Supplementary Material to "Consistent Local Spectrum (LCM) Inference for Predictive Return Regressions"

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C Proofs of Technical Lemmas

This online supplementary material provides proofs of the technical lemmas in the Appendix.

C.1 Proof of Lemma B.1

Proof. First, **(a)** and **(c)** follow directly from AV (2021, Theorems 4(a) and 4(c)), since the assumptions and specification of the regressors in this paper readily follow their framework.[1]

For **(b)**, let us first define $\widehat{e}_t = \widehat{e}_t^{(1)} + \widehat{e}_t^{(2)}$, where

$$\widehat{e}_t^{(1)} \equiv (1-L)^{\widehat{d}_1} a + \boldsymbol{\mathcal{B}}' \boldsymbol{\mathcal{Q}}(L)(1-L)^{\widehat{d}_1} \boldsymbol{x}_{t-1} + (1-L)^{\widehat{d}_1} \xi_{t-1}^{(-d_1)}, \quad \widehat{e}_t^{(2)} \equiv (1-L)^{\widehat{d}_1} \eta_t, \tag{C.1}$$

for which the component $\widehat{e}_t^{(1)}$ is equivalent to the case without cointegration considered by AV (2021, Theorem 4(b)) due to Assumptions D1-D3 and C. By applying the decomposition (C.1), we have

$$\widehat{\boldsymbol{F}}^c_{\widehat{u}\widehat{e}}(\ell, m) - \widehat{\boldsymbol{F}}_{\widehat{u}\widehat{e}}(\ell, m) = \widehat{\boldsymbol{F}}^c_{\widehat{c}\widehat{e}}(\ell, m) = \widehat{\boldsymbol{F}}^{(c,1)}_{\widehat{c}\widehat{e}}(\ell, m) + \widehat{\boldsymbol{F}}^{(c,2)}_{\widehat{c}\widehat{e}}(\ell, m), \tag{C.2}$$

where $\widehat{\boldsymbol{F}}^{(c,1)}_{\widehat{c}\widehat{e}}(\ell, m)$ and $\widehat{\boldsymbol{F}}^{(c,2)}_{\widehat{c}\widehat{e}}(\ell, m)$ are the TDACs between $\widehat{\boldsymbol{c}}_{t-1}$ and $\widehat{e}_t^{(1)}$, respectively, $\widehat{e}_t^{(2)}$. Now, by applying AV (2021, Theorem 4(b)) and AVOA (2020, Lemma A.12(b)), we have

$$\lambda_m^{-1} \widehat{\boldsymbol{F}}^{(c,1)}_{\widehat{c}\widehat{e}}(\ell, m) \le O_p^+((m/n)^{\underline{d}_x}/\ell^{1+\epsilon}) \quad \text{and} \quad \boldsymbol{w}_{\widehat{c}}(\lambda_j, i) = O_p(\lambda_j^{d_i}), \tag{C.3}$$

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[1] While AV (2021) state their results for $\underline{d} = \min(d_1, \underline{d}_x)$ rather than $\underline{d}_x$ to maintain simplicity in their framework, it is clear that their results apply to $\underline{d}_x$, since the parameter appears when using the differencing operator on $\boldsymbol{u}_{t-1}$ and $\boldsymbol{c}_{t-1}$.

for $i = 2, \ldots, k+1$. Moreover, we can write $\widehat{\theta}_1 = \widehat{d}_1 - d_1 = O_p(1/\sqrt{m_d})$ and

$$\widehat{e}_t^{(2)} = (1-L)^{\widehat{d}_1 - d_1}(1-L)^{d_1}\eta_t \equiv (1-L)^{\widehat{\theta}_1}\widetilde{e}_t^{(2)}, \tag{C.4}$$

using Assumption F, such that by AVOA (2020, Lemmas A.8 and A.9(a)), it follows that,

$$w_{\widehat{e}}^{(2)}(\lambda_j)) = w_{\widetilde{e}}^{(2)}(\lambda_j))\Big(1 + O_p\big(\ln(n)/\sqrt{m_d}\big) + O_p\big(\ln(n)^2/m_d\big)\Big), \quad \text{with} \tag{C.5}$$

$$w_{\widetilde{e}}^{(2)}(\lambda_j) = \lambda_j^{d_1} e^{-(\pi/2)\mathrm{i}d_1} w_\eta(\lambda_j) + O_p\big(\lambda_j^{d_1}\ln(n)/j^{-1/2}\big) + O_p\big(n^{-d_1-1}\big), \tag{C.6}$$

and, furthermore, by AVOA (2020, Lemma A.12(b)) that $w_{\widehat{e}}^{(2)}(\lambda_j) = O_p(\lambda_j^{d_1})$ when $\ln(n)/j^{1/2} \to 0$. Hence, since $0 < d_i \le d_1 + d_i$, $i = 2, \ldots, k+1$, we may further write

$$\widehat{\boldsymbol{F}}_{\widehat{c}\widehat{e}}^{(c,2)}(\ell, m) \le \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+(\lambda_j^{\underline{d}_x}) \le \frac{2\pi m^{1+\underline{d}_x}}{n^{1+\underline{d}_x}}\sum_{j=\ell}^{m} O_p^+\left(\left(\frac{j}{m}\right)^{\underline{d}_x}\frac{1}{j^{1+\epsilon}}\right) \le O_p^+\left(\left(\frac{m}{n}\right)^{1+\underline{d}_x}\frac{1}{\ell^{1+\epsilon}}\right), \tag{C.7}$$

for some arbitrarily small $\epsilon > 0$, using $|\sum_{j=\ell}^{m} O_p(j^{-p})| \le O_p^+(\ell^{-p})$ for some $p > 1$ by Varneskov (2017, Lemma C.4). The stated result follows by combining bounds for $\widehat{\boldsymbol{F}}_{\widehat{c}\widehat{e}}^{(c,1)}(\ell, m)$ and $\widehat{\boldsymbol{F}}_{\widehat{c}\widehat{e}}^{(c,2)}(\ell, m)$.

For **(d)**, by applying the same decomposition as for **(b)**, we have

$$\widehat{\boldsymbol{G}}_{\widehat{u}\widehat{e}}^{c}(\ell_G, m_G) - \widehat{\boldsymbol{G}}_{\widehat{u}\widehat{e}}(\ell_G, m_G) = \widehat{\boldsymbol{G}}_{\widehat{c}\widehat{e}}^{c}(\ell_G, m_G) = \widehat{\boldsymbol{G}}_{\widehat{c}\widehat{e}}^{(c,1)}(\ell_G, m_G) + \widehat{\boldsymbol{G}}_{\widehat{c}\widehat{e}}^{(c,2)}(\ell_G, m_G), \tag{C.8}$$

where, again, the DFT bounds in (C.3) apply to $w_{\widehat{e}}^{(2)}(\lambda_j)$ and $\boldsymbol{w}_{\widehat{c}}(\lambda_j, i)$. Moreover, by AV (2021, Theorem 4(c)), we have

$$\widehat{\boldsymbol{G}}_{\widehat{c}\widehat{e}}^{(c,1)}(\ell_G, m_G) \le O_p^+((m_G/n)^{\underline{d}_x}/\ell_G^{1+\epsilon}). \tag{C.9}$$

Next, using, again, $0 < d_i \le d_1 + d_i$, $i = 2, \ldots, k+1$, we may similarly write

$$\begin{aligned}\widehat{\boldsymbol{G}}_{\widehat{c}\widehat{e}}^{(c,2)}(\ell_G, m_G) &\le \frac{1}{m_G - \ell_G + 1}\sum_{j=\ell_G}^{m_G} O_p^+(\lambda_j^{\underline{d}_x}) \\ &\le \frac{K m_G^{\underline{d}_x}}{n^{\underline{d}_x}}\sum_{j=\ell_G}^{m_G} O_p^+\left(\left(\frac{j}{m_G}\right)^{\underline{d}_x}\frac{1}{j^{1+\epsilon}}\right) \le O_p^+\left(\left(\frac{m_G}{n}\right)^{\underline{d}_x}\frac{1}{\ell_G^{1+\epsilon}}\right),\end{aligned} \tag{C.10}$$

using $m_G/(m_G - \ell_G + 1) \le K$ and Varneskov (2017, Lemma C.4). The stated result follows by combining asymptotic bounds for $\widehat{\boldsymbol{G}}_{\widehat{c}\widehat{e}}^{(c,1)}(\ell_G, m_G)$ and $\widehat{\boldsymbol{G}}_{\widehat{c}\widehat{e}}^{(c,2)}(\ell_G, m_G)$.

For **(e)**, recall $\widehat{\eta}_t^{(d_1)} = \widehat{e}_t - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m)'\widehat{\boldsymbol{u}}_{t-1}$ and let us define

$$\widehat{\eta}_t^{(d_1,1)} = \widehat{e}_t^{(1)} - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m)'\widehat{\boldsymbol{u}}_{t-1}, \quad \widehat{\tau}_{t-1}^{(1)} = (\widehat{\boldsymbol{\mathcal{B}}}_c(\ell, m) - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m))'\widehat{\boldsymbol{u}}_{t-1}^{c}, \quad \widehat{\tau}_{t-1}^{(2)} = \widehat{\boldsymbol{\mathcal{B}}}(\ell, m)\widehat{\boldsymbol{c}}_{t-1}, \tag{C.11}$$

such that we can use $\widehat{e}_t = \widehat{e}_t^{(1)} + \widehat{e}_t^{(2)}$ to decompose $\widehat{\eta}_t^{(d_1)} = \widehat{\eta}_t^{(d_1,1)} + \widehat{e}_t^{(2)}$ and write

$$\widehat{\eta}_t^{(d_1,c)} = \widehat{e}_t - \widehat{\boldsymbol{\mathcal{B}}}_c(\ell,m)'\widehat{\boldsymbol{u}}_{t-1}^c = \widehat{\eta}_t^{(d_1,1)} + \widehat{e}_t^{(2)} - \widehat{\tau}_{t-1}^{(1)} - \widehat{\tau}_{t-1}^{(2)} = \widehat{\eta}_t^{(d_1)} - \widehat{\tau}_{t-1}^{(1)} - \widehat{\tau}_{t-1}^{(2)}. \tag{C.12}$$

The main difference between this decomposition and the corresponding in AV (2021, Theorem 4) is the presence of $\widehat{e}_t^{(2)}$ and the fact that we have $d_1 \neq 0$ for $\eta_t^{(d_1)}$. Hence, we need to distinguish between cases without cointegration in scenarios (ii) and (iii), where ξ_{t-1} is driving the limit, and scenario (iv), where $\xi_{t-1} = 0$, $\forall t \geq 1$. In both inference regimes, we have $d_1 = b > 0$. In the first case, we have fractionally differenced $\xi_{t-1}^{(-d_1)}$ such that the resulting error process is ξ_{t-1}, asymptotically.

Before treating the subtleties of the two inference regimes, we provide bounds on the error terms that are common to both. To this end, we use (C.12) and make the decomposition,

$$\begin{aligned}\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\eta}}^{(d_1,c)}(\ell_G,m_G) - \widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\eta}}^{(d_1)}(\ell_G,m_G) = \widehat{\boldsymbol{G}}_{\widehat{\tau}\widehat{\tau}}^{(1,1)}(\ell_G,m_G) + \widehat{\boldsymbol{G}}_{\widehat{\tau}\widehat{\tau}}^{(2,2)}(\ell_G,m_G) + 2\widehat{\boldsymbol{G}}_{\widehat{\tau}\widehat{\tau}}^{(1,2)}(\ell_G,m_G)\\ - 2\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,1)}(\ell_G,m_G) - 2\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,2)}(\ell_G,m_G),\end{aligned} \tag{C.13}$$

where the first three terms are (trimmed) long-run (co)variance estimates for $\widehat{\tau}_{t-1}^{(1)}$ and $\widehat{\tau}_{t-1}^{(2)}$, and the final two terms are their respective long-run covariances with $\widehat{\eta}_t^{(d_1)}$. Let us further write,

$$\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,i)}(\ell_G,m_G) = \widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,i,1)}(\ell_G,m_G) + \widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,i,2)}(\ell_G,m_G), \quad i = 1,2, \tag{C.14}$$

to indicate the decomposition of $\widehat{\eta}_t^{(d_1)}$ into $\widehat{\eta}_t^{(d_1,1)}$ and $\widehat{e}_t^{(2)}$. Now, since **(a)** and **(b)** yield,

$$\widehat{\boldsymbol{\mathcal{B}}}_c(\ell,m) - \widehat{\boldsymbol{\mathcal{B}}}(\ell,m) \leq O_p^+\left((m/n)^{\underline{d}_x}/\ell^{1+\epsilon}\right), \tag{C.15}$$

we may use equations (A.23), (A.26) and (A.31) in AVOA (2020) to show

$$\begin{aligned}\widehat{\boldsymbol{G}}_{\widehat{\tau}\widehat{\tau}}^{(1,1)}(\ell_G,m_G) &\leq O_p^+\left(\left(\frac{m}{n}\right)^{2\underline{d}_x}\frac{1}{\ell^{2(1+\epsilon)}}\right)\times\left(1+\left(\frac{m_G}{n}\right)^{\underline{d}_x}\frac{1}{\ell_G^{1+\epsilon}}\right),\\ \widehat{\boldsymbol{G}}_{\widehat{\tau}\widehat{\tau}}^{(2,2)}(\ell_G,m_G) &\leq O_p^+\left(\left(\frac{m}{n}\right)^{2\underline{d}_x}\frac{1}{\ell^{1+\epsilon}}\right),\\ \widehat{\boldsymbol{G}}_{\widehat{\tau}\widehat{\tau}}^{(1,2)}(\ell_G,m_G) &\leq O_p^+\left(\left(\frac{m}{n}\right)^{\underline{d}_x}\frac{1}{\ell^{(1+\epsilon)}}\right)\times\left(\left(\frac{m_G}{n}\right)^{\underline{d}_x}\frac{1}{\ell_G^{1+\epsilon}}\right),\end{aligned}$$

where, again, the bounds are restated with $\underline{d}_x$ rather than $\underline{d} = \min(d_1, \underline{d}_x)$ as in AVOA (2020), since the parameter appears when applying the fractional differencing operator to $\boldsymbol{u}_{t-1}$ and $\boldsymbol{c}_{t-1}$.

The case without cointegration. Here, $\widehat{e}_t^{(1)}$ will drive the asymptotic limit and $\widehat{e}_t^{(2)}$ will be a lower order error term. Moreover, the former corresponds to the case considered by AV (2021, Theorem 4), with ξ_{t-1} acting as the (regression) error process. Hence, by invoking equations (A.27) and (A.30) of

AVOA (2020), it follows for the two long-run covariance terms involving $\widehat{\eta}_t^{(d_1,1)}$ that,

$$\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,1,1)}(\ell_G, m_G) \leq O_p^+\left(\left(\frac{m}{n}\right)^{\underline{d}_x}\frac{1}{\ell^{1+\epsilon}}\right),$$
$$\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,2,1)}(\ell_G, m_G) \leq O_p^+\left(\left(\frac{m_G}{n}\right)^{\underline{d}_x}\frac{1}{\ell_G^{1+\epsilon}}\right) \times \left(1 + m^{-1/2}\right).$$

This implies that in order to complete the proof, we need to establish corresponding asymptotic bounds for the remaining terms, $\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,1,2)}(\ell_G, m_G)$ and $\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,2,2)}(\ell_G, m_G)$, i.e., the long-run covariances involving $\widehat{e}_t^{(2)}$. To this end, let us use the discrete Fourier transform bounds in (C.3) and (C.5)-(C.6), $\widehat{\boldsymbol{\mathcal{B}}}(\ell, m) = O_p(1)$, uniformly by AV (2021, Theorem 1), and $0 < d_i \leq d_1 + d_i$ to write,

$$\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,2,2)}(\ell_G, m_G) \leq \frac{1}{m_G - \ell_G + 1}\sum_{j=\ell_G}^{m_G} O_p^+(\lambda_j^{\underline{d}_x}) \leq O_p^+\left(\left(\frac{m_G}{n}\right)^{\underline{d}_x}\frac{1}{\ell_G^{1+\epsilon}}\right), \tag{C.16}$$

similarly to (C.10). For the last term, make the decomposition,

$$\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,1,2)}(\ell_G, m_G) = (\widehat{\boldsymbol{\mathcal{B}}}_c(\ell, m) - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m))'\left(\widehat{\boldsymbol{G}}_{\widehat{u}\widehat{e}}^{(2)}(\ell_G, m_G) + \widehat{\boldsymbol{G}}_{\widehat{c}\widehat{e}}^{(c,2)}(\ell_G, m_G)\right), \tag{C.17}$$

where $\widehat{\boldsymbol{G}}_{\widehat{c}\widehat{e}}^{(c,2)}(\ell_G, m_G) \leq O_p^+((m_G/n)^{\underline{d}_x} 1/\ell_G^{1+\epsilon})$ by (C.10). Moreover, for $i = 2, \ldots, k+1$, since

$$\boldsymbol{w}_{\widehat{u}}(\lambda_j, i) = \boldsymbol{w}_u(\lambda_j, i) + O_p\left(\frac{n^{1/2-d_i}}{j^{1-d_i}}\right) + O_p\left(\frac{\ln(n)n^{1/2}}{m_d^{1/2} j}\right), \quad \boldsymbol{w}_u(\lambda_j, i) = O_p(1), \tag{C.18}$$

by equations (A.8), (A.60), (A.65) and Lemma A.6(a) in AVOA (2020), we may write

$$\begin{aligned}
\widehat{\boldsymbol{G}}_{\widehat{u}\widehat{e}}^{(2)}(\ell_G, m_G) &\leq \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O_p^+(\lambda_j^{d_1}) + \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O_p^+\left(\frac{\lambda_j^{d_1+\underline{d}_x} n^{1/2}}{j}\right) + \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O_p^+\left(\frac{\lambda_j^{d_1}\ln(n)n^{1/2}}{m_d^{1/2} j}\right) \\
&\leq O_p^+(1) + O_p^+\left(\left(\frac{m_G}{n}\right)^{\underline{d}_x}\frac{n^{1/2}}{m_G^{1-\epsilon}\ell_G^{1+\epsilon}}\right) + O_p^+\left(\frac{n^{1/2}\ln(n)}{m_G^{1-\epsilon} m_d^{1/2}\ell_G^{1+\epsilon}}\right),
\end{aligned} \tag{C.19}$$

for some arbitrarily small $\epsilon > 0$, using $d_1 \geq 0$ and Varneskov (2017, Lemma C.4). Hence, by combining bounds, $n^{1/2}/m_G \to 0$, Lemmas B.1(a)-(b) in the absence of cointegration in conjunction with the continuous mapping theorem, we have $\widehat{\boldsymbol{G}}_{\widehat{u}\widehat{e}}^{(2)}(\ell_G, m_G) \leq O_p^+(1)$ and, thus,

$$\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,1,2)}(\ell_G, m_G) \leq O_p^+((m/n)^{\underline{d}_x}/\ell^{1+\epsilon}). \tag{C.20}$$

Consequently, by collecting bounds for all components in (C.14),

$$\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,1)}(\ell_G, m_G) + \widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\tau}}^{(d_1,2)}(\ell_G, m_G) \leq O_p^+((m_G/n)^{\underline{d}_x}/\ell_G^{1+\epsilon}) + O_p^+((m/n)^{\underline{d}_x}/\ell^{1+\epsilon}), \tag{C.21}$$

which, together with bounds for the remaining terms in equation (C.13), provides the requisite result when cointegration is absent, that is, for the inference scenarios (ii)-(iii).

The case with cointegration. By **(a)** and **(b)**, whose rates are independent of cointegration, there is no difference between the treatment of the terms in (C.13) and (C.14) except for $\widehat{\boldsymbol{G}}^{(d_1,1,1)}_{\widetilde{\eta}\widehat{\tau}}(\ell_G, m_G)$ and $\widehat{\boldsymbol{G}}^{(d_1,2,1)}_{\widetilde{\eta}\widehat{\tau}}(\ell_G, m_G)$, that is, the covariance terms involving $\eta^{(d_1,1)}_t$. Hence, let us define

$$\widetilde{e}^{(1)}_t = \widehat{e}^{(1)}_t - (1-L)^{\widehat{d}_1}\xi^{(-d_1)}_{t-1} \equiv \widehat{e}^{(1)}_t - \widehat{\xi}_{t-1}, \quad \widetilde{\eta}^{(d_1,1)}_t = \widetilde{e}^{(1)}_t - \widehat{\boldsymbol{\mathcal{B}}}(\ell,m)'\widehat{\boldsymbol{u}}_{t-1}, \tag{C.22}$$

such that the triangle inequality delivers:

$$\left|\widehat{\boldsymbol{G}}^{(d_1,1,1)}_{\widehat{\eta}\widehat{\tau}}(\ell_G, m_G) - \widehat{\boldsymbol{G}}^{(d_1,1,1)}_{\widetilde{\eta}\widehat{\tau}}(\ell_G, m_G)\right| \le O^+_p\left(\widehat{\boldsymbol{G}}^{(d_1,1,1)}_{\widehat{\xi}\widehat{\tau}}(\ell_G, m_G)\right), \tag{C.23}$$

$$\left|\widehat{\boldsymbol{G}}^{(d_1,2,1)}_{\widehat{\eta}\widehat{\tau}}(\ell_G, m_G) - \widehat{\boldsymbol{G}}^{(d_1,2,1)}_{\widetilde{\eta}\widehat{\tau}}(\ell_G, m_G)\right| \le O^+_p\left(\widehat{\boldsymbol{G}}^{(d_1,2,1)}_{\widehat{\xi}\widehat{\tau}}(\ell_G, m_G)\right). \tag{C.24}$$

This implies that the result *with cointegration* follows from the result *without cointegration* as well as establishing (and verifying) the bounds on the long-run covariance terms between $\widehat{\xi}_{t-1}$ and the errors $\widehat{\tau}^{(1)}_{t-1}$ and $\widehat{\tau}^{(2)}_{t-1}$. To this end, we may use AVOA (2020, Lemma A.12(b)) to write

$$w_{\widehat{\xi}}(\lambda_j) = O_p(1) + O_p\left(\frac{n^{1/2-d_1}}{j^{1-d_1}}\right) + O_p\left(\frac{\ln(n)n^{1/2}}{m_d^{1/2} j}\right), \tag{C.25}$$

$$w^{(1)}_{\widehat{\tau}}(\lambda_j) = \left(\widehat{\boldsymbol{\mathcal{B}}}_c(\ell,m) - \widehat{\boldsymbol{\mathcal{B}}}(\ell,m)\right)'\left(\boldsymbol{w}_{\widehat{u}}(\lambda_j) + \boldsymbol{w}_{\widehat{c}}(\lambda_j)\right), \quad w^{(2)}_{\widehat{\tau}}(\lambda_j) = \widehat{\boldsymbol{\mathcal{B}}}(\ell,m)'\boldsymbol{w}_{\widehat{c}}(\lambda_j), \tag{C.26}$$

where the components in $w^{(1)}_{\widehat{\tau}}(\lambda_j)$ and $w^{(2)}_{\widehat{\tau}}(\lambda_j)$ are described by (C.3), (C.15) and (C.18). Hence, it suffices to study $\widehat{\boldsymbol{G}}_{\widehat{u}\widehat{\xi}}(\ell_G, m_G)$ and $\widehat{\boldsymbol{G}}_{\widehat{c}\widehat{\xi}}(\ell_G, m_G)$. First, for the latter,

$$\begin{aligned}\widehat{\boldsymbol{G}}_{\widehat{c}\widehat{\xi}}(\ell_G, m_G) &\le \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O^+_p(\lambda_j^{\underline{d}_x}) + \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O^+_p\left(\frac{\lambda_j^{d_1+\underline{d}_x} n^{1/2}}{j}\right) + \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O^+_p\left(\frac{\lambda_j^{\underline{d}_x}\ln(n)n^{1/2}}{m_d^{1/2} j}\right) \\ &\le O^+_p\left(\left(\frac{m_G}{n}\right)^{\underline{d}_x}\frac{1}{\ell_G^{1+\epsilon}}\right) + O^+_p\left(\left(\frac{m_G}{n}\right)^{\underline{d}_x}\frac{n^{1/2}}{m_G^{1-\epsilon}\ell_G^{1+\epsilon}}\right) + O^+_p\left(\frac{n^{1/2}\ln(n)}{m_G^{1-\epsilon}m_d^{1/2}\ell_G^{1+\epsilon}}\right),\end{aligned} \tag{C.27}$$

by the same arguments used for $\widehat{\boldsymbol{G}}^{(2)}_{\widehat{u}\widehat{e}}(\ell_G, m_G)$ in (C.19). Similarly, we have

$$\begin{aligned}\widehat{\boldsymbol{G}}_{\widehat{u}\widehat{\xi}}(\ell_G, m_G) &\le \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O^+_p(1) + \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O^+_p\left(\frac{\lambda_j^{d_1} n^{1/2}}{j}\right) + \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O^+_p\left(\frac{\ln(n)n^{1/2}}{m_d^{1/2} j}\right) \\ &\quad + \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O^+_p\left(\frac{n}{j^2}\left(\lambda_j^{d_1+\underline{d}_x} + \lambda_j^{d_1}\ln(n)/\sqrt{m_d} + \ln(n)^2/m_d\right)\right) \\ &\le O^+_p(1) + O^+_p\left(\frac{n^{1/2}}{m_G^{1-\epsilon}\ell_G^{1+\epsilon}}\right) + O^+_p\left(\frac{n^{1/2}\ln(n)}{m_G^{1-\epsilon}m_d^{1/2}\ell_G^{1+\epsilon}}\right)\end{aligned}$$

$$+ O_p^+\left(\frac{n}{m_G\,\ell_G^2}\left(\left(\frac{m_G}{n}\right)^{d_x}+\frac{\ln(n)}{\sqrt{m_d}}+\frac{\ln(n)^2}{m_d}\right)\right), \tag{C.28}$$

which is $O_p^+(1)$ by Assumption T-G. Hence, by combining results, we have

$$\widehat{\boldsymbol{G}}^{(b,2,1)}_{\widehat{\xi}\widehat{\tau}}(\ell_G, m_G) \leq O_p^+\left(\left(\frac{m_G}{n}\right)^{d_x}\frac{1}{\ell_G^{1+\epsilon}}\right), \quad \widehat{\boldsymbol{G}}^{(b,1,1)}_{\widehat{\xi}\widehat{\tau}}(\ell_G, m_G) \leq O_p^+\left(\left(\frac{m}{n}\right)^{d_x}\frac{1}{\ell^{1+\epsilon}}\right), \tag{C.29}$$

which, together with the remaining bounds for the case *without cointegration*, provides the requisite result for the case *with cointegration*, i.e., scenario (iv), thereby concluding the proof. □

C.2 Proof of Lemma B.2

Proof. First, for **(a)**, we may use $d_1 = b$, (C.5) and (C.6) to write

$$w^{(2)}_{\widehat{e}}(\lambda_j) = \lambda_j^{d_1} e^{-(\pi/2)\mathrm{i}d_1} w_\eta(\lambda_j) + O_p\big(\lambda_j^{d_1}\ln(n) m_d^{-1/2}\big) + O_p\big(\lambda_j^{d_1}\ln(n) j^{-1/2}\big) + O_p\big(n^{-d_1-1}\big). \tag{C.30}$$

Moreover, let us define the related long-run covariance measure,

$$\widetilde{\boldsymbol{G}}^{(2)}_{\widetilde{e}\widetilde{e}}(\ell_G, m_G) \equiv \frac{1}{m_G - \ell_G + 1}\sum_{j=\ell_G}^{m_G} \lambda_j^{2d_1}\Re\Big(w_\eta(\lambda_j)\bar{w}_\eta(\lambda_j)\Big), \tag{C.31}$$

and subsequently make the error decomposition,

$$\begin{aligned}
&\widehat{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(\ell_G, m_G) - \widetilde{\boldsymbol{G}}^{(2)}_{\widetilde{e}\widetilde{e}}(\ell_G, m_G) \leq \frac{K\ln(n)}{m_G}\sum_{j=\ell_G}^{m_G} O_p^+\big(\lambda_j^{2d_1}\big(j^{-1/2}+m_d^{-1/2}\big)\big) + \frac{K}{m_G n^{d_1+1}}\sum_{j=\ell_G}^{m_G} O_p^+\big(\lambda_j^{d_1}\big)\\
&\leq \ln(n)\left(\frac{m_G}{n}\right)^{2d_1}\left(\frac{m_G^\epsilon}{m_G^{1/2}}+\frac{m_G^\epsilon}{m_d^{1/2}}\right)\sum_{j=\ell_G}^{m_G} O_p^+\left(\left(\frac{j}{m_G}\right)^{2d_1} j^{-(1+\epsilon)}\right)\\
&\quad+\left(\frac{m_G}{n}\right)^{d_1}\frac{m_G^\epsilon}{n}\sum_{j=\ell_G}^{m_G} O_p^+\left(\left(\frac{j}{m_G}\right)^{d_1} j^{-(1+\epsilon)}\right)\\
&\leq O_p^+\left(\frac{\ln(n)}{\ell_G^{1+\epsilon}}\left(\frac{m_G}{n}\right)^{2d_1}\left(\frac{m_G^\epsilon}{m_G^{1/2}}+\frac{m_G^\epsilon}{m_d^{1/2}}\right)\right) + O_p^+\left(\left(\frac{m_G}{n}\right)^{d_1}\frac{m_G^\epsilon}{n\,\ell_G^{1+\epsilon}}\right),
\end{aligned} \tag{C.32}$$

for some arbitrarily small $\epsilon > 0$, using Varneskov (2017, Lemma C.4) and that the remaining cross-product error terms arising from the product of the decomposition in (C.30) are of strictly lower asymptotic order by the tuning parameters being $\ell_G \asymp n^{\nu_G}$, $m_G \asymp n^{\kappa_G}$ and $m_d \asymp n^{\varrho}$, with $0 < \nu_G < \kappa_G < \varrho \leq 1$ in Assumptions F and T-G. Together with $0 \leq d_1 \leq 1$, this implies

$$\lambda_{m_G}^{-2d_1}\Big(\widehat{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(\ell_G, m_G) - \widetilde{\boldsymbol{G}}^{(2)}_{\widetilde{e}\widetilde{e}}(\ell_G, m_G)\Big) \leq o_p^+(1). \tag{C.33}$$

Hence, we continue by examining $\widetilde{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(\ell_G, m_G)$. By definition, we have

$$\widetilde{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(\ell_G, m_G) - \widetilde{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(1, m_G) = -\sum_{j=1}^{\ell_G - 1} \frac{\lambda_j^{2d_1} \Re\Big(w_\eta(\lambda_j)\bar{w}_\eta(\lambda_j)\Big)}{m_G - \ell_G + 1} = \frac{-\ell_G \widetilde{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(1, \ell_G - 1)}{m_G - \ell_G + 1}. \tag{C.34}$$

Let $l \in \{\ell_G - 1, m_G\}$ be either of the two generic sequences of integers, then we adopt exactly the same arguments used to establish Christensen & Varneskov (2017, Equation (B.7)) (see also Robinson & Marinucci (2003, p. 361) and the steps for Lobato (1997, Theorem 1)) to show

$$\lambda_l^{-2d_1} \left| \widetilde{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(1, l) - G_{\eta\eta}/(1 + 2d_1) \right| \leq o_p^+(1), \tag{C.35}$$

noting that $\widetilde{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(1, l)$ corresponds to $\lambda_m^{-1} \boldsymbol{F}^*_{zz}(1, m)$ in their notation, and $b = d_1$ to $-d_i$. Hence, by combining equations (C.34) and (C.35), we can establish the following bound,

$$\lambda_{m_G}^{-2d_1} \left(\widetilde{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(\ell_G, m_G) - \widetilde{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(1, m_G) \right) \leq O_p^+ \left(\left(\frac{\lambda_{\ell_G}}{\lambda_{m_G}} \right)^{2d_1} \frac{\ell_G}{m_G} \right) = o_p^+(1). \tag{C.36}$$

The requisite result, thus, follows by combining results (C.34), (C.35) and (C.36).

Next, for **(b)**, we will study the properties of $\widehat{\boldsymbol{G}}^{(d_1,1)}_{\widehat{\eta}\widehat{\eta}}(\ell_G, m_G)$ under the scenarios (ii)-(iii) as well as the cointegration setting (iv). To this end, let us write

$$\widehat{\eta}_t^{(d_1,1)} = \widehat{\eta}_t^{(d_1,1,1)} + \widehat{\eta}_t^{(d_1,1,2)} + \widehat{\eta}_t^{(d_1,1,3)} + \widehat{\eta}_t^{(d_1,1,4)} \tag{C.37}$$

where, by addition and subtraction, the components are defined as,

$$\begin{aligned} &\widehat{\eta}_t^{(d_1,1,1)} = (1-L)^{\widehat{d}_1} \xi_{t-1}^{(-d_1)}, \quad \widehat{\eta}_t^{(d_1,1,2)} = \widehat{\boldsymbol{\mathcal{B}}}(\ell, m)' \big(\boldsymbol{u}_{t-1} - \widehat{\boldsymbol{u}}_{t-1}\big), \quad \widehat{\eta}_t^{(d_1,1,3)} = \big(\boldsymbol{\mathcal{B}} - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m)\big)' \boldsymbol{u}_{t-1}, \\ &\widehat{\eta}_t^{(d_1,1,4)} = \widehat{e}_t^{(1)} - \boldsymbol{\mathcal{B}}' \boldsymbol{u}_{t-1} = (1-L)^{\widehat{d}_1} a + \boldsymbol{\mathcal{B}}' \left(\boldsymbol{\mathcal{Q}}(L)(1-L)^{\widehat{d}_1} \boldsymbol{x}_{t-1} - \boldsymbol{u}_{t-1} \right). \end{aligned}$$

This resembles the decomposition in the proof of AVOA (2020, Lemma A.9(b)) (cf., their equation (A.77)) and we rely on similar arguments. Next, we will establish results for the discrete Fourier transforms of each term in the decomposition. First, by AVOA (2020, Lemma A.9(a)), we have

$$w_{\widehat{\eta}}^{(d_1,1,1)}(\lambda_j) = \begin{cases} w_\xi(\lambda_j) \left(1 + O_p\big(\ln(n)/\sqrt{m_d}\big)\right) & \text{under models (ii) and (iii),} \\ 0 & \text{under model (iv).} \end{cases} \tag{C.38}$$

Second, by combining $\widehat{\boldsymbol{\mathcal{B}}}(\ell, m) = O_p(1)$ and (C.18), we have

$$w_{\widehat{\eta}}^{(d_1,1,2)}(\lambda_j) = O_p\left(\frac{n^{1/2 - \underline{d}_x}}{j^{1 - \underline{d}_x}} \right) + O_p\left(\frac{\ln(n) n^{1/2}}{m_d^{1/2} j} \right). \tag{C.39}$$

Third, by applying Theorem 1 and (C.18), we readily have,

$$w_{\widehat{\eta}}^{(d_1,1,3)}(\lambda_j) = \begin{cases} O_p\big(1/\sqrt{m}\big) & \text{under models (ii) and (iii),} \\ O_p\big(\lambda_m^{d_1}/\sqrt{m}\big) & \text{under model (iv).} \end{cases} \tag{C.40}$$

Fourth, by applying equation (A.63) and Lemmas A.6(a)-(c) of AVOA (2020) (as on their page 32), it follows that,

$$w_{\widehat{\eta}}^{(d_1,1,4)}(\lambda_j) = O_p\left(\frac{n^{1/2-d_1}}{j^{1-d_1}}\right). \tag{C.41}$$

Hence, using $b = d_1 \leq \underline{d}_x$, we may combine results to show,

$$w_{\widehat{\eta}}^{(d_1,1)}(\lambda_j) = w_\xi(\lambda_j) + O_p\big(\ln(n)/\sqrt{m_d}\big) + O_p\big(1/\sqrt{m}\big) + O_p\left(\frac{n^{1/2-d_1}}{j^{1-d_1}}\right) + O_p\left(\frac{\ln(n)n^{1/2}}{m_d^{1/2}j}\right) \tag{C.42}$$

under models (ii)-(iii), and, similarly, that

$$w_{\widehat{\eta}}^{(d_1,1)}(\lambda_j) = O_p\big(\lambda_m^{d_1}/\sqrt{m}\big) + O_p\left(\frac{n^{1/2-d_1}}{j^{1-d_1}}\right) + O_p\left(\frac{\ln(n)n^{1/2}}{m_d^{1/2}j}\right) \tag{C.43}$$

under model (iv). We are now ready to study the asymptotic properties of the long-run covariance estimate $\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\eta}}^{(d_1,1)}(\ell_G, m_G)$ in the two inference regimes, with and without cointegration.

The case without cointegration. The discrete Fourier transform in (C.42) allows us to write,

$$\begin{aligned} &\widehat{\boldsymbol{G}}_{\widehat{\eta}\widehat{\eta}}^{(d_1,1)}(\ell_G, m_G) - \widehat{\boldsymbol{G}}_{\xi\xi}(\ell_G, m_G) \leq O_p^+\big(\ln(n)/\sqrt{m_d}\big) + O_p^+\big(1/\sqrt{m}\big) \\ &+ \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O_p^+\left(\frac{n^{1/2}}{j}\left(\lambda_j^{d_1} + \frac{\ln(n)}{m_d^{1/2}}\right)\right) + \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O_p^+\left(\frac{n}{j^2}\left(\lambda_j^{2d_1} + \frac{\ln(n)^2}{m_d} + \frac{\ln(n)\lambda_j^{d_1}}{m_d^{1/2}}\right)\right) \\ &\leq O_p^+\big(\ln(n)/\sqrt{m_d}\big) + O_p^+\big(1/\sqrt{m}\big) + O_p^+\left(\frac{n^{1/2}}{m_G^{1-\epsilon}\ell_G^{1+\epsilon}}\left(\left(\frac{m_G}{n}\right)^{d_1} + \frac{\ln(n)}{m_d^{1/2}}\right)\right) \\ &+ O_p^+\left(\frac{n}{m_G\ell_G^2}\left(\left(\frac{m_G}{n}\right)^{2d_1} + \frac{\ln(n)^2}{m_d} + \frac{\ln(n)}{m_d^{1/2}}\left(\frac{m_G}{n}\right)^{d_1}\right)\right) \end{aligned} \tag{C.44}$$

similarly to (C.28), for some arbitrarily small $\epsilon > 0$, using, again, Varneskov (2017, Lemma C.4) and that the remaining cross-product terms of the errors are of strictly lower order by the tuning parameters satisfying $\ell_G \asymp n^{\nu_G}$, $m_G \asymp n^{\kappa_G}$ and $m_d \asymp n^{\varrho}$, with $0 < \nu_G < \kappa_G < \varrho \leq 1$ in Assumptions F and T-G. Next, as for equation (C.34), we have, by definition,

$$\left|\widehat{\boldsymbol{G}}_{\xi\xi}^{(d_1,1)}(\ell_G, m_G) - \widehat{\boldsymbol{G}}_{\xi\xi}(1, m_G)\right| = \left|\frac{\ell_G \widehat{\boldsymbol{G}}_{\xi\xi}(1, \ell_G - 1)}{m_G - \ell_G + 1}\right| \leq O_p^+\big(\ell_G/m_G\big). \tag{C.45}$$

Finally, since Christensen & Varneskov (2017, Lemma 6) provides $|\widehat{\boldsymbol{G}}_{\xi\xi}(1,m_G) - G_{\xi\xi}| \leq o_p^+(1)$, we can combine this with (C.44), (C.45) and the triangle inequality to show $|\widehat{\boldsymbol{G}}^{(d_1,1)}_{\widehat{\eta}\widehat{\eta}}(\ell_G,m_G) - G_{\xi\xi}| \leq o_p^+(1)$.

The case with cointegration. The discrete Fourier transform in (C.43) allows us to write,

$$\begin{aligned}
&\widehat{\boldsymbol{G}}^{(d_1,1)}_{\widehat{\eta}\widehat{\eta}}(\ell_G,m_G) \leq O_p^+\big(\lambda_m^{2d_1}/m\big) \\
&+ \frac{K\lambda_m^{d_1}}{m_G\sqrt{m}}\sum_{j=\ell_G}^{m_G} O_p^+\left(\frac{n^{1/2}}{j}\left(\lambda_j^{d_1} + \frac{\ln(n)}{m_d^{1/2}}\right)\right) + \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O_p^+\left(\frac{n}{j^2}\left(\lambda_j^{2d_1} + \frac{\ln(n)^2}{m_d} + \frac{\ln(n)\lambda_j^{d_1}}{m_d^{1/2}}\right)\right) \\
&\leq O_p^+\left(\left(\frac{m}{n}\right)^{2d_1}\frac{1}{m}\right) + O_p^+\left(\frac{n^{1/2}}{m_G^{1-\epsilon}\ell_G^{1+\epsilon}\sqrt{m}}\left(\frac{m}{n}\right)^{d_1}\left(\left(\frac{m_G}{n}\right)^{d_1} + \frac{\ln(n)}{m_d^{1/2}}\right)\right) \\
&+ O_p^+\left(\frac{n}{m_G\ell_G^2}\left(\left(\frac{m_G}{n}\right)^{2d_1} + \frac{\ln(n)^2}{m_d} + \frac{\ln(n)}{m_d^{1/2}}\left(\frac{m_G}{n}\right)^{d_1}\right)\right), \qquad \text{(C.46)}
\end{aligned}$$

using the same arguments as for (C.44). Now, by invoking the regularity conditions in Assumption T-G, we have $\lambda_{m_G}^{-2d_1}\widehat{\boldsymbol{G}}^{(d_1,1)}_{\widehat{\eta}\widehat{\eta}}(\ell_G,m_G) \leq o_p^+(1)$, providing the requisite result.

For **(c)**, since we have by the Cauchy-Schwarz inequality,

$$\left|\widehat{\boldsymbol{G}}^{(d_1,1,2)}_{\widehat{\eta}\widehat{e}}(\ell_G,m_G)\right| \leq \sqrt{\widehat{\boldsymbol{G}}^{(d_1,1)}_{\widehat{\eta}\widehat{\eta}}(\ell_G,m_G)\widehat{\boldsymbol{G}}^{(2)}_{\widehat{e}\widehat{e}}(\ell_G,m_G)}, \qquad \text{(C.47)}$$

the convergence results follow by invoking **(a)** and **(b)**, concluding the proof. □

C.3 Proof of Lemma B.3

Proof. **(a)** follows by the Taylor expansion in equation (A.63) of AVOA (2020) in conjunction with their Lemmas A.6(a)-(c), since $\gamma_x > 0$. **(b)-(d)** follow by Lemmas A.8 and A.9(a) in AVOA (2020). □

C.4 Proof of Lemma B.4

Proof. First, for **(a)**, we have by Lemma B.3 that,

$$w_{\check{e}}(\lambda_j) = O_p\big((j/n)^{\psi}\big) + O_p\big((j/n)^{\psi}n^{1/2}j^{-1}\big) + O_p\big(\ln(n)m_d^{-1/2}n^{1/2}j^{-1}\big), \qquad \text{(C.48)}$$

since $\psi \geq 0$ and $\psi \leq \gamma_x$. Hence, by (C.3), we can write $\widehat{\boldsymbol{F}}^c_{\widehat{u}\check{e}}(\ell,m) = \widehat{\boldsymbol{F}}_{\widehat{u}\check{e}}(\ell,m) + \widehat{\boldsymbol{F}}_{\widehat{c}\check{e}}(\ell,m)$, where,

$$\begin{aligned}
\widehat{\boldsymbol{F}}_{\widehat{c}\check{e}}(\ell,m) &\leq \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+\left(\lambda_j^{\underline{d}_x+\psi}\right) + \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+\left(\lambda_j^{\underline{d}_x+\psi}\frac{n^{1/2}}{j}\right) + \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+\left(\frac{\ln(n)n^{1/2}}{m_d^{1/2}j}\right) \\
&\leq O_p^+\left(\left(\frac{m}{n}\right)^{1+\underline{d}_x+\psi}\frac{1}{\ell^{1+\epsilon}}\right) + O_p^+\left(\left(\frac{m}{n}\right)^{\underline{d}_x+\psi}\frac{m^\epsilon}{n^{1/2}\ell^{1+\epsilon}}\right) + O_p^+\left(\frac{m^\epsilon}{m_d^{1/2}n^{1/2}\ell^{1+\epsilon}}\right), \qquad \text{(C.49)}
\end{aligned}$$

which provides the requisite result, since the second and third asymptotic bound are dominated by the first bound when $\sqrt{n}/m \to 0$, $m/m_d \to 0$, $\underline{d}_x \leq 1$ and $\psi \leq 1$.

For **(b)**. The proof proceeds similarly to that of B.1(e). First, let us write $\breve{\eta}_t^{(d_1)} = \breve{e}_t - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x)'\widehat{\boldsymbol{u}}_{t-1}$, where $\widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x)$ is defined in equation (C.62) and corresponds to $\widehat{\boldsymbol{\mathcal{B}}}_c(\ell, m, \widehat{\gamma}_x)$, but free from endogeneity-related bias. Moreover, using the notation from equation (B.4), define,

$$\breve{e}_t^{(1)} \equiv \breve{e}_t^{(1)} + \breve{e}_t^{(2)} + \breve{e}_t^{(3)}, \quad \breve{e}_t^{(2)} \equiv \breve{e}_t^{(4)}, \quad \breve{\eta}_t^{(d_1,1)} = \breve{e}_t^{(1)} - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x)'\widehat{\boldsymbol{u}}_{t-1}, \quad \breve{\eta}_t^{(d_1)} = \breve{\eta}_t^{(d_1,1)} + \breve{e}_t^{(2)}, \tag{C.50}$$

such that, with $\breve{\tau}_{t-1}^{(1)} = (\widehat{\boldsymbol{\mathcal{B}}}_c(\ell, m, \widehat{\gamma}_x) - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x))'\widehat{\boldsymbol{u}}_{t-1}^c$ and $\breve{\tau}_{t-1}^{(2)} = \widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x)\widehat{\boldsymbol{c}}_{t-1}$, we have,

$$\breve{\eta}_t^{(d_1,c)} = \breve{e}_t - \widehat{\boldsymbol{\mathcal{B}}}_c(\ell, m, \widehat{\gamma}_x)'\widehat{\boldsymbol{u}}_{t-1}^c = \breve{\eta}_t^{(d_1,1)} + \breve{e}_t^{(2)} - \breve{\tau}_{t-1}^{(1)} - \breve{\tau}_{t-1}^{(2)} = \breve{\eta}_t^{(d_1)} - \breve{\tau}_{t-1}^{(1)} - \breve{\tau}_{t-1}^{(2)}. \tag{C.51}$$

Next, using these, we make the decomposition,

$$\begin{aligned}\widehat{\boldsymbol{G}}_{\breve{\eta}\breve{\eta}}^{(d_1,c)}(\ell_G, m_G) - \widehat{\boldsymbol{G}}_{\breve{\eta}\breve{\eta}}^{(d_1)}(\ell_G, m_G) &= \widehat{\boldsymbol{G}}_{\breve{\tau}\breve{\tau}}^{(1,1)}(\ell_G, m_G) + \widehat{\boldsymbol{G}}_{\breve{\tau}\breve{\tau}}^{(2,2)}(\ell_G, m_G) + 2\widehat{\boldsymbol{G}}_{\breve{\tau}\breve{\tau}}^{(1,2)}(\ell_G, m_G) \\ &\quad - 2\widehat{\boldsymbol{G}}_{\breve{\eta}\breve{\tau}}^{(d_1,1)}(\ell_G, m_G) - 2\widehat{\boldsymbol{G}}_{\breve{\eta}\breve{\tau}}^{(d_1,2)}(\ell_G, m_G),\end{aligned} \tag{C.52}$$

where the first three terms are long-run (co)variance estimates for $\breve{\tau}_{t-1}^{(1)}$ and $\breve{\tau}_{t-1}^{(2)}$, and the final two terms are their respective long-run covariances with $\breve{\eta}_t^{(d_1)}$. Next, let us consider,

$$w_{\breve{\tau}}^{(1)}(\lambda_j) = \left(\widehat{\boldsymbol{\mathcal{B}}}_c(\ell, m, \widehat{\gamma}_x) - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x)\right)\left(\boldsymbol{w}_{\widehat{u}}(\lambda_j) + \boldsymbol{w}_{\widehat{c}}(\lambda_j)\right) \leq O_p^+\left((m/n)^{\underline{d}_x + \psi}/\ell^{1+\epsilon}\right), \tag{C.53}$$

using equations (C.3), (C.18) and (B.8). Hence, we readily have

$$\widehat{\boldsymbol{G}}_{\breve{\tau}\breve{\tau}}^{(1,1)}(\ell_G, m_G) \leq O_p^+\left((m/n)^{2(\underline{d}_x+\psi)}/\ell^{2(1+\epsilon)}\right). \tag{C.54}$$

Similarly, by Theorem 3 and (C.18), $w_{\breve{\tau}}^{(2)}(\lambda_j) \leq O_p^+\left(\lambda_m^{\psi}\lambda_j^{\underline{d}_x}\right)$ such that, by Assumption T-G-d_1,

$$\widehat{\boldsymbol{G}}_{\breve{\tau}\breve{\tau}}^{(2,2)}(\ell_G, m_G) \leq \frac{\lambda_m^{2\psi}K}{m_G}\sum_{j=\ell_G}^{m_G} O_p^+\left(\lambda_j^{2\underline{d}_x}\right) \leq O_p^+\left((m/n)^{2(\underline{d}_x+\psi)}m^{\epsilon}/\ell_G^{1+\epsilon}\right). \tag{C.55}$$

Next, since $\widehat{\boldsymbol{G}}_{\breve{\eta}\breve{\eta}}^{(d_1)}(\ell_G, m_G) = O_p\left((m_G/n)^{2\psi}\right)$ by Lemma B.9(c), the Cauchy-Schwarz inequality yields,

$$\begin{aligned}\widehat{\boldsymbol{G}}_{\breve{\tau}\breve{\tau}}^{(1,2)}(\ell_G, m_G) &\leq O_p^+\left((m/n)^{2(\underline{d}_x+\psi)}m^{\epsilon/2}/(\ell^{1+\epsilon}\ell_G^{(1+\epsilon)/2})\right), \\ \widehat{\boldsymbol{G}}_{\breve{\eta}\breve{\tau}}^{(d_1,1)}(\ell_G, m_G) &\leq O_p^+\left((m/n)^{2\psi+\underline{d}_x}/\ell^{1+\epsilon}\right), \\ \widehat{\boldsymbol{G}}_{\breve{\eta}\breve{\tau}}^{(d_1,2)}(\ell_G, m_G) &\leq O_p^+\left((m/n)^{2\psi+\underline{d}_x}m^{\epsilon/2}/\ell_G^{(1+\epsilon)/2}\right).\end{aligned}$$

Hence, since $0 < \underline{d}_x \leq 1$ and $\psi \geq 0$, the dominant asymptotic bounds are provided by the covariance terms $\widehat{\boldsymbol{G}}_{\breve{\eta}\breve{\tau}}^{(d_1,1)}(\ell_G, m_G)$ and $\widehat{\boldsymbol{G}}_{\breve{\eta}\breve{\tau}}^{(d_2,2)}(\ell_G, m_G)$, thereby establishing the requisite result. □

C.5 Proof of Lemma B.5

Proof. First, for **(a)**, we may combine (C.18) and Lemma B.3(a) to write,

$$\widehat{\boldsymbol{F}}^{(1)}_{\widehat{u}\breve{e}}(\ell,m) \leq \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+\left(\lambda_j^{\gamma_x}\frac{n^{1/2}}{j}\right) + \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+\left(\lambda_j^{\gamma_x+\underline{d}_x}\frac{n}{j^2}\right) + \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+\left(\lambda_j^{\gamma_x}\frac{\ln(n)n}{m_d^{1/2}j^2}\right)$$
$$\leq O_p^+\left(\left(\frac{m}{n}\right)^{\gamma_x}\frac{m^{\epsilon}}{n^{1/2}\ell^{1+\epsilon}}\right) + O_p^+\left(\left(\frac{m}{n}\right)^{\gamma_x+\underline{d}_x}\frac{1}{\ell^2}\right) + O_p^+\left(\left(\frac{m}{n}\right)^{\gamma_x}\frac{\ln(n)}{m_d^{1/2}\ell^2}\right). \tag{C.56}$$

Since the third term is dominated by the second as $m/m_d \to 0$, $\sqrt{n}/m \to 0$ and $\underline{d}_x \leq 1$, this gives the requisite bound. Second, for **(b)**, we may combine (C.18) and Lemma B.3(b) to write,

$$\widehat{\boldsymbol{F}}^{(1)}_{\widehat{u}\breve{e}}(\ell,m) - \widehat{\boldsymbol{F}}^{(2,1)}_{\widehat{u}\breve{e}}(\ell,m) \leq \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+\left(\frac{n^{1/2}}{j}\left(\lambda_j^{\psi} + \frac{\ln(n)}{\sqrt{m_d}} + \lambda_j^{\psi+\underline{d}_x} + \lambda_j^{\psi}\frac{\ln(n)}{\sqrt{m_d}}\right)\right)$$
$$+ \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+\left(\frac{n}{j^2}\left(\lambda_j^{\psi+\underline{d}_x} + \frac{\ln(n)^2}{m_d} + \lambda_j^{\underline{d}_x}\frac{\ln(n)}{\sqrt{m_d}} + \lambda_j^{\psi}\frac{\ln(n)}{\sqrt{m_d}}\right)\right)$$
$$\leq O_p^+\left(\frac{m^{\epsilon}}{n^{1/2}\ell^{1+\epsilon}}\left(\left(\frac{m}{n}\right)^{\psi} + \frac{\ln(n)}{\sqrt{m_d}}\right)\right) +$$
$$+ O_p^+\left(\frac{1}{\ell^2}\left(\left(\frac{m}{n}\right)^{\psi+\underline{d}_x} + \frac{\ln(n)^2}{m_d} + \left(\frac{m}{n}\right)^{\underline{d}_x}\frac{\ln(n)}{\sqrt{m_d}} + \left(\frac{m}{n}\right)^{\psi}\frac{\ln(n)}{\sqrt{m_d}}\right)\right)$$
$$= O_p^+\left(\frac{m^{\epsilon}}{n^{1/2}\ell^{1+\epsilon}}\left(\frac{m}{n}\right)^{\psi}\right) + O_p^+\left(\frac{1}{\ell^2}\left(\frac{m}{n}\right)^{\psi+\underline{d}_x}\right) \tag{C.57}$$

using, similarly, $m/m_d \to 0$, $\sqrt{n}/m \to 0$ and $0 \leq \psi \leq 1$ for the final equality. This gives the requisite bound. Finally, **(c)** and **(d)** follow by the same arguments as **(b)**, concluding the proof. ☐

C.6 Proof of Lemma B.6

Proof. First, for **(a)**, recall that,

$$\widehat{\boldsymbol{F}}^{(2,1)}_{\widehat{u}\breve{e}}(\ell,m) = \frac{2\pi}{n}\sum_{j=\ell}^{m}\lambda_j^{\psi}\Re\big(e^{(\pi/2)\psi \mathrm{i}}\boldsymbol{w}_u(\lambda_j)\overline{\boldsymbol{w}}_u(\lambda_j)\big)\boldsymbol{\mathcal{B}}.$$

Hence, using $\Re\big(e^{(\pi/2)\psi\mathrm{i}}\big) = \cos(\pi\psi/2)$, we seek to establish an error bound for the decomposition,

$$\lambda_m^{-1-\psi}\big(\widehat{\boldsymbol{F}}^{(2,1)}_{\widehat{u}\breve{e}}(\ell,m) - \widetilde{\boldsymbol{F}}^{(2,1)}_{\widehat{u}\breve{e}}(\ell,m)\big) = \frac{\cos(\psi\pi/2)}{m}\sum_{j=\ell}^{m}\Re\big(\boldsymbol{I}_{uu}(\lambda_j)\big)\boldsymbol{\mathcal{B}}\left(\left(\frac{j}{m}\right)^{\psi} - \frac{1}{1+\psi}\right)$$
$$= \frac{\cos(\psi\pi/2)}{m}\sum_{j=\ell}^{m}\boldsymbol{G}_{uu}\boldsymbol{\mathcal{B}}\left(\left(\frac{j}{m}\right)^{\psi} - \frac{1}{1+\psi}\right)$$

$$+\frac{\cos(\psi\pi/2)}{m}\sum_{j=\ell}^{m}\big(\Re\big(\boldsymbol{I}_{uu}(\lambda_j)\big)-\boldsymbol{G}_{uu}\big)\boldsymbol{\mathcal{B}}\left(\left(\frac{j}{m}\right)^{\psi}-\frac{1}{1+\psi}\right)\equiv\boldsymbol{\mathcal{E}}_1+\boldsymbol{\mathcal{E}}_2. \tag{C.58}$$

First, for $\boldsymbol{\mathcal{E}}_1$, we have

$$\|\boldsymbol{\mathcal{E}}_1\|\leq K\left|\frac{1}{m}\sum_{j=\ell}^{m}\left(\left(\frac{j}{m}\right)^{\psi}-\frac{1}{1+\psi}\right)\right|\leq O\big(m^{-1}\big), \tag{C.59}$$

using Shimotsu & Phillips (2005, Lemma 5.4). Next, for $\boldsymbol{\mathcal{E}}_2$, we may use Assumptions D1-D3 to invoke Theorem 2 and Corollary 1 in Hannan (1970, pp. 248-249)[2], providing,

$$\mathbb{E}\big[\Re\big(\boldsymbol{I}_{uu}(\lambda_j)\big)\big]=\Re\big(\boldsymbol{f}_{uu}(\lambda_j)\big)+O\big(n^{-1}\big),\quad \|\boldsymbol{f}_{uu}(\lambda_j)-\boldsymbol{G}_{uu}\|=O(\lambda_j^{\varpi}),$$
$$\mathbb{V}\big[\Re\big(\boldsymbol{I}_{uu}(\lambda_j)\big)\big]=\Re\big(\boldsymbol{f}_{uu}(\lambda_j)\otimes\boldsymbol{f}_{uu}(\lambda_j)\big)+O\big(n^{-1}\big),\quad \mathrm{Cov}\,\big[\Re\big(\boldsymbol{I}_{uu}(\lambda_j),\Re\big(\boldsymbol{I}_{uu}(\lambda_k)\big)\big]=O\big(n^{-1}\big),$$

for $j,k=1,\ldots,m$, $m/n\to 0$ and $j\neq k$. Hence, we obtain the following bound,

$$\mathbb{E}[\boldsymbol{\mathcal{E}}_2]\leq\frac{K}{m}\sum_{j=\ell}^{m}\big\|\boldsymbol{I}_{uu}(\lambda_j)-\boldsymbol{G}_{uu}\big\|\times\left|\left(\frac{j}{m}\right)^{\psi}-\frac{1}{1+\psi}\right|\leq O\big(n^{-1}\big)+O\big(\lambda_m^{\varpi}\big). \tag{C.60}$$

Moreover, using the (co-)periodogram second moment results, we have

$$\begin{aligned}\mathbb{V}[\boldsymbol{\mathcal{E}}_2]&=\cos(\pi\psi/2)^2\frac{1}{m^2}\sum_{j=\ell}^{m}\left(\left(\frac{j}{m}\right)^{\psi}-\frac{1}{1+\psi}\right)\sum_{k=\ell}^{m}\left(\left(\frac{k}{m}\right)^{\psi}-\frac{1}{1+\psi}\right)\\&\quad\times\mathbb{E}\big[(\boldsymbol{I}_{uu}(\lambda_j)-\boldsymbol{G}_{uu})\otimes(\boldsymbol{I}_{uu}(\lambda_k)-\boldsymbol{G}_{uu})\big]\\&\leq K\big(1+O\big(n^{-1}\big)\big)\left|\frac{1}{m}\sum_{j=\ell}^{m}\left(\left(\frac{j}{m}\right)^{\psi}-\frac{1}{1+\psi}\right)\right|^2+\frac{1}{m^2}\sum_{j=\ell}^{m}\left|\left(\frac{j}{m}\right)^{\psi}-\frac{1}{1+\psi}\right|^2O\big(\lambda_m^{\varpi}(1+\lambda_m^{\varpi})\big)\\&\leq O\big(m^{-2}\big)+O\big(m^{-1}\lambda_m^{\varpi}\big).\end{aligned} \tag{C.61}$$

The moment results for $\boldsymbol{\mathcal{E}}_2$ readily imply $\boldsymbol{\mathcal{E}}_2\leq O_p^{+}\big(m^{-1}\big)+O_p^{+}\big(m^{-1/2}\lambda_m^{\varpi/2}\big)+O_p^{+}\big(\lambda_m^{\varpi}\big)$, which, together with the asymptotic bound in equation (C.59), provides the requisite result.

Finally, **(b)** and **(c)** follow directly from AVOA (2020, Lemma A.3), concluding the proof. □

C.7 Proof of Lemma B.7

Proof. We have $\widehat{\boldsymbol{F}}_{\hat{u}\breve{e}}^{(i,1)}(\ell,m)-\widehat{\boldsymbol{F}}_{\hat{u}\breve{e}}^{(i,1)}(1,m)=-\widehat{\boldsymbol{F}}_{\hat{u}\breve{e}}^{(i,1)}(1,\ell-1)$, $i=3,4$. Hence, the results follow by applying AVOA (2020, Lemma A.1(d)) for both **(a)** and **(b)**. □

[2]See also Parzen (1957, Theorem 4) and Brockwell & Davis (1991, Theorem 10.3.2) for similar results.

C.8 Proof of Lemma B.8

Proof. First, recall that $\check{\boldsymbol{v}}_t = (\check{e}_t, \widehat{\boldsymbol{u}}'_{t-1})'$, then, by invoking Lemmas B.1(a) and B.4(a) as well as the continuous mapping theorem, while recalling that $\psi = \gamma_x - d_1$,

$$\sqrt{m}\lambda_m^{-\gamma_x}\left(\widehat{\boldsymbol{\mathcal{B}}}_c(\ell, m, \widehat{\gamma}_x) - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x)\right) \leq O_p^+\left((m/n)^{\underline{d}_x - d_1}\sqrt{m}/\ell^{1+\epsilon}\right), \tag{C.62}$$

for some arbitrarily small $\epsilon > 0$, where $\widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x) = \widehat{\boldsymbol{F}}_{\widehat{u}\widehat{u}}(\ell, m)^{-1}\widehat{\boldsymbol{F}}_{\widehat{u}\check{e}}(\ell, m)$. Hence, we continue working with the estimate without endogeneity, $\check{\boldsymbol{v}}_t$. Next, using the definitions in Lemma B.6, we have

$$\sqrt{m}\lambda_m^{-1-\gamma_x}\left(\widehat{\boldsymbol{F}}_{\widehat{u}\check{e}}(\ell, m) - \widetilde{\boldsymbol{F}}_{\widehat{u}\check{e}}^{(2,1)}(\ell, m) - \widehat{\boldsymbol{F}}_{\widehat{u}\check{e}}^{(3,1)}(\ell, m) - \widehat{\boldsymbol{F}}_{\widehat{u}\check{e}}^{(4,1)}(\ell, m)\right) = \sum_{i=1}^{6}\boldsymbol{\mathcal{A}}_i, \tag{C.63}$$

where the asymptotic bounds on the right-hand-side error terms are,

$$\boldsymbol{\mathcal{A}}_1 \leq O_p^+\left(\left(\frac{n}{m}\right)^{1/2+d_1}\frac{m^\epsilon}{\ell^{1+\epsilon}}\right), \quad \boldsymbol{\mathcal{A}}_2 \leq O_p^+\left(\left(\frac{m}{n}\right)^{\underline{d}_x - d_1}\frac{n}{m^{1/2}\ell^2}\right), \quad \boldsymbol{\mathcal{A}}_3 \leq O_p^+\left(\left(\frac{n}{m}\right)^{d_1}\frac{1}{m^{1/2}}\right),$$
$$\boldsymbol{\mathcal{A}}_4 \leq O_p^+\left(\left(\frac{m}{n}\right)^{\varpi/2 - d_1}\right), \quad \boldsymbol{\mathcal{A}}_5 \leq O_p^+\left(\left(\frac{m}{n}\right)^{\varpi - d_1} m^{1/2}\right), \quad \boldsymbol{\mathcal{A}}_6 \leq O_p^+\left(\left(\frac{n}{m}\right)^{d_1}\left(\frac{\ell}{m}\right)^{\psi}\frac{\ell^{1+\varpi}}{m^{1/2}n^{\varpi}}\right),$$

using Lemma B.5 for $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathcal{A}}_2$; Lemma B.6 for $\boldsymbol{\mathcal{A}}_3$, $\boldsymbol{\mathcal{A}}_4$ and $\boldsymbol{\mathcal{A}}_5$; and Lemma B.7 for $\boldsymbol{\mathcal{A}}_6$. These are all $o_p(1)$ by Assumption T-d_1, $0 < d_1 < 1$ and the mutual consistency condition. Moreover, we have

$$\begin{aligned}
\widehat{\boldsymbol{F}}_{\widehat{u}\widehat{u}}(\ell, m) - \widehat{\boldsymbol{F}}_{uu}(\ell, m) &\leq \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+\left(\frac{n^{1/2}}{j}\left(\lambda_j^{\underline{d}_x} + \frac{\ln(n)}{\sqrt{m_d}}\right)\right) \\
&\leq \frac{2\pi}{n}\sum_{j=\ell}^{m} O_p^+\left(\frac{n}{j^2}\left(\lambda_j^{2\underline{d}_x} + \lambda_j^{\underline{d}_x}\frac{\ln(n)}{\sqrt{m_d}} + \frac{\ln(n)^2}{m_d}\right)\right) \\
&\leq O_p^+\left(\left(\frac{m}{n}\right)^{\underline{d}_x}\frac{m^\epsilon}{n^{1/2}\ell^{1+\epsilon}}\right) + O_p^+\left(\left(\frac{m}{n}\right)^{2\underline{d}_x}\frac{1}{\ell^2}\right),
\end{aligned} \tag{C.64}$$

using (C.18), $\underline{d}_x \leq 1$, $m/m_d \to 0$ and $\sqrt{m}/n \to 0$. Hence

$$\sqrt{m}\lambda_m^{-1-\gamma_x}\left(\widehat{\boldsymbol{F}}_{\widehat{u}\widehat{u}}(\ell, m) - \widehat{\boldsymbol{F}}_{uu}(\ell, m)\right) \leq O_p^+\left(\left(\frac{n}{m}\right)^{1/2+\gamma_x - \underline{d}_x}\frac{m^\epsilon}{\ell^{1+\epsilon}}\right) + O_p^+\left(\left(\frac{m}{n}\right)^{2\underline{d}_x - \gamma_x}\frac{n}{m^{1/2}\ell^2}\right),$$

which is $o_p(1)$ using, again, the conditions in Assumption T-d_1. Furthermore, we have $\lambda_m^{-1}\widehat{\boldsymbol{F}}_{uu}(\ell, m) \xrightarrow{\mathbb{P}} \boldsymbol{G}_{uu}$ by AVOA (2020, Lemmas A.1 and A.2). Hence, we can combine results to write,

$$\sqrt{m}\,\lambda_m^{-\gamma_x}\left(\widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x) - \lambda_m^{\psi}c(\psi)\boldsymbol{\mathcal{B}}\right) = \sqrt{m}\,\lambda_m^{-\gamma_x}\widehat{\boldsymbol{F}}_{uu}(\ell, m)^{-1}\left(\widehat{\boldsymbol{F}}_{\widehat{u}\check{e}}^{(3,1)}(\ell, m) + \widehat{\boldsymbol{F}}_{\widehat{u}\check{e}}^{(4,1)}(\ell, m)\right) + o_p(1).$$

The final results in **(a)** and **(b)** follow by applying Lemmas B.6(b) and (c) with scale factors $\sqrt{m}\lambda_m^{-\psi}$ and $\sqrt{m}\lambda_m^{-\gamma_x}$, respectively, to the right-hand-side terms. In models (ii) and (iii), the limit for $\widehat{\boldsymbol{F}}_{\widehat{u}\check{e}}^{(3,1)}(\ell, m)$

dominates the corresponding for $\widehat{\boldsymbol{F}}^{(4,1)}_{\widehat{u}\check{e}}(\ell, m)$, which becomes a higher-order error. In contrast, in model (iv), we have $\widehat{\boldsymbol{F}}^{(3,1)}_{\widehat{u}\check{e}}(\ell, m) = \mathbf{0}$ and the limit is driven by $\widehat{\boldsymbol{F}}^{(4,1)}_{\widehat{u}\check{e}}(\ell, m)$. The requisite central limit theorems, thus, follow by the continuous mapping theorem and Slutsky's theorem.

Besides requiring additional tuning parameter restrictions in Assumption T-d_1 as well as the conditions $0 < d_1 < 1$ and $2d_1 < \varpi$, there are no differences between this and the corresponding treatment of the mutual consistency condition in Theorem 1, concluding this proof. □

C.9 Proof of Lemma B.9

Proof. First, **(a)** follows by the same arguments as Lemma B.2(a). Second, for **(b)**, we follow the same steps as in the proof of Lemma B.2(b), implying we will study the properties of $\widehat{\boldsymbol{G}}^{(d_1,1)}_{\check{\eta}\check{\eta}}(\ell_G, m_G)$ under the model scenarios (ii)-(iii) as well as the cointegration setting (iv). To this end, let us write

$$\check{\eta}_t^{(d_1,1)} = \check{\eta}_t^{(d_1,1,1)} + \check{\eta}_t^{(d_1,1,2)} + \check{\eta}_t^{(d_1,1,3)} + \check{\eta}_t^{(d_1,1,4)} \tag{C.65}$$

where, by addition and subtraction, the components are defined as,

$$\check{\eta}_t^{(d_1,1,1)} = (1-L)^{\widehat{\gamma}_x - \gamma_x}\xi_{t-1}^{(\psi)}, \quad \check{\eta}_t^{(d_1,1,2)} = \widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x)'\big(\boldsymbol{u}_{t-1} - \widehat{\boldsymbol{u}}_{t-1}\big),$$
$$\check{\eta}_t^{(d_1,1,3)} = \big(\boldsymbol{\mathcal{B}} - \widehat{\boldsymbol{\mathcal{B}}}(\ell, m, \widehat{\gamma}_x)\big)'\boldsymbol{u}_{t-1}, \quad \check{\eta}_t^{(d_1,1,4)} = (1-L)^{\widehat{\gamma}_x}a + \boldsymbol{\mathcal{B}}'\left((1-L)^{\widehat{\gamma}_x - d_1}\boldsymbol{u}_{t-1} - \boldsymbol{u}_{t-1}\right).$$

The proof for the case $\psi = 0$ follows by the same arguments as Lemma B.2(b). Hence, we focus on the case $\psi > 0$ and proceed by initially establishing results for the respective discrete Fourier transforms of each term in the decomposition. First, by Lemma B.3(c), we have

$$w_{\check{\eta}}^{(d_1,1,1)}(\lambda_j) = \begin{cases} \lambda_j^{\psi} e^{-(\pi/2)\psi \mathrm{i}} w_{\xi}(\lambda_j) + O_p\Big(\frac{n^{1/2-\psi}}{j^{1-\psi}}\Big) + O_p\Big(\frac{\ln(n) n^{1/2}}{m_d^{1/2} j}\Big), & \text{in models (ii) and (iii)}, \\ 0, & \text{in model (iv)}. \end{cases} \tag{C.66}$$

Second, by combining $\widehat{\boldsymbol{\mathcal{B}}}(\ell, m) = O_p(\lambda_m^{\psi})$ in Theorem 3 and equation (C.18), we have,

$$w_{\check{\eta}}^{(d_1,1,2)}(\lambda_j) = O_p\left(\lambda_m^{\psi} \frac{n^{1/2-\underline{d}_x}}{j^{1-\underline{d}_x}}\right) + O_p\left(\lambda_m^{\psi} \frac{\ln(n) n^{1/2}}{m_d^{1/2} j}\right). \tag{C.67}$$

Third, by applying Theorem 3 and equation (C.18), we have,

$$w_{\check{\eta}}^{(d_1,1,3)}(\lambda_j) = \begin{cases} O_p\big(\lambda_m^{\psi} m^{-1/2}\big), & \text{in model (ii)}, \\ \big(1 - c(\psi)\lambda_m^{\psi}\big)\boldsymbol{\mathcal{B}}' \boldsymbol{w}_u(\lambda_j) + O_p\big(\lambda_m^{\psi} m^{-1/2}\big), & \text{in model (iii)}, \\ \big(1 - c(\psi)\lambda_m^{\psi}\big)\boldsymbol{\mathcal{B}}' \boldsymbol{w}_u(\lambda_j) + O_p\big(\lambda_m^{\gamma_x} m^{-1/2}\big), & \text{in model (iv)}. \end{cases} \tag{C.68}$$

Fourth, by the same arguments as given when establishing Lemma B.2(a)-(b), we have

$$w_{\breve{\eta}}^{(d_1,1,4)}(\lambda_j) = \begin{cases} O_p\Big(\frac{n^{1/2-\gamma_x}}{j^{1-\gamma_x}}\Big), & \text{in model (ii)}, \\ -\big(1-\lambda_j^{\psi} e^{-(\pi/2)\psi \mathrm{i}}\big)\boldsymbol{\mathcal{B}}'\boldsymbol{w}_u(\lambda_j) + O_p\Big(\frac{n^{1/2-\psi}}{j^{1-\psi}}\Big) + O_p\Big(\frac{\ln(n)n^{1/2}}{m_d^{1/2} j}\Big), & \text{in (iii)-(iv)}. \end{cases} \tag{C.69}$$

Hence, in model (ii), it follows by $0 < \underline{d}_x < 1$, $\psi \geq 0$ and $\psi \leq \gamma_x$,

$$w_{\breve{\eta}}^{(d_1,1)}(\lambda_j) = \lambda_j^{\psi} e^{-(\pi/2)\psi \mathrm{i}} w_{\xi}(\lambda_j) + O_p\left(\frac{n^{1/2}}{j}\left(\lambda_j^{\psi} + \lambda_j^{\underline{d}_x}\lambda_m^{\psi} + \frac{\ln(n)}{m_d^{1/2}}\right)\right) + O_p\left(\frac{\lambda_m^{\psi}}{m^{1/2}}\right). \tag{C.70}$$

Similarly, in model (iii), we have

$$\begin{aligned} w_{\breve{\eta}}^{(d_1,1)}(\lambda_j) &= \lambda_j^{\psi} e^{-(\pi/2)\psi \mathrm{i}} w_{\xi}(\lambda_j) + O_p\left(\frac{n^{1/2}}{j}\left(\lambda_j^{\psi} + \lambda_j^{\underline{d}_x}\lambda_m^{\psi} + \frac{\ln(n)}{m_d^{1/2}}\right)\right) + O_p\left(\frac{\lambda_m^{\psi}}{m^{1/2}}\right) \\ &\quad + \left(\lambda_j^{\psi} e^{-(\pi/2)\psi \mathrm{i}} - c(\psi)\lambda_m^{\psi}\right)\boldsymbol{\mathcal{B}}'\boldsymbol{w}_u(\lambda_j), \end{aligned} \tag{C.71}$$

and model (iv) exhibits a DFT on the same form, but having $w_{\xi}(\lambda_j) = 0$ as well as the error term $O_p\big(\lambda_m^{\psi} m^{-1/2}\big)$ being replaced by one of order $O_p\big(\lambda_m^{\gamma_x} m^{-1/2}\big)$, with $\psi \leq \gamma_x$. Hence, we will explicitly treat model (iii) in the following since the same arguments may readily be applied to establish the corresponding results for models (ii) and (iv). To this end, define

$$\begin{aligned} \widetilde{G}_{\breve{\xi}\breve{\xi}}(\ell_G, m_G) &= \frac{1}{m_G - \ell_G + 1}\sum_{j=\ell_G}^{m_G} \lambda_j^{2\psi}\Re\big(I_{\xi\xi}(\lambda_j)\big), \\ \widetilde{B}_{uu}(\ell_G, m_G) &= \frac{1}{m_G - \ell_G + 1}\sum_{j=\ell_G}^{m_G}\left(\lambda_j^{2\psi} + c(\psi)^2\lambda_m^{2\psi} - 2\lambda_j^{\psi}\lambda_m^{\psi}\cos(\pi\psi/2)c(\psi)\right)\Re\Big(\boldsymbol{\mathcal{B}}'\boldsymbol{I}_{uu}(\lambda_j)\boldsymbol{\mathcal{B}}\Big). \end{aligned}$$

Moreover, since Assumptions D1-D3 and M together with Theorem 2 and Corollary 1 in Hannan (1970, pp. 248-249) imply $\mathbb{E}[\boldsymbol{I}_{u\xi}(\lambda_j)] = O(n^{-1})$ and $\mathbb{V}[\boldsymbol{I}_{u\xi}(\lambda_j)] = O(n^{-1})$, for $j = 1, \ldots, m$, we have,

$$\begin{aligned} &\widehat{G}_{\breve{\eta}\breve{\eta}}^{(d_1,1)}(\ell_G, m_G) - \widetilde{G}_{\breve{\xi}\breve{\xi}}(\ell_G, m_G) - \widetilde{B}_{uu}(\ell_G, m_G) \leq O_p^+\left(\lambda_m^{2\psi} m^{-1/2}\right) + O_p^+\left(\lambda_m^{2\psi} n^{-1/2}\right) \\ &\quad + \frac{K\lambda_m^{\psi}}{m_G}\sum_{j=\ell_G}^{m_G} O_p^+\left(\frac{n^{1/2}}{j}\left(\lambda_m^{\psi} + \frac{\ln(n)}{m_d^{1/2}}\right)\right) + \frac{K}{m_G}\sum_{j=\ell_G}^{m_G} O_p^+\left(\frac{n}{j^2}\left(\lambda_m^{2\psi} + \frac{\ln(n)^2}{m_d} + \frac{\ln(n)\lambda_m^{\psi}}{m_d^{1/2}}\right)\right) \\ &\leq O_p^+\left(\lambda_m^{2\psi} m^{-1/2}\right) + O_p^+\left(\frac{n^{1/2}}{m_G^{1-\epsilon}\ell_G^{1+\epsilon}}\left(\frac{m_G}{n}\right)^{\psi}\left(\left(\frac{m_G}{n}\right)^{\psi} + \frac{\ln(n)}{m_d^{1/2}}\right)\right) \\ &\quad + O_p^+\left(\frac{n}{m_G\ell_G^2}\left(\left(\frac{m_G}{n}\right)^{2\psi} + \frac{\ln(n)^2}{m_d} + \frac{\ln(n)}{m_d^{1/2}}\left(\frac{m_G}{n}\right)^{\psi}\right)\right), \end{aligned} \tag{C.72}$$

similarly to (C.28), for some arbitrarily small $\epsilon > 0$, using, again, $0 < \underline{d}_x \leq 1$, $\psi \geq 0$, Varneskov (2017, Lemma C.4) and that the remaining cross-product terms of the errors are of strictly lower order by the tuning parameters satisfying $\ell_G \asymp n^{\nu_G}$, $m_G \asymp n^{\kappa_G}$ and $m_d \asymp n^{\varrho}$, with $0 < \nu_G < \kappa_G < \varrho \leq 1$ in Assumptions F-d_1 and T-d_1. Hence, by invoking the latter, we have

$$\lambda_m^{-2\psi}\Big(\widehat{G}_{\breve{\eta}\breve{\eta}}^{(d_1,1)}(\ell_G, m_G) - \widetilde{G}_{\breve{\xi}\breve{\xi}}(\ell_G, m_G) - \widetilde{B}_{uu}(\ell_G, m_G)\Big) = o_p(1). \tag{C.73}$$

Finally, by the definition $c(\psi) = \cos(\pi\psi/2)/(1+\psi)$ and the same arguments for Lemma B.2(a),

$$\lambda_m^{-2\psi}\left(\widetilde{G}_{\breve{\xi}\breve{\xi}}(\ell_G, m_G) - G_{\psi\psi}/(1+2\psi)\right) = o_p(1), \tag{C.74}$$

$$\lambda_m^{-2\psi}\left(\widetilde{B}_{uu}(\ell_G, m_G) - \big(1/(1+2\psi) - c(\psi)^2\big)\boldsymbol{\mathcal{B}}'\boldsymbol{G}_{uu}\boldsymbol{\mathcal{B}}\right) = o_p(1). \tag{C.75}$$

This delivers the requisite convergence result for model (iii), and equivalent arguments establish the corresponding results for models (ii) and (iv), which appear as special cases of the limit.

For **(c)**, since we have by the Cauchy-Schwarz inequality,

$$\left|\widehat{\boldsymbol{G}}_{\breve{\eta}\breve{e}}^{(d_1,1,2)}(\ell_G, m_G)\right| \leq \sqrt{\widehat{\boldsymbol{G}}_{\breve{\eta}\breve{\eta}}^{(d_1,1)}(\ell_G, m_G)\widehat{\boldsymbol{G}}_{\breve{e}\breve{e}}^{(2)}(\ell_G, m_G)}, \tag{C.76}$$

the convergence results follow by invoking **(a)** and **(b)**, concluding the proof. □

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