Online Supplementary Material for
"On Multiple Structural Breaks in Distribution: An Empirical Characteristic Function Approach"


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This appendix provides the proofs for the main results in this paper. Section A provides the proofs of theorems in Sections 2 to 4. Section B provides the proofs of technical lemmas. Section C provides additional results. Throughout the appendix, we let

$$\psi_j^0(u) = E(e^{\mathbf{i}u'Y_t})$$

denote the true CF of $Y_t$ for $t \in [T_{j-1}^0 + 1, T_j^0]$,

$$\tilde{\psi}_j(u) = \frac{1}{T_j - T_{j-1}} \sum_{t=T_{j-1}+1}^{T_j} e^{\mathbf{i}u'Y_t}$$

denote the ECF of $Y_t$ for $t \in [T_{j-1} + 1, T_j]$,

$$\tilde{\psi}_j^0(u) = \frac{1}{T_j^0 - T_{j-1}^0} \sum_{t=T_{j-1}^0+1}^{T_j^0} e^{\mathbf{i}u'Y_t}$$

denote the ECF of $Y_t$ for $t \in [T_{j-1}^0 + 1, T_j^0]$, and

$$\hat{\psi}_j(u) = \frac{1}{\hat{T}_j - \hat{T}_{j-1}} \sum_{t=\hat{T}_{j-1}+1}^{\hat{T}_j} e^{\mathbf{i}u'Y_t}$$

denote the feasible ECF of $Y_t$ for $t \in [\hat{T}_{j-1} + 1, \hat{T}_j]$. We use $\mathcal{C} \in (0, \infty)$ to denote a generic positive constant that may vary from case to case.

We first state some technical lemmas.

**Lemma** A.1 *Suppose Assumptions A.1-A.3 hold. Then for any partition $\{T_j\}_{j=1}^M$ and any specified number of breaks $M$,*

$$\frac{1}{T} \int_{\mathbb{R}^d} \left| \sum_{t=T_{j-1}+1}^{T_j} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u = O_P(1),$$

*for each $j = 1, ..., M$.*

**Lemma** A.2 *Suppose Assumptions A.1-A.5 hold. Then*

$$\sup_{\eta\in[0,C]}\int_{\mathbb{R}^d}\left|v_T\sum_{t=T_{j-1}^0+1}^{T_{j-1}^0+\lfloor\eta v_T^{-2}\rfloor}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u=O_P(1),$$

*for each* $j=1,...,M^0+1$*, where* $v_T=O(T^{-a})$ *with* $a\in(0,1/2)$*.*

**Lemma** A.3 *Suppose Assumptions A.1-A.3 hold, and let* $d_t(u)=\hat{\psi}_k(u)-\psi_j^0(u)$ *for* $t\in[\hat{T}_{k-1}+1,\hat{T}_k]\cap[T_{j-1}^0+1,T_j^0]$ *with* $k,j=1,2,...,M^0+1$ *denote the difference between the feasible ECF and the true CF. Then*

$$\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\mathrm{Re}\left[\varepsilon_t(u)d_t(u)^*\right]W(u)\mathrm{d}u=O_P(T^{-1/2}).$$

**Lemma** A.4 *Suppose Assumptions A.1-A.3 hold, and* $\lim_{T\to\infty}\hat{r}_j\neq r_j^0$ *for some* $j=1,...,M^0$*. Let* $d_t(u)=\hat{\psi}_k(u)-\psi_j^0(u)$ *for* $t\in[\hat{T}_{k-1}+1,\hat{T}_k]\cap[T_{j-1}^0+1,T_j^0]$ *with* $k,j=1,2,...,M^0+1$ *denote the difference between the feasible ECF and the true CF. Then for any* $0<c_0<1$*, there exists a* $\delta>0$*, such that*

$$P\left(\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|d_t(u)\right|^2 W(u)\mathrm{d}u>\delta\int_{\mathbb{R}^d}\left|\psi_j^0(u)-\psi_{j+1}^0(u)\right|^2 W(u)\mathrm{d}u\right)>c_0.$$

**Lemma** A.5 *Suppose that there are* $M^0$ *structural breaks in the distribution of* $\{Y_t\}$*. If the model is under-specified, i.e., the estimated break points* $M<M^0$*, then we have that the estimated break fractions* $\{\hat{r}_k\}_{k=1}^M$ *are consistent for* $M$ *breaks contained in the collection true break fractions* $\{r_j^0\}_{j=1}^{M^0}$*, such that*

$$\hat{r}_k\xrightarrow{p}r_j^0,$$

*for any* $k=1,...,M$ *and some corresponding* $j=1,...,M^0$*.*

**Lemma** A.6 *Suppose that the number of breaks* $M$ *is bounded from above by a finite integer* $M_{\max}$*, and Assumptions A.1-A.4 hold. Then, as* $T\to\infty$*,*

$$\max_{M^0\leq M\leq M_{\max}}\left|\hat{\sigma}^2(M)-\hat{\sigma}^2(M^0)\right|=O_P(T^{-1}),$$

*where* $\hat{\sigma}^2(M)=T^{-1}\mathrm{SSGR}_M(\hat{r}_1,...,\hat{r}_M)$ *and* $\{\hat{r}_j\}_{j=1}^M$ *is the collection of estimated break fractions.*

# A Proofs of Theorems

**Proof of Theorem 2.1** Following the notations in the paper, for $j=1,...,M^0+1$, let

$$\varepsilon_t(u)=e^{\mathbf{i}u'Y_t}-\psi_j^0(u),\text{ for }t\in[T_{j-1}^0+1,T_j^0],$$

$$\tilde{\varepsilon}_t(u)=e^{\mathbf{i}u'Y_t}-\tilde{\psi}_j^0(u),\text{ for }t\in[T_{j-1}^0+1,T_j^0],$$

$$\hat{\varepsilon}_t(u)=e^{\mathbf{i}u'Y_t}-\hat{\psi}_j(u),\text{ for }t\in[\hat{T}_{j-1}+1,\hat{T}_j].$$

Since $\{\hat{T}_1, ..., \hat{T}_{M^0}\}$ minimize the objective function in (2.2) for any partition $\{T_j\}_{j=1}^{M^0}$, it holds that

$$\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}|\hat{\varepsilon}_t(u)|^2 W(u)\mathrm{d}u \leq \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}|\tilde{\varepsilon}_t(u)|^2 W(u)\mathrm{d}u. \tag{A.1}$$

By the definition of $\varepsilon_t(u)$ and $\tilde{\varepsilon}_t(u)$, it follows

$$\begin{aligned}\tilde{\varepsilon}_t(u) &= e^{\mathbf{i}u'Y_t} - \frac{1}{T_j^0 - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0} e^{\mathbf{i}u'Y_t} \\ &= \varepsilon_t(u) - \frac{1}{T_j^0 - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u)\end{aligned}$$

for $t \in [T_{j-1}^0 + 1, T_j^0]$. Let $\widetilde{\Xi}(u) = [\tilde{\varepsilon}_1(u), \tilde{\varepsilon}_2(u), ..., \tilde{\varepsilon}_T(u)]'$, $\Xi(u) = [\varepsilon_1(u), \varepsilon_2(u), ..., \varepsilon_T(u)]'$, and $\mathcal{Z}^0 = \mathrm{diag}(\mathbf{1}_{T_1^0}, \mathbf{1}_{T_2^0-T_1^0}, ..., \mathbf{1}_{T-T_{M^0}^0})$, where $\mathbf{1}_n$ denotes an $n \times 1$ vector with each element being 1. We have the following matrix representation

$$\widetilde{\Xi}(u) = \Xi(u) - \mathcal{Z}^0(\mathcal{Z}^{0\prime}\mathcal{Z}^0)^{-1}\mathcal{Z}^{0\prime}\Xi(u),$$

Therefore,

$$\begin{aligned}\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}|\tilde{\varepsilon}_t(u)|^2 W(u)\mathrm{d}u &= \frac{1}{T}\int_{\mathbb{R}^d}\left[\widetilde{\Xi}(u)^*\widetilde{\Xi}(u)\right]W(u)\mathrm{d}u \\ &= \frac{1}{T}\int_{\mathbb{R}^d}\left[\Xi(u)^*\Xi(u)\right]W(u)\mathrm{d}u - \frac{1}{T}\int_{\mathbb{R}^d}\left[\Xi(u)^*\mathcal{Z}^0(\mathcal{Z}^{0\prime}\mathcal{Z}^0)^{-1}\mathcal{Z}^{0\prime}\Xi(u)\right]W(u)\mathrm{d}u.\end{aligned}$$

Let $P^0 = \mathcal{Z}^0(\mathcal{Z}^{0\prime}\mathcal{Z}^0)^{-1}\mathcal{Z}^{0\prime}$, then it is straightforward to show that $P^0$ is an $T \times T$ block-diagonal matrix with $M^0 + 1$ diagonally partitions $P_j^0 = (T_j^0 - T_{j-1}^0)^{-1}\mathbf{1}_{(T_j^0-T_{j-1}^0)\times(T_j^0-T_{j-1}^0)}$, where $\mathbf{1}_{(T_j^0-T_{j-1}^0)\times(T_j^0-T_{j-1}^0)}$ is a $(T_j^0 - T_{j-1}^0) \times (T_j^0 - T_{j-1}^0)$ square matrix with all entries being 1, for $j = 1, 2, ..., M^0 + 1$. Then it follows

$$\begin{aligned}&\frac{1}{T}\int_{\mathbb{R}^d}\left[\Xi(u)^*\mathcal{Z}^0(\mathcal{Z}^{0\prime}\mathcal{Z}^0)^{-1}\mathcal{Z}^{0\prime}\Xi(u)\right]W(u)\mathrm{d}u \\ =\ &\frac{1}{T}\sum_{j=1}^{M^0+1}\frac{1}{T_j^0 - T_{j-1}^0}\int_{\mathbb{R}^d}\left|\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\ =\ &O_P(T^{-1}),\end{aligned}$$

by Lemma A.1. Then (A.1) implies

$$\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}|\hat{\varepsilon}_t(u)|^2 W(u)\mathrm{d}u \leq \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}|\varepsilon_t(u)|^2 W(u)\mathrm{d}u - O_P(T^{-1}). \tag{A.2}$$

Let $d_t(u) = \hat{\psi}_k(u) - \psi_j^0(u)$ for $t \in [\hat{T}_{k-1} + 1, \hat{T}_k] \cap [T_{j-1}^0 + 1, T_j^0]$ with $k, j = 1, 2, ..., M^0 + 1$ denote the difference between the feasible ECF and the true CF. By the fact that $\hat{\varepsilon}_t(u) = \varepsilon_t(u) - d_t(u)$, then

$$
\begin{aligned}
&\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|\hat{\varepsilon}_t(u)\right|^2 W(u)\mathrm{d}u \\
&= \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u + \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|d_t(u)\right|^2 W(u)\mathrm{d}u - \frac{2}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\mathrm{Re}\left[\varepsilon_t(u)d_t(u)^*\right]W(u)\mathrm{d}u.
\end{aligned}
$$

Based on Lemmas $A.3$ and $A.4$, we know that if some break fraction $r_j^0$ is not consistently estimated, then the following inequality holds.

$$
\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|\hat{\varepsilon}_t(u)\right|^2 W(u)\mathrm{d}u \geq \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u + \delta\int_{\mathbb{R}^d}\left|\psi_j^0(u) - \psi_{j+1}^0(u)\right|^2 W(u)\mathrm{d}u + o_P(1)
$$

with probability no less than some $0 < c_0 < 1$. This is contradictory to the inequality in $(A.2)$, which holds with probability 1 for all $T$. Hence, all breaks are consistently estimated. ■

**Proof of Theorem 2.2** The proof is quite similar to the proof of Proposition 2 in Bai and Perron (1998). Hence we omit it. ■

**Proof of Theorem 2.3** Without loss of generality, we assume $\hat{T}_{j-1} < T_{j-1}^0 < T_j^0 < \hat{T}_j$. Then, it follows

$$
\begin{aligned}
\sqrt{T}\left[\hat{\psi}_j(u) - \psi_j^0(u)\right] &= \frac{\sqrt{T}}{\hat{T}_j - \hat{T}_{j-1}}\sum_{t=\hat{T}_{j-1}+1}^{\hat{T}_j}\left[e^{\mathbf{i}u'Y_t} - \psi_j^0(u)\right] \\
&= \frac{T_j^0 - T_{j-1}^0}{\hat{T}_j - \hat{T}_{j-1}}\left\{\frac{\sqrt{T}}{T_j^0 - T_{j-1}^0}\sum_{t=\hat{T}_{j-1}+1}^{T_{j-1}^0}\left[e^{\mathbf{i}u'Y_t} - \psi_j^0(u)\right] + \frac{\sqrt{T}}{T_j^0 - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0}\left[e^{\mathbf{i}u'Y_t} - \psi_j^0(u)\right]\right. \\
&\quad \left. + \frac{\sqrt{T}}{T_j^0 - T_{j-1}^0}\sum_{t=T_j^0+1}^{\hat{T}_j}\left[e^{\mathbf{i}u'Y_t} - \psi_j^0(u)\right]\right\} \\
&= C_j\left[D_1(u) + D_2(u) + D_3(u)\right], \text{ say.}
\end{aligned}
$$

By Theorem 2.2, it follows that $C_j = 1 + o_P(1)$. By the triangle inequality

$$
\begin{aligned}
\sup_{u\in\mathbb{R}^d}\left|D_1(u)\right| &\leq \sup_{u\in\mathbb{R}^d}\left[\frac{\sqrt{T}}{T_j^0 - T_{j-1}^0}\sum_{t=\hat{T}_{j-1}+1}^{T_{j-1}^0}\left|e^{\mathbf{i}u'Y_t} - \psi_j^0(u)\right|\right] \\
&\leq \frac{2(T_{j-1}^0 - \hat{T}_{j-1})}{\sqrt{T}(r_j^0 - r_{j-1}^0)},
\end{aligned}
$$

where the last equality is by the fact that

$$
\left|e^{\mathbf{i}u'Y_t} - \psi_j^0(u)\right| \leq \left|e^{\mathbf{i}u'Y_t}\right| + \left|\psi_j^0(u)\right| \leq 2.
$$

Given $P(|\hat{T}_j - T_j^0| > \delta) < \eta$, then

$$P\left(\frac{T_{j-1}^0 - \hat{T}_{j-1}}{\sqrt{T}(r_j^0 - r_{j-1}^0)} > \frac{\delta}{\sqrt{T}(r_j^0 - r_{j-1}^0)}\right) < \eta,$$

for any $\eta > 0$ and $T$ sufficiently large. Thus, we have $\sup_{u\in\mathbb{R}^d} |D_1(u)| = O_P(T^{-1/2})$. Analogously, it follows that $\sup_{u\in\mathbb{R}^d} |D_3(u)| = O_P(T^{-1/2})$. At last, we consider $D_2(u)$.

$$\begin{aligned} D_2(u) &= \frac{\sqrt{T}}{T_j^0 - T_{j-1}^0} \sum_{t=T_{j-1}^0+1}^{T_j^0} \left[e^{\mathbf{i}u'Y_t} - \psi_j^0(u)\right] \\ &= (r_j^0 - r_{j-1}^0)^{-1/2} \left[\frac{1}{\sqrt{T_j^0 - T_{j-1}^0}} \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u)\right] \\ &\Rightarrow (r_j^0 - r_{j-1}^0)^{-1/2} B^{(j)}(u,1), \end{aligned}$$

by Assumption A.4. ■

**Proof of Theorem 2.4** We assume the magnitude of shifts depends on $T$, i.e., $\Delta_{T,j}(u) = \psi_{j+1}^0(u) - \psi_j^0(u) = v_T\Delta_j(u)$. Let $\mathrm{SSGR}_{M^0}(T_j^0 + \lfloor \eta v_T^{-2} \rfloor)$ and $\mathrm{SSGR}_{M^0}(T_j^0)$ denote the sum of squared generalized residuals based on partition $\{T_1^0, ..., T_j^0 + \lfloor \eta v_T^{-2} \rfloor, ..., T_{M^0}^0\}$ and the true break dates $\{T_1^0, ..., T_{M^0}^0\}$, respectively. We consider the process $\mathrm{SSGR}_{M^0}(T_j^0 + \lfloor \eta v_T^{-2} \rfloor) - \mathrm{SSGR}_{M^0}(T_j^0)$ indexed by $\eta$, where $\eta$ is a real number such that $\eta \in [-C, C]$. For notational simplicity, in the rest of this proof, we suppress the subscript $M^0$, and write them as $\mathrm{SSGR}(T_j^0 + \lfloor \eta v_T^{-2} \rfloor)$ and $\mathrm{SSGR}(T_j^0)$.

By the definition of $\tilde{\psi}_j^0(u)$, it follows

$$\tilde{\psi}_j^0(u) - \psi_j^0(u) = \frac{1}{T_j^0 - T_{j-1}^0} \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u), \tag{A.3}$$

for all $j = 1, ..., M^0 + 1$. Let $T_j \equiv T_j(\eta) = T_j^0 + \lfloor \eta v_T^{-2} \rfloor$. For notational simplicity, we suppress the dependence of $T_j$ on $\eta$. Given $v_T = T^{-a}$ with $a \in (0, 1/2)$, it follows

$$\sup_{\eta\in[-C,C]} T_j/T = r_j^0 + o(1).$$

Furthermore, we let

$$\tilde{\psi}_j(u) = \frac{1}{T_j - T_{j-1}^0} \sum_{t=T_{j-1}^0+1}^{T_j} e^{\mathbf{i}u'Y_t},$$

$$\tilde{\psi}_{j+1}(u) = \frac{1}{T_{j+1}^0 - T_j} \sum_{t=T_j+1}^{T_{j+1}^0} e^{\mathbf{i}u'Y_t},$$

denote the ECFs for the $j$-th and $(j+1)$-th regimes under the partition $\{T_1^0, ..., T_j^0 + \lfloor \eta v_T^{-2} \rfloor, ..., T_{M^0}^0\}$.

When $\eta > 0$, it follows $T_{j-1}^0 < T_j^0 < T_j < T_{j+1}^0$. Then it is straightforward to show

$$\tilde{\psi}_j(u) - \psi_j^0(u) = \frac{T_j - T_j^0}{T_j - T_{j-1}^0}\left[\psi_{j+1}^0(u) - \psi_j^0(u)\right] + \frac{1}{T_j - T_{j-1}^0}\left[\sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) + \sum_{t=T_j^0+1}^{T_j} \varepsilon_t(u)\right], \tag{A.4}$$

and

$$\tilde{\psi}_{j+1}(u) - \psi_{j+1}^0(u) = \frac{1}{T_{j+1}^0 - T_j}\sum_{t=T_j+1}^{T_{j+1}^0} \varepsilon_t(u). \tag{A.5}$$

Then

$$\begin{aligned}
&\mathrm{SSGR}(T_j^0 + \lfloor \eta v_T^{-2} \rfloor) - \mathrm{SSGR}(T_j^0) \\
=& \sum_{t=T_{j-1}^0+1}^{T_j^0} \int_{\mathbb{R}^d} \left[\left|e^{\mathbf{i}u'Y_t} - \tilde{\psi}_j(u)\right|^2 - \left|e^{\mathbf{i}u'Y_t} - \tilde{\psi}_j^0(u)\right|^2\right] W(u)\mathrm{d}u \\
&+ \sum_{t=T_j^0+1}^{T_j} \int_{\mathbb{R}^d} \left[\left|e^{\mathbf{i}u'Y_t} - \tilde{\psi}_j(u)\right|^2 - \left|e^{\mathbf{i}u'Y_t} - \tilde{\psi}_{j+1}^0(u)\right|^2\right] W(u)\mathrm{d}u \\
&+ \sum_{t=T_j+1}^{T_{j+1}^0} \int_{\mathbb{R}^d} \left[\left|e^{\mathbf{i}u'Y_t} - \tilde{\psi}_{j+1}(u)\right|^2 - \left|e^{\mathbf{i}u'Y_t} - \tilde{\psi}_{j+1}^0(u)\right|^2\right] W(u)\mathrm{d}u \\
=& S_1(\eta) + S_2(\eta) + S_3(\eta).
\end{aligned}$$

Consider $S_1(\eta)$. By (A.3) and (A.4), and $\psi_{j+1}^0(u) - \psi_j^0(u) = v_T\Delta_j(u)$,

$$\begin{aligned}
S_1(\eta) =& \sum_{t=T_{j-1}^0+1}^{T_j^0} \int_{\mathbb{R}^d} \left[\left|\tilde{\psi}_j(u)\right|^2 - \left|\tilde{\psi}_j^0(u)\right|^2 - 2\mathrm{Re}\left\{e^{\mathbf{i}u'Y_t}\left[\tilde{\psi}_j(u) - \tilde{\psi}_j^0(u)\right]\right\}^*\right] W(u)\mathrm{d}u \\
=& (T_j^0 - T_{j-1}^0)\int_{\mathbb{R}^d}\left|\left[\tilde{\psi}_j(u) - \psi_j^0(u)\right] - \left[\tilde{\psi}_j^0(u) - \psi_j^0(u)\right]\right|^2 W(u)\mathrm{d}u \\
=& (T_j^0 - T_{j-1}^0)\int_{\mathbb{R}^d}\left|\frac{T_j - T_j^0}{T_j - T_{j-1}^0}v_T\Delta_j(u) + \frac{T_j^0 - T_j}{(T_j^0 - T_{j-1}^0)(T_j - T_{j-1}^0)}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u) + \frac{1}{T_j - T_{j-1}^0}\sum_{t=T_j^0+1}^{T_j}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\
\leq& 3(T_j^0 - T_{j-1}^0)\int_{\mathbb{R}^d}\left|\frac{T_j - T_j^0}{T_j - T_{j-1}^0}v_T\Delta_j(u)\right|^2 W(u)\mathrm{d}u \\
&+ 3(T_j^0 - T_{j-1}^0)\int_{\mathbb{R}^d}\left|\frac{T_j^0 - T_j}{(T_j^0 - T_{j-1}^0)(T_j - T_{j-1}^0)}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\
&+ 3(T_j^0 - T_{j-1}^0)\int_{\mathbb{R}^d}\left|\frac{1}{T_j - T_{j-1}^0}\sum_{t=T_j^0+1}^{T_j}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\
=& S_{11}(\eta) + S_{12}(\eta) + S_{13}(\eta), \text{ say.}
\end{aligned}$$

Given Assumption A.5(i) and $T_j = T_j^0 + \lfloor \eta v_T^{-2} \rfloor$

$$\sup_{\eta\in[0,C]} |S_{11}(\eta)| = \sup_{\eta\in[0,C]} \left| \frac{3(T_j^0 - T_{j-1}^0)\left(\lfloor \eta v_T^{-1} \rfloor\right)^2}{(T_j - T_{j-1}^0)^2} \right| \int_{\mathbb{R}^d} |\Delta_j(u)|^2 W(u)\mathrm{d}u = O(T^{-1}v_T^{-2}).$$

By Lemma A.1,

$$\sup_{\eta\in[0,C]} |S_{12}(\eta)| = \sup_{\eta\in[0,C]} \left| \frac{3(\lfloor \eta v_T^{-2} \rfloor)^2}{(r_j^0 - r_{j-1}^0)(T_j - T_{j-1}^0)^2} \right| \left[ \frac{1}{T} \int_{\mathbb{R}^d} \left| \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u \right] = O_P(T^{-2}v_T^{-4}).$$

By Lemma A.2,

$$\sup_{\eta\in[0,C]} |S_{13}(\eta)| = \sup_{\eta\in[0,C]} \left| \frac{3(T_j^0 - T_{j-1}^0)v_T^{-2}}{(T_j^0 + \lfloor \eta v_T^{-2} \rfloor - T_{j-1}^0)^2} \right| \left[ \int_{\mathbb{R}^d} \left| v_T \sum_{t=T_j^0+1}^{T_j^0+\lfloor \eta v_T^{-2} \rfloor} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u \right] = O_P(T^{-1}v_T^{-2}).$$

Hence, $\sup_{\eta\in[0,C]} |S_1(\eta)| = O_P(T^{-1}v_T^{-2}) = o_P(1)$ given $v_T = T^{-a}$ with $a \in (0, \frac{1}{2})$.

Next, we show $\sup_{\eta\in[0,C]} |S_3(\eta)| = o_P(1)$ can be established analogously. By (A.3) and (A.5),

$$\begin{aligned}
S_3(\eta) &= \sum_{t=T_j+1}^{T_{j+1}^0} \int_{\mathbb{R}^d} \left\{ \left|\tilde{\psi}_{j+1}(u)\right|^2 - \left|\tilde{\psi}_{j+1}^0(u)\right|^2 - 2\mathrm{Re}\left\{ e^{\mathbf{i}u'Y_t} \left[ \tilde{\psi}_{j+1}(u) - \tilde{\psi}_{j+1}^0(u) \right]^* \right\} \right\} W(u)\mathrm{d}u \\
&= -(T_{j+1}^0 - T_j) \int_{\mathbb{R}^d} \left| \left[ \tilde{\psi}_{j+1}(u) - \psi_{j+1}^0(u) \right] - \left[ \tilde{\psi}_{j+1}^0(u) - \psi_{j+1}^0(u) \right] \right|^2 W(u)\mathrm{d}u \\
&= -(T_{j+1}^0 - T_j) \int_{\mathbb{R}^d} \left| \frac{1}{T_{j+1}^0 - T_j} \sum_{t=T_j+1}^{T_{j+1}^0} \varepsilon_t(u) - \frac{1}{T_{j+1}^0 - T_j^0} \sum_{t=T_j^0+1}^{T_{j+1}^0} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u \\
&= -(T_{j+1}^0 - T_j) \int_{\mathbb{R}^d} \left| \frac{1}{T_{j+1}^0 - T_j} \left[ \sum_{t=T_j^0+1}^{T_{j+1}^0} \varepsilon_t(u) - \sum_{t=T_j^0+1}^{T_j} \varepsilon_t(u) \right] - \frac{1}{T_{j+1}^0 - T_j^0} \sum_{t=T_j^0+1}^{T_{j+1}^0} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u \\
&= -(T_{j+1}^0 - T_j) \int_{\mathbb{R}^d} \left| \frac{T_j - T_j^0}{(T_{j+1}^0 - T_j)(T_{j+1}^0 - T_j^0)} \sum_{t=T_j^0+1}^{T_{j+1}^0} \varepsilon_t(u) - \frac{1}{T_{j+1}^0 - T_j} \sum_{t=T_j^0+1}^{T_j} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u.
\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}
\sup_{\eta\in[0,C]} |S_3(\eta)| &\leq \sup_{\eta\in[0,C]} \left| \frac{2T^{-1}(\lfloor \eta v_T^{-2} \rfloor)^2}{(T_{j+1}^0 - T_j)(r_{j+1}^0 - r_j^0)^2} \right| \left[ \frac{1}{T} \int_{\mathbb{R}^d} \left| \sum_{t=T_j^0+1}^{T_{j+1}^0} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u \right] \\
&\quad + \sup_{\eta\in[0,C]} \left| \frac{2v_T^{-2}}{(T_{j+1}^0 - T_j)} \right| \int_{\mathbb{R}^d} \left| v_T \sum_{t=T_j^0+1}^{T_j^0+\lfloor \eta v_T^{-2} \rfloor} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u \\
&= O_P(T^{-2}v_T^{-4}) + O_P(T^{-1}v_T^{-2}).
\end{aligned}$$

Now, we consider $S_2(\eta)$. Note that for $T_j^0+1 \le t \le T_j$, $e^{\mathbf{i}u'Y_t} = \psi_{j+1}^0(u) + \varepsilon_t(u)$. Then

$$\begin{aligned}
S_2(\eta) &= \sum_{t=T_j^0+1}^{T_j} \int_{\mathbb{R}^d} \left[\left|e^{\mathbf{i}u'Y_t} - \tilde{\psi}_j(u)\right|^2 - \left|e^{\mathbf{i}u'Y_t} - \tilde{\psi}_{j+1}^0(u)\right|^2\right] W(u)\mathrm{d}u \\
&= \sum_{t=T_j^0+1}^{T_j} \int_{\mathbb{R}^d} \left[\left|\tilde{\psi}_j(u)\right|^2 - \left|\tilde{\psi}_{j+1}^0(u)\right|^2 - 2\mathrm{Re}\left\{e^{\mathbf{i}u'Y_t}\left[\tilde{\psi}_j(u) - \tilde{\psi}_{j+1}^0(u)\right]^*\right\}\right] W(u)\mathrm{d}u \\
&= \sum_{t=T_j^0+1}^{T_j} \int_{\mathbb{R}^d} \left[\left|\tilde{\psi}_j(u)\right|^2 - \left|\tilde{\psi}_{j+1}^0(u)\right|^2 - 2\mathrm{Re}\left\{\left[\psi_{j+1}^0(u) + \varepsilon_t(u)\right]\left[\tilde{\psi}_j(u) - \tilde{\psi}_{j+1}^0(u)\right]^*\right\}\right] W(u)\mathrm{d}u \\
&= \sum_{t=T_j^0+1}^{T_j} 2\int_{\mathbb{R}^d} \mathrm{Re}\left\{\varepsilon_t(u)\left[\tilde{\psi}_{j+1}^0(u) - \tilde{\psi}_j(u)\right]^*\right\} W(u)\mathrm{d}u \\
&\quad + \lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d} \left[\left|\tilde{\psi}_j(u) - \psi_{j+1}^0(u)\right|^2 - \left|\tilde{\psi}_{j+1}^0(u) - \psi_{j+1}^0(u)\right|^2\right] W(u)\mathrm{d}u \\
&= S_{21}(\eta) + S_{22}(\eta).
\end{aligned}$$

By (A.3), (A.4), and the fact that $\psi_{j+1}^0(u) - \psi_j^0(u) = v_T\Delta_j(u)$,

$$\begin{aligned}
S_{21}(\eta) &= \sum_{t=T_j^0+1}^{T_j} 2\int_{\mathbb{R}^d} \mathrm{Re}\left\{\varepsilon_t(u)\left[\tilde{\psi}_{j+1}^0(u) - \tilde{\psi}_j(u)\right]^*\right\} W(u)\mathrm{d}u \\
&= \sum_{t=T_j^0+1}^{T_j} 2\int_{\mathbb{R}^d} \mathrm{Re}\left\{\varepsilon_t(u)\left[\left\{\tilde{\psi}_{j+1}^0(u) - \psi_{j+1}^0(u)\right\} - \left\{\tilde{\psi}_j(u) - \psi_j^0(u)\right\} + \left\{\psi_{j+1}^0(u) - \psi_j^0(u)\right\}\right]^*\right\} W(u)\mathrm{d}u \\
&= 2\int_{\mathbb{R}^d} \mathrm{Re}\left\{\left[\sum_{t=T_j^0+1}^{T_j} \varepsilon_t(u)\right]\left[\frac{1}{T_{j+1}^0 - T_j^0} \sum_{t=T_j^0+1}^{T_{j+1}^0} \varepsilon_t(u)\right]^*\right\} W(u)\mathrm{d}u \\
&\quad + 2\int_{\mathbb{R}^d} \mathrm{Re}\left\{\left[\sum_{t=T_j^0+1}^{T_j} \varepsilon_t(u)\right]\left[\frac{T_j^0 - T_{j-1}^0}{T_j - T_{j-1}^0} v_T\Delta_j(u)\right]^*\right\} W(u)\mathrm{d}u \\
&\quad - 2\int_{\mathbb{R}^d} \mathrm{Re}\left\{\left[\sum_{t=T_j^0+1}^{T_j} \varepsilon_t(u)\right]\left(\frac{1}{T_j - T_{j-1}^0}\left[\sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) + \sum_{t=T_j^0+1}^{T_j} \varepsilon_t(u)\right]\right)^*\right\} W(u)\mathrm{d}u \\
&= S_{211}(\eta) + S_{212}(\eta) - S_{213}(\eta), \text{ say.}
\end{aligned}$$

By the Cauchy-Schwarz inequality, Lemmas A.1 and A.2,

$$\begin{aligned}
\sup_{\eta\in[0,C]} |S_{211}(\eta)| &\le \frac{2v_T^{-1}T^{-1/2}}{r_{j+1}^0 - r_j^0} \sup_{\eta\in[0,C]} \left(\int_{\mathbb{R}^d} \left|v_T \sum_{t=T_j^0+1}^{T_j^0+\lfloor\eta v_T^{-2}\rfloor} \varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right)^{1/2} \left(\frac{1}{T}\int_{\mathbb{R}^d} \left|\sum_{t=T_j^0+1}^{T_{j+1}^0} \varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right)^{1/2} \\
&= O_P(T^{-1/2}v_T^{-1}),
\end{aligned}$$

and

$$
\begin{aligned}
\sup_{\eta\in[0,C]}|S_{213}(\eta)| &\leq 2\sup_{\eta\in[0,C]}\left|\int_{\mathbb{R}^d}\operatorname{Re}\left\{\left[\sum_{t=T_j^0+1}^{T_j}\varepsilon_t(u)\right]\left[\frac{1}{T_j-T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u)\right]^*\right\}W(u)\mathrm{d}u\right| \\
&\quad +2\sup_{\eta\in[0,C]}\left(\frac{1}{T_j-T_{j-1}^0}\int_{\mathbb{R}^d}\left|\sum_{t=T_j^0+1}^{T_j}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right) \\
&\leq \sup_{\eta\in[0,C]}\frac{2v_T^{-1}T^{1/2}}{T_j-T_{j-1}^0}\left(\int_{\mathbb{R}^d}\left|v_T\sum_{t=T_j^0+1}^{T_j^0+\lfloor\eta v_T^{-2}\rfloor}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right)^{1/2}\left(\frac{1}{T}\int_{\mathbb{R}^d}\left|\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right)^{1/2} \\
&\quad +\sup_{\eta\in[0,C]}\frac{2v_T^{-2}}{T_j-T_{j-1}^0}\int_{\mathbb{R}^d}\left|v_T\sum_{t=T_j^0+1}^{T_j^0+\lfloor\eta v_T^{-2}\rfloor}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\
&= O_P(T^{-1/2}v_T^{-1})+O_P(T^{-1}v_T^{-2}).
\end{aligned}
$$

For $S_{212}(\eta)$, consider the limiting behavior of $2\int_{\mathbb{R}^d}\operatorname{Re}\left\{\left[v_T\sum_{t=T_j^0+1}^{T_j^0+\lfloor\eta v_T^{-2}\rfloor}\varepsilon_t(u)\right][\Delta_j(u)]^*\right\}W(u)\mathrm{d}u$. By Assumption A.5(ii) and the continuous mapping theorem,

$$
2\int_{\mathbb{U}}\operatorname{Re}\left\{\left[v_T\sum_{t=T_j^0+1}^{T_j^0+\lfloor\eta v_T^{-2}\rfloor}\varepsilon_t(u)\right][\Delta_j(u)]^*\right\}W(u)\mathrm{d}u \Rightarrow 2\int_{\mathbb{U}}\operatorname{Re}\left\{\mathcal{G}_1^{(j)}(u,\eta)\Delta_j(u)^*\right\}W(u)\mathrm{d}u,
$$

for any compact subset $\mathbb{U}$ of $\mathbb{R}^d$. It remains to show that $2\int_{\mathbb{U}^c}\operatorname{Re}\left\{\left[v_T\sum_{t=T_j^0+1}^{T_j^0+\lfloor\eta v_T^{-2}\rfloor}\varepsilon_t(u)\right][\Delta_j(u)]^*\right\}W(u)\mathrm{d}u$ is asymptotically negligible. Note that for each fixed $\eta$, and constant $\iota>0$, there exists $\mathbb{U}^c$ small enough such that

$$
\begin{aligned}
E\left[2\int_{\mathbb{U}^c}\operatorname{Re}\left\{\left[v_T\sum_{t=T_j^0+1}^{T_j^0+\lfloor\eta v_T^{-2}\rfloor}\varepsilon_t(u)\right][\Delta_j(u)]^*\right\}W(u)\mathrm{d}u\right]^2 &\leq 4E\int_{\mathbb{U}^c}\left|v_T\sum_{t=T_j^0+1}^{T_j^0+\lfloor\eta v_T^{-2}\rfloor}\varepsilon_t(u)\right|^2|\Delta_j(u)|^2 W(u)\mathrm{d}u \\
&\leq \mathcal{C}\int_{\mathbb{U}^c}W(u)\mathrm{d}u<\iota,
\end{aligned}
$$

where the second to last inequality holds due to Assumption A.5(i) and the mixing condition in Assumption A.1. For tightness, let $0<\eta_1<\eta_2<C$, and some $q>2$,

$$
\begin{aligned}
&E\left|2\int_{\mathbb{U}^c}\operatorname{Re}\left\{\left[v_T\sum_{t=T_j^0+1}^{T_j^0+\lfloor\eta_2 v_T^{-2}\rfloor}\varepsilon_t(u)\right][\Delta_j(u)]^*\right\}W(u)\mathrm{d}u-2\int_{\mathbb{U}^c}\operatorname{Re}\left\{\left[v_T\sum_{t=T_j^0+1}^{T_j^0+\lfloor\eta_1 v_T^{-2}\rfloor}\varepsilon_t(u)\right][\Delta_j(u)]^*\right\}W(u)\mathrm{d}u\right|^q \\
\leq\ & 2^qE\int_{\mathbb{U}^c}\left|v_T\sum_{t=T_j^0+\lfloor\eta_1 v_T^{-2}\rfloor}^{T_j^0+\lfloor\eta_2 v_T^{-2}\rfloor}\varepsilon_t(u)\right|^q|\Delta_j(u)|^q W(u)\mathrm{d}u
\end{aligned}
$$

$$\leq \quad 2^q v_T^q \mathcal{C} \left[(\eta_2-\eta_1)v_T^{-2}\right]^{q/2} \sup_u |\Delta_j(u)|^q \int_{\mathbb{U}^c} W(u)\mathrm{d}u < \mathcal{C}(\eta_2-\eta_1)^q,$$

by Theorem 2 of Yokoyama (1980) given $E[\varepsilon_t(u)] = 0$ and $\sup_u |\varepsilon_t(u)| < \mathcal{C}$. By Theorem 15.6 of Billingsley (1968),

$$2\int_{\mathbb{R}^d} \mathrm{Re}\left\{\left[v_T \sum_{t=T_j^0+1}^{T_j^0+\lfloor \eta v_T^{-2}\rfloor} \varepsilon_t(u)\right][\Delta_j(u)]^*\right\} W(u)\mathrm{d}u \Rightarrow 2\int_{\mathbb{R}^d} \mathrm{Re}\left\{\mathcal{G}_1^{(j)}(u,\eta)\Delta_j(u)^*\right\} W(u)\mathrm{d}u.$$

Then it follows

$$\begin{aligned} S_{212}(\eta) &= 2\int_{\mathbb{R}^d} \mathrm{Re}\left\{\left[v_T \sum_{t=T_j^0+1}^{T_j^0+\lfloor \eta v_T^{-2}\rfloor} \varepsilon_t(u)\right]\left[\frac{r_j^0 - r_{j-1}^0}{r_j^0 - r_{j-1}^0 + T^{-1}\lfloor \eta v_T^{-2}\rfloor}\Delta_j(u)\right]^*\right\} W(u)\mathrm{d}u \\ &\Rightarrow 2\int_{\mathbb{R}^d} \mathrm{Re}\left\{\mathcal{G}_1^{(j)}(u,\eta)\Delta_j(u)^*\right\} W(u)\mathrm{d}u. \end{aligned}$$

Now, we consider $S_{22}(\eta)$. By (A.3) and (A.4)

$$\begin{aligned} S_{22}(\eta) &= \lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d}\left[\left|\tilde{\psi}_j(u) - \psi_{j+1}^0(u)\right|^2 - \left|\tilde{\psi}_{j+1}^0(u) - \psi_{j+1}^0(u)\right|^2\right] W(u)\mathrm{d}u \\ &= \lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d}\left[\left|\left[\tilde{\psi}_j(u) - \psi_j^0(u)\right] - \left[\psi_{j+1}^0(u) - \psi_j^0(u)\right]\right|^2 - \left|\frac{1}{T_{j+1}^0 - T_j^0}\sum_{t=T_j^0+1}^{T_{j+1}^0}\varepsilon_t(u)\right|^2\right] W(u)\mathrm{d}u \\ &= \lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d}\left|\frac{T_{j-1}^0 - T_j^0}{T_j - T_{j-1}^0} v_T\Delta_j(u) + \frac{1}{T_j - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u) + \frac{1}{T_j - T_{j-1}^0}\sum_{t=T_j^0+1}^{T_j}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\ &\quad -\lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d}\left|\frac{1}{T_{j+1}^0 - T_j^0}\sum_{t=T_j^0+1}^{T_{j+1}^0}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\ &= \lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d}\left|\frac{T_{j-1}^0 - T_j^0}{T_j - T_{j-1}^0} v_T\Delta_j(u)\right|^2 W(u)\mathrm{d}u + \lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d}\left|\frac{1}{T_j - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\ &\quad +\lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d}\left|\frac{1}{T_j - T_{j-1}^0}\sum_{t=T_j^0+1}^{T_j}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u - \lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d}\left|\frac{1}{T_{j+1}^0 - T_j^0}\sum_{t=T_j^0+1}^{T_{j+1}^0}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\ &\quad +2\lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d}\mathrm{Re}\left\{\left[\frac{T_{j-1}^0 - T_j^0}{T_j - T_{j-1}^0} v_T\Delta_j(u)\right]\left[\frac{1}{T_j - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u)\right]^*\right\} W(u)\mathrm{d}u \\ &\quad +2\lfloor \eta v_T^{-2}\rfloor \int_{\mathbb{R}^d}\mathrm{Re}\left\{\left[\frac{T_{j-1}^0 - T_j^0}{T_j - T_{j-1}^0} v_T\Delta_j(u)\right]\left[\frac{1}{T_j - T_{j-1}^0}\sum_{t=T_j^0+1}^{T_j}\varepsilon_t(u)\right]^*\right\} W(u)\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
& +2\lfloor \eta v_T^{-2} \rfloor \int_{\mathbb{R}^d} \mathrm{Re} \left\{ \left[ \frac{1}{T_j - T_{j-1}^0} \sum_{t=T_{j-1}^0}^{T_j^0} \varepsilon_t(u) \right] \left[ \frac{1}{T_j - T_{j-1}^0} \sum_{t=T_j^0+1}^{T_j} \varepsilon_t(u) \right]^* \right\} W(u) \mathrm{d}u \\
= \quad & S_{221}(\eta) + S_{222}(\eta) + S_{223}(\eta) - S_{224}(\eta) + S_{225}(\eta) + S_{226}(\eta) + S_{227}(\eta).
\end{aligned}
$$

It is straightforward to see that

$$
\begin{aligned}
S_{221}(\eta) \quad &= \quad \frac{(T_j^0 - T_{j-1}^0)^2}{(T_j - T_{j-1}^0)^2} \lfloor \eta \rfloor \int_{\mathbb{R}^d} |\Delta_j(u)|^2 W(u) \mathrm{d}u \\
&\Rightarrow \quad \eta \int_{\mathbb{R}^d} |\Delta_j(u)|^2 W(u) \mathrm{d}u.
\end{aligned}
$$

By Lemma A.1,

$$
\begin{aligned}
\sup_{\eta \in [0,C]} |S_{222}(\eta)| \quad &= \quad \sup_{\eta \in [0,C]} \frac{\lfloor \eta v_T^{-2} \rfloor (T_j^0 - T_{j-1}^0)}{(T_j - T_{j-1}^0)^2} \int_{\mathbb{R}^d} \left| \frac{1}{\sqrt{T_j^0 - T_{j-1}^0}} \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) \right|^2 W(u) \mathrm{d}u \\
&= \quad O_P(T^{-1} v_T^{-2}),
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{\eta \in [0,C]} |S_{224}(\eta)| \quad &= \quad \sup_{\eta \in [0,C]} \frac{\lfloor \eta v_T^{-2} \rfloor}{T_{j+1}^0 - T_j^0} \int_{\mathbb{R}^d} \left| \frac{1}{\sqrt{T_{j+1}^0 - T_j^0}} \sum_{t=T_{j1}^0+1}^{T_{j+1}^0} \varepsilon_t(u) \right|^2 W(u) \mathrm{d}u \\
&= \quad O_P(T^{-1} v_T^{-2}).
\end{aligned}
$$

By Lemma A.2,

$$
\begin{aligned}
\sup_{\eta \in [0,C]} |S_{223}(\eta)| \quad &= \quad \sup_{\eta \in [0,C]} \frac{v_T^{-2} \lfloor \eta v_T^{-2} \rfloor}{(T_j - T_{j-1}^0)^2} \int_{\mathbb{R}^d} \left| v_T \sum_{t=T_j^0+1}^{T_j^0 + \lfloor \eta v_T^{-2} \rfloor} \varepsilon_t(u) \right|^2 W(u) \mathrm{d}u \\
&= \quad O_P(T^{-2} v_T^{-4}).
\end{aligned}
$$

Furthermore, by the Cauchy-Schwarz inequality, Lemmas A.1, and A.2,

$$
|S_{225}(\eta)| \quad \leq \quad 2\lfloor \eta v_T^{-2} \rfloor \sqrt{\int_{\mathbb{R}^d} \left| \frac{T_{j-1}^0 - T_j^0}{T_j - T_{j-1}^0} v_T \Delta_j(u) \right|^2 W(u) \mathrm{d}u} \sqrt{\int_{\mathbb{R}^d} \left| \frac{1}{T_j - T_{j-1}^0} \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) \right|^2 W(u) \mathrm{d}u},
$$

$$
|S_{226}(\eta)| \quad \leq \quad 2\lfloor \eta v_T^{-2} \rfloor \sqrt{\int_{\mathbb{R}^d} \left| \frac{T_{j-1}^0 - T_j^0}{T_j - T_{j-1}^0} v_T \Delta_j(u) \right|^2 W(u) \mathrm{d}u} \sqrt{\int_{\mathbb{R}^d} \left| \frac{1}{T_j - T_{j-1}^0} \sum_{t=T_j^0+1}^{T_j} \varepsilon_t(u) \right|^2 W(u) \mathrm{d}u},
$$

and

$$|S_{227}(\eta)| \quad \leq \quad 2\lfloor \eta v_T^{-2} \rfloor \sqrt{\int_{\mathbb{R}^d} \left| \frac{1}{T_j - T_{j-1}^0} \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u} \sqrt{\int_{\mathbb{R}^d} \left| \frac{1}{T_j - T_{j-1}^0} \sum_{t=T_j^0+1}^{T_j} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u}.$$

Hence, $\sup_{\eta\in[0,C]} |S_{225}(\eta)| = O_P(T^{-1/2}v_T^{-1})$, $\sup_{\eta\in[0,C]} |S_{226}(\eta)| = O_P(T^{-1}v_T^{-2})$, and $\sup_{\eta\in[0,C]} |S_{227}(\eta)| = O_P(T^{-3/2}v_T^{-3})$. Therefore, we have

$$S_{22}(\eta) \quad \Rightarrow \quad \eta \int_{\mathbb{R}^d} |\Delta_j(u)|^2 W(u)\mathrm{d}u.$$

Combining $S_{21}(\eta)$ and $S_{22}(\eta)$, we have

$$S_2(\eta) \Rightarrow 2 \int_{\mathbb{R}^d} \mathrm{Re}\left\{ \mathcal{G}_1^{(j)}(u,\eta)\Delta_j(u)^* \right\} W(u)\mathrm{d}u + \eta \int_{\mathbb{R}^d} |\Delta_j(u)|^2 W(u)\mathrm{d}u.$$

As a result, for $0 < \eta < \mathcal{C}$, we have shown

$$\begin{aligned} &\mathrm{SSGR}(T_j^0 + \lfloor \eta v_T^{-2} \rfloor) - \mathrm{SSGR}(T_j^0) \\ &\Rightarrow 2 \int_{\mathbb{R}^d} \mathrm{Re}\left\{ \mathcal{G}_1^{(j)}(u,\eta)\Delta_j(u)^* \right\} W(u)\mathrm{d}u + \eta \int_{\mathbb{R}^d} |\Delta_j(u)|^2 W(u)\mathrm{d}u. \end{aligned}$$

Following analogous steps, we can show that

$$\begin{aligned} &\mathrm{SSGR}(T_j^0 + \lfloor \eta v_T^{-2} \rfloor) - \mathrm{SSGR}(T_j^0) \\ &\Rightarrow -2 \int_{\mathbb{R}^d} \mathrm{Re}\left\{ \mathcal{G}_2^{(j)}(u,-\eta)\Delta_j(u)^* \right\} W(u)\mathrm{d}u - \eta \int_{\mathbb{R}^d} |\Delta_j(u)|^2 W(u)\mathrm{d}u, \end{aligned}$$

when $\eta < 0$. Combing the obtained results, we have

$$\mathrm{SSGR}(T_j^0) - \mathrm{SSGR}(T_j^0 + \lfloor \eta v_T^{-2} \rfloor) \Rightarrow \Lambda^{(j)}(\eta),$$

where

$$\Lambda^{(j)}(\eta) = \begin{cases} 2\int_{\mathbb{R}^d} \mathrm{Re}\left\{ \mathcal{G}^{(j)}(u,\eta)\Delta_j(u)^* \right\} W(u)\mathrm{d}u - |\eta| \int_{\mathbb{R}^d} |\Delta_j(u)|^2 W(u)\mathrm{d}u, & \text{if } \eta < 0; \\ -2\int_{\mathbb{R}^d} \mathrm{Re}\left\{ \mathcal{G}^{(j)}(u,\eta)\Delta_j(u)^* \right\} W(u)\mathrm{d}u - |\eta| \int_{\mathbb{R}^d} |\Delta_j(u)|^2 W(u)\mathrm{d}u, & \text{if } \eta > 0; \end{cases}$$

By the continuous mapping theorem for an argmax function (see Kim and Pollard, 1990), we have

$$v_T^2(\hat{T}_j - T_j^0) \xrightarrow{d} \arg\max_{\eta} \Lambda^{(j)}(\eta).$$

■

**Proof of Theorem 3.1** Under $\mathbb{H}_0 : \phi_t(u) = \phi^0(u)$, it follows

$$e^{\mathbf{i}u'Y_t} = \phi^0(u) + \varepsilon_t(u),$$

for all $t = 1, ..., T$. We let

$$\tilde{\psi}^{(R)}(u) = \frac{1}{T}\sum_{t=1}^{T} e^{\mathbf{i}u'Y_t} = \phi^0(u) + \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t(u)$$

denote the ECF under the restricted model and let

$$\tilde{\psi}_j^{(U)}(u) = \frac{1}{T_j - T_{j-1}}\sum_{t=T_{j-1}+1}^{T_j} e^{\mathbf{i}u'Y_t} = \phi^0(u) + \frac{1}{T_j - T_{j-1}}\sum_{t=T_{j-1}+1}^{T_j}\varepsilon_t(u)$$

denote the ECF of $j$-th regime under the unrestricted model, where $\{T_j\}_{j=1}^{M}$ is the collection of break dates under $\mathbb{H}_A$. By definition,

$$\begin{aligned}
F_T(r_1, ..., r_M) &= \mathrm{SSGR}_0 - \mathrm{SSGR}_M(r_1, ..., r_M) \\
&= \sum_{j=1}^{M+1}\sum_{t=T_{j-1}+1}^{T_j}\int_{\mathbb{R}^d}\left|e^{\mathbf{i}u'Y_t} - \tilde{\psi}^{(R)}(u)\right|^2 - \left|e^{\mathbf{i}u'Y_t} - \tilde{\psi}_j^{(U)}(u)\right|^2 W(u)\mathrm{d}u \\
&= \sum_{j=1}^{M+1}\sum_{t=T_{j-1}+1}^{T_j}\int_{\mathbb{R}^d}\left[\left|\tilde{\psi}^{(R)}(u)\right|^2 - \left|\tilde{\psi}_j^{(U)}(u)\right|^2 - 2\mathrm{Re}\left\{e^{\mathbf{i}u'Y_t}\tilde{\psi}^{(R)}(u)^*\right\} + 2\mathrm{Re}\left\{e^{\mathbf{i}u'Y_t}\tilde{\psi}_j^{(U)}(u)^*\right\}\right] W(u)\mathrm{d}u \\
&= \sum_{j=1}^{M+1}(T_j - T_{j-1})\int_{\mathbb{R}^d}\left|\tilde{\psi}^{(R)}(u)\right|^2 + \left|\tilde{\psi}_j^{(U)}(u)\right|^2 - 2\mathrm{Re}\left[\tilde{\psi}_j^{(U)}(u)\tilde{\psi}^{(R)}(u)\right]^* W(u)\mathrm{d}u \\
&= \sum_{j=1}^{M+1}(T_j - T_{j-1})\int_{\mathbb{R}^d}\left|\tilde{\psi}_j^{(U)}(u) - \tilde{\psi}^{(R)}(u)\right|^2 W(u)\mathrm{d}u.
\end{aligned}$$

Under $\mathbb{H}_0$, we have

$$\begin{aligned}
F_T(r_1, ..., r_M) &= \sum_{j=1}^{M+1}(T_j - T_{j-1})\int_{\mathbb{R}^d}\left|\frac{1}{T_j - T_{j-1}}\sum_{t=T_{j-1}+1}^{T_j}\varepsilon_t(u) - \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\
&= \sum_{j=1}^{M+1}(r_j - r_{j-1})\int_{\mathbb{R}^d}\left|\frac{1}{r_j - r_{j-1}}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}\varepsilon_t(u) - \frac{1}{\sqrt{T}}\sum_{t=1}^{T_{j-1}}\varepsilon_t(u)\right] - \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\
&= \sum_{j=1}^{M+1}\frac{1}{r_j - r_{j-1}}\int_{\mathbb{R}^d}\left|\left[\mathcal{L}_T(u, r_j) - \mathcal{L}_T(u, r_{j-1})\right] - (r_j - r_{j-1})\mathcal{L}_T(u, 1)\right|^2 W(u)\mathrm{d}u,
\end{aligned}$$

where we let $\mathcal{L}_T(u, r_j) \equiv T^{-1/2}\sum_{t=1}^{T_j}\varepsilon_t(u)$. Define the following

$$\begin{aligned}
\mathcal{A}_{1,T}(r_1, ..., r_M) &= \sum_{j=1}^{M+1}\frac{1}{r_j - r_{j-1}}\int_{\mathbb{U}}\left|\left[\mathcal{L}_T(u, r_j) - \mathcal{L}_T(u, r_{j-1})\right] - (r_j - r_{j-1})\mathcal{L}_T(u, 1)\right|^2 W(u)\mathrm{d}u, \\
\mathcal{A}_{2,T}(r_1, ..., r_M) &= \sum_{j=1}^{M+1}\frac{1}{r_j - r_{j-1}}\int_{\mathbb{U}^c}\left|\left[\mathcal{L}_T(u, r_j) - \mathcal{L}_T(u, r_{j-1})\right] - (r_j - r_{j-1})\mathcal{L}_T(u, 1)\right|^2 W(u)\mathrm{d}u,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{A}_1(r_1,...,r_M) &= \sum_{j=1}^{M+1}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{U}}\left|[B(u,r_j)-B(u,r_{j-1})]-(r_j-r_{j-1})B(u,1)\right|^2 W(u)\mathrm{d}u,\\
\mathcal{A}_2(r_1,...,r_M) &= \sum_{j=1}^{M+1}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{U}^c}\left|[B(u,r_j)-B(u,r_{j-1})]-(r_j-r_{j-1})B(u,1)\right|^2 W(u)\mathrm{d}u,
\end{aligned}
$$

where $\mathbb{U}$ is any compact subset of $\mathbb{R}^d$ and $\mathbb{U}^c$ is its complement set. Obviously, $F_T(r_1,...,r_M)=\mathcal{A}_{1,T}(r_1,...,r_M)+\mathcal{A}_{2,T}(r_1,...,r_M)$. Furthermore, we denote $F(r_1,...,r_M)\equiv\mathcal{A}_1(r_1,...,r_M)+\mathcal{A}_2(r_1,...,r_M)$. Now, we show that $\sup_{\{r_1,...,r_M\}}F_T(r_1,...,r_M)\xrightarrow{d}\sup_{\{r_1,...,r_M\}}F(r_1,...,r_M)$.

Note that for any fixed constant $\iota>0$, and fixed collection of breaks $\{r_1,...,r_M\}\in\Pi_\epsilon$, there exists a compact subset $\mathbb{U}$ that depend on $\iota$, such that

$$
\begin{aligned}
E\left[\mathcal{A}_{2,T}(r_1,...,r_M)\right] &= \sum_{j=1}^{M}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{U}^c}E\left|[\mathcal{L}_T(u,r_j)-\mathcal{L}_T(u,r_{j-1})]-(r_j-r_{j-1})\mathcal{L}_T(u,1)\right|^2 W(u)\mathrm{d}u\\
&\le \sum_{j=1}^{M}\frac{1}{(r_j-r_{j-1})}\int_{\mathbb{U}^c}2E\left|\mathcal{L}_T(u,r_j)-\mathcal{L}_T(u,r_{j-1})\right|^2+2E\left|(r_j-r_{j-1})\mathcal{L}_T(u,1)\right|^2 W(u)\mathrm{d}u\\
&= 2\sum_{j=1}^{M}\left[\int_{\mathbb{U}^c}E\left|\frac{1}{\sqrt{T(r_j-r_{j-1})}}\sum_{t=1}^{\lfloor T(r_j-r_{j-1})\rfloor}\varepsilon_t(u)\right|^2+(r_j-r_{j-1})E\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_t(u)\right|^2\right]W(u)\mathrm{d}u\\
&\le 2M\mathcal{C}\int_{\mathbb{U}^c}W(u)\mathrm{d}u<\iota,
\end{aligned}
$$

where the last equality is by the stationarity condition in Assumption A.4*, and the second to last inequality is by the mixing condition in Assumption A.1, the fact that $|r_j-r_{j-1}|<1$ for all $j$, and the boundedness of $\varepsilon_t(u)$. An analogous result holds for $E\left[\mathcal{A}_2(r_1,...,r_M)\right]$. Under Assumption A.4*, $T^{-1/2}\mathcal{L}(u,r_j)\Rightarrow B(u,r_j)$ on $\mathbb{U}\times[0,1]$ for any compact subset $\mathbb{U}$ of $\mathbb{R}^d$. Thus, by the continuous mapping theorem, $\mathcal{A}_{1,T}(r_1,...,r_M)\Rightarrow\mathcal{A}_1(r_1,...,r_M)$.

Hence, for each fixed $\{r_1,...,r_M\}$,

$$
F_T(r_1,...,r_M)\xrightarrow{d}F(r_1,...,r_M).
$$

Now, we show tightness. Without loss generality, we consider $M=1$. Let $0<r_1<s_1<1$, and $q>2$, it follows

$$
\begin{aligned}
& E\left|F_T(s_1)-F_T(r_1)\right|^q\\
=\;& E\left|\frac{1}{s_1(1-s_1)}\int_{\mathbb{R}^d}\left|\mathcal{L}_T(u,s_1)-s_1\mathcal{L}_T(u,1)\right|^2W(u)\mathrm{d}u-\frac{1}{r_1(1-r_1)}\int_{\mathbb{R}^d}\left|\mathcal{L}_T(u,r_1)-r_1\mathcal{L}_T(u,1)\right|^2W(u)\mathrm{d}u\right|^q\\
\le\;& \frac{2^{q-1}}{s_1^q(1-s_1)^q}E\left|\int_{\mathbb{R}^d}\left\{\left|\mathcal{L}_T(u,s_1)-s_1\mathcal{L}_T(u,1)\right|^2-\left|\mathcal{L}_T(u,r_1)-r_1\mathcal{L}_T(u,1)\right|^2\right\}W(u)\mathrm{d}u\right|^q\\
& +2^{q-1}\left[\frac{(s_1-r_1)(s_1+r_1-1)}{s_1r_1(1-s_1)(1-r_1)}\right]^qE\left|\int_{\mathbb{R}^d}\left|\mathcal{L}_T(u,r_1)-r_1\mathcal{L}_T(u,1)\right|^2W(u)\mathrm{d}u\right|^q\\
=\;& \mathcal{I}_1(s_1,r_1)+\mathcal{I}_2(s_1,r_1),\ \text{say.}
\end{aligned}
$$

Consider $\mathcal{I}_1(s_1, r_1)$. By the Cauchy-Schwarz inequality,

$$
\begin{aligned}
& \left|\int_{\mathbb{R}^d} \left\{|\mathcal{L}_T(u,s_1) - s_1\mathcal{L}_T(u,1)|^2 - |\mathcal{L}_T(u,r_1) - r_1\mathcal{L}_T(u,1)|^2\right\} W(u)\mathrm{d}u\right| \\
= \; & \left|\int_{\mathbb{R}^d} \mathrm{Re}\left\{[\mathcal{L}_T(u,s_1) - \mathcal{L}_T(u,r_1) - (s_1 - r_1)\mathcal{L}_T(u,1)]\,[\mathcal{L}_T(u,s_1) + \mathcal{L}_T(u,r_1) - (s_1 + r_1)\mathcal{L}_T(u,1)]^*\right\} W(u)\mathrm{d}u\right| \\
\leq \; & \left[\int_{\mathbb{R}^d} |\mathcal{L}_T(u,s_1) - \mathcal{L}_T(u,r_1) - (s_1 - r_1)\mathcal{L}_T(u,1)|^2 W(u)\mathrm{d}u\right]^{1/2} \\
& \times \left[\int_{\mathbb{R}^d} |\mathcal{L}_T(u,s_1) + \mathcal{L}_T(u,r_1) - (s_1 + r_1)\mathcal{L}_T(u,1)|^2 W(u)\mathrm{d}u\right]^{1/2}.
\end{aligned}
$$

Then, it follows

$$
\begin{aligned}
\mathcal{I}_1(s_1, r_1) \leq \; & \frac{2^{q-1}}{s_1^q(1-s_1)^q} E\left\{\left[\int_{\mathbb{R}^d} |\mathcal{L}_T(u,s_1) - \mathcal{L}_T(u,r_1) - (s_1 - r_1)\mathcal{L}_T(u,1)|^2 W(u)\mathrm{d}u\right]^{q/2}\right. \\
& \left. \times \left[\int_{\mathbb{R}^d} |\mathcal{L}_T(u,s_1) + \mathcal{L}_T(u,r_1) - (s_1 + r_1)\mathcal{L}_T(u,1)|^2 W(u)\mathrm{d}u\right]^{q/2}\right\} \\
\leq \; & \frac{2^{q-1}}{s_1^q(1-s_1)^q} \left(E\left[\int_{\mathbb{R}^d} |\mathcal{L}_T(u,s_1) - \mathcal{L}_T(u,r_1) - (s_1 - r_1)\mathcal{L}_T(u,1)|^2 W(u)\mathrm{d}u\right]^q\right)^{1/2} \\
& \times \left(E\left[\int_{\mathbb{R}^d} |\mathcal{L}_T(u,s_1) + \mathcal{L}_T(u,r_1) - (s_1 + r_1)\mathcal{L}_T(u,1)|^2 W(u)\mathrm{d}u\right]^q\right)^{1/2} \\
\leq \; & \frac{2^{q-1}}{s_1^q(1-s_1)^q} \left(E\int_{\mathbb{R}^d} |\mathcal{L}_T(u,s_1) - \mathcal{L}_T(u,r_1) - (s_1 - r_1)\mathcal{L}_T(u,1)|^{2q} W(u)\mathrm{d}u\right)^{1/2} \\
& \times \left(E\int_{\mathbb{R}^d} |\mathcal{L}_T(u,s_1) + \mathcal{L}_T(u,r_1) - (s_1 + r_1)\mathcal{L}_T(u,1)|^{2q} W(u)\mathrm{d}u\right)^{1/2} \\
= \; & \frac{2^{q-1}}{s_1^q(1-s_1)^q} \mathcal{I}_{11}(s_1, r_1)\mathcal{I}_{12}(s_1, r_1),
\end{aligned}
$$

where the second inequality is by the Cauchy-Schwarz inequality, and the last inequality is by Jensen's inequality. By stationarity,

$$
\begin{aligned}
& E\int_{\mathbb{R}^d} |\mathcal{L}_T(u,s_1) - \mathcal{L}_T(u,r_1) - (s_1 - r_1)\mathcal{L}_T(u,1)|^{2q} W(u)\mathrm{d}u \\
= \; & T^{-q}\int_{\mathbb{R}^d} E\left|\sum_{t=1}^{\lfloor T(s_1-r_1)\rfloor} \varepsilon_t(u) - (s_1 - r_1)\sum_{t=1}^{T}\varepsilon_t(u)\right|^{2q} W(u)\mathrm{d}u \\
\leq \; & 2^{2q-1}T^{-q}\left[\int_{\mathbb{R}^d} E\left|\sum_{t=1}^{\lfloor T(s_1-r_1)\rfloor} \varepsilon_t(u)\right|^{2q} W(u)\mathrm{d}u + \int_{\mathbb{R}^d} E\left|(s_1 - r_1)\sum_{t=1}^{T}\varepsilon_t(u)\right|^{2q} W(u)\mathrm{d}u\right] \\
\leq \; & 2^{2q-1}T^{-q}\mathcal{C}\left[T(s_1 - r_1)\right]^q + 2^{2q-1}T^{-q}\mathcal{C}(s_1 - r_1)^{2q}T^q \\
\leq \; & \mathcal{C}(s_1 - r_1)^q,
\end{aligned}
$$

where the second to last inequality is by Theorem 2 of Yokoyama (1980) and Assumption A.2. Hence,

$\mathcal{I}_{11}(s_1,r_1) \leq \mathcal{C}(s_1-r_1)^{q/2}$. By analogous arguments, given $0<r_1<s_1<1$, we have $\sup_{s_1,r_1}\mathcal{I}_{12}(s_1,r_1) \leq \mathcal{C}$. Finally, for $\mathcal{I}_2(s_1,r_1)$, we have

$$\begin{aligned} E\left|\int_{\mathbb{R}^d} \left|\mathcal{L}_T(u,r_1) - r_1\mathcal{L}_T(u,1)\right|^2 W(u)\mathrm{d}u\right|^q &\leq \int_{\mathbb{R}^d} E\left|\mathcal{L}_T(u,r_1) - r_1\mathcal{L}_T(u,1)\right|^{2q} W(u)\mathrm{d}u \\ &\leq 2^{2q-1}\int_{\mathbb{R}^d} E\left|\mathcal{L}_T(u,r_1)\right|^{2q} W(u)\mathrm{d}u + 2^{2q-1}r_1^{2q}\int_{\mathbb{R}^d} E\left|\mathcal{L}_T(u,1)\right|^{2q} W(u)\mathrm{d}u \\ &\leq 2^{2q-1}T^{-q}\mathcal{C}(Tr_1)^q + 2^{2q-1}r_1^{2q}T^{-q}\mathcal{C}T^q \\ &\leq \mathcal{C}, \end{aligned}$$

given $0<r_1<1$. Besides, given $0<r_1<s_1<1$,

$$\left[\frac{(s_1-r_1)(s_1+r_1-1)}{s_1r_1(1-s_1)(1-r_1)}\right]^q \leq \mathcal{C}(s_1-r_1)^q.$$

Combining the results, we have

$$E\left|F_T(s_1) - F_T(r_1)\right|^q \leq \mathcal{C}(s_1-r_1)^{q/2}.$$

By Theorem 15.6 of Billingsley (1968),

$$\sup_{r_1} F_T(r_1) \xrightarrow{d} \sup_{r_1} F(r_1),$$

when $M=1$. When $M>1$,

$$\sup_{\{r_1,\ldots,r_M\}} F_T(r_1,\ldots,r_M) \xrightarrow{d} \sup_{\{r_1,\ldots,r_M\}} F(r_1,\ldots,r_M),$$

where

$$\begin{aligned} F(r_1,\ldots,r_M) &= \sum_{j=1}^{M+1}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{R}^d}\left|[B(u,r_j)-B(u,r_{j-1})]-(r_j-r_{j-1})B(u,1)\right|^2 W(u)\mathrm{d}u \\ &= \sum_{j=1}^{M+1}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{R}^d}\left|\mathcal{B}(u,r_j)-\mathcal{B}(u,r_{j-1})\right|^2 W(u)\mathrm{d}u, \end{aligned}$$

where $\mathcal{B}(u,r) \equiv B(u,r) - rB(u,1)$ is a generalized Brownian bridge. ■

**Proof of Theorem 3.2** Let $\{T_j\}_{j=1}^M$ be the specified collection of break dates under $\mathbb{H}_A$. By proof of Theorem 3.1,

$$\begin{aligned} F_T(r_1,\ldots,r_M) &= \sum_{j=1}^{M+1}(T_j-T_{j-1})\int_{\mathbb{R}^d}\left|\tilde{\psi}_j^{(U)}(u)-\tilde{\psi}^{(R)}(u)\right|^2 W(u)\mathrm{d}u \\ &= \sum_{j=1}^{M+1}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{R}^d}\left|\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}e^{\mathbf{i}u'Y_t}-\frac{r_j}{\sqrt{T}}\sum_{t=1}^{T}e^{\mathbf{i}u'Y_t}\right)-\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T_{j-1}}e^{\mathbf{i}u'Y_t}-\frac{r_{j-1}}{\sqrt{T}}\sum_{t=1}^{T}e^{\mathbf{i}u'Y_t}\right)\right|^2 W(u)\mathrm{d}u. \end{aligned}$$

Let $\{T_k^0\}_{k=1}^{M^0}$ denote the collection of true breaks. Under $\mathbb{H}_A(a_T): \psi_k^0(u) = \phi^0(u) + a_T\theta_k(u)$ for the $k$-th regime, $k=1,...,M^0+1$, then

$$e^{\mathbf{i}u'Y_t} = \phi^0(u) + a_T\theta_k(u) + \varepsilon_t(u),$$

for $t = T^0_{k-1}+1,...,T^0_k$, and $k=1,...,M^0+1$. Given $a_T = T^{-1/2}$, it follows

$$\begin{aligned}
\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j} e^{\mathbf{i}u'Y_t} &= \frac{T_j}{\sqrt{T}}\phi^0(u) + \frac{1}{\sqrt{T}}\sum_{k=1}^{l}\sum_{t=T^0_{k-1}+1}^{T^0_k}[a_T\theta_k(u)+\varepsilon_t(u)] + \frac{1}{\sqrt{T}}\sum_{t=T^0_l+1}^{T_j}[a_T\theta_{l+1}(u)+\varepsilon_t(u)] \\
&= r_j\sqrt{T}\phi^0(u) + \left[\sum_{k=1}^{l}(r^0_k - r^0_{k-1})\theta_k(u) + (r_j - r^0_l)\theta_{l+1}(u)\right] \\
&\quad + \left[\sum_{k=1}^{l}\frac{\sqrt{r^0_k - r^0_{k-1}}}{\sqrt{T^0_k - T^0_{k-1}}}\sum_{t=T^0_{k-1}+1}^{T^0_k}\varepsilon_t(u) + \frac{\sqrt{r^0_{l+1}-r^0_l}}{\sqrt{T^0_{l+1}-T^0_l}}\sum_{t=T^0_l+1}^{T_j}\varepsilon_t(u)\right],
\end{aligned}$$

where $r^0_l < r_j < r^0_{l+1}$, and

$$\frac{r_j}{\sqrt{T}}\sum_{t=1}^{T} e^{\mathbf{i}u'Y_t} = r_j\sqrt{T}\phi^0(u) + r_j\sum_{k=1}^{M^0+1}(r^0_k - r^0_{k-1})\theta_k(u) + r_j\sum_{k=1}^{M^0+1}\frac{\sqrt{r^0_k - r^0_{k-1}}}{\sqrt{T^0_k - T^0_{k-1}}}\sum_{t=T^0_{k-1}+1}^{T^0_k}\varepsilon_t(u).$$

By Assumption A.4,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j} e^{\mathbf{i}u'Y_t} - \frac{r_j}{\sqrt{T}}\sum_{t=1}^{T} e^{\mathbf{i}u'Y_t} \Rightarrow G(u,r_j) + \Gamma(u,r_j),$$

where

$$G(u,r_j) = \left[\sum_{k=1}^{l}(r^0_k - r^0_{k-1})^{1/2}B^{(k)}(u,1) + (r^0_{l+1}-r^0_l)^{1/2}B^{(l+1)}\left(u, \frac{r_j - r^0_l}{r^0_{l+1}-r^0_l}\right)\right] - r_j\left[\sum_{k=1}^{M^0+1}(r^0_k - r^0_{k-1})^{1/2}B^{(k)}(u,1)\right],$$

and

$$\Gamma(u,r_j) = \left[\sum_{k=1}^{l}(r^0_k - r^0_{k-1})\theta_k(u) + (r_j - r^0_l)\theta_{l+1}(u)\right] - r_j\left[\sum_{k=1}^{M^0+1}(r^0_k - r^0_{k-1})\theta_k(u)\right].$$

Note that $\Gamma(u,r_j)$ is continuous for $r_j$ and $\int_{\mathbb{R}^d}|\Gamma(u,r_j)|^2 W(u)\mathrm{d}u < \infty$ for each $j$. Given Assumptions A.1-A.4, by analogous arguments to those in the proof of Theorem 3.1,

$$\sup_{\{r_1,...,r_M\}\in\Pi_\epsilon} F_T(r_1,...,r_M) \xrightarrow{d} \sup_{\{r_1,...,r_M\}\in\Pi_\epsilon} F^A(r_1,...,r_M),$$

where

$$F^A(r_1,...,r_M) = \sum_{j=1}^{M+1}\frac{1}{r_j - r_{j-1}}\int_{\mathbb{R}^d}|G(u,r_j) - G(u,r_{j-1}) + \Gamma(u,r_j) - \Gamma(u,r_{j-1})|^2 W(u)\mathrm{d}u.$$

■

**Proof of Theorem 3.3** Let the reconstructed sample obtained via the moving block bootstrap be $\{Y_1^\star, Y_2^\star, ..., Y_T^\star\}$, where $Y_t^\star = Y_{t_i}$ for some $1 \leq t_i \leq T$. Under $\mathbb{H}_0$ of no structural breaks,

$$e^{\mathbf{i}u'Y_t^\star} = \phi_0(u) + \varepsilon_t^\star(u),$$

where $\phi_0(u) = E(e^{\mathbf{i}u'Y_t})$ is the time-invariant CF of $Y_t$ for all $t$. It implies that resampling $Y_t$ is equivalent to resampling the generalized error function $\varepsilon_t(u)$ under $\mathbb{H}_0$. Consider the following joint test statistic based on the bootstrap sample

$$\begin{aligned}
F_T^\star(r_1, ..., r_M) &= \mathrm{SSGR}_0^\star - \mathrm{SSGR}_M^\star \\
&= \sum_{j=1}^{M+1} (T_j - T_{j-1}) \int_{\mathbb{R}^d} \left| \frac{1}{T_j - T_{j-1}} \sum_{s=T_{j-1}+1}^{T_j} e^{\mathbf{i}u'Y_s^\star} - \frac{1}{T} \sum_{s=1}^{T} e^{\mathbf{i}u'Y_s^\star} \right|^2 W(u)\mathrm{d}u \\
&= \sum_{j=1}^{M+1} (T_j - T_{j-1}) \int_{\mathbb{R}^d} \left| \frac{1}{T_j - T_{j-1}} \sum_{s=T_{j-1}+1}^{T_j} \varepsilon_s^\star(u) - \frac{1}{T} \sum_{s=1}^{T} \varepsilon_s^\star(u) \right|^2 W(u)\mathrm{d}u,
\end{aligned}$$

where

$$\mathrm{SSGR}_0^\star = \sum_{t=1}^{T} \int_{\mathbb{R}^d} \left| e^{\mathbf{i}u'Y_t^\star} - \frac{1}{T} \sum_{s=1}^{T} e^{\mathbf{i}u'Y_s^\star} \right|^2 W(u)\mathrm{d}u,$$

and

$$\mathrm{SSGR}_M^\star = \sum_{j=1}^{M+1} \sum_{t=T_{j-1}+1}^{T_j} \int_{\mathbb{R}^d} \left| e^{\mathbf{i}u'Y_t^\star} - \frac{1}{T_j - T_{j-1}} \sum_{s=T_{j-1}+1}^{T_j} e^{\mathbf{i}u'Y_s^\star} \right|^2 W(u)\mathrm{d}u.$$

By analogous steps as in the proof of Theorem 3.1, we have

$$F_T^\star(r_1, ..., r_M) = \sum_{j=1}^{M+1} \frac{1}{r_j - r_{j-1}} \int_{\mathbb{R}^d} \left| \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^{T_j} \varepsilon_t^\star(u) - \frac{r_j}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_t^\star(u) \right] - \left[ \frac{1}{\sqrt{T}} \sum_{t=1}^{T_{j-1}} \varepsilon_t^\star(u) - \frac{r_{j-1}}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_t^\star(u) \right] \right|^2 W(u)\mathrm{d}u.$$

Let $E^\star[\bar{\varepsilon}^\star(u)] \equiv E^\star[T^{-1} \sum_{t=1}^{T} \varepsilon_t^\star(u)]$ be the expectation of the sample average of $\varepsilon_t^\star(u)$ conditioning on the observable sample, we have

$$\begin{aligned}
\frac{1}{\sqrt{T}} \sum_{t=1}^{T_j} \varepsilon_t^\star(u) - \frac{r_j}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_t^\star(u) &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T_j} [\varepsilon_t^\star(u) - E^\star[\bar{\varepsilon}^\star(u)]] - \frac{r_j}{\sqrt{T}} \sum_{t=1}^{T} [\varepsilon_t^\star(u) - E^\star[\bar{\varepsilon}^\star(u)]] \\
&= \mathcal{L}_T^\star(u, r_j) - r_j \mathcal{L}_T^\star(u, 1),
\end{aligned}$$

where

$$\mathcal{L}_T^\star(u, r_j) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T_j} [\varepsilon_t^\star(u) - E^\star[\bar{\varepsilon}^\star(u)]] .$$

Then, it follows

$$F_T^\star(r_1,...,r_M) \quad = \quad \sum_{j=1}^{M+1} \frac{1}{r_j - r_{j-1}} \int_{\mathbb{R}^d} \left| [\mathcal{L}_T^\star(u,r_j) - r_j \mathcal{L}_T^\star(u,1)] - [\mathcal{L}_T^\star(u,r_{j-1}) - r_{j-1}\mathcal{L}_T^\star(u,1)] \right|^2 W(u)\mathrm{d}u.$$

Note that proof of Theorem 3.1 implies

$$F_T(r_1,...,r_M) \quad = \quad \sum_{j=1}^{M+1} \frac{1}{r_j - r_{j-1}} \int_{\mathbb{R}^d} \left| [\mathcal{L}_T(u,r_j) - r_j \mathcal{L}_T(u,1)] - [\mathcal{L}_T(u,r_{j-1}) - r_{j-1}\mathcal{L}_T(u,1)] \right|^2 W(u)\mathrm{d}u,$$

where

$$\mathcal{L}_T(u,r_j) = T^{-1/2} \sum_{t=1}^{T_j} \varepsilon_t(u) \Rightarrow B(u,r_j)$$

under Assumption A.4*. To show the validity of the bootstrap test statistic, it suffices to show that

$$\mathcal{L}_T^\star(u,r_j) \Rightarrow^\star B(u,r_j),$$

in probability over $\mathbb{U} \times [0,1]$. We first establish (a) the convergence of $\mathcal{L}_T^\star(u,r_j)$ for each fixed $u \in \mathbb{U}$ and then show that (b) $\mathcal{L}_T^\star(u,r_j)$ is stochastically equicontinuous in $\mathbb{U}$.

We use Theorem 2 of Calhoun (2018) to establish (a). By Assumption A.1(ii) and $\sup_u |e^{\mathbf{i}u'Y_t}| = 1$, for each fixed $u \in \mathbb{U}$, $\varepsilon_t(u) = e^{\mathbf{i}u'Y_t} - \phi_0(u)$ is $L_2$ Near-Epoch-Dependent process of size $-1/2$ since the process $\{Y_t\}$ is strong mixing with mixing coefficient $\alpha(s) = O(s^{-\frac{q}{q-2}})$ for some $q > 2$. Hence, condition 1 in Theorem 2 of Calhoun (2018) is satisfied. Note that $E[\varepsilon_t(u)] = 0$, and $\varepsilon_t(u)$ is uniformly $L_q$ bounded. Besides,

$$\sqrt{T} \left| \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t(u) \right| \to \Omega(u,u)^{1/2},$$

under Assumption A.4*. Then, conditions 2 and 3 in Theorem 2 of Calhoun (2018) are satisfied. Finally, condition 4 in Theorem 2 of Calhoun (2018) holds under Assumption A.6. Furthermore,

$$\frac{1}{T} \sum_{s=1}^{T_j} \sum_{t=1}^{T_j} \mathrm{cov}\left[\varepsilon_t(u), \varepsilon_s(u)\right] \to r_j \Omega(u,u)$$

under Assumption A.4*. Then, by Theorem 2 of Calhoun (2018),

$$\mathcal{L}_T^\star(u,r_j) \Rightarrow^\star B(u,r_j),$$

for each fixed $u \in \mathbb{U}$.

Now, we show (b). Note that

$$\begin{aligned} E^\star[\bar{\varepsilon}^\star(u)] \quad &= \quad \frac{1}{T - l_T + 1} \sum_{t=1}^{T} \varepsilon_t(u) - \frac{1}{T - l_T + 1} \sum_{k=1}^{l_T - 1} \left(1 - \frac{k}{l_T}\right) [\varepsilon_k(u) + \varepsilon_{T-k+1}(u)] \\ &= \quad \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t(u) + O_P(l_T T^{-1}). \end{aligned}$$

Then $\mathcal{L}_T^\star(u,r_j) = T^{-1/2}\sum_{t=1}^{T_j}\varepsilon_t^\star(u) - r_j T^{-1/2}\sum_{t=1}^{T}\varepsilon_t(u) + O_P(l_T T^{-1/2})$.

Let $\mathbb{D}_{u,\delta}$ be an open ball in $\mathbb{U}$ centered at $u$ such that $\|v-u\|<\delta$ for all $v\in\mathbb{D}_{u,\delta}$. By the mean value theorem,

$$\sum_{t=1}^{T_j}\left[\varepsilon_t^\star(u) - \varepsilon_t^\star(v)\right] = \sum_{t=1}^{T_j}\Upsilon_t(\tilde{u})'(u-v)$$

for some $\tilde{u}$ lying in between $u$ and $v$, where

$$\Upsilon_t(u) = \frac{\mathrm{d}\varepsilon_t^\star(u)}{\mathrm{d}u} = \mathbf{i}\left[Y_t^\star e^{\mathbf{i}u'Y_t^\star} - E\left(Y_t^\star e^{\mathbf{i}u'Y_t^\star}\right)\right]$$

We note that $\tilde{u}$ can depend on $j$. Then

$$\begin{aligned}
& \lim_{\delta\to 0}\limsup_{T\to\infty} P\left[\sup_{u\in\mathbb{U}}\sup_{v\in\mathbb{D}_{u,\delta}}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}\left[\varepsilon_t^\star(u)-\varepsilon_t^\star(v)\right]\right| > \epsilon\right] \\
= & \lim_{\delta\to 0}\limsup_{T\to\infty} P\left[\sup_{u\in\mathbb{U}}\sup_{v\in\mathbb{D}_{u,\delta}}\left|\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}\Upsilon_t(\tilde{u})'(u-v)\right| > \epsilon\right] \\
\leq & \lim_{\delta\to 0}\limsup_{T\to\infty} P\left[\sup_{\bar{u}\in\mathbb{U}}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}\Upsilon_t(\tilde{u})\right\| > \epsilon/\delta\right] \\
= & 0,
\end{aligned}$$

for any $\epsilon>0$, where the last equality holds since

$$\sup_{\bar{u}\in\mathbb{U}}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}\Upsilon_t(\tilde{u})\right\| = \sup_{\bar{u}\in\mathbb{U}}\left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}\left[Y_t^\star e^{\mathbf{i}\tilde{u}'Y_t^\star} - E\left(Y_t^\star e^{\mathbf{i}\tilde{u}'Y_t^\star}\right)\right]\right\| = O_P(1),$$

under the mixing condition and moment restriction in Assumptions A.1(ii) and A.6(ii). Analogous results can be obtained for $r_j T^{-1/2}\sum_{t=1}^{T}\varepsilon_t(u)$. Then, it follows $T^{-1/2}\mathcal{L}_T^\star(u,r_j) \Rightarrow^\star B(u,r_j)$, in probability for each $j$. Given $\mathbb{U}\times[0,1]$ is a compact set, by the continuous mapping theorem, and analogous arguments in the proof of Theorem 3.1,

$$F_T^\star(r_1,...,r_M) \Rightarrow^\star F(r_1,...,r_M),$$

in probability.

Next, we show that the proposed MBB is asymptotically valid under $\mathbb{H}_A$. Let $\phi_t(u)\equiv E(e^{\mathbf{i}u'Y_t})$ denote the true time-varying CF and $\phi_t^\star(u)\equiv E(e^{\mathbf{i}u'Y_t^\star})$ be the CF of $Y_t^\star$ based on the bootstrap sample. Given $e^{\mathbf{i}u'Y_t^\star} = \phi_t^\star(u) + \varepsilon_t^\star(u)$ under $\mathbb{H}_A$, we have

$$\begin{aligned}
F_T^\star(r_1,...,r_M) &= \mathrm{SSGR}_0^\star - \mathrm{SSGR}_M^\star \\
&= \sum_{j=1}^{M+1}(T_j - T_{j-1})\int_{\mathbb{R}^d}\left|\frac{1}{T_j - T_{j-1}}\sum_{s=T_{j-1}+1}^{T_j} e^{\mathbf{i}u'Y_s^\star} - \frac{1}{T}\sum_{s=1}^{T} e^{\mathbf{i}u'Y_s^\star}\right|^2 W(u)\mathrm{d}u \\
&\leq 2\sum_{j=1}^{M+1}(T_j - T_{j-1})\int_{\mathbb{R}^d}\left|\frac{1}{T_j - T_{j-1}}\sum_{s=T_{j-1}+1}^{T_j}\varepsilon_s^\star(u) - \frac{1}{T}\sum_{s=1}^{T}\varepsilon_s^\star(u)\right|^2 W(u)\mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{j=1}^{M+1}(T_j-T_{j-1})\int_{\mathbb{R}^d}\left|\frac{1}{T_j-T_{j-1}}\sum_{s=T_{j-1}+1}^{T_j}\phi_s^\star(u)-\frac{1}{T}\sum_{s=1}^{T}\phi_s^\star(u)\right|^2 W(u)\mathrm{d}u \\
=\;& 2\mathcal{R}_1(r_1,...,r_M)+2\mathcal{R}_2(r_1,...,r_M),\ \text{say.}
\end{aligned}
$$

By the mixing condition in Assumption A.1(ii) and the analogous proof in Lemma A.1, we can show $\sup_{r_1,...,r_M}|\mathcal{R}_1(r_1,...,r_M)|=O_P(1)$. We consider $\mathcal{R}_2(r_1,...,r_M)$. By analogous steps, we have

$$
\begin{aligned}
\mathcal{R}_2(r_1,...,r_M) &= \sum_{j=1}^{M+1}(T_j-T_{j-1})\int_{\mathbb{R}^d}\left|\frac{1}{T_j-T_{j-1}}\sum_{s=T_{j-1}+1}^{T_j}\phi_s^\star(u)-\frac{1}{T}\sum_{s=1}^{T}\phi_s^\star(u)\right|^2 W(u)\mathrm{d}u \\
&= \sum_{j=1}^{M+1}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{R}^d}\left|\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}\phi_t^\star(u)-\frac{r_j}{\sqrt{T}}\sum_{t=1}^{T}\phi_t^\star(u)\right]-\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T_{j-1}}\phi_t^\star(u)-\frac{r_{j-1}}{\sqrt{T}}\sum_{t=1}^{T}\phi_t^\star(u)\right]\right|^2 W(u)\mathrm{d}u \\
&\leq 2\sum_{j=1}^{M+1}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{R}^d}\left|\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}\left\{\phi_t^\star(u)-E^\star[\bar{\phi}^\star(u)]\right\}-\frac{r_j}{\sqrt{T}}\sum_{t=1}^{T}\left\{\phi_t^\star(u)-E^\star[\bar{\phi}^\star(u)]\right\}\right]\right|^2 W(u)\mathrm{d}u \\
&\quad+2\sum_{j=1}^{M+1}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{R}^d}\left|\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T_{j-1}}\left\{\phi_t^\star(u)-E^\star[\bar{\phi}^\star(u)]\right\}-\frac{r_{j-1}}{\sqrt{T}}\sum_{t=1}^{T}\left\{\phi_t^\star(u)-E^\star[\bar{\phi}^\star(u)]\right\}\right]\right|^2 W(u)\mathrm{d}u \\
&= 2\sum_{j=1}^{M+1}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{R}^d}\left|\mathcal{N}_T^\star(u,r_j)-r_j\mathcal{N}_T^\star(u,1)\right|^2 W(u)\mathrm{d}u \\
&\quad+2\sum_{j=1}^{M+1}\frac{1}{r_j-r_{j-1}}\int_{\mathbb{R}^d}\left|\mathcal{N}_T^\star(u,r_{j-1})-r_{j-1}\mathcal{N}_T^\star(u,1)\right|^2 W(u)\mathrm{d}u,
\end{aligned}
$$

where $E^\star[\bar{\phi}^\star(u)]\equiv E^\star[T^{-1}\sum_{t=1}^{T}\phi_t^\star(u)]$ be the expectation of the sample average of $\phi_t^\star(u)$ conditioning on the observable sample, and $\mathcal{N}_T^\star(u,r_j)\equiv T^{-1/2}\sum_{t=1}^{T_j}\left\{\phi_t^\star(u)-E^\star[\bar{\phi}^\star(u)]\right\}$. Let $\mathcal{X}_i(u)=\phi_i(u)+\cdots+\phi_{i+l_T-1}(u)$ for the $i$-th block, $i=1,...,N$, and $\mathcal{X}_k^\star(u)=\phi_{(k-1)l_T+1}^\star(u)+\cdots+\phi_{kl_T}^\star(u)$ for the $k$-th resampled block, $k=1,...,K$. Conditioning on the data, $\mathcal{X}_1^\star(u),\mathcal{X}_2^\star(u),...,\mathcal{X}_K^\star(u)$ are independent and identically distributed. Given $T=l_TK$, and $N=T-l_T+1$, then it follows

$$
\begin{aligned}
E^\star\left[\mathcal{N}_T^\star(u,r_j)\right] &= \sqrt{T}E^\star\left[\frac{1}{l_TK}\sum_{t=1}^{\lfloor l_TKr_j\rfloor}\phi_t^\star(u)\right]-\sqrt{T}r_jE^\star\left[\frac{1}{l_TK}\sum_{t=1}^{l_TK}\phi_t^\star(u)\right] \\
&= \frac{\sqrt{T}}{l_TK}\sum_{k=1}^{\lfloor Kr_j\rfloor}E^\star\left[\mathcal{X}_i^\star(u)\right]-\frac{\sqrt{T}r_j}{l_TK}\sum_{k=1}^{K}E^\star\left[\mathcal{X}_i^\star(u)\right] \\
&= \frac{\sqrt{T}}{l_TK}\sum_{k=1}^{\lfloor Kr_j\rfloor}\frac{1}{N}\sum_{i=1}^{N}\mathcal{X}_i(u)-\frac{\sqrt{T}r_j}{l_TK}\sum_{k=1}^{K}\frac{1}{N}\sum_{i=1}^{N}\mathcal{X}_i(u) \\
&= \frac{\sqrt{T}r_j}{Nl_T}\sum_{i=1}^{N}\mathcal{X}_i(u)-\frac{\sqrt{T}r_j}{Nl_T}\sum_{i=1}^{N}\mathcal{X}_i(u) \\
&= 0,
\end{aligned}
$$

for all $\{r_j\}_{j=1}^{M+1}$. And

$$
\begin{aligned}
E^\star\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}\left\{\phi_t^\star(u)-E^\star[\bar{\phi}^\star(u)]\right\}\right]^2 &= \mathrm{var}^\star\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T_j}\phi_t^\star(u)\right]\\
&= \frac{1}{T}\mathrm{var}^\star\left[\sum_{k=1}^{\lfloor Kr_j\rfloor}\mathcal{X}_k^\star(u)\right]\\
&= \frac{Kr_j}{T}\mathrm{var}^\star\left[\mathcal{X}_k^\star(u)\right]\\
&= \frac{Kr_j}{T}\frac{1}{N}\sum_{i=1}^{N}|\mathcal{X}_i(u)|^2-\frac{Kr_j}{T}\left|\frac{1}{N}\sum_{i=1}^{N}\mathcal{X}_i(u)\right|^2\\
&= O_P(l_T).
\end{aligned}
$$

Hence, $\mathcal{N}_T^\star(u,r_j)=O_{P^\star}(l_T^{1/2})$ for all $u$ and $r_j$. The uniform results can be established in a similar way as in the proof of Theorem 3.1. Then, we have that under $\mathbb{H}_A$, the bootstrap joint test statistic $\sup F_T^\star=O_{P^\star}(l_T)$.

Note that Theorem 3.2 implies that our test statistic

$$
\begin{aligned}
\sup F_T(r_1,...,r_M) &= \mathrm{SSGR}_0-\mathrm{SSGR}_M(\hat{r}_1,...,\hat{r}_M)\\
&= O_P(Ta_T^2),
\end{aligned}
$$

under $\mathbb{H}_A(a_T)$. Hence, if $Ta_T^2l_T^{-1}\to\infty$, we have that $P^\star(\sup F>\sup F^\star)\to 1$ as $T\to\infty$. ■

**Proof of Theorem 4.1** Under the null hypothesis of $M$ breaks, we let $D(T_j,T_k)$ be the SSGR using the data within the segment specified by $[T_j+1,T_k]$ for $0\le j<k\le M+1$. Then it is obvious that

$$
\mathrm{SSGR}_M(T_1^0,...,T_M^0) = \sum_{k=1}^{M+1}D(T_{k-1}^0,T_k^0),
$$

and

$$
\mathrm{SSGR}_{M+1}\left(T_1^0,...,T_{j-1}^0,\tau,T_j^0,...,T_M^0\right) = \sum_{k=1}^{j-1}D(T_{k-1}^0,T_k^0)+D(T_{j-1}^0,\tau)+D(\tau,T_j^0)+\sum_{k=j+1}^{M+1}D(T_{k-1}^0,T_k^0),
$$

for some $1<j<M$. We note that

$$
\begin{aligned}
\mathrm{SSGR}_{M+1}\left(T_1^0,...,T_{j-1}^0,\tau,T_j^0,...,T_M^0\right) &\equiv \mathrm{SSGR}_{M+1}\left(\tau,T_1^0,...,T_M^0\right)\\
&= D(1,\tau)+D(\tau,T_1^0)+\sum_{k=2}^{M+1}D(T_{k-1}^0,T_k^0),
\end{aligned}
$$

when $j=1$, and

$$
\begin{aligned}
\mathrm{SSGR}_{M+1}\left(T_1^0,...,T_{j-1}^0,\tau,T_j^0,...,T_M^0\right) &\equiv \mathrm{SSGR}_{M+1}\left(T_1^0,...,T_M^0,\tau\right)\\
&= \sum_{k=1}^{M}D(T_{k-1}^0,T_k^0)+D(T_M^0,\tau)+D(\tau,T),
\end{aligned}
$$

when $j = M$. Let

$$\begin{aligned} F_T^0(M+1|M) &= \mathrm{SSGR}_M\left(T_1^0,...,T_M^0\right) - \min_{1\le j\le M+1}\inf_{\tau\in\Lambda_{j,\epsilon}^0}\mathrm{SSGR}_{M+1}\left(T_1^0,...,T_{j-1}^0,\tau,T_j^0,...,T_M^0\right) \\ &= \max_{1\le j\le M+1}\sup_{\tau\in\Lambda_{j,\epsilon}^0}\left[\mathrm{SSGR}_M\left(T_1^0,...,T_M^0\right) - \mathrm{SSGR}_{M+1}\left(T_1^0,...,T_{j-1}^0,\tau,T_j^0,...,T_M^0\right)\right] \\ &= \max_{1\le j\le M+1}\sup_{\tau\in\Lambda_{j,\epsilon}^0}\left[D(T_{j-1}^0,T_j^0) - D(T_{j-1}^0,\tau) - D(\tau,T_j^0)\right], \end{aligned}$$

where

$$\Lambda_{j,\epsilon}^0 = \left\{\tau : T_{j-1}^0 + \lfloor (T_j^0 - T_{j-1}^0)\epsilon\rfloor \le \tau \le T_j^0 - \lfloor (T_j^0 - T_{j-1}^0)\epsilon\rfloor\right\},$$

for some arbitrarily small $\epsilon > 0$. It is straightforward to see that

$$\sup_{\tau\in\Lambda_{j,\epsilon}^0}\left[D(T_{j-1}^0,T_j^0) - D(T_{j-1}^0,\tau) - D(\tau,T_j^0)\right]$$

is equivalent to the sup-$F$ test statistic in Theorem 3.1 in a subsample specified by $[T_{j-1}^0, T_j^0]$ with $M = 1$ in the alternative hypothesis.

It follows

$$\begin{aligned} & D(T_{j-1}^0,T_j^0) - D(T_{j-1}^0,\tau) - D(\tau,T_j^0) \\ = \quad & (\tau - T_{j-1}^0)\int_{\mathbb{R}^d}\left|\frac{1}{T_j^0 - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u) - \frac{1}{\tau - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{\tau}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\ & +(T_j^0 - \tau)\int_{\mathbb{R}^d}\left|\frac{1}{T_j^0 - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u) - \frac{1}{T_j^0 - \tau}\sum_{t=\tau+1}^{T_j^0}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u. \end{aligned}$$

Let $r = \frac{\tau - T_{j-1}^0}{T_j^0 - T_{j-1}^0}$. Given Assumption A.4, we have for $t \in [T_{j-1}^0, T_j^0]$,

$$\frac{1}{\sqrt{T_j^0 - T_{j-1}^0}}\sum_{t=T_{j-1}^0+1}^{\tau}\varepsilon_t(u) = \frac{1}{\sqrt{T_j^0 - T_{j-1}^0}}\sum_{t=T_{j-1}^0+1}^{T_{j-1}^0+\lfloor (T_j^0 - T_{j-1}^0)r\rfloor}\varepsilon_t(u) \Rightarrow B^{(j)}(u,r).$$

Hence,

$$\begin{aligned} & D(T_{j-1}^0,T_j^0) - D(T_{j-1}^0,\tau) - D(\tau,T_j^0) \\ = \quad & \frac{1}{\tau - T_{j-1}^0}\int_{\mathbb{R}^d}\left|\frac{\tau - T_{j-1}^0}{T_j^0 - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u) - \sum_{t=T_{j-1}^0+1}^{\tau}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \\ & +\frac{1}{T_j^0 - \tau}\int_{\mathbb{R}^d}\left|\frac{T_j^0 - \tau}{T_j^0 - T_{j-1}^0}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u) - \sum_{t=\tau+1}^{T_j^0}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lfloor r(T_j^0 - T_{j-1}^0)\rfloor} \int_{\mathbb{R}^d} \left| r \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) - \sum_{t=T_{j-1}^0+1}^{\tau} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u \\
&\quad + \frac{1}{\lfloor (1-r)(T_j^0 - T_{j-1}^0)\rfloor} \int_{\mathbb{R}^d} \left| (1-r) \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) - \left[ \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) - \sum_{t=T_{j-1}^0+1}^{\tau} \varepsilon_t(u) \right] \right|^2 W(u)\mathrm{d}u \\
&= \frac{1}{r} \int_{\mathbb{R}^d} \left| r \frac{1}{\sqrt{T_j^0 - T_{j-1}^0}} \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) - \frac{1}{\sqrt{T_j^0 - T_{j-1}^0}} \sum_{t=T_{j-1}^0+1}^{\tau} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u \\
&\quad + \frac{1}{1-r} \int_{\mathbb{R}^d} \left| r \frac{1}{\sqrt{T_j^0 - T_{j-1}^0}} \sum_{t=T_{j-1}^0+1}^{T_j^0} \varepsilon_t(u) - \frac{1}{\sqrt{T_j^0 - T_{j-1}^0}} \sum_{t=T_{j-1}^0+1}^{\tau} \varepsilon_t(u) \right|^2 W(u)\mathrm{d}u \\
&\Rightarrow \frac{1}{r(1-r)} \int_{\mathbb{R}^d} \left| B^{(j)}(u,r) - rB^{(j)}(u,1) \right|^2 W(u)\mathrm{d}u.
\end{aligned}
$$

It follows that

$$
\sup_{\tau \in \Lambda_{j,\epsilon}^0} \left[ D(T_{j-1}^0, T_j^0) - D((T_{j-1}^0, \tau) - D(\tau, T_j^0) \right] \xrightarrow{d} \sup_{\epsilon \le r \le 1-\epsilon} \int_{\mathbb{R}^d} \frac{\left| B^{(j)}(u,r) - rB^{(j)}(u,1) \right|^2}{r(1-r)} W(u)\mathrm{d}u.
$$

Hence, we have

$$
\begin{aligned}
F_T^0(M+1|M) &= \max_{1 \le j \le M+1} \sup_{\tau \in \Lambda_{j,\epsilon}^0} \left[ D(T_{j-1}^0, T_j^0) - D(T_{j-1}^0, \tau) - D(\tau, T_j^0) \right] \\
&\xrightarrow{d} \max_{1 \le j \le M+1} \sup_{\epsilon \le r \le 1-\epsilon} \int_{\mathbb{R}^d} \frac{\left| \mathcal{B}^{(j)}(u,r) \right|^2}{r(1-r)} W(u)\mathrm{d}u,
\end{aligned}
$$

where $\mathcal{B}^{(j)}(u,r) \equiv B^{(j)}(u,r) - rB^{(j)}(u,1)$ is a generalized Brownian bridge under the $j$-th subsample, $j = 1, ..., M+1$. Under the null hypothesis, Theorem 2.2 indicates that $\hat{r}_j = r_j^0 + O_P(T^{-1})$. Based on this result, we know the above equation also holds with $T_{j-1}^0$ and $T_j^0$ replaced by $\hat{T}_{j-1}$ and $\hat{T}_j$, respectively. Hence, the limiting distribution of $F_T(M+1|M)$ is the same as $F_T^0(M+1|M)$. ■

**Proof of Theorem 4.2** Let $\mathcal{M} = \{1, 2, ..., M_{\max}\}$. We divide $\mathcal{M}$ into three subsets: $\mathcal{M}_0 = \{M \in \mathcal{M} : M = M^0\}$, $\mathcal{M}_- = \{M \in \mathcal{M} : M < M^0\}$, and $\mathcal{M}_+ = \{M \in \mathcal{M} : M > M^0\}$. $\mathcal{M}_0, \mathcal{M}_-$, and $\mathcal{M}_+$ denote the subsets of $\mathcal{M}$ which correctly estimate, under-estimate, and over-estimate the true number of breaks. We prove the theorem by showing that neither the under-fitted model nor the over-fitted model can minimize the information criterion function, i.e., $P(\min_{M \in (\mathcal{M}_- \cup \mathcal{M}_+)} IC(M) > IC(M^0)) \to 1$ as $T \to \infty$, where

$$
IC(M) \quad \equiv \ln\left[\hat{\sigma}^2(M)\right] + \rho_T(M+1),
$$

with $\hat{\sigma}^2(M) = T^{-1}\mathrm{SSGR}_M(\hat{r}_1, ..., \hat{r}_M)$.

By Theorems 2.1-2.3, under Assumption A.7, it immediately follows that

$$
IC(M^0) = \ln\left[\hat{\sigma}^2(M^0)\right] + \rho_T(M^0+1)
$$

$$= \ln\left[\frac{1}{T}\sum_{j=1}^{M^0+1}\sum_{t=\hat{T}_{j-1}+1}^{\hat{T}_j}\int_{\mathbb{R}^d}\left|e^{\mathbf{i}u'Y_t}-\hat{\psi}_j(u)\right|^2 W(u)\mathrm{d}u\right]+o(1)$$

$$= \ln\left[\frac{1}{T}\sum_{j=1}^{M^0+1}\sum_{t=T^0_{j-1}+1}^{T^0_j}\int_{\mathbb{R}^d}\left|e^{\mathbf{i}u'Y_t}-\psi^0_j(u)\right|^2 W(u)\mathrm{d}u\right]+o_P(1)$$

$$= \ln\left[\frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d}\left|\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right]+o_P(1),$$

where $\varepsilon_t(u)$ is as defined in the proof of Theorem 2.1. We now consider the cases of under- and over-fitted models separately.

When the model is under-fitted, i.e., $M < M^0$, Lemma $A$.5 implies that the estimated break fractions $\{\hat{r}_1, ..., \hat{r}_M\}$ are consistent for $M$ breaks in the collection of true break fractions $\{r^0_j\}_{j=1}^{M^0}$. Then there must exist $M^0 - M$ break fractions in $\{r^0_j\}_{j=1}^{M^0}$ that cannot be identified.

Without loss of generality, we assume the $j$-th break $r^0_j$ is not consistently estimated. Then with some positive probability $0 < c_0 < 1$ there exists an $\kappa > 0$ such that no estimated break fraction falls in the interval $[T(r^0_j - \kappa), T(r^0_j + \kappa)]$. Suppose this interval is classified into the $k$-th regime, i.e., $\hat{T}_{k-1} \le T^0_j - \lfloor \kappa T\rfloor + 1 < T^0_j + \lfloor \kappa T\rfloor \le \hat{T}_k$. Then, following the proof of Lemma $A$.4, we have

$$\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|\hat{\varepsilon}_t(u)\right|^2 W(u)\mathrm{d}u \ge \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u + \delta\int_{\mathbb{R}^d}\left|\psi^0_j(u)-\psi^0_{j+1}(u)\right|^2 W(u)\mathrm{d}u + o_P(1),$$

for any $M < M^0$, where $\delta = \frac{1}{2}\kappa$. It follows

$$\begin{aligned}\min_{M\in\mathcal{M}_-} IC(M) - IC\left(M^0\right) \quad &= \quad \min_{M\in\mathcal{M}_-}\left[\ln\left(\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|\hat{\varepsilon}_t(u)\right|^2 W(u)\mathrm{d}u\right)+\rho_T(M+1)\right]\\ &\quad -\left[\ln\left(\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right)+\rho_T(M^0+1)+o_P(1)\right]\\ &\ge \quad \ln\left[1+\frac{\delta\int_{\mathbb{R}^d}\left|\psi^0_j(u)-\psi^0_{j+1}(u)\right|^2 W(u)\mathrm{d}u}{\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u}\right]+o_P(1)\end{aligned}$$

Therefore, for any positive probability $0 < c_0 < 1$, there exists a $\kappa = 2\delta$ such that

$$P\left(\min_{M\in\mathcal{M}_-} IC(M) - IC\left(M_0\right) > \Delta\right) \quad = \quad c_0,$$

for $T$ sufficiently large, where $0 < \Delta < \lim_{T\to\infty}\ln\left[1+\frac{\delta\int_{\mathbb{R}^d}\left|\psi^0_j(u)-\psi^0_{j+1}(u)\right|^2 W(u)\mathrm{d}u}{\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u}\right]$. Then by Assumption A.7 and Slutsky Lemma, we have

$$P(\min_{M\in\mathcal{M}_-} IC(M) > IC(M^0)) \to 1,$$

as $T\to\infty$.

When the model is over-fitted, by Assumption A.7 and Lemma $A$.6, we have

$$
\begin{aligned}
& P\left(\min_{M\in\mathcal{M}_+} IC(M) > IC(M^0)\right) \\
&= P\left(\min_{M\in\mathcal{M}_+} \left[\ln(\hat{\sigma}^2(M)/\hat{\sigma}^2(M^0)) + T\rho_T(M-M^0)\right] > 0\right) \\
&\to 1 \text{ as } T\to\infty,
\end{aligned}
$$

given $M > M^0$. Therefore, we have proved that

$$
P\left(\min_{M\in\mathcal{M}_-\cup\mathcal{M}_+} IC(M) > IC(M^0)\right) \to 1 \text{ as } T\to\infty.
$$

■

# B Proofs of Technical Lemmas

**Proof of Lemma** A.1 Given $M$ is a finite number, without loss of generality, we let $j=1$. By the definition of $\varepsilon_t(u)$, we have

$$
\begin{aligned}
E\left[\frac{1}{T}\int_{\mathbb{R}^d}\left|\sum_{t=1}^{T_1}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right] &= \int_{\mathbb{R}^d}\frac{1}{T}\sum_{s=1}^{T_1}\sum_{t=1}^{T_1}\mathrm{Re}\left\{E\left[\varepsilon_t(u)\varepsilon_s(u)^*\right]\right\}W(u)\mathrm{d}u \\
&\le \frac{1}{T}\sum_{s=1}^{T_1}\sum_{t=1}^{T_1}\int_{\mathbb{R}^d}\left|\mathrm{cov}\left(e^{\mathbf{i}u'Y_t}, e^{\mathbf{i}u'Y_s}\right)\right| W(u)\mathrm{d}u \\
&\le 8r_1\left(\sup_{u\in\mathbb{R}^d}\max_{1\le t\le T_1}|e^{\mathbf{i}u'Y_t}|_q\right)^2\frac{1}{T_1}\sum_{s=1}^{T_1}\sum_{t=1}^{T_1}\alpha(s-t)^{(q-2)/q}\int_{\mathbb{R}^d}W(u)\mathrm{d}u \\
&\le 8r_1\sum_{l=-\infty}^{\infty}\alpha(l)^{(q-2)/q}\int_{\mathbb{R}^d}W(u)\mathrm{d}u \\
&\le \mathcal{C}.
\end{aligned}
$$

The second inequality is due to the Davydov's inequality and the fact that $r_1 = T_1/T$. The second inequality holds since

$$
\sup_{u\in\mathbb{R}^d}\max_{1\le t\le T_1}|e^{\mathbf{i}u'Y_t}|_q = 1
$$

for any $q>2$. And the last inequality holds due to Assumption A.1(ii) and Assumption A.2. Hence, by the Markov inequality,

$$
\frac{1}{T}\int_{\mathbb{R}^d}\left|\sum_{t=1}^{T_1}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u = O_P(1).
$$

■

**Proof of Lemma** A.2 Without loss of generality, we consider $j=1$. Let

$$
K_T(\eta) \equiv \left[\int_{\mathbb{R}^d}\left|\upsilon_T\sum_{t=1}^{\lfloor\eta\upsilon_T^{-2}\rfloor}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right]^{1/2},
$$

we want to show that $\sup_{\eta\in[0,C]}\left[K_T(\eta)\right]^2 = O_P(1)$. Given $[0,C]$ is compact set, it suffices to show (i) $K_T(\eta)=O_P(1)$ for each fixed $\eta$; and (ii) $K_T(\eta)$ is asymptotically tight.

For (i), by the definition of $\varepsilon_t(u)$, for each $\eta\in[0,C]$,

$$
\begin{aligned}
E\left[K_T(\eta)\right]^2 &= v_T^2\int_{\mathbb{R}^d} E\left|\sum_{t=1}^{\lfloor \eta v_T^{-2}\rfloor}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\\
&= v_T^2\int_{\mathbb{R}^d}\sum_{s=1}^{\lfloor \eta v_T^{-2}\rfloor}\sum_{t=1}^{\lfloor \eta v_T^{-2}\rfloor}\mathrm{Re}\left\{E\left[\varepsilon_t(u)\varepsilon_s(u)^*\right]\right\}W(u)\mathrm{d}u\\
&\le v_T^2\sum_{s=1}^{\lfloor \eta v_T^{-2}\rfloor}\sum_{t=1}^{\lfloor \eta v_T^{-2}\rfloor}\int_{\mathbb{R}^d}\left|\mathrm{cov}\left(e^{\mathbf{i}u'Y_t},e^{\mathbf{i}u'Y_s}\right)\right|W(u)\mathrm{d}u\\
&\le 8\eta\left(\sup_{u\in\mathbb{R}^d}\max_{1\le t\le\lfloor \eta v_T^{-2}\rfloor}|e^{\mathbf{i}u'Y_t}|_q\right)^2\eta^{-1}v_T^2\sum_{s=1}^{\lfloor \eta v_T^{-2}\rfloor}\sum_{t=1}^{\lfloor \eta v_T^{-2}\rfloor}\alpha(s-t)^{(q-2)/q}\int_{\mathbb{R}^d}W(u)\mathrm{d}u\\
&\le 8\eta\sum_{l=-\infty}^{\infty}\alpha(l)^{(q-2)/q}\int_{\mathbb{R}^d}W(u)\mathrm{d}u\\
&\le \mathcal{C},
\end{aligned}
$$

by analogous arguments in the proof of Lemma A.1. Hence, by the Markov inequality,

$$
\left[K_T(\eta)\right]^2 = O_P(1),
$$

for any fixed $\eta$.

Now, we show (ii). Without loss of generality, consider $0\le\eta_1<\eta_2\le C$. By the Minkowski inequality,

$$
|K_T(\eta_2)-K_T(\eta_1)| \le \left[\int_{\mathbb{R}^d}\left|v_T\sum_{t=\lfloor \eta_1 v_T^{-2}\rfloor+1}^{\lfloor \eta_2 v_T^{-2}\rfloor}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right]^{1/2}.
$$

Let $q>2$, for any $\epsilon>0$,

$$
\begin{aligned}
P(|K_T(\eta_2)-K_T(\eta_1)|\ge\epsilon) &\le P\left(\left[\int_{\mathbb{R}^d}\left|v_T\sum_{t=\lfloor \eta_1 v_T^{-2}\rfloor+1}^{\lfloor \eta_2 v_T^{-2}\rfloor}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right]^{q/2}\ge\epsilon^q\right)\\
&\le \frac{E\left[\int_{\mathbb{R}^d}\left|v_T\sum_{t=1}^{\lfloor(\eta_2-\eta_1)v_T^{-2}\rfloor}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right]^{q/2}}{\epsilon^q},
\end{aligned}
$$

where the last equality is by stationarity and the Markov inequality. By Jensen's inequality,

$$
E\left[\int_{\mathbb{R}^d}\left|v_T\sum_{t=1}^{\lfloor(\eta_2-\eta_1)v_T^{-2}\rfloor}\varepsilon_t(u)\right|^2 W(u)\mathrm{d}u\right]^{q/2} \le v_T^q\int_{\mathbb{R}^d}E\left|\sum_{t=1}^{\lfloor(\eta_2-\eta_1)v_T^{-2}\rfloor}\varepsilon_t(u)\right|^q W(u)\mathrm{d}u
$$

$$\begin{aligned} &\leq v_T^q \mathcal{C}\left[(\eta_2-\eta_1)v_T^{-2}\right]^{q/2} \\ &= \mathcal{C}|\eta_2-\eta_1|^{q/2}, \end{aligned}$$

where the last equality is by Theorem 2 of Yokoyama (1980) given the mixing condition in Assumption A.1(ii), $\sup_{u\in\mathbb{R}^d}|\varepsilon_t(u)|<\mathcal{C}$, and the integrability of $W(u)$ in Assumption A.2. Hence, by Theorem 15.6 of Billingsley (1968), $K_T(\eta)$ is tight. Thus, (ii) is established. It follows

$$\sup_{\eta\in[0,C]} K_T(\eta)=O_P(1).$$

That completes the proof. ■

**Proof of Lemma** A.3 By the definition of $d_t(u)=\hat{\psi}_k(u)-\psi_j^0(u)$, for $t\in[\hat{T}_{k-1}+1,\hat{T}_k]\cap[T_{j-1}^0+1,T_j^0]$ with $k,j=1,2,...,M^0+1$,

$$\begin{aligned} \frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d}\mathrm{Re}\left[\varepsilon_t(u)d_t(u)^*\right]W(u)\mathrm{d}u &= \frac{1}{T}\sum_{k=1}^{M^0+1}\sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k}\int_{\mathbb{R}^d}\mathrm{Re}\left[\varepsilon_t(u)\hat{\psi}_k(u)^*\right]W(u)\mathrm{d}u \\ &\quad -\frac{1}{T}\sum_{j=1}^{M^0+1}\sum_{t=T_{j-1}^0+1}^{T_j^0}\int_{\mathbb{R}^d}\mathrm{Re}\left[\varepsilon_t(u)\psi_j^0(u)^*\right]W(u)\mathrm{d}u \\ &\equiv Q_1-Q_2. \end{aligned}$$

Consider $Q_1$. By the definition of $\hat{\psi}_k(u)$,

$$\begin{aligned} Q_1 &= \frac{1}{T}\sum_{k=1}^{M^0+1}\sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k}\int_{\mathbb{R}^d}\mathrm{Re}\left\{\varepsilon_t(u)\left[\frac{1}{\hat{T}_k-\hat{T}_{k-1}}\sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k}\left(\phi_t^0(u)+\varepsilon_t(u)\right)\right]^*\right\}W(u)\mathrm{d}u \\ &= \frac{1}{T}\sum_{k=1}^{M^0+1}\int_{\mathbb{R}^d}\mathrm{Re}\left\{\left[\sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k}\varepsilon_t(u)\right]\left[\frac{1}{\hat{T}_k-\hat{T}_{k-1}}\sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k}\phi_t^0(u)\right]^*\right\}W(u)\mathrm{d}u \\ &\quad +\frac{1}{T}\sum_{k=1}^{M^0+1}\frac{1}{\hat{T}_k-\hat{T}_{k-1}}\int_{\mathbb{R}^d}\left|\sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k}\varepsilon_t(u)\right|^2W(u)\mathrm{d}u \\ &\equiv Q_{11}+Q_{12}, \end{aligned}$$

where $\phi_t^0(u)\equiv E(e^{\mathbf{i}u'Y_t})$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} |Q_{11}| &\leq \frac{1}{T}\sum_{k=1}^{M^0+1}\left[\int_{\mathbb{R}^d}\left|\sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k}\varepsilon_t(u)\right|^2W(u)\mathrm{d}u\right]^{1/2}\left[\int_{\mathbb{R}^d}\left|\frac{1}{\hat{T}_k-\hat{T}_{k-1}}\sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k}\phi_t^0(u)\right|^2W(u)\mathrm{d}u\right]^{1/2} \\ &\leq \frac{1}{\sqrt{T}}\sum_{k=1}^{M^0+1}\left[\frac{1}{T}\int_{\mathbb{R}^d}\left|\sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k}\varepsilon_t(u)\right|^2W(u)\mathrm{d}u\right]^{1/2}\left[\int_{\mathbb{R}^d}\left[\frac{1}{\hat{T}_k-\hat{T}_{k-1}}\sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k}\left|\phi_t^0(u)\right|\right]^2W(u)\mathrm{d}u\right]^{1/2} \\ &= O_P(T^{-1/2}), \end{aligned}$$

where the last equality holds by Lemma A.1 and the fact that

$$\frac{1}{\hat{T}_k - \hat{T}_{k-1}} \sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k} \left|\phi_t^0(u)\right| \quad \le \quad \frac{1}{\hat{T}_k - \hat{T}_{k-1}} \sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k} E\left|e^{\mathbf{i}u'Y_t}\right| = 1.$$

Analogously, $Q_{12} = O_P(T^{-1})$ by Lemma A.1. Hence, $Q_1 = O_P(T^{-1/2})$. Consider $Q_2$.

$$\begin{aligned} Q_2 &= \frac{1}{T}\sum_{j=1}^{M^0+1}\sum_{t=T_{j-1}^0+1}^{T_j^0}\int_{\mathbb{R}^d}\mathrm{Re}\left[\varepsilon_t(u)\psi_j^0(u)^*\right]W(u)\mathrm{d}u \\ &= \frac{1}{\sqrt{T}}\sum_{j=1}^{M^0+1}\int_{\mathbb{R}^d}\mathrm{Re}\left\{\left[\frac{1}{\sqrt{T}}\sum_{t=T_{j-1}^0+1}^{T_j^0}\varepsilon_t(u)\right]\psi_j^0(u)^*\right\}W(u)\mathrm{d}u \\ &= O_P(T^{-1/2}), \end{aligned}$$

by Assumption A.4. Hence, we establish the desired result. ■

**Proof of Lemma** A.4 If there exists a break, say $r_j^0$, which is not consistently estimated, then for some positive probability $0 < c_0 < 1$, there exists a $\tau > 0$ such that no estimated break dates fall into the interval $\left[T_j^0 - \lfloor\kappa T\rfloor + 1, T_j^0 + \lfloor\kappa T\rfloor\right]$ for a subsequence of $T$. Suppose this interval is classified into the $k$-th regime, i.e., $\hat{T}_{k-1} \le T_j^0 - \lfloor\kappa T\rfloor + 1 < T_j^0 + \lfloor\kappa T\rfloor \le \hat{T}_k$, then

$$d_t(u) = \begin{cases} \hat{\psi}_k(u) - \psi_j^0(u), & \text{for } t \in \left[T_j^0 - \lfloor\kappa T\rfloor + 1, T_j^0\right]; \\ \hat{\psi}_k(u) - \psi_{j+1}^0(u), & \text{for } t \in \left[T_j^0 + 1, T_j^0 + \lfloor\kappa T\rfloor\right]. \end{cases} .$$

By the definition of $d_t$,

$$\begin{aligned} &\frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=1}^{T}\left|d_t(u)\right|^2 W(u)\mathrm{d}u \\ &\ge \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=T_j^0-\lfloor\kappa T\rfloor+1}^{T_j^0+\lfloor\kappa T\rfloor}\left|d_t(u)\right|^2 W(u)\mathrm{d}u \\ &= \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=T_j^0-\lfloor\kappa T\rfloor+1}^{T_j^0}\left|d_t(u)\right|^2 W(u)\mathrm{d}u + \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=T_j^0+1}^{T_j^0+\lfloor\kappa T\rfloor}\left|d_t(u)\right|^2 W(u)\mathrm{d}u \\ &= \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=T_j^0-\lfloor\kappa T\rfloor+1}^{T_j^0}\left|\hat{\psi}_k(u) - \psi_j^0(u)\right|^2 W(u)\mathrm{d}u + \frac{1}{T}\int_{\mathbb{R}^d}\sum_{t=T_j^0+1}^{T_j^0+\lfloor\kappa T\rfloor}\left|\hat{\psi}_k(u) - \psi_{j+1}^0(u)\right|^2 W(u)\mathrm{d}u \\ &= \tau\int_{\mathbb{R}^d}\left[\left|\hat{\psi}_k(u) - \psi_j^0(u)\right|^2 + \left|\hat{\psi}_k(u) - \psi_{j+1}^0(u)\right|^2\right]W(u)\mathrm{d}u \\ &\ge \frac{1}{2}\tau\int_{\mathbb{R}^d}\left|\psi_{j+1}^0(u) - \psi_j^0(u)\right|^2 W(u)\mathrm{d}u, \end{aligned}$$

by the triangle inequality. Define $\delta = \frac{1}{2}\eta$, we then get the desired result. ■

**Proof of Lemma** A.5 Without loss of generality, we assume that the true number of breaks is $M^0 = 2$

and the corresponding break fractions are $r_1^0$ and $r_2^0$, such that $r_1^0 < r_2^0$. Suppose (2.2) is solved by setting the number of breaks at $M = 1$. Denoting the estimated break fractions by $\hat{r}$, we want to show that $\hat{r}$ is consistent for $r_1^0$ or $r_2^0$. Let $r \in (\epsilon, 1-\epsilon)$, consider the following process

$$\begin{aligned} S_T(r) &= \frac{1}{T}\mathrm{SSGR}(rT) \\ &= \frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d} |\tilde{\varepsilon}_t(u,r)|^2 W(u)\mathrm{d}u, \end{aligned}$$

where $\tilde{\varepsilon}_t(u,r) = e^{\mathbf{i}u'Y_t} - \tilde{\psi}_k(u,r)$ for with $\tilde{\psi}_k(u,r) = \frac{1}{T_k - T_{k-1}}\sum_{t=T_{k-1}+1}^{T_k} e^{\mathbf{i}u'Y_t}$ being the ECF for $t \in [T_{k-1}, T_k]$, $k = 1, 2$. Following the convention that $T_0 = 0$ and $T_2 = T$, we note that $T_1 = \lfloor Tr \rfloor$.

Let $\tilde{d}_t(u,r) = \tilde{\psi}_k(u,r) - \psi_j^0(u)$ for $t \in [T_{k-1}+1, T_k] \cap [T_{j-1}^0 + 1, T_j^0]$ with $k = 1, 2$ and $j = 1, 2, 3$. Then it follows

$$\begin{aligned} S_T(r) &= \frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d} \left|\varepsilon_t(u) - \tilde{d}_t(u,r)\right|^2 W(u)\mathrm{d}u \\ &= \frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d} |\varepsilon_t(u)|^2 W(u)\mathrm{d}u + \frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d} \left|\tilde{d}_t(u,r)\right|^2 W(u)\mathrm{d}u \\ &\quad -\frac{2}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d} \mathrm{Re}\left[\varepsilon_t(u)\tilde{d}_t(u,r)^*\right] W(u)\mathrm{d}u \\ &= \frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d} |\varepsilon_t(u)|^2 W(u)\mathrm{d}u + S_T^{(1)}(r) - 2S_T^{(2)}(r), \end{aligned}$$

where

$$S_T^{(1)}(r) \equiv \frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d} \left|\tilde{d}_t(u,r)\right|^2 W(u)\mathrm{d}u,$$

and

$$S_T^{(2)}(r) \equiv \frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d} \mathrm{Re}\left[\varepsilon_t(u)\tilde{d}_t(u,r)^*\right] W(u)\mathrm{d}u.$$

Apparently, the first term $\frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d} |\varepsilon_t(u)|^2 W(u)\mathrm{d}u$ does not dependent on $r$. Then, it suffices to consider $S_T^{(1)}(r)$ and $S_T^{(2)}(r)$.

We first show that $S_T^{(2)}(r) = o_P(1)$ for all $r \in (\epsilon, 1-\epsilon)$.

$$\begin{aligned} S_T^{(2)}(r) &= \frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d} \mathrm{Re}\left[\varepsilon_t(u)\tilde{d}_t(u)^*\right] W(u)\mathrm{d}u \\ &= \frac{1}{T}\sum_{k=1}^{2}\sum_{t=T_{k-1}}^{T_k}\int_{\mathbb{R}^d} \mathrm{Re}\left[\varepsilon_t(u)\tilde{\psi}_k(u,r)^*\right] W(u)\mathrm{d}u \\ &\quad -\frac{1}{T}\sum_{j=1}^{3}\sum_{t=T_{j-1}^0}^{T_j^0}\int_{\mathbb{R}^d} \mathrm{Re}\left[\varepsilon_t(u)\psi_j^0(u)^*\right] W(u)\mathrm{d}u \\ &= Q_1(r) + Q_2, \text{ say.} \end{aligned}$$

Following analogous treatment in the proof of Lemma A.3, we have $Q_1(r) = O_P(T^{-1/2})$ for all $r \in (\epsilon, 1-\epsilon)$ and $Q_2 = O_P(T^{-1/2})$. Thus, $S_T^{(2)}(r) = o_P(1)$ for all $r \in (\epsilon, 1-\epsilon)$.

Now, we consider $S_T^{(1)}(r)$. When $r = r_1^0$, we have $\tilde{\psi}_1(u, r_1^0) = \psi_1^0(u) + \frac{1}{T_1^0}\sum_{t=1}^{T_1^0}\varepsilon_t(u)$, and $\tilde{\psi}_2(u, r_1^0) = \frac{1}{T-T_1^0}\sum_{t=T_1^0+1}^{T} e^{\mathbf{i}u'Y_t} = \frac{T_2^0-T_1^0}{T-T_1^0}\psi_2^0(u) + \frac{T-T_2^0}{T-T_1^0}\psi_3^0(u) + \frac{1}{T-T_1^0}\sum_{t=T_1^0+1}^{T}\varepsilon_t(u)$. Then, it follows

$$\begin{aligned}
S_T^{(1)}(r_1^0) &= \frac{1}{T}\sum_{t=1}^{T}\int_{\mathbb{R}^d}\left|\tilde{d}_t(u, r_1^0)\right|^2 W(u)\mathrm{d}u \\
&= \frac{1}{T}\int_{\mathbb{R}^d}\left[T_1^0\left|\tilde{\psi}_1(u, r_1^0) - \psi_1^0(u)\right|^2 + (T_2^0 - T_1^0)\left|\tilde{\psi}_2(u, r_1^0) - \psi_2^0(u)\right|^2 + (T - T_2^0)\left|\tilde{\psi}_2(u, r_1^0) - \psi_3^0(u)\right|^2\right] W(u)\mathrm{d}u \\
&= \int_{\mathbb{R}^d}\left\{r_1^0\left|\frac{1}{T_1^0}\sum_{t=1}^{T_1^0}\varepsilon_t(u)\right|^2 + (r_2^0 - r_1^0)\left|\frac{T-T_2^0}{T-T_1^0}\left[\psi_3^0(u) - \psi_2^0(u)\right] + \frac{1}{T-T_1^0}\sum_{t=T_1^0+1}^{T}\varepsilon_t(u)\right|^2\right. \\
&\quad \left. + (1 - r_2^0)\left|\frac{T_2^0 - T_1^0}{T - T_1^0}\left[\psi_2^0(u) - \psi_3^0(u)\right] + \frac{1}{T-T_1^0}\sum_{t=T_1^0+1}^{T}\varepsilon_t(u)\right|^2\right\} W(u)\mathrm{d}u \\
&= \frac{(1-r_2^0)(r_2^0 - r_1^0)}{1 - r_1^0}\int_{\mathbb{R}^d}|\psi_2^0(u) - \psi_3^0(u)|^2 W(u)\mathrm{d}u + o_P(1),
\end{aligned}$$

where the last equality is due to Assumption A.4(ii). Analogously, when $r = r_2^0$, we can show that

$$S_T^{(1)}(r_2^0) = \frac{r_1^0(r_2^0 - r_1^0)}{r_2^0}\int_{\mathbb{R}^d}|\psi_1^0(u) - \psi_2^0(u)|^2 W(u)\mathrm{d}u + o_P(1).$$

Without loss of generality, we assume

$$\frac{(1-r_2^0)(r_2^0 - r_1^0)}{1 - r_1^0}\left|\psi_2^0(u) - \psi_3^0(u)\right|^2 < \frac{r_1^0(r_2^0 - r_1^0)}{r_2^0}\left|\psi_1^0(u) - \psi_2^0(u)\right|^2. \tag{B.1}$$

That implies that $r_1^0$ is the asymptotic minimizer for $S_T(r)$ relatively to $r_2^0$. Now we divide the set $(\epsilon, 1-\epsilon)$ into three subsets $(\epsilon, r_1^0]$, $(r_1^0, r_2^0)$, and $[r_2^0, 1-\epsilon)$.

When $r \in (\epsilon, r_1^0]$, $T_1 < T_1^0$. It follows $\tilde{\psi}_1(u, r) = \psi_1^0(u) + \frac{1}{T_1}\sum_{t=1}^{T_1}\varepsilon_t(u)$, and $\tilde{\psi}_2(u, r) = \frac{1}{T-T_1}\sum_{t=T_1+1}^{T} e^{\mathbf{i}u'Y_t} = \frac{T_1^0 - T_1}{T-T_1}\psi_1^0(u) + \frac{T_2^0 - T_1^0}{T-T_1}\psi_2^0(u) + \frac{T-T_2^0}{T-T_1}\psi_3^0(u) + \frac{1}{T-T_1}\sum_{t=T_1+1}^{T}\varepsilon_t(u)$. Then, we have

$$\begin{aligned}
& S_T(r) - S_T(r_1^0) \\
=\ & S_T^{(1)}(r) - S_T^{(1)}(r_1^0) + o_P(1) \\
=\ & \frac{1}{T}\int_{\mathbb{R}^d}\left[T_1\left|\tilde{\psi}_1(u, r) - \psi_1^0(u)\right|^2 + (T_1^0 - T_1)\left|\tilde{\psi}_2(u, r) - \psi_1^0(u)\right|^2\right. \\
& \left. + (T_2^0 - T_1^0)\left|\tilde{\psi}_2(u, r) - \psi_2^0(u)\right|^2 + (T - T_2^0)\left|\tilde{\psi}_2(u, r) - \psi_3^0(u)\right|^2\right] W(u)\mathrm{d}u \\
& - \frac{(1-r_2^0)(r_2^0 - r_1^0)}{(1 - r_1^0)}\int_{\mathbb{R}^d}|\psi_2^0(u) - \psi_3^0(u)|^2 W(u)\mathrm{d}u + o_P(1) \\
=\ & \int_{\mathbb{R}^d}\left[r\left|\frac{1}{T_1}\sum_{t=1}^{T_1}\varepsilon_t(u)\right|^2 + (r_1^0 - r)\left|\frac{T_1^0 - T}{T - T_1}\psi_1^0(u) + \frac{T_2^0 - T_1^0}{T - T_1}\psi_2^0(u) + \frac{T - T_2^0}{T - T_1}\psi_3^0(u) + \frac{1}{T - T_1}\sum_{t=T_1+1}^{T}\varepsilon_t(u)\right|^2\right.
\end{aligned}$$

$$
\begin{aligned}
& +(r_2^0-r_1^0)\left|\frac{T_1^0-T_1}{T-T_1}\psi_1^0(u)+\frac{T_2^0-T_1^0-T+T_1}{T-T_1}\psi_2^0(u)+\frac{T-T_2^0}{T-T_1}\psi_3^0(u)+\frac{1}{T-T_1}\sum_{t=T_1+1}^{T}\varepsilon_t(u)\right|^2 \\
& \left.+(1-r_2^0)\left|\frac{T_1^0-T_1}{T-T_1}\psi_1^0(u)+\frac{T_2^0-T_1^0}{T-T_1}\psi_2^0(u)+\frac{T_1-T_2^0}{T-T_1}\psi_3^0(u)+\frac{1}{T-T_1}\sum_{t=T_1+1}^{T}\varepsilon_t(u)\right|^2\right] W(u)\mathrm{d}u \\
& -\frac{(1-r_2^0)(r_2^0-r_1^0)}{(1-r_1^0)}\int_{\mathbb{R}^d}|\psi_2^0(u)-\psi_3^0(u)|^2W(u)\mathrm{d}u+o_P(1) \\
= & \int_{\mathbb{R}^d}\left[(r_1^0-r)\left|\frac{1-r_1^0}{1-r}\left[\psi_2^0(u)-\psi_1^0(u)\right]+\frac{1-r_2^0}{1-r}\left[\psi_3^0(u)-\psi_2^0(u)\right]\right|^2\right. \\
& +(r_2^0-r_1^0)\left|\frac{r_1^0-r}{1-r}\left[\psi_1^0(u)-\psi_2^0(u)\right]+\frac{1-r_2^0}{1-r}\left[\psi_3^0(u)-\psi_2^0(u)\right]\right|^2 \\
& \left.+(1-r_2^0)\left|\frac{r_1^0-r}{1-r}\left[\psi_1^0(u)-\psi_2^0(u)\right]+\frac{r_2^0-r}{1-r}\left[\psi_2^0(u)-\psi_3^0(u)\right]\right|^2\right]W(u)\mathrm{d}u \\
& -\frac{(1-r_2^0)(r_2^0-r_1^0)}{(1-r_1^0)}\int_{\mathbb{R}^d}|\psi_2^0(u)-\psi_3^0(u)|^2W(u)\mathrm{d}u+o_P(1) \\
= & \int_{\mathbb{R}^d}\left\{\frac{(1-r_1^0)(r_1^0-r)}{1-r}\left|\psi_1^0(u)-\psi_2^0(u)\right|^2+\frac{(1-r_2^0)(r_2^0-r)}{1-r}\left|\psi_2^0(u)-\psi_3^0(u)\right|^2\right. \\
& \left.+2\frac{(r_1^0-r)(1-r_2^0)}{1-r}\left[\psi_1^0(u)-\psi_2^0(u)\right]\left[\psi_2^0(u)-\psi_3^0(u)\right]^*\right\}W(u)\mathrm{d}u \\
& -\frac{(1-r_2^0)(r_2^0-r_1^0)}{(1-r_1^0)}\int_{\mathbb{R}^d}|\psi_2^0(u)-\psi_3^0(u)|^2W(u)\mathrm{d}u+o_P(1) \\
= & \int_{\mathbb{R}^d}\left\{\frac{(1-r_1^0)(r_1^0-r)}{1-r}\left|\psi_1^0(u)-\psi_2^0(u)\right|^2+\frac{(1-r_2^0)^2(r_1^0-r)}{(1-r)(1-r_1^0)}\left|\psi_2^0(u)-\psi_3^0(u)\right|^2\right. \\
& \left.+2\frac{(r_1^0-r)(1-r_2^0)}{1-r}\left[\psi_1^0(u)-\psi_2^0(u)\right]\left[\psi_2^0(u)-\psi_3^0(u)\right]^*\right\}W(u)\mathrm{d}u+o_P(1) \\
\xrightarrow{p} & \frac{r_1^0-r}{(1-r)(1-r_1^0)}\int_{\mathbb{R}^d}\left|(1-r_1^0)[\psi_1^0(u)-\psi_2^0(u)]+(1-r_2^0)[\psi_2^0(u)-\psi_3^0(u)]\right|^2W(u)\mathrm{d}u\geq 0
\end{aligned}
$$

for distinct $\psi_1^0(u),\psi_2^0(u),\psi_3^0(u)$ and nonnegative $W(u)$.

For $r\in[r_2^0,1-\epsilon)$, by symmetry, we have $\lim_{T\to\infty}\left[S_T(r)-S_T(r_2^0)\right]\geq 0$ by analogous derviations as above. Then, by (B.1),

$$
\begin{aligned}
\lim_{T\to\infty}\left[S_T(r)-S_T(r_1^0)\right] &= \lim_{T\to\infty}\left[S_T(r)-S_T(r_2^0)\right]+\lim_{T\to\infty}\left[S_T(r_2^0)-S_T(r_1^0)\right] \\
&> 0,
\end{aligned}
$$

for all $r\in[r_2^0,1-\epsilon)$.

For $r\in(r_1^0,r_2^0)$, by tedious but analogous derivation,

$$
\begin{aligned}
S_T(r)-S_T(r_1^0) = & \int_{\mathbb{R}^d}\left[\frac{r_1^0(r-r_1^0)}{r}\left|\psi_1^0(u)-\psi_2^0(u)\right|^2+\frac{(r_2^0-r)(1-r_2^0)}{1-r}\left|\psi_2^0(u)-\psi_3^0(u)\right|^2\right]W(u)\mathrm{d}u \\
& -\frac{(1-r_2^0)(r_2^0-r_1^0)}{(1-r_1^0)}\int_{\mathbb{R}^d}|\psi_2^0(u)-\psi_3^0(u)|^2W(u)\mathrm{d}u+o_P(1)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} \left[ \frac{r_1^0(r-r_1^0)}{r} \left|\psi_1^0(u)-\psi_2^0(u)\right|^2 - \frac{(r-r_1^0)(1-r_2^0)^2}{(1-r)(1-r_1^0)} \left|\psi_2^0(u)-\psi_3^0(u)\right|^2 \right] W(u)\mathrm{d}u + o_P(1) \\
&= \frac{(r-r_1^0)r_2^0}{r} \int_{\mathbb{R}^d} \left[ \frac{r_1^0}{r_2^0} \left|\psi_1^0(u)-\psi_2^0(u)\right|^2 - \frac{r(1-r_2^0)^2}{r_2^0(1-r)(1-r_1^0)} \left|\psi_2^0(u)-\psi_3^0(u)\right|^2 \right] W(u)\mathrm{d}u + o_P(1) \\
&> \frac{(r-r_1^0)r_2^0}{r} \int_{\mathbb{R}^d} \left[ \frac{1-r_2^0}{1-r_1^0} \left|\psi_2^0(u)-\psi_3^0(u)\right|^2 - \frac{r(1-r_2^0)^2}{r_2^0(1-r)(1-r_1^0)} \left|\psi_2^0(u)-\psi_3^0(u)\right|^2 \right] W(u)\mathrm{d}u + o_P(1) \\
&= \frac{(r-r_1^0)r_2^0(1-r_2^0)}{r(1-r_1^0)} \left[ 1 - \frac{r(1-r_2^0)}{r_2^0(1-r)} \right] \int_{\mathbb{R}^d} \left|\psi_2^0(u)-\psi_3^0(u)\right|^2 W(u)\mathrm{d}u + o_P(1) \\
&= \frac{(r-r_1^0)r_2^0(1-r_2^0)}{r(1-r_1^0)} \left[ \frac{r_2^0-r}{r_2^0(1-r)} \right] \int_{\mathbb{R}^d} \left|\psi_2^0(u)-\psi_3^0(u)\right|^2 W(u)\mathrm{d}u + o_P(1) \\
&\xrightarrow{p} \frac{(r-r_1^0)(1-r_2^0)(r_2^0-r)}{r(1-r_1^0)(1-r)} \int_{\mathbb{R}^d} \left|\psi_2^0(u)-\psi_3^0(u)\right|^2 W(u)\mathrm{d}u > 0,
\end{aligned}
$$

where the first inequality is by (B.1) and the last one is by $r \in (r_1^0, r_2^0)$. Therefore, we have shown that $S_T(r)$ has a unique asymptotic global minimum at $r_1^0$ under (B.1). Besides, given $\hat{r}$ is the global minimizer for $S_T(r)$, we have $S_T(\hat{r}) \leq S_T(r_1)$ for all $T$. Thus, and the consistency of $\hat{r} = \hat{T}_1/T \xrightarrow{p} r_1^0$ holds. Analogously, we can show that $\hat{r} = \hat{T}_1/T \xrightarrow{p} r_2^0$ if we assume that $r_2^0$ is an asymptotic minimizer for $S_T(r)$ relatively to $r_1^0$. We note that the proof for $M^0 > 2$ and $M > 1$ is virtually quite similar, but much more tedious. For space, we neglect it. ■

**Proof of Lemma** A.6 When $M \geq M^0$, following the proof of Theorem 2.1, we can show that the collection $\{\hat{r}_k\}_{k=1}^M$ contains at least $M^0$ distinct estimated break fractions, say $\hat{r}_{k_1} < \cdots < \hat{r}_{k_{M^0}}$, such that $\hat{r}_{k_j} - r_j^0 = O_P(T^{-1})$ for $j = 1, 2, ..., M^0$, where $\{k_j\}_{j=1}^{M^0}$ is a subset of $\{k\}_{k=1}^M$.

Let $\hat{\sigma}^2(M) = T^{-1}\mathrm{SSGR}_M(\hat{r}_1, ..., \hat{r}_M)$, then

$$
\begin{aligned}
\hat{\sigma}^2(M) &= \frac{1}{T} \sum_{k=1}^{M+1} \sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k} \int_{\mathbb{R}^d} \left| e^{\mathbf{i}u'Y_t} - \hat{\psi}_k(u) \right|^2 W(u)\mathrm{d}u \\
&= \sum_{k=1}^{M+1} \hat{\sigma}_k^2(M),
\end{aligned}
$$

where $\hat{\psi}_k(u) = \frac{1}{\hat{T}_k - \hat{T}_{k-1}} \sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k} e^{\mathbf{i}u'Y_t}$ is the feasible ECF of $Y_t$ for $t \in [\hat{T}_{k-1}+1, \hat{T}_k]$ and

$$
\hat{\sigma}_k^2(M) \equiv \frac{1}{T} \sum_{t=\hat{T}_{k-1}+1}^{\hat{T}_k} \int_{\mathbb{R}^d} \left| e^{\mathbf{i}u'Y_t} - \hat{\psi}_k(u) \right|^2 W(u)\mathrm{d}u. \tag{B.2}
$$

Consider the break fractions $\hat{r}_{k_1} < \cdots < \hat{r}_{k_{M^0}}$, such that $\hat{r}_{k_j} - r_j^0 = O_P(T^{-1})$ for $j = 1, 2, ..., M^0$. Note that they divide the sample into $M^0 + 1$ segments, the $j$-th of which is $[\hat{T}_{k_{j-1}} + 1, \hat{T}_{k_j}]$, for $j = 1, ..., M^0 + 1$. Here, we let $k_0 = 0$, and $k_{M^0+1} = M + 1$. Following the convention, it follows $\hat{T}_{k_0} = \hat{T}_0 = 0$, and $\hat{T}_{k_{M^0+1}} = \hat{T}_{M+1} = T$.

It is obvious that each interval $[\hat{T}_{k_{j-1}} + 1, \hat{T}_{k_j}]$ can contain $L_j$ sub-intervals specified by $\hat{r}_{k_{j-1}} <$

$\hat{r}_{k_{j-1}+1} \cdots < \hat{r}_{k_j-1} < \hat{r}_{k_j}$, where $L_j \geq 1$. Then, we have

$$\hat{\sigma}^2(M) = \sum_{j=1}^{M^0+1} \sum_{l=1}^{L_j} \hat{\sigma}^2_{k_{j-1}+l}(M),$$

such that $k_{j-1} + L_j = k_j$ for $L_j \geq 1$ and $j = 1, .., M^0 + 1$.

Now, consider the SSGR under $M^0$ breaks.

$$\begin{aligned}\hat{\sigma}^2(M^0) &= \frac{1}{T}\text{SSGR}(M^0) \\ &= \sum_{l=1}^{M^0+1} \frac{1}{T} \sum_{t=\hat{T}_{l-1}+1}^{\hat{T}_l} \int_{\mathbb{R}^d} \left| e^{\mathbf{i}u'Y_t} - \hat{\psi}_l(u) \right|^2 W(u)\mathrm{d}u,\end{aligned}$$

where $\{\hat{T}_l\}_{l=1}^{M^0}$ is the collection of estimated breaks and $\hat{\psi}_l(u)$ is the feasible ECF given $\{\hat{T}_l\}_{l=1}^{M^0}$. Denote

$$\hat{\varsigma}^2_l(M_0) = \frac{1}{T} \sum_{t=\hat{T}_{l-1}+1}^{\hat{T}_l} \int_{\mathbb{R}^d} \left| e^{\mathbf{i}u'Y_t} - \hat{\psi}_l(u) \right|^2 W(u)\mathrm{d}u. \tag{B.3}$$

Then

$$\begin{aligned}\hat{\sigma}^2(M_0) - \hat{\sigma}^2(M) &= \sum_{j=1}^{M^0+1} \left[ \hat{\varsigma}^2_j(M_0) - \sum_{l=1}^{L_j} \hat{\sigma}^2_{k_{j-1}+l}(M) \right] \\ &= \sum_{j=1}^{M^0+1} \left[ \tilde{\varsigma}^2_j(M_0) - \sum_{l=1}^{L_j} \tilde{\sigma}^2_{k_{j-1}+l}(M) \right] + O_P(T^{-1}),\end{aligned}$$

where $\tilde{\varsigma}^2_j(M_0)$ and $\tilde{\sigma}^2_{k_{j-1}+l}(M)$ are defined as (B.2) and (B.3) with each $\hat{T}_{k_j}$ and $\hat{T}_j$ replaced by the corresponding true break date $T^0_j$ for $j = 1, ..., M^0$ . The last equality holds since $\hat{r}_{k_j} - r^0_j = O_P(T^{-1})$ for the estimated break fractions $\{\hat{r}_{k_j}\}_{j=1}^{M_0}$ under $M$ breaks and $\hat{r}_j - r^0_j = O_P(T^{-1})$ for the estimated break fractions $\{\hat{r}_j\}_{j=1}^{M^0}$ under $M^0$ breaks.

Now, it remains to show

$$\tilde{\varsigma}^2_j(M_0) - \sum_{l=1}^{L_j} \tilde{\sigma}^2_{k_{j-1}+l}(M) = O_P(T^{-1}).$$

Consider the $j$-th segment specified by the true break dates $[T^0_{j-1} + 1, T^0_j]$, then

$$\tilde{\varsigma}^2_j(M_0) = \frac{1}{T} \sum_{t=T^0_{j-1}+1}^{T^0_j} \int_{\mathbb{R}^d} \left| e^{\mathbf{i}u'Y_t} - \tilde{\psi}^0_j(u) \right|^2 W(u)\mathrm{d}u,$$

Given

$$e^{\mathbf{i}u'Y_t} = \psi^0_j(u) + \varepsilon_t(u),$$

for $t \in [T^0_{j-1} + 1, T^0_j]$. Obviously, $T\tilde{\varsigma}^2_j(M_0)$ is equivalent to the SSGR for a sample with no breaks, and $T\sum_{l=1}^{L_j} \tilde{\sigma}^2_{k_{j-1}+l}$ is equivalent to the SSGR by setting the number of breaks at $L^j$ for that sample. Following analogous arguments in the proof of Theorem 3.1, $T\tilde{\varsigma}_j^2(M_0) - T\sum_{l=1}^{L_j}\tilde{\sigma}^2_{k_{j-1}+l}$ is equivalent to the sup-$F$ test in Theorem 3.1. Thus,

$$\tilde{\varsigma}_j^2(M_0) - \sum_{l=1}^{L_j}\tilde{\sigma}^2_{k_{j-1}+l} = O_P(T^{-1}),$$

for all $j = 1, ..., M^0$. Therefore,

$$\hat{\sigma}^2(M_0) - \hat{\sigma}^2(M) = O_P(T^{-1}),$$

for all $M^0 \leq M \leq M_{\max}$, where $M_{\max}$ is a finite integer. ■

# C Additional Results

As mentioned in the paper, the trimodality phenomenon in the finite sample theory exists in our estimation for break fractions. When the considered DGPs have a small signal-to-noise ratio, Figure C.1 provides the histograms of estimated break fractions under DGPs.P1 and P2. We decrease the break size from 1 to 0.5 in mean under DGP.P1 and the break size from 2 to 1.5 in variance under DGP.P2, and the trimodality appears corresponding histograms when the sample size is small.

Table C.1 shows the effect of the penalty coefficients on the performance of our IC. We provide the results of our IC with the penalty coefficient $c_\rho = 1, 1.5$, and 2, under DGPs.P1-P5. Tables C.2 and C3 evaluate the impact of weighting functions on the finite sample power of our test. We report the empirical rejection rates of our test under DGPs.P1-P5 with different values $c$ for the uniform weighting function $U(-c, c)$ and different values $b$ for the normal weighting function $N(0, b)$ in Tables C.2 and C3, respectively.

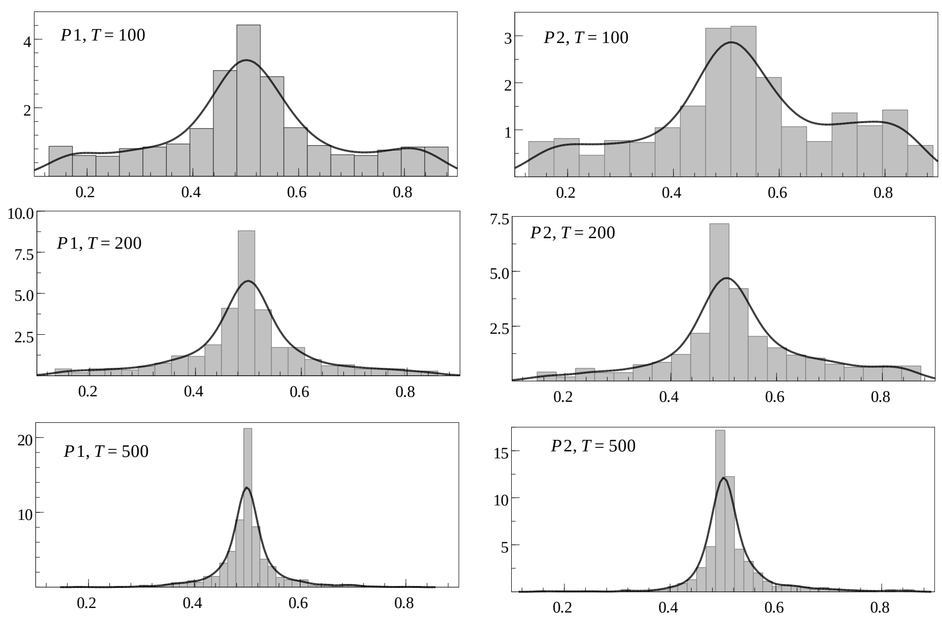


Figure C.1: Histograms of estimated break fractions under DGPs.P1 and P2 with small break size

Table C.1 Performance of our IC with different penalty coefficients

| DGP | $T$ | $c_\rho = 1$ | | $c_\rho = 1.5$ | | $c_\rho = 2$ | |
|---|---|---|---|---|---|---|---|
| | | Num | Perc | Num | Perc | Num | Perc |
| P1 | 100 | 1.254 | 79.4 | 0.925 | 88.9 | 0.732 | 72.8 |
| | 200 | 1.140 | 88.9 | 1.003 | 99.3 | 0.982 | 98.2 |
| | 500 | 1.072 | 94.0 | 1.004 | 99.6 | 1.000 | 100 |
| P2 | 100 | 0.842 | 66.4 | 0.358 | 35.6 | 0.105 | 10.5 |
| | 200 | 1.014 | 92.3 | 0.734 | 73.4 | 0.422 | 42.2 |
| | 500 | 1.025 | 97.7 | 1.00 | 100 | 0.991 | 99.1 |
| P3 | 100 | 0.980 | 61.0 | 0.415 | 39.3 | 0.179 | 17.9 |
| | 200 | 1.070 | 87.1 | 0.769 | 75.6 | 0.526 | 52.6 |
| | 500 | 1.039 | 96.5 | 1.004 | 99.4 | 0.992 | 99.0 |
| P4 | 100 | 1.093 | 91.1 | 0.583 | 57.1 | 0.107 | 10.7 |
| | 200 | 1.081 | 92.2 | 1.004 | 99.6 | 0.955 | 95.5 |
| | 500 | 1.048 | 95.7 | 1.001 | 99.9 | 1.00 | 100 |
| P5 | 100 | 1.774 | 68.1 | 1.032 | 3.10 | 0.948 | 1.0 |
| | 200 | 2.062 | 94.1 | 1.912 | 90.8 | 1.081 | 8.0 |
| | 500 | 2.034 | 96.6 | 2.002 | 99.8 | 2.000 | 100 |

Notes: (i) The main entries report the results based on 1000 replications. (ii) "Num" and "Perc" denote the "average number of breaks" and the "percentage of correct selection".

Table C.2: Power of tests under DGPs.P1-P5 with uniform weighting function

| | | $U_1, 1$ | | $U_1, 2$ | | $U_2, 1$ | | $U_2, 2$ | | $U_5, 1$ | | $U_5, 2$ | | $U_{10}, 1$ | | $U_{10}, 2$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% |
| P1 | $T = 100$ | 95.8 | 99.0 | 79.1 | 92.0 | 85.6 | 93.8 | 65.4 | 80.8 | 51.6 | 69.4 | 39.2 | 58.4 | 33.2 | 51.8 | 26.8 | 44.8 |
| | $T = 200$ | 100 | 100 | 96.0 | 99.4 | 98.2 | 100 | 88.6 | 98.2 | 91.6 | 97.4 | 76.8 | 89.2 | 72.8 | 88.4 | 57.0 | 75.8 |
| | $T = 500$ | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| P2 | $T = 100$ | 74.8 | 83.4 | 60.4 | 76.2 | 66.2 | 77.0 | 55.6 | 69.6 | 33.6 | 50.2 | 31.2 | 44.8 | 23.4 | 36.6 | 19.8 | 34.6 |
| | $T = 200$ | 98.8 | 99.6 | 95.4 | 98.4 | 94.4 | 97.6 | 89.8 | 94.4 | 72.0 | 82.4 | 62.4 | 75.4 | 50.2 | 65.4 | 40.6 | 52.8 |
| | $T = 500$ | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| P3 | $T = 100$ | 34.2 | 45.2 | 23.6 | 36.8 | 78.2 | 86.4 | 67.0 | 78.6 | 90.6 | 95.6 | 85.0 | 93.4 | 90.4 | 95.4 | 84.0 | 90.2 |
| | $T = 200$ | 59.8 | 70.8 | 44.8 | 58.2 | 97.4 | 98.8 | 95.0 | 97.6 | 99.6 | 99.6 | 99.2 | 99.4 | 99.8 | 100 | 99.4 | 100 |
| | $T = 500$ | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| P4 | $T = 100$ | 69.8 | 88.0 | 62.4 | 82.4 | 99.0 | 99.8 | 95.2 | 99.8 | 100 | 100 | 98.6 | 100 | 98.6 | 99.8 | 94.0 | 98.2 |
| | $T = 200$ | 97.4 | 99.2 | 90.0 | 97.8 | 100 | 100 | 99.4 | 100 | 100 | 100 | 99.8 | 100 | 100 | 100 | 99.8 | 100 |
| | $T = 500$ | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| P5 | $T = 100$ | 84.4 | 98.8 | 78.6 | 98.2 | 97.6 | 100 | 97.8 | 100 | 58.2 | 93.6 | 99.8 | 100 | 65.0 | 92.8 | 92.0 | 98.4 |
| | $T = 200$ | 98.2 | 99.8 | 97.4 | 99.8 | 99.6 | 100 | 100 | 100 | 93.8 | 99.6 | 100 | 100 | 97.8 | 100 | 100 | 100 |
| | $T = 500$ | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |

Notes: $U_c, M$ denotes our joint test with uniform$(-c, c)$ weighting function for the alternative hypothesis of $M$ breaks. The main entries report the percentage of rejections..

Table C.3: Power of tests under DGPs.P1-P5 with normal weighting function

| | | $N(0,1),1$ | | $N(0,1),2$ | | $N(0,2),1$ | | $N(0,2),2$ | | $N(0,5),1$ | | $N(0,5),2$ | | $N(0,10),1$ | | $N(0,10),2$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% | 5% | 10% |
| P1 | $T=100$ | 88.6 | 96.6 | 65.2 | 85.1 | 81.7 | 93.9 | 57.3 | 78.6 | 69.8 | 85.6 | 48.0 | 67.6 | 55.5 | 76.6 | 34.8 | 58.6 |
| | $T=200$ | 99.7 | 100 | 84.6 | 97.9 | 98.3 | 99.9 | 84.4 | 97.2 | 96.4 | 99.8 | 78.2 | 93.8 | 92.8 | 98.9 | 71.9 | 91.0 |
| | $T=500$ | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| P2 | $T=100$ | 67.9 | 81.1 | 55.7 | 71.3 | 62.6 | 77.8 | 49.8 | 64.1 | 53.7 | 68.9 | 42.1 | 56.0 | 39.5 | 57.1 | 28.7 | 43.7 |
| | $T=200$ | 98.0 | 99.3 | 91.7 | 96.3 | 95.7 | 98.2 | 90.3 | 95.3 | 92.4 | 95.2 | 83.4 | 90.4 | 84.9 | 92.0 | 74.5 | 85.9 |
| | $T=500$ | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| P3 | $T=100$ | 63.7 | 77.1 | 49.3 | 64.3 | 81.9 | 90.6 | 67.9 | 81.7 | 89.0 | 95.6 | 79.4 | 89.9 | 91.7 | 96.5 | 82.6 | 91.5 |
| | $T=200$ | 95.5 | 97.4 | 89.8 | 94.8 | 99.3 | 99.8 | 97.9 | 99.1 | 99.8 | 99.8 | 98.9 | 99.5 | 99.9 | 100 | 99.2 | 99.8 |
| | $T=500$ | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| P4 | $T=100$ | 90.7 | 98.3 | 76.3 | 94.4 | 96.3 | 99.8 | 88.9 | 98.9 | 97.8 | 100 | 92.5 | 99.6 | 97.9 | 100 | 92.1 | 99.7 |
| | $T=200$ | 100 | 100 | 98.6 | 100 | 99.9 | 100 | 99.4 | 100 | 100 | 100 | 99.8 | 100 | 100 | 100 | 99.6 | 100 |
| | $T=500$ | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| P5 | $T=100$ | 86.0 | 99.0 | 86.2 | 99.1 | 88.6 | 99.7 | 95.9 | 100 | 79.8 | 99.5 | 98.2 | 100 | 56.3 | 95.4 | 97.9 | 100 |
| | $T=200$ | 99.5 | 100 | 99.9 | 100 | 99.5 | 100 | 100 | 100 | 99.4 | 100 | 100 | 100 | 98.7 | 99.9 | 100 | 100 |
| | $T=500$ | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |

Notes: $N(0,b),M$ denotes our joint test with normal weighting function with variance $b$ in each dimension for the alternative hypothesis of $M$ breaks. The main entries report the percentage of rejections.

# Additional References


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