Online Supplementary Material to
"A wild bootstrap for dependent data"

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This document contains supplementary results published online alongside the main paper, "A wild bootstrap for dependent data". All references to section numbers are references to sections in the main paper. Note that there is no overlap between the labelling of equations below, asymptotic results as well as equations in the main text. We organized this online appendix as follows. First, in Appendix C1, we state an auxiliary lemma and its proof, which is useful for the proof of results in Section 2.4. Second, in Appendix C2, we state Lemmas C2.1 and C2.2, and their proofs, which are utilized to justify the moment conditions of examples of external random variables appearing in Section 2.5. Finally, in Appendix C3, we provide the proofs of results in Theorems 2.1 and 2.2.

Appendix C1: Auxiliary lemma for the proofs of results in Section 2.4

Lemma C1.1. *Let $\{Y_{Nt}^*, t = 1, 2, \ldots, N\}$ be a sequence of the WBDD pseudo-time series, we have that*

(a) $W_N^* = N^{-1/2} \sum_{j=1}^{Q} B_j \cdot \left(\sqrt{\ell} u_j\right) \equiv N^{-1} \sum_{j=1}^{Q} B_j^*$, *where* $B_j = \tilde{B}_j - \frac{1}{Q} \sum_{j=1}^{Q} \tilde{B}_j$, *with*

$$\tilde{B}_j = h\left(\bar{X}_N\right)' \frac{1}{\|w_\ell\|_2} \sum_{i=1}^{\ell} w_\ell\left(i\right) X_{N,i+j-1} = h\left(\bar{X}_N\right)' \frac{\|w_\ell\|_1}{\|w_\ell\|_2} \sum_{i=1}^{\ell} \frac{w_\ell\left(i\right)}{\|w_\ell\|_1} X_{N,i+j-1},$$

implying that $B_j = h\left(\bar{X}_N\right)' \frac{\|w_\ell\|_1}{\|w_\ell\|_2} \left(\sum_{i=1}^{\ell} \frac{w_\ell(i)}{\|w_\ell\|_1} X_{N,i+j-1} - \bar{X}_{\ell,w} \right)$.

(b) $\sigma_N^{*2} = Var^*\left(W_N^*\right) = \frac{Q}{N}\left(\ell Var\left(u\right)\right) \left(\frac{1}{Q} \sum_{j=1}^{Q} B_j^2 \right) = h\left(\bar{X}_N\right)' \check{\sigma}_{\ell,WBDD}^2 h\left(\bar{X}_N\right)$, *where*

$$\check{\sigma}_{\ell,WBDD}^2 \equiv \frac{Q}{N}\left(\ell Var\left(u\right)\right) \frac{1}{Q} \frac{\|w_\ell\|_1^2}{\|w_\ell\|_2^2} \sum_{j=1}^{Q} \left(\sum_{i=1}^{\ell} \frac{w_\ell\left(i\right)}{\|w_\ell\|_1} X_{N,i+j-1} - \bar{X}_{\ell,w} \right) \left(\sum_{i=1}^{\ell} \frac{w_\ell\left(i\right)}{\|w_\ell\|_1} X_{N,i+j-1} - \bar{X}_{\ell,w} \right)'.$$

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Proof of Lemma C1.1 part (a). Given the definition of W_N^* and equation (3), we have

$$\begin{aligned} W_N^* &= N^{-1/2}\sum_{t=1}^{N}\left(Y_{Nt}-\bar{Y}_{\ell,w}\right)\eta_t \\ &= N^{-1/2}\sum_{t=1}^{N}\left(\sum_{j=1}^{Q}\frac{w_\ell\left(t-j+1\right)}{\|w_\ell\|_2}\sqrt{\ell}u_j\right)\left(Y_{Nt}-\bar{Y}_{\ell,w}\right) \\ &= N^{-1/2}\sum_{t=1}^{N}\left(\sum_{j=1}^{Q}\left(\frac{w_\ell\left(t-j+1\right)}{\|w_\ell\|_2}\left(Y_{Nt}-\bar{Y}_{\ell,w}\right)\right)\sqrt{\ell}u_j\right). \end{aligned}$$

Given that $w_\ell\left(j\right)=0$ if $j\notin\{1,2,\ldots,\ell\}$, we can write

$$\begin{aligned} W_N^* &= N^{-1/2}\sum_{j=1}^{Q}\left(\sum_{i=1}^{\ell}\frac{w_\ell\left(i\right)}{\|w_\ell\|_2}\left(Y_{N,i+j-1}-\bar{Y}_{\ell,w}\right)\right)\sqrt{\ell}u_j \\ &= N^{-1/2}\sum_{j=1}^{Q}\left(\sum_{i=1}^{\ell}\frac{w_\ell\left(i\right)}{\|w_\ell\|_2}Y_{N,i+j-1}-\bar{Y}_{\ell,w}\sum_{i=1}^{\ell}\frac{w_\ell\left(i\right)}{\|w_\ell\|_2}\right)\sqrt{\ell}u_j \\ &= N^{-1/2}\sum_{j=1}^{Q}\left(\sum_{i=1}^{\ell}\frac{w_\ell\left(i\right)}{\|w_\ell\|_2}Y_{N,i+j-1}-\bar{Y}_{\ell,w}\frac{\|w_\ell\|_1}{\|w_\ell\|_2}\right)\sqrt{\ell}u_j. \end{aligned}$$

Thus, using the definition of $Y_{N,t}$ (see equation (1)) it follows that

$$\begin{aligned} W_N^* &= N^{-1/2}\sum_{j=1}^{Q}h\left(\bar{X}_N\right)'\left(\sum_{i=1}^{\ell}\frac{w_\ell\left(i\right)}{\|w_\ell\|_2}\left(X_{N,i+j-1}-\bar{X}_N\right)-\left(\bar{X}_{\ell,w}-\bar{X}_N\right)\frac{\|w_\ell\|_1}{\|w_\ell\|_2}\right)\sqrt{\ell}u_j \\ &= N^{-1/2}\sum_{j=1}^{Q}\underbrace{h\left(\bar{X}_N\right)'\left(\sum_{i=1}^{\ell}\frac{w_\ell\left(i\right)}{\|w_\ell\|_2}X_{N,i+j-1}-\bar{X}_{\ell,w}\frac{\|w_\ell\|_1}{\|w_\ell\|_2}\right)}_{\equiv B_j=\tilde{B}_j-\frac{1}{Q}\sum_{j=1}^{Q}\tilde{B}_j}\sqrt{\ell}u_j. \end{aligned}$$

Proof of Lemma C1.1 part (b). Given part (a) of Lemma C1.1, we can write

$$\begin{aligned} \sigma_N^{*2} = Var^*\left(W_N^*\right) &= Var^*\left(N^{-1/2}\sum_{j=1}^{Q}B_j\cdot\left(\sqrt{\ell}u_j\right)\right) \\ &= N^{-1}\sum_{j=1}^{Q}B_j^2 Var^*\left(\sqrt{\ell}u_j\right) \\ &= h\left(\bar{X}_N\right)'\check{\sigma}^2_{\ell,WBDD}h\left(\bar{X}_N\right), \end{aligned} \tag{C1.1}$$

where we used the fact that $B_j=h\left(\bar{X}_N\right)'\frac{\|w_\ell\|_1}{\|w_\ell\|_2}\left(\sum_{i=1}^{\ell}\frac{w_\ell(i)}{\|w_\ell\|_1}X_{N,i+j-1}-\bar{X}_{\ell,w}\right)$, and the definition of $\check{\sigma}^2_{\ell,WBDD}$ (given in part (b) of Lemma C1.1)).

Appendix C2: Auxiliary lemma for the proofs of results in Section 2.5

Lemma C2.1. *Let* $u_j = \ell^{-1} \sum_{i=(j-1)l+1}^{j\ell} \tilde{v}_i$, $j = 1, \ldots, Q$, *where* $\tilde{v}_i = v_i - \mathbb{E}(v_i)$ *with* $v_i \sim$ *i.i.d, we have that* $\mathbb{E}(u_j) = 0$, $\ell \mathbb{E}\left(u_j^2\right) = \mathbb{E}\left(v_i - \mathbb{E}(v_i)\right)^2$ *and* $\ell \mathbb{E}\left(u_j^3\right) = \ell^{-1} \mathbb{E}\left(v_i - \mathbb{E}(v_i)\right)^3$.

Proof of Lemma C2.1. Given the definition of u_j, $\tilde{v}_i$, the fact that v_i are i.i.d and using the linearity property of $\mathbb{E}(\cdot)$, we can write

$$
\begin{aligned}
\mathbb{E}(u_j) &= \mathbb{E}\left(\ell^{-1} \sum_{i=(j-1)l+1}^{j\ell} \tilde{v}_i\right) \\
&= \ell^{-1} \sum_{i=(j-1)l+1}^{j\ell} \mathbb{E}(\tilde{v}_i) = \ell^{-1} \sum_{i=(j-1)l+1}^{j\ell} \underbrace{\mathbb{E}\left(v_i - \mathbb{E}(v_i)\right)}_{=0} \\
&= 0.
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
\ell \mathbb{E}\left(u_j^2\right) &= \ell \mathbb{E}\left(\ell^{-1} \sum_{i=(j-1)l+1}^{j\ell} \tilde{v}_i\right)^2 \\
&= \ell^{-1}\left(\sum_{i=(j-1)l+1}^{j\ell} \mathbb{E}(\tilde{v}_i)^2 + \underbrace{\sum_{i \neq i'} \mathbb{E}(\tilde{v}_i)\, \mathbb{E}(\tilde{v}_{i'})}_{=0}\right) \\
&= \mathbb{E}\left(v_i - \mathbb{E}(v_i)\right)^2.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\ell \mathbb{E}\left(u_j^3\right) &= \ell \mathbb{E}\left(\ell^{-1} \sum_{i=(j-1)l+1}^{j\ell} \tilde{v}_i\right)^3 \\
&= \ell^{-2}\left(\sum_{i=(j-1)l+1}^{j\ell} \mathbb{E}(\tilde{v}_i)^3\right) \\
&= \ell^{-1} \mathbb{E}\left(v_i - \mathbb{E}(v_i)\right)^3.
\end{aligned}
$$

Lemma C2.2. *Consider the following class of two-point distributions* V_α *indexed on a parameter* $\alpha > 0$*, such that*

$$
V_\alpha = \begin{cases} \alpha, & \text{wih prob } p = \frac{1}{1+\alpha^2} \\ -\frac{1}{\alpha}, & \text{with prob } 1-p = \frac{\alpha^2}{1+\alpha^2} \end{cases}.
$$

We have that $\mathbb{E}(V_\alpha) = 0$, $\mathbb{E}(V_\alpha^2) = 1$ *and* $\mathbb{E}(V_\alpha^3) = \alpha - \frac{1}{\alpha}$.

Proof of Lemma C2.2. Given the definition of V_α, we have

$$\begin{aligned}\mathbb{E}(V_\alpha) &= \alpha p + \left(-\frac{1}{\alpha}\right)(1-p) \\ &= \frac{\alpha}{1+\alpha^2} - \frac{\alpha^2}{\alpha\left(1+\alpha^2\right)} \\ &= \frac{\alpha^2-\alpha^2}{\alpha\left(1+\alpha^2\right)} = 0.\end{aligned}$$

Next, we can write

$$\begin{aligned}\mathbb{E}(V_\alpha^2) &= \alpha^2 p + \left(-\frac{1}{\alpha}\right)^2(1-p) \\ &= \frac{\alpha^2}{1+\alpha^2} + \frac{\alpha^2}{\alpha^2\left(1+\alpha^2\right)} \\ &= \frac{1+\alpha^2}{1+\alpha^2} = 1.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathbb{E}(V_\alpha^3) &= \alpha^3 p + \left(-\frac{1}{\alpha}\right)^3(1-p) \\ &= \frac{\alpha^3}{1+\alpha^2} - \frac{\alpha^2}{\alpha^3\left(1+\alpha^2\right)} \\ &= \frac{\alpha^4-1}{\alpha\left(1+\alpha^2\right)} = \frac{\left(\alpha^2-1\right)\left(\alpha^2+1\right)}{\alpha\left(1+\alpha^2\right)} = \alpha - \frac{1}{\alpha}.\end{aligned}$$

Proof of Examples 2.3-2.7. First, note that the desired results for Examples 2.3 and 2.4, follow directly by using Lemma C2.1 and the fact that $\mathbb{E}(v_i) = 0$ and $\mathbb{E}(v_i^2) = 1$. Second, for Example 2.5, we use Lemma C2.2, where we let $v_i = V_\alpha$ with

$$\alpha = \frac{\ell + \sqrt{\ell^2+4}}{2}. \tag{C2.1}$$

Implying that

$$\begin{aligned}-\frac{1}{\alpha} &= -\frac{2}{\ell+\sqrt{\ell^2+4}} = -\frac{2\left(\ell-\sqrt{\ell^2+4}\right)}{\left(\ell+\sqrt{\ell^2+4}\right)\left(\ell-\sqrt{\ell^2+4}\right)} \\ &= -\frac{2\left(\ell-\sqrt{\ell^2+4}\right)}{\ell^2-\left(\ell^2+4\right)} \\ &= \frac{\ell-\sqrt{\ell^2+4}}{2},\end{aligned} \tag{C2.2}$$

and

$$
\begin{aligned}
p &= \frac{1}{1+\alpha^2} = \frac{1}{1+\left(\frac{\ell+\sqrt{\ell^2+4}}{2}\right)^2} = \frac{1}{\frac{4+\ell^2+2\ell\sqrt{\ell^2+4}+\ell^2+4}{4}} \\
&= \frac{4}{2\ell^2+8+2\ell\sqrt{\ell^2+4}} = \frac{2}{\ell^2+4+\ell\sqrt{\ell^2+4}} \\
&= \frac{2}{\ell^2+4+\ell\sqrt{\ell^2+4}} = \frac{2}{\sqrt{\ell^2+4}}\frac{1}{\ell+\sqrt{\ell^2+4}}\frac{\ell-\sqrt{\ell^2+4}}{\ell-\sqrt{\ell^2+4}} \\
&= \frac{2\left(\ell-\sqrt{\ell^2+4}\right)}{\sqrt{\ell^2+4}\,(\ell^2-(\ell^2+4))} = -\frac{\ell-\sqrt{\ell^2+4}}{2\sqrt{\ell^2+4}} \\
&= \frac{\sqrt{\ell^2+4}-\ell}{2\sqrt{\ell^2+4}}. \qquad \text{(C2.3)}
\end{aligned}
$$

Given (C2.1) and (C2.2), note that

$$\alpha - \frac{1}{\alpha} = \frac{\ell+\sqrt{\ell^2+4}}{2} + \frac{\ell-\sqrt{\ell^2+4}}{2} = \ell.$$

Consequently, we have $\mathbb{E}(v_i) = 0$, $\mathbb{E}(v_i^2) = 1$ and $\mathbb{E}(v_i^3) = \ell$. Thus, the requisite result follows given Lemma C2.1.

Next, for Example 2.6, notice that when $v_i \sim \text{i.i.d.}\Gamma(\alpha,\beta)$, the kth moment is given by

$$\mathbb{E}(v_i^k) = \frac{(\alpha+k-1)\ldots\alpha}{\beta^k},$$

implying that

$$\mathbb{E}(v_i) = \frac{\alpha}{\beta}, \quad \mathbb{E}(v_i^2) = \frac{(\alpha+1)\,\alpha}{\beta^2} \quad \text{and} \quad \mathbb{E}(v_i^3) = \frac{(\alpha+2)\,(\alpha+1)\,\alpha}{\beta^3}.$$

Hence, we have

$$\mathbb{E}(\tilde{v}_i) = 0, \quad \mathbb{E}(\tilde{v}_i^2) = \frac{\alpha}{\beta^2} \quad \text{and} \quad \mathbb{E}(\tilde{v}_i^3) = 2\frac{\alpha}{\beta^3}.$$

Therefore, by letting the parameters $\alpha = \frac{4}{\ell^2}$ and $\beta = \frac{2}{\ell}$, it follows that $\mathbb{E}(u_j) = 0$, $\ell\mathbb{E}(u_j^2) = 1$ and $\ell\mathbb{E}(u_j^3) = 1$.

Finally, for Example 2.7, given that by definition $u_j = v_j - \mathbb{E}(v_j)$, it follows that $\mathbb{E}(u_j) = 0$. Next, using the property of multinomial distribution, we have

$$
\begin{aligned}
\mathbb{E}(v_j) &= \frac{N}{\ell}\frac{1}{Q}, \\
\mathbb{E}(v_j^2) &= \frac{N}{\ell}\frac{1}{Q} + \frac{N}{\ell}\left(\frac{N}{\ell}-1\right)\frac{1}{Q^2}, \\
\mathbb{E}(v_j^3) &= \frac{N}{\ell}\frac{1}{Q} + 3\frac{N}{\ell}\left(\frac{N}{\ell}-1\right)\frac{1}{Q^2} + \frac{N}{\ell}\left(\frac{N}{\ell}-1\right)\left(\frac{N}{\ell}-2\right)\frac{1}{Q^3}.
\end{aligned}
$$

Thus, we can deduce that

$$\begin{aligned}\ell\mathbb{E}(u_j^2) &= \ell\left(\mathbb{E}(v_j^2)-(\mathbb{E}(v_j))^2\right)\\ &= \ell\left[\frac{N}{\ell}\frac{1}{Q}+\frac{N}{\ell}\left(\frac{N}{\ell}-1\right)\frac{1}{Q^2}-\left(\frac{N}{\ell}\frac{1}{Q}\right)^2\right]\\ &= \frac{N}{Q}+\frac{N}{Q}\left(\frac{N}{Q}\frac{1}{\ell}-\frac{1}{Q}\right)-\frac{1}{\ell}\left(\frac{N}{Q}\right)^2 \to 1,\end{aligned}$$

as $N\to\infty$, $\ell\to\infty$ such that $\ell/N=o(1)$ and $Q=N-\ell+1$. Similarly, we have

$$\begin{aligned}\ell\mathbb{E}(u_j^3) &= \ell\left[\mathbb{E}(v_j^3)-3\mathbb{E}(v_j^2)\mathbb{E}(v_j)+2\left[\mathbb{E}(v_j)\right]^3\right]\\ &= \ell\left[\frac{N}{\ell}\frac{1}{Q}+3\frac{N}{\ell}\left(\frac{N}{\ell}-1\right)\frac{1}{Q^2}+\frac{N}{\ell}\left(\frac{N}{\ell}-1\right)\left(\frac{N}{\ell}-2\right)\frac{1}{Q^3}\right]\\ &\quad+\ell\left[-3\left(\frac{N}{\ell}\frac{1}{Q}+\frac{N}{\ell}\left(\frac{N}{\ell}-1\right)\frac{1}{Q^2}\right)\left(\frac{N}{\ell}\frac{1}{Q}\right)+2\left(\frac{N}{\ell}\frac{1}{Q}\right)^3\right]\\ &= \frac{N}{Q}+3\frac{N}{Q}\left(\frac{N}{Q}\frac{1}{\ell}-\frac{1}{Q}\right)+\frac{N}{Q}\left(\frac{N}{Q}\frac{1}{\ell}-\frac{1}{Q}\right)\left(\frac{N}{Q}\frac{1}{\ell}-\frac{2}{Q}\right)\\ &\quad-3\left(\frac{N}{Q}+\frac{N}{Q}\left(\frac{N}{Q}\frac{1}{\ell}-\frac{1}{Q}\right)\right)\left(\frac{N}{Q}\frac{1}{\ell}\right)+2\frac{1}{\ell^2}\left(\frac{N}{Q}\right)^2.\end{aligned}$$

Hence, $\ell\mathbb{E}(u_j^3)\to 1$, as $N\to\infty$, $\ell\to\infty$ such that $\ell/N=o(1)$ and $Q=N-\ell+1$.

Appendix C3: Proof of Theorems 2.1 and 2.2.

Without lost of generality, in the proofs (of Theorems 2.1 and 2.2) for simplicity we will consider $\{X_{Nt}\}$ to be real-valued. The results for the multivariate case follow directly by showing that the assumptions are satisfied for linear combinations $\lambda' X_{Nt}$ for any nonzero $\lambda\in\mathbb{R}^d$.

Proof of Theorem 2.1 part (a). Recall that from part (b) of Lemma C1.1, we have $\sigma_N^{*2}=\left(h\left(\bar{X}_N\right)\right)^2\check{\sigma}^2_{\ell,WBDD}$, next using Theorem 3.1 of Künsch (1989), it follows that

$$\sigma_N^{*2}=\left(h\left(\bar{X}_N\right)\right)^2\underbrace{\frac{Q}{N}\ell Var\left(u\right)\sum_{\tau=-\ell+1}^{\ell-1}\frac{\upsilon_\ell\left(\tau\right)}{\upsilon_\ell\left(0\right)}\sum_{t=1}^{N-|\tau|}\beta_{N,t,\tau}\left(X_{Nt}-\bar{X}_{\ell,w}\right)\left(X_{N,t+|\tau|}-\bar{X}_{\ell,w}\right)}_{=\check{\sigma}^2_{\ell,WBDD}}. \quad \text{(C3.1)}$$

Given (C3.1), the fact that $\frac{Q}{N}\to 1$, $\ell Var\left(u\right)\to 1$, and $h\left(\bar{X}_N\right)-h\left(\bar{\mu}_N\right)\xrightarrow{P}0$, the rest of the proof contains two steps. In (1) we show that $\tilde{\sigma}_N^2-Var\left(S_N\right)\xrightarrow{P}0$, and in (2) we show that $\check{\sigma}^2_{\ell,WBDD}-\left(\tilde{\sigma}_N^2+U_N\right)\xrightarrow{P}0$, where $\tilde{\sigma}_N^2$ is defined as follows

$$\tilde{\sigma}_N^2=\frac{Q}{N}\ell Var\left(u\right)\sum_{\tau=-\ell+1}^{\ell-1}\frac{\upsilon_\ell\left(\tau\right)}{\upsilon_\ell\left(0\right)}\sum_{t=1}^{N-|\tau|}\beta_{N,t,\tau}\left(X_{Nt}-\mu_{Nt}\right)\left(X_{N,t+|\tau|}-\mu_{N,t+|\tau|}\right). \quad \text{(C3.2)}$$

For step 1, we also have two steps.

(i) We show that $\lim_{N\to\infty}\left|\mathbb{E}\left(\left(\frac{Q}{N}\ell Var\left(u\right)\right)^{-1}\tilde{\sigma}_N^2\right) - Var\left(S_N\right)\right| = 0.$

(ii) We show that $Var\left(\tilde{\sigma}_N^2\right) \to 0.$

Define $Z_{Nt} \equiv X_{Nt} - \mu_{Nt}$ and $R_{N,t}\left(\tau\right) = \mathbb{E}\left(Z_{Nt}Z_{N,t+\tau}\right)$. Given the definition of $\tilde{\sigma}_N^2$, we can write

$$\mathbb{E}\left(\left(\frac{Q}{N}\ell Var\left(u\right)\right)^{-1}\tilde{\sigma}_N^2\right) = \sum_{t=1}^{N}\beta_{N,t,0}R_{N,t}\left(0\right) + 2\sum_{\tau=1}^{\ell-1}\frac{\upsilon_\ell\left(\tau\right)}{\upsilon_\ell\left(0\right)}\sum_{t=1}^{N-\tau}\beta_{N,t,\tau}R_{N,t}\left(\tau\right),$$

and note that

$$Var\left(S_N\right) = \frac{1}{N}\sum_{t=1}^{N}R_{N,t}\left(0\right) + \frac{2}{N}\sum_{\tau=1}^{\ell-1}\sum_{t=1}^{N-\tau}R_{N,t}\left(\tau\right) + \frac{2}{N}\sum_{\tau=l}^{N-1}\sum_{t=1}^{N-\tau}R_{N,t}\left(\tau\right).$$

Then using the triangle inequality we have,

$$\begin{aligned}\left|\mathbb{E}\left(\left(\frac{Q}{N}\ell Var\left(u\right)\right)^{-1}\tilde{\sigma}_N^2\right) - Var\left(S_N\right)\right| &\leq \sum_{t=1}^{N}\left|\beta_{N,t,0} - N^{-1}\right|R_{Nt}\left(0\right) + \sum_{t=1}^{N}\left|\beta_{N,t,0} - N^{-1}\right|R_{Nt}\left(0\right) \\ &\quad + 2\sum_{\tau=1}^{\ell-1}\left|\frac{\upsilon_\ell\left(\tau\right)}{\upsilon_\ell\left(0\right)}\sum_{t=1}^{N-\tau}\beta_{N,t,\tau} - N^{-1}\right|R_{N,t}\left(\tau\right) + 2\sum_{\tau=l}^{N-1}N^{-1}\sum_{t=1}^{N-\tau}\left|R_{N,t}\left(\tau\right)\right| \\ &= o\left(1\right),\end{aligned}$$

where we used the same argument like Gonçalves and White (2002) to bound the terms in their equation (A.3). Specifically it is due to the assumed size conditions on α_k and υ_k and because, $\left|R_{N,t}\left(\tau\right)\right| \leq \Delta\left(5\alpha_{\left[\frac{\tau}{4}\right]}^{\left(\frac{1}{2}-\frac{1}{r}\right)} + \upsilon_{\left[\frac{\tau}{4}\right]}\right)$ (see Gallant and White, 1988, pp. 109-110).

To show that $Var\left(\tilde{\sigma}_N^2\right) \to 0$, define

$$\tilde{R}_{N,0}\left(\tau\right) = \sum_{t=1}^{N-|\tau|}\beta_{N,t,\tau}Z_{Nt}Z_{N,t+|\tau|},$$

and write

$$Var\left(\left(\frac{Q}{N}\ell Var\left(u\right)\right)^{-1}\tilde{\sigma}_N^2\right) = \sum_{\tau=-\ell+1}^{\ell-1}\sum_{\lambda=-\ell+1}^{\ell-1}\frac{\upsilon_\ell\left(\tau\right)\upsilon_\ell\left(\lambda\right)}{\upsilon_\ell^2\left(0\right)}Cov\left(\tilde{R}_{N,0}\left(\tau\right),\tilde{R}_{N,0}\left(\lambda\right)\right).$$

We show that $Var\left(\tilde{R}_{N,0}\left(\tau\right)\right) = O\left(\frac{1}{N}\right)$, which by Cauchy-Schwarz inequality implies that $Var\left(\tilde{\sigma}_N^2\right) =$

$O\left(\frac{\ell^2}{N}\right)$, since we have $\sum_{\tau=-\ell+1}^{\ell-1}\sum_{\lambda=-\ell+1}^{\ell-1}\frac{\upsilon_\ell(\tau)\upsilon_\ell(\lambda)}{\upsilon_\ell^2(0)} = \ell^2$, $\frac{Q}{N}\to 1$, and $\ell Var\left(u\right)\to 1$. Note that we can write,

$$\begin{aligned}
Var\left(\tilde{R}_{N,0}\left(\tau\right)\right) &= \sum_{t=1}^{N-|\tau|}\beta_{N,t,\tau}^2 Var\left(Z_{Nt}Z_{N,t+|\tau|}\right)\\
&+2\sum_{t=1}^{N-|\tau|}\sum_{s=t+1}^{N-|\tau|}\beta_{N,t,\tau}\beta_{N,s,\tau}Cov\left(Z_{Nt}Z_{N,t+|\tau|},Z_{Ns}Z_{N,s+|\tau|}\right)\\
&\leq \frac{1}{Q^2}\sum_{t=1}^{N-|\tau|}Var\left(Z_{Nt}Z_{N,t+|\tau|}\right)+\frac{2}{Q^2}\sum_{t=1}^{N-|\tau|}\sum_{s=t+1}^{N-|\tau|}Cov\left(Z_{Nt}Z_{N,t+|\tau|},Z_{Ns}Z_{N,s+|\tau|}\right)\\
&+\frac{2}{Q^2}\sum_{t=1}^{N-|\tau|}\sum_{s=t+|\tau|+1}^{N-|\tau|}Cov\left(Z_{Nt}Z_{N,t+|\tau|},Z_{Ns}Z_{N,s+|\tau|}\right)
\end{aligned}$$

given that $\beta_{N,t,\tau}\leq\frac{1}{Q}$ for all t and τ.

$$\begin{aligned}
Q^2 Var\left(\tilde{R}_{N,0}\left(\tau\right)\right) &\leq KN\left\{\Delta^2+\sum_{k=1}^{\infty}\alpha_{\left[\frac{k}{4}\right]}^{\frac{1}{2}-\frac{1}{r}}+\sum_{k=1}^{\infty}\upsilon^{\left[\frac{k}{4}\right]}+\sum_{k=1}^{\infty}\upsilon_{\left[\frac{k}{4}\right]}^{\frac{r-2}{2(r-1)}}\right\}\\
&+KN\left(|\tau|\,\alpha_{\left[\frac{|\tau|}{4}\right]}^{2\left(\frac{1}{2}-\frac{1}{r}\right)}+|\tau|\,\upsilon_{\left[\frac{|\tau|}{4}\right]}^2+2\,|\tau|\,\alpha_{\left[\frac{k}{4}\right]}^{\frac{1}{2}-\frac{1}{r}}\upsilon^{\left[\frac{|\tau|}{4}\right]}\right).
\end{aligned}$$

Thus, using arguments similar to that of Gonçalves and White (2002) to bound the terms in their equation (A.4), it follows that $Var\left(\tilde{R}_{N,0}\left(\tau\right)\right)\leq K\frac{N}{Q^2}$. Hence, $Var\left(\tilde{R}_{N,0}\left(\tau\right)\right)=O\left(\frac{1}{N}\right)$.

For step 2, define $S_{N,1}=\frac{Q}{N}\ell Var\left(u\right)\sum_{\tau=-\ell+1}^{\ell-1}\frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)}\sum_{t=1}^{N-|\tau|}\beta_{N,t,\tau}X_{Nt}X_{N,t+|\tau|}$, thus given (C3.1) and (C3.2), it follows that

$$\begin{aligned}
\check{\sigma}_{\ell,WBDD}^2 &= S_{N,1}+\frac{Q}{N}\ell Var\left(u\right)\sum_{\tau=-\ell+1}^{\ell-1}\frac{\upsilon_\ell\left(\tau\right)}{\upsilon_\ell\left(0\right)}\sum_{t=1}^{N-|\tau|}\beta_{N,t,\tau}\left(-\bar{X}_{\ell,w}X_{Nt}-\bar{X}_{\ell,w}X_{N,t+|\tau|}+\bar{X}_{\ell,w}^2\right), \text{ and}\\
\tilde{\sigma}_N^2 &= S_{N,1}+\frac{Q}{N}\ell Var\left(u\right)\sum_{\tau=-\ell+1}^{\ell-1}\frac{\upsilon_\ell\left(\tau\right)}{\upsilon_\ell\left(0\right)}\sum_{t=1}^{N-|\tau|}\beta_{N,t,\tau}\left(-\mu_{N,t+|\tau|}X_{Nt}-\mu_{Nt}X_{N,t+|\tau|}+\mu_{Nt}\mu_{N,t+|\tau|}\right).
\end{aligned}$$

Then, we have $\left(\frac{Q}{N}\ell Var\left(u\right)\right)^{-1}\left(\check{\sigma}_{\ell,WBDD}^2-\left(\tilde{\sigma}_N^2+U_N\right)\right)=A_{N1}+A_{N2}+A_{N3}+A_{N4}$, where

$$\left(\frac{Q}{N}\ell Var\left(u\right)\right)^{-1}\left(\check{\sigma}_{\ell,WBDD}^2-\tilde{\sigma}_N^2\right)=\sum_{\tau=-\ell+1}^{\ell-1}\frac{\upsilon_\ell\left(\tau\right)}{\upsilon_\ell\left(0\right)}\sum_{t=1}^{N-|\tau|}\beta_{N,t,\tau}\left(\begin{array}{c}-\bar{X}_{\ell,w}Z_{Nt}-\bar{X}_{\ell,w}\mu_{Nt}-\bar{X}_{\ell,w}Z_{N,t+|\tau|}\\-\bar{X}_{\ell,w}\mu_{N,t+|\tau|}+\mu_{N,t+|\tau|}Z_{Nt}\\+\mu_{Nt}Z_{N,t+|\tau|}+\bar{X}_{\ell,w}^2+\mu_{N,t+|\tau|}\mu_{Nt}\end{array}\right),$$

by adding and subtracting appropriately, we can write

$$\left(\frac{Q}{N}\ell Var\left(u\right)\right)^{-1}\left(\check{\sigma}_{\ell,WBDD}^2-\tilde{\sigma}_N^2\right)=A_{N1}+A_{N2}+A_{N3}+A_{N4},$$

where

$$A_{N1} = -\left(\bar{X}_{\ell,w} - \bar{\mu}_{\ell,w}\right) \sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \sum_{t=1}^{N-|\tau|} \beta_{N,t,\tau} \left(Z_{Nt} + Z_{N,t+|\tau|}\right),$$

$$A_{N2} = \sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \sum_{t=1}^{N-|\tau|} \beta_{N,t,\tau} \left(\mu_{Nt} - \bar{\mu}_{\ell,w}\right) Z_{N,t+|\tau|},$$

$$A_{N3} = \sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \sum_{t=1}^{N-|\tau|} \beta_{N,t,\tau} \left(\mu_{N,t+|\tau|} - \bar{\mu}_{\ell,w}\right) Z_{N,t},$$

$$A_{N4} = \sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \sum_{t=1}^{N-|\tau|} \beta_{N,t,\tau} \left(\bar{X}_{\ell,w}^2 - \left(\mu_{Nt} + \mu_{N,t+|\tau|}\right) \bar{X}_{\ell,w} + \mu_{Nt}\mu_{N,t+|\tau|}\right),$$

with $\bar{\mu}_{\ell,w} = \sum_{t=1}^{N} a_N(t)\,\mu_{Nt}$. We have that

$$\begin{aligned}
A_{N4} &= \sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \sum_{t=1}^{N-|\tau|} \beta_{N,t,\tau} \begin{pmatrix} \left(\bar{X}_{\ell,w} - \bar{\mu}_{\ell,w}\right)^2 + 2\left(\bar{X}_{\ell,w} - \bar{\mu}_{\ell,w}\right)\bar{\mu}_{\ell,w} \\ -\left(\mu_{Nt} + \mu_{N,t+|\tau|}\right)\left(\bar{X}_{\ell,w} - \bar{\mu}_{\ell,w}\right) + \bar{\mu}_{\ell,w}^2 \\ -\left(\mu_{Nt} + \mu_{N,t+|\tau|}\right)\bar{\mu}_{\ell,w} + \mu_{Nt}\mu_{N,t+|\tau|} \end{pmatrix} \\
&= U_N + \left(\bar{X}_{\ell,w} - \bar{\mu}_{\ell,w}\right)^2 \sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \sum_{t=1}^{N-|\tau|} \beta_{N,t,\tau} \\
&\quad + \left(\bar{X}_{\ell,w} - \bar{\mu}_{\ell,w}\right) \sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \sum_{t=1}^{N-|\tau|} \beta_{N,t,\tau} \left(2\bar{\mu}_{\ell,w} - \left(\mu_{Nt} + \mu_{N,t+|\tau|}\right)\right) \\
&= U_N + A_{N4}^{'},
\end{aligned}$$

where $\left(\frac{Q}{N}\ell Var(u)\right)^{-1} U_N = \sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \sum_{t=1}^{N-|\tau|} \beta_{N,t,\tau} \left(\mu_{Nt} - \bar{\mu}_{\ell,w}\right)\left(\mu_{N,t+|\tau|} - \bar{\mu}_{\ell,w}\right)$.

The rest of the proof follows closely that for Theorem 2.1 of Gonçalves and White (2002), however for completeness, we present the relevant details. We now show that $\bar{X}_{\ell,w} - \bar{\mu}_{\ell,w} = o_p\left(\ell^{-1}\right)$. Define $\phi_{Nt}(x) = \omega_{Nt}$, where $\omega_{Nt} \equiv \sum_{j=1}^{Q} \frac{w_\ell(t-j+1)}{\|w_\ell\|_1}$, and note that $\phi_{Nt}(\cdot)$ is uniformly Lipschitz continuous. Next, write $\bar{X}_{\ell,w} - \bar{\mu}_{\ell,w} = N^{-1}\sum_{t=1}^{N} Y_{Nt}$, where $Y_{Nt} = \phi_{Nt}(Z_{Nt})$ is a mean zero NED array on $\{V_t\}$ of the same size as Z_{Nt} by Theorem 17.12 of Davidson (1994), satisfying the same moment conditions. Hence, results follow by using the same argument as in Gonçalves and White (2002). In particular, by Lemma A.1 of Gonçalves and White (2002) $\left\{Y_{N,t}, \bar{\mathcal{F}}^t\right\}$ is a L_2-mixingale of size $-\frac{3r-2}{3(r-2)}$, and thus of size $-1/2$, with uniformly bounded constants, and by Lemma A.2 of Gonçalves and White (2002) $\mathbb{E}\left(\max_{1\leq j\leq N}\left(\sum_{t=1}^{j} Y_{Nt}\right)^2\right) = O(N)$. By Chebyshev's inequality, for ϵ, $P\left[\ell\left(\bar{X}_{\ell,w} - \bar{\mu}_{\ell,w}\right) > 0\right] \leq \frac{\ell^2}{\epsilon^2 Q^2}\mathbb{E}\left(\sum_{t=1}^{N} Y_{Nt}\right)^2 = O\left(\frac{\ell^2 N}{Q^2}\right) = o(1)$, if $\ell = o\left(N^{1/2}\right)$. This implies $A_{N4}^{'} = o_p(1)$ and similarly $A_{N1} = o_p(1)$, given that we have $\sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \sum_{t=1}^{N-|\tau|} \beta_{N,t,\tau}\left(Z_{Nt} + Z_{N,t+|\tau|}\right) = O_p(\ell)$.

To prove that $A_{N3} = o_p(1)$, define $\mathcal{Y}_{Nt\tau} = \omega_{Nt\tau}\left(\mu_{N,t+|\tau|} - \bar{\mu}_{\ell,w}\right) Z_{N,t} = \phi_{Nt\tau}(Z_{N,t})$, where $\omega_{Nt\tau} \equiv$

$\frac{1}{\upsilon_l(\tau)} \sum_{j=1}^{Q} w_\ell (t-j+1) w_\ell (t-j+1+|\tau|)$ with $\tau < j$, and $\phi_{Nt\tau}(\cdot)$ is uniformly Lipschitz continuous. Arguing as in Gonçalves and White (2002), $\{\mathcal{Y}_{Nt\tau}, \bar{\mathcal{F}}^t\}$ is a L_2-mixingale of size $-1/2$, with uniformly, with mixingale constants $c^{\mathcal{Y}}_{Nt\tau} \le K \max\{\|w_\ell\|_{3r}, 1\}$ which are bounded uniformly in N, t, and τ. Thus,

$$
\begin{aligned}
P\left[\left|\sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \frac{1}{Q} \sum_{t=1}^{N-|\tau|} \mathcal{Y}_{Nt\tau}\right| \ge \epsilon\right] &\le \frac{1}{Q\epsilon}\left[\sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \mathbb{E}\left|\sum_{t=1}^{N-|\tau|} \mathcal{Y}_{Nt\tau}\right|\right] \\
&\le \frac{1}{Q\epsilon}\left[\sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \mathbb{E}\left(\left(\sum_{t=1}^{N-|\tau|} \mathcal{Y}_{Nt\tau}\right)^2\right)^{1/2}\right] \\
&\le \frac{1}{Q\epsilon}\left[\sum_{\tau=-\ell+1}^{\ell-1} \frac{\upsilon_\ell(\tau)}{\upsilon_\ell(0)} \left(K \sum_{t=1}^{N-|\tau|} \left(c^{\mathcal{Y}}_{Nt\tau}\right)^2\right)^{1/2}\right] K\frac{\ell N^{1/2}}{Q} \\
&= o(1)
\end{aligned}
$$

where the first inequality holds by Markov's inequality, the second inequality holds by Jensen's inequality, the third inequality holds by Lemma A.2 of Gonçalves and White (2002) applied to $\{\mathcal{Y}_{Nt\tau}\}$ for each τ, and the last inequality holds by the uniform boundedness of $c^{\mathcal{Y}}_{Nt\tau}$. The proof of $A_{N2} = o_p(1)$ follows similarly.

Proof of Theorem 2.1 part (b) Immediate from the proof of part (a) of Theorem 3.1.

Proof of Theorem 2.1 part (c) Immediate from the proof of part (b) of Theorem 3.1.

Proof of Theorem 2.2. First note that the assumed conditions are sufficient to ensure that S_N is asymptotically normal. Thus, a Taylor expansion of H around $\bar{\mu}_N$ confirms that $W_N \to^d N(0, \sigma^2_\infty)$. Therefore, to prove our result, we just need to show that the WBDD distribution is approximately close to $\Phi(x/\sigma_\infty)$. Define $Z_{Nt} \equiv X_{Nt} - \mu_{Nt}$ and its WBDD analogue $Z^*_{Nt} = X^*_{Nt} - \mu^*_{Nt}$. Note that, we can write

$$
W^*_N = N^{1/2}\left(\bar{Z}^*_N - \mathbb{E}^*\left(\bar{Z}^*_N\right)\right) = \sum_{j=1}^{Q} z^*_{Nj},
$$

where $z^*_{Nj} = \mathcal{Z}_{Nj} u_j - \mathbb{E}^*(\mathcal{Z}_{Nj} u_j)$, with $\mathcal{Z}_{Nj} \equiv h\left(\bar{X}_N\right) \tilde{\mathcal{Z}}_{Nj}$ such that $\tilde{\mathcal{Z}}_{Nj} = \left(\frac{\ell}{N}\right)^{1/2} \left(\sum_{i=1}^{\ell} \frac{w_\ell(i)}{\|w_\ell\|_2} Z_{N,i+j-1} - \bar{Z}_{\ell,w} \frac{\|w_\ell\|_1}{\|w_\ell\|_2}\right)$ Also note that $\mathbb{E}^*\left(z^*_{Nj}\right) = 0$ and that

$$
Var^*\left(\sum_{j=1}^{Q} z^*_{Nj}\right) = Var^*\left(W^*_N\right) \xrightarrow{P} \sigma^2_\infty,
$$

by Corollary 2.1. Moreover, since $z^*_{N1}, \ldots, z^*_{NQ}$ are conditionally independent, by the Berry-Esseen bound, for some small $\delta > 0$ and for some constant $K > 0$ (which changes from line to line),

$$
\sup_{x \in \mathbb{R}} \left|P^*\left(W^*_N \le x\right) - \Phi\left(x/\sigma_\infty\right)\right| \le K \sum_{j=1}^{Q} \mathbb{E}^*\left|z^*_{Nj}\right|^{2+\delta},
$$

which converges to zero in probability as $\ell \to \infty$, $N \to \infty$ such that $\ell = o\left(N^{1/2}\right)$. Hence, we may further write

$$\begin{aligned}
\sum_{j=1}^{Q} \mathbb{E}^* \left|z_{Nj}^*\right|^{2+\delta} &= \sum_{j=1}^{Q} \mathbb{E}^* \left| \left(\frac{\ell}{N}\right)^{1/2} \left(\mathcal{Z}_{Nj} u_j - \mathbb{E}^* \left(\mathcal{Z}_{Nj} u_j\right)\right) \right|^{2+\delta} \\
&\leq 2 \left(\frac{1}{N}\right) \left(\frac{\ell}{N}\right)^{\delta/2} \sum_{j=1}^{Q} \ell \mathbb{E}^* \left|\mathcal{Z}_{Nj} u_j\right|^{2+\delta} \\
&= 2 \left| \left(h\left(\bar{X}_N\right) - h\left(\bar{\mu}_N\right)\right) + h\left(\bar{\mu}_N\right)\right|^{2+\delta} \left(\frac{1}{N}\right) \left(\frac{\ell}{N}\right)^{\delta/2} \sum_{j=1}^{Q} \left|\tilde{\mathcal{Z}}_{Nj}\right|^{2+\delta} \underbrace{\left(\ell \mathbb{E}\left|u_j\right|^{2+\delta}\right)}_{\to C_\delta < \infty} \\
&\leq K \underbrace{\left(\left|\left(\bar{X}_N - \bar{\mu}_N\right)\right|^{2+\delta} + \left|h\left(\bar{\mu}_N\right)\right|^{2+\delta}\right)}_{=O_p(1)} \underbrace{\left(\frac{1}{N}\right) \left(\frac{\ell}{N}\right)^{\delta/2} \sum_{j=1}^{Q} \left|\tilde{\mathcal{Z}}_{Nj}\right|^{2+\delta}}_{\equiv \tilde{\mathcal{Z}}_{N,\delta}}, \qquad \text{(C3.3)}
\end{aligned}$$

where the first inequality follows from the C_r and the Jensen inequalities, whereas the second inequality follows from triangular and C_r inequalities, given Assumption WBDD and the Lipschitz condition on $h(\cdot)$. Next, note that

$$\begin{aligned}
\mathbb{E}\left|\tilde{\mathcal{Z}}_{N,\delta}\right| &= \mathbb{E}\left| \left(\frac{1}{N}\right) \left(\frac{\ell}{N}\right)^{\delta/2} \sum_{j=1}^{Q} \left|\tilde{\mathcal{Z}}_{Nj}\right|^{2+\delta} \right| \\
&\leq \left(\frac{1}{N}\right) \left(\frac{\ell}{N}\right)^{\delta/2} \sum_{j=1}^{Q} \mathbb{E}\left|\tilde{\mathcal{Z}}_{Nj}\right|^{2+\delta} \\
&= \left(\frac{\ell}{N}\right)^{\delta/2} \frac{N^{-1}}{\|w_\ell\|_2^{2+\delta}} \sum_{j=1}^{Q} \mathbb{E}\left| \sum_{i=1}^{\ell} w_\ell(i) Z_{N,i+j-1} - \|w_\ell\|_1 \bar{Z}_{\ell,w} \right|^{2+\delta} \\
&\leq \left(\frac{\ell}{N}\right)^{\delta/2} \frac{N^{-1}}{\|w_\ell\|_2^{2+\delta}} \sum_{j=1}^{Q} \left(\left\| \sum_{i=1}^{\ell} w_\ell(i) Z_{N,i+j-1} \right\|_{2+\delta} + \left\| \|w_\ell\|_1 \bar{Z}_{\ell,w} \right\|_{2+\delta} \right)^{2+\delta}, \qquad \text{(C3.4)}
\end{aligned}$$

where the first inequality follows from the triangle inequality, whereas the second inequality uses the Minkowski inequality. Under our assumptions,

$$\begin{aligned}
\left\| \sum_{i=1}^{\ell} w_\ell(i) Z_{N,i+j-1} \right\|_{2+\delta} &\leq \underbrace{\max_{1 \leq i \leq \ell} w_\ell(i)}_{\leq 1} \left\| \sum_{i=1}^{\ell} Z_{N,i+j-1} \right\|_{2+\delta} \\
&\leq \left\| \max_{1 \leq t \leq \ell} \left| \sum_{i=j}^{j+t-1} Z_{N,i} \right| \right\|_{2+\delta} \leq K \left(\sum_{i=j}^{j+\ell-1} c_{Ni}^{\in} \right)^{1/2} \leq K \ell^{1/2},
\end{aligned}$$

by Lemmas A.3 and A.4 of Gonçalves and White (2002), given that c_{Ni} are uniformly bounded. Similarly, $\left\| \|w_\ell\|_1 \bar{Z}_{\ell,w} \right\|_{2+\delta} = O\left(\ell^{1/2}\right)$, which from (C3.3) and (C3.4) implies $\sum_{j=1}^{Q} \mathbb{E}^* \left|z_{Nj}^*\right|^{2+\delta} = O\left(\left(\frac{\ell}{N}\right)^{\delta/2}\right) = o(1)$, since $\ell^{1/2} / \|w_\ell\|_2 = O(1)$.

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