How large is the jump discontinuity in the diffusion coefficient of a time-homogeneous diffusion?

Online Appendix

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Abstract

We consider high-frequency observations from a one-dimensional time-homogeneous diffusion process Y. We assume that the diffusion coefficient σ is continuously differentiable in y, but with a jump discontinuity at some level y, say $y = 0$. We first study sign-constrained kernel estimators of functions of the left and right limits of σ at 0. These functions intricately depend on both limits. We propose a method to extricate these functions by searching for bandwidths where the kernel estimators are stable by iteration. We finally provide an estimator of the discontinuity jump size. We prove its convergence in probability and discuss its rate of convergence. A Monte Carlo study shows the finite sample properties of this estimator.

Keywords: Ito diffusion processes; Discontinuous diffusion coefficient; Fixed points; Local times; Skew Brownian motion;

In this Online Appendix we provide the proofs of Propositions 3 and 8, and a discussion that explains why it is possible to study the asymptotic properties of estimators with the same methodology as if there were no jumps.

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Proof of Proposition 3

To prove the stable convergence in law of Z^n, we use the same ideas as those introduced in Section 5 and 6 in Jacod (1998). Note that, by polarization, it is enough to prove the proposition when h is a 1-dimensional function such that $\lambda_\theta(H_{\theta,h}) = 0$, which we assume in the sequel. Let

$$f_{i,n} = h\left(\sqrt{n}X_{(i-1)/n}, \sqrt{n}\left(X_{i/n} - X_{(i-1)/n}\right)\right) \tag{1}$$

so that

$$Z_t^n = \frac{1}{n^{1/4}}U(h)_t^n = \frac{1}{n^{1/4}}\sum_{i=1}^{\lfloor nt \rfloor} f_{i,n}.$$

Let us first introduce some notation and provide two lemmas. For a Lebesgue-integrable function f, we have

$$P_t^\theta f(x) = \int p_\theta(t,x,y) f(y)dy = P_t f(x) + \theta\left(P_t f_+(-|x|) - P_t f_-(|x|)\right)$$

where $f_+(x) = f(x)1_{\{x\geq 0\}}$, $f_-(x) = f(x)1_{\{x<0\}}$, and $P = P^0$ is the Brownian semi-group defined by $P_t f(x) = \int p(t, y-x) f(y)dy$. Note that

$$\lambda_\theta(f) = \lambda(f) + \theta\left(\lambda(f_+) - \lambda(f_-)\right).$$

We also define

$$\beta_\gamma^\theta(f) = \beta_\gamma(f) + |\theta|\left(\beta_\gamma(f_+) + \beta_\gamma(f_-)\right)$$

and therefore, $\beta_\gamma^\theta(f) \leq 2\beta_\gamma(f)$.

Remark 1 *Since $P_t^\theta f(\cdot)$ may be written as a linear combination of $P_t f(\cdot)$, $P_t f_+(-|\cdot|)$ and $P_t f_-(|\cdot|)$, and since $\lambda_\theta(f)$ may be also written as a linear combination of $\lambda(f)$, $\lambda(f_+)$ and $\lambda(f_-)$ with the same coefficients, all inequalities in Lemmas* 3.1, 3.2 *and* 3.3 *in Jacod (1998) hold with P_t replaced by P_t^θ, $\lambda(f)$ replaced by $\lambda_\theta(f)$ and $\beta_\gamma(f)$ replaced by $\beta_\gamma^\theta(f)$ or $2\beta_\gamma(f)$.*

For a Lebesgue-integrable function f such that $\lambda_\theta(f) = 0$, let

$$F_{\theta,n}(f)(x) = \sum_{j=0}^{w_n} P_j^\theta f(x)$$

where $w_n = [n^\beta]$, $\beta \in (0, 1/2)$ and $[x]$ denotes the integer part of a real x.

Lemma 1 *i) Assume that f is a bounded Borel function on $\mathbb{R}$ such that $\lambda_\theta(f) = 0$ and $\beta_1(f) < \infty$.*

Then

$$|F_{\theta,n}(f)(x)| \leq K \log n.$$

ii) Assume that f is a bounded Borel function on $\mathbb{R}$ such that $\lambda_\theta(f) = 0$ and, for some $\gamma \geq 0$, $\beta_{1+\gamma}(f) < \infty$, then

$$|F_{\theta,n}(f)(x)| \leq |f(x)| + K \log n \left(\frac{1}{1 + |xn^{-\beta/2}|^\gamma} + \frac{1}{1 + |x|^\gamma} \right)$$

and

$$|P^\theta_{w_n+1} f(x)| \leq K n^{-\beta} \left(\frac{1}{1 + |xn^{-\beta/2}|^\gamma} + \frac{1}{1 + |x|^\gamma} \right).$$

It follows that $\sup_{x\in\mathbb{R}} |P^\theta_{w_n+1} f(x)| \leq K n^{-\beta}$ and if, for some $\gamma > 1$ such that $\beta_{1+\gamma}(f) < \infty$, that

$$\lambda_\theta(|P^\theta_{w_n+1} f|) \leq K n^{-\beta/2}.$$

Proof.

i) By Remark 1, the inequality is derived in the same way as Eq. (5.8) in Jacod (1998) with $\delta = 0$ and $\alpha = 1/2$ (using Lemma 3.1 in Jacod (1998)). Note that the condition $\lambda_\theta(f) = 0$ is essential here.

ii) By Remark 1, the first inequality is derived in the same way as Eq. (5.7) in Jacod (1998) with $\delta = 0$ and $\alpha = 1/2$ (using Lemma 3.1 in Jacod (1998)). For the second inequality, use Eq. (5.9) in Jacod (1998) with $\delta = 0$ and $\alpha = 1/2$. It is important to note that $\lambda_\theta(P^\theta_{w_n+1} f(x)) = 0$ because λ_θ is the invariant measure of the semi-group $\left(P^\theta_t\right)_{t\geq 0}$.

■

Lemma 2 *Assume that g is a bounded Borel function on $\mathbb{R}$ such that $\beta^\theta_1(g) < \infty$.*

If $p \geq 1$,

$$\mathbb{E}^\theta \left[\left| \frac{1}{\sqrt{n}} V(g)^n_t \right|^p \right] \leq K \left(\sup_{x\in\mathbb{R}} |g(x)|^p + \lambda_\theta(|g|)^p \right).$$

If p is an even integer,

$$\mathbb{E}^\theta \left[\sup_{t\in[0,1]} \left| \frac{1}{\sqrt{n}} V(g)^n_t \right|^p \right] \leq K \left(\sup_{x\in\mathbb{R}} |g(x)|^{p-1} + \lambda_\theta(|g|)^{p-1} \right) \left(\frac{\sup_{x\in\mathbb{R}} |g(x)| + \beta^\theta_1(g) \log n}{\sqrt{n}} + |\lambda_\theta(g)| \right).$$

Proof.

The proof follows the same lines as in the proof of Lemma 1 in Jacod (2000) using Remark 1. Just note that the second inequality in Jacod (2000) is given for $\mathbb{E}^\theta\left[\left|n^{-1/2}V(g)^n_t\right|^p\right]$ instead of $\mathbb{E}^\theta\left[\sup_{t\in[0,1]}\left|n^{-1/2}V(g)^n_t\right|^p\right]$, but the extension is obvious with the method used in Jacod (2000).

■

We can now begin the proof of Proposition 3.

Step 1: Let

$$\tilde{Z}^n_t = n^{-1/4}\left[\sum_{i=1}^{\lfloor nt\rfloor}\left(f_{i,n} - H_{\theta,h}(\sqrt{n}X_{(i-1)/n}) + g_{i,n}\right)\right]$$

where $f_{i,n}$ is given in Eq. (1)

$$\begin{aligned} g_{i,n} &= \sum_{j=0}^{w_n}\left(\mathbb{E}^\theta\left[H_{\theta,h}(\sqrt{n}X_{(i+j)/n})\middle|\mathcal{F}_{i/n}\right] - \mathbb{E}^\theta\left[H_{\theta,h}(\sqrt{n}X_{(i+j)/n})\middle|\mathcal{F}_{(i-1)/n}\right]\right)\\ &= \sum_{j=0}^{w_n}\left(\mathbb{E}^\theta\left[f_{(i+j+1),n}\middle|\mathcal{F}_{i/n}\right] - \mathbb{E}^\theta\left[f_{(i+j+1),n}\middle|\mathcal{F}_{(i-1)/n}\right]\right). \end{aligned}$$

Note that $\tilde{Z}^n$ is a locally square-integrable martingale with respect to the filtration $\left(\mathcal{F}_{\lfloor nt\rfloor/n}\right)_{t\geq 0}$ since

$$\mathbb{E}^\theta\left[f_{i,n}\middle|\sqrt{n}X_{(i-1)/n} = x\right] = \int p_\theta(1,x,y)\,h(x,y-x)\,dy = H_{\theta,h}(x).$$

Let us prove that

$$\sup_{t\in[0,1]}|Z^n_t - \tilde{Z}^n_t| \xrightarrow{P} 0.$$

Note that

$$\sum_{j=0}^{w_n}\mathbb{E}^\theta\left[H_{\theta,h}(\sqrt{n}X_{(i+j)/n})\middle|\mathcal{F}_{i/n}\right] = F_{\theta,n}(H_{\theta,h})(\sqrt{n}X_{i/n}) = \sum_{j=1}^{w_n+1}\mathbb{E}^\theta\left[f_{i+j,n}\middle|\mathcal{F}_{i/n}\right]$$

and

$$\begin{aligned} &\sum_{j=0}^{w_n}\mathbb{E}^\theta\left[H_{\theta,h}(\sqrt{n}X_{(i+j)/n})\middle|\mathcal{F}_{(i-1)/n}\right]\\ =\ & F_{\theta,n}(H_{\theta,h})(\sqrt{n}X_{(i-1)/n}) + \mathbb{E}^\theta\left[H_{\theta,h}(\sqrt{n}X_{(i+w_n)/n})\middle|\mathcal{F}_{(i-1)/n}\right] - H_{\theta,h}(\sqrt{n}X_{(i-1)/n})\\ =\ & \sum_{j=1}^{w_n+1}\mathbb{E}^\theta\left[f_{i-1+j,n}\middle|\mathcal{F}_{(i-1)/n}\right] + \mathbb{E}^\theta\left[f_{w_n+1+i,n}\middle|\mathcal{F}_{(i-1)/n}\right] - \mathbb{E}^\theta\left[f_{i,n}\middle|\mathcal{F}_{(i-1)/n}\right]. \end{aligned}$$

It follows that

$$
\begin{aligned}
g_{i,n} &= F_{\theta,n}(H_{\theta,h})(\sqrt{n}X_{i/n}) - F_{\theta,n}(H_{\theta,h})(\sqrt{n}X_{(i-1)/n}) - \mathbb{E}^{\theta}\left[H_{\theta,h}(\sqrt{n}X_{(i+w_n)/n}) \middle| \mathcal{F}_{(i-1)/n}\right] + H_{\theta,h}(\sqrt{n}X_{(i-1)/n}) \\
&= \sum_{j=1}^{w_n+1} \mathbb{E}^{\theta}\left[f_{i+j,n} \middle| \mathcal{F}_{i/n}\right] - \sum_{j=1}^{w_n+1} \mathbb{E}^{\theta}\left[f_{i-1+j,n} \middle| \mathcal{F}_{(i-1)/n}\right] - \mathbb{E}^{\theta}\left[f_{w_n+1+i,n} \middle| \mathcal{F}_{(i-1)/n}\right] + \mathbb{E}^{\theta}\left[f_{i,n} \middle| \mathcal{F}_{(i-1)/n}\right]
\end{aligned}
$$

and therefore

$$
Z_t^n = n^{-1/4}\left[\sum_{i=1}^{\lfloor nt\rfloor}\left(f_{i,n} - H_{\theta,h}(\sqrt{n}X_{(i-1)/n}) + g_{i,n}\right) + H_{n,t} + I_{n,t}\right]
$$

where

$$
\begin{aligned}
H_{n,t} &= F_{\theta,n}(H_{\theta,h})(\sqrt{n}X_0) - F_{\theta,n}(H_{\theta,h})(\sqrt{n}X_{\lfloor nt\rfloor/n}) \\
I_{n,t} &= \sum_{i=1}^{\lfloor nt\rfloor} \mathbb{E}^{\theta}\left[H_{\theta,h}(\sqrt{n}X_{(i+w_n)/n}) \middle| \mathcal{F}_{(i-1)/n}\right] \\
&= \sum_{i=1}^{\lfloor nt\rfloor} P^{\theta}_{w_n+1} H_{\theta,h}(\sqrt{n}X_{(i-1)/n}) = V(P^{\theta}_{w_n+1}H_{\theta,h})^n_t.
\end{aligned}
$$

Since $H_{\theta,h}\left(x\right) = \mathbb{E}^{\theta}_x\left[h(x, X_1 - x)\right]$ and h satisfies Condition 4 with $\mathbb{E}^{\theta}_x\left[e^{a|X_1-x|}\right] < \infty$, we have

$$
\beta_1\left(H_{\theta,h}\right) = \int |x||H_{\theta,h}\left(x\right)|dx \le K\int |x||\bar{h}\left(x\right)|dx = K\beta_1\left(\bar{h}\right) < \infty.
$$

By Lemma 1 i), we have

$$
n^{-1/4}\left|F_{\theta,n}(H_{\theta,h})(x)\right| \le Kn^{-1/4}\log n \to 0
$$

and then

$$
n^{-1/4}\sup_{t\in[0,1]}|H_{n,t}| \xrightarrow{P} 0.
$$

We have

$$
\beta_1^{\theta}\left(P^{\theta}_{w_n+1}H_{\theta,h}\right) \le 2\beta_1\left(P^{\theta}_{w_n+1}H_{\theta,h}\right) = 2\int |x||P^{\theta}_{w_n+1}H_{\theta,h}\left(x\right)|dx.
$$

Since $\lambda_{\theta}(H_{\theta,h}) = 0$, by Lemma 1 ii), we deduce that, for some $\gamma > 2$,

$$
|P^{\theta}_{w_n+1}H_{\theta,h}| \le Kn^{-\beta}\left(\frac{1}{1+|xn^{-\beta/2}|^{\gamma}} + \frac{1}{1+|x|^{\gamma}}\right),
$$

and

$$
\begin{aligned}
&\int |x||P^{\theta}_{w_n+1}H_{\theta,h}\left(x\right)|dx \\
\leq\ & Kn^{-\beta/2}\int\left(\frac{|n^{-\beta/2}x|}{1+|xn^{-\beta/2}|^{\gamma}}\right)dx+Kn^{-\beta}\int\left(\frac{|x|}{1+|x|^{\gamma}}\right)dx\leq K.
\end{aligned}
$$

By Lemma 2 (with $p=2$ and $g=P^{\theta}_{w_n+1}H_{\theta,h}$), we have

$$
\mathbb{E}^{\theta}\left[\sup_{t\in[0,1]}\left|\frac{1}{\sqrt{n}}V(P^{\theta}_{w_n+1}H_{\theta,h})^n_t\right|^2\right]\leq K\left(n^{-\beta}+n^{-\beta/2}\right)\left(\frac{n^{-\beta}+\log n}{\sqrt{n}}\right)\leq Kn^{-\beta/2}\frac{\log n}{\sqrt{n}}
$$

since $\lambda_\theta(P^{\theta}_{w_n+1}H_{\theta,h})=0$ (because $\lambda_\theta(H_{\theta,h})=0$ and λ_θ is the invariant measure of the semi-group $\left(P^{\theta}_t\right)_{t\geq 0}$).

Therefore,

$$
\mathbb{E}^{\theta}\left[n^{-1/2}\sup_{t\in[0,1]}I^2_{n,t}\right]\leq Kn^{-\beta/2}\log n\to 0,
$$

and then

$$
n^{-1/4}\sup_{t\in[0,1]}|I_{n,t}|\overset{P}{\to}0.
$$

Step 2: By Theorem 3.2 in Jacod (1997), it is now sufficient to prove that

$$
\begin{aligned}
&\sup_{s\in[0,1]}\left|\sum_{i=1}^{\lfloor nt\rfloor}\mathbb{E}^{\theta}\left[\Delta^n_i\tilde{Z}\middle|\mathcal{F}_{(i-1)/n}\right]\right|\overset{P}{\to}0\\
&\sum_{i=1}^{\lfloor nt\rfloor}\mathbb{E}^{\theta}\left[(\Delta^n_i\tilde{Z})^2\middle|\mathcal{F}_{(i-1)/n}\right]\overset{P}{\to}\lambda_\theta(H_{\theta,h^2}+2\bar{H}_{\theta,h,\Phi_h})L_t\\
&\sum_{i=1}^{\lfloor nt\rfloor}\mathbb{E}^{\theta}\left[(\Delta^n_iZ)^2\,\mathbb{I}_{\{|\Delta^n_iZ|>\varepsilon\}}\middle|\mathcal{F}_{(i-1)/n}\right]\overset{P}{\to}0\\
&\sum_{i=1}^{\lfloor nt\rfloor}\mathbb{E}^{\theta}\left[\Delta^n_i\tilde{Z}\Delta^n_iW\middle|\mathcal{F}_{(i-1)/n}\right]\overset{P}{\to}0\\
&\sum_{i=1}^{\lfloor nt\rfloor}\mathbb{E}^{\theta}\left[\Delta^n_i\tilde{Z}\Delta^n_iM\middle|\mathcal{F}_{(i-1)/n}\right]\overset{P}{\to}0,
\end{aligned}
$$

for any bounded $\left(\mathcal{F}^W_t\right)_{t\geq 0}$-martingale M such that, for all $s\in[0,1]$, the cross variation satisfies $P\left(\langle M,Y\rangle_s=0\right)=1$.

The first condition is clearly satisfied since $\tilde{Z}^n$ is a locally square-integrable martingale with respect to the filtration $\left(\mathcal{F}_{\lfloor nt \rfloor/n}\right)_{t\geq 0}$ and will not be discussed.

i) Let us first study

$$\sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}^\theta \left[(\Delta_i^n \tilde{Z})^2 \middle| \mathcal{F}_{(i-1)/n} \right].$$

Note that

$$\begin{aligned} n^{1/4}\Delta_i^n \tilde{Z} &= f_{i,n} + \sum_{j=1}^{w_n+1} \mathbb{E}^\theta \left[f_{i+j,n} \middle| \mathcal{F}_{i/n} \right] - \sum_{j=1}^{w_n+1} \mathbb{E}^\theta \left[f_{i-1+j,n} \middle| \mathcal{F}_{(i-1)/n} \right] - \mathbb{E}^\theta \left[f_{w_n+1,n} \middle| \mathcal{F}_{(i-1)/n} \right] \\ &= f_{i,n} + F_{i,n} - F_{i-1,n} - \mathbb{E}^\theta \left[f_{w_n+1,n} \middle| \mathcal{F}_{(i-1)/n} \right] \end{aligned}$$

with

$$F_{i,n} = \sum_{j=1}^{w_n+1} \mathbb{E}^\theta \left[f_{i+j,n} \middle| \mathcal{F}_{i/n} \right].$$

Since

$$\frac{1}{n^{1/4}} \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}^\theta \left[f_{w_n+1,n} \middle| \mathcal{F}_{(i-1)/n} \right] = \frac{1}{n^{1/4}} \sum_{i=1}^{\lfloor nt \rfloor} P_{w_n+1}^\theta H_{\theta,h}(\sqrt{n} X_{(i-1)/n}) = n^{-1/4} I_{n,t}$$

we only consider

$$m_{i,n} = f_{i,n} + F_{i,n} - F_{i-1,n}$$

and study

$$\begin{aligned} & \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}^\theta \left[m_{i,n}^2 \middle| \mathcal{F}_{(i-1)/n} \right] \\ = & \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}^\theta \left[(f_{i,n} + F_{i,n} - F_{i-1,n})^2 \middle| \mathcal{F}_{(i-1)/n} \right] \\ = & \sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}^\theta \left[f_{i,n}^2 + (F_{i,n} - F_{i-1,n})^2 + 2 f_{i,n}(F_{i,n} - F_{i-1,n}) \middle| \mathcal{F}_{(i-1)/n} \right]. \end{aligned}$$

We have

a)

$$\begin{aligned}
\mathbb{E}^{\theta}\left[f_{i,n}F_{i,n} \middle| \mathcal{F}_{(i-1)/n}\right] &= \mathbb{E}^{\theta}\left[f_{i,n} \sum_{j=1}^{w_n+1} \mathbb{E}_x^{\theta}\left[f_{i+j,n} \middle| \mathcal{F}_{i/n}\right] \middle| \mathcal{F}_{(i-1)/n}\right] \\
&= \mathbb{E}^{\theta}\left[\sum_{j=1}^{w_n+1} \mathbb{E}_x^{\theta}\left[f_{i,n}f_{i+j,n} \middle| \mathcal{F}_{i/n}\right] \middle| \mathcal{F}_{(i-1)/n}\right] \\
&= \sum_{j=1}^{w_n+1} \mathbb{E}^{\theta}\left[f_{i,n}f_{i+j,n} \middle| \mathcal{F}_{(i-1)/n}\right],
\end{aligned}$$

b)

$$\begin{aligned}
\mathbb{E}^{\theta}\left[f_{i,n}F_{i-1,n} \middle| \mathcal{F}_{(i-1)/n}\right] &= \mathbb{E}^{\theta}\left[f_{i,n} \sum_{j=1}^{w_n+1} \mathbb{E}^{\theta}\left[f_{i+j-1,n} \middle| \mathcal{F}_{(i-1)/n}\right] \middle| \mathcal{F}_{(i-1)/n}\right] \\
&= F_{i-1,n}\mathbb{E}^{\theta}\left[f_{i,n} \middle| \mathcal{F}_{(i-1)/n}\right],
\end{aligned}$$

c)

$$\begin{aligned}
\mathbb{E}^{\theta}\left[F_{i,n}F_{i-1,n} \middle| \mathcal{F}_{(i-1)/n}\right] &= F_{i-1,n}\mathbb{E}^{\theta}\left[F_{i,n} \middle| \mathcal{F}_{(i-1)/n}\right] \\
&= F_{i-1,n}\left(-\mathbb{E}^{\theta}\left[f_{i,n} \middle| \mathcal{F}_{(i-1)/n}\right] + F_{i-1,n} + \mathbb{E}^{\theta}\left[f_{w_n+1+i,n} \middle| \mathcal{F}_{(i-1)/n}\right]\right),
\end{aligned}$$

d)

$$\begin{aligned}
& \mathbb{E}^{\theta}\left[(F_{i,n}-F_{i-1,n})^2 \middle| \mathcal{F}_{(i-1)/n}\right] \\
= \;& \mathbb{E}^{\theta}\left[F_{i,n}^2 \middle| \mathcal{F}_{(i-1)/n}\right] + F_{i-1,n}^2 - 2F_{i-1,n}\left(-\mathbb{E}^{\theta}\left[f_{i,n} \middle| \mathcal{F}_{(i-1)/n}\right] + F_{i-1,n} + \mathbb{E}^{\theta}\left[f_{w_n+1+i,n} \middle| \mathcal{F}_{(i-1)/n}\right]\right) \\
= \;& \mathbb{E}^{\theta}\left[F_{i,n}^2 \middle| \mathcal{F}_{(i-1)/n}\right] - F_{i-1,n}^2 + 2F_{i-1,n}\mathbb{E}^{\theta}\left[f_{i,n} \middle| \mathcal{F}_{(i-1)/n}\right] - 2F_{i-1,n}\mathbb{E}^{\theta}\left[f_{w_n+1+i,n} \middle| \mathcal{F}_{(i-1)/n}\right].
\end{aligned}$$

Hence we have (assuming without loss of generality that $t = 1$)

$$\begin{aligned}
& \sum_{i=1}^{n} \mathbb{E}^{\theta}\left[m_{i,n}^2 \middle| \mathcal{F}_{(i-1)/n} \right] \\
= \; & n^{-1/2} \sum_{i=1}^{n} \mathbb{E}^{\theta}\left[f_{i,n}^2 \middle| \mathcal{F}_{(i-1)/n} \right] \\
& + n^{-1/2} \sum_{i=1}^{n} \Big[\mathbb{E}^{\theta}\left[F_{i,n}^2 \middle| \mathcal{F}_{(i-1)/n} \right] - F_{i-1,n}^2 + 2F_{i-1,n}\mathbb{E}^{\theta}\left[f_{i,n} \middle| \mathcal{F}_{(i-1)/n} \right] - 2F_{i-1,n}\mathbb{E}^{\theta}\left[f_{w_n+1+i,n} \middle| \mathcal{F}_{(i-1)/n} \right] \Big] \\
& + 2n^{-1/2} \sum_{i=1}^{n} \Big[\mathbb{E}^{\theta}\left[f_{i,n}F_{i,n} \middle| \mathcal{F}_{(i-1)/n} \right] - F_{i-1,n}\mathbb{E}^{\theta}\left[f_{i,n} \middle| \mathcal{F}_{(i-1)/n} \right] \Big] \\
= \; & n^{-1/2} \sum_{i=1}^{n} \mathbb{E}^{\theta}\left[f_{i,n}^2 \middle| \mathcal{F}_{(i-1)/n} \right] + 2n^{-1/2} \sum_{i=1}^{n} \mathbb{E}^{\theta}\left[f_{i,n}F_{i,n} \middle| \mathcal{F}_{(i-1)/n} \right] \\
& + n^{-1/2} \sum_{i=1}^{n} \Big[\mathbb{E}^{\theta}\left[F_{i,n}^2 \middle| \mathcal{F}_{(i-1)/n} \right] - F_{i-1,n}^2 \Big] - 2n^{-1/2} \sum_{i=1}^{n} F_{i-1,n}\mathbb{E}^{\theta}\left[f_{w_n+1+i,n} \middle| \mathcal{F}_{(i-1)/n} \right].
\end{aligned}$$

Let us consider

$$n^{-1/2} \sum_{i=1}^{n} \mathbb{E}^{\theta}\left[f_{i,n}^2 \middle| \mathcal{F}_{(i-1)/n} \right] = n^{-1/2} \sum_{i=1}^{n} \mathbb{E}^{\theta}\left[h^2\left(\sqrt{n}X_{(i-1)/n}, \sqrt{n}\left(X_{i/n} - X_{(i-1)/n} \right) \right) \middle| \mathcal{F}_{(i-1)/n} \right].$$

Note that h^2 satisfies Condition 2, i.e. h^2 is a Borel function on $\mathbb{R}^2$ such that the functions H_{θ,h^2}, H_{θ,h^4} satisfy Condition 1, i.e. H_{θ,h^2}, H_{θ,h^4} are bounded functions and $\beta_2(H_{\theta,h^2}) < \infty$ and $\beta_2(H_{\theta,h^4}) < \infty$. Indeed, since h satisfies Condition 4, we have

$$\begin{aligned}
H_{\theta,h^2}(x) &= \mathbb{E}_x^{\theta}\left[h^2(x, X_1 - x) \right] \le K\bar{h}^2(x) \le K \\
H_{\theta,h^4}(x) &= \mathbb{E}_x^{\theta}\left[h^4(x, X_1 - x) \right] \le K\bar{h}^4(x) \le K
\end{aligned}$$

and

$$\begin{aligned}
\beta_2(H_{\theta,h^2}) &= \int |x|^2 H_{\theta,h^2}(x)\, dx \le K \int |x|^2 \bar{h}^2(x)\, dx \le K \int |x|^2 \bar{h}(x)\, dx \le K\beta_2(\bar{h}) < \infty \\
\beta_2(H_{\theta,h^4}) &= \int |x|^2 |H_{\theta,h^4}(x)|\, dx \le K \int |x|^2 \bar{h}^4(x)\, dx \le K \int |x|^2 \bar{h}(x)\, dx \le K\beta_2(\bar{h}) < \infty.
\end{aligned}$$

We deduce by Proposition 2 that

$$n^{-1/2} \sum_{i=1}^{n} \mathbb{E}^{\theta}\left[f_{i,n}^2 \middle| \mathcal{F}_{(i-1)/n} \right] \xrightarrow{P} \lambda_{\theta}(H_{\theta,h^2})L_1.$$

Let us now consider

$$n^{-1/2}\sum_{i=1}^{n}\mathbb{E}^{\theta}\left[\left. f_{i,n}F_{i,n}\right|\mathcal{F}_{(i-1)/n}\right].$$

We have

$$\begin{aligned}
& n^{-1/2}\sum_{i=1}^{n}\mathbb{E}^{\theta}\left[\left. f_{i,n}F_{i,n}\right|\mathcal{F}_{(i-1)/n}\right] \\
= \;& n^{-1/2}\sum_{i=1}^{n}\mathbb{E}^{\theta}\left[\left. h\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\left(X_{i/n}-X_{(i-1)/n}\right)\right)F_{\theta,n}(H_{\theta,h})\left(\sqrt{n}X_{i/n}\right)\right|\mathcal{F}_{(i-1)/n}\right] \\
= \;& n^{-1/2}\sum_{i=1}^{n}\mathbb{E}^{\theta}\left[\left. h_0\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\left(X_{i/n}-X_{(i-1)/n}\right)\right)\right|\mathcal{F}_{(i-1)/n}\right] \\
& +n^{-1/2}\sum_{i=1}^{n}\mathbb{E}^{\theta}\left[\left. h\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\left(X_{i/n}-X_{(i-1)/n}\right)\right)\left(F_{\theta,n}(H_{\theta,h})\left(\sqrt{n}X_{i/n}\right)-F_{\theta}(H_{\theta,h})\left(\sqrt{n}X_{i/n}\right)\right)\right|\mathcal{F}_{(i-1)/n}\right]
\end{aligned}$$

where

$$h_0\left(x,y\right)=h\left(x,y\right)F_{\theta}(H_{\theta,h})\left(x+y\right).$$

Note that h_0 satisfies Condition 2, i.e. h_0 is a Borel function on $\mathbb{R}^2$ such that the functions H_{θ,h_0}, H_{θ,h_0^2} satisfy Condition 1, i.e. H_{θ,h_0}, H_{θ,h_0^2} are bounded functions and $\beta_2(H_{\theta,h_0})<\infty$ and $\beta_2(H_{\theta,h_0^2})<\infty$. Indeed, by Remark 1 and Eq. (3.3) in Lemma 3.1 in Jacod (1998), we have

$$\left|F_{\theta}(H_{\theta,h})\left(x\right)\right|\leq K\left(1+|x|\right)$$

since $\beta_2(H_{\theta,h})<\infty$. Since h satisfies Condition 4, we also have

$$|h_0(x,y)|\leq\bar{h}\left(x\right)e^{a|y|}\left(1+|x|+|y|\right)$$

and then

$$\begin{aligned}
\left|H_{\theta,h_0}\left(x\right)\right| &= \left|\mathbb{E}_x^{\theta}\left[h_0(x,X_1-x)\right]\right|\leq K|x|\bar{h}\left(x\right)\leq K \\
H_{\theta,h_0^2}\left(x\right) &= \mathbb{E}_x^{\theta}\left[h_0^2(x,X_1-x)\right]\leq Kx^2\bar{h}^2\left(x\right)\leq K
\end{aligned}$$

and

$$\begin{aligned}
\beta_2(H_{\theta,h_0}) &= \int|x|^2H_{\theta,h_0}\left(x\right)dx\leq K\int|x|^3\bar{h}\left(x\right)dx\leq K\beta_3(\bar{h})<\infty \\
\beta_2(H_{\theta,h_0^2}) &= \int|x|^2\left|H_{\theta,h_0^2}\left(x\right)\right|dx\leq K\int|x|^4\bar{h}^2\left(x\right)dx\leq K\int|x|^3\bar{h}\left(x\right)dx\leq K\beta_3(\bar{h})<\infty.
\end{aligned}$$

We deduce by Proposition 2 that

$$
\begin{aligned}
& n^{-1/2}\sum_{i=1}^{n}\mathbb{E}^{\theta}\left[h\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\left(X_{i/n}-X_{(i-1)/n}\right)\right)F_{\theta}(H_{\theta,h})\left(\sqrt{n}X_{i/n}\right)\right|\mathcal{F}_{(i-1)/n}\Big] \\
= \;& n^{-1/2}\sum_{i=1}^{n}\mathbb{E}^{\theta}\left[h_0\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\left(X_{i/n}-X_{(i-1)/n}\right)\right)\right|\mathcal{F}_{(i-1)/n}\Big] \\
& \xrightarrow{P}\lambda_{\theta}(H_{\theta,h_0})L_1=\lambda_{\theta}(\bar{H}_{\theta,h,\Phi_h})L_1.
\end{aligned}
$$

By Remark 1 and Eq. (3.3) in Lemma 3.1 in Jacod (1998), we have

$$
\left|F_{\theta,n}(H_{\theta,h})\,(x)-F_{\theta}(H_{\theta,h})\,(x)\,\right|\leq\sum_{i=w_n+1}^{\infty}K\frac{1}{i^{3/2}}\left(1+|x|\right)\leq K\frac{1}{w_n^{1/2}}\left(1+|x|\right).
$$

It follows in the same way that

$$
n^{-1/2}\sum_{i=1}^{n}\mathbb{E}^{\theta}\left[h\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\left(X_{i/n}-X_{(i-1)/n}\right)\right)\left[F_{\theta,n}(H_{\theta,h})-F_{\theta}(H_{\theta,h})\right]\left(\sqrt{n}X_{i/n}\right)\right|\mathcal{F}_{(i-1)/n}\Big]\xrightarrow{P}0.
$$

We finally deduce that

$$
n^{-1/2}\sum_{i=1}^{n}\mathbb{E}^{\theta}\left[f_{i,n}F_{i,n}\right|\mathcal{F}_{(i-1)/n}\Big]\xrightarrow{P}\lambda_{\theta}(\bar{H}_{\theta,h,\Phi_h})L_1.
$$

Let us now consider

$$
n^{-1/2}\sum_{i=1}^{n}F_{i-1,n}\mathbb{E}^{\theta}\left[f_{w_n+1+i,n}\right|\mathcal{F}_{(i-1)/n}\Big].
$$

We have

$$
\begin{aligned}
F_{i-1,n} &= F_{\theta,n}(H_{\theta,h})(\sqrt{n}X_{(i-1)/n}) \\
\mathbb{E}^{\theta}\left[f_{w_n+1+i,n}\right|\mathcal{F}_{(i-1)/n}\Big] &= P^{\theta}_{w_n+1}H_{\theta,h}(\sqrt{n}X_{(i-1)/n})
\end{aligned}
$$

and therefore

$$
n^{-1/2}\sum_{i=1}^{n}F_{i-1,n}\mathbb{E}^{\theta}\left[f_{w_n+1+i,n}\right|\mathcal{F}_{(i-1)/n}\Big]=n^{-1/2}\sum_{i=1}^{n}F_{\theta,n}(H_{\theta,h})(\sqrt{n}X_{(i-1)/n})P^{\theta}_{w_n+1}H_{\theta,h}(\sqrt{n}X_{(i-1)/n}).
$$

By Lemma 1 i) and ii)

$$
\left|F_{\theta,n}(H_{\theta,h})\,(x)\,\right|\leq K\log(n)
$$

and

$$
|P^{\theta}_{w_n+1}H_{\theta,h}(x)|\leq Kn^{-\beta}\left(\frac{1}{1+|xn^{-\beta/2}|^{\gamma}}+\frac{1}{1+|x|^{\gamma}}\right).
$$

By Remark 1 and Theorem 4.1 a) in Jacod (1998), we therefore deduce that

$$n^{-1/2}\sum_{i=1}^{n} F_{i-1,n}\mathbb{E}^{\theta}\left[f_{w_n+1+i,n}\middle|\, \mathcal{F}_{(i-1)/n}\right] \xrightarrow{P} 0.$$

Let us now consider

$$n^{-1/2}\sum_{i=1}^{n}\left[\mathbb{E}^{\theta}\left[F_{i,n}^2\middle|\, \mathcal{F}_{(i-1)/n}\right] - F_{i-1,n}^2\right].$$

First note that

$$\begin{aligned} F_{i-1,n}^2 &= F_{\theta,n}^2(H_{\theta,h})(\sqrt{n}X_{(i-1)/n}) \\ \mathbb{E}^{\theta}\left[F_{i,n}^2\middle|\, \mathcal{F}_{(i-1)/n}\right] &= \mathbb{E}^{\theta}\left[F_{\theta,n}^2(H_{\theta,h})(\sqrt{n}X_{i/n})\middle|\, \mathcal{F}_{(i-1)/n}\right] \\ &= P_1^{\theta}F_{\theta,n}^2(H_{\theta,h})(\sqrt{n}X_{(i-1)/n}) \end{aligned}$$

and

$$n^{-1/2}\sum_{i=1}^{n}\left[\mathbb{E}^{\theta}\left[F_{i,n}^2\middle|\, \mathcal{F}_{(i-1)/n}\right] - F_{i-1,n}^2\right] = n^{-1/2}\sum_{i=1}^{n}\left[\left(P_1^{\theta}F_{\theta,n}^2(H_{\theta,h}) - F_{\theta,n}^2(H_{\theta,h})\right)(\sqrt{n}X_{(i-1)/n})\right].$$

Let

$$g_n = P_1^{\theta}F_{\theta,n}^2(H_{\theta,h}) - F_{\theta,n}^2(H_{\theta,h}).$$

We have $\lambda_\theta(g_n) = 0$ since λ_θ is the invariant measure of the semi-group $\left(P_t^{\theta}\right)_{t\geq 0}$. By Lemma 1 ii), we have

$$F_{\theta,n}^2(H_{\theta,h})(x) \leq 2\bar{h}^2\,(x) + K(\log n)^2\left(\frac{1}{1+|xn^{-\beta/2}|^{2\gamma}} + \frac{1}{1+|x|^{2\gamma}}\right)$$

and it is easily deduced that (see e.g. Lemma 3.2 in Jacod (1998)),

$$P_1^{\theta}F_{\theta,n}^2(H_{\theta,h})(x) \leq 2P_1^{\theta}\bar{h}^2\,(x) + K(\log n)^2\left(\frac{1}{1+|xn^{-\beta/2}|^{2\gamma}} + \frac{1}{1+|x|^{2\gamma}}\right).$$

Then, with $\gamma > 1$, $\sup_x |g_n\,(x)\,| \leq K(\log n)^2$, $\lambda_\theta(|g_n|) \leq K(\log n)^2 n^{\beta/2}$, $\lambda_\theta(g_n^2) \leq K(\log n)^4 n^{\beta}$ and $\beta_1(g_n) \leq K(\log n)^2 n^{\beta}$. By choosing $\beta < 1/3$, we deduce from Remark 1 and Theorem 4.1 in Jacod (1998) that

$$n^{-1/2}\sum_{i=1}^{n}\left(P_1^{\theta}F_{\theta,n}^2(H_{\theta,h}) - F_{\theta,n}^2(H_{\theta,h})\right)(\sqrt{n}X_{(i-1)/n}) \xrightarrow{P} 0.$$

Finally we can conclude that

$$\sum_{i=1}^{n}\mathbb{E}^{\theta}\left[(\Delta_i^n\tilde{Z})^2\middle|\, \mathcal{F}_{(i-1)/n}\right] \xrightarrow{P} \lambda_\theta(H_{\theta,h^2} + 2\bar{H}_{\theta,h,\Phi_h})L_1.$$

ii) We have to prove that

$$\sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}^{\theta}\left[(\Delta_i^n Z)^2 \, \mathbb{I}_{\{|\Delta_i^n Z|>\varepsilon\}} \right| \left. \mathcal{F}_{(i-1)/n} \right] \xrightarrow{P} 0.$$

By Condition 4, we have

$$|\Delta_i^n Z|^6 \leq K n^{-3/2} e^{a\sqrt{n}|X_{i/n}-X_{(i-1)/n}|}$$

and

$$\mathbb{E}^{\theta}\left[|\Delta_i^n Z|^6 \right| \left. \mathcal{F}_{(i-1)/n} \right] \leq K n^{-3/2}$$

and

$$\mathbb{E}^{\theta}\left[(\Delta_i^n Z)^2 \, \mathbb{I}_{\{|\Delta_i^n Z|>\varepsilon\}} \right| \left. \mathcal{F}_{(i-1)/n} \right] \leq K \varepsilon^{-4} n^{-3/2}.$$

The result easily follows.

iii) We have to prove that

$$\sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}^{\theta}\left[\Delta_i^n \tilde{Z} \Delta_i^n W \right| \left. \mathcal{F}_{(i-1)/n} \right] \xrightarrow{P} 0.$$

First note that

$$\mathbb{E}^{\theta}\left[\Delta_i^n \tilde{Z} \Delta_i^n W \right| \left. \mathcal{F}_{(i-1)/n} \right] = n^{-1/4} \mathbb{E}^{\theta}\left[(f_{i,n} + F_{i,n}) \Delta_i^n W \right| \left. \mathcal{F}_{(i-1)/n} \right]$$

since $\mathbb{E}^{\theta}\left[\Delta_i^n W \right| \left. \mathcal{F}_{(i-1)/n} \right] = 0$. We have

$$\begin{aligned}
& \mathbb{E}^{\theta}\left[F_{i,n} \Delta_i^n W \right| \left. \mathcal{F}_{(i-1)/n} \right] \\
= \;& \mathbb{E}^{\theta}\left[F_{\theta,n}(H_{\theta,h})(\sqrt{n} X_{i/n}) \left(\Delta_i^n X - \theta \Delta_i^n L \right) \right| \left. \mathcal{F}_{(i-1)/n} \right] \\
= \;& \frac{1}{\sqrt{n}} \mathbb{E}^{\theta}\left[F_{\theta,n}(H_{\theta,h})(\sqrt{n} X_{i/n}) \left(\sqrt{n} \Delta_i^n X - \theta \sqrt{n} \Delta_i^n L \right) \right| \left. \mathcal{F}_{(i-1)/n} \right].
\end{aligned}$$

Note that, for some $\gamma > 0$, $\beta_{1+\gamma}(H_{\theta,h}) < \infty$ since

$$\beta_{1+\gamma}(H_{\theta,h}) = \int |x|^{1+\gamma} |H_{\theta,h}(x)| dx \leq K \int |x|^{1+\gamma} |\bar{h}(x)| dx = K \beta_{1+\gamma}(\bar{h}) < \infty.$$

Therefore, we have by Lemma 1

$$
\begin{aligned}
& \left|F_{\theta,n}(H_{\theta,h})(\sqrt{n}X_{i/n})\left(\sqrt{n}\Delta_i^n X-\theta\sqrt{n}\Delta_i^n L\right)\right| \\
\leq\ & |H_{\theta,h}(\sqrt{n}X_{(i-1)/n}+\sqrt{n}\Delta_i^n X)|\left(\left|\sqrt{n}\Delta_i^n X\right|+|\theta|\sqrt{n}\Delta_i^n L\right) \\
& +K\log n\left(\frac{1+|\sqrt{n}\Delta_i^n X|^{\gamma}}{1+|\sqrt{n}X_{(i-1)/n}n^{-\beta/2}|^{\gamma}}+\frac{1+|\sqrt{n}\Delta_i^n X|^{\gamma}}{1+|\sqrt{n}X_{(i-1)/n}|^{\gamma}}\right)\left(\left|\sqrt{n}\Delta_i^n X\right|+|\theta|\sqrt{n}\Delta_i^n L\right)
\end{aligned}
$$

since

$$
\begin{aligned}
\frac{1}{1+|\sqrt{n}X_{i/n}n^{-\beta/2}|^{\gamma}} &\leq K\frac{1+|\sqrt{n}\Delta_i^n X|^{\gamma}}{1+|\sqrt{n}X_{(i-1)/n}n^{-\beta/2}|^{\gamma}} \\
\frac{1}{1+|\sqrt{n}X_{i/n}|^{\gamma}} &\leq K\frac{1+|\sqrt{n}\Delta_i^n X|^{\gamma}}{1+|\sqrt{n}X_{(i-1)/n}|^{\gamma}}.
\end{aligned}
$$

Let

$$
\begin{aligned}
f_1\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^n X,\sqrt{n}\Delta_i^n L\right) &= |H_{\theta,h}(\sqrt{n}X_{(i-1)/n}+\sqrt{n}\Delta_i^n X)|\left(\left|\sqrt{n}\Delta_i^n X\right|+|\theta|\sqrt{n}\Delta_i^n L\right) \\
f_{2,n}\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^n X,\sqrt{n}\Delta_i^n L\right) &= \left(\frac{1+|\sqrt{n}\Delta_i^n X|^{\gamma}}{1+|\sqrt{n}X_{(i-1)/n}n^{-\beta/2}|^{\gamma}}+\frac{1+|\sqrt{n}\Delta_i^n X|^{\gamma}}{1+|\sqrt{n}X_{(i-1)/n}|^{\gamma}}\right) \\
& \quad\times\left(\left|\sqrt{n}\Delta_i^n X\right|+|\theta|\sqrt{n}\Delta_i^n L\right).
\end{aligned}
$$

We have, by Hölder inequality that, for $p>1$ and $q>1$, such that $p^{-1}+q^{-1}=1$,

$$
\begin{aligned}
& \mathbb{E}^{\theta}\left[\left|f_1\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^n X,\sqrt{n}\Delta_i^n L\right)\right|\Big|\,\mathcal{F}_{(i-1)/n}\right] \\
\leq\ & \left(\mathbb{E}^{\theta}\left[|H_{\theta,h}(\sqrt{n}X_{(i-1)/n}+\sqrt{n}\Delta_i^n X)|^q\Big|\,\mathcal{F}_{(i-1)/n}\right]\right)^{1/q}\left(\mathbb{E}^{\theta}\left[\left(\left|\sqrt{n}\Delta_i^n X\right|+|\theta|\sqrt{n}\Delta_i^n L\right)^p\Big|\,\mathcal{F}_{(i-1)/n}\right]\right)^{1/p} \\
\leq\ & K\left(\mathbb{E}^{\theta}\left[|H_{\theta,h}(\sqrt{n}X_{(i-1)/n}+\sqrt{n}\Delta_i^n X)|^q\Big|\,\mathcal{F}_{(i-1)/n}\right]\right)^{1/q}
\end{aligned}
$$

by Eq. (4). Note that

$$
\mathbb{E}^{\theta}\left[|H_{\theta,h}(\sqrt{n}X_{(i-1)/n}+\sqrt{n}\Delta_i^n X)|^q\Big|\,\mathcal{F}_{(i-1)/n}\right]=P_1^{\theta}|H_{\theta,h}|^q\left(\sqrt{n}X_{(i-1)/n}\right).
$$

By Remark 1 and Eq. (3.2) of Lemma 3.1 in Jacod (1998),

$$
\begin{aligned}
P_1^{\theta}|H_{\theta,h}|^q(x) &\leq K\frac{\beta_{1+\gamma}\left(|H_{\theta,h}|^q\right)}{1+|x|^{\gamma}}+\frac{\lambda_{\theta}\left(|H_{\theta,h}|^q\right)}{\sqrt{2\pi}}e^{-x^2/2} \\
&\leq K\frac{\beta_{1+\gamma}\left(|H_{\theta,h}|^q\right)}{1+|x|^{\gamma}}
\end{aligned}
$$

with

$$
\beta_{1+\gamma}\left(|H_{\theta,h}|^q\right)=\int|x|^{1+\gamma}|H_{\theta,h}|^q(x)\,dx\leq K\int|x|^{1+\gamma}|\bar{h}|(x)\,dx=K\beta_{1+\gamma}\left(\bar{h}\right).
$$

Moreover, since $H_{\theta,h}$ is a bounded function (h satisfies Condition 4), this is also the case for $|H_{\theta,h}|^q$

and we can conclude that $P_1^\theta|H_{\theta,h}|^q$ is also bounded. Therefore $\left(P_1^\theta|H_{\theta,h}|^q\right)^{1/q}$ satisfies Condition 1 if $\beta_2(\left(P_1^\theta|H_{\theta,h}|^q\right)^{1/q}) < \infty$, but this is the case, since we can choose $\gamma > 3$, $q > 1$ such that

$$\beta_2\left(\left(P_1^\theta|H_{\theta,h}|^q\right)^{1/q}\right) \leq K\int x^2\frac{1}{1+|x|^{\gamma/q}}dx < \infty.$$

It follows that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\left(P_1^\theta|H_{\theta,h}|^q\right)^{1/q}\left(\sqrt{n}X_{(i-1)/n}\right)\xrightarrow{P}\lambda_\theta(\left(P_1^\theta|H_{\theta,h}|^q\right)^{1/q})L_t.$$

Let

$$\begin{aligned} G_{\theta,f_1}\left(\sqrt{n}X_{(i-1)/n}\right) &= \mathbb{E}^\theta\left[f_1\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^nX,\sqrt{n}\Delta_i^nL\right)\middle|\mathcal{F}_{(i-1)/n}\right] \\ &= \mathbb{E}^\theta\left[|H_{\theta,h}(\sqrt{n}X_{(i-1)/n}+\sqrt{n}\Delta_i^nX)|\left(\left|\sqrt{n}\Delta_i^nX\right|+|\theta|\sqrt{n}\Delta_i^nL\right)\middle|\mathcal{F}_{(i-1)/n}\right]. \end{aligned}$$

Note that G_{θ,f_1} satisfies Condition 1 since G_{θ,f_1} is bounded ($H_{\theta,h}$ is bounded) and $\beta_2\left(G_{\theta,f_1}\right)\leq K\beta_2\left(H_{\theta,h}\right) < \infty$. By Proposition 2 i), we deduce that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}G_{\theta,f_1}\left(\sqrt{n}X_{(i-1)/n}\right)\xrightarrow{P}\lambda_\theta(G_{\theta,f_1})L_t$$

and it follows that

$$n^{-1/4}\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\mathbb{E}^\theta\left[f_1\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^nX,\sqrt{n}\Delta_i^nL\right)\middle|\mathcal{F}_{(i-1)/n}\right]\xrightarrow{P}0.$$

Note now that

$$\begin{aligned} &\mathbb{E}^\theta\left[f_{2,n}\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^nX,\sqrt{n}\Delta_i^nL\right)\middle|\mathcal{F}_{(i-1)/n}\right] \\ \leq\ & K\left(\frac{1}{1+|\sqrt{n}X_{(i-1)/n}n^{-\beta/2}|^\gamma}+\frac{1}{1+|\sqrt{n}X_{(i-1)/n}|^\gamma}\right). \end{aligned}$$

Let

$$g_n\left(x\right) = n^{-1/4}\left(\frac{1}{1+|xn^{-\beta/2}|^\gamma}+\frac{1}{1+|x|^\gamma}\right).$$

We have

$$\lambda_\theta\left(g_n\right) = n^{-1/4}\int\left(\frac{1}{1+|xn^{-\beta/2}|^\gamma}+\frac{1}{1+|x|^\gamma}\right)dx \leq Kn^{\beta/2-1/4}\to 0$$

and, as $n\to\infty$,

$$\frac{g_n\left(x\sqrt{n}\right)}{\sqrt{n}}\to 0.$$

It follows from Remark 1 and Theorem 4.1 in Jacod (1998) that

$$n^{-1/4}\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\mathbb{E}^{\theta}\left[\, f_2\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^nX,\sqrt{n}\Delta_i^nL\right)\right|\mathcal{F}_{(i-1)/n}\Big]\xrightarrow{P}0.$$

Let

$$f_3\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^nX,\sqrt{n}\Delta_i^nL\right)=h\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^nX\right)\left(\sqrt{n}\Delta_i^nX-\theta\sqrt{n}\Delta_i^nL\right).$$

We have

$$\begin{aligned}
& \mathbb{E}^{\theta}\left[\, f_{i,n}\Delta_i^nW\right|\mathcal{F}_{(i-1)/n}\Big] \\
= & \ \mathbb{E}^{\theta}\left[\, f_{i,n}\left(\Delta_i^nX-\theta\Delta_i^nL\right)\right|\mathcal{F}_{(i-1)/n}\Big] \\
= & \ \frac{1}{\sqrt{n}}\mathbb{E}^{\theta}\left[h\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^nX\right)\left(\sqrt{n}\Delta_i^nX-\theta\sqrt{n}\Delta_i^nL\right)\right|\mathcal{F}_{(i-1)/n}\Big] \\
= & \ \frac{1}{\sqrt{n}}\mathbb{E}^{\theta}\left[\, f_3\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^nX,\sqrt{n}\Delta_i^nL\right)\right|\mathcal{F}_{(i-1)/n}\Big]
\end{aligned}$$

where f_3 satisfies Condition 3, i.e. f_3 is a Borel function on $\mathbb{R}^3$ such that the functions G_{θ,f_3}, G_{θ,f_3^2} satisfy Condition 1, i.e. G_{θ,f_3}, G_{θ,f_3^2} are bounded functions and $\beta_2(G_{\theta,f_3})<\infty$ and $\beta_2(G_{\theta,f_3^2})<\infty$. Indeed, since h satisfies Condition 4, we have

$$\begin{aligned}
|G_{\theta,f_3}(x)| &= |\mathbb{E}_x^{\theta}\left[f_3(x,X_1-x,L_1)\right]|\le K\bar{h}(x)\le K \\
G_{\theta,f_3^2}(x) &= \mathbb{E}_x^{\theta}\left[f_3^2(x,X_1-x,L_1)\right]\le K\bar{h}^2(x)\le K
\end{aligned}$$

and

$$\begin{aligned}
\beta_2(G_{\theta,f_3}) &= \int|x|^2G_{\theta,f_3}(x)\,dx\le K\int|x|^2\bar{h}(x)\,dx\le K\beta_2(\bar{h})<\infty \\
\beta_2(G_{\theta,f_3^2}) &= \int|x|^2|G_{\theta,f_3^2}(x)|dx\le K\int|x|^2\bar{h}^2(x)\,dx\le K\int|x|^2\bar{h}(x)\,dx\le K\beta_2(\bar{h})<\infty.
\end{aligned}$$

By Proposition 2 iii), we deduce that

$$\sum_{i=1}^{\lfloor nt\rfloor}\mathbb{E}^{\theta}\left[\, f_{i,n}\Delta_i^nW\right|\mathcal{F}_{(i-1)/n}\Big]\xrightarrow{P}\lambda_{\theta}(G_{\theta,f_3})L_t$$

and it follows that

$$n^{-1/4}\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}\mathbb{E}^{\theta}\left[\, f_3\left(\sqrt{n}X_{(i-1)/n},\sqrt{n}\Delta_i^nX,\sqrt{n}\Delta_i^nL\right)\right|\mathcal{F}_{(i-1)/n}\Big]\xrightarrow{P}0.$$

Therefore, we can conclude that

$$\sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}^{\theta}\left[\Delta_i^n \tilde{Z} \Delta_i^n W \middle| \mathcal{F}_{(i-1)/n} \right] \xrightarrow{P} 0.$$

iv) We finally have to prove that

$$\sum_{i=1}^{\lfloor nt \rfloor} \mathbb{E}^{\theta}\left[\Delta_i^n \tilde{Z} \Delta_i^n M \middle| \mathcal{F}_{(i-1)/n} \right] \xrightarrow{P} 0$$

where M is a bounded $\left(\mathcal{F}_t^W\right)_{t\geq 0}$-martingale such that, for all $s \in [0,1]$, the cross variation satisfies $P\left(\langle M, Y\rangle_s = 0\right) = 1$.

If M is a square integrable $\left(\mathcal{F}_t^W\right)_{t\in[0,1]}$-martingale such that for all $s \in [0,1]$ the cross variation satisfies $P\left(\langle M, Y\rangle_s = 0\right) = 1$, then M is constant. Indeed, since M is a square integrable $\left(\mathcal{F}_t^W\right)_{t\in[0,1]}$-martingale, by the martingale representation theorem, we have

$$M_t = M_0 + \int_0^t \eta_s dW_s$$

where η is a $\left(\mathcal{F}_t^W\right)_{t\in[0,1]}$-progressively measurable process such that $P(\int_0^1 \eta_s^2 ds < \infty) = 1$. The condition that, for all s, $P\left(\langle M, Y\rangle_s = 0\right) = 1$ writes $P(\int_0^s \eta_u \sigma\left(Y_u\right) du = 0) = 1$. Since σ is positive, it implies that $P(\int_0^1 \mathbb{I}_{\{|\eta_s|>0\}} ds = 0) = 1$. We can therefore conclude that $\mathbb{E}^{\theta}\left[\Delta_i^n \tilde{Z} \Delta_i^n M \middle| \mathcal{F}_{(i-1)/n} \right] = 0$ a.s.

Proof of Proposition 8

Let

$$\begin{aligned} f_{i,n}(c) &= \mathbb{I}_{\{-c/\sqrt{n} < X_{(i-1)/n} < 0, X_{i/n} < 0\}} \left\{ \sqrt{n} \left| X_{i/n} - X_{(i-1)/n} \right| - \varphi_{-\theta}\left(c\right) \right\} \\ &= h_c\left(\sqrt{n} X_{(i-1)/n}, \sqrt{n}\left(X_{i/n} - X_{(i-1)/n} \right) \right) \end{aligned}$$

with

$$h_c\left(x, y\right) = \mathbb{I}_{\{-c<x<0, y+x<0\}} \left\{ |y| - \varphi_{-\theta}\left(c\right) \right\}$$

and

$$Z^n(c) = \frac{1}{n^{1/4}} \sum_{i=1}^{n} f_{i,n}(c).$$

Let $0 < C < \infty$. We want to prove that, for any $\eta > 0$,

$$\frac{1}{n^{1/20}(\log n)^{1+\eta}} \sup_{c\in[0,C]} |Z^n(c)| \xrightarrow{P} 0.$$

It will follow that, for $0 < \underline{C} < \bar{C} < \infty$,

$$\frac{n^{1/5}}{(\log n)^{1+\eta}} \sup_{c\in[\underline{C},\bar{C}]} \left|B_-^n(c) - \varphi_{-\theta}(c)\right| \xrightarrow{P} 0,$$

and then

$$\frac{n^{1/5}}{(\log n)^{1+\eta}} \sup_{c\in[\underline{C},\bar{C}]} \left|A_-^n(c) - A_{\theta,-}(c)\right| \xrightarrow{P} 0.$$

Since $c_-^n \xrightarrow{P} c_{\theta,-}$ and $\hat{\theta}_n \xrightarrow{P} \theta$, we will deduce that

$$\lim_{n\to\infty} \frac{n^{1/5}}{(\log n)^{1+\eta}} \left|\hat{\sigma}_-^n - \sigma_-\right| \stackrel{P}{=} 0.$$

And in the same way, we will have

$$\lim_{n\to\infty} \frac{n^{1/5}}{(\log n)^{1+\eta}} \left|\hat{\sigma}_+^n - \sigma_+\right| \stackrel{P}{=} 0$$

and therefore

$$\lim_{n\to\infty} \frac{n^{1/5}}{(\log n)^{1+\eta}} \left|\hat{\delta}^n - \delta\right| \stackrel{P}{=} 0.$$

We first begin with a lemma.

Lemma 3 *i) Let us assume that for some constants $0 < c < d$ such that $|d-c| < 1$,*

$$|g_{c,d}(x)| \le K\left(\mathbb{I}_{\{-d<x<-c\}} + \mathbb{I}_{\{-c<x<0\}}|d-c|\right),$$

then

$$\sup_{0\le t\le 1} \mathbb{E}^\theta\left[\left(\frac{1}{\sqrt{n}}V(g_{c,d})_t^n\right)^2\right] \le K\left(\frac{|d-c|}{\sqrt{n}} + |d-c|^2\right).$$

ii) Let us assume that for some constants $0 < c < d$ such that $|d-c| < 1$ and $\gamma \ge 1$

$$|h_{c,d}(x,y)| \le K\left(\mathbb{I}_{\{-d<x<-c\}}(1+|y|^\gamma) + \mathbb{I}_{\{-c<x<0\}}|d-c|\right)$$

and that $H_{\theta,h_{c,d}}(x)=0$ *for all* x, *then*

$$\sup_{0\le t\le 1}\mathbb{E}^\theta\left[\left(\frac{1}{n^{1/4}}U(h_{c,d})_t^n\right)^4\right]\le K\left(\frac{|d-c|}{\sqrt{n}}+|d-c|^2\right).$$

Proof.

i) By Eq. (3.10) in Jacod (1998), we have

$$\begin{aligned}\mathbb{E}_x^\theta\left[\frac{1}{\sqrt{n}}V(|g_{c,d}|)_t^n\right] &\le K\left(\frac{|g_{c,d}(\sqrt{n}x)|}{\sqrt{n}}+\lambda_\theta(|g_{c,d}|)\right)\\ &\le K\left(\frac{\mathbb{I}_{\{-d<\sqrt{n}x<-c\}}}{\sqrt{n}}+|d-c|\right)\end{aligned}$$

and therefore, for large n,

$$\mathbb{E}^\theta\left[\frac{1}{\sqrt{n}}V(|g_{c,d}|)_t^n\right]\le K|d-c|.$$

Moreover

$$\left(\frac{1}{\sqrt{n}}V(g_{c,d})_t^n\right)^2=\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{n}}\left(V(g_{c,d}^2)_t^n\right)\right)+\frac{2}{n}\sum_{1\le i<j\le\lfloor nt\rfloor}g_{c,d}\left(\sqrt{n}X_{(i-1)/n}\right)g_{c,d}\left(\sqrt{n}X_{(j-1)/n}\right)$$

and

$$\begin{aligned}|g_{c,d}(x)|^2 &\le K\left(\mathbb{I}_{\{-d<x<-c\}}+\mathbb{I}_{\{-d<x<-c\}}|d-c|+\mathbb{I}_{\{-c<x<0\}}|d-c|^2\right)\\ &\le K\left(\mathbb{I}_{\{-d<x<-c\}}+\mathbb{I}_{\{-c<x<0\}}|d-c|\right).\end{aligned}$$

By the Markov property, we deduce that

$$\begin{aligned}&\mathbb{E}^\theta\left[\left(\frac{1}{\sqrt{n}}V(g_{c,d})_t^n\right)^2\right]\\ \le\ &\frac{K}{\sqrt{n}}|d-c|+K\mathbb{E}^\theta\left[\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt\rfloor}|g_{c,d}|\left(\sqrt{n}X_{(i-1)/n}\right)\left(\frac{\mathbb{I}_{\{-d<\sqrt{n}X_{(i-1)/n}<-c\}}}{\sqrt{n}}+|d-c|\right)\right)\right]\\ \le\ &\frac{K}{\sqrt{n}}|d-c|+\frac{K}{\sqrt{n}}|d-c|+K|d-c|^2\\ \le\ &K\left(\frac{|d-c|}{\sqrt{n}}+|d-c|^2\right)\end{aligned}$$

and the result follows.

ii) $M^n=n^{-1/4}U(h_{c,d})_t^n$ is a martingale with respect to the filtration $\left(\mathcal{F}_{\lfloor nt\rfloor/n}\right)_{t\ge0}$ since $H_{\theta,h_{c,d}}(x)=0$, with optional and predictable brackets given by

$$[M^n, M^n] = \frac{1}{\sqrt{n}} U(h^2_{c,d})^n, \quad \langle M^n, M^n \rangle = \frac{1}{\sqrt{n}} V(H_{\theta, h^2_{c,d}})^n.$$

Note that

$$N^n = ([M^n, M^n] - \langle M^n, M^n \rangle) = \frac{1}{\sqrt{n}} U(h^2_{c,d} - H_{\theta, h^2_{c,d}})^n$$

is also a martingale with respect to the filtration $\left(\mathcal{F}_{\lfloor nt \rfloor / n}\right)_{t \geq 0}$.

By the Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}^\theta \left[\sup_{0 \leq t \leq 1} |M^n_t|^4 \right] \leq K \mathbb{E}^\theta \left[[M^n, M^n]^2_1 \right].$$

Moreover

$$[M^n, M^n]^2_1 \leq 2 \left((N^n_1)^2 + \langle M^n, M^n \rangle^2_1 \right).$$

Since

$$H_{\theta, h^2_{c,d}}(x) = \int p_\theta(1, x, y) \, h^2_{c,d}(x, y - x) dy$$

and

$$h^2_{c,d}(x,y) \leq K \left(\mathbb{I}_{\{-d<x<-c\}} \left(1 + |y|^{2\gamma}\right) + \mathbb{I}_{\{-c<x<0\}} |d-c|^2 + \mathbb{I}_{\{-d<x<-c\}} \left(1 + |y|^\gamma\right) |d-c| \right)$$

it follows that

$$H_{\theta, h^2_{c,d}}(x) \leq K \left(\mathbb{I}_{\{-d<x<-c\}} + \mathbb{I}_{\{-c<x<0\}} |d-c| \right).$$

By i), we have

$$\mathbb{E}^\theta \left[\langle M^n, M^n \rangle^2_1 \right] = \mathbb{E}^\theta \left[\left(\frac{1}{\sqrt{n}} V(H_{\theta, h^2_{c,d}})^n \right)^2 \right] \leq K \left(\frac{|d-c|}{\sqrt{n}} + |d-c|^2 \right).$$

Doob's inequality yields

$$\mathbb{E}^\theta \left[\sup_{0 \leq t \leq 1} |M^n_t|^2 \right] \leq 4 \mathbb{E}^\theta \left[\langle M^n, M^n \rangle_1 \right],$$

and by using the same arguments as in the beginning of the proof of i), we have

$$\mathbb{E}^\theta \left[\sup_{0 \leq t \leq 1} |M^n_t|^2 \right] \leq K |d-c|.$$

Since

$$
\begin{aligned}
& |h^2_{c,d}(x,y) - H_{\theta,h^2_{c,d}}(x)| \\
\leq\ & K\left(\mathbb{I}_{\{-d<x<-c\}}\left(1+|y|^{2\gamma}\right) + \mathbb{I}_{\{-c<x<0\}}|d-c|^2 + \mathbb{I}_{\{-d<x<-c\}}\left(1+|y|^{\gamma}\right)|d-c|\right) \\
& + K\left(\mathbb{I}_{\{-d<x<-c\}} + \mathbb{I}_{\{-c<x<0\}}|d-c|\right) \\
\leq\ & K\left(\mathbb{I}_{\{-d<x<-c\}}\left(1+|y|^{2\gamma}\right) + \mathbb{I}_{\{-c<x<0\}}|d-c|\right),
\end{aligned}
$$

we deduce in the same way that

$$
\sqrt{n}\mathbb{E}^{\theta}\left[\sup_{0\leq t\leq 1}|N^n_t|^2\right] \leq K\,|d-c|\,.
$$

It follows that

$$
\sup_{0\leq t\leq 1}\mathbb{E}^{\theta}\left[\left(\frac{1}{n^{1/4}}U(h_{c,d})^n_t\right)^4\right] \leq \mathbb{E}^{\theta}\left[\sup_{0\leq t\leq 1}|M^n_t|^4\right] \leq K\left(\frac{|d-c|}{\sqrt{n}} + |d-c|^2\right).
$$

The proof is complete.

■

Step 1: An upper bound for $\mathbb{E}^{\theta}[(Z_n(d)-Z_n(c))^4]$.

Recall that

$$
Z^n(c) = \frac{1}{n^{1/4}}\sum_{i=1}^{n} f_{i,n}(c)
$$

with

$$
f_{i,n}(c) = \mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0, X_{i/n}<0\}}\left\{\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right| - \varphi_{-\theta}(c)\right\}.
$$

Define

$$
\begin{aligned}
k_c(x) &= \mathbb{E}^{\theta}\left[f_{i,n}(c)\middle|\sqrt{n}X_{(i-1)/n}=x\right] \\
&= \mathbb{I}_{\{-c<x<0\}}\left\{\left[\mathbb{E}[|Z|\mathbb{I}_{\{Z<-x\}}] - \theta\mathbb{E}[|Z-2x|\mathbb{I}_{\{Z<x\}}]\right] - \varphi_{-\theta}(c)\left[\Phi(-x)-\theta\Phi(x)\right]\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
g_{i,n}(c) &= \sum_{j=0}^{w_n}\left(\mathbb{E}^{\theta}\left[k_c(\sqrt{n}X_{(i+j)/n})\middle|\mathcal{F}_{i/n}\right] - \mathbb{E}^{\theta}\left[k_c(\sqrt{n}X_{(i+j)/n})\middle|\mathcal{F}_{(i-1)/n}\right]\right) \\
&= \sum_{j=0}^{w_n}\left(\mathbb{E}^{\theta}\left[f_{(i+j+1),n}(c)\middle|\mathcal{F}_{i/n}\right] - \mathbb{E}^{\theta}\left[f_{(i+j+1),n}(c)\middle|\mathcal{F}_{(i-1)/n}\right]\right)
\end{aligned}
$$

with $w_n = [n^\beta]$ and $\beta \in (0, 1/2)$. Note that $k_c(x) = H_{\theta,h_1}(x)$ where h_1 is defined in the proof of Proposition 7. Therefore $\lambda_\theta(k_c) = \lambda_\theta(H_{\theta,h_1}) = 0$.

We have

$$Z^n(c) = n^{-1/4}\sum_{i=1}^n f_{i,n}(c) = Y^n(c) + W^n(c) + n^{-1/4}H_n(c) + n^{-1/4}I_n(c)$$

where

$$\begin{aligned}
Y^n(c) &= n^{-1/4}\sum_{i=1}^n \left[f_{i,n}(c) - k_c(\sqrt{n}X_{(i-1)/n})\right] \\
W^n(c) &= n^{-1/4}\sum_{i=1}^n g_{i,n}(c) \\
H_n(c) &= F_{\theta,n}(k_c)(\sqrt{n}X_0) - F_{\theta,n}(k_c)(\sqrt{n}X_1) \\
I_n(c) &= \sum_{i=1}^n \mathbb{E}^\theta\left[k_c(\sqrt{n}X_{(i+w_n)/n})\middle|\mathcal{F}_{(i-1)/n}\right] \\
&= \sum_{i=1}^n P^\theta_{w_n+1}k_c(\sqrt{n}X_{(i-1)/n}) = V(P^\theta_{w_n+1}k_c)^n_1,
\end{aligned}$$

with

$$F_{\theta,n}(k_c)(\sqrt{n}X_{i/n}) = \sum_{j=0}^{w_n}\mathbb{E}^\theta\left[k_c(\sqrt{n}X_{(i+j)/n})\middle|\mathcal{F}_{i/n}\right] = \sum_{j=1}^{w_n+1}\mathbb{E}^\theta\left[f_{i+j,n}(c)\middle|\mathcal{F}_{i/n}\right].$$

Let, for $c < d$,

$$k_{c,d}(x) = k_d(x) - k_c(x) \quad \text{and} \quad f_{i,n}(c,d) = f_{i,n}(d) - f_{i,n}(c).$$

Then, since φ_θ is Lipschitz on $\left(\underline{C}, \bar{C}\right)$ for any $0 < \underline{C} < \bar{C} < \infty$, we have

$$\begin{aligned}
|k_{c,d}(x)| &\le K\left(\mathbb{I}_{\{-d<x<-c\}} + \mathbb{I}_{\{-c<x<0\}}|d-c|\right) \\
\beta_1(k_{c,d}) &\le K|d-c| \\
\beta_{1+\gamma}(k_{c,d}) &\le K\left(|d-c| + |d-c|^{1+\gamma}\right) \le K|d-c|.
\end{aligned}$$

We now study each part of

$$\begin{aligned}
Z^n(d) - Z^n(c) &= (Y^n(d) - Y^n(c)) + (W^n(d) - W^n(c)) \\
&\quad + n^{-1/4}(H_n(d) - H_n(c)) + n^{-1/4}(I_n(d) - I_n(c)).
\end{aligned}$$

A) By Remark 1, Lemma 3.1 and Eq. (3.2) in Jacod (1998) (noting that $\lambda_\theta(k_{c,d})=0$), we have

$$\begin{aligned}|F_{\theta,n}(k_{c,d})(x)| &\leq |k_{c,d}(x)|+K\log n\left(\frac{\beta_1(k_{c,d})}{1+\left|xn^{-\beta/2}\right|^\gamma}+\frac{\beta_{1+\gamma}(k_{c,d})}{1+|x|^\gamma}\right)\\ &\leq |k_{c,d}(x)|+K\log n\,|d-c|.\end{aligned}$$

Then

$$|H_n(d)-H_n(c)|\leq\left|k_{c,d}\left(\sqrt{n}x_0\right)\right|+\left|k_{c,d}\left(\sqrt{n}X_1\right)\right|+K\log n\,|d-c|$$

and

$$|H_n(d)-H_n(c)|^4\leq K\left(\left|k_{c,d}\left(\sqrt{n}x_0\right)\right|^4+\left|k_{c,d}\left(\sqrt{n}X_1\right)\right|^4+(\log n)^4|d-c|^4\right).$$

Moreover by Lemma 3.1 in Jacod (1998)

$$\mathbb{E}^\theta\left[\left|k_{c,d}\left(\sqrt{n}X_1\right)\right|^4\right]\leq K\frac{\lambda_\theta\left(k_{c,d}^4\right)}{\sqrt{n}}\leq K\frac{|d-c|}{\sqrt{n}}.$$

It follows that

$$n^{-1}\mathbb{E}^\theta\left[|H_n(d)-H_n(c)|^4\right]\leq K\left(\frac{|d-c|}{n^{3/2}}+|d-c|^4n^{-1}(\log n)^4\right).$$

B) By Remark 1, Lemma 3.1 and Eq. (3.2) in Jacod (1998) (noting that $\lambda_\theta\left(P^\theta_{w_n+1}k_{c,d}\right)=0$), we have

$$\begin{aligned}\left|P^\theta_{w_n+1}(k_{c,d})(x)\right| &\leq Kn^{-\beta}\left(\frac{\beta_1(k_{c,d})}{1+\left|xn^{-\beta/2}\right|^\gamma}+\frac{\beta_{1+\gamma}(k_{c,d})}{1+|x|^\gamma}\right)\\ &\leq Kn^{-\beta}|d-c|\left(\frac{1}{1+\left|xn^{-\beta/2}\right|^\gamma}+\frac{1}{1+|x|^\gamma}\right).\end{aligned}$$

Let

$$G_{n,c,d}(x)=P^\theta_{w_n+1}(k_{c,d})(x).$$

Note that $\lambda_\theta(G_{n,c,d})=\lambda_\theta\left(P^\theta_{w_n+1}(k_{c,d})\right)=\lambda_\theta(k_{c,d})=0$ since λ_θ is the invariant measure of the semi-group $\left(P^\theta_t\right)_{t\geq0}$. We also have (since γ may be chosen larger than 2)

$$\begin{aligned}\|G_{n,c,d}\| &\leq Kn^{-\beta}|d-c|\\ \lambda_\theta(|G_{n,c,d}|) &\leq Kn^{-\beta/2}|d-c|\\ \beta_1(G_{n,c,d}) &\leq K|d-c|.\end{aligned}$$

Since

$$n^{-1/4}(I_n(d)-I_n(c))=n^{1/4}\frac{1}{\sqrt{n}}V(G_{n,c,d})_1^n$$

and by Lemma 2, we have

$$
\begin{aligned}
& \mathbb{E}^{\theta}\left[\left(\frac{1}{\sqrt{n}}V\left(G_{n,c,d}\right)_1^n\right)^4\right] \\
\leq\ & K\left|d-c\right|^3\left(n^{-3\beta}+n^{-3\beta/2}\right)\left(\left|d-c\right|\frac{\log n+n^{-\beta}}{\sqrt{n}}\right) \\
\leq\ & K\left|d-c\right|^4 n^{-3\beta/2-1/2}
\end{aligned}
$$

and it follows that

$$
\mathbb{E}^{\theta}\left[n^{-1}|I_n(d)-I_n(c)|^4\right]\leq K\left|d-c\right|^4(\log n)n^{-3\beta/2+1/2}.
$$

C) Let

$$
\hat{F}_{\theta,n}(k_c)(\sqrt{n}X_{i/n})=\sum_{j=1}^{w_n+1}\mathbb{E}^{\theta}\left[k_c(\sqrt{n}X_{(i+j)/n})\middle|\mathcal{F}_{i/n}\right]
$$

such that

$$
\begin{aligned}
g_{i,n}(c) &= \sum_{j=0}^{w_n}\left(\mathbb{E}^{\theta}\left[k_c(\sqrt{n}X_{(i+j)/n})\middle|\mathcal{F}_{i/n}\right]-\mathbb{E}^{\theta}\left[k_c(\sqrt{n}X_{(i+j)/n})\middle|\mathcal{F}_{(i-1)/n}\right]\right) \\
&= F_{\theta,n}(k_c)\left(\sqrt{n}X_{i/n}\right)-\hat{F}_{\theta,n}(k_c)\left(\sqrt{n}X_{(i-1)/n}\right).
\end{aligned}
$$

By Remark 1, Lemma 3.1 and Eq. (3.2) in Jacod (1998) (noting that $\lambda_\theta\left(k_{c,d}\right)=0$), we have

$$
\begin{aligned}
\left|\hat{F}_{\theta,n}(k_{c,d})\left(x\right)\right| &\leq K\log n\left(\frac{\beta_1\left(k_{c,d}\right)}{1+\left|xn^{-\beta/2}\right|^{\gamma}}+\frac{\beta_{1+\gamma}\left(k_{c,d}\right)}{1+\left|x\right|^{\gamma}}\right) \\
&\leq K\log n\left|d-c\right|.
\end{aligned}
\tag{2}
$$

Let

$$
\begin{aligned}
h_{n,c,d}(x,y) &= F_{\theta,n}(k_{c,d})\left(x+y\right)-\hat{F}_{\theta,n}(k_{c,d})\left(x\right) \\
H_{\theta,h^2_{n,c,d}}\left(x\right) &= \int p_\theta\left(1,x,y\right)h^2_{n,c,d}(x,y)dy \\
&= \int p_\theta\left(1,x,y\right)F^2_{\theta,n}(k_{c,d})\left(x+y\right)dy-\hat{F}^2_{\theta,n}(k_{c,d})\left(x\right).
\end{aligned}
$$

Then

$$
W^n_{c,d}=n^{-1/4}\sum_{i=1}^{n}\left(g_{i,n}(d)-g_{i,n}(c)\right)
$$

is a square integrable martingale with respect to the filtration $\left(\mathcal{F}_{\lfloor nt\rfloor/n}\right)_{t\geq 0}$, with optional and predictable brackets given by

$$
[W^n_{c,d},W^n_{c,d}]=\frac{1}{\sqrt{n}}U(h^2_{n,c,d})^n,\quad\left\langle W^n_{c,d},W^n_{c,d}\right\rangle=\frac{1}{\sqrt{n}}V(H_{\theta,h^2_{n,c,d}})^n.
$$

Since

$$|k_{c,d}(x)| \leq K\left(\mathbb{I}_{\{-d<x<-c\}} + \mathbb{I}_{\{-c<x<0\}}\,|d-c|\right),$$

we have, by Eq. (2),

$$\begin{aligned}
&\int p_\theta\,(1,x,y)\,F^2_{\theta,n}(k_{c,d})\,(x+y)\,dy \\
\leq\ & K\Pr\left(X_1+x\in(-d,-c)\right)+K\Pr\left(X_1+x\in(-c,0)\right)|d-c|+K\left(\log n\right)^2|d-c|^2\left(\frac{1}{1+\left|xn^{-\beta/2}\right|^\gamma}\right)^2
\end{aligned}$$

using also Remark 1 and Lemma 3.2 in Jacod (1998). Then (since γ may be chosen larger than 2)

$$\left\|H_{\theta,h^2_{n,c,d}}\right\| \leq K\,|d-c| + K\left(\log n\right)^2|d-c|^2$$

$$\lambda_\theta(|H_{\theta,h^2_{n,c,d}}|) \leq K\,|d-c| + K\left(\log n\right)^2|d-c|^2\,n^{\beta/2}$$

and by Lemma 2, we have

$$\begin{aligned}
\mathbb{E}^\theta\left[\left\langle W^n_{c,d},W^n_{c,d}\right\rangle_1^2\right] &= \mathbb{E}^\theta\left[\left(\frac{1}{\sqrt{n}}V(H_{\theta,h^2_{n,c,d}})^n\right)^2\right] \\
&\leq K\left(|d-c|^2+\left(\log n\right)^4|d-c|^4\,n^\beta\right).
\end{aligned}$$

The Burkholder-Davis-Gundy inequality provides

$$\mathbb{E}^\theta\left[|W^n_{c,d}|^4\right] \leq K\mathbb{E}^\theta\left[[W^n_{c,d},W^n_{c,d}]^2\right].$$

The jumps of $W^n_{c,d}$ are bounded by $K\log n\,|d-c|\,n^{-1/4}$. The martingale

$$M^n_{c,d} = [W^n_{c,d},W^n_{c,d}] - \left\langle W^n_{c,d},W^n_{c,d}\right\rangle$$

has n jumps, all bounded by $K\left(\log n\right)^2|d-c|^2\,n^{-1/2}$. It follows that $[M^n_{c,d},M^n_{c,d}]^2 \leq K\left(\log n\right)^4|d-c|^4$ and

$$\begin{aligned}
\mathbb{E}^\theta\left[|W^n_{c,d}|^4\right] &\leq K\left(\mathbb{E}^\theta\left[[M^n_{c,d},M^n_{c,d}]^2\right]+\mathbb{E}^\theta\left[\left\langle W^n_{c,d},W^n_{c,d}\right\rangle_1^2\right]\right) \\
&\leq K\left(|d-c|^2+\left(\log n\right)^4|d-c|^4\,n^\beta\right).
\end{aligned}$$

The choice for β that lets the upper bounds of B) and C) be equivalent is $\beta = -3\beta/2+1/2$, i.e. $\beta = 1/5$.

D) Let

$$h_{c,d}(x,y) = f_{c,d}(x,y) - k_{c,d}(x)$$

where

$$\begin{aligned} f_{c,d}(x,y) &= \mathbb{I}_{\{-d<x<-c,y+x<0\}}\,|y| - \mathbb{I}_{\{-d<x<-c,y+x<0\}}\varphi_{-\theta}\,(d) \\ &\quad -\mathbb{I}_{\{-c<x<0,y+x<0\}}\,(\varphi_{-\theta}\,(d) - \varphi_{-\theta}\,(c))\,. \end{aligned}$$

We have

$$\begin{aligned} Y^n\,(d) - Y^n\,(c) &= n^{-1/4}\sum_{i=1}^{n}\left[f_{i,n}(c,d) - k_{c,d}(\sqrt{n}X_{(i-1)/n})\right] \\ &= n^{-1/4}\sum_{i=1}^{n} h_{c,d}(\sqrt{n}X_{(i-1)/n}, \sqrt{n}\left(X_{i/n} - X_{(i-1)/n}\right)). \end{aligned}$$

Note that

$$|k_{c,d}(x)| \leq K\left(\mathbb{I}_{\{-d<x<-c\}} + \mathbb{I}_{\{-c<x<0\}}\,|d-c|\right)$$

and that for some $\gamma \geq 1$

$$|h_{c,d}(x,y)| \leq K\left(\mathbb{I}_{\{-d<x<-c\}}\,(1+|y|^{\gamma}) + \mathbb{I}_{\{-c<x<0\}}\,|d-c|\right).$$

Let us now remark that $H_{\theta,h_{c,d}}\,(x) = 0$ for all x. As in Proposition 7, let us consider

$$\begin{aligned} h_c\,(x,y) &= h_{c,\sigma-}\,(x,y) - \varphi_{-\theta}\,(c)\,h_{c,k-}\,(x,y) \\ &= \mathbb{I}_{\{-c<x<0,y+x<0\}}|y| - \varphi_{-\theta}\,(c)\,\mathbb{I}_{\{-c<x<0,y+x<0\}}. \end{aligned}$$

It is important to note that $k_c(x) = H_{\theta,h_c}\,(x)$. We have

$$\begin{aligned} & h_d\,(x,y) - h_c\,(x,y) \\ =\;& \mathbb{I}_{\{-d<x<-c,y+x<0\}}\,|y| - \varphi_{-\theta}\,(d)\,\mathbb{I}_{\{-d<x<0,y+x<0\}} + \varphi_{-\theta}\,(c)\,\mathbb{I}_{\{-c<x<0,y+x<0\}} \\ =\;& \mathbb{I}_{\{-d<x<-c,y+x<0\}}\,|y| - \mathbb{I}_{\{-d<x<-c,y+x<0\}}\varphi_{-\theta}\,(d) - \mathbb{I}_{\{-c<x<0,y+x<0\}}\,(\varphi_{-\theta}\,(d) - \varphi_{-\theta}\,(c)) \\ =\;& f_{c,d}(x,y) \end{aligned}$$

and it follows that

$$H_{\theta,h_{c,d}}\,(x) = H_{\theta,h_d}\,(x) - H_{\theta,h_c}\,(x) - (k_d(x) - k_c(x)) = 0.$$

By Lemma 3 and since $H_{\theta,h_{c,d}}\,(x) = 0$ for all x, we have

$$\mathbb{E}^{\theta}\left[\left(\frac{1}{n^{1/4}}U(h_{c,d})_t^n\right)^4\right] \leq K\left(\frac{|d-c|}{\sqrt{n}} + |d-c|^2\right).$$

E) Putting A, B, C, D together, we have

$$
\begin{aligned}
& \mathbb{E}^{\theta}\left[\left(Z_n(d)-Z_n(c)\right)^4\right] \\
\leq\; & K\frac{|d-c|}{n^{3/2}}+K\,|d-c|^4\,n^{-1}\,(\log n)^4+K\,|d-c|^4\,n^{1/5}\log n \\
& +K\left(|d-c|^2+(\log n)^4\,|d-c|^4\,n^{1/5}\right)+K\left(\frac{|d-c|}{\sqrt{n}}+|d-c|^2\right)
\end{aligned}
$$

and then

$$
\begin{aligned}
& \mathbb{E}^{\theta}\left[\left(Z_n(d)-Z_n(c)\right)^4\right] \\
\leq\; & K\left(\frac{|d-c|}{\sqrt{n}}+|d-c|^2+(\log n)^4\,|d-c|^4\,n^{1/5}\right) \\
\leq\; & K\left(\frac{|d-c|}{\sqrt{n}}+(\log n)^4\,|d-c|^2\,n^{1/5}\right)
\end{aligned}
$$

or equivalently

$$
\frac{1}{n^{1/5}\,(\log n)^4}\mathbb{E}^{\theta}\left[\left(Z_n(d)-Z_n(c)\right)^4\right]\leq K\left(\frac{|d-c|}{n^{7/10}\,(\log n)^4}+|d-c|^2\right).
$$

<u>Step 2:</u> We have

$$
f_{i,n}(c)=h_{1,i,n}\left(c\right)-h_{2,i,n}\left(c\right)
$$

with

$$
\begin{aligned}
h_{1,i,n}\left(c\right) &= \mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0,X_{i/n}<0\}}\sqrt{n}\left|X_{i/n}-X_{(i-1)/n}\right| \\
h_{2,i,n}\left(c\right) &= \mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<0,X_{i/n}<0\}}\varphi_{-\theta}\left(c\right).
\end{aligned}
$$

Both functions are positive and $h_{1,i,n}$ is increasing with respect to c. For $0\leq c\leq d$, we have

$$
\begin{aligned}
f_{i,n}(c) &= h_{1,i,n}\left(c\right)-h_{2,i,n}\left(c\right) \\
&\leq h_{1,i,n}\left(d\right)-h_{2,i,n}\left(c\right) \\
&= f_{i,n}(d)-\left(h_{2,i,n}\left(c\right)-h_{2,i,n}\left(d\right)\right) \\
&\leq f_{i,n}(d)+h_{2,i,n}\left(d\right)
\end{aligned}
$$

and

$$
\sum_{i=1}^{n}f_{i,n}(c)\leq\sum_{i=1}^{n}f_{i,n}(d)+\sum_{i=1}^{n}h_{2,i,n}\left(d\right)
$$

and

$$\sum_{i=1}^{n} f_{i,n}(c) \leq \left|\sum_{i=1}^{n} f_{i,n}(d)\right| + \sum_{i=1}^{n} h_{2,i,n}\left(d\right).$$

Moreover

$$-\sum_{i=1}^{n} f_{i,n}(c) \leq \sum_{i=1}^{n} h_{2,i,n}\left(c\right) \leq \left|\sum_{i=1}^{n} f_{i,n}(d)\right| + \sum_{i=1}^{n} h_{2,i,n}\left(d\right) \vee h_{2,i,n}\left(c\right)$$

since

$$-f_{i,n}(c) = h_{2,i,n}\left(c\right) - h_{1,i,n}\left(c\right) \leq h_{2,i,n}\left(c\right) \leq h_{2,i,n}\left(d\right) \vee h_{2,i,n}\left(c\right),$$

and therefore

$$\left|\sum_{i=1}^{n} f_{i,n}(c)\right| \leq \left|\sum_{i=1}^{n} f_{i,n}(d)\right| + \sum_{i=1}^{n} h_{2,i,n}\left(d\right) \vee h_{2,i,n}\left(c\right).$$

Using the same type of arguments, we have, for $(j-1)p \leq c \leq jp$,

$$\begin{aligned}
\left|\sum_{i=1}^{n} f_{i,n}(c) - \sum_{i=1}^{n} f_{i,n}((j-1)p)\right| &\leq \left|\sum_{i=1}^{n} f_{i,n}(jp) - \sum_{i=1}^{n} f_{i,n}((j-1)p)\right| \\
&\quad + \sum_{i=1}^{n} \left(h_{2,i,n}\left(jp\right) - h_{2,i,n}\left((j-1)p\right)\right) \vee \left(h_{2,i,n}\left(c\right) - h_{2,i,n}\left((j-1)p\right)\right) \\
&\leq \left|\sum_{i=1}^{n} f_{i,n}(jp) - \sum_{i=1}^{n} f_{i,n}((j-1)p)\right| \\
&\quad + \sum_{i=1}^{n} \sup_{c\in((j-1)p,jp]} \left|\left(h_{2,i,n}\left(c\right) - h_{2,i,n}\left((j-1)p\right)\right)\right|.
\end{aligned}$$

We deduce that

$$\sup_{0\leq c\leq mp}\left|\sum_{i=1}^{n} f_{i,n}(c)\right| \leq 3\max_{j\leq m}\left|\sum_{i=1}^{n} f_{i,n}(jp)\right| + \max_{j\leq m}\sum_{i=1}^{n} \sup_{c\in((j-1)p,jp]} \left|\left(h_{2,i,n}\left(c\right) - h_{2,i,n}\left((j-1)p\right)\right)\right|.$$

Then, with $jp \leq C$, $(j-1)p \leq c \leq jp$,

$$\begin{aligned}
& h_{2,i,n}\left(c\right) - h_{2,i,n}\left((j-1)p\right) \\
= \; & \mathbb{I}_{\{-c/\sqrt{n}<X_{(i-1)/n}<-(j-1)p/\sqrt{n},X_{i/n}<0\}}\varphi_{-\theta}\left(c\right) + \mathbb{I}_{\{-(j-1)p/\sqrt{n}<X_{(i-1)/n}<0,X_{i/n}<0\}}\left(\varphi_{-\theta}\left(c\right) - \varphi_{-\theta}\left((j-1)p\right)\right)
\end{aligned}$$

and

$$\left|h_{2,i,n}\left(c\right) - h_{2,i,n}\left((j-1)p\right)\right| \leq \left[\sup_{c\in[0,C]} \varphi_{-\theta}\left(c\right)\right]\mathbb{I}_{\{-jp/\sqrt{n}<X_{(i-1)/n}<-(j-1)p/\sqrt{n}\}} + Kp\mathbb{I}_{\{-C/\sqrt{n}<X_{(i-1)/n}<0\}}$$

and it follows

$$\max_{j\leq m}\sum_{i=1}^{n}\sup_{c\in((j-1)p,jp]}\left|\left(h_{2,i,n}\left(c\right)-h_{2,i,n}\left((j-1)p\right)\right)\right|$$
$$\leq\quad\left[\sup_{c\in[0,C]}\varphi_{-\theta}\left(c\right)\right]\max_{j\leq m}\sum_{i=1}^{n}\mathbb{I}_{\{-jp/\sqrt{n}<X_{(i-1)/n}<-(j-1)p/\sqrt{n}\}}+K\sum_{i=1}^{n}p\mathbb{I}_{\{-C/\sqrt{n}<X_{(i-1)/n}<0\}}.$$

Using Proposition 2, we have

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{I}_{\{-C/\sqrt{n}<X_{(i-1)/n}<0\}}\overset{P}{\to}\left(1-\theta\right)CL_1.$$

Moreover if we assume that $p\to 0$ as $n\to\infty$ such that $\sqrt{n}p\to\infty$, then, using the same arguments as in the proof of Theorem 1.1 in Jacod (1998), we get

$$\frac{1}{\sqrt{np}}\max_{j\leq m}\sum_{i=1}^{n}\mathbb{I}_{\{-jp/\sqrt{n}<X_{(i-1)/n}<-(j-1)p/\sqrt{n}\}}\overset{P}{\to}\left(1-\theta\right)\sup_{c\in[0,C]}L_1^c$$

where L_1^c is the symmetric local time of X at level c.

Therefore

$$\frac{\sup_{c\in[0,C]}\varphi_{-\theta}\left(c\right)\max_{j\leq m}\sum_{i=1}^{n}\mathbb{I}_{\{-jp/\sqrt{n}<X_{(i-1)/n}<-(j-1)p/\sqrt{n}\}}+\sum_{i=1}^{n}p\mathbb{I}_{\{-C/\sqrt{n}<X_{(i-1)/n}<0\}}}{\sqrt{np}\log n}\overset{P}{\to}0,$$

and it follows

$$P\left(\sup_{0\leq c\leq mp}\left|Z^n(c)\right|\leq 3\max_{j\leq m}\left|Z^n(jp)\right|+pn^{1/4}\log n\right)\to 1.$$

Step 3:

If $0<\varepsilon<1$ and

$$\frac{\varepsilon}{n^{7/10}\left(\log n\right)^4}\leq\left|d-c\right|,$$

we get from Step 1

$$\frac{1}{n^{1/5}\left(\log n\right)^4}\mathbb{E}^{\theta}\left[\left(Z_n(d)-Z_n(c)\right)^4\right]\leq\frac{K}{\varepsilon}\left|d-c\right|^2.$$

Assume that p is a number such that $\varepsilon n^{-7/10}/(\log n)^4 \leq p$. Consider the random variables

$$\frac{1}{n^{1/20}\log n}\left(Z^n(ip) - Z^n((i-1)p)\right), \quad i = 1, ..., m.$$

By Theorem 12.2 in Billingsley (1968), we have

$$P\left(\max_{i\leq m}\frac{1}{n^{1/20}\log n}\left|Z^n(ip)\right| > \lambda\right) \leq \frac{K}{\varepsilon\lambda^4}m^2p^2.$$

Let

$$\mathcal{A}_n = \left\{\sup_{0\leq c\leq mp}|Z^n(c)| \leq 3\max_{j\leq m}|Z^n(jp)| + pn^{1/4}\log n\right\}.$$

From Step 2, we have $P(\mathcal{A}_n) \to 1$ if $\sqrt{n}p \to \infty$. Then

$$\begin{aligned} & P\left(\frac{1}{n^{1/20}\log n}\sup_{0\leq c\leq mp}\left|Z^n(c)\right| > 4\varepsilon\right) \\ \leq\ & P\left(\frac{1}{n^{1/20}\log n}\sup_{0\leq c\leq mp}\left|Z^n(c)\right| > 4\varepsilon, \mathcal{A}_n\right) + P\left(\mathcal{A}_n^c\right). \end{aligned}$$

If

$$\frac{\varepsilon}{n^{7/10}(\log n)^4} \leq \frac{\log n}{n^{1/2}} \leq p \leq \frac{\varepsilon}{n^{1/5}}$$

then

$$\begin{aligned} P\left(\frac{1}{n^{1/20}\log n}\sup_{0\leq c\leq mp}\left|Z^n(c)\right| > 4\varepsilon, \mathcal{A}_n\right) &\leq P\left(\frac{1}{n^{1/20}\log n}\max_{j\leq m}|Z^n(jp)| > \varepsilon\right) \\ &\leq \frac{K}{\varepsilon^5}m^2p^2. \end{aligned}$$

Let $\eta > 0$, we get

$$P\left(\frac{1}{n^{1/20}\log n}\sup_{0\leq c\leq mp}\left|Z^n(c)\right| > 4\varepsilon'(\log n)^\eta, \mathcal{A}_n\right) \leq \frac{K}{\varepsilon'^5}\frac{C^2}{(\log n)^{5\eta}}$$

where $mp \leq C$ or equivalently

$$P\left(\frac{1}{n^{1/20}\log(n)^{1+\eta}}\sup_{0\leq c\leq mp}\left|Z^n(c)\right| > 4\varepsilon, \mathcal{A}_n\right) \leq \frac{K}{\varepsilon^5}\frac{C^2}{(\log n)^{5\eta}}.$$

Therefore if $mp \to C$ as $n \to \infty$, then

$$P\left(\frac{1}{n^{1/20}}\frac{1}{\log(n)^{1+\eta}}\sup_{0\leq c\leq C}\left|Z^n(c)\right| > \varepsilon\right) \to 0$$

and the result follows.

Some comments on the one-dimensional time-homogeneous diffusion with finite activity jumps case

Let us assume that process Y is the solution of the one-dimensional time-homogeneous stochastic differential equation defined by Eq. (1) with

$$b(y) = \frac{1}{2}\sigma'(y)\sigma(y)\mathbb{I}_{\{y\neq 0\}}.$$

Let us denote by $N_t = \int_0^t \int_{|x|\geq 1} N(dt, dx)$ the number of jumps of Y occurring between time 0 and t.

The local time at level 0 of the semi-martingale Y is defined as

$$L_t(Y) = |Y_t| - |y_0| - \int_0^t sgn(Y_{s-})dY_s - \sum_{0<s\leq t} \{|Y_s| - |Y_{s-}| - sgn(Y_{s-})\Delta Y_s\}$$

where $\Delta Y_s = Y_s - Y_{s-}$ (see e.g. Definition p. 216 in Protter (2005)). Note that, since Y is a one-dimensional time-homogeneous diffusion with finite activity jumps,

$$L_t(Y) =_{a.s.} \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{I}_{\{|Y_s|\leq\varepsilon\}} d[Y]_s^c$$

by Corollaries 2 and 3 in p. 229-230 of Protter (2005).

By the Ito-Tanaka-Meyer formula for semi-martingales, we have

$$S(Y_t) = S(y_0) + W_t + \frac{1}{2}\left(\frac{1}{\sigma_+} - \frac{1}{\sigma_-}\right) L_t(Y) + \sum_{0<s\leq t} \Delta S(Y_s)$$

where

$$S(y) = \int_0^y \frac{1}{\sigma(x)}dx, \quad y \in \mathbb{R}\backslash\{0\}, \quad S(0) = 0,$$

can be written as the difference of two convex functions. By definition of the local time of $S(Y)$ at level 0, we also have

$$|S(Y_t)| = |S(y_0)| + \int_0^t sgn(S(Y_{s-}))dS(Y_s) + \sum_{0<s\leq t} \{\Delta|S(Y_s)| - sgn(S(Y_{s-}))\Delta S(Y_s)\} + L_t(S(Y)).$$

Since $sgn(S(Y_{s-})) = sgn(Y_{s-})$, we deduce that

$$|S(Y_t)| = |S(y_0)| + \int_0^t sgn(Y_{s-})dW_s + \sum_{0<s\leq t} \Delta|S(Y_s)| + L_t(S(Y)).$$

Now, if we consider the function $y \to |S(y)|$ (which can also be written as the difference of two convex functions), we derive by the Ito-Tanaka-Meyer formula that

$$|S(Y_t)| = |S(y_0)| + \int_0^t sgn(Y_{s-})dW_s + \sum_{0<s\le t} \Delta|S(Y_s)| + \frac{1}{2}\left(\frac{1}{\sigma_+} + \frac{1}{\sigma_-}\right) L_t\left(Y\right).$$

We therefore conclude that

$$L_t\left(S(Y)\right) = \frac{1}{2}\left(\frac{1}{\sigma_+} + \frac{1}{\sigma_-}\right) L_t\left(Y\right)$$

(as in the case where the finite activity jump component of Y does not exist, see the proof of Proposition 1) and that

$$S(Y_t) = S(y_0) + W_t + \theta L_t\left(S(Y)\right) + \sum_{0<s\le t} \Delta S\left(Y_s\right).$$

Let $X_t = S(Y_t)$, we get

$$X_t = x_0 + W_t + \theta L_t\left(X\right) + \sum_{0<s\le t} \Delta X_s.$$

We can derive from the Markov property of (Y_t) (and (X_t)) that, if $N_{i/n} - N_{(i-1)/n} = 0$, $(X_{(i-1)/n}, \left(X_{i/n} - X_{(i-1)/n}\right))$ has the same distribution as $(X_{(i-1)/n}, \tilde{X}_{1/n} - X_{(i-1)/n})$ where $(\tilde{X}_t)$ is an SBM with parameter θ such that $\tilde{X}_0 = 0$ and that is independent of $X_{(i-1)/n}$.

Let us now consider the estimators

$$\begin{aligned} A_-^n\left(c, u_n\right) &= \frac{\sum_{i=1}^n \mathbb{I}_{\{S\left(-c/\sqrt{n}\right) < X_{(i-1)/n} < 0, X_{i/n} < 0, |S^{-1}\left(X_{i/n}\right) - S^{-1}(X_{(i-1)/n})| \le u_n\}} \sqrt{n}\left|S^{-1}\left(X_{i/n}\right) - S^{-1}\left(X_{(i-1)/n}\right)\right|}{\sum_{i=1}^n \mathbb{I}_{\{S\left(-c/\sqrt{n}\right) < X_{(i-1)/n} < 0, X_{i/n} < 0, |S^{-1}\left(X_{i/n}\right) - S^{-1}(X_{(i-1)/n})| \le u_n\}}}, \\ A_+^n\left(c, u_n\right) &= \frac{\sum_{i=1}^n \mathbb{I}_{\{0 < X_{(i-1)/n} < S\left(c/\sqrt{n}\right), X_{i/n} > 0, |S^{-1}\left(X_{i/n}\right) - S^{-1}(X_{(i-1)/n})| \le u_n\}} \sqrt{n}\left|S^{-1}\left(X_{i/n}\right) - S^{-1}\left(X_{(i-1)/n}\right)\right|}{\sum_{i=1}^n \mathbb{I}_{\{0 < X_{(i-1)/n} < S\left(c/\sqrt{n}\right), X_{i/n} > 0, |S^{-1}\left(X_{i/n}\right) - S^{-1}(X_{(i-1)/n})| \le u_n\}}}. \end{aligned}$$

Since

$$\begin{aligned} & \mathbb{I}_{\{|S^{-1}\left(X_{i/n}\right) - S^{-1}(X_{(i-1)/n})| \le u_n\}} \\ = & \ \mathbb{I}_{\{N_{i/n} - N_{(i-1)/n} = 0\}} + \mathbb{I}_{\{|S^{-1}\left(X_{i/n}\right) - S^{-1}(X_{(i-1)/n})| \le u_n, N_{i/n} - N_{(i-1)/n} > 0\}} \\ & - \mathbb{I}_{\{|S^{-1}\left(X_{i/n}\right) - S^{-1}(X_{(i-1)/n})| > u_n, N_{i/n} - N_{(i-1)/n} = 0\}}, \end{aligned}$$

we deduce from the discussion given in Section 2 that, for all n large enough, $A_-^n\left(c, u_n\right)$ and $A_+^n\left(c, u_n\right)$ are respectively equal to

$$\begin{aligned} A_-^n\left(c\right) &= \frac{\sum_{i=1}^n \mathbb{I}_{\{S\left(-c/\sqrt{n}\right) < X_{(i-1)/n} < 0, X_{i/n} < 0, N_{i/n} - N_{(i-1)/n} = 0\}} \sqrt{n}\left|S^{-1}\left(X_{i/n}\right) - S^{-1}\left(X_{(i-1)/n}\right)\right|}{\sum_{i=1}^n \mathbb{I}_{\{S\left(-c/\sqrt{n}\right) < X_{(i-1)/n} < 0, X_{i/n} < 0, N_{i/n} - N_{(i-1)/n} = 0\}}} \\ A_+^n\left(c\right) &= \frac{\sum_{i=1}^n \mathbb{I}_{\{0 < X_{(i-1)/n} < S\left(c/\sqrt{n}\right), X_{i/n} > 0, N_{i/n} - N_{(i-1)/n} = 0\}} \sqrt{n}\left|S^{-1}\left(X_{i/n}\right) - S^{-1}\left(X_{(i-1)/n}\right)\right|}{\sum_{i=1}^n \mathbb{I}_{\{0 < X_{(i-1)/n} < S\left(c/\sqrt{n}\right), X_{i/n} > 0, N_{i/n} - N_{(i-1)/n} = 0\}}} \end{aligned}$$

with probability approaching 1. The number of intervals $((i-1)/n, i/n]$ for which $N_{i/n} - N_{(i-1)/n} > 0$ is a.s. finite. We can therefore study the asymptotic properties of our estimators by assuming that the finite activity jump component of Y in Eq. (1) of the paper does not exist. Note however that the presence of jumps changes the path of (X_t) and $(L_t(X))$ and so the values of the estimators are different.

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