Supplement to "Simultaneous Confidence Bands for Conditional Value-at-Risk and Expected Shortfall"

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Abstract

The supplementary materials consist of three parts. Section S1 presents a set of auxiliary lemmas that are necessary for the proofs of main theorems. Second S2 discusses the extension to nonparametric models. Section S3 includes the proof of Proposition C.1. Throughout the supplementary materials, we use the generic notation $X_{1:n} \leq \ldots \leq X_{n:n}$ to denote the order statistics of an untruncated sequence of random variables $\{X_t\}_{t=1}^n$. This is to distinguish from the notation $X_{(k)}$ in the main paper, which denotes the $(k+1)$-largest value of $\{X_t\}_{t=d_n}^n$, the truncated version of $\{X_t\}_{t=1}^n$. In addition, we use $\mathbb{I}(\cdot)$ to denote the indicator function.

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S1 Auxiliary Lemmas

Lemma S1.1. *Let ε be a random variable with distribution function $F_\varepsilon(x) = \Pr(\varepsilon \le x)$. Define the function U_ε be the left-continuous inverse of $1/(1-F_\varepsilon)$. Suppose that U_ε satisfies Assumption 3.4 (i) with $\gamma_R \in (0,1)$. Then for any $\epsilon > 0$ and $0 < \delta < 1/\gamma_R - \rho_R/\gamma_R - 1$, there exists $x_0 = x_0(\epsilon, \delta, \gamma_R, \rho_R)$ such that for all $x \ge x_0$,*

$$\begin{aligned} & \left| \frac{1}{A_R(1/(1-F_\varepsilon(x)))} \left[\frac{E(\varepsilon|\varepsilon > x)}{x} - \frac{1}{1-\gamma_R} \right] - \frac{1}{(1-\gamma_R)(1-\gamma_R-\rho_R)} \right| \\ \le \quad & \epsilon \left[\frac{1}{\rho_R(1-\gamma_R)} + \frac{1}{\gamma_R(1-\gamma_R-\rho_R-\delta\gamma_R)} \right]. \end{aligned}$$

Proof: By Theorem 2.3.9 of de Haan and Ferreira (2006), Assumption 3.4 (i) is equivalent to

$$\lim_{x\to\infty} \frac{\frac{1-F_\varepsilon(xy)}{1-F_\varepsilon(x)} - y^{-1/\gamma_R}}{A_0(x)} = y^{-1/\gamma_R} \frac{y^{\rho_R/\gamma_R} - 1}{\gamma_R \rho_R}$$

for all $y > 0$ with $A_0(x) = A_R(1/(1-F_\varepsilon(x)))$, which is eventually positive or negative. In addition, $\lim_{x\to\infty} A_0(x) = 0$. Define $\varepsilon_+ = \max(\varepsilon, 0)$, then the distribution of ε_+ is $F_+(x) = 0$ if $x < 0$ and $F_+(x) = F_\varepsilon(x)$ if $x \ge 0$. So $\bar{F}_+ = 1 - F_+$ satisfies that for all $y > 0$

$$\lim_{x\to\infty} \frac{\bar{F}_+(xy)/\bar{F}_+(x) - y^{-1/\gamma_R}}{A_{0+}(x)} = y^{-1/\gamma_R} \frac{y^{\rho_R/\gamma_R} - 1}{\rho_R/\gamma_R}$$

with $A_{0+}(x) = A_0(x)/\gamma_R^2$. It implies that $\bar{F}_+ \in 2RV_{-1/\gamma_R, \rho_R/\gamma_R}$. By Lemma 2.1 of Pan, Leng, and Hu (2013), for any $\epsilon > 0$ and $0 < \delta < 1/\gamma_R - \rho_R/\gamma_R - 1$, there exists $x_0 = x_0(\epsilon, \delta, \gamma_R, \rho_R)$ such that for all $x \ge x_0$ and $y \ge 1$,

$$\left| \frac{\bar{F}_+(xy)/\bar{F}_+(x) - y^{-1/\gamma_R}}{A_{0+}(x)} - y^{-1/\gamma_R} \frac{y^{\rho_R/\gamma_R} - 1}{\rho_R/\gamma_R} \right| \le \epsilon y^{-1/\gamma_R} \left[\left| \frac{y^{\rho_R/\gamma_R} - 1}{\rho_R/\gamma_R} \right| + y^{\rho_R/\gamma_R} \max(y^\delta, y^{-\delta}) \right]. \quad \text{(S1.1)}$$

Since

$$\begin{aligned} & \frac{1}{A_R(1/(1-F_\varepsilon(x)))} \left[\frac{E(\varepsilon|\varepsilon > x)}{x} - \frac{1}{1-\gamma_R} \right] \\ = \quad & \frac{1}{A_R(1/(1-F_\varepsilon(x)))} \left[\int_x^\infty \frac{\bar{F}_+(z)}{x\bar{F}_+(x)} dz - \frac{\gamma_R}{1-\gamma_R} \right] \\ = \quad & \frac{1}{A_R(1/(1-F_\varepsilon(x)))} \int_1^\infty \left[\bar{F}_+(xy)/\bar{F}_+(x) - y^{-1/\gamma_R} \right] dy, \end{aligned}$$

and

$$\int_1^\infty y^{-1/\gamma_R}\frac{y^{\rho_R/\gamma_R}-1}{\rho_R/\gamma_R}dy=\frac{\gamma_R^2}{(1-\gamma_R)(1-\gamma_R-\rho_R)},$$

it follows from (S1.1) that

$$\begin{aligned}
&\left|\frac{1}{A_R(1/(1-F_\varepsilon(x)))}\left[\frac{E(\varepsilon|\varepsilon>x)}{x}-\frac{1}{1-\gamma_R}\right]-\frac{1}{(1-\gamma_R)(1-\gamma_R-\rho_R)}\right|\\
\leq\;& \frac{1}{\gamma_R^2}\int_1^\infty\left|\frac{\bar{F}_+(xy)/\bar{F}_+(x)-y^{-1/\gamma_R}}{A_{0+}(x)}-y^{-1/\gamma_R}\frac{y^{\rho_R/\gamma_R}-1}{\rho_R/\gamma_R}\right|dy\\
\leq\;& \frac{\epsilon}{\gamma_R^2}\int_1^\infty\left|y^{-1/\gamma_R}\left[\left|\frac{y^{\rho_R/\gamma_R}-1}{\rho_R/\gamma_R}\right|+y^{\rho_R/\gamma_R}\max(y^\delta,y^{-\delta})\right]\right|dy\\
\leq\;& \epsilon\left[\frac{1}{\rho_R(1-\gamma_R)}+\frac{1}{\gamma_R(1-\gamma_R-\rho_R-\delta\gamma_R)}\right]
\end{aligned}$$

for all $x\geq x_0$. □

Lemmas S1.2 – S1.5 provide essential results for theoretical investigations of left- and right-tails simultaneously, which may be of independent interest.

Lemma S1.2. *Let $U_1,\ldots,U_n$ be i.i.d. from a standard uniform distribution, the uniform distribution on $[0,1]$. Denote $U_{1:n}\leq U_{2:n}\leq\cdots\leq U_{n:n}$ as the n-th order statistics of $\{U_i\}_{i=1}^n$. Then, as $n\to\infty$, $k_1\to\infty$, $k_2\to\infty$, $k_1/n\to 0$ and $k_2/n\to 0$,*

$$\begin{pmatrix}(U_{n-k_1:n}-b_{k_1,n})/a_{k_1,n}\\(U_{k_2+1:n}-b'_{k_2,n})/a'_{k_2,n}\end{pmatrix}$$

is asymptotically two-dimensional standard normal with

$$b_{k_1,n}=(n-k_1-2)/(n-2),\quad a_{k_1,n}=\sqrt{b_{k_1,n}(1-b_{k_1,n})/(n-2)},$$
$$b'_{k_2,n}=k_2/(n-2),\quad a'_{k_2,n}=\sqrt{b'_{k_2,n}(1-b'_{k_2,n})/(n-2)}.$$

Proof: The joint density of $U_{n-k_1:n}$ and $U_{k_2+1:n}$ is

$$f_{U_{n-k_1:n};U_{k_2+1:n}}(x,y)=\frac{n!}{k_1!(n-k_1-k_2-2)!k_2!}(1-x)^{k_1}(x-y)^{n-k_1-k_2-2}y^{k_2},$$

for $0<y<x<1$. By multivariate change of variables, the joint density of

$$\left((U_{n-k_1:n}-b_{k_1,n})/a_{k_1,n},(U_{k_2+1:n}-b'_{k_2,n})/a'_{k_2,n}\right)^\top$$

is given by

$$
\begin{aligned}
f(u,v) &= \frac{n!}{k_1!(n-k_1-k_2-2)!k_2!} a_{k_1,n} a'_{k_2,n} (1-b_{k_1,n})^{k_1} b'^{k_2}_{k_2,n} (b_{k_1,n}-b'_{k_2,n})^{n-k_1-k_2-2} \\
&\quad \times \left(1-\frac{a_{k_1,n}}{1-b_{k_1,n}}u\right)^{k_1} \left(1+\frac{a'_{k_2,n}}{b'_{k_2,n}}v\right)^{k_2} \left(1+\frac{a_{k_1,n}u-a'_{k_2,n}v}{b_{k_1,n}-b'_{k_2,n}}\right)^{n-k_1-k_2-2}. \qquad \text{(S1.2)}
\end{aligned}
$$

By Stirling's formula, we have that

$$
\begin{aligned}
&\frac{n!}{k_1!(n-k_1-k_2-2)!k_2!} a_{k_1,n} a'_{k_2,n} (1-b_{k_1,n})^{k_1} b'^{k_2}_{k_2,n} (b_{k_1,n}-b'_{k_2,n})^{n-k_1-k_2-2} \\
\sim\ &\frac{\sqrt{2\pi n} n^n e^{-n}}{\sqrt{2\pi} k_1^{k_1+1/2} e^{-k_1} \sqrt{2\pi} k_2^{k_2+1/2} e^{-k_2} \sqrt{2\pi} (n-k_1-k_2-2)^{n-k_1-k_2-3/2} e^{-(n-k_1-k_2-2)}} \\
&\times (n-2)^{-2} (n-k_1-2)^{1/2} (n-k_2-2)^{1/2} (1-b_{k_1,n})^{k_1+1/2} b'^{k_2+1/2}_{k_2,n} (b_{k_1,n}-b'_{k_2,n})^{n-k_1-k_2-2} \\
=\ &(2\pi)^{-1} e^{-2} \left(\frac{n}{n-2}\right)^n n^{1/2} (n-2)^{-1} (n-k_1-k_2-2)^{-1/2} (n-k_1-2)^{1/2} (n-k_2-2)^{1/2} \\
\to\ &(2\pi)^{-1}
\end{aligned}
$$

as $n, k_1, k_2 \to \infty$ under the condition that $k_1/n \to 0$ and $k_2/n \to 0$.

Note that $a_{k_1,n}/(1-b_{k_1,n}) = O(k_1^{-1/2})$, $a'_{k_2,n}/b'_{k_2,n} = O(k_2^{-1/2})$, $a_{k_1,n}/(b_{k_1,n}-b'_{k_2,n}) = O(k_1^{1/2} n^{-1})$ and $a'_{k_2,n}/(b_{k_1,n}-b'_{k_2,n}) = O(k_2^{1/2} n^{-1})$, we can get that

$$
\begin{aligned}
&\log\left[\left(1-\frac{a_{k_1,n}}{1-b_{k_1,n}}u\right)^{k_1} \left(1+\frac{a'_{k_2,n}}{b'_{k_2,n}}v\right)^{k_2} \left(1+\frac{a_{k_1,n}u-a'_{k_2,n}v}{b_{k_1,n}-b'_{k_2,n}}\right)^{n-k_1-k_2-2}\right] \\
=\ &k_1\left[-\frac{a_{k_1,n}}{1-b_{k_1,n}}u - \frac{1}{2}\left(\frac{a_{k_1,n}}{1-b_{k_1,n}}u\right)^2 + O\left(k_1^{-3/2}\right)\right] + k_2\left[\frac{a'_{k_2,n}}{b'_{k_2,n}}v - \frac{1}{2}\left(\frac{a'_{k_2,n}}{b'_{k_2,n}}v\right)^2 + O\left(k_2^{-3/2}\right)\right] \\
&+(n-k_1-k_2-2)\left[\frac{a_{k_1,n}u-a'_{k_2,n}v}{b_{k_1,n}-b'_{k_2,n}} - \frac{1}{2}\left(\frac{a_{k_1,n}u-a'_{k_2,n}v}{b_{k_1,n}-b'_{k_2,n}}\right)^2 + O\left(k_1^{3/2}n^{-3} + k_2^{3/2}n^{-3}\right)\right],
\end{aligned}
$$

so the coefficients of u and v vanish to 0. The coefficients of $-u^2/2$ and $-v^2/2$ are

$$
\begin{aligned}
k_1\left(\frac{a_{k_1,n}}{1-b_{k_1,n}}\right)^2 + (n-k_1-k_2-2)\left(\frac{a_{k_1,n}}{b_{k_1,n}-b'_{k_2,n}}\right)^2 &\to 1 \\
k_2\left(\frac{a'_{k_2,n}}{b'_{k_2,n}}\right)^2 + (n-k_1-k_2-2)\left(\frac{a'_{k_2,n}}{b_{k_1,n}-b'_{k_2,n}}\right)^2 &\to 1
\end{aligned}
$$

as $n \to \infty$. Finally, as the coefficient of uv is

$$(n-k_1-k_2-2)\frac{a_{k_1,n}a'_{k_2,n}}{(b_{k_1,n}-b'_{k_2,n})^2} \to 0$$

as $n \to \infty$, we finish the proof of this lemma. Note that this lemma also indicates that $U_{n-k_1:n}$ and $U_{k_2+1:n}$ are asymptotically independent as long as $k_1/n \to 0$ and $k_2/n \to 0$. □

Lemma S1.3. *Let $\zeta_1, \ldots, \zeta_n$ be i.i.d. with distribution function $F_\zeta(y) = 1-1/y$, $y \geq 1$. Denote $\zeta_{1:n} \leq \zeta_{2:n} \leq \cdots \leq \zeta_{n:n}$ as the n-th order statistics of $\{\zeta_i\}_{i=1}^n$. Then, as $n \to \infty$, $k_1 \to \infty$, $k_2 \to \infty$, $k_1/n \to 0$ and $k_2/n \to 0$,*

$$\begin{pmatrix} k_1^{1/2}(k_1 n^{-1}\zeta_{n-k_1:n}-1) \\ k_2^{-1/2}n[(n-k_2)n^{-1}\zeta_{k_2+1:n}-1] \end{pmatrix}$$

is asymptotically two-dimensional standard normal.

Proof: Define $U_i = 1-1/\zeta_i$ for $i = 1, \ldots, n$, then $U_1, \ldots, U_n$ are i.i.d. from the standard uniform distribution. Denote $U_{1:n} \leq U_{2:n} \leq \cdots \leq U_{n:n}$ as the n-th order statistics of $\{U_i\}_{i=1}^n$, then $U_{i:n} = 1-1/\zeta_{i:n}$ and $\zeta_{i:n} = 1/(1-U_{i:n})$ for $i = 1, \ldots, n$ as $1-1/x$ is strictly increasing in $x \in [1, \infty)$. Thus, $\left(\zeta_{n-k_1:n}, \zeta_{k_2+1:n}\right)^T$ has the same joint distribution as $\left(1/(1-U_{n-k_1:n}), 1/(1-U_{k_2+1:n})\right)^T$.

By multivariate change of variables and according to the proof of Lemma S1.2, the joint density of $\left(k_1^{1/2}(k_1 n^{-1}\zeta_{n-k_1:n}-1), k_2^{-1/2}n[(n-k_2)n^{-1}\zeta_{k_2+1:n}-1]\right)^T$ is

$$\begin{aligned} f(u',v') &= \frac{n!}{k_1!(n-k_1-k_2-2)!k_2!}(n-k_2)n^{-n-1}k_1^{k_1+1/2}k_2^{k_2+1/2}(n-k_1-k_2)^{n-k_1-k_2-2}\left(1+k_1^{-1/2}u'\right)^{-n+k_2} \\ &\quad \times\left(1+k_2^{-1/2}v'\right)^{k_2}\left(1+k_2^{1/2}n^{-1}v'\right)^{-n+k_1}\left[1+\frac{k_1^{-1/2}(n-k_2)u'-k_1k_2^{1/2}n^{-1}v'}{n-k_1-k_2}\right]^{n-k_1-k_2-2}. \end{aligned}$$

By Stirling's formula, we have that

$$\begin{aligned} &\frac{n!}{k_1!(n-k_1-k_2-2)!k_2!}(n-k_2)n^{-n-1}k_1^{k_1+1/2}k_2^{k_2+1/2}(n-k_1-k_2)^{n-k_1-k_2-2} \\ \sim\ &\frac{\sqrt{2\pi n}n^n e^{-n}}{\sqrt{2\pi}k_1^{k_1+1/2}e^{-k_1}\sqrt{2\pi}k_2^{k_2+1/2}e^{-k_2}\sqrt{2\pi}(n-k_1-k_2-2)^{n-k_1-k_2-3/2}e^{-(n-k_1-k_2-2)}} \\ &\times(n-k_2)n^{-n-1}k_1^{k_1+1/2}k_2^{k_2+1/2}(n-k_1-k_2)^{n-k_1-k_2-2} \\ =\ &(2\pi)^{-1}e^{-2}\left(\frac{n-k_1-k_2}{n-k_1-k_2-2}\right)^{n-k_1-k_2-2}n^{-1/2}(n-k_2)(n-k_1-k_2-2)^{-1/2}, \end{aligned}$$

which converges to $(2\pi)^{-1}$ as $n, k_1, k_2 \to \infty$ under the condition that $k_1/n \to 0$ and $k_2/n \to 0$.

Using the same approach as in the proof of Lemma S1.2, we expand

$$\begin{aligned}
&\log\Big\{ \left(1+k_1^{-1/2}u'\right)^{-n+k_2}\left(1+k_2^{-1/2}v'\right)^{k_2}\left(1+k_2^{1/2}n^{-1}v'\right)^{-n+k_1} \\
&\qquad \times \left[1+\frac{k_1^{-1/2}(n-k_2)u'-k_1k_2^{1/2}n^{-1}v'}{n-k_1-k_2}\right]^{n-k_1-k_2-2}\Big\} \\
= &-(n-k_2)\left[k_1^{-1/2}u'-\frac{1}{2}\left(k_1^{-1/2}u'\right)^2+O\left(k_1^{-3/2}\right)\right]+k_2\left[k_2^{-1/2}v'-\frac{1}{2}\left(k_2^{-1/2}v'\right)^2+O\left(k_2^{-3/2}\right)\right] \\
&-(n-k_1)\left[k_2^{1/2}n^{-1}v'-\frac{1}{2}\left(k_2^{1/2}n^{-1}v'\right)^2+O(k_2^{3/2}n^{-3})\right] \\
&+(n-k_1-k_2-2)\Big\{\frac{k_1^{-1/2}(n-k_2)u'-k_1k_2^{1/2}n^{-1}v'}{n-k_1-k_2}-\frac{1}{2}\left[\frac{k_1^{-1/2}(n-k_2)u'-k_1k_2^{1/2}n^{-1}v'}{n-k_1-k_2}\right]^2 \\
&\qquad +O\left(k_1^{-3/2}+k_1^3k_2^{3/2}n^{-6}\right)\Big\}.
\end{aligned}$$

First, the coefficients of u' and v' are

$$\begin{aligned}
&-(n-k_2)k_1^{-1/2}+(n-k_1-k_2-2)\frac{k_1^{-1/2}(n-k_2)}{n-k_1-k_2}=-\frac{2k_1^{-1/2}(n-k_2)}{n-k_1-k_2}\to 0,\\
&k_2^{1/2}-(n-k_1)k_2^{1/2}n^{-1}-(n-k_1-k_2-2)\frac{k_1k_2^{1/2}n^{-1}}{n-k_1-k_2}=\frac{2k_1k_2^{1/2}n^{-1}}{n-k_1-k_2}\to 0
\end{aligned}$$

as $n \to \infty$. Second, the coefficients of $-u'^2/2$ and $-v'^2/2$ are

$$\begin{aligned}
&-(n-k_2)k_1^{-1}+(n-k_1-k_2-2)\frac{k_1^{-1}(n-k_2)^2}{(n-k_1-k_2)^2}\to 1,\\
&1-(n-k_1)k_2n^{-2}+(n-k_1-k_2-2)\frac{k_1^2k_2n^{-2}}{(n-k_1-k_2)^2}\to 1
\end{aligned}$$

as $n \to \infty$. Finally, as the coefficient of $u'v'$ is

$$(n-k_1-k_2-2)\frac{k_1^{1/2}k_2^{1/2}n^{-1}(n-k_2)}{(n-k_1-k_2)^2}\to 0$$

as $n \to \infty$, we finish the proof of this lemma. □

Lemma S1.4. *Let $\zeta_1, \ldots, \zeta_n$ be i.i.d. with distribution function $F_\zeta(y) = 1 - 1/y$, $y \geq 1$. Denote $\zeta_{1:n} \leq \zeta_{2:n} \leq \cdots \leq \zeta_{n:n}$ as the n-th order statistics of $\{\zeta_i\}_{i=1}^n$. Suppose $k_2 \to \infty$ such that $k_2/n \to 0$ as $n \to \infty$. Then $(n-k_2)n^{-1}\zeta_{k_2+1:n} \to 1$ with probability 1 as $n \to \infty$.*

Proof: Denote $p_{2n} = F_\zeta(n/(n-k_2) - e_{2n}) = (k_2/(n-k_2) - e_{2n})/(n/(n-k_2) - e_{2n})$ with $e_{2n} = C_1 k_2^{1/2} n^{-2/3}$, where $C_1 > 1$ is a constant. First note that

$$\Pr\left(\zeta_{k_2:n} < n/(n-k_2) - e_{2n}\right) \le \Pr\left(\sum_{i=1}^{n} \mathbb{I}(\zeta_i \le n/(n-k_2) - e_{2n}) \ge k_2\right)$$
$$= \Pr\left(\sum_{i=1}^{n} \mathbb{I}(\zeta_i \le n/(n-k_2) - e_{2n}) - np_{2n} \ge k_2 - np_{2n}\right).$$

By simple algebra, there exists a sufficient large N_1 such that $k_2 - np_{2n} = \frac{(n-k_2)e_{2n}}{n/(n-k_2)-e_{2n}} > C_2 n e_{2n}$ for some $C_2 > 0$ and for $n \ge N_1$. By Bernstein inequality, we have that for $n \ge N_1$,

$$\Pr\left(\zeta_{k_2:n} < n/(n-k_2) - e_{2n}\right) \le \exp\left\{-\frac{(k_2 - np_{2n})^2}{2\left\{np_{2n} + (k_2 - np_{2n})/3\right\}}\right\} \le \exp\left[-\frac{C_2^2 n^2 e_{2n}^2}{2\left(np_{2n} + C_2 n e_{2n}/3\right)}\right].$$

The last inequality is due to the fact that the function $x^2/[2(np_{2n} + x/3)]$ is strictly increasing in $x > 0$. There exists a sufficient large $N_2 \ge N_1$ such that $\frac{C_2^2 n^2 e_{2n}^2}{2(np_{2n} + C_2 n e_{2n}/3)} > C_3 n^{2/3}$ for some constant $C_3 > 0$ and $n \ge N_2$. Thus,

$$\sum_{n \ge N_2} \Pr\left(\zeta_{k_2:n} < n/(n-k_2) - e_{2n}\right) < \infty.$$

Similarly, we can show that $\sum_{n \ge N_2} \Pr\left(\zeta_{k_2:n} > n/(n-k_2) + e_{2n}\right) < \infty$. Finally, we apply the Borel-Cantelli lemma to conclude that $(n-k_2)n^{-1}\zeta_{k_2+1:n} \to 1$ with probability 1 as $n \to \infty$. □

Lemma S1.5. *Denote $\varepsilon_{1:n} \le \varepsilon_{2:n} \le \cdots \le \varepsilon_{n:n}$ as the n-th order statistics of the i.i.d. residuals $\{\varepsilon_i\}_{i=1}^n$. Define $\hat{\gamma}_{\varepsilon,R} = k_1^{-1} \sum_{i=0}^{k_1} \log\left(\varepsilon_{n-i:n}/\varepsilon_{n-k_1:n}\right)$ and $\hat{\gamma}_{\varepsilon,L} = k_2^{-1} \sum_{i=0}^{k_2} \log\left(\varepsilon_{i+1:n}/\varepsilon_{k_2+1:n}\right)$. Then under Assumption 3.4 (i) and (ii), as $n \to \infty$, $k_1 \to \infty$, $k_2 \to \infty$, $k_1/n \to 0$ and $k_2/n \to 0$,*

$$\begin{pmatrix} k_1^{1/2} \gamma_R^{-1} \left[\varepsilon_{n-k_1:n}/U_\varepsilon(n/k_1) - 1\right] \\ k_1^{1/2} \gamma_R^{-1} (\hat{\gamma}_{\varepsilon,R} - \gamma_R) \\ k_2^{1/2} \gamma_L^{-1} \left[-\varepsilon_{k_2+1:n}/U_{-\varepsilon}(n/k_2) - 1\right] \\ k_2^{1/2} \gamma_L^{-1} (\hat{\gamma}_{\varepsilon,L} - \gamma_L) \end{pmatrix}$$

is asymptotically four-dimensional standard normal.

Proof: Let $\zeta_1, \ldots, \zeta_n$ be i.i.d. with distribution function $F_\zeta(y) = 1 - 1/y$, $y \ge 1$, and denote $\zeta_{1:n} \le \zeta_{2:n} \le \cdots \le \zeta_{n:n}$ as the n-th order statistics of $\{\zeta_i\}_{i=1}^n$. Then $(U_\varepsilon(\zeta_1), \ldots, U_\varepsilon(\zeta_n))$ has the same joint distribution as $(\varepsilon_1, \ldots, \varepsilon_n)$ and $(U_\varepsilon(\zeta_{1:n}), \ldots, U_\varepsilon(\zeta_{n:n}))$ has the same joint distribution as $(\varepsilon_{1:n}, \ldots, \varepsilon_{n:n})$.

Thus, it is sufficient to prove the result for

$$\begin{pmatrix} k_1^{1/2}\gamma_R^{-1}\left[U_\varepsilon(\zeta_{n-k_1:n})/U_\varepsilon(n/k_1)-1\right] \\ k_1^{1/2}\gamma_R^{-1}(\hat{\gamma}_{\zeta,R}-\gamma_R) \\ k_2^{1/2}\gamma_L^{-1}\left[-U_\varepsilon(\zeta_{k_2+1:n})/U_{-\varepsilon}(n/k_2)-1\right] \\ k_2^{1/2}\gamma_L^{-1}(\hat{\gamma}_{\zeta,L}-\gamma_L) \end{pmatrix}$$

where $\hat{\gamma}_{\zeta,R} = k_1^{-1}\sum_{i=0}^{k_1}\log\left[U_\varepsilon(\zeta_{n-i:n})/U_\varepsilon(\zeta_{n-k_1:n})\right]$ and $\hat{\gamma}_{\zeta,L} = k_2^{-1}\sum_{i=0}^{k_2}\log\left[U_\varepsilon(\zeta_{i+1:n})/U_\varepsilon(\zeta_{k_2+1:n})\right]$.

According to Lemma 2.1 of Pan, Leng, and Hu (2013), the second-order condition on $U_\varepsilon(\cdot)$ in Assumption 3.4 (i) implies that for any $\epsilon > 0$ and $\delta > 0$, there exists a $x_0 = x_0(\epsilon,\delta)$ such that for all $x \geq x_0$ and $xy \geq x_0$,

$$\left|\frac{U_\varepsilon(xy)/U_\varepsilon(x)-y^{\gamma_R}}{A_R(x)} - y^{\gamma_R}\frac{y^{\rho_R}-1}{\rho_R}\right| \leq \epsilon y^{\gamma_R}\left[\left|\frac{y^{\rho_R}-1}{\rho_R}\right| + y^{\rho_R}\max(y^{\delta},y^{-\delta})\right]. \tag{S1.3}$$

According to Lemma 3.2.1 of de Haan and Ferreira (2006), $\zeta_{n-k_1:n} \to \infty$ with probability 1. By Lemma S1.4, $(n-k_2)n^{-1}\zeta_{k_2+1:n} \to 1$ with probability 1. Applying (S1.3) with $x = \zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}]$ and $y = (n-k_2)n^{-1}\zeta_{k_2+1:n}$, we have that

$$\begin{aligned} &\frac{U_\varepsilon(\zeta_{n-k_1:n})}{U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n})]} \\ = \; & [(n-k_2)n^{-1}\zeta_{k_2+1:n}]^{\gamma_R} + O(1)A_R(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}]) \\ & \times\Bigg\{[(n-k_2)n^{-1}\zeta_{k_2+1:n}]^{\gamma_R}\left|\frac{[(n-k_2)n^{-1}\zeta_{k_2+1:n}]^{\rho_R}-1}{\rho_R}\right| \\ & + [(n-k_2)n^{-1}\zeta_{k_2+1:n}]^{\gamma_R+\rho_R}\max([(n-k_2)n^{-1}\zeta_{k_2+1:n}]^{\delta},[(n-k_2)n^{-1}\zeta_{k_2+1:n}]^{-\delta})\Bigg\} \end{aligned}$$

with probability 1. According to Lemma S1.3, $(n-k_2)n^{-1}\zeta_{k_2+1:n} = 1+O_p(k_2^{1/2}n^{-1})$ and $k_1n^{-1}\zeta_{n-k_1:n} = 1+O_p(k_1^{-1/2})$. Thus,

$$\frac{U_\varepsilon(\zeta_{n-k_1:n})}{U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])} = 1+O_p(k_2^{1/2}n^{-1})+O_p(1)A_R(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}]).$$

By the fact that $A_R(\cdot)$ is regularly varying, and Potter's inequality (de Haan and Ferreira, 2006, Proposition B.1.9), we can show that

$$A_R(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])/A_R(n/k_1) \to 1$$

in probability. This results in that

$$
\begin{aligned}
&\frac{U_\varepsilon(\zeta_{n-k_1:n})}{U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])} \\
= & \ 1+O_p(k_2^{1/2}n^{-1})+O_p(1)A_R(n/k_1)\frac{A_R(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])}{A_R(n/k_1)} \\
= & \ 1+O_p(k_2^{1/2}n^{-1})+O_p(1)A_R(n/k_1) \\
= & \ 1+o_p(k_1^{-1/2})+O_p(k_2^{1/2}n^{-1}),
\end{aligned}
$$

where the last equality is due to that $k_1^{1/2}A_R(n/k_1)\to 0$. Applying (S1.3) with $x=n/k_1$ and $xy=\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}]$, we obtain

$$
\begin{aligned}
&\frac{U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])}{U_\varepsilon(n/k_1)} \\
= & \left[\frac{k_1}{n}\frac{\zeta_{n-k_1:n}}{(n-k_2)n^{-1}\zeta_{k_2+1:n}}\right]^{\gamma_R}+O(1)A_R(n/k_1)\left[\left[\frac{k_1}{n}\frac{(n-k_2)n^{-1}\zeta_{n-k_1:n}}{\zeta_{k_2+1:n}}\right]^{\gamma_R}\frac{\left[\frac{k_1}{n}\frac{\zeta_{n-k_1:n}}{(n-k_2)n^{-1}\zeta_{k_2+1:n}}\right]^{\rho_R}-1}{\rho_R}\right. \\
& \left.+\left[\frac{k_1}{n}\frac{\zeta_{n-k_1:n}}{(n-k_2)n^{-1}\zeta_{k_2+1:n}}\right]^{\gamma_R+\rho_R}\max\left(\left[\frac{k_1}{n}\frac{\zeta_{n-k_1:n}}{(n-k_2)n^{-1}\zeta_{k_2+1:n}}\right]^{\delta},\left[\frac{k_1}{n}\frac{\zeta_{n-k_1:n}}{(n-k_2)n^{-1}\zeta_{k_2+1:n}}\right]^{-\delta}\right)\right] \\
= & \left(\frac{k_1}{n}\zeta_{n-k_1:n}\right)^{\gamma_R}[1+O_p(k_2^{1/2}n^{-1})]+o_p(k_1^{-1/2}).
\end{aligned}
$$

As $k_1^{1/2}(k_1n^{-1}\zeta_{n-k_1:n}-1)$ converges to standard normal distribution, we have that

$$
k_1^{1/2}\gamma_R^{-1}\left[\frac{U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])}{U_\varepsilon(n/k_1)}-1\right]
$$

is asymptotically standard normal and thus $U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])/U_\varepsilon(n/k_1)=1+O_p(k_1^{-1/2})$. Write

$$
\begin{aligned}
&k_1^{1/2}\gamma_R^{-1}\left[U_\varepsilon(\zeta_{n-k_1:n})/U_\varepsilon(n/k_1)-1\right] \\
= & \ k_1^{1/2}\gamma_R^{-1}\left\{\frac{U_\varepsilon(\zeta_{n-k_1:n})}{U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])}\frac{U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])}{U_\varepsilon(n/k_1)}-1\right\} \\
= & \ k_1^{1/2}\gamma_R^{-1}\left\{\left[1+o_p(k_1^{-1/2})+O_p(k_2^{1/2}n^{-1})\right]\frac{U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])}{U_\varepsilon(n/k_1)}-1\right\} \\
= & \ k_1^{1/2}\gamma_R^{-1}\left\{\frac{U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])}{U_\varepsilon(n/k_1)}-1\right\}+o_p(1),
\end{aligned}
$$

which indicates that $k_1^{1/2}\gamma_R^{-1}\left[U_\varepsilon(\zeta_{n-k_1:n})/U_\varepsilon(n/k_1)-1\right]$ converges to the same asymptotic distribution as $k_1^{1/2}\gamma_R^{-1}\left[U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])/U_\varepsilon(n/k_1)-1\right]$, and they are all asymptotically standard normal.

According to the proof of Theorem 3.2.5 of (de Haan and Ferreira, 2006), we know that

$$k_1^{1/2}\gamma_R^{-1}(\hat{\gamma}_{\zeta,R}-\gamma_R)=k_1^{1/2}\left[k_1^{-1}\sum_{i=0}^{k_1-1}\log\left(\frac{\zeta_{n-i:n}}{\zeta_{n-k_1:n}}\right)-1\right]+o_p(1),$$

and $k_1^{1/2}\left[k_1^{-1}\sum_{i=0}^{k_1-1}\log\left(\zeta_{n-i:n}/\zeta_{n-k_1:n}\right)-1\right]$ converges to standard normal in distribution. Thus, as have that $k_1^{1/2}\gamma_R^{-1}(\hat{\gamma}_{\zeta,R}-\gamma_R)$ has the same asymptotic distribution as

$$k_1^{1/2}\left[k_1^{-1}\sum_{i=0}^{k_1-1}\log\left(\zeta_{n-i:n}/\zeta_{n-k_1:n}\right)-1\right].$$

So it suffices to show that

$$\begin{pmatrix} k_1^{1/2}\gamma_R^{-1}\left[U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])/U_\varepsilon(n/k_1)-1\right] \\ k_1^{1/2}\{k_1^{-1}\sum_{i=0}^{k_1-1}\log\left(\zeta_{n-i:n}/\zeta_{n-k_1:n}\right)-1\} \\ k_2^{1/2}\gamma_L^{-1}\left[-U_\varepsilon(\zeta_{k_2+1:n})/U_{-\varepsilon}(n/k_2)-1\right] \\ k_2^{1/2}\gamma_L^{-1}(\hat{\gamma}_{\zeta,L}-\gamma_L) \end{pmatrix}$$

is asymptotically four-dimensional standard normal.

By Rényi's representation (Rényi, 1953), $\{\zeta_{n-i:n}/\zeta_{n-k_1:n}\}_{i=0}^{k_1-1}$, $\zeta_{n-k_1:n}/\zeta_{k_2+1:n}$ and $\{\zeta_{i:n}\}_{i=1}^{k_2+1}$ are independent. It reduces to show asymptotic normality for

$$k_1^{1/2}\gamma_R^{-1}\left[U_\varepsilon(\zeta_{n-k_1:n}/[(n-k_2)n^{-1}\zeta_{k_2+1:n}])/U_\varepsilon(n/k_1)-1\right],$$

$k_1^{1/2}[k_1^{-1}\sum_{i=0}^{k_1-1}\log\left(\zeta_{n-i:n}/\zeta_{n-k_1:n}\right)-1]$ and

$$\begin{pmatrix} k_2^{1/2}\gamma_L^{-1}\left[-U_\varepsilon(\zeta_{k_2+1:n})/U_{-\varepsilon}(n/k_2)-1\right] \\ k_2^{1/2}\gamma_L^{-1}(\hat{\gamma}_{\zeta,L}-\gamma_L) \end{pmatrix},$$

separately. The asymptotic normality of the first two terms has been shown above. For the last random vector, its asymptotic normality can be shown similarly using Assumption 3.4 (ii) on $U_{-\varepsilon}(\cdot)$. Thus we complete the proof of this lemma. □

The following lemma is a slightly modified version of Theorem 5.1.4 of de Haan and Ferreira (2006) and Proposition 1 of Hoga (2019). Let $\xrightarrow{\mathscr{D}}$ denotes weak convergence of stochastic processes, and $D([a,b])$ be the space of càdlàg functions on $[a,b]$ equipped with the Skorohod metric.

Lemma S1.6. *Under Assumption 3.4 (i) and (ii), let $d_n \to \infty$ and $d_n = o(\min(k_1^{1/2}, k_2^{1/2}))$. Then for any $\iota > 0$ and $\upsilon \in [0, 1/2)$, as $n \to \infty$, $k_1 \to \infty$, $k_2 \to \infty$, $k_1/n \to 0$ and $k_2/n \to 0$,*

$$x^{-\upsilon} k_1^{1/2} \left[k_1^{-1} \sum_{t=d_n}^{n} \mathbb{I}\big(\varepsilon_t > x^{-\gamma_R} U_\varepsilon(n/k_1)\big) - x \right] \xrightarrow{\mathscr{D}} x^{-\upsilon} W_1(x), \tag{S1.4}$$

and

$$x^{-\upsilon} k_2^{1/2} \left[k_2^{-1} \sum_{t=d_n}^{n} \mathbb{I}\big(-\varepsilon_t > x^{-\gamma_L} U_{-\varepsilon}(n/k_2)\big) - x \right] \xrightarrow{\mathscr{D}} x^{-\upsilon} W_2(x), \tag{S1.5}$$

in $D([0, 1+\iota])$, where $W_1(\cdot)$ and $W_2(\cdot)$ are standard Brownian motions.

Define the operator $E_t(\cdot) = E(\cdot|\mathscr{I}_t)$. The next lemma is similar to Proposition 2 of Hoga (2019).

Lemma S1.7. *Suppose that Assumptions 2.1, 3.1, 3.2, 3.3, 3.4 and 3.5 hold. Then for all $\iota > 0$ and $\upsilon \in [0, 1/2)$,*

$$\sup_{x\in(0,1+\iota]} x^{-\upsilon} k_1^{1/2} \left| k_1^{-1} \sum_{t=d_n}^{n} \left[\mathbb{I}\big(\hat{\varepsilon}_t > x^{-\gamma_R} U_\varepsilon(n/k_1)\big) - \mathbb{I}\big(\varepsilon_t > x^{-\gamma_R} U_\varepsilon(n/k_1)\big) \right] \right| = o_p(1) \tag{S1.6}$$

and

$$\sup_{x\in(0,1+\iota]} x^{-\upsilon} k_2^{1/2} \left| k_2^{-1} \sum_{t=d_n}^{n} \left[\mathbb{I}\big(-\hat{\varepsilon}_t > x^{-\gamma_L} U_{-\varepsilon}(n/k_2)\big) - \mathbb{I}\big(-\varepsilon_t > x^{-\gamma_L} U_{-\varepsilon}(n/k_2)\big) \right] \right| = o_p(1). \tag{S1.7}$$

Proof: We only present the proof of (S1.6) as (S1.7) can be proven in the same manner.

For $t = 1, \ldots, n$, denote $\hat{\varepsilon}_t(\theta) = [R_t - m(\tilde{\mathscr{I}}_{t-1}, \theta)]/\sigma(\tilde{\mathscr{I}}_{t-1}, \theta)$, then $\hat{\varepsilon}_t = \hat{\varepsilon}_t(\hat{\theta})$. Let $\eta > 0$ and $N_n(\eta) = \{\theta : \|\theta - \theta_0\| \le \eta n^{-\upsilon_0/2}\}$.

For $t = d_n, \ldots, n$, denote $\Pi_{1,n,t} = \sup_{\theta\in N_n(\eta)} \left\| \frac{\partial m(\mathscr{I}_{t-1},\theta)}{\partial \theta} \right\|$, $\Pi_{2,n,t} = \sup_{\theta\in N_n(\eta)} \left\| \frac{\partial \sigma(\mathscr{I}_{t-1},\theta)}{\partial \theta} \right\|$, $\Pi_{3,n,t} = \sup_{\theta\in N_n(\eta)} \left| m(\mathscr{I}_{t-1}, \theta) - m(\tilde{\mathscr{I}}_{t-1}, \theta) \right|$, and $\Pi_{4,n,t} = \sup_{\theta\in N_n(\eta)} \left| \sigma(\tilde{\mathscr{I}}_{t-1}, \theta) - \sigma(\mathscr{I}_{t-1}, \theta) \right|$. Then $\Pi_{1,n,t}$,

$\Pi_{2,n,t}$, $\Pi_{3,n,t}$ and $\Pi_{4,n,t}$ are all $\mathscr{I}_{t-1}$-measurable for $t = d_n, \ldots, n$. Write

$$\begin{aligned}\hat{\varepsilon}_t(\theta) &= \varepsilon_t\left[1+\frac{\sigma(\mathscr{I}_{t-1},\theta)-\sigma(\mathscr{I}_{t-1},\theta_0)}{\sigma(\mathscr{I}_{t-1},\theta_0)}\right]^{-1}\left[1+\frac{\sigma(\tilde{\mathscr{I}}_{t-1},\theta)-\sigma(\mathscr{I}_{t-1},\theta)}{\sigma(\mathscr{I}_{t-1},\theta)}\right]^{-1} \\ &\quad +\left[\frac{m(\mathscr{I}_{t-1},\theta_0)-m(\mathscr{I}_{t-1},\theta)}{\sigma(\mathscr{I}_{t-1},\theta)}+\frac{m(\mathscr{I}_{t-1},\theta)-m(\tilde{\mathscr{I}}_{t-1},\theta)}{\sigma(\mathscr{I}_{t-1},\theta)}\right]\left[1+\frac{\sigma(\tilde{\mathscr{I}}_{t-1},\theta)-\sigma(\mathscr{I}_{t-1},\theta)}{\sigma(\mathscr{I}_{t-1},\theta)}\right]^{-1}\end{aligned}$$

If $\theta \in N_n(\eta)$,

$$\begin{aligned}\left|\frac{\sigma(\mathscr{I}_{t-1},\theta)-\sigma(\mathscr{I}_{t-1},\theta_0)}{\sigma(\mathscr{I}_{t-1},\theta_0)}\right| &\le c^{-1}|\theta-\theta_0|\sup_{\theta\in N_n(\eta)}\left\|\frac{\partial\sigma(\mathscr{I}_{t-1},\theta)}{\partial\theta}\right\| = c^{-1}|\theta-\theta_0|\Pi_{2,n,t}, \\ \left|\frac{\sigma(\tilde{\mathscr{I}}_{t-1},\theta)-\sigma(\mathscr{I}_{t-1},\theta)}{\sigma(\mathscr{I}_{t-1},\theta)}\right| &\le c^{-1}\sup_{\theta\in N_n(\eta)}\left|\sigma(\tilde{\mathscr{I}}_{t-1},\theta)-\sigma(\mathscr{I}_{t-1},\theta)\right| = c^{-1}\Pi_{4,n,t}, \\ \left|\frac{m(\mathscr{I}_{t-1},\theta_0)-m(\mathscr{I}_{t-1},\theta)}{\sigma(\mathscr{I}_{t-1},\theta)}\right| &\le c^{-1}|\theta-\theta_0|\sup_{\theta\in N_n(\eta)}\left\|\frac{\partial m(\mathscr{I}_{t-1},\theta)}{\partial\theta}\right\| = c^{-1}|\theta-\theta_0|\Pi_{1,n,t}, \\ \left|\frac{m(\mathscr{I}_{t-1},\theta)-m(\tilde{\mathscr{I}}_{t-1},\theta)}{\sigma(\mathscr{I}_{t-1},\theta)}\right| &\le c^{-1}\sup_{\theta\in N_n(\eta)}\left|m(\mathscr{I}_{t-1},\theta)-m(\tilde{\mathscr{I}}_{t-1},\theta)\right| = c^{-1}\Pi_{3,n,t}.\end{aligned}$$

For any real η_0 and $x > 0$, define $A_t(x,\theta) = \mathbb{I}\left(\hat{\varepsilon}_t(\theta) > x^{-\gamma_R}U_\varepsilon(n/k_1)\right)$, $A_t(x) = \mathbb{I}(\varepsilon_t > x^{-\gamma_R}U_\varepsilon(n/k_1))$ and

$$\begin{aligned}A_t(x,\eta,\eta_0) = \mathbb{I}\Big(&\varepsilon_t\left(1+\eta_0 n^{-\upsilon_0/2}\Pi_{2,n,t}\right)\left(1+\eta_0\Pi_{4,n,t}\right) \\ &+\left(\eta_0 n^{-\upsilon_0/2}\Pi_{1,n,t}+\eta_0\Pi_{3,n,t}\right)\left(1+\eta_0\Pi_{4,n,t}\right) > x^{-\gamma_R}U_\varepsilon(n/k_1)\Big).\end{aligned}$$

Let $\epsilon_0 \in (0, 1/10)$, then using the same technique as in Lemma A2 of Hoga (2019), we can show that there exists $\eta_0 > 0$ such that for $\theta \in N_n(\eta)$,

$$w_t A_t(x,\eta,-\eta_0) \le w_t A_t(x,\theta) \le w_t A_t(x,\eta,\eta_0)$$

for $t = d_n, \ldots, n$ and $x \in [0, 1+\iota]$, where

$$w_t = \mathbb{I}\left(\max\left(\frac{\eta_0 n^{-\upsilon_0/2}\Pi_{1,n,t}}{U_\varepsilon(n/k_1)}, \eta_0 n^{-\upsilon_0/2}\Pi_{2,n,t}, \frac{\eta_0\Pi_{3,n,t}}{U_\varepsilon(n/k_1)}, \eta_0\Pi_{4,n,t}\right) < \epsilon_0\right),$$

and for $x \in [0, 1+\iota]$, there exists a positive C_1 such that

$$\frac{n}{k_1} w_t\left|E_{t-1}[A_t(x,\eta,\eta_0)] - A_t(x)\right| \le C_1 w_t x \max\left(\frac{\eta_0 n^{-\upsilon_0/2}\Pi_{1,n,t}}{U_\varepsilon(n/k_1)}, \eta_0 n^{-\upsilon_0/2}\Pi_{2,n,t}, \frac{\eta_0\Pi_{3,n,t}}{U_\varepsilon(n/k_1)}, \eta_0\Pi_{4,n,t}\right).$$

In addition, by Markov's inequality, for any $\epsilon > 0$,

$$\begin{aligned}
\Pr\left(\sum_{t=d_n}^{n}(1-w_t)\geq\epsilon\right) &\leq \epsilon^{-1}E\left|\sum_{t=d_n}^{n}(1-w_t)\right|\leq\epsilon^{-1}\sum_{i=d_n}^{n}E|1-w_t| \\
&\leq \epsilon^{-1}\sum_{t=d_n}^{n}\Bigg[\Pr\left(\frac{\eta_0 n^{-\upsilon_0/2}\Pi_{1,n,t}}{U_\varepsilon(n/k_1)}\geq\epsilon_0\right)+\Pr\left(\eta_0 n^{-\upsilon_0/2}\Pi_{2,n,t}\geq\epsilon_0\right) \\
&\quad +\Pr\left(\frac{\eta_0\Pi_{3,n,t}}{U_\varepsilon(n/k_1)}\geq\epsilon_0\right)+\Pr\left(\eta_0\Pi_{4,n,t}\geq\epsilon_0\right)\Bigg] \\
&\leq \epsilon^{-1}\sum_{t=d_n}^{n}\Bigg(\epsilon_0^{-\upsilon_1}\eta_0^{\upsilon_1}n^{-\upsilon_0\upsilon_1/2}\left\{[U_\varepsilon(n/k_1)]^{-\upsilon_1}E\left|\Pi_{1,n,t}\right|^{\upsilon_1}+E\left|\Pi_{2,n,t}\right|^{\upsilon_1}\right\} \\
&\quad +\epsilon_0^{-1}\eta_0\left\{[U_\varepsilon(n/k_1)]^{-1}E\left|\Pi_{3,n,t}\right|+E\left|\Pi_{4,n,t}\right|\right\}\Bigg) \\
&= o(1), \qquad\qquad \text{(S1.8)}
\end{aligned}$$

which indicates that $\sum_{t=d_n}^{n}(1-w_t)=o_p(1)$.

Similarly to Lemma 3 of Hoga (2019), we can show that for any $\iota > 0$ and $\upsilon\in[0,1/2)$,

$$\sup_{x\in(0,1+\iota]}k_1^{-1/2}x^{-\upsilon}\left|\sum_{t=d_n}^{n}[A_t(x,\eta,\eta_0)-A_t(x)]\right|=o_p(1),$$

which can conclude the result of this lemma based on Assumption 3.1 that $\hat\theta\in N_n(\eta)$ with probability approaching 1 as $n\to\infty$. □

Define $\hat F_n^{(R)}(x)=k_1^{-1}\sum_{t=d_n}^{n}\mathbb{I}\left(\hat\varepsilon_t>x\hat\varepsilon_{(k_1)}\right)$ and $\hat F_n^{(L)}(x)=k_2^{-1}\sum_{t=d_n}^{n}\mathbb{I}\left(-\hat\varepsilon_t>-x\hat\varepsilon_{(n-k_2-d_n)}\right)$ as the right- and left-tail empirical process. We have the following asymptotic properties concerning $\hat F_n^{(R)}(x)$ and $\hat F_n^{(L)}(x)$.

Lemma S1.8. *Suppose that Assumptions 2.1, 3.1, 3.2, 3.3, 3.4 and 3.5 hold. Then for any $\upsilon\in[0,1/2)$ under a Skorohod construction, as $n\to\infty$,*

$$\sup_{x\geq1}x^{\upsilon/\gamma_R}\left|k_1^{1/2}\left[\hat F_n^{(R)}(x)-x^{-1/\gamma_R}\right]-\left[W_1(x^{-1/\gamma_R})-x^{-1/\gamma_R}W_1(1)\right]\right|\to0$$

and

$$\sup_{x\geq1}x^{\upsilon/\gamma_L}\left|k_2^{1/2}\left[\hat F_n^{(L)}(x)-x^{-1/\gamma_L}\right]-\left[W_2(x^{-1/\gamma_L})-x^{-1/\gamma_L}W_2(1)\right]\right|\to0$$

almost surely. Here $W_1(x)$ and $W_2(x)$ are standard Brownian motions same as those in Lemma S1.6.

Proof: First, Lemmas S1.6 and S1.7 implies

$$x^{-\upsilon}k_1^{1/2}\left[k_1^{-1}\sum_{i=d_n}^{n}\mathbb{I}\big(\hat{\varepsilon}_i > x^{-\gamma_R}U_\varepsilon(n/k_1)\big)-x\right]\xrightarrow{\mathscr{D}} x^{-\upsilon}W_1(x) \tag{S1.9}$$

in $D([0,1+\iota])$ as $n\to\infty$. By Skorohod's representation theorem, on a suitable probability space,

$$\sup_{x\in[0,1+\iota]} x^{-\upsilon}\left|k_1^{1/2}\left[k_1^{-1}\sum_{i=d_n}^{n}\mathbb{I}\big(\hat{\varepsilon}_i > x^{-\gamma_R}U_\varepsilon(n/k_1)\big)-x\right]-W_1(x)\right|\xrightarrow{a.s.}0. \tag{S1.10}$$

This indicates that

$$\sup_{x\in[0,1+\iota]}\left|k_1^{1/2}\left[k_1^{-1}\sum_{i=d_n}^{n}\mathbb{I}\big(\hat{\varepsilon}_i > x^{-\gamma_R}U_\varepsilon(n/k_1)\big)-x\right]-W_1(x)\right|\xrightarrow{a.s.}0$$

as $n\to\infty$. Equivalently,

$$\sup_{x\in[0,1+\iota]}\left|k_1^{1/2}\left[k_1^{-1}\sum_{i=d_n}^{n}\mathbb{I}\left(\left(\frac{\hat{\varepsilon}_i}{U_\varepsilon(n/k_1)}\right)^{-1/\gamma_R}<x\right)-x\right]-W_1(x)\right|\xrightarrow{a.s.}0.$$

By Vervaat's Lemma (de Haan and Ferreira, 2006, Lemma A.0.2), it follows that

$$\sup_{x\in[0,1+\iota]}\left|k_1^{1/2}\left[\left(\frac{\hat{\varepsilon}_{(\lfloor xk_1\rfloor)}}{U_\varepsilon(n/k_1)}\right)^{-1/\gamma_R}-x\right]+W_1(x)\right|\xrightarrow{a.s.}0. \tag{S1.11}$$

Take $x=1$ in (S1.11), we obtain that

$$\left|k_1^{1/2}\left[\left(\frac{\hat{\varepsilon}_{(k_1)}}{U_\varepsilon(n/k_1)}\right)^{-1/\gamma_R}-1\right]+W_1(1)\right|\xrightarrow{a.s.}0, \tag{S1.12}$$

which implies that $\hat{\varepsilon}_{(k_1)}/U_\varepsilon(n/k_1)=1+o(1)$ almost surely. Replacing $x^{-\gamma_R}$ with $x_n=x\hat{\varepsilon}_{(k_1)}/U_\varepsilon(n/k_1)$ in (S1.10), we have that

$$\sup_{x\geq 1} x_n^{\upsilon/\gamma_R}\left|k_1^{1/2}\left[k_1^{-1}\sum_{i=d_n}^{n}\mathbb{I}\big(\hat{\varepsilon}_i > x\hat{\varepsilon}_{(k_1)}\big)-x_n^{-1/\gamma_R}\right]-W_1(x_n^{-1/\gamma_R})\right|\xrightarrow{a.s.}0. \tag{S1.13}$$

As $x_n^{-1/\gamma_R} = x^{-1/\gamma_R}[1+o(1)]$ uniformly in $x \geq 1$,

$$\sup_{x\geq 1} \left| x_n^{\upsilon/\gamma_R} / x^{\upsilon/\gamma_R} \right| = \left| \hat{\varepsilon}_{(k_1)} / U_\varepsilon(n/k_1) \right|^{\upsilon/\gamma_R} = 1 + o(1) \tag{S1.14}$$

almost surely. In addition, by (S1.12),

$$x^{\upsilon/\gamma_R} k_1^{1/2} \left(x_n^{-1/\gamma_R} - x^{-1/\gamma_R} \right) = -x^{-(1-\upsilon)/\gamma_R} W_1(1) + o(1) \tag{S1.15}$$

uniformly in $x \geq 1$ almost surely. Furthermore,

$$\sup_{x\geq 1} x^{\upsilon/\gamma_R} \left| W_1(x_n^{-1/\gamma_R}) - W_1(x^{-1/\gamma_R}) \right| \xrightarrow{a.s.} 0. \tag{S1.16}$$

Substituting (S1.14), (S1.15) and (S1.16) into (S1.13) yields that

$$\sup_{x\geq 1} x^{\upsilon/\gamma_R} \left| k_1^{1/2} \left[\hat{F}_n^{(R)}(x) - x^{-1/\gamma_R} \right] - \left[W_1(x^{-1/\gamma_R}) - x^{-1/\gamma_R} W_1(1) \right] \right| \xrightarrow{a.s.} 0. \tag{S1.17}$$

The proof of the result for $\hat{F}_n^{(L)}(x)$ is similar and thus is omitted here. □

Lemma S1.9. *(i) (i.i.d. residuals) Under Assumption 3.4 (i) and (ii), let $\varepsilon_{(k)}$ be the $(k+1)$-largest value of $\{\varepsilon_t\}_{t=d_n}^n$ for $k = 0, 1, \cdots, n-d_n$, where $d_n \to \infty$ satisfies $d_n = o(\min(k_1^{-1/2} n, k_2^{-1/2} n))$. Define $\hat{\gamma}_{\varepsilon,R} = k_1^{-1} \sum_{i=0}^{k_1} \log\left(\varepsilon_{(i)}/\varepsilon_{(k_1)}\right)$ and $\hat{\gamma}_{\varepsilon,L} = k_2^{-1} \sum_{i=0}^{k_2} \log\left(\varepsilon_{(n-i-d_n)}/\varepsilon_{(n-k_2-d_n)}\right)$. Then as $n \to \infty$, $k_1 \to \infty$, $k_2 \to \infty$, $k_1/n \to 0$ and $k_2/n \to 0$,*

$$\begin{pmatrix} k_1^{1/2} \gamma_R^{-1} \left[\varepsilon_{(k_1)} / U_\varepsilon(n/k_1) - 1 \right] \\ k_1^{1/2} \gamma_R^{-1} (\hat{\gamma}_{\varepsilon,R} - \gamma_R) \\ k_2^{1/2} \gamma_L^{-1} \left[-\varepsilon_{(n-k_2-d_n)} / U_{-\varepsilon}(n/k_2) - 1 \right] \\ k_2^{1/2} \gamma_L^{-1} (\hat{\gamma}_{\varepsilon,L} - \gamma_L) \end{pmatrix}$$

is asymptotically four-dimensional standard normal.

(ii) (standardized residuals) Under Assumptions 2.1, 3.1, 3.2, 3.3, 3.4 and *3.5,*

$$\begin{pmatrix} k_1^{1/2} \gamma_R^{-1} \left[\hat{\varepsilon}_{(k_1)} / U_\varepsilon(n/k_1) - 1 \right] \\ k_1^{1/2} \gamma_R^{-1} (\hat{\gamma}_R - \gamma_R) \\ k_2^{1/2} \gamma_L^{-1} \left[-\hat{\varepsilon}_{(n-k_2-d_n)} / U_{-\varepsilon}(n/k_2) - 1 \right] \\ k_2^{1/2} \gamma_L^{-1} (\hat{\gamma}_L - \gamma_L) \end{pmatrix}$$

is asymptotically four-dimensional standard normal as $n \to \infty$.

Proof: (i) Applying (S1.3) with $x = (n - d_n + 1)/k_1$ and $y = n/(n - d_n + 1)$, we can show that

$$U_\varepsilon(n/k_1)/U_\varepsilon((n - d_n + 1)/k_1) = 1 + o(k_1^{-1/2}) + O(d_n n^{-1}). \tag{S1.18}$$

Using a similar argument on $U_{-\varepsilon}(\cdot)$, we get that

$$U_{-\varepsilon}(n/k_2)/U_{-\varepsilon}((n - d_n + 1)/k_2) = 1 + o(k_2^{-1/2}) + O(d_n n^{-1}). \tag{S1.19}$$

Then the result of (i) follows from Lemma S1.5, (S1.18) and (S1.19).

(ii) According to the proof of Lemma S1.8,

$$k_1^{1/2}\gamma_R^{-1}\left[\hat{\varepsilon}_{(k_1)}/U_R(n/k_1) - 1\right] = k_1^{1/2}\gamma_R^{-1}\left[\varepsilon_{(k_1)}/U_R(n/k_1) - 1\right] + o_p(1).$$

Similarly, we can conclude that

$$k_2^{1/2}\gamma_L^{-1}\left[-\hat{\varepsilon}_{(n-k_2-d_n)}/U_{-\varepsilon}(n/k_2) - 1\right] = k_2^{1/2}\gamma_L^{-1}\left[-\varepsilon_{(n-k_2-d_n)}/U_{-\varepsilon}(n/k_2) - 1\right] + o_p(1).$$

Note $\hat{\gamma}_R = k_1^{-1}\sum_{i=0}^{k_1}\log\left(\hat{\varepsilon}_{(i)}/\hat{\varepsilon}_{(k_1)}\right) = \int_1^\infty \hat{F}^{(R)}(s)\frac{ds}{s}$ with $\hat{F}_n^{(R)}(x) = k_1^{-1}\sum_{t=d_n}^n \mathbb{I}\left(\hat{\varepsilon}_t > x\hat{\varepsilon}_{(k_1)}\right)$, and $\hat{\gamma}_L = k_2^{-1}\sum_{i=0}^{k_2}\log\left(\frac{-\hat{\varepsilon}_{(n-i-d_n)}}{-\hat{\varepsilon}_{(n-k_2-d_n)}}\right) = \int_1^\infty \hat{F}^{(L)}(s)\frac{ds}{s}$ with $\hat{F}_n^{(L)}(x) = k_2^{-1}\sum_{t=d_n}^n \mathbb{I}\left(-\hat{\varepsilon}_t > -x\hat{\varepsilon}_{(n-k_2-d_n)}\right)$.

According to Lemma S1.6 and S1.8,

$$k_2^{1/2}\gamma_R^{-1}(\hat{\gamma}_R - \gamma_R) = k_2^{1/2}\gamma_R^{-1}(\hat{\gamma}_{\varepsilon,R} - \gamma_R) + o_p(1),$$

and

$$k_2^{1/2}\gamma_L^{-1}(\hat{\gamma}_L - \gamma_L) = k_2^{1/2}\gamma_L^{-1}(\hat{\gamma}_{\varepsilon,L} - \gamma_L) + o_p(1).$$

Then the result of part (ii) follows from part (i) and we complete the proof of this lemma. □

Lemma S1.10. *Suppose $\tau_l \le \tau_u$ satisfies $\tau_l \to 0$ as $n \to \infty$ and $\tau_u = \Delta\tau_l$ for some constant $\Delta \ge 1$. Assume $k_1 \to \infty$, $k_2 \to \infty$, $k_1/n \to 0$ and $k_2/n \to 0$ as $n \to \infty$. There exist two constants $c_1, c_2 \in [0,1)$ such that $\lim_{n\to\infty} n\tau_l/k_1 = c_1$ and $\lim_{n\to\infty} n\tau_l/k_2 = c_2$. In addition, $\lim_{n\to\infty} k_1^{-1/2}\log(n\tau_l/k_1) = 0$ and $\lim_{n\to\infty} k_2^{-1/2}\log(n\tau_l/k_2) = 0$.*

(i) (i.i.d. residuals) Under the conditions of Lemma S1.9 (i), define $\hat{\varepsilon}_R(\tau) = \varepsilon_{(k_1)}(n\tau/k_1)^{-\hat{\gamma}_{\varepsilon,R}}$ *and* $\hat{\varepsilon}_L(\tau) = \varepsilon_{(n-k_2-d_n)}(n\tau/k_2)^{-\hat{\gamma}_{\varepsilon,L}}$. *Then uniformly in* $\tau \in [\tau_l, \tau_u]$,

$$\frac{\hat{\varepsilon}_R(\tau)}{U_\varepsilon(1/\tau)} - 1 = \left[\frac{\varepsilon_{(k_1)}}{U_\varepsilon(n/k_1)} - 1\right] + \log\left(\frac{k_1}{n\tau}\right)(\hat{\gamma}_{\varepsilon,R} - \gamma_R) + o_p(k_1^{-1/2}\log(n\tau/k_1)), \tag{S1.20}$$

and

$$\frac{-\hat{\varepsilon}_L(\tau)}{U_{-\varepsilon}(1/\tau)} - 1 = \left[\frac{-\varepsilon_{(n-k_2-d_n)}}{U_{-\varepsilon}(n/k_2)} - 1\right] + \log\left(\frac{k_2}{n\tau}\right)(\hat{\gamma}_{\varepsilon,L} - \gamma_L) + o_p(k_2^{-1/2}\log(n\tau/k_2)). \tag{S1.21}$$

(ii) (standardized residuals) Suppose that Assumptions 2.1, 3.1, 3.2, 3.3, 3.4 and 3.5 hold, we have uniformly in $\tau \in [\tau_l, \tau_u]$,

$$\frac{\hat{Q}_\varepsilon(1-\tau)}{Q_\varepsilon(1-\tau)} - 1 = \left[\frac{\hat{\varepsilon}_{(k_1)}}{U_\varepsilon(n/k_1)} - 1\right] + \log\left(\frac{k_1}{n\tau}\right)(\hat{\gamma}_R - \gamma_R) + o_p(k_1^{-1/2}\log(n\tau/k_1)), \tag{S1.22}$$

and

$$\frac{\hat{Q}_\varepsilon(\tau)}{Q_\varepsilon(\tau)} - 1 = \left[\frac{-\hat{\varepsilon}_{(n-k_2-d_n)}}{U_{-\varepsilon}(n/k_2)} - 1\right] + \log\left(\frac{k_2}{n\tau}\right)(\hat{\gamma}_L - \gamma_L) + o_p(k_2^{-1/2}\log(n\tau/k_2)). \tag{S1.23}$$

Proof: (i) We only show the proof for $\hat{\varepsilon}_R(\tau)$, the proof of (S1.21) is similar and thus is omitted here.

By simple algebra,

$$\begin{aligned}
& \hat{\varepsilon}_R(\tau)/U_\varepsilon(1/\tau) - 1 \\
= \quad & \frac{\left[\varepsilon_{(k_1)} - U_\varepsilon(n/k_1)\right](n\tau/k_1)^{-\hat{\gamma}_{\varepsilon,R}}}{U_\varepsilon(1/\tau)} + \frac{\left[(n\tau/k_1)^{-\hat{\gamma}_{\varepsilon,R}} - (n\tau/k_1)^{-\gamma_R}\right]U_\varepsilon(n/k_1)}{U_\varepsilon(1/\tau)} \\
& + \frac{U_\varepsilon(n/k_1)(n\tau/k_1)^{-\gamma_R} - U_\varepsilon(1/\tau)}{U_\varepsilon(1/\tau)} \\
=: \quad & \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Under Assumption 3.4 (i), Lemma 2.1 of Pan, Leng, and Hu (2013) indicates that for any $\epsilon, \delta > 0$, there exists $x_0 = x_0(\epsilon, \delta) > 0$ such that for all $x \geq x_0$ and $xy \geq x_0$,

$$\left|\frac{U_\varepsilon(xy)/U_\varepsilon(x) - y^{\gamma_R}}{A_R(x)} - y^{\gamma_R}\frac{y^{\rho_R} - 1}{\rho_R}\right| \leq \epsilon y^{\gamma_R}\left[\left|\frac{y^{\rho_R} - 1}{\rho_R}\right| + y^{\rho_R}\max(y^\delta, y^{-\delta})\right],$$

which immediately implies that for all $x \geq x_0$ and $xy \geq x_0$,

$$\left|\frac{y^{-\gamma_R}U_\varepsilon(xy)/U_\varepsilon(x) - 1}{A_R(x)}\right| \leq \left|\frac{y^{\rho_R} - 1}{\rho_R}\right| + \epsilon\left[\left|\frac{y^{\rho_R} - 1}{\rho_R}\right| + y^{\rho_R}\max(y^\delta, y^{-\delta})\right].$$

Take $x = n/k_1$, $y = k_1/(n\tau)$ and choose $\rho_R + \delta < 0$, we have that for all $n/k_1 \geq x_0$ and $1/\tau \geq x_0$,

$$\begin{aligned}
&\left|\frac{(n\tau/k_1)^{\gamma_R} U_\varepsilon(1/\tau)/U_\varepsilon(n/k_1) - 1}{A_R(n/k_1)}\right| \\
\leq \;& \left|\frac{\left(\frac{k_1}{n\tau}\right)^{\rho_R} - 1}{\rho_R}\right| + \epsilon\left[\left|\frac{\left(\frac{k_1}{n\tau}\right)^{\rho_R} - 1}{\rho_R}\right| + \left(\frac{k_1}{n\tau}\right)^{\rho_R} \max\left(\left(\frac{k_1}{n\tau}\right)^{\delta}, \left(\frac{k_1}{n\tau}\right)^{-\delta}\right)\right],
\end{aligned}$$

where the right-hand side is uniformly bounded for $\tau \in [\tau_l, \tau_u]$. Furthermore, by the condition that $\lim_{n\to\infty} k_1^{1/2} A_R(n/k_1) = 0$, we have for sufficient large n, $(n\tau/k_1)^{\gamma_R} U_\varepsilon(1/\tau)/U_\varepsilon(n/k_1) = 1 + o(k_1^{-1/2})$ uniformly in $\tau \in [\tau_l, \tau_u]$, which indicates that

$$\mathrm{III} = \frac{U_\varepsilon(n/k_1)}{U_\varepsilon(1/\tau)}\left(\frac{n\tau}{k_1}\right)^{-\gamma_R} - 1 = o(k_1^{-1/2}) \quad \text{uniformly in } \tau \in [\tau_l, \tau_u].$$

According to Lemma S1.5, $\hat{\gamma}_{\varepsilon,R} - \gamma_R = O_p(k_1^{-1/2})$. For any $\tau \in [\tau_l, \tau_u]$, by the mean value theorem and the fact that $\frac{\partial}{\partial a} x^a = x^a \log x$, there exists a $\nu = \nu(\tau) \in [0, 1]$ such that

$$\left(\frac{n\tau}{k_1}\right)^{-\hat{\gamma}_{\varepsilon,R}} - \left(\frac{n\tau}{k_1}\right)^{-\gamma_R} = \left(\frac{k_1}{n\tau}\right)^{\gamma_R}\left[\left(\frac{k_1}{n\tau}\right)^{\hat{\gamma}_{\varepsilon,R} - \gamma_R} - 1\right] = \left(\frac{k_1}{n\tau}\right)^{\gamma_R} (\hat{\gamma}_{\varepsilon,R} - \gamma_R)\left(\frac{k_1}{n\tau}\right)^{\nu(\hat{\gamma}_{\varepsilon,R} - \gamma_R)} \log\left(\frac{k_1}{n\tau}\right),$$

where $\left(\frac{k_1}{n\tau}\right)^{\nu(\hat{\gamma}_{\varepsilon,R} - \gamma_R)}$ satisfies $\left|\left(\frac{k_1}{n\tau}\right)^{\nu(\hat{\gamma}_{\varepsilon,R} - \gamma_R)} - 1\right| = \left|\exp\left[\nu(\hat{\gamma}_{\varepsilon,R} - \gamma_R)\log\left(\frac{k_1}{n\tau}\right)\right] - 1\right| = o_p(1)$ uniformly in $\tau \in [\tau_l, \tau_u]$ as $\lim_{n\to\infty} k_1^{-1/2} \log(n\tau_l/k_1) = 0$. Note that

$$\left|\mathrm{II} - \log\left(\frac{k_1}{n\tau}\right)(\hat{\gamma}_{\varepsilon,R} - \gamma_R)\right| \leq \left[|\mathrm{III} + 1|\left|\left(\frac{k_1}{n\tau}\right)^{\nu(\hat{\gamma}_{\varepsilon,R} - \gamma_R)} - 1\right| + |\mathrm{III}|\right]\left|\log\left(\frac{k_1}{n\tau}\right)(\hat{\gamma}_{\varepsilon,R} - \gamma_R)\right|,$$

we arrive at

$$\mathrm{II} = \log\left(\frac{k_1}{n\tau}\right)(\hat{\gamma}_{\varepsilon,R} - \gamma_R) + o_p(k_1^{-1/2} \log(n\tau/k_1)) \quad \text{uniformly in } \tau \in [\tau_l, \tau_u].$$

Finally, as $\varepsilon_{(k_1)}/U_\varepsilon(n/k_1) - 1 = O_p(k_1^{-1/2})$, then uniformly in $\tau \in [\tau_l, \tau_u]$,

$$\begin{aligned}
\mathrm{I} &= \frac{\left[\varepsilon_{(k_1)} - U_\varepsilon(n/k_1)\right](n\tau/k_1)^{-\hat{\gamma}_{\varepsilon,R}}}{U_\varepsilon(1/\tau)} = \left[\frac{\varepsilon_{(k_1)}}{U_\varepsilon(n/k_1)} - 1\right]\frac{U_\varepsilon(n/k_1)}{U_\varepsilon(1/\tau)}\left(\frac{n\tau}{k_1}\right)^{-\hat{\gamma}_{\varepsilon,R}} \\
&= \left[\frac{\varepsilon_{(k_1)}}{U_\varepsilon(n/k_1)} - 1\right]\frac{U_\varepsilon(n/k_1)}{U_\varepsilon(1/\tau)}\left\{\left[\left(\frac{n\tau}{k_1}\right)^{-\hat{\gamma}_{\varepsilon,R}} - \left(\frac{n\tau}{k_1}\right)^{-\gamma_R}\right] + \left(\frac{n\tau}{k_1}\right)^{-\gamma_R}\right\} \\
&= (1 + \mathrm{II} + \mathrm{III})\left[\varepsilon_{(k_1)}/U_\varepsilon(n/k_1) - 1\right] = \left[\varepsilon_{(k_1)}/U_\varepsilon(n/k_1) - 1\right] + o_p(k_1^{-1/2}),
\end{aligned}$$

from which we conclude (S1.20).

(ii) It is noted that $U_\varepsilon(1/\tau) = Q_\varepsilon(1-\tau)$ and $U_{-\varepsilon}(1/\tau) = -Q_\varepsilon(\tau)$. According to Lemma S1.9 (ii) and carrying out the same approach as in the proof of part (i), with $\varepsilon_{(k_1)}$, $\varepsilon_{(n-k_2-d_n)}$, $\hat{\gamma}_{\varepsilon,R}$, $\hat{\gamma}_{\varepsilon,L}$, $\hat{\varepsilon}_R(\tau)$ and $\hat{\varepsilon}_L(\tau)$ replaced by $\hat{\varepsilon}_{(k_1)}$, $\hat{\varepsilon}_{(n-k_2-d_n)}$, $\hat{\gamma}_R$, $\hat{\gamma}_L$, $\hat{Q}_\varepsilon(1-\tau)$ and $\hat{Q}_\varepsilon(\tau)$, we can finish the proof of this part.

□

Let $\ell^\infty([1,\Delta])$ be the collection of all bounded functions on $[1,\Delta]$.

Lemma S1.11. *(i.i.d. residuals) Assume the conditions of Lemma S1.10 (i) hold.*

(i) If $c_1 = c_2 = 0$, assume there exists a positive constant c_0 such that $\lim_{n\to\infty} k_1/k_2 = c_0$. Denote for $u \in [1,\Delta]$,

$$
\begin{aligned}
S_{1U,n}(u) &= \frac{k_1^{1/2}}{\log(k_1/(nu\tau_l))} \log\left[\frac{\hat{\varepsilon}_R(u\tau_l)}{U_\varepsilon(1/(u\tau_l))}\right], \\
S_{1D,n}(u) &= \frac{k_2^{1/2}}{\log(k_2/(nu\tau_l))} \log\left[\frac{-\hat{\varepsilon}_L(u\tau_l)}{U_{-\varepsilon}(1/(u\tau_l))}\right] \text{ and} \\
S_{1R,n}(u) &= \frac{k_1^{1/2}}{\log(k_1/(nu\tau_l))} \left\{\log\left[\frac{\hat{\varepsilon}_R(u\tau_l)}{-\hat{\varepsilon}_L(u\tau_l)}\right] - \log\left[\frac{U_\varepsilon(1/(u\tau_l))}{U_{-\varepsilon}(1/(u\tau_l))}\right]\right\},
\end{aligned}
$$

then $S_{1U,n}(u) \xrightarrow{\mathscr{D}} Z_U$, $S_{1D,n}(u) \xrightarrow{\mathscr{D}} Z_D$ and $S_{1R,n}(u) \xrightarrow{\mathscr{D}} Z_U - c_0^{1/2} Z_D$ in $\ell^\infty([1,\Delta])$, where $Z_U \sim \mathscr{N}(0,\gamma_R^2)$ and $Z_D \sim \mathscr{N}(0,\gamma_L^2)$ are independent.

(ii) If $c_1 > 0$ and $c_2 > 0$, denote

$$
\begin{aligned}
S_{2U,n}(u) &= k_1^{1/2} \log\left[\frac{\hat{\varepsilon}_R(u\tau_l)}{U_\varepsilon(1/(u\tau_l))}\right], \\
S_{2D,n}(u) &= k_2^{1/2} \log\left[\frac{-\hat{\varepsilon}_L(u\tau_l)}{U_{-\varepsilon}(1/(u\tau_l))}\right] \text{ and} \\
S_{2R,n}(u) &= k_1^{1/2} \left\{\log\left[\frac{\hat{\varepsilon}_R(u\tau_l)}{-\hat{\varepsilon}_L(u\tau_l)}\right] - \log\left[\frac{U_\varepsilon(1/(u\tau_l))}{U_{-\varepsilon}(1/(u\tau_l))}\right]\right\}.
\end{aligned}
$$

Then $S_{2U,n}(u) \xrightarrow{\mathscr{D}} G_{1,U}(u)$, $S_{2D,n}(u) \xrightarrow{\mathscr{D}} G_{1,D}(u)$ and $S_{2R,n}(u) \xrightarrow{\mathscr{D}} G_{1,U}(u) - c_1^{-1/2} c_2^{1/2} G_{1,D}(u)$ in $\ell^\infty([1,\Delta])$, where $G_{1,U}(u)$ and $G_{1,D}(u)$ are two independent centered Gaussian processes with covariance functions

$$
\begin{aligned}
\mathrm{Cov}(G_{1,U}(u_1), G_{1,U}(u_2)) &= \gamma_R^2 [1 + \log(c_1 u_1)\log(c_1 u_2)], \\
\mathrm{Cov}(G_{1,D}(u_1), G_{1,D}(u_2)) &= \gamma_L^2 [1 + \log(c_2 u_1)\log(c_2 u_2)].
\end{aligned}
$$

Proof: (i) When $c_1 = c_2 = 0$, Lemma S1.10 (i) indicates that uniformly in $u \in [1, \Delta]$,

$$S_{1U,n}(u) = \frac{k_1^{1/2}}{\log(k_1/(nu\tau_l))}\left[\frac{\hat{\varepsilon}_R(u\tau_l)}{U_\varepsilon\left(1/(u\tau_l)\right)} - 1\right] + o_p(1) = k_2^{1/2}(\hat{\gamma}_{\varepsilon,L} - \gamma_L) + o_p(1),$$
$$S_{1D,n}(u) = \frac{k_1^{1/2}}{\log(k_1/(nu\tau_l))}\left[\frac{-\hat{\varepsilon}_L(u\tau_l)}{U_{-\varepsilon}\left(1/(u\tau_l)\right)} - 1\right] + o_p(1) = k_2^{1/2}(\hat{\gamma}_{\varepsilon,L} - \gamma_L) + o_p(1).$$

By simple algebra, uniformly in $u \in [1, \Delta]$,

$$S_{1R,n}(u) = S_{1U,n}(u) - \left(\frac{k_1}{k_2}\right)^{1/2} \frac{\log(k_2/(nu\tau_l))}{\log(k_1/(nu\tau_l))} S_{1D,n}(u).$$

According to Lemma S1.9 (i), $\left(k_1^{1/2}\gamma_R^{-1}(\hat{\gamma}_{\varepsilon,R} - \gamma_R), k_2^{1/2}\gamma_L^{-1}(\hat{\gamma}_{\varepsilon,L} - \gamma_L)\right)$ is asymptotically bivariate standard normal. The results follow immediately as $k_1/k_2 \to c_0$ and $\log(k_2/(nu\tau_l))/\log(k_1/(nu\tau_l)) \to 1$.

(ii) When $c_1 > 0$ and $c_2 > 0$, according to Lemma S1.10 (i), we have uniformly in $u \in [1, \Delta]$,

$$\begin{aligned} S_{2U,n}(u) &= k_1^{1/2} \log\left[\hat{\varepsilon}_R(u\tau_l)/U_\varepsilon(1/(u\tau_l)) - 1 + 1\right] \\ &= k_1^{1/2}\left[\hat{\varepsilon}_R(u\tau_l)/U_\varepsilon(1/(u\tau_l)) - 1\right] + o_p(1) \\ &= k_1^{1/2}\left[\frac{\varepsilon_{(k_1)}}{U_\varepsilon\left(n/k_1\right)} - 1\right] + \left[\log\left(\frac{k_1}{n\tau_l}\right) - \log(u)\right] k_1^{1/2}(\hat{\gamma}_{\varepsilon,R} - \gamma_R) + o_p(1). \end{aligned}$$

For all $u_1, \ldots, u_m \in [1, \Delta]$ with any fixed $m \in \mathbb{N}$, Lemma S1.9 (i) and continuous mapping theorem yields $\left(S_{2U,n}(u_1), \ldots, S_{2U,n}(u_m)\right)$ converges in distribution to $(G_{1,U}(u_1), \ldots, G_{1,U}(u_m))$, which follows a m-dimensional normal distribution with covariance function

$$\mathrm{Cov}(G_{1,U}(u_1), G_{1,U}(u_2)) = \gamma_R^2\left[1 + \log(c_1 u_1)\log(c_1 u_2)\right].$$

In order to show $S_{2U,n}(u) \xrightarrow{\mathscr{D}} G_{1,U}(u)$ in $\ell^\infty([1, \Delta])$, it is sufficient to show that $S'_{2U,n}(u) \xrightarrow{\mathscr{D}} G_{1,U}(u)$ in $\ell^\infty([1, \Delta])$, where $S'_{2U,n}(u) = k_1^{1/2}\left[\frac{\varepsilon_{(k_1)}}{U_\varepsilon(n/k_1)} - 1\right] + \left[\log\left(\frac{k_1}{n\tau_l}\right) - \log(u)\right] k_1^{1/2}(\hat{\gamma}_{\varepsilon,R} - \gamma_R)$.

According to Theorem 1.5.4 of van der Vaart and Wellner (1996), it is enough to show the asymptotic tightness of $S'_{2U,n}(u)$. Regarding Theorem 1.5.7 of van der Vaart and Wellner (1996), it suffices to show that

$$\limsup_{n\to\infty} \Pr\left(\sup_{|u_1 - u_2| \le \delta} |S'_{2U,n}(u_1) - S'_{2U,n}(u_2)| \ge \epsilon\right) \to 0 \tag{S1.24}$$

as $\delta \to 0$ for any positive ϵ. It is mentioned that

$$\sup_{0\le u_2-u_1\le\delta} |S_{2U,n}(u_1)-S_{2U,n}(u_2)| \le \sup_{0\le u_2-u_1\le\delta} \log(u_2/u_1)\left|k_1^{1/2}(\hat\gamma_{\varepsilon,R}-\gamma_R)\right| \le \log(1+\delta)\left|k_1^{1/2}(\hat\gamma_{\varepsilon,R}-\gamma_R)\right|.$$

Thus, we have that

$$\begin{aligned} & \Pr\left(\sup_{0\le u_2-u_1\le\delta} |S'_{2U,n}(u_1)-S'_{2U,n}(u_2)| \ge \epsilon\right) \le \Pr\left(\log(1+\delta)\left|k_1^{1/2}(\hat\gamma_{\varepsilon,R}-\gamma_R)\right| \ge \epsilon\right) \\ = & \Pr\left(\left|k_1^{1/2}(\hat\gamma_{\varepsilon,R}-\gamma_R)\right| \ge \epsilon/\log(1+\delta)\right) \to 0 \end{aligned}$$

as $n\to\infty$ and $\delta\to 0$ for any positive ϵ. Similarly, we can show that

$$\Pr\left(\sup_{0\le u_1-u_2\le\delta} |S'_{2U,n}(u_1)-S'_{2U,n}(u_2)| \ge \epsilon\right) \to 0$$

as $n\to\infty$ and $\delta\to 0$ for any positive ϵ. Finally, we obtain (S1.24) and thus $S_{2U,n}(u) \xrightarrow{\mathscr{D}} G_{1,U}(u)$ in $\ell^\infty([1,\Delta])$. The result of $S_{2D,n}(u) \xrightarrow{\mathscr{D}} G_{1,D}(u)$ in $\ell^\infty([1,\Delta])$ can be proved similarly.

Now we consider $S_{2R,n}(u)$. By simple algebra, $S_{2R,n}(u) = S_{2U,n}(u) - (k_1/k_2)^{1/2} S_{2D,n}(u)$ uniformly in $u\in[1,\Delta]$. According to Lemma S1.9 (i), it can be shown that for all $u_1,\ldots,u_m \in [1,\Delta]$ with any fixed $m\in\mathbb{N}$, $\big(S_{2R,n}(u_1),\ldots,S_{2R,n}(u_m)\big)$ converges in distribution to $(G_{1,U}(u_1)-c_1^{-1/2}c_2^{1/2}G_{1,D}(u_1),\ldots,G_{1,U}(u_m)-c_1^{-1/2}c_2^{1/2}G_{1,D}(u_m))$. The asymptotic tightness of $S_{2R,n}(u)$ can be proved in the same manner as for $S_{2U,n}(u)$. Thus, we finish the proof of this lemma. □

Lemma S1.12. *(standardized residuals) Suppose the conditions of Lemma S1.10 (ii) hold.*

(i) If $c_1 = c_2 = 0$, assume there exists a positive constant c_0 such that $\lim_{n\to\infty} k_1/k_2 = c_0$. Denote for $u\in[1,\Delta]$,

$$\begin{aligned} S_{3U,n}(u) &= \frac{k_1^{1/2}}{\log(k_1/(nu\tau_l))}\log\left[\frac{\hat Q_\varepsilon(1-u\tau_l)}{Q_\varepsilon(1-u\tau_l)}\right], \\ S_{3D,n}(u) &= \frac{k_2^{1/2}}{\log(k_2/(nu\tau_l))}\log\left[\frac{\hat Q_\varepsilon(u\tau_l)}{Q_\varepsilon(u\tau_l)}\right] \text{ and} \\ S_{3R,n}(u) &= \frac{k_1^{1/2}}{\log(k_1/(nu\tau_l))}\left\{\log\left[\frac{\hat Q_\varepsilon(1-u\tau_l)}{-\hat Q_\varepsilon(u\tau_l)}\right]-\log\left[\frac{Q_\varepsilon(1-u\tau_l)}{-Q_\varepsilon(u\tau_l)}\right]\right\}. \end{aligned}$$

Then $S_{3U,n}(u) \xrightarrow{\mathscr{D}} Z_U$, $S_{3D,n}(u) \xrightarrow{\mathscr{D}} Z_D$ and $S_{3R,n}(u) \xrightarrow{\mathscr{D}} Z_U - c_0^{1/2}Z_D$ in $\ell^\infty([1,\Delta])$, where $Z_U \sim \mathscr{N}(0,\gamma_R^2)$ and $Z_D \sim \mathscr{N}(0,\gamma_L^2)$ are independently distributed.

(ii) If $c_1 > 0$ and $c_2 > 0$, denote

$$S_{4U,n}(u) = k_1^{1/2} \log\left[\frac{\hat{Q}_\varepsilon(1-u\tau_l)}{Q_\varepsilon(1-u\tau_l)}\right],$$
$$S_{4D,n}(u) = k_2^{1/2} \log\left[\frac{\hat{Q}_\varepsilon(u\tau_l)}{Q_\varepsilon(u\tau_l)}\right] \text{ and}$$
$$S_{4R,n}(u) = k_1^{1/2} \left\{\log\left[\frac{\hat{Q}_\varepsilon(1-u\tau_l)}{-\hat{Q}_\varepsilon(u\tau_l)}\right] - \log\left[\frac{Q_\varepsilon(1-u\tau_l)}{-Q_\varepsilon(u\tau_l)}\right]\right\}.$$

Then $S_{4U,n}(u) \xrightarrow{\mathscr{D}} G_{1,U}(u)$, $S_{4D,n}(u) \xrightarrow{\mathscr{D}} G_{1,D}(u)$ and $S_{4R,n}(u) \xrightarrow{\mathscr{D}} G_{1,U}(u) - c_1^{-1/2}c_2^{1/2}G_{1,D}(u)$ in $\ell^\infty([1,\Delta])$, where $G_{1,U}(u)$ and $G_{1,D}(u)$ are two independent centered Gaussian processes with covariance functions

$$\text{Cov}(G_{1,U}(u_1), G_{1,U}(u_2)) = \gamma_R^2 [1 + \log(c_1u_1)\log(c_1u_2)],$$
$$\text{Cov}(G_{1,D}(u_1), G_{1,D}(u_2)) = \gamma_L^2 [1 + \log(c_2u_1)\log(c_2u_2)].$$

Proof: Regarding Lemma S1.9 (ii) and Lemma S1.10 (ii), the proof of this lemma is similar to that of Lemma S1.11 and thus is omitted here. □

Lemma S1.13. *Denote $M_{\varepsilon,R}(\tau) = E[\varepsilon_t|\varepsilon_t > U_\varepsilon(1/\tau)]$ and $M_{\varepsilon,L}(\tau) = E[\varepsilon_t|\varepsilon_t < -U_{-\varepsilon}(1/\tau)]$.*

(i) (i.i.d. residuals) Under the conditions assumed in Lemma S1.10 (i), define $\hat{M}_{\varepsilon,R}(\tau) = \hat{\varepsilon}_R(\tau)/(1-\hat{\gamma}_{\varepsilon,R})$ and $\hat{M}_{\varepsilon,L}(\tau) = \hat{\varepsilon}_L(\tau)/(1-\hat{\gamma}_{\varepsilon,L})$. Then uniformly in $\tau \in [\tau_l, \tau_u]$,

$$\frac{\hat{M}_{\varepsilon,R}(\tau)}{M_{\varepsilon,R}(\tau)} - 1 = \left[\frac{\varepsilon_{(k_1)}}{U_\varepsilon(n/k_1)} - 1\right] + \left[\log\left(\frac{k_1}{n\tau}\right) + \frac{1}{1-\gamma_R}\right](\hat{\gamma}_{\varepsilon,R} - \gamma_R) + o_p(k_1^{-1/2}\log(n\tau/k_1)), \quad \text{(S1.25)}$$

and

$$\frac{\hat{M}_{\varepsilon,L}(\tau)}{M_{\varepsilon,L}(\tau)} - 1 = \left[\frac{-\varepsilon_{(n-k_2-d_n)}}{U_{-\varepsilon}(n/k_2)} - 1\right] + \left[\log\left(\frac{k_2}{n\tau}\right) + \frac{1}{1-\gamma_L}\right](\hat{\gamma}_{\varepsilon,L} - \gamma_L) + o_p(k_2^{-1/2}\log(n\tau/k_2)). \quad \text{(S1.26)}$$

(ii) (standardized residuals) Suppose that Assumption 2.1, 3.1, 3.2, 3.3, 3.4 and 3.5 hold, we have uniformly in $\tau \in [\tau_l, \tau_u]$,

$$\begin{aligned}&\frac{\hat{E}[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]}{E[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]} - 1 \\ = \;& \left[\frac{\varepsilon_{(k_1)}}{U_\varepsilon(n/k_1)} - 1\right] + \left[\log\left(\frac{k_1}{n\tau}\right) + \frac{1}{1-\gamma_R}\right](\hat{\gamma}_{\varepsilon,R} - \gamma_R) + o_p(k_1^{-1/2}\log(n\tau/k_1)), \quad \text{(S1.27)}\end{aligned}$$

and

$$\begin{aligned}&\frac{\hat{E}[\varepsilon|\varepsilon<Q_\varepsilon(\tau)]}{E[\varepsilon|\varepsilon<Q_\varepsilon(\tau)]}-1\\ =\;&\left[\frac{-\varepsilon_{(n-k_2-d_n)}}{U_{-\varepsilon}(n/k_2)}-1\right]+\left[\log\left(\frac{k_2}{n\tau}\right)+\frac{1}{1-\gamma_L}\right](\hat{\gamma}_{\varepsilon,L}-\gamma_L)+o_p(k_2^{-1/2}\log(n\tau/k_2)).\end{aligned}\tag{S1.28}$$

Proof: We only show the proof for $\hat{M}_{\varepsilon,R}(\tau)$, the proof of (S1.26) is similar and thus is omitted.

Write

$$\frac{\hat{M}_{\varepsilon,R}(\tau)}{M_{\varepsilon,R}(\tau)}-1=\frac{\hat{\varepsilon}_R(\tau)}{U_\varepsilon(1/\tau)}\frac{U_\varepsilon(1/\tau)/(1-\gamma_R)}{M_{\varepsilon,R}(\tau)}\frac{1-\gamma_R}{1-\hat{\gamma}_{\varepsilon,R}}-1.$$

According to Lemma S1.10 (i), we have uniformly in $\tau\in[\tau_l,\tau_u]$,

$$\frac{\hat{\varepsilon}_R(\tau)}{U_\varepsilon(1/\tau)}=1+\left[\frac{\varepsilon_{(k_1)}}{U_\varepsilon(n/k_1)}-1\right]+\log\left(\frac{k_1}{n\tau}\right)(\hat{\gamma}_{\varepsilon,R}-\gamma_R)+o_p(k_1^{-1/2}\log(n\tau/k_1)).\tag{S1.29}$$

From Lemma S1.9 (i), we have that

$$\frac{1-\gamma_R}{1-\hat{\gamma}_{\varepsilon,R}}=\frac{1}{1-(\hat{\gamma}_{\varepsilon,R}-\gamma_R)/(1-\gamma_R)}=1+(\hat{\gamma}_{\varepsilon,R}-\gamma_R)/(1-\gamma_R)+o_p(k_1^{-1/2}).\tag{S1.30}$$

By Lemma S1.1, we have that for any $\epsilon>0$ and $0<\delta<1/\gamma_R-\rho_R/\gamma_R-1$, there exists $x_0=x_0(\epsilon,\delta,\gamma_R,\rho_R)$ such that for all $x\geq x_0$,

$$\begin{aligned}&\left|\frac{1}{A_R(1/(1-F_\varepsilon(x)))}\left[\frac{E(\varepsilon_t|\varepsilon_t>x)}{x}-\frac{1}{1-\gamma_R}\right]-\frac{1}{(1-\gamma_R)(1-\gamma_R-\rho_R)}\right|\\ \leq\;&\epsilon\left[\frac{1}{\rho_R(1-\gamma_R)}+\frac{1}{\gamma_R(1-\gamma_R-\rho_R-\delta\gamma_R)}\right].\end{aligned}$$

Thus, for sufficient large n such that $U_\varepsilon(1/\tau_u)\geq x_0$, we have uniformly for all $\tau\in[\tau_l,\tau_u]$,

$$\begin{aligned}&\left|\frac{1}{A_R(1/(1-F_\varepsilon(U_\varepsilon(1/\tau))))}\left[\frac{M_{\varepsilon,R}(\tau)}{U_\varepsilon(1/\tau)}-\frac{1}{1-\gamma_R}\right]-\frac{1}{(1-\gamma_R)(1-\gamma_R-\rho_R)}\right|\\ \leq\;&\epsilon\left[\frac{1}{\rho_R(1-\gamma_R)}+\frac{1}{\gamma_R(1-\gamma_R-\rho_R-\delta\gamma_R)}\right].\end{aligned}$$

As $F_\varepsilon(U_\varepsilon(1/\tau))=F_\varepsilon(F_\varepsilon^{\leftarrow}(1-\tau))\geq 1-\tau$, then for sufficient large n, there exists a positive constant

C_1, such that $1/(1-F_\varepsilon(U_\varepsilon(1/\tau))) \geq 1/\tau \geq C_1 n/k_1$ for all $\tau \in [\tau_l, \tau_u]$, which implies

$$A_R(1/(1-F_\varepsilon(U_\varepsilon(1/\tau)))) = O(A_R(n/k_1)) = o(k_1^{-1/2}) \quad \text{uniformly in } \tau \in [\tau_l, \tau_u].$$

Thus we conclude that

$$\frac{U_\varepsilon(1/\tau)/(1-\gamma_R)}{M_{\varepsilon,R}(\tau)} = 1 + o(k_1^{-1/2}) \quad \text{uniformly in } \tau \in [\tau_l, \tau_u]. \tag{S1.31}$$

Combining (S1.29), (S1.30), and (S1.31), we have shown (S1.25).

(ii) It is noted that $E[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)] = M_{\varepsilon,R}(\tau)$ and $E[\varepsilon|\varepsilon < Q_\varepsilon(\tau)] = M_{\varepsilon,L}(\tau)$. According to Lemma S1.9 (ii) and carrying out the same approach as in the proof of part (i), with $\varepsilon_{(k_1)}$, $\varepsilon_{(n-k_2-d_n)}$, $\hat{\gamma}_{\varepsilon,R}$, $\hat{\gamma}_{\varepsilon,L}$, $\hat{M}_{\varepsilon,R}(\tau)$ and $\hat{M}_{\varepsilon,L}(\tau)$ replaced by $\hat{\varepsilon}_{(k_1)}$, $\hat{\varepsilon}_{(n-k_2-d_n)}$, $\hat{\gamma}_R$, $\hat{\gamma}_L$, $\hat{E}[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]$ and $\hat{E}[\varepsilon|\varepsilon < Q_\varepsilon(\tau)]$, we can finish the proof of this part. □

Lemma S1.14. *(i.i.d. residuals) Assume the conditions of Lemma S1.13 (i) hold.*

(i) If $c_1 = c_2 = 0$, assume there exists a positive constant c_0 such that $\lim_{n\to\infty} k_1/k_2 = c_0$. Denote for $u \in [1, \Delta]$,

$$W_{1U,n}(u) = \frac{k_1^{1/2}}{\log(k_1/(nu\tau_l))} \log\left[\frac{\hat{M}_{\varepsilon,R}(\tau)}{M_{\varepsilon,R}(\tau)}\right],$$
$$W_{1D,n}(u) = \frac{k_2^{1/2}}{\log(k_2/(nu\tau_l))} \log\left[\frac{\hat{M}_{\varepsilon,L}(\tau)}{M_{\varepsilon,L}(\tau)}\right] \text{ and}$$
$$W_{1R,n}(u) = \frac{k_1^{1/2}}{\log(k_1/(nu\tau_l))} \left\{\log\left[\frac{-\hat{M}_{\varepsilon,L}(\tau)}{\hat{M}_{\varepsilon,L}(\tau)}\right] - \log\left[\frac{-M_{\varepsilon,R}(\tau)}{M_{\varepsilon,L}(\tau)}\right]\right\},$$

then $W_{1U,n}(u) \xrightarrow{\mathscr{D}} Z_U$, $W_{1D,n}(u) \xrightarrow{\mathscr{D}} Z_D$ and $W_{1R,n}(u) \xrightarrow{\mathscr{D}} Z_U - c_0^{1/2} Z_D$ in $\ell^\infty([1,\Delta])$, where $Z_U \sim \mathcal{N}(0, \gamma_R^2)$ and $Z_D \sim \mathcal{N}(0, \gamma_L^2)$ are independently distributed.

(ii) If $c_1 > 0$ and $c_2 > 0$, denote

$$W_{2U,n}(u) = k_1^{1/2} \log\left[\frac{\hat{M}_{\varepsilon,R}(\tau)}{M_{\varepsilon,R}(\tau)}\right],$$
$$W_{2D,n}(u) = k_2^{1/2} \log\left[\frac{\hat{M}_{\varepsilon,L}(\tau)}{M_{\varepsilon,L}(\tau)}\right] \text{ and}$$
$$W_{2R,n}(u) = k_1^{1/2} \left\{\log\left[\frac{-\hat{M}_{\varepsilon,L}(\tau)}{\hat{M}_{\varepsilon,L}(\tau)}\right] - \log\left[\frac{-M_{\varepsilon,R}(\tau)}{M_{\varepsilon,L}(\tau)}\right]\right\}.$$

Then $W_{2U,n}(u) \xrightarrow{\mathscr{D}} G_{2,U}(u)$, $W_{2D,n}(u) \xrightarrow{\mathscr{D}} G_{2,D}(u)$ *and* $W_{2R,n}(u) \xrightarrow{\mathscr{D}} G_{2,U}(u) - c_1^{-1/2} c_2^{1/2} G_{2,D}(u)$ *in* $\ell^\infty([1,\Delta])$, *where* $G_{2,U}(u)$ *and* $G_{2,D}(u)$ *are two independent centered Gaussian processes with covariance functions*

$$\begin{aligned}
\mathrm{Cov}(G_{2,U}(u_1), G_{2,U}(u_2)) &= \gamma_R^2 \left[1 + \left\{(1-\gamma_R)^{-1} - \log(c_1 u_1)\right\}\left\{(1-\gamma_R)^{-1} - \log(c_1 u_2)\right\}\right], \\
\mathrm{Cov}(G_{2,D}(u_1), G_{2,D}(u_2)) &= \gamma_L^2 \left[1 + \left\{(1-\gamma_L)^{-1} - \log(c_2 u_1)\right\}\left\{(1-\gamma_L)^{-1} - \log(c_2 u_2)\right\}\right].
\end{aligned}$$

Proof: The proof of this lemma, based on Lemmas S1.9 (i) and S1.13 (i), is similar to that of Lemma S1.11 and thus is omitted. □

Lemma S1.15. *(standardized residuals) Assume the conditions of Lemma S1.13 (ii) hold.*

(i) If $c_1 = c_2 = 0$, *assume there exists a positive constant* c_0 *such that* $\lim_{n\to\infty} k_1/k_2 = c_0$. *Denote for* $u \in [1,\Delta]$,

$$\begin{aligned}
W_{3U,n}(u) &= \frac{k_1^{1/2}}{\log(k_1/(nu\tau_l))} \log\left\{\frac{\hat{E}[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]}{E[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]}\right\}, \\
W_{3D,n}(u) &= \frac{k_2^{1/2}}{\log(k_2/(nu\tau_l))} \log\left\{\frac{\hat{E}[\varepsilon|\varepsilon < Q_\varepsilon(\tau)]}{E[\varepsilon|\varepsilon < Q_\varepsilon(\tau)]}\right\} \text{ and} \\
W_{3R,n}(u) &= \frac{k_1^{1/2}}{\log(k_1/(nu\tau_l))} \left(\log\left\{\frac{\hat{E}[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]}{-\hat{E}[\varepsilon|\varepsilon < Q_\varepsilon(\tau)]}\right\} - \log\left\{\frac{E[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]}{-E[\varepsilon|\varepsilon < Q_\varepsilon(\tau)]}\right\}\right),
\end{aligned}$$

then $W_{3U,n}(u) \xrightarrow{\mathscr{D}} Z_U$, $W_{3D,n}(u) \xrightarrow{\mathscr{D}} Z_D$ *and* $W_{3R,n}(u) \xrightarrow{\mathscr{D}} Z_U - c_0^{1/2} Z_D$ *in* $\ell^\infty([1,\Delta])$, *where* $Z_U \sim \mathcal{N}(0, \gamma_R^2)$ *and* $Z_D \sim \mathcal{N}(0,\gamma_L^2)$ *are independently distributed.*

(ii) If $c_1 > 0$ *and* $c_2 > 0$, *denote*

$$\begin{aligned}
W_{4U,n}(u) &= k_1^{1/2} \log\left\{\frac{\hat{E}[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]}{E[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]}\right\}, \\
W_{4D,n}(u) &= k_2^{1/2} \log\left\{\frac{\hat{E}[\varepsilon|\varepsilon < Q_\varepsilon(\tau)]}{E[\varepsilon|\varepsilon < Q_\varepsilon(\tau)]}\right\} \text{ and} \\
W_{4R,n}(u) &= k_1^{1/2} \left(\log\left\{\frac{\hat{E}[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]}{-\hat{E}[\varepsilon|\varepsilon < Q_\varepsilon(\tau)]}\right\} - \log\left\{\frac{E[\varepsilon|\varepsilon > Q_\varepsilon(1-\tau)]}{-E[\varepsilon|\varepsilon < Q_\varepsilon(\tau)]}\right\}\right).
\end{aligned}$$

Then $W_{4U,n}(u) \xrightarrow{\mathscr{D}} G_{2,U}(u)$, $W_{4D,n}(u) \xrightarrow{\mathscr{D}} G_{2,D}(u)$ *and* $W_{4R,n}(u) \xrightarrow{\mathscr{D}} G_{2,U}(u) - c_1^{-1/2} c_2^{1/2} G_{2,D}(u)$ *in* $\ell^\infty([1,\Delta])$, *where* $G_{2,U}(u)$ *and* $G_{2,D}(u)$ *are two independent centered Gaussian processes with covariance functions*

$$\begin{aligned}
\mathrm{Cov}(G_{2,U}(u_1), G_{2,U}(u_2)) &= \gamma_R^2 \left[1 + \left\{(1-\gamma_R)^{-1} - \log(c_1 u_1)\right\}\left\{(1-\gamma_R)^{-1} - \log(c_1 u_2)\right\}\right], \\
\mathrm{Cov}(G_{2,D}(u_1), G_{2,D}(u_2)) &= \gamma_L^2 \left[1 + \left\{(1-\gamma_L)^{-1} - \log(c_2 u_1)\right\}\left\{(1-\gamma_L)^{-1} - \log(c_2 u_2)\right\}\right].
\end{aligned}$$

Proof: The proof of this lemma, based on Lemma S1.9 (ii) and Lemmas S1.13 (ii), is similar to that of Lemma S1.11 and thus is omitted. □

Lemma S1.16. *Suppose Assumptions 2.1, 3.1, 3.2 and 3.3 hold, then*

$$m(\tilde{\mathscr{I}}_n, \hat{\theta}) - m(\mathscr{I}_n, \theta_0) = O_p(n^{-\upsilon_0/2}) \ \textit{and} \ \sigma(\tilde{\mathscr{I}}_n, \hat{\theta}) - \sigma(\mathscr{I}_n, \theta_0) = O_p(n^{-\upsilon_0/2}).$$

Proof: Write $m(\tilde{\mathscr{I}}_n, \hat{\theta}) - m(\mathscr{I}_n, \theta_0) = m(\tilde{\mathscr{I}}_n, \hat{\theta}) - m(\mathscr{I}_n, \hat{\theta}) + m(\mathscr{I}_n, \hat{\theta}) - m(\mathscr{I}_n, \theta_0)$, where $m(\tilde{\mathscr{I}}_n, \hat{\theta}) - m(\mathscr{I}_n, \hat{\theta}) = O_p(n^{-\upsilon_0/2})$ according to Assumption 3.3. It remains to show that $m(\mathscr{I}_n, \hat{\theta}) - m(\mathscr{I}_n, \theta_0) = O_p(n^{-\upsilon_0/2})$. By the mean value theorem, there exists a $\theta^* \in \Theta_0$ between θ_0 and $\hat{\theta}$, such that

$$\left| m(\mathscr{I}_n, \hat{\theta}) - m(\mathscr{I}_n, \theta_0) \right| = \left| \frac{\partial m(\mathscr{I}_n, \theta)}{\partial \theta} \bigg|_{\theta=\theta^*} (\hat{\theta} - \theta_0) \right| \leq \sup_{\theta \in \Theta_0} \left\| \frac{\partial m(\mathscr{I}_n, \theta)}{\partial \theta} \right\| \cdot \|\hat{\theta} - \theta_0\|,$$

which is $O_p(n^{-\upsilon_0/2})$ according to Assumption 3.1, 3.2 (iii) and Markov's inequality.

The proof of $\sigma(\tilde{\mathscr{I}}_n, \hat{\theta}) - \sigma(\mathscr{I}_n, \theta_0) = O_p(n^{-\upsilon_0/2})$ is similar and thus is omitted. □

S2 Extension to nonparametric models

In this section, we investigate the extension of our methodologies and theoretical results to the nonparametric model

$$R_t = m(\mathbf{X}_t) + \sigma(\mathbf{X}_t)\varepsilon_t, \tag{S2.1}$$

where $\mathbf{X}_t$ is a d-dimensional random vector which may include lagged returns $\{R_{t-\ell}\}_{\ell=1}^{p}$ for some positive integer p, $m(\cdot)$ and $\sigma(\cdot)$ are nonparametric functions defined on the range of $\mathbf{X}_t$.

Let $\hat{m}(\cdot)$ and $\hat{\sigma}(\cdot)$ be nonparametric estimators of $m(\cdot)$ and $\sigma(\cdot)$, and $\hat{\varepsilon}_t^* = (R_t - \hat{m}(\mathbf{X}_t))/\hat{\sigma}(\mathbf{X}_t)$ for $t = 1, \ldots, n$ be the corresponding standardized nonparametric residuals, which are utilized to derive estimations of CVaR and CES in the same manner as in Section 2.2 with $\hat{\varepsilon}_t$ replaced by $\hat{\varepsilon}_t^*$.

By observing the technical derivations in the proofs of main theorems, the only adjustment needed to obtain theoretical results in Sections 3 and 4 of this paper is to prove the same results of Lemma S1.7 for $\hat{\varepsilon}_t^*$, which quantifies the difference between the tail empirical distributions of $\hat{\varepsilon}_t^*$ and ε_t. We have the following lemma to justify this point.

Lemma S2.1. *Consider the model* $R_t = m(\mathbf{X}_t) + \sigma(\mathbf{X}_t)\varepsilon_t$*, where* $\{\mathbf{X}_t\}_{t=1}^{\infty}$ *is a strictly stationary process,* ε_t *is independent of* $\mathbf{X}_t$*,* $m(\mathbf{X}_t)$ *and* $\sigma(\mathbf{X}_t)$ *are measurable with respect to* $\mathscr{I}_{t-1}$*, the* σ*-algebra generated by* $\varepsilon_1, \ldots, \varepsilon_{t-1}$*. In addition,* $\inf_{\mathbf{x}\in\mathscr{G}} \sigma(\mathbf{x}) > c$ *for some* $c > 0$*, where* $\mathscr{G}$ *is the support set of* $\mathbf{X}_t$*. Let* $\hat{m}(\cdot)$ *and* $\hat{\sigma}(\cdot)$ *be nonparametric estimators of* $m(\cdot)$ *and* $\sigma(\cdot)$ *that satisfy the following conditions:*

(i) $\hat{m}(\cdot)$ *and* $\hat{\sigma}(\cdot)$ *are measurable with respect to* $\mathscr{I}_{t-1}$*;*

(ii) $\sup_{\mathbf{x}\in\mathscr{G}} |\hat{m}(\mathbf{x}) - m(\mathbf{x})| = O_p(n^{-\upsilon_0/2})$ *and* $\sup_{\mathbf{x}\in\mathscr{G}} |\hat{\sigma}(\mathbf{x}) - \sigma(\mathbf{x})| = O_p(n^{-\upsilon_0/2})$ *for some positive* $\upsilon_0 \leq 1$*;*

(iii) $E\left[\sup_{\mathbf{x}\in\mathscr{G}} |\hat{m}(\mathbf{x}) - m(\mathbf{x})|\right] = O(n^{-\upsilon_2})$ *and* $E\left[\sup_{\mathbf{x}\in\mathscr{G}} |\hat{\sigma}(\mathbf{x}) - \sigma(\mathbf{x})|\right] = O(n^{-\upsilon_2})$ *for some positive* υ_2*.*

Denote $\hat{\varepsilon}_t^* = (R_t - \hat{m}(\mathbf{X}_t))/\hat{\sigma}(\mathbf{X}_t)$ *for* $t = 1, \ldots, n$*. Suppose that Assumptions 3.4 and 3.5 hold. Then for all* $\iota > 0$ *and* $\upsilon \in [0, 1/2)$*,*

$$\sup_{x\in(0,1+\iota]} x^{-\upsilon} k_1^{1/2} \left| k_1^{-1} \sum_{t=d_n}^{n} \left[\mathbb{I}\left(\hat{\varepsilon}_t^* > x^{-\gamma_R} U_\varepsilon(n/k_1)\right) - \mathbb{I}\left(\varepsilon_t > x^{-\gamma_R} U_\varepsilon(n/k_1)\right)\right] \right| = o_p(1) \tag{S2.2}$$

and

$$\sup_{x\in(0,1+\iota]} x^{-\upsilon} k_2^{1/2} \left| k_2^{-1} \sum_{t=d_n}^{n} \left[\mathbb{I}\left(-\hat{\varepsilon}_t^* > x^{-\gamma_L} U_{-\varepsilon}(n/k_2)\right) - \mathbb{I}\left(-\varepsilon_t > x^{-\gamma_L} U_{-\varepsilon}(n/k_2)\right)\right] \right| = o_p(1). \tag{S2.3}$$

Proof: We only show the proof of (S2.2) as (S2.3) can be proven in the same manner.

Denote $\Pi_{1,n}^* = \sup_{\mathbf{x}\in\mathscr{G}} \|\hat{m}(\mathbf{x}) - m(\mathbf{x})\|$ and $\Pi_{2,n}^* = \sup_{\mathbf{x}\in\mathscr{G}} \|\hat{\sigma}(\mathbf{x}) - \sigma(\mathbf{x})\|$, then $\Pi_{1,n}^*$ and $\Pi_{2,n}^*$ are $\mathscr{I}_{t-1}$-measurable. By simple algebra, we have that

$$\hat{\varepsilon}_t^* = \varepsilon_t \left[1 + \frac{\hat{\sigma}(\mathbf{X}_t) - \sigma(\mathbf{X}_t)}{\sigma(\mathbf{X}_t)}\right]^{-1} + \left[\frac{m(\mathbf{X}_t) - \hat{m}(\mathbf{X}_t)}{\sigma(\mathbf{X}_t)}\right]\left[1 + \frac{\hat{\sigma}(\mathbf{X}_t) - \sigma(\mathbf{X}_t)}{\sigma(\mathbf{X}_t)}\right]^{-1},$$

where

$$\begin{aligned} \left|\frac{m(\mathbf{X}_t) - \hat{m}(\mathbf{X}_t)}{\sigma(\mathbf{X}_t)}\right| &\leq c^{-1} \sup_{\mathbf{x}\in\mathscr{G}} \|m(\mathbf{x}) - \hat{m}(\mathbf{x})\| = c^{-1}\Pi_{1,n}^* \text{ and} \\ \left|\frac{\hat{\sigma}(\mathbf{X}_t) - \sigma(\mathbf{X}_t)}{\sigma(\mathbf{X}_t)}\right| &\leq c^{-1} \sup_{\mathbf{x}\in\mathscr{G}} \|m(\mathbf{x}) - \hat{m}(\mathbf{x})\| = c^{-1}\Pi_{2,n}^*. \end{aligned}$$

Similar as in the proof of Lemma S1.7, for any real η_0 and $x > 0$, define $\hat{A}_t^*(x) = \mathbb{I}\left(\hat{\varepsilon}_t^* > x^{-\gamma_R} U_\varepsilon(n/k_1)\right)$, $A_t^*(x) = \mathbb{I}(\varepsilon_t > x^{-\gamma_R} U_\varepsilon(n/k_1))$ and

$$A_t^*(x, \eta_0) = \mathbb{I}\left(\varepsilon_t \left(1 + \eta_0 \Pi_{2,n}^*\right) + \left(\eta_0 \Pi_{1,n}^*\right)\left(1 + \eta_0 \Pi_{2,n}^*\right) > x^{-\gamma_R} U_\varepsilon(n/k_1)\right)$$

for $t = d_n, \ldots, n$. Let $\epsilon_0 \in (0, 1/10)$, we can show that there exists $\eta_0 > 0$ such that

$$w^* A_t^*(x, -\eta_0) \le w^* \hat{A}_t^*(x) \le w^* A_t^*(x, \eta_0)$$

for $t = d_n, \ldots, n$ and $x \in [0, 1+\iota]$, where

$$w^* = \mathbb{I}\left(\max\left(\frac{\eta_0 \Pi_{1,n}^*}{U_\varepsilon(n/k_1)}, \eta_0 \Pi_{2,n}^*\right) < \epsilon_0\right),$$

and for $x \in [0, 1+\iota]$, there exists a positive constant C_2 such that

$$\frac{n}{k_1} w^* \left| E_{t-1}[A_t^*(x, \eta_0)] - A_t^*(x) \right| \le C_2 w^* x \max\left(\frac{\eta_0 \Pi_{1,n}^*}{U_\varepsilon(n/k_1)}, \eta_0 \Pi_{2,n}^*\right).$$

In addition, by Markov's inequality, for any $\epsilon > 0$,

$$\Pr(1 - w^* \ge \epsilon) \quad \le \quad \epsilon^{-1} E|1 - w^*| \le \epsilon^{-1} \left[\Pr\left(\frac{\eta_0 \Pi_{1,n}^*}{U_\varepsilon(n/k_1)} \ge \epsilon_0\right) + \Pr\left(\eta_0 \Pi_{2,n}^* \ge \epsilon_0\right)\right] = o(1),$$

which indicates that $1 - w^* = o_p(1)$.

Now we proceed to show that for any $\iota > 0$ and $\upsilon \in [0, 1/2)$,

$$\sup_{x \in (0, 1+\iota]} x^{-\upsilon} k_1^{-1/2} \left| \sum_{t=d_n}^{n} \left[A_t^*(x, \eta_0) - A_t^*(x)\right] \right| = o_p(1). \tag{S2.4}$$

We adapt the proof of Lemma 3 of Hoga (2019). Denote $B_t^*(x, \eta_0) = A_t^*(x, \eta_0) - A_t^*(x)$ for $t = d_n, \ldots, n$. Let $\varrho_n = \lfloor k_1^{\upsilon_2} \log(n) \rfloor$ and set $\iota = 0$ without loss of generality. Decompose $(0, 1] = \sum_{j=0}^{\infty} (x_{j+1}, x_j]$, where $x_j = x_{j,n} = e^{-j/\varrho_n}$. It is noticed that for $x \in (x_{j+1}, x_j]$,

$$\begin{aligned}
x^{-\upsilon} k_1^{-1/2} \sum_{t=d_n}^{n} B_t^*(x, \eta_0) \quad &\le \quad x_{j+1}^{-\upsilon} k_1^{-1/2} \sum_{t=d_n}^{n} \left\{B_t^*(x_j, \eta_0) - E_{t-1}[B_t^*(x_j, \eta_0)]\right\} \\
&\quad + x_{j+1}^{-\upsilon} k_1^{-1/2} \sum_{t=d_n}^{n} E_{t-1}[B_t^*(x_j, \eta_0)] + x_{j+1}^{-\upsilon} k_1^{-1/2} \sum_{t=d_n}^{n} \left[A_t^*(x_j) - A_t^*(x_{j+1})\right] \quad \text{and} \\
x^{-\upsilon} k_1^{-1/2} \sum_{t=d_n}^{n} B_t^*(x, \eta_0) \quad &\ge \quad x_j^{-\upsilon} k_1^{-1/2} \sum_{t=d_n}^{n} \left\{B_t^*(x_{j+1}, \eta_0) - E_{t-1}[B_t^*(x_{j+1}, \eta_0)]\right\} \\
&\quad + x_j^{-\upsilon} k_1^{-1/2} \sum_{t=d_n}^{n} E_{t-1}[B_t^*(x_{j+1}, \eta_0)] + x_j^{-\upsilon} k_1^{-1/2} \sum_{t=d_n}^{n} \left[A_t^*(x_{j+1}) - A_t^*(x_j)\right].
\end{aligned}$$

Thus, it is sufficient to show that

$$\max_{j\geq 0}\left|x_{j+1}^{-\upsilon}k_1^{-1/2}\sum_{t=d_n}^{n}\left\{B_t^*(x_j,\eta_0)-E_{t-1}[B_t^*(x_j,\eta_0)]\right\}\right| = o_p(1), \tag{S2.5}$$

$$\max_{j\geq 0}\left|x_{j+1}^{-\upsilon}k_1^{-1/2}\sum_{t=d_n}^{n}E_{t-1}[B_t^*(x_j,\eta_0)]\right| = o_p(1), \tag{S2.6}$$

$$\max_{j\geq 0}\left|x_{j+1}^{-\upsilon}k_1^{-1/2}\sum_{t=d_n}^{n}\left[A_t^*(x_j)-A_t^*(x_{j+1})\right]\right| = o_p(1). \tag{S2.7}$$

We use a similar approach as in the proof of Lemma 3 of Hoga (2019) to show (S2.5). Note that $\{w^*\{B_t^*(x_j,\eta_0)-E_{t-1}[B_t^*(x_j,\eta_0)]\},\mathscr{I}_t\}$ is a martingale difference sequence, then for any $\epsilon>0$,

$$\begin{aligned}
&\Pr\left(\max_{j\geq 0}\left|x_{j+1}^{-\upsilon}k_1^{-1/2}\sum_{t=d_n}^{n}w^*\left\{B_t^*(x_j,\eta_0)-E_{t-1}[B_t^*(x_j,\eta_0)]\right\}\right|\geq\epsilon\right)\\
\leq\;& \sum_{j=0}^{\infty}\Pr\left(\left|x_{j+1}^{-\upsilon}k_1^{-1/2}\sum_{t=d_n}^{n}w^*\left\{B_t^*(x_j,\eta_0)-E_{t-1}[B_t^*(x_j,\eta_0)]\right\}\right|\geq\epsilon\right)\\
\leq\;& \sum_{j=0}^{\infty}\epsilon^{-2}k_1^{-1}x_{j+1}^{-2\upsilon}E\left(\left|\sum_{t=d_n}^{n}w^*\left\{B_t^*(x_j,\eta_0)-E_{t-1}[B_t^*(x_j,\eta_0)]\right\}\right|^2\right)\\
\leq\;& C_3\sum_{j=0}^{\infty}k_1^{-1}x_{j+1}^{-2\upsilon}\sum_{t=d_n}^{n}E\left(\left|w^*\left\{B_t^*(x_j,\eta_0)-E_{t-1}[B_t^*(x_j,\eta_0)]\right\}\right|^2\right)\\
\leq\;& C_3\sum_{j=0}^{\infty}k_1^{-1}x_{j+1}^{-2\upsilon}\sum_{t=d_n}^{n}\left|E[w^*B_t^*(x_j,\eta_0)]\right|\\
\leq\;& C_3\sum_{j=0}^{\infty}n^{-1}x_jx_{j+1}^{-2\upsilon}\sum_{t=d_n}^{n}E\left[\max\left(\frac{\eta_0\Pi_{1,n}^*}{U_\varepsilon(n/k_1)},\eta_0\Pi_{2,n}^*\right)\right]\\
\leq\;& C_3\sum_{j=0}^{\infty}\frac{e^{-j/\varrho_n}}{e^{-(j+1)2\upsilon/\varrho_n}}\left\{E\left[\frac{\Pi_{1,n}^*}{U_\varepsilon(n/k_1)}\right]+E(\Pi_{2,n}^*)\right\}\\
=\;& O\left(\frac{k^{\upsilon_2}\log(n)}{n^{\upsilon_2}}\right)=o(1),
\end{aligned}$$

where C_3 is a positive constant, the fifth step is due to the fact that $B_t^*(x_j,\eta_0)\in\{0,1\}$ or $\{-1,0\}$ and the last step uses $E\left[\sup_{\mathbf{x}\in\mathscr{G}}|\hat{m}(\mathbf{x})-m(\mathbf{x})|\right]=O(n^{-\upsilon_2})$ and $E\left[\sup_{\mathbf{x}\in\mathscr{G}}|\hat{\sigma}(\mathbf{x})-\sigma(\mathbf{x})|\right]=O(n^{-\upsilon_2})$ for some positive υ_2.

For (S2.6) and (S2.7), they can be proved similarly as in Hoga (2019). Thus, we conclude (S2.4). The remaining proof is similar to that of Lemma S1.7 and thus is omitted here. □

S3 Proof of Proposition C.1

Proof: It is clear that for all three models, $m(\mathscr{I}_{t-1}, \theta)$ and $\sigma(\mathscr{I}_{t-1}, \theta)$ are differentiable with respect to θ.

(i) For the ARMA-GARCH model, $m(\mathscr{I}_{t-1}, \theta) = \mu + \phi R_{t-1} + \psi e_{t-1}$ and $\sigma^2(\mathscr{I}_{t-1}, \theta) = \omega + \omega_1 e_{t-1}^2 + \omega_2 \sigma_{t-1}^2$, which can be written as

$$m(\mathscr{I}_{t-1}, \theta) = \tfrac{\mu}{1-\phi} + (\phi + \psi) \textstyle\sum_{j=1}^{\infty} \phi^{j-1} e_{t-j} \ \text{ and } \ \sigma^2(\mathscr{I}_{t-1}, \theta) = \tfrac{\omega}{1-\omega_2} + \omega_1 \textstyle\sum_{j=1}^{\infty} \omega_2^{j-1} e_{t-j}^2.$$

When $\omega > 0$, $0 < \omega_1 < 1$ and $0 < \omega_2 < 1$, $\sigma^2(\mathscr{I}_{t-1}, \theta)$ is bounded below by $\frac{\omega}{1-\omega_2} > 0$, and it is easy to verify that Assumption 3.2 is satisfied if $E(|R_t|^{\upsilon_1}) < \infty$.

Under information truncation,

$$m(\tilde{\mathscr{I}}_{t-1}, \theta) = \tfrac{\mu}{1-\phi} + (\phi + \psi) \textstyle\sum_{j=1}^{t-1} \phi^{j-1} e_{t-j} \ \text{ and } \ \sigma^2(\tilde{\mathscr{I}}_{t-1}, \theta) = \tfrac{\omega}{1-\omega_2} + \omega_1 \textstyle\sum_{j=1}^{t-1} \omega_2^{j-1} e_{t-j}^2.$$

Thus, $m(\tilde{\mathscr{I}}_{t-1}, \theta)$ and $\sigma(\tilde{\mathscr{I}}_{t-1}, \theta)$ are measurable with respect to $\mathscr{I}_{t-1}$. In addition, as

$$\begin{aligned} m(\tilde{\mathscr{I}}_{t-1}, \theta) - m(\mathscr{I}_{t-1}, \theta) &= -(\phi + \psi) \sum_{j=t}^{\infty} \phi^{j-1} e_{t-j} \ \text{ and} \\ \sigma^2(\tilde{\mathscr{I}}_{t-1}, \theta) - \sigma^2(\mathscr{I}_{t-1}, \theta) &= -\omega_1 \sum_{j=t}^{\infty} \omega_2^{j-1} e_{t-j}^2, \end{aligned}$$

it is clear that $\left|m(\tilde{\mathscr{I}}_n, \theta) - m(\mathscr{I}_n, \theta)\right| = (\phi+\psi) \left|\sum_{j=n}^{\infty} \phi^{j-1} e_{n-j}\right| \le (\phi+\psi) \sum_{j=n}^{\infty} \phi^{j-1} \left|e_{n-j}\right| = O_p(n^{-\upsilon_0/2})$. Similarly, as $\sigma^2(\mathscr{I}_{t-1}, \theta) > \omega/(1-\omega_2)$ and $\sigma^2(\tilde{\mathscr{I}}_{t-1}, \theta) > \omega/(1-\omega_2)$, we have $\left|\sigma(\tilde{\mathscr{I}}_n, \theta) - \sigma(\mathscr{I}_n, \theta)\right| \le \left|\sigma^2(\tilde{\mathscr{I}}_n, \theta) - \sigma^2(\mathscr{I}_n, \theta)\right| / \{2\omega/(1-\omega_2)\} = O_p(n^{-\upsilon_0/2})$. Finally,

$$\sum_{t=d_n}^{n} E\left|m(\tilde{\mathscr{I}}_{t-1}, \theta) - m(\mathscr{I}_{t-1}, \theta)\right| \le (\phi + \psi) \sum_{t=d_n}^{n} \sum_{j=t}^{\infty} \phi^{j-1} E|e_{t-j}| = o(1)$$

and

$$\begin{aligned} \sum_{t=d_n}^{n} E\left|\sigma(\tilde{\mathscr{I}}_{t-1}, \theta) - \sigma(\mathscr{I}_{t-1}, \theta)\right| &\le \sum_{t=d_n}^{n} E\left|\sigma^2(\tilde{\mathscr{I}}_{t-1}, \theta) - \sigma^2(\mathscr{I}_{t-1}, \theta)\right| / \{2\omega/(1-\omega_2)\} \\ &\le \omega_1 \sum_{t=d_n}^{n} \sum_{j=t}^{\infty} \omega_2^{j-1} E|e_{t-j}^2| / \{2\omega/(1-\omega_2)\} = o(1). \end{aligned}$$

Thus, Assumption 3.3 is also satisfied by the ARMA-GARCH model.

(ii) For the TGARCH model, as $m(\mathscr{I}_{t-1},\theta)=\mu$ is a constant, so the assumptions on $m(\mathscr{I}_{t-1},\theta)$ and $m(\tilde{\mathscr{I}}_{t-1},\theta)$ are clearly satisfied. We can rewrite $\sigma^2(\mathscr{I}_{t-1},\theta)$ as

$$\sigma^2(\mathscr{I}_{t-1},\theta)=\frac{\omega}{1-\omega_2}+(\omega_0 N_{t-1}+\omega_1)\sum_{j=1}^{\infty}\omega_2^{j-1}\sigma_{t-j}^2\varepsilon_{t-1}^2,$$

from which we can easily check the conditions on $\sigma(\mathscr{I}_{t-1},\theta)$ and $\sigma(\tilde{\mathscr{I}}_{t-1},\theta)$ similarly as in the ARMA-GARCH model. Thus, Assumptions 3.2 and 3.3 are satisfied by the TGARCH model.

(iii) In the GARCH-in-Mean model, the variance function $\sigma^2(\mathscr{I}_{t-1},\theta)$ is the same as in the ARMA-GARCH model, so it is sufficient to check the conditions for $m(\mathscr{I}_{t-1},\theta)=\mu+\phi X_{t-1}+\psi\sigma_{t-1}^2$. Rewrite

$$m(\mathscr{I}_{t-1},\theta) \;=\; \mu+\phi X_{t-1}+\psi\sigma_{t-1}^2=\mu+\sum_{j=1}^{\infty}\psi_1^{j-1}\eta_{t-j}+\psi\left(\frac{\omega}{1-\omega_2}+\omega_1\sum_{j=1}^{\infty}\omega_2^{j-1}e_{t-j-1}^2\right).$$

Using similar techniques as in the proof of part (i), we can show that Assumptions 3.2 and 3.3 are satisfied.

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