Omega_i - \frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} f_{0t} f_{0t}'\right)(\hat{\mathbf{H}}')^{-1}(\hat{\lambda}_i - \check{\lambda}_{0i})\right\| \\
&\quad \lesssim \max_{1\le i\le N}\|\hat{\lambda}_i - \check{\lambda}_{0i}\| \cdot \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right\|^2} \cdot \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}(l_{it}^{(2)} - \mathsf{f}_{it}(0)) f_{0t} f_{0t}'\right\|^2} \\
&\qquad = O_P(T^{1/(2m)}/T^{1/2}) \cdot O_P(1/\sqrt{Th^2}) \cdot \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}(l_{it}^{(2)} - \mathsf{f}_{it}(0)) f_{0t} f_{0t}'\right\|^2}.
\end{aligned}$$

Moreover, similar to the proof of the previous step, we can show that

$$\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T}(l_{it}^{(2)} - \mathsf{f}_{it}(0)) f_{0t} f_{0t}'\right\|^2 = O_P(1/(Th^2)).$$

Thus, by Assumption 3(vii)

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}\left(\Omega_i - \frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} f_{0t} f_{0t}'\right)(\hat{\mathbf{H}}')^{-1}(\hat{\lambda}_i - \check{\lambda}_{0i})\right\| = O_P\left(\frac{1}{T}\cdot\frac{T^{1/(2m)}}{T^{0.5-2c}}\right) = o_P\left(\frac{1}{T}\right). \quad \text{(A.42)}$$

Step 4: By Lemma S1 and Assumption 3(vii), we can write

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)} f_{0t}\right) &= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} \tilde{l}_{it}^{(1)} f_{0t}\right) + o_P(h^q) \\
&\quad \frac{1}{T}\cdot\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} l_{it}^{(2)} Z_{it} f_{0t}'\right)\Omega_i^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \tilde{l}_{it}^{(1)} f_{0t}\right) + o_P(T^{-1}). \quad \text{(A.43)}
\end{aligned}$$

First, by Lemma S1 and the mixing property, we have

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}l_{it}^{(2)}Z_{it}f_{0t}'\right)\Omega_i^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{l}_{it}^{(1)}f_{0t}\right)\right]=\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathbb{E}[l_{it}^{(2)}\tilde{l}_{is}^{(1)}Z_{it}f_{0t}'\Omega_i^{-1}f_{0s}]\right)$$

$$=\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[l_{it}^{(2)}\tilde{l}_{it}^{(1)}Z_{it}f_{0t}'\Omega_i^{-1}f_{0t}]\right)+\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\mathbb{E}[l_{it}^{(2)}\tilde{l}_{is}^{(1)}Z_{it}f_{0t}'\Omega_i^{-1}f_{0s}]\right)$$

$$=(\tau-0.5)\cdot\frac{1}{N}\sum_{i=1}^{N}\omega_{T,i}^{(1)}+\frac{1}{N}\sum_{i=1}^{N}\omega_{T,i}^{(2)}+O(Th^q)=-b_1+o(1),$$

where

$$\omega_{T,i}^{(1)}=\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[\mathsf{f}_{it}(0|X_{it})Z_{it}\big]f_{0t}'\Omega_i^{-1}f_{0t},$$

$$\omega_{T,i}^{(2)}=\frac{1}{T}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\left(\tau\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}]-\mathbb{E}\left[\int_{\infty}^{0}\mathsf{f}_{i,ts}(0,v|X_{it},X_{is})dv\cdot Z_{it}\right]\right)f_{0t}'\Omega_i^{-1}f_{0s},$$

and

$$b_1=-(\tau-0.5)\cdot\lim_{N,T\to\infty}\frac{1}{N}\sum_{i=1}^{N}\omega_{T,i}^{(1)}-\lim_{N,T\to\infty}\frac{1}{N}\sum_{i=1}^{N}\omega_{T,i}^{(2)}.$$

Second, similar to the proof the Lemma A6 of Galvao and Kato (2016), we can show that

$$\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}l_{it}^{(2)}Z_{it}f_{0t}'\right)\Omega_i^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{l}_{it}^{(1)}f_{0t}\right)\right)=o(1).$$

Thus, we have

$$\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}l_{it}^{(2)}Z_{it}f_{0t}'\right)\Omega_i^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{l}_{it}^{(1)}f_{0t}\right)=-b_1+o_P(1).$$

Finally, the last term on the right-hand side of (A.39) can be shown to be $o_P(\|\hat{\beta}-\beta_0\|)+o_P(T^{-1})$. Then the desired result follows from (A.39) to (A.43) □

Lemma 8. *Under Assumptions 1 to 4,*

$$[S^{\beta f'}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda f'}](\hat{F}-\check{F}_0)=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{A}_t(\mathbf{H}_0)^{-1}\Psi_0'e_{it}-\frac{d_1}{N}+o_P(T^{-1}).$$

Proof. **Step 1:** Write

$$
\begin{aligned}
& [S^{\beta f'} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^{\lambda f'}](\hat{F} - \check{F}_0) \\
= \;& \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N} l_{it}^{(2)} X_{it}\lambda_{0i}'\right)(\hat{\mathbf{H}})^{-1}(\hat{f}_t - \check{f}_{0t}) - \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}\Phi_i(\hat{\mathbf{H}})^{-1}(l_{it}^{(2)}\check{f}_{0t}\check{\lambda}_{0i}' - l_{it}^{(1)})\right)(\hat{f}_t - \check{f}_{0t}) \\
= \;& \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}[l_{it}^{(2)} X_{it}\lambda_{0i}' - l_{it}^{(2)}\Phi_i f_{0t}\lambda_{0i}' + l_{it}^{(1)}\Phi_i]\right)(\hat{\mathbf{H}})^{-1}(\hat{f}_t - \check{f}_{0t}) \\
= \;& \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}[l_{it}^{(2)} Z_{it}\lambda_{0i}' + l_{it}^{(1)}\Phi_i]\right)(\hat{\mathbf{H}})^{-1}(\hat{f}_t - \check{f}_{0t}) \\
= \;& \frac{1}{T}\sum_{t=1}^{T}\mathbf{A}_t(\hat{\mathbf{H}})^{-1}(\hat{f}_t - \check{f}_{0t}) + \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}\left\{(l_{it}^{(2)} Z_{it} - \mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}])\lambda_{0i}' + l_{it}^{(1)}\Phi_i\right\}\right)(\hat{\mathbf{H}})^{-1}(\hat{f}_t - \check{f}_{0t}).
\end{aligned}
$$

Step 2: By Lemma 1, we have $\hat{f}_t - \check{f}_{0t} = \hat{\Psi}'\bar{e}_t$, thus,

$$
\frac{1}{T}\sum_{t=1}^{T}\mathbf{A}_t(\hat{\mathbf{H}})^{-1}(\hat{f}_t - \check{f}_{0t}) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{A}_t(\mathbf{H}_0)^{-1}\Psi_0' e_{it} + \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{A}_t\left[(\hat{\mathbf{H}})^{-1}\hat{\Psi}' - (\mathbf{H}_0)^{-1}\Psi_0'\right]e_{it}.
$$

It is easy to show that

$$
\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{A}_t\left[(\hat{\mathbf{H}})^{-1}\hat{\Psi}' - (\mathbf{H}_0)^{-1}\Psi_0'\right]e_{it}\right\| \leq O_P((NT)^{-1/2})\cdot\left(\|\hat{\mathbf{H}} - \mathbf{H}_0\| + \|\hat{\Psi} - \Psi_0\|\right) = o_P(T^{-1}).
$$

Therefore,

$$
\frac{1}{T}\sum_{t=1}^{T}\mathbf{A}_t(\hat{\mathbf{H}})^{-1}(\hat{f}_t - \check{f}_{0t}) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{A}_t(\mathbf{H}_0)^{-1}\Psi_0' e_{it} + o_P(T^{-1}). \tag{A.44}
$$

Step 3: Since $\|\hat{\mathbf{H}} - \mathbf{H}_0\| = O_P(T^{-1/2})$ and $\|\hat{\Psi} - \Psi_0\| = O_P(T^{-1/2})$, we have

$$
\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{i=1}^{N}\left\{(l_{it}^{(2)} Z_{it} - \mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}])\lambda_{0i}' + l_{it}^{(1)}\Phi_i\right\}\right)(\hat{\mathbf{H}})^{-1}(\hat{f}_t - \check{f}_{0t}) \\
& = (1+o_P(1))\cdot\frac{1}{N}\cdot\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left\{(l_{it}^{(2)} Z_{it} - \mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}])\lambda_{0i}' + l_{it}^{(1)}\Phi_i\right\}\right)(\mathbf{H}_0)^{-1}\Psi_0'\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right).
\end{aligned} \tag{A.45}
$$

Next, by Lemma S1 and Assumption 3(iv), it can be shown that

$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left\{(l_{it}^{(2)}Z_{it}-\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}])\lambda_{0i}'+l_{it}^{(1)}\Phi_i\right\}\right)(\mathbf{H}_0)^{-1}\Psi_0'\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right)\right]$$
$$=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}\left[\left(l_{it}^{(2)}Z_{it}\lambda_{0i}'+l_{it}^{(1)}\Phi_i\right)(\mathbf{H}_0)^{-1}\Psi_0'e_{it}\right]$$
$$=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}\lambda_{0i}'(\mathbf{H}_0)^{-1}\Psi_0'e_{it}]+o(1)=-d_1+o(1). \quad \text{(A.46)}$$

Step 4: Define

$$\mathcal{Z}_t=\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left\{(l_{it}^{(2)}Z_{it}-\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}])\lambda_{0i}'+l_{it}^{(1)}\Phi_i\right\}\right)(\mathbf{H}_0)^{-1}\Psi_0'\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right).$$

To complete the proof, it remains to show that

$$\left\|\mathrm{Var}\left(\frac{1}{T}\sum_{t=1}^{T}\mathcal{Z}_t\right)\right\|=o(1).$$

Note that

$$\left\|\mathrm{Var}\left(\frac{1}{T}\sum_{t=1}^{T}\mathcal{Z}_t\right)\right\|\leq\frac{1}{Th^2}\cdot\mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}h(\mathcal{Z}_t-\mathbb{E}[\mathcal{Z}_t)]\right\|^2.$$

By Assumption 3(iv), $\{\mathcal{Z}_1,\ldots,\mathcal{Z}_T\}$ is $\alpha-$mixing. Thus, by Theorem 3 of Yoshihara (1978) it suffices to show that $\mathbb{E}\|h\mathcal{Z}_t\|^4<\infty$ for all t. By the Cauchy-Schwarz inequality,

$$\mathbb{E}\|h\mathcal{Z}_t\|^4\lesssim\sqrt{\mathbb{E}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}h\left\{(l_{it}^{(2)}Z_{it}-\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}])\lambda_{0i}'+l_{it}^{(1)}\Phi_i\right\}\right\|^8}\cdot\sqrt{\mathbb{E}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right\|^8}=O(1).$$

Thus, we have

$$\left\|\mathrm{Var}\left(\frac{1}{T}\sum_{t=1}^{T}\mathcal{Z}_t\right)\right\|=O\left(\frac{1}{Th^2}\right)=o(1).$$

This completes the proof. □

Lemma 9. *Under Assumptions 1 to 4,*

$$\sum_{i=1}^{N}\sum_{h=1}^{r}(S_*^{\beta\lambda'\lambda_{ih}}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S_*^{\lambda\lambda'\lambda_{ih}})(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{\Lambda}-\check{\Lambda}_0)=-\frac{2b_2}{T}+o_P(T^{-1})+o_P(\|\hat{\beta}-\beta_0\|).$$

Proof. Let Φ_{ik} be the kth row of $\Phi_i=\Xi_i\Omega_i^{-1}$. The kth element of $-\sum_{i=1}^{N}\sum_{h=1}^{r}(S_*^{\beta\lambda'\lambda_{ih}}-\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S_*^{\lambda\lambda'\lambda_{ih}})(\hat{\lambda}_{ih}-$

$\check{\lambda}_{0i,h})(\hat{\Lambda}-\check{\Lambda}_0)$ can be written as

$$\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{h=1}^{r} l_{it}^{(3)}(*)(X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*)f_t^{*'}f_{th}^*(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{\lambda}_i-\check{\lambda}_{0i}) \\
= & \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}(\hat{\lambda}_i-\check{\lambda}_{0i})' l_{it}^{(3)}(*)(X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*)f_t^* f_t^{*'}(\hat{\lambda}_i-\check{\lambda}_{0i}) \\
= & \frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\check{\lambda}_{0i})'\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)(X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*)f_t^* f_t^{*'}\right)(\hat{\lambda}_i-\check{\lambda}_{0i}).
\end{aligned}$$

Step 1: Note that

$$\begin{aligned}
& \frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)(X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*)f_t^* f_t^{*'} \\
& \quad = \hat{\mathbf{H}}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)} Z_{it,k} f_{0t} f_{0t}'\right)\hat{\mathbf{H}}' + \frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)(X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}F_t^*)(f_t^* f_t^{*'}-\hat{\mathbf{H}}f_{0t}f_{0t}'\hat{\mathbf{H}}) \\
& \qquad + \hat{\mathbf{H}}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)\Phi_{ik}(f_{0t}-(\hat{\mathbf{H}})^{-1}f_t^*)f_{0t}f_{0t}'\right)\hat{\mathbf{H}}' + \hat{\mathbf{H}}\left(\frac{1}{T}\sum_{t=1}^{T}(l_{it}^{(3)}(*)-l_{it}^{(3)})Z_{it,k}f_{0t}f_{0t}'\right)\hat{\mathbf{H}}'.
\end{aligned}$$

Thus, by Lemma 2, Lemma 5 and Lemma S1 we have

$$\begin{aligned}
& \left\|\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\check{\lambda}_{0i})'\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}(*)(X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*)f_t^* f_t^{*'}\right)(\hat{\lambda}_i-\check{\lambda}_{0i})\right. \\
& \qquad\qquad \left. - \frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\check{\lambda}_{0i})'\hat{\mathbf{H}}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)} Z_{it,k} f_{0t} f_{0t}'\right)\hat{\mathbf{H}}'(\hat{\lambda}_i-\check{\lambda}_{0i})\right\| \\
& = O_P(T^{-1+1/m})\cdot O_P(T^{1/(2m)}/\sqrt{Th^6}) + O_P(T^{-1+1/m})\cdot\max_i \frac{1}{T}\sum_{t=1}^{T}\|Z_{it}\|\cdot\|\hat{\beta}-\beta_0\|/h^3 = o_P(T^{-1})+o_P(\|\hat{\beta}-\beta_0\|),
\end{aligned}$$

where we have used the fact that

$$|l_{it}^{(3)}(*)-l_{it}^{(3)}| \lesssim \left(\|\hat{\beta}-\beta_0\|+\|\hat{f}_t-\check{f}_{0t}\|+\|\hat{\lambda}_i-\check{\lambda}_{0i}\|\right)/h^3,$$

and thus

$$\begin{aligned}
& \max_i \left\|\frac{1}{T}\sum_{t=1}^{T}(l_{it}^{(3)}(*)-l_{it}^{(3)})Z_{it,k}f_{0t}f_{0t}'\right\| \\
& \qquad \lesssim \max_i \frac{1}{T}\sum_{t=1}^{T}\|Z_{it}\|\cdot\|\hat{\beta}-\beta_0\|/h^3 + \max_t\|\hat{f}_t-\check{f}_{0t}\|/h^3 + \max_i\|\hat{\lambda}_i-\check{\lambda}_{0i}\|/h^3 \\
& \qquad\qquad = \max_i \frac{1}{T}\sum_{t=1}^{T}\|Z_{it}\|\cdot\|\hat{\beta}-\beta_0\|/h^3 + O_P(T^{1/(2m)}/\sqrt{Th^6}).
\end{aligned}$$

Step 2: Recall that

$$\mathbf{C}_{i,k} = -\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\mathsf{f}_{it}^{(1)}(0|X_{it})Z_{it,k}]f_{0t}f_{0t}'.$$

First, by Lemma 5,

$$\left\|\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\check{\lambda}_{0i})'\hat{\mathbf{H}}\left(\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}Z_{it,k}f_{0t}f_{0t}'\right)\hat{\mathbf{H}}'(\hat{\lambda}_i-\check{\lambda}_{0i})-\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\check{\lambda}_{0i})'\hat{\mathbf{H}}\mathbf{C}_{i,k}\hat{\mathbf{H}}'(\hat{\lambda}_i-\check{\lambda}_{0i})\right\|$$
$$\leq O_P(T^{-1+1/m})\cdot\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}Z_{it,k}f_{0t}f_{0t}'-\mathbf{C}_{i,k}\right\|.$$

Second, by Lemma S1 and Theorem 3 of Yoshihara (1978),

$$\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}Z_{it,k}f_{0t}f_{0t}'-\mathbf{C}_{i,k}\right\|\leq\frac{1}{\sqrt{Th^4}}\sqrt{\mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}h^2(l_{it}^{(3)}Z_{it,k}-\mathbb{E}[l_{it}^{(3)}Z_{it,k}])f_{0t}f_{0t}'\right\|^2}+o(T^{-1})=O(1/\sqrt{Th^4}).$$

Thus, we have

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{h=1}^{r}l_{it}^{(3)}(*)(X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*)f_t^{*'}f_{th}^*(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{\lambda}_i-\check{\lambda}_{0i})$$
$$=\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\check{\lambda}_{0i})'\hat{\mathbf{H}}\mathbf{C}_{i,k}\hat{\mathbf{H}}'(\hat{\lambda}_i-\check{\lambda}_{0i})+o_P(T^{-1}) \quad \text{(A.47)}$$

because $O_P(T^{-1+1/m}/\sqrt{Th^4})=o_P(T^{-1})$ by Assumption 3(iii).

Step 3: By Lemma 6 we can write

$$\hat{\lambda}_i-\check{\lambda}_{0i}=\tilde{\Omega}_i^{-1}\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(1)}\check{f}_{0t}+g_i+\bar{O}_P(\|\hat{\beta}-\beta_0\|)+\bar{O}_P(T^{1/(2m)}/(Th^2)).$$

where

$$g_i=\tilde{\Omega}_i^{-1}\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(1)}(\hat{f}_t-\check{f}_{0t})-\tilde{\Omega}_i^{-1}\left(\tilde{\Omega}_i-\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}\check{f}_{0t}\check{f}_{0t}'\right)(\hat{\lambda}_i-\check{\lambda}_{0i})-\tilde{\Omega}_i^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}\check{f}_{0t}(\hat{f}_t-\check{f}_{0t})'\right)\check{\lambda}_{0i}.$$

By the proof of Lemma 7, it can be shown that

$$\frac{1}{N}\sum_{i=1}^{N}\|g_i\|=O_P\left(\frac{1}{T^{1-1/m}h}\right).$$

Thus,

$$\left\| \frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i - \check{\lambda}_{0i})'\hat{\mathbf{H}}\mathbf{C}_{i,k}\hat{\mathbf{H}}'(\hat{\lambda}_i - \check{\lambda}_{0i}) - \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)} f_{0t}'\right)\Omega_i^{-1}\mathbf{C}_{i,k}\Omega_i^{-1}\left(\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(1)} f_{0t}\right)\right\|$$
$$= O_P\left(\max_{1\le i\le N}\|\hat{\lambda}_i - \check{\lambda}_{0i}\|\right)\cdot\frac{1}{N}\sum_{i=1}^{N}\|g_i\| + o_P(\|\hat{\beta}-\beta_0\|) + O_P\left(\max_{1\le i\le N}\|\hat{\lambda}_i - \check{\lambda}_{0i}\|\right)\cdot O_P(T^{1/(2m)}/(Th^2))$$
$$= O_P\left(\frac{1}{T^{1.5-1.5/m}h} + \frac{1}{T^{1.5-1/m}h^2}\right) + o_P(\|\hat{\beta}-\beta_0\|) = o_P(T^{-1}) + o_P(\|\hat{\beta}-\beta_0\|).$$

Thus, we can write

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i - \check{\lambda}_{0i})'\hat{\mathbf{H}}\mathbf{C}_{i,k}\hat{\mathbf{H}}'(\hat{\lambda}_i - \check{\lambda}_{0i})$$
$$= \frac{1}{T}\cdot\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} l_{it}^{(1)} f_{0t}'\right)\Omega_i^{-1}\mathbf{C}_{i,k}\Omega_i^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} l_{it}^{(1)} f_{0t}\right) + o_P(T^{-1}) + o_P(\|\hat{\beta}-\beta_0\|). \quad \text{(A.48)}$$

Step 4: First, by Lemma S1,

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} l_{it}^{(1)} f_{0t}'\right)\Omega_i^{-1}\mathbf{C}_{i,k}\Omega_i^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} l_{it}^{(1)} f_{0t}\right)\right]$$
$$= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}\left(l_{it}^{(1)}\right)^2 f_{0t}'\Omega_i^{-1}\mathbf{C}_{i,k}\Omega_i^{-1}f_{0t} + \frac{1}{N}\sum_{i=1}^{N}\cdot\frac{1}{T}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\mathbb{E}\left[l_{it}^{(1)}l_{is}^{(1)}\right] f_{0t}'\Omega_i^{-1}\mathbf{C}_{i,k}\Omega_i^{-1}f_{0s}$$
$$= \frac{1}{N}\sum_{i=1}^{N}\omega_{T,i}^{(3)} + \frac{1}{N}\sum_{i=1}^{N}\omega_{T,i}^{(4)} + o(1) = 2b_2 + o(1),$$

where

$$\omega_{T,i,k}^{(3)} = \tau(1-\tau)\cdot\frac{1}{T}\sum_{t=1}^{T} f_{0t}'\Omega_i^{-1}\mathbf{C}_{i,k}\Omega_i^{-1}f_{0t}$$

$$\omega_{T,i,k}^{(4)} = \frac{1}{T}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\{\mathbb{E}[\mathbf{1}\{u_{it}\le 0, u_{is}\le 0\}] - \tau^2\} f_{0t}'\Omega_i^{-1}\mathbf{C}_{i,k}\Omega_i^{-1}f_{0s}$$

and

$$b_{2,k} = 0.5\lim_{N,T\to\infty}\frac{1}{N}\sum_{i=1}^{N}\omega_{T,i,k}^{(3)} + 0.5\lim_{N,T\to\infty}\frac{1}{N}\sum_{i=1}^{N}\omega_{T,i,k}^{(4)}.$$

Step 5: Finally, note that by Theorem 3 of Yoshihara (1978),

$$\left\|\mathrm{Var}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}l_{it}^{(1)}f_{0t}'\right)\Omega_i^{-1}\mathbf{C}_{i,k}\Omega_i^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}l_{it}^{(1)}f_{0t}\right)\right]\right\|$$
$$\lesssim \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\left\|\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}l_{it}^{(1)}f_{0t}'\right)\Omega_i^{-1}\mathbf{C}_{i,k}\Omega_i^{-1}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}l_{it}^{(1)}f_{0t}\right)\right\|^2 \lesssim \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}l_{it}^{(1)}f_{0t}'\right\|^4$$
$$= O(N^{-1}) = o(1),$$

it then follows from (A.47) and (A.48) that

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{h=1}^{r}l_{it}^{(3)}(*)(X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*)f_t^{*'}f_{th}^*(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{\lambda}_i-\check{\lambda}_{0i}) = -2b_{2,k}+o_P(T^{-1})+o_P(\|\hat{\beta}-\beta_0\|),$$

and the desired result follows. □

Lemma 10. *Under Assumptions 1 to 4,*

$$\sum_{t=1}^{T}\sum_{h=1}^{r}(S_*^{\beta f' f_{th}} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S_*^{\lambda f' f_{th}})(\hat{f}_{th}-\check{f}_{0t,h})(\hat{F}-\check{F}_0) = -\frac{2d_2}{N} + o_P(T^{-1}) + o_P(\|\hat{\beta}-\beta_0\|).$$

Proof. **Step 1:**

Note that the kth element of $-\sum_{t=1}^{T}\sum_{h=1}^{r}(S_*^{\beta f' f_{th}} - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S_*^{\lambda f' f_{th}})(\hat{f}_{th}-\check{f}_{0t,h})(\hat{F}-\check{F}_0)$ can be written as:

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T}\sum_{h=1}^{r}\left(\frac{1}{N}\sum_{i=1}^{N}l_{it}^{(3)}(*)[X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*](\lambda_i^*)'\lambda_{ih}^*\right)(\hat{f}_{th}-\check{f}_{0t,h})(\hat{f}_t-\check{f}_{0t})\\
=\ &\frac{1}{T}\sum_{t=1}^{T}(\hat{f}_t-\check{f}_{0t})'\left(\frac{1}{N}\sum_{i=1}^{N}l_{it}^{(3)}(*)[X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*]\lambda_i^*\lambda_i^{*'}\right)(\hat{f}_t-\check{f}_{0t})\\
&+2\cdot\frac{1}{T}\sum_{t=1}^{T}(\hat{f}_t-\check{f}_{0t})'\left(\frac{1}{N}\sum_{i=1}^{N}l_{it}^{(2)}(*)\lambda_i^*\Phi_{ik}(\hat{\mathbf{H}})^{-1}\right)(\hat{f}_t-\check{f}_{0t}).
\end{aligned}$$

Step 2:

First, by Lemma 2,

$$\left\|\frac{1}{T}\sum_{t=1}^{T}(\hat{f}_t-\check{f}_{0t})'\left(\frac{1}{N}\sum_{i=1}^{N}l_{it}^{(3)}(*)[X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*]\lambda_i^*\lambda_i^{*'}\right)(\hat{f}_t-\check{f}_{0t}) - \frac{1}{T}\sum_{t=1}^{T}(\hat{f}_t-\check{f}_{0t})'(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}(\hat{f}_t-\check{f}_{0t})\right\|$$
$$\leq O_P(\log T/T)\cdot\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{N}\sum_{i=1}^{N}l_{it}^{(3)}(*)[X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*]\lambda_i^*\lambda_i^{*'} - (\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}\right\|.$$

Second, write

$$\frac{1}{N}\sum_{i=1}^{N} l_{it}^{(3)}(*)[X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*]\lambda_i^*\lambda_i^{*'} = \frac{1}{N}\sum_{i=1}^{N} l_{it}^{(3)} Z_{it,k}\check{\lambda}_{0i}(\check{\lambda}_{0i})'+\frac{1}{N}\sum_{i=1}^{N} l_{it}^{(3)}[\Phi_{ik}f_{0t}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*]\check{\lambda}_{0i}(\check{\lambda}_{0i})'$$
$$+\frac{1}{N}\sum_{i=1}^{N} l_{it}^{(3)}(*)[X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*]\left(\lambda_i^*(\lambda_i^*)'-\check{\lambda}_{0i}(\check{\lambda}_{0i})'\right)+\frac{1}{N}\sum_{i=1}^{N}\left(l_{it}^{(3)}(*)-l_{it}^{(3)}\right)[X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*]\check{\lambda}_{0i}(\check{\lambda}_{0i})',$$

it then follows that

$$\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{N}\sum_{i=1}^{N} l_{it}^{(3)}(*)[X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*]\lambda_i^*\lambda_i^{*'}-(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}\right\| \leq$$
$$O_P(1)\cdot\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{N}\sum_{i=1}^{N} l_{it}^{(3)} Z_{it,k}\lambda_{0i}\lambda_{0i}'-\mathbf{D}_{t,k}\right\|+O_P\left(\max_t\|\hat{f}_t-\check{f}_{0t}\|/h^3\right)+O_P\left(\max_i\|\hat{\lambda}_i-\check{\lambda}_{0i}\|/h^3\right)+O_P(\|\hat{\beta}-\beta_0\|/h^3),$$

where we have used Lemma S1, Lemma 4, and the fact that

$$|l_{it}^{(3)}(*)-l_{it}^{(3)}| \lesssim \left(\|\hat{\beta}-\beta_0\|\cdot\|X_{it}\|+\|\hat{f}_t-\check{f}_{0t}\|+\|\hat{\lambda}_i-\check{\lambda}_{0i}\|\right)/h^3$$

and

$$\max_t \frac{1}{N}\sum_{i=1}^{N}\|X_{it}\|^2 = O_P(1).$$

Similar to the proof of Lemma 9, it can be shown that

$$\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{N}\sum_{i=1}^{N} l_{it}^{(3)} Z_{it,k}\lambda_{0i}\lambda_{0i}'-\mathbf{D}_{t,k}\right\| = O_P(1/\sqrt{Nh^4}).$$

Therefore, by Lemma 2 and Lemma 5,

$$\frac{1}{T}\sum_{t=1}^{T}(\hat{f}_t-\check{f}_{0t})'\left(\frac{1}{N}\sum_{i=1}^{N} l_{it}^{(3)}(*)[X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*]\lambda_i^*\lambda_i^{*'}\right)(\hat{f}_t-\check{f}_{0t}) = \frac{1}{T}\sum_{t=1}^{T}(\hat{f}_t-\check{f}_{0t})'(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}(\hat{f}_t-\check{f}_{0t})$$
$$+O_P\left(\frac{T^{1/m}}{T}\right)\cdot O_P\left(\frac{T^{1/(2m)}}{\sqrt{Th^6}}\right)+O_P\left(\frac{T^{1/m}}{Th^3}\right)\cdot O_P(\|\hat{\beta}-\beta_0\|)$$
$$=\frac{1}{T}\sum_{t=1}^{T}(\hat{f}_t-\check{f}_{0t})'(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}(\hat{f}_t-\check{f}_{0t})+o_P(T^{-1})+o_P(\|\hat{\beta}-\beta_0\|). \quad \text{(A.49)}$$

Step 3: By Lemma 2 we can write

$$\frac{1}{T}\sum_{t=1}^{T}(\hat{f}_t-\check{f}_{0t})'(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}(\hat{f}_t-\check{f}_{0t}) = \frac{1}{N}\cdot\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right)'\hat{\Psi}(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}\hat{\Psi}'\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right)$$

$$= \frac{1}{N}\cdot\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right)'\Psi_0(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}\Psi_0'\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right) + \frac{1}{N}\cdot o_P\left(\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right\|^2\right)$$

$$= \frac{1}{N}\cdot\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right)'\Psi_0(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}\Psi_0'\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right) + o_P(N^{-1}).$$

Define

$$\mathcal{Z}_t = \left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right)'\Psi_0(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}\Psi_0'\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e_{it}\right).$$

First, it is easy to show that

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[\mathcal{Z}_t] = \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathrm{Tr}\left\{\mathbb{E}[e_{it}e_{it}']\Psi_0(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}\Psi_0'\right\}.$$

Second, we have

$$\mathrm{Var}\left(\frac{1}{T}\sum_{t=1}^{T}\mathcal{Z}_t\right) = \frac{1}{T}\cdot\mathbb{E}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\mathcal{Z}_t-\mathbb{E}[\mathcal{Z}_t])\right|^2.$$

Since the process $\{\mathcal{Z}_1,\ldots,\mathcal{Z}_T\}$ is $\alpha-$mixing, it then follows from $\mathbb{E}|\mathcal{Z}_t|^4 \lesssim \mathbb{E}\|N^{-1/2}\sum_{i=1}^{N}e_{it}\|^8 < \infty$ that

$$\mathbb{E}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(\mathcal{Z}_t-\mathbb{E}[\mathcal{Z}_t])\right|^2 = O(1)$$

and thus

$$\mathrm{Var}\left(\frac{1}{T}\sum_{t=1}^{T}\mathcal{Z}_t\right) = O(T^{-1}) = o(1).$$

Therefore, we have

$$\frac{1}{T}\sum_{t=1}^{T}(\hat{f}_t-\check{f}_{0t})'\left(\frac{1}{N}\sum_{i=1}^{N}l_{it}^{(3)}(*)[X_{it,k}-\Phi_{ik}(\hat{\mathbf{H}})^{-1}f_t^*]\lambda_i^*\lambda_i^{*\prime}\right)(\hat{f}_t-\check{f}_{0t})$$
$$= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathrm{Tr}\left\{\mathbb{E}[e_{it}e_{it}']\Psi_0(\mathbf{H}_0')^{-1}\mathbf{D}_{t,k}(\mathbf{H}_0)^{-1}\Psi_0'\right\} + o_P(T^{-1}) + o_P(\|\hat{\beta}-\beta_0\|). \quad \text{(A.50)}$$

Step 4: Similarly, we can show that:

$$\frac{1}{T}\sum_{t=1}^{T}(\hat{f}_t-\check{f}_{0t})'\left(\frac{1}{N}\sum_{i=1}^{N}l_{it}^{(2)}(*)\lambda_i^*\Phi_{ik}(\hat{\mathbf{H}})^{-1}\right)(\hat{f}_t-\check{f}_{0t})$$
$$=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathrm{Tr}\left\{\mathbb{E}[e_{it}e_{it}']\Psi_0(\mathbf{H}_0')^{-1}\mathbf{B}_{t,k}(\mathbf{H}_0)^{-1}\Psi_0'\right\}+o_P(T^{-1})+o_P(\|\hat{\beta}-\beta_0\|).$$

This completes the proof. □

Lemma 11. *Under Assumptions 1 to 4,*

$$\sum_{i=1}^{N}\sum_{h=1}^{r}S_*^{\beta f'\lambda_{ih}}(\hat{\lambda}_{ih}-\check{\lambda}_{0,ih})(\hat{F}-\check{F}_0)=o_P(T^{-1})+o_P(\|\hat{\beta}-\beta_0\|)$$

$$\sum_{t=1}^{T}\sum_{h=1}^{r}S_*^{\beta\lambda' f_{th}}(\hat{f}_{th}-\check{f}_{0t,h})(\hat{\Lambda}-\check{\Lambda}_0)=o_P(T^{-1})+o_P(\|\hat{\beta}-\beta_0\|),$$

$$\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\sum_{i=1}^{N}\sum_{h=1}^{r}S_*^{\lambda f'\lambda_{ih}}(\hat{\lambda}_{ih}-\check{\lambda}_{0,ih})(\hat{F}-\check{F}_0)=o_P(T^{-1})+o_P(\|\hat{\beta}-\beta_0\|),$$

$$\bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}\sum_{t=T}^{N}\sum_{h=1}^{r}S_*^{\lambda\lambda' f_{th}}(\hat{f}_{th}-\tilde{f}_{0t,h})(\hat{\Lambda}-\check{\Lambda}_0)=o_P(T^{-1})+o_P(\|\hat{\beta}-\beta_0\|).$$

Proof. To save space, we only prove the first result. The proof of the other results are similar. Write

$$\sum_{i=1}^{N}\sum_{h=1}^{r}S_*^{\beta f'\lambda_{ih}}(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{F}-\check{F}_0)=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}[l_{it}^{(2)}(\beta^*,\lambda_i^*,f_t^*)X_{it}-l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^*)X_{it}(\lambda_i^*)'f_t^*](\hat{\lambda}_i-\check{\lambda}_{0i})'(\hat{f}_t-\check{f}_{0t}).$$

Let $R_{it}(*)=l_{it}^{(2)}(\beta^*,\lambda_i^*,f_t^*)X_{it}-l_{it}^{(3)}(\beta^*,\lambda_i^*,f_t^*)X_{it}(\lambda_i^*)'f_t^*$, and $R_{it}=l_{it}^{(2)}X_{it}-l_{it}^{(3)}X_{it}\lambda_{0i}'f_{0t}$, then by Lemma 1 and Lemma 4 we have

$$\sum_{i=1}^{N}\sum_{h=1}^{r}S_*^{\beta f'\lambda_{ih}}(\hat{\lambda}_{ih}-\check{\lambda}_{0i,h})(\hat{F}-\check{F}_0)=\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}R_{it}(*)(\hat{f}_t-\check{f}_{0t})'\right)(\hat{\lambda}_i-\check{\lambda}_{0i})$$
$$=\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T}R_{it}(\hat{f}_t-\check{f}_{0t})'\right)(\hat{\lambda}_i-\check{\lambda}_{0i})+O_P\left(\max_i\left\|\hat{\lambda}_i-\check{\lambda}_{0i}\right\|\right)\cdot O_P\left(\max_t\left\|\hat{f}_t-\check{f}_{0t}\right\|\right)\cdot O_P\left(\max_{i,t}\|R_{it}(*)-R_{it}\|\right).$$

Similar to the proof of Lemma 10, it can be shown that

$$\max_{i,t}\|R_{it}(*)-R_{it}\|=O_P(T^{1/(2m)}/(T^{1/2}h^3))+O_P(\|\hat{\beta}-\beta_0\|/h^3),$$

thus, by Lemma 2 and Lemma 6,

$$O_P\left(\max_i \left\|\hat{\lambda}_i - \check{\lambda}_{0i}\right\|\right) \cdot O_P\left(\max_t \left\|\hat{f}_t - \check{f}_{0t}\right\|\right) \cdot O_P\left(\max_{i,t} \|R_{it}(*) - R_{it}\|\right)$$
$$= O_P\left(\frac{T^{1/m}}{T} \cdot \frac{T^{1/(2m)}}{T^{1/2}h^3}\right) + O_P\left(\frac{T^{1/m}}{Th^3}\right) O_P(\|\hat{\beta} - \beta_0\|) = o_P(T^{-1}) + o_P(\|\hat{\beta} - \beta_0\|).$$

Therefore,

$$\sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\beta f' \lambda_{ih}} (\hat{\lambda}_{ih} - \check{\lambda}_{0i,h})(\hat{F} - \check{F}_0) = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} R_{it}(\hat{f}_t - \check{f}_{0t})'\right)(\hat{\lambda}_i - \check{\lambda}_{0i}) + o_P(T^{-1}) + o_P(\|\hat{\beta} - \beta_0\|). \tag{A.51}$$

Note that

$$\left\|\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} R_{it}(\hat{f}_t - \check{f}_{0t})'\right)(\hat{\lambda}_i - \check{\lambda}_{0i})\right\| \leq \max_i \left\|\hat{\lambda}_i - \check{\lambda}_{0i}\right\| \cdot \frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T} R_{it}(\hat{f}_t - \check{f}_{0t})'\right\|.$$

Similar to Step 2 of the proof of Lemma 7, we can show that

$$\mathbb{E}\left\|\frac{1}{T}\sum_{t=1}^{T} R_{it}(\hat{F}_t - \tilde{F}_{0t})'\right\| = O(T^{-1}h^{-2}),$$

it then follows from Lemma 6 that

$$\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{t=1}^{T} R_{it}(\hat{f}_t - \check{f}_{0t})'\right)(\hat{\lambda}_i - \check{\lambda}_{0i}) = O_P\left(\frac{T^{1/(2m)}}{T^{3/2}h^2}\right) = o_P(T^{-1}). \tag{A.52}$$

Thus, it follows from (A.51) and (A.52) that

$$\sum_{i=1}^{N}\sum_{h=1}^{r} S_*^{\beta f' \lambda_{ih}} (\hat{\lambda}_{ih} - \tilde{\lambda}_{0,ih})(\hat{F} - \tilde{F}_0) = o_P(T^{-1}) + o_P(\|\hat{\beta} - \beta_0\|).$$

The proofs of the other results are similar and thus are omitted. □

Proof of Theorem 2

Proof. It follows from (A.38) and Lemma 7 to Lemma 11 that

$$\Delta(\hat{\beta} - \beta_0) = -\left[S^\beta - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^\lambda\right] - \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} \mathbf{A}_t(\mathbf{H}_0)^{-1}\Psi_0' e_{it} + \frac{b}{T} + \frac{d}{N} + o_P(T^{-1}) + o_P(\|\hat{\beta} - \beta_0\|).$$

Since

$$S^\beta - \bar{S}^{\beta\lambda'}(\bar{S}^{\lambda\lambda'})^{-1}S^\lambda = -\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} l_{it}^{(1)} Z_{it},$$

it then follows from Assumption 3(vii) that

$$\sqrt{NT}(\hat{\beta}-\beta_0) = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\Delta^{-1}(l_{it}^{(1)}Z_{it}-\mathbf{A}_t(\mathbf{H}_0)^{-1}\Psi_0'e_{it})+\Delta^{-1}(\kappa b+\kappa^{-1}d)+o_P(1)+o_P(\sqrt{NT}\|\hat{\beta}-\beta_0\|).$$

Define $W_{it}^* = l_{it}^{(1)}Z_{it} - \mathbf{A}_t(\mathbf{H}_0)^{-1}\Psi_0'e_{it}$ and $\bar{W}_i^* = T^{-1/2}\sum_{t=1}^{T}W_{it}^*$, it remains to show that

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}W_{it}^* = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{W}_i^* \xrightarrow{d} \mathcal{N}(0,\mathbf{V}). \tag{A.53}$$

First, by Lemma S1, we have $\mathbb{E}[\bar{W}_i^*] = O(h^q) = o(T^{-1})$, thus

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\bar{W}_i^* = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\bar{W}_i^* - \mathbb{E}[\bar{W}_i^*]\right) + o(1). \tag{A.54}$$

Second, by the mixing property and Theorem 3 of Yoshihara (1978)

$$\mathbb{E}\left\|\bar{W}_i^* - \mathbb{E}[\bar{W}_i^*]\right\|^3 = \mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}(W_{it}^* - \mathbb{E}[W_{it}^*])\right\|^3 < \infty.$$

Third, since $\bar{W}_1^*,\ldots,\bar{W}_N^*$ are independent, it follows from Lyapunov's CLT that

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\bar{W}_i^* - \mathbb{E}[\bar{W}_i^*]\right) \xrightarrow{d} \mathcal{N}(0,\mathbf{V}^*) \tag{A.55}$$

where

$$\begin{aligned}\mathbf{V}^* = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left[(\bar{W}_i^* - \mathbb{E}[\bar{W}_i^*])(\bar{W}_i^* - \mathbb{E}[\bar{W}_i^*])'\right] &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}\left[(W_{it}^* - \mathbb{E}[W_{it}^*])(W_{it}^* - \mathbb{E}[W_{it}^*])'\right] \\ &+ \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\mathbb{E}\left[(W_{it}^* - \mathbb{E}[W_{it}^*])(W_{is}^* - \mathbb{E}[W_{is}^*])'\right].\end{aligned}$$

Finally, by Lemma S1 it is easy to show that

$$\begin{aligned}\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}\left[(W_{it}^* - \mathbb{E}[W_{it}^*])(W_{it}^* - \mathbb{E}[W_{it}^*])'\right] &= \tau(1-\tau)\cdot\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[Z_{it}Z_{it}'] \\ &+ \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{A}_t(\mathbf{H}_0)^{-1}\Psi_0'\mathbb{E}[e_{it}e_{it}']\Psi_0(\mathbf{H}_0')^{-1}\mathbf{A}_t' + o(1) = \mathbf{V}_1 + o(1),\end{aligned}$$

and

$$
\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\mathbb{E}\left[(W_{it}^{*}-\mathbb{E}[W_{it}^{*}])(W_{is}^{*}-\mathbb{E}[W_{is}^{*}])'\right]\\
&=\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\mathbb{E}[(\tau^2-\mathbb{E}[\mathbf{1}\{u_{it}\leq 0\}|X_{it},X_{is}]-\mathbb{E}[\mathbf{1}\{u_{is}\leq 0\}|X_{it},X_{is}]+\mathbb{E}[\mathbf{1}\{u_{it}\leq 0\},u_{is}\leq 0\}|X_{it},X_{is}])Z_{it}Z_{is}']\\
&\qquad -\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\mathbb{E}[(\tau-\mathbb{E}[\mathbf{1}\{u_{it}\leq 0\}|X_{it},X_{is}])Z_{it}e_{is}]\Psi_0(\mathbf{H}_0')^{-1}\mathbf{A}_s'\\
&\qquad -\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\mathbf{A}_t(\mathbf{H}_0)^{-1}\Psi_0'\mathbb{E}[e_{it}Z_{is}'(\tau-\mathbb{E}[\mathbf{1}\{u_{is}\leq 0\}|X_{it},X_{is}])]\\
&\qquad\qquad +\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\mathbf{A}_t(\mathbf{H}_0)^{-1}\Psi_0'\mathbb{E}[e_{it}e_{is}]\Psi_0(\mathbf{H}_0')^{-1}\mathbf{A}_s'+o(1)=\mathbf{V}_2+o(1).
\end{aligned}
$$

Thus, we have $\mathbf{V}^*=\mathbf{V}+o(1)$, and (A.53) follows from (A.54) and (A.55). This completes the proof. □

A.4 Proof of Theorem 3

Lemma 12. *Under Assumptions 1 to 4, we have*
(i) $\max_{1\leq i\leq N}\|\hat{\Xi}_i-\tilde{\Xi}_i\|=O_P(T^{1/(2m)}/(T^{0.5}h))$, $\max_{1\leq i\leq N}\|\hat{\Omega}_i-\tilde{\Omega}_i\|=O_P(T^{1/(2m)}/(T^{0.5}h))$, $\max_{1\leq i\leq N}\|\hat{\Phi}_i-\Phi_i\hat{\mathbf{H}}^{-1}\|=O_P(T^{1/(2m)}/(T^{0.5}h))$, $\max_{i,t}\|\hat{Z}_{it}-Z_{it}\|=O_P(T^{1/(2m)}/(T^{0.5}h))$.
(ii) $\max_{i,t}\|\hat{e}_{it}-e_{it}\|=O_P(T^{1/(2m)}/\sqrt{T})$.

Proof. **Step 1:** Adding the subtracting terms, we have

$$\hat{\Xi}_i-\tilde{\Xi}_i=\frac{1}{T}\sum_{t=1}^{T}l^{(2)}(\hat{u}_{it})X_{it}\hat{f}_t'-\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}X_{it}f_{0t}'\hat{\mathbf{H}}'+\frac{1}{T}\sum_{t=1}^{T}\left(l_{it}^{(2)}X_{it}-\mathbb{E}[\mathsf{f}_{it}(0|X_{it})X_{it}]\right)f_{0t}'\cdot\hat{\mathbf{H}}'.$$

First, by Theorem 3 of Yoshihara (1978)

$$\mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}h\left(l_{it}^{(2)}X_{it}-\mathbb{E}[\mathsf{f}_{it}(0|X_{it})X_{it}]\right)f_{0t}'\right\|^{J}<M$$

for any $2m>J\geq 2$. It then follows from Lemma 2.2.2 of van der Vaart and Wellner (1996) that

$$\max_{1\leq i\leq N}\left\|\frac{1}{T}\sum_{t=1}^{T}\left(l_{it}^{(2)}X_{it}-\mathbb{E}[\mathsf{f}_{it}(0|X_{it})X_{it}]\right)f_{0t}'\right\|=O_P\left(\frac{N^{1/J}}{\sqrt{T}h}\right)=O_P\left(\frac{T^{1/(2m)}}{T^{0.5}h}\right).$$

Second,

$$\left\|\frac{1}{T}\sum_{t=1}^{T}l^{(2)}(\hat{u}_{it})X_{it}\hat{f}_t'-\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(2)}X_{it}f_{0t}'\hat{\mathbf{H}}'\right\|\leq\left\|\frac{1}{T}\sum_{t=1}^{T}l^{(2)}(\hat{u}_{it})X_{it}(\hat{f}_t-\check{f}_{0t})'\right\|+\left\|\frac{1}{T}\sum_{t=1}^{T}\left(l^{(2)}(\hat{u}_{it})-l_{it}^{(2)}\right)X_{it}f_{0t}'\right\|\cdot\left\|\hat{\mathbf{H}}'\right\|.$$

The first term on the right-hand side of the above inequality is $O_P(1/\sqrt{Nh^2})$ by Lemma 2. For the second term, using Taylor expansion and Lemma 5 we can write

$$\frac{1}{T}\sum_{t=1}^{T}\left(l^{(2)}(\hat{u}_{it})-l_{it}^{(2)}\right)X_{it}f_{0t}' = -\frac{1}{T}\sum_{t=1}^{T}\left\{l_{it}^{(3)}[X_{it}'(\hat{\beta}-\beta_0)]X_{it}f_{0t}' + l_{it}^{(3)}[\tilde{f}_{0t}'(\hat{\lambda}_i-\tilde{\lambda}_{0i})]X_{it}f_{0t}' + l_{it}^{(3)}[\tilde{\lambda}_{0i}'(\hat{f}_t-\tilde{f}_{0t})]X_{it}f_{0t}'\right\} + \bar{O}_P\left(T^{1/m}/(Th^3)\right).$$

By Theorem 2,

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}[X_{it}'(\hat{\beta}-\beta_0)]X_{it}f_{0t}'\right\| = O_P\left(\frac{1}{\sqrt{NT}}\cdot\frac{1}{h^2}\right) = o_P\left(\frac{1}{\sqrt{T}h}\right).$$

Next, by Lemma 2 and Lemma 5,

$$\begin{aligned}\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}[\tilde{f}_{0t}'(\hat{\lambda}_i-\tilde{\lambda}_{0i})]X_{it}f_{0t}'\right\| &\le \max_{1\le i\le N}\max_{1\le k\le r}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}X_{it}f_{0t}'\tilde{f}_{0t,k}\right\|\cdot\max_{1\le i\le N}\|\hat{\lambda}_i-\tilde{\lambda}_{0i}\| \\ &= O_P\left(\frac{T^{1/(2m)}}{T^{0.5}}\right)\cdot\max_{1\le i\le N}\max_{1\le k\le r}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}X_{it}f_{0t}'\tilde{f}_{0t,k}\right\|.\end{aligned}$$

Similar to the proof of Lemma 4, it can be shown that

$$\max_{1\le i\le N}\max_{1\le k\le r}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}X_{it}f_{0t}'\tilde{f}_{0t,k}\right\| = O_P(1).$$

Thus, we have

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}[\tilde{f}_{0t}'(\hat{\lambda}_i-\tilde{\lambda}_{0i})]X_{it}f_{0t}'\right\| = O_P\left(\frac{T^{1/(2m)}}{T^{0.5}}\right) = o_P\left(\frac{T^{1/(2m)}}{T^{0.5}h}\right).$$

Similarly,

$$\begin{aligned}\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}[\tilde{\lambda}_{0i}'(\hat{f}_t-\tilde{f}_{0t})]X_{it}f_{0t}'\right\| &\le \max_{1\le k\le r}\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}l_{it}^{(3)}(\hat{f}_{t,k}-\tilde{f}_{0t,k})X_{it}f_{0t}'\tilde{\lambda}_{0i,k}\right\| \\ &\lesssim \max_{1\le t\le T}\|\hat{f}_{0t}-\tilde{f}_{0t}\|\cdot\max_{1\le i\le N}\frac{1}{T}\sum_{t=1}^{T}|l_{it}^{(3)}|\cdot\|X_{it}f_{0t}'\|.\end{aligned}$$

Since $\mathbb{E}[|l_{it}^{(3)}|\cdot\|X_{it}f_{0t}'\|] = \bar{O}(h^{-1})$ by Lemma S1, and it can be shown that

$$\max_{1\le i\le N}\frac{1}{T}\sum_{t=1}^{T}\left\{|l_{it}^{(3)}|\cdot\|X_{it}f_{0t}'\| - \mathbb{E}[|l_{it}^{(3)}|\cdot\|X_{it}f_{0t}'\|]\right\} = O_P\left(\frac{T^{1/(2m)}}{T^{0.5}h^2}\right),$$

and it follows from Lemma 2 that

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l_{it}^{(3)}[\tilde{\lambda}_{0i}'(\hat{f}_t-\tilde{f}_{0t})]X_{it}f_{0t}'\right\| = O_P(T^{1/(2m)}/\sqrt{N})\cdot O_P(h^{-1}) = o_P\left(\frac{T^{1/(2m)}}{T^{0.5}h}\right).$$

Combining all the above results, we have

$$\max_{1\le i\le N}\|\hat{\Xi}_i-\tilde{\Xi}_i\| = O_P\left(\frac{T^{1/(2m)}}{T^{0.5}h}\right).$$

It can be shown in a similar way that

$$\max_{1\le i\le N}\|\hat{\Omega}_i-\tilde{\Omega}_i\| = O_P\left(\frac{T^{1/(2m)}}{T^{0.5}h}\right).$$

Moreover, by the definition of $\hat{\Phi}_i$ and Φ_i,

$$\max_{1\le i\le N}\|\hat{\Phi}_i-\Phi_i\hat{\mathbf{H}}^{-1}\| = \max_{1\le i\le N}\left\|\hat{\Xi}_i\hat{\Omega}_i^{-1}-\Xi_i\Omega_i^{-1}\hat{\mathbf{H}}^{-1}\right\| = \max_{1\le i\le N}\left\|\hat{\Xi}_i\hat{\Omega}_i^{-1}-\tilde{\Xi}_i\tilde{\Omega}_i^{-1}\right\|,$$

thus, it follows from the above results and the fact that $\tilde{\Omega}_i$ is positive definite with probability approaching 1 that

$$\max_{1\le i\le N}\|\hat{\Phi}_i-\Phi_i\hat{\mathbf{H}}^{-1}\| = O_P\left(\frac{T^{1/(2m)}}{T^{0.5}h}\right).$$

Finally, by the definitions of Z_{it} and $\hat{Z}_{it}$ and the results above,

$$\max_{i,t}\left\|\hat{Z}_{it}-Z_{it}\right\| \lesssim \max_i\left\|\tilde{\Xi}_i-\hat{\Xi}_i\right\| + \max_i\left\|\tilde{\Omega}_i^{-1}-\hat{\Omega}_i^{-1}\right\| + \max_t\left\|\hat{f}_t-\check{f}_{0t}\right\| = O_P\left(\frac{T^{1/(2m)}}{T^{0.5}h}\right).$$

Step 2: Define $X_i=(X_{i1},\ldots,X_{iT})'$, and $\hat{\Gamma}_i'=(\hat{F}'\hat{F})^{-1}\hat{F}'X_i$. Then we have $\hat{e}_{it}=X_{it}-\hat{\Gamma}_i\hat{f}_t$, and $e_{it}=X_{it}-\Gamma_i f_{0t}=X_{it}-\Gamma_i\hat{\mathbf{H}}^{-1}\check{f}_{0t}$. It follows that

$$\max_{i,t}\|\hat{e}_{it}-e_{it}\| \lesssim \max_i\|\hat{\Gamma}_i-\Gamma_i\hat{\mathbf{H}}^{-1}\| + \max_t\|\hat{f}_t-\check{f}_{0t}\|.$$

Write

$$\hat{\Gamma}_i' = \left(\frac{1}{T}\sum_{t=1}^{T}\hat{f}_t\hat{f}_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\hat{f}_tX_{it}'\right) = \left(\frac{1}{T}\sum_{t=1}^{T}\hat{f}_t\hat{f}_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\hat{f}_tf_{0t}'\right)\Gamma_i' + \left(\frac{1}{T}\sum_{t=1}^{T}\hat{f}_t\hat{f}_t'\right)^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}\hat{f}_te_{it}'\right).$$

By Lemma 2, we have

$$\left(\frac{1}{T}\sum_{t=1}^{T}\hat{f}_t\hat{f}_t'\right)^{-1} = (\hat{\mathbf{H}}')^{-1}\left(\frac{1}{T}\sum_{t=1}^{T}f_{0t}f_{0t}'\right)^{-1}\hat{\mathbf{H}}^{-1} + O_P(N^{-1/2})$$

and

$$\frac{1}{T}\sum_{t=1}^{T}\hat{f}_tf_{0t}' = \hat{\mathbf{H}}\cdot\frac{1}{T}\sum_{t=1}^{T}f_{0t}f_{0t}' + O_P(N^{-1/2}).$$

Thus, it follows that

$$\max_i \|\hat{\Gamma}_i - \Gamma_i \hat{\mathbf{H}}^{-1}\| \leq O_P(N^{-1/2}) + \max_i \left\| \frac{1}{T}\sum_{t=1}^T \hat{f}_t e_{it}' \right\|.$$

Next, by the proof of Lemma 2,

$$\max_i \left\| \frac{1}{T}\sum_{t=1}^T \hat{f}_t e_{it}' \right\| \leq \max_i \left\| \frac{1}{T}\sum_{t=1}^T f_{0t} e_{it}' \right\| + O_P(N^{-1/2}) = O_P(N^{1/(2m)}/\sqrt{T}) + O_P(N^{-1/2}).$$

It then follows from the above results and Lemma 2 that $\max_{i,t}\|\hat{e}_{it} - e_{it}\| = O_P(T^{1/(2m)}/\sqrt{T})$. □

Proof of Theorem 3

Proof. **Step 1:** We first show that $\hat{\Delta} = \Delta + o_P(1)$.

Note that by the definition of Δ and Lemma S1,

$$\|\hat{\Delta} - \Delta\| \leq \left\| \hat{\Delta} - \frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T l_{it}^{(2)} Z_{it} Z_{it} \right\| + \left\| \frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T \left\{ l_{it}^{(2)} Z_{it} Z_{it} - \mathbb{E}[l_{it}^{(2)} Z_{it} Z_{it}] \right\} \right\| + o(1).$$

Similar to Step 2 in the proof of Lemma 4, it can be shown that

$$\frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T \left\{ l_{it}^{(2)} Z_{it} Z_{it} - \mathbb{E}[l_{it}^{(2)} Z_{it} Z_{it}] \right\} = o_P(1).$$

Thus, it remains to show that

$$\left\| \hat{\Delta} - \frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T l_{it}^{(2)} Z_{it} Z_{it} \right\| = o_P(1).$$

By the definition of $\hat{\Delta}$, we have

$$\left\| \hat{\Delta} - \frac{1}{NT}\sum_{i=1}^N \sum_{t=1}^T l_{it}^{(2)} Z_{it} Z_{it} \right\| \lesssim \max_{i,t} \left\| l^{(2)}(\hat{u}_{it}) - l_{it}^{(2)} \right\| + \max_{i,t} \left\| \hat{Z}_{it} - Z_{it} \right\| / h.$$

First, by Lemma 2 and Lemma 5,

$$\begin{aligned} \max_{i,t} \left\| l^{(2)}(\hat{u}_{it}) - l_{it}^{(2)} \right\| &\lesssim \left(\|\hat{\beta} - \beta_0\| \cdot \|X_{it}\| + \max_i \left\| \hat{\lambda}_i - \lambda_{0i} \right\| + \max_t \left\| \hat{f}_t - f_{0t} \right\| \right) / h^2 \\ &= O_P(T^{1/(2m)}/\sqrt{Th^4}) = o_P(1). \end{aligned}$$

Second, by Lemma 12

$$\max_{i,t} \left\| \hat{Z}_{it} - Z_{it} \right\| / h = O_P\left(\frac{T^{1/(2m)}}{T^{0.5} h^2} \right) = o_P(1).$$

Therefore, the desired result follows.

Step 2: Next, we show that $\hat{b} = b + o_P(1)$.

By definitions, it suffices to show that

$$\max_{1\le i\le N}\left\|\hat{\omega}^{(1)}_{T,i} - \omega^{(1)}_{T,i}\right\| = o_P(1), \quad \max_{1\le i\le N}\left\|\hat{\omega}^{(2)}_{T,i} - \omega^{(2)}_{T,i}\right\| = o_P(1),$$

and

$$\max_{1\le i\le N}\left\|\hat{\omega}^{(3)}_{T,i,k} - \omega^{(3)}_{T,i,k}\right\| = o_P(1), \quad \max_{1\le i\le N}\left\|\hat{\omega}^{(4)}_{T,i,k} - \omega^{(4)}_{T,i,k}\right\| = o_P(1)$$

for $k \le r$.

First, similar to the proof of the previous step, we have

$$\begin{aligned}
\max_{1\le i\le N}\left\|\hat{\omega}^{(1)}_{T,i} - \omega^{(1)}_{T,i}\right\| &= \max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T} l^{(2)}(\hat{u}_{it})\hat{Z}_{it}\cdot \hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_t - \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\big[\mathsf{f}_{it}(0|X_{it})Z_{it}\big] f_{0t}'\Omega_i^{-1}f_{0t}\right\| \\
\le \max_{1\le i\le N}&\left\|\frac{1}{T}\sum_{t=1}^{T} l^{(2)}(\hat{u}_{it})\hat{Z}_{it}\cdot \hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_t - \frac{1}{T}\sum_{t=1}^{T} l^{(2)}_{it}Z_{it}\cdot f_{0t}'\Omega_i^{-1}f_{0t}\right\| + \max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}\Big(l^{(2)}_{it}Z_{it} - \mathbb{E}\big[\mathsf{f}_{it}(0|X_{it})Z_{it}\big]\Big) f_{0t}'\Omega_i^{-1}f_{0t}\right\| \\
&\lesssim \max_{i,t}\left\|l^{(2)}(\hat{u}_{it}) - l^{(2)}_{it}\right\| + \max_{i,t}\left\|\hat{Z}_{it} - Z_{it}\right\|/h + \max_{i,t}\left\|\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_t - f_{0t}'\Omega_i^{-1}f_{0t}\right\|/h + o_P(1) \\
&= \max_{i,t}\left\|\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_t - f_{0t}'\Omega_i^{-1}f_{0t}\right\|/h + o_P(1).
\end{aligned}$$

Note that by Lemma 2 and Lemma 12,

$$\begin{aligned}
&\max_{i,t}\left\|\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_t - f_{0t}'\Omega_i^{-1}f_{0t}\right\|/h = \max_{i,t}\left\|\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_t - f_{0t}'\hat{\mathbf{H}}'(\hat{\mathbf{H}}')^{-1}\Omega_i^{-1}(\hat{\mathbf{H}})^{-1}\hat{\mathbf{H}}f_{0t}\right\|/h \\
&= \max_{i,t}\left\|\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_t - \check{f}_{0t}'\tilde{\Omega}_i^{-1}\check{f}_{0t}\right\|/h \lesssim \max_{t}\|\hat{f}_t - \check{f}_{0t}\|/h + \max_{i}\|\hat{\Omega}_i^{-1} - \tilde{\Omega}_i^{-1}\|/h = O_P\left(\frac{T^{1/(2m)}}{T^{0.5}h^2}\right) = o_P(1).
\end{aligned}$$

Thus, it follows that

$$\max_{1\le i\le N}\left\|\hat{\omega}^{(1)}_{T,i} - \omega^{(1)}_{T,i}\right\| = o_P(1).$$

Second, by Lemma 12

$$\begin{aligned}
\max_{1\le i\le N}\left\|\hat{\omega}^{(3)}_{T,i} - \omega^{(3)}_{T,i}\right\| &= \tau(1-\tau)\cdot\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}\hat{f}_t'\hat{\Omega}_i^{-1}\hat{\mathbf{C}}_{i,k}\hat{\Omega}_i^{-1}\hat{f}_t - \frac{1}{T}\sum_{t=1}^{T} f_{0t}'\Omega_i^{-1}\mathbf{C}_{i,k}\Omega_i^{-1}f_{0t}\right\| \\
&\lesssim \max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T}\hat{f}_t'\hat{\Omega}_i^{-1}\hat{\mathbf{C}}_{i,k}\hat{\Omega}_i^{-1}\hat{f}_t - \frac{1}{T}\sum_{t=1}^{T}\check{f}_{0t}'\tilde{\Omega}_i^{-1}\cdot\hat{\mathbf{H}}\mathbf{C}_{i,k}\hat{\mathbf{H}}'\cdot\tilde{\Omega}_i^{-1}\check{f}_{0t}\right\| \\
\le \max_{t}\|\hat{f}_t - \check{f}_{0t}\|/h^2 + \max_{i}\|\hat{\Omega}_i^{-1} - \tilde{\Omega}_i^{-1}\|/h^2 &+ \max_{i}\left\|\hat{\mathbf{C}}_{i,k} - \hat{\mathbf{H}}\mathbf{C}_{i,k}\hat{\mathbf{H}}'\right\| = \max_{i}\left\|\hat{\mathbf{C}}_{i,k} - \hat{\mathbf{H}}\mathbf{C}_{i,k}\hat{\mathbf{H}}'\right\| + o_P(1).
\end{aligned}$$

By the definitions of $\hat{\mathbf{C}}_{i,k}$ and $\mathbf{C}_{i,k}$, it follows from Lemma 12 that

$$\max_i \left\|\hat{\mathbf{C}}_{i,k} - \hat{\mathbf{H}}\mathbf{C}_{i,k}\hat{\mathbf{H}}'\right\| = \max_i \left\|\frac{1}{T}\sum_{t=1}^{T} l^{(3)}(\hat{u}_{it})\hat{Z}_{it,k}\hat{f}_t\hat{f}_t' - \frac{1}{T}\sum_{t=1}^{T} -\mathbb{E}[\mathsf{f}_{it}^{(1)}(0|X_{it})Z_{it,k}]\hat{\mathbf{H}}f_{0t}f_{0t}'\hat{\mathbf{H}}'\right\|$$

$$\leq \max_i \left\|\frac{1}{T}\sum_{t=1}^{T}\left(l^{(3)}(\hat{u}_{it})\hat{Z}_{it,k}\hat{f}_t\hat{f}_t' - l_{it}^{(3)}Z_{it,k}\check{f}_{0t}\check{f}_{0t}'\right)\right\| + \max_i \left\|\frac{1}{T}\sum_{t=1}^{T}\left(l_{it}^{(3)}Z_{it,k} + \mathbb{E}[\mathsf{f}_{it}^{(1)}(0|X_{it})Z_{it,k}]\right) f_{0t}f_{0t}'\right\| \cdot \|\hat{\mathbf{H}}\|^2$$

$$\lesssim \max_t \|\hat{f}_t - \check{f}_{0t}\|/h^2 + \max_{i,t}\|\hat{u}_{it} - u_{it}\|/h^3 + \max_{i,t}\|\hat{Z}_{it} - Z_{it}\|/h^2 + o_P(1) = O_P\left(\frac{T^{1/(2m)}}{\sqrt{Th^6}}\right) + o_P(1) = o_P(1).$$

Thus, we have

$$\max_{1\leq i\leq N}\left\|\hat{\omega}_{T,i}^{(3)} - \omega_{T,i}^{(3)}\right\| = o_P(1).$$

Third, by the proof Lemma 7 and Assumption 2(iv), it can be shown that

$$\omega_{T,i}^{(2)} = \frac{1}{T}\sum_{t=1}^{T-L}\sum_{s=t+1}^{t+L}\mathbb{E}\left[l_{it}^{(2)}Z_{it}l_{it}^{(1)}\right]\cdot f_{0t}'\Omega_i^{-1}f_{0s} + \frac{1}{T}\sum_{t=L+1}^{T}\sum_{s=t-L}^{t-1}\mathbb{E}\left[l_{it}^{(2)}Z_{it}l_{is}^{(1)}\right]\cdot f_{0t}'\Omega_i^{-1}f_{0s} + \bar{O}_P(Th^q) + O(\alpha^L).$$

Since $0 < \alpha < 1$ and $L \to \infty$ as $T \to \infty$, by the definition of $\hat{\omega}_{T,i}^{(2)}$, it follows that

$$\max_{1\leq i\leq N}\left\|\hat{\omega}_{T,i}^{(2)} - \omega_{T,i}^{(2)}\right\| \leq \max_{1\leq i\leq N}\left\|\frac{1}{T}\sum_{t=1}^{T-L}\sum_{s=t+1}^{t+L}\left\{l^{(2)}(\hat{u}_{it})\hat{Z}_{it}l^{(1)}(\hat{u}_{is})\cdot\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_s - \mathbb{E}\left[l_{it}^{(2)}Z_{it}l_{is}^{(1)}\right]\cdot f_{0t}'\Omega_i^{-1}f_{0s}\right\}\right\|$$

$$+ \max_{1\leq i\leq N}\left\|\frac{1}{T}\sum_{t=L+1}^{T}\sum_{s=t-L}^{t-1}\left\{l^{(2)}(\hat{u}_{it})\hat{Z}_{it}l^{(1)}(\hat{u}_{is})\cdot\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_s - \mathbb{E}\left[l_{it}^{(2)}Z_{it}l_{is}^{(1)}\right]\cdot f_{0t}'\Omega_i^{-1}f_{0s}\right\}\right\| + o_P(1).$$

Note that

$$\max_{1\leq i\leq N}\left\|\frac{1}{T}\sum_{t=1}^{T-L}\sum_{s=t+1}^{t+L}\left\{l^{(2)}(\hat{u}_{it})\hat{Z}_{it}l^{(1)}(\hat{u}_{is})\cdot\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_s - \mathbb{E}\left[l_{it}^{(2)}Z_{it}l_{it}^{(1)}\right]\cdot f_{0t}'\Omega_i^{-1}f_{0s}\right\}\right\| \leq$$

$$\max_{1\leq i\leq N}\left\|\frac{1}{T}\sum_{t=1}^{T-L}\sum_{s=t+1}^{t+L}\left\{l^{(2)}(\hat{u}_{it})\hat{Z}_{it}l^{(1)}(\hat{u}_{is})\cdot\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_s - l_{it}^{(2)}Z_{it}l_{it}^{(1)}\cdot f_{0t}'\Omega_i^{-1}f_{0s}\right\}\right\|$$

$$+ \max_{1\leq i\leq N}\left\|\frac{1}{T}\sum_{t=1}^{T-L}\sum_{s=t+1}^{t+L}\left\{l_{it}^{(2)}Z_{it}l_{it}^{(1)} - \mathbb{E}\left[l_{it}^{(2)}Z_{it}l_{it}^{(1)}\right]\right\}\cdot f_{0t}'\Omega_i^{-1}f_{0s}\right\|.$$

It can be shown that the second term on the right-hand side of the above inequality is $O_P(LT^{1/(2m)}/(T^{0.5}h)) =$

$o_P(1)$. For the first term, similar to the proof for $\hat{\omega}_{T,i}^{(1)}$ we have

$$\max_{1\le i\le N}\left\|\frac{1}{T}\sum_{t=1}^{T-L}\sum_{s=t+1}^{t+L}\left\{l^{(2)}(\hat{u}_{it})\hat{Z}_{it}l^{(1)}(\hat{u}_{is})\cdot\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_s-l_{it}^{(2)}Z_{it}l_{it}^{(1)}\cdot f_{0t}'\Omega_i^{-1}f_{0s}\right\}\right\|$$
$$\lesssim L\left(\max_{i,t}\left\|l^{(2)}(\hat{u}_{it})-l_{it}^{(2)}\right\|+\max_{i,t}\left\|l^{(1)}(\hat{u}_{it})-l_{it}^{(1)}\right\|/h+\max_{i,t}\left\|\hat{Z}_{it}-Z_{it}\right\|/h+\max_{t,s}\left\|\hat{f}_t'\hat{\Omega}_i^{-1}\hat{f}_s-f_{0t}'\Omega_i^{-1}f_{0s}\right\|/h\right)$$
$$=O_P\left(\frac{LT^{1/(2m)}}{T^{0.5}h^2}\right)=o_P(1).$$

Thus, we have

$$\max_{1\le i\le N}\left\|\hat{\omega}_{T,i}^{(2)}-\omega_{T,i}^{(2)}\right\|=o_P(1).$$

Finally, we show can that

$$\max_{1\le i\le N}\left\|\hat{\omega}_{T,i}^{(4)}-\omega_{T,i}^{(4)}\right\|=O_P\left(\frac{LT^{1/(2m)}}{T^{0.5}h^3}\right)=o_P(1)$$

in a similar way. Thus, we can conclude that $\hat{b}=b+o_P(1)$.

Step 3: Finally, we show that $\hat{d}=d+o_P(1)$, which follows from $\hat{d}_1=d_1+o_P(1)$ and $\hat{d}_{2,k}=d_{2,k}+o_P(1)$ for all $k\le r$.

First, by the definitions of d_1 and $\hat{d}_1$, we need to show that

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}l^{(2)}(\hat{u}_{it})\hat{Z}_{it}\hat{\lambda}_i'\hat{\Psi}'\hat{e}_{it}-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}\lambda_{0i}'(\mathbf{H}_0)^{-1}\Psi_0'e_{it}]\right\|=o_P(1).$$

We have

$$\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}l^{(2)}(\hat{u}_{it})\hat{Z}_{it}\hat{\lambda}_i'\hat{\Psi}'\hat{e}_{it}-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}\lambda_{0i}'(\mathbf{H}_0)^{-1}\Psi_0'e_{it}]\right\|$$
$$\le\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}l^{(2)}(\hat{u}_{it})\hat{Z}_{it}\hat{\lambda}_i'\hat{\Psi}'\hat{e}_{it}-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}l_{it}^{(2)}Z_{it}\lambda_{0i}'(\mathbf{H}_0)^{-1}\Psi_0'e_{it}\right\|$$
$$+\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}l_{it}^{(2)}Z_{it}\lambda_{0i}'(\mathbf{H}_0)^{-1}\Psi_0'e_{it}-\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}\lambda_{0i}'(\mathbf{H}_0)^{-1}\Psi_0'e_{it}]\right\|.$$

It is easy to see that the second term on the right-hand side of the above inequality is $O_P(1/\sqrt{NTh^2})+$

$O_P(h^q) = o_P(1)$. For the first term, by Lemma 2, Lemma 5 and Lemma 12 we have

$$\left\| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} l^{(2)}(\hat{u}_{it}) \hat{Z}_{it} \hat{\lambda}_i' \hat{\Psi}' \hat{e}_{it} - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} l_{it}^{(2)} Z_{it} \lambda_{0i}' (\mathbf{H}_0)^{-1} \Psi_0' e_{it} \right\|$$
$$\leq \max_{i,t} \|l^{(2)}(\hat{u}_{it}) - l_{it}^{(2)}\| + \max_{i,t} \|\hat{Z}_{it} - Z_{it}\|/h + \max_{i} \|\hat{\lambda}_i - \tilde{\lambda}_i\|/h + \|\hat{\Psi} - \Psi_0\|/h + \max_{i,t} \|\hat{e}_{it} - e_{it}\|/h$$
$$= O_P\left(\frac{T^{1/(2m)}}{\sqrt{Th^4}} \right) = o_P(1).$$

Therefore, we can conclude that $\hat{d}_1 = d_1 + o_P(1)$.

Second, by the definitions of $d_{2,k}$ and $\hat{d}_{2,k}$, we need to show that

$$\left\| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{e}_{it}' \hat{\Psi} \hat{\mathbf{B}}_{t,k} \hat{\Psi}' \hat{e}_{it} - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathrm{Tr} \left\{ \mathbb{E}[e_{it} e_{it}'] \cdot \Psi_0 (\mathbf{H}_0')^{-1} \mathbf{B}_{t,k} (\mathbf{H}_0)^{-1} \Psi_0' \right\} \right\| = o_P(1),$$

and

$$\left\| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{e}_{it}' \hat{\Psi} \hat{\mathbf{D}}_{t,k} \hat{\Psi}' \hat{e}_{it} - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathrm{Tr} \left\{ \mathbb{E}[e_{it} e_{it}'] \cdot \Psi_0 (\mathbf{H}_0')^{-1} \mathbf{D}_{t,k} (\mathbf{H}_0)^{-1} \Psi_0' \right\} \right\| = o_P(1).$$

To save space, we only prove the first result. The proof of the second result is similar. Note that

$$\left\| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{e}_{it}' \hat{\Psi} \hat{\mathbf{B}}_{t,k} \hat{\Psi}' \hat{e}_{it} - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathrm{Tr} \left\{ \mathbb{E}[e_{it} e_{it}'] \cdot \Psi_0 (\mathbf{H}_0')^{-1} \mathbf{B}_{t,k} (\mathbf{H}_0)^{-1} \Psi_0' \right\} \right\|$$
$$\leq \left\| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{e}_{it}' \hat{\Psi} \hat{\mathbf{B}}_{t,k} \hat{\Psi}' \hat{e}_{it} - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} e_{it}' \Psi_0 (\mathbf{H}_0')^{-1} \mathbf{B}_{t,k} (\mathbf{H}_0)^{-1} \Psi_0' e_{it} \right\| +$$
$$\left\| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} e_{it}' \Psi_0 (\mathbf{H}_0')^{-1} \mathbf{B}_{t,k} (\mathbf{H}_0)^{-1} \Psi_0' e_{it} - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \mathrm{Tr} \left\{ \mathbb{E}[e_{it} e_{it}'] \cdot \Psi_0 (\mathbf{H}_0')^{-1} \mathbf{B}_{t,k} (\mathbf{H}_0)^{-1} \Psi_0' \right\} \right\|.$$

It is easy to see that the second term on the right-hand side of the above inequality is $O_P(1/\sqrt{NT})$. For the first term, we have

$$\left\| \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} \hat{e}_{it}' \hat{\Psi} \hat{\mathbf{B}}_{t,k} \hat{\Psi}' \hat{e}_{it} - \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} e_{it}' \Psi_0 (\mathbf{H}_0')^{-1} \mathbf{B}_{t,k} (\mathbf{H}_0)^{-1} \Psi_0' e_{it} \right\|$$
$$\lesssim \max_{i,t} \|\hat{e}_{it} - e_{it}\|/h + \|\hat{\Psi} - \Psi_0\|/h + \max_{t} \left\| \hat{\mathbf{B}}_{t,k} - (\mathbf{H}_0')^{-1} \mathbf{B}_{t,k} (\mathbf{H}_0)^{-1} \right\|$$
$$= \max_{t} \left\| \hat{\mathbf{B}}_{t,k} - (\mathbf{H}_0')^{-1} \mathbf{B}_{t,k} (\mathbf{H}_0)^{-1} \right\| + O_P(T^{1/(2m)}/\sqrt{Th^2}).$$

By the definitions of $\hat{\mathbf{B}}_{t,k}$ and $\mathbf{B}_{t,k}$, it follows that

$$\max_t \left\| \hat{\mathbf{B}}_{t,k} - (\mathbf{H}_0')^{-1}\mathbf{B}_{t,k}(\mathbf{H}_0)^{-1} \right\| = \max_t \left\| \frac{1}{N}\sum_{i=1}^N l^{(2)}(\hat{u}_{it})\hat{\lambda}_i\hat{\Phi}_{i,k} - \frac{1}{N}\sum_{i=1}^N \mathsf{f}_{it}(0)(\mathbf{H}_0')^{-1}\lambda_{0i}\Phi_{i,k}(\mathbf{H}_0)^{-1} \right\|$$

$$\leq \max_t \left\| \frac{1}{N}\sum_{i=1}^N l^{(2)}(\hat{u}_{it})\hat{\lambda}_i\hat{\Phi}_{i,k} - \frac{1}{N}\sum_{i=1}^N l_{it}^{(2)}\tilde{\lambda}_{0i}\Phi_{i,k}(\mathbf{H}_0)^{-1} \right\| + \max_t \left\| \frac{1}{N}\sum_{i=1}^N [l_{it}^{(2)} - \mathsf{f}_{it}(0)]\tilde{\lambda}_{0i}\Phi_{i,k} \right\| \cdot \|\mathbf{H}_0^{-1}\|$$

$$\lesssim \max_{i,t} \|l^{(2)}(\hat{u}_{it}) - l_{it}^{(2)}\| + \max_i \|\hat{\lambda}_i - \tilde{\lambda}_i\|/h + \max_i \|\hat{\Phi}_i - \Phi_i\hat{\mathbf{H}}^{-1}\|/h + \|\mathbf{H}_0 - \mathbf{H}^{-1}\|/h + o_P(1)$$

$$= O_P\left(\frac{T^{1/(2m)}}{\sqrt{Th^4}}\right) + o_P(1) = o_P(1).$$

Combining all the above results gives that $\hat{d}_{2,k} = d_{2,k} + o_P(1)$ for all $k \leq r$. This completes the proof of Theorem 3. □

A.5 Proof of Theorem 4

Proof. First, similar to the proof of Theorem 3, we have

$$\max_{1\leq t\leq T} \left\| \hat{\mathbf{A}}_t - \mathbf{A}_t\mathbf{H}_0^{-1} \right\| = \max_{1\leq t\leq T} \left\| \frac{1}{N}\sum_{i=1}^N l^{(2)}(\hat{u}_{it})\hat{Z}_{it}\hat{\lambda}_i' - \frac{1}{N}\sum_{i=1}^N \mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}]\lambda_{0i}'\mathbf{H}_0^{-1} \right\|$$

$$\leq \max_{1\leq t\leq T} \left\| \frac{1}{N}\sum_{i=1}^N \left\{ l^{(2)}(\hat{u}_{it})\hat{Z}_{it}\hat{\lambda}_i' - l_{it}^{(2)}Z_{it}\tilde{\lambda}_{0i}' \right\} \right\| + \max_{1\leq t\leq T} \left\| \frac{1}{N}\sum_{i=1}^N \left\{ l_{it}^{(2)}Z_{it} - \mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}] \right\} \lambda_{0i}' \right\| \cdot \|\mathbf{H}_0^{-1}\|$$

$$\lesssim \max_{i,t} \|l^{(2)}(\hat{u}_{it}) - l_{it}^{(2)}\| + \max_{i,t} \|\hat{Z}_{it} - Z_{it}\|/h + \max_i \|\hat{\lambda}_i - \tilde{\lambda}_i\|/h + O_P\left(\frac{T^{1/(2m)}}{T^{0.5}h^2}\right) = O_P\left(\frac{T^{1/(2m)}}{T^{0.5}h^2}\right).$$

Second, it follows that

$$\max_{i,t} \|\hat{W}_{it} - W_{it}^*\| = \max_{i,t} \left\| l^{(1)}(\hat{u}_{it})\hat{Z}_{it} - \hat{\mathbf{A}}_t\hat{\Psi}'\hat{e}_{it} - [l_{it}^{(1)}Z_{it} - \mathbf{A}_t(\mathbf{H}_0)^{-1}\Psi_0'e_{it}] \right\| \lesssim \max_{i,t} \left\| l^{(1)}(\hat{u}_{it}) - l_{it}^{(1)} \right\|$$

$$+ \max_{i,t} \left\| \hat{Z}_{it} - Z_{it} \right\| + \max_t \left\| \hat{\mathbf{A}}_t - \mathbf{A}_t\mathbf{H}_0^{-1} \right\| + \|\hat{\Psi} - \Psi_0\| + \max_{i,t} \|\hat{e}_{it} - e_{it}\| = O_P\left(\frac{T^{1/(2m)}}{T^{0.5}h^2}\right).$$

Third, similar to the proof of Theorem 3, we have

$$\begin{aligned}
\left\|\hat{\mathbf{V}}_2 - \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s\neq t}^{T}\mathbb{E}[W_{it}^{*}W_{is}^{*'}]\right\| &\leq \left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T-L}\sum_{s=t+1}^{t+L}\left\{\hat{W}_{it}\hat{W}_{is}' - \mathbb{E}[W_{it}^{*}W_{is}^{*'}]\right\}\right\| \\
&+ \left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1+L}^{T}\sum_{s=t-L}^{t-1}\left\{\hat{W}_{it}\hat{W}_{is}' - \mathbb{E}[W_{it}^{*}W_{is}^{*'}]\right\}\right\| + o_P(1) \\
\leq \left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T-L}\sum_{s=t+1}^{t+L}\left\{\hat{W}_{it}\hat{W}_{is}' - W_{it}^{*}W_{is}^{*'}\right\}\right\| &+ \left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T-L}\sum_{s=t+1}^{t+L}\left\{W_{it}^{*}W_{is}^{*'} - \mathbb{E}[W_{it}^{*}W_{is}^{*'}]\right\}\right\| \\
+ \left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1+L}^{T}\sum_{s=t-L}^{t-1}\left\{\hat{W}_{it}\hat{W}_{is}' - W_{it}^{*}W_{is}^{*'}\right\}\right\| &+ \left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1+L}^{T}\sum_{s=t-L}^{t-1}\left\{W_{it}^{*}W_{is}^{*'} - \mathbb{E}[W_{it}^{*}W_{is}^{*'}]\right\}\right\| \\
&\lesssim L\cdot\max_{i,t}\|\hat{W}_{it} - W_{it}^{*}\| + O_P(L/\sqrt{NT}) = O_P\left(\frac{LT^{1/(2m)}}{T^{0.5}h^2}\right) + o_P(1) = o_P(1).
\end{aligned}$$

By the proof of Theorem 2, it follows that $\hat{\mathbf{V}}_2 = \mathbf{V}_2 + o_P(1)$.

Finally, we can show that $\hat{\mathbf{V}}_1 = \mathbf{V}_1 + o_P(1)$ in a similar way. This completes the proof. □

B Some Auxiliary Lemmas

Lemma S1. *Let M be a generic bounded constant. Under Assumptions 1 to 3, it can be shown that (i) $\sup_{u\in\mathbb{R}} l^{(j)}(u)\cdot h^{j-1} \leq M$ for $j = 1,\ldots,4$;*
(ii)

$$\mathbb{E}[l_{it}^{(1)}X_{it}] = O(h^q),\ \mathbb{E}[l_{it}^{(1)}] = O(h^q),\ \mathbb{E}[l_{it}^{(2)}X_{it}] = \mathbb{E}[\mathsf{f}_{it}(0|X_{it})X_{it}] + O(h^q),\ \mathbb{E}[l_{it}^{(2)}] = \mathsf{f}_{it}(0) + O(h^q),$$

$$\mathbb{E}[l_{it}^{(3)}X_{it}] = -\mathbb{E}[\mathsf{f}_{it}^{(1)}(0|X_{it})X_{it}] + O(h^{q-1}),\quad \mathbb{E}[l_{it}^{(2)}l_{it}^{(1)}X_{it}] = (\tau - 0.5)\cdot\mathbb{E}[\mathsf{f}_{it}(0|X_{it})X_{it}] + o(1),$$

$$\mathbb{E}[l_{it}^{(2)}l_{is}^{(1)}X_{it}] = \tau\mathbb{E}[\mathsf{f}_{it}(0|X_{it})Z_{it}] - \mathbb{E}\left[\int_{\infty}^{0}\mathsf{f}_{i,ts}(0,v|X_{it},X_{is})dv\cdot Z_{it}\right] + o(1),$$

$$\mathbb{E}\left[\left(l_{it}^{(1)}\right)^2\right] = \tau(1-\tau) + o(1),\ \mathbb{E}\left[l_{it}^{(1)}l_{is}^{(1)}\right] = \mathbb{E}[\mathbf{1}\{u_{it}\leq 0, u_{is}\leq 0\}] - \tau^2 + o(1),$$

$$\mathbb{E}\left[\left(l_{it}^{(1)}\right)^2 Z_{it}Z_{it}'\right] = \tau(1-\tau)\cdot\mathbb{E}[Z_{it}Z_{it}'] + o(1).$$

(iii) $\max_{i,t}\sup_{\beta\in\mathcal{B},\lambda_i\in\mathcal{A}}\mathbb{E}[l^{(3)}(Y_{it} - \beta'X_{it} - \lambda_i\tilde{f}_{0t})|X_{it}] \leq M$ almost surely.

Proof. The proof of the above results follows from standard calculations of nonparametric kernel estimators, and can be found in Horowitz (1998) or Galvao and Kato (2016). Thus, it is omitted. □

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