Online Supplementary Material to "Trend Extraction from Economic Time Series with Missing Observations by Generalized Hodrick–Prescott Filters"*

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B.1 Introduction

This document provides online supplementary material to Yamada (2021). In Section B.2, the Matlab user-defined functions referred to in Yamada (2021) are provided. Section B.3 presents the figures referred to in Section 5 of Yamada (2021). Note that the equation numbers referred to in this document are the same as those in Yamada (2021).

B.2 Matlab functions

In this section, we provide five Matlab user-defined functions. We note that among such functions, `calcxhat_nast` requires CVX, a package for specifying and solving convex programs (CVX Research, Inc., 2011; Grant and Boyd, 2008).

B.2.1 A function to make $\boldsymbol{D}_n$ in (6)

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```
function D_n=makeD_n(tau_n)
   n=length(tau_n);
   D1=diff(eye(n));
   D2=D1(1:n-2,1:n-1);
   invDelta=diag(ones(n-1,1)./diff(tau_n));
   D_n=D2*invDelta*D1;
end
```

B.2.2 A function to calculate $\widehat{x}_n$ in (8)

```
function xhat_n=calcxhat_n(tau_n,y_n,lambda_n)
   D_n=makeD_n(tau_n);
   n=length(tau_n);
   xhat_n=(eye(n)+lambda_n*D_n'*D_n)\y_n;
end
```

B.2.3 A function to calculate the solution of the convex problem given by (26)–(27)

```
function xhat_nast=calcxhat_nast(tau_n,y_n,c)
   D_n=makeD_n(tau_n);
   n=length(tau_n);
   cvx_clear
   cvx_begin
      variables xhat_nast(n)
      minimize(sum((D_n*xhat_nast).^2))
      subject to
         sum((y_n-xhat_nast).^2)<=c
   cvx_end
end
```

B.2.4 A function to calculate $\widehat{x}_T$ in (11) and $S\widehat{x}_T$ in (12)

```
function [xhat_T,Sxhat_T]=calcxhat_T(tau_n,y_n,lambda_T)
   T=tau_n(end);
```

```
    I=eye(T);
    D_T=diff(I,2);
    S=I(tau_n,:);
    xhat_T=(S'*S+lambda_T*D_T'*D_T)\(S'*y_n);
    Sxhat_T=S*xhat_T;
end
```

B.2.5 A function to calculate $\widehat{\psi}$ in (A.38) and $\widehat{\phi}$ in (A.42)

```
function [psihat,phihat]=calcxhat_T2(tau_n,y_n,lambda_T)
    T=tau_n(end);
    n=length(tau_n);
    I_T=eye(T);
    I_n=eye(n);
    D_T=diff(I_T,2);
    tau_n_c=setdiff((1:T)',tau_n);
    S=I_T(tau_n,:);
    S_p=I_T(tau_n_c,:);
    R=eye(T-2)-D_T*S_p'*inv(S_p*D_T'*D_T*S_p')*S_p*D_T';
    F=R*D_T*S';
    psihat=(I_n+lambda_T*F'*F)\y_n;
    phihat=-inv(S_p*D_T'*D_T*S_p')*S_p*D_T'*D_T*S'*psihat;
end
```

B.3 Figures

In this section, we present the figures referred to in Section 5 of Yamada (2021). Figures B.1–B.5 correspond to the case where $n = 90$. Likewise, Figures B.6–B.10 (resp. Figures B.11–B.15) correspond to the case where $n = 50$ (resp. $n = 30$). Recall that Figures 5–9 in Yamada (2021) correspond to the case where $n = 70$.

From the figures in this material, we may confirm that the results shown in Yamada (2021) are also observable even for $n = 90, 50, 30$. For example, we may observe that $\widehat{\boldsymbol{x}}_n$ and $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ are almost the same. Nevertheless, we remark that the deviations between the gHP_T filter and the HP filter increase as n/T decreases. See Figure 9 in Yamada (2021) and Figures B.5, B.10, and B.15 in this material.

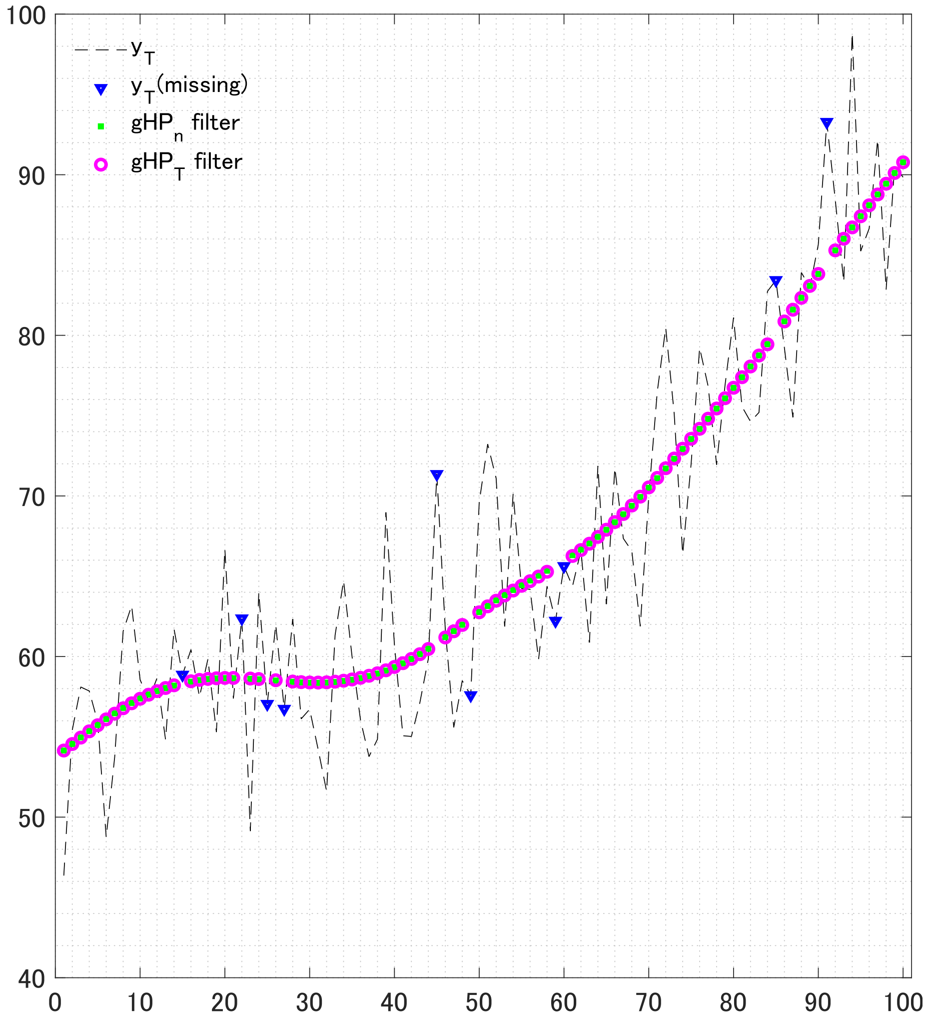

Figure B.1: For the explanation of $\mathtt{y_T}$, see Figure 5 in Yamada (2021). $\mathtt{y_T}$`(missing)` denotes $10(= 100 - 90)$ missing observations selected randomly from $\{y_2, \dots, y_{T-1}\}$. $\mathtt{gHP_T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 1600$. $\mathtt{gHP_T}$ `filter` denotes $\widehat{\boldsymbol{x}}_n$ in (8) estimated with $\lambda_n = 1491.99$, which is specified so that $\|\boldsymbol{y}_n - \widehat{\boldsymbol{x}}_n\|^2 = \|\boldsymbol{y}_n - \boldsymbol{S}\widehat{\boldsymbol{x}}_T\|^2$.

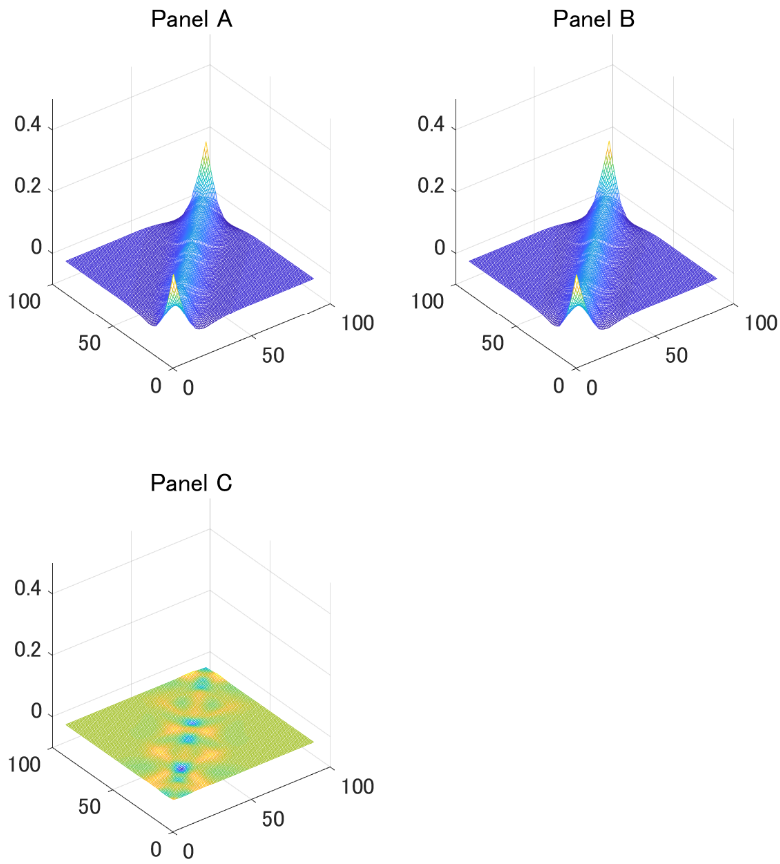

Figure B.2: Panel A (resp. Panel B) plots the smoother matrix corresponding to $\widehat{\boldsymbol{x}}_n$ (resp. $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$) in Figure B.1. Panel C plots their difference.

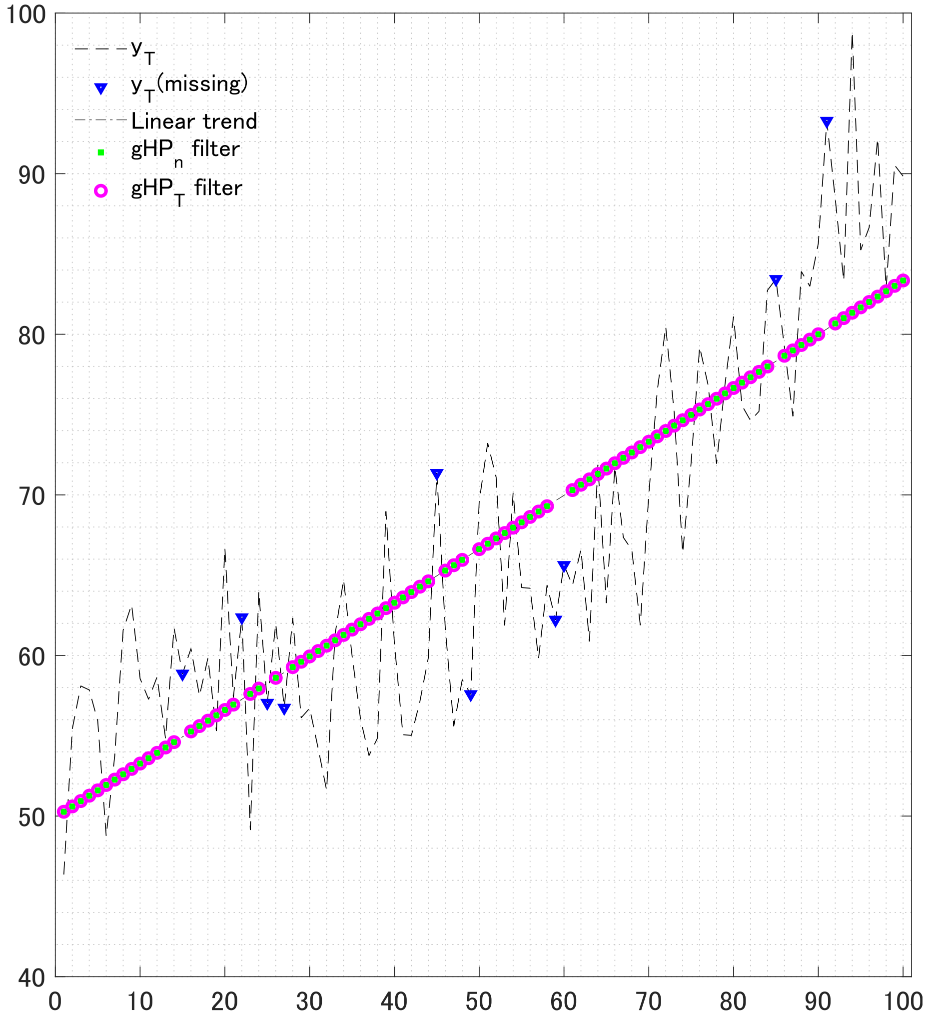

Figure B.3: For the explanation of $\texttt{y}_\texttt{T}$ and $\texttt{y}_\texttt{T}\texttt{(missing)}$, see Figure B.1. $\texttt{gHP}_\texttt{n}$ `filter` denotes $\widehat{\boldsymbol{x}}_n$ in (8) estimated with $\lambda_n = 10^8$ and $\texttt{gHP}_\texttt{T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 10^8$. `Linear trend` denotes $\boldsymbol{P}\boldsymbol{y}_n[= \boldsymbol{\Pi}_n(\boldsymbol{\Pi}_n'\boldsymbol{\Pi}_n)^{-1}\boldsymbol{\Pi}_n'\boldsymbol{y}_n]$.

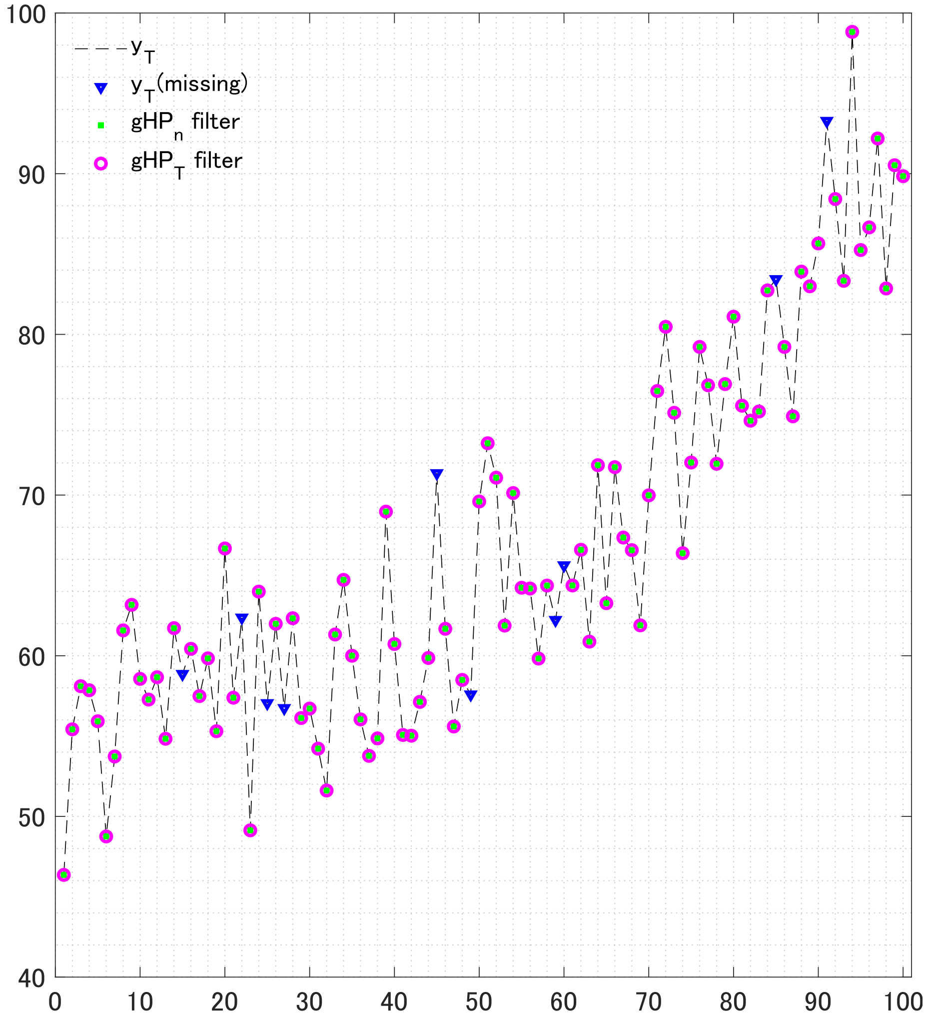

Figure B.4: For the explanation of $\mathtt{y_T}$ and $\mathtt{y_T(missing)}$, see Figure B.1. $\mathtt{gHP_n}$ `filter` denotes $\widehat{\boldsymbol{x}}_n$ in (8) estimated with $\lambda_n = 10^{-4}$ and $\mathtt{gHP_T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 10^{-4}$.

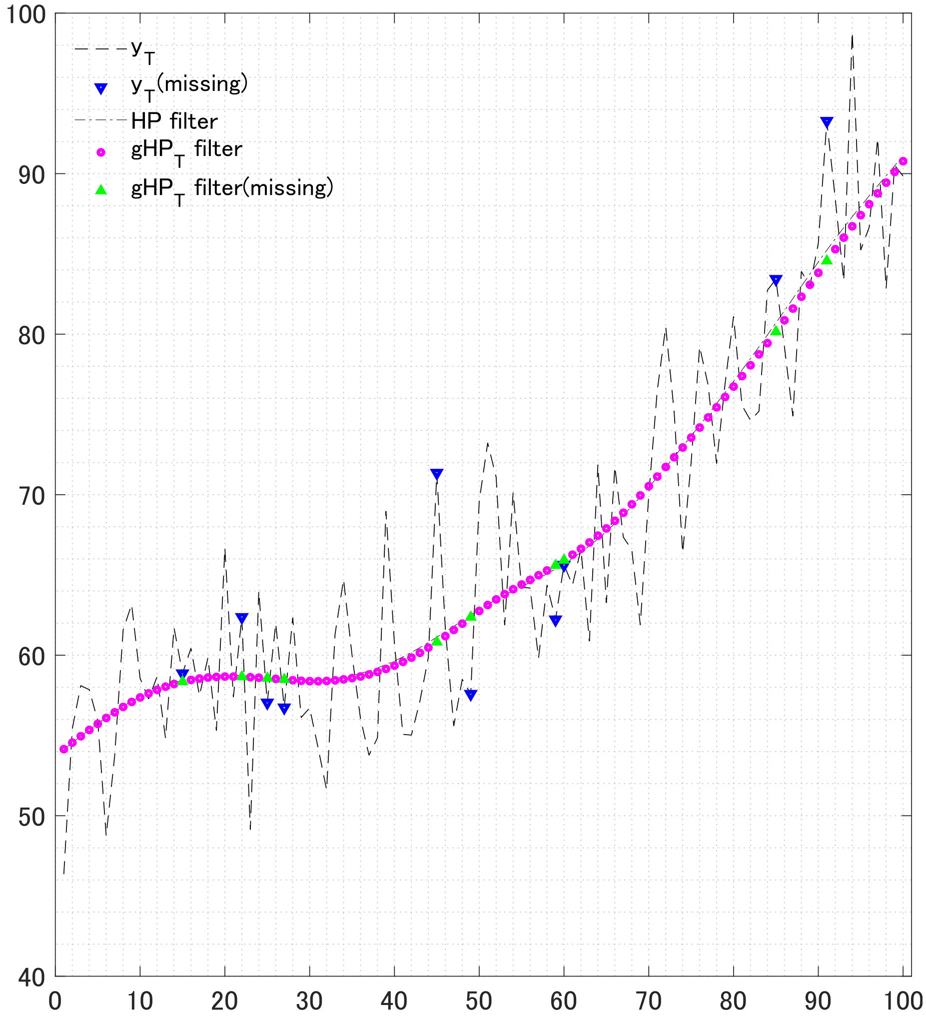

Figure B.5: For the explanation of $\mathtt{y_T}$ and $\mathtt{y_T(missing)}$, see Figure B.1. `HP filter` denotes $\widehat{\boldsymbol{x}}$ in (16), which is estimated with $\lambda = 1600$ from not only available observations but also missing observations. $\mathtt{gHP_T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 1600$ and $\mathtt{gHP_T}$ `filter(missing)` denotes $\boldsymbol{S}_\perp\widehat{\boldsymbol{x}}_T$ in (13) estimated with $\lambda_T = 1600$.

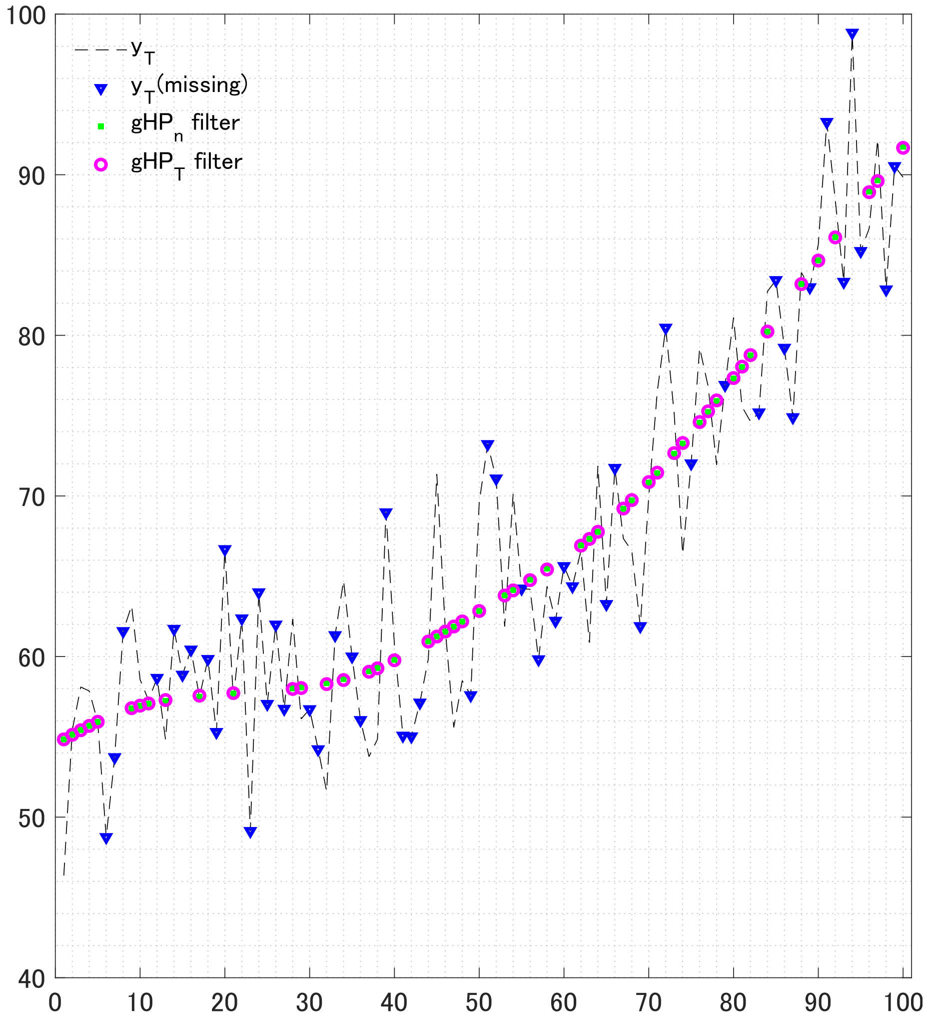

Figure B.6: For the explanation of $\mathtt{y_T}$, see Figure 5 in Yamada (2021). $\mathtt{y_T(missing)}$ denotes $50(= 100 - 50)$ missing observations selected randomly from $\{y_2, \dots, y_{T-1}\}$. $\mathtt{gHP_T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 1600$. $\mathtt{gHP_T}$ `filter` denotes $\widehat{\boldsymbol{x}}_n$ in (8) estimated with $\lambda_n = 870.14$, which is specified so that $\|\boldsymbol{y}_n - \widehat{\boldsymbol{x}}_n\|^2 = \|\boldsymbol{y}_n - \boldsymbol{S}\widehat{\boldsymbol{x}}_T\|^2$.

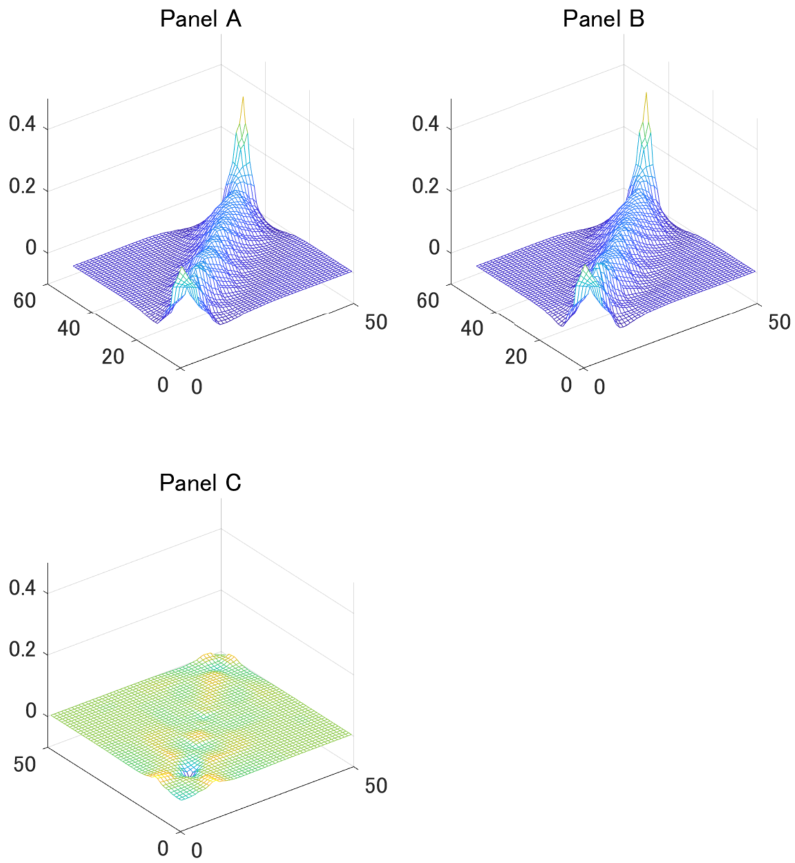

Figure B.7: Panel A (resp. Panel B) plots the smoother matrix corresponding to $\widehat{\boldsymbol{x}}_n$ (resp. $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$) in Figure B.6. Panel C plots their difference.

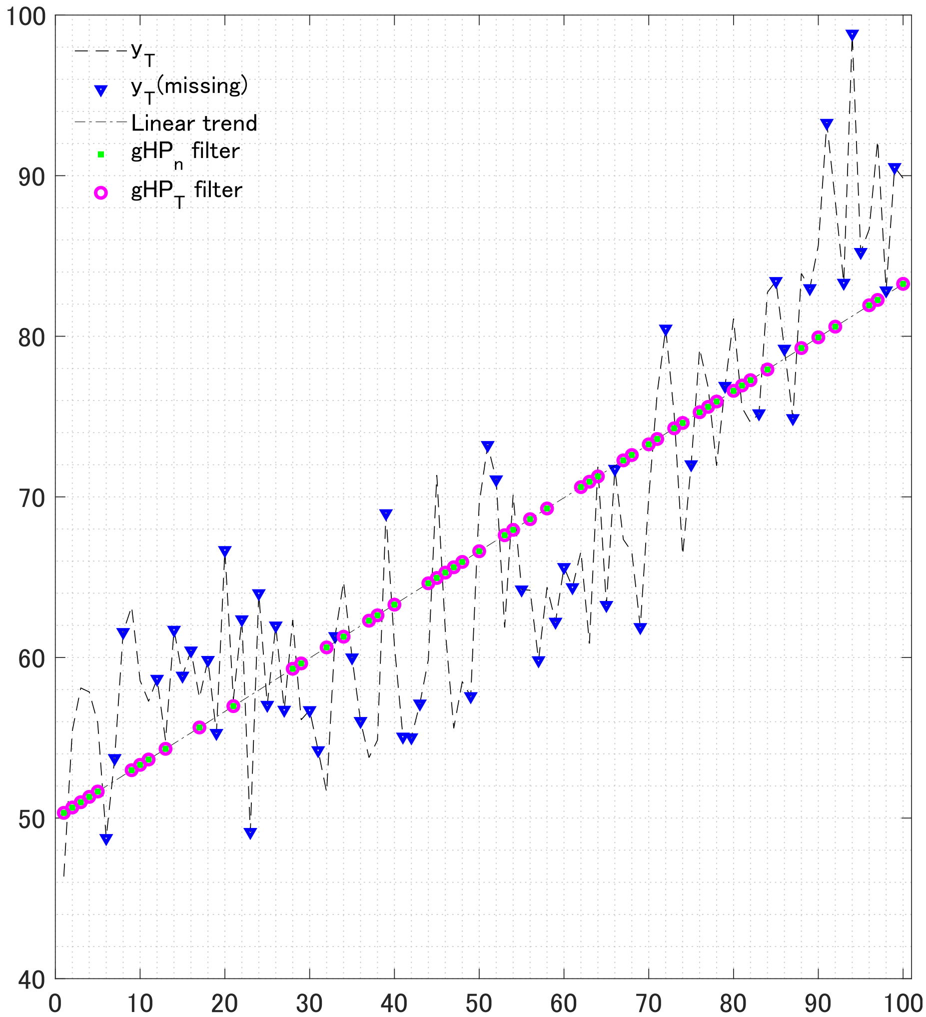

Figure B.8: For the explanation of $\mathtt{y_T}$ and $\mathtt{y_T(missing)}$, see Figure B.6. $\mathtt{gHP_n}$ `filter` denotes $\widehat{\boldsymbol{x}}_n$ in (8) estimated with $\lambda_n = 10^8$ and $\mathtt{gHP_T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 10^8$. `Linear trend` denotes $\boldsymbol{P}\boldsymbol{y}_n[= \boldsymbol{\Pi}_n(\boldsymbol{\Pi}_n'\boldsymbol{\Pi}_n)^{-1}\boldsymbol{\Pi}_n'\boldsymbol{y}_n]$.

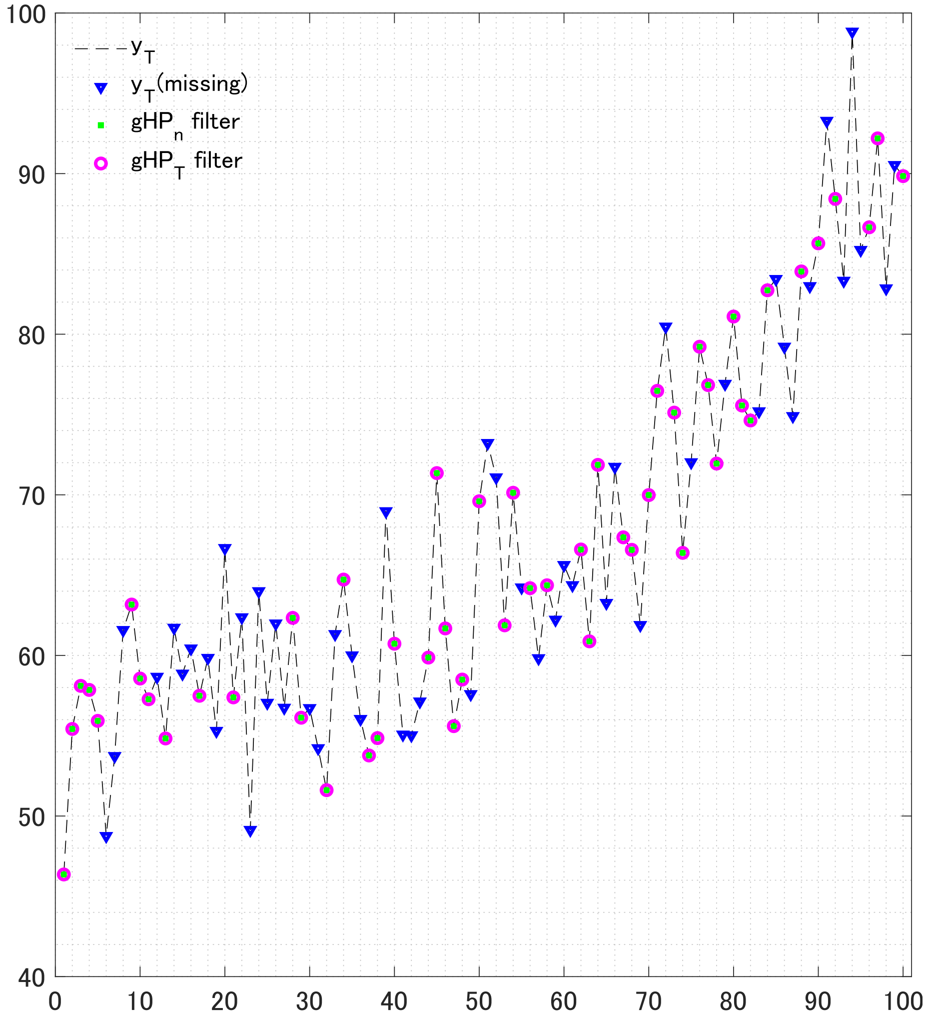

Figure B.9: For the explanation of $\mathtt{y_T}$ and $\mathtt{y_T(missing)}$, see Figure B.6. $\mathtt{gHP_n}$ `filter` denotes $\widehat{\boldsymbol{x}}_n$ in (8) estimated with $\lambda_n = 10^{-4}$ and $\mathtt{gHP_T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 10^{-4}$.

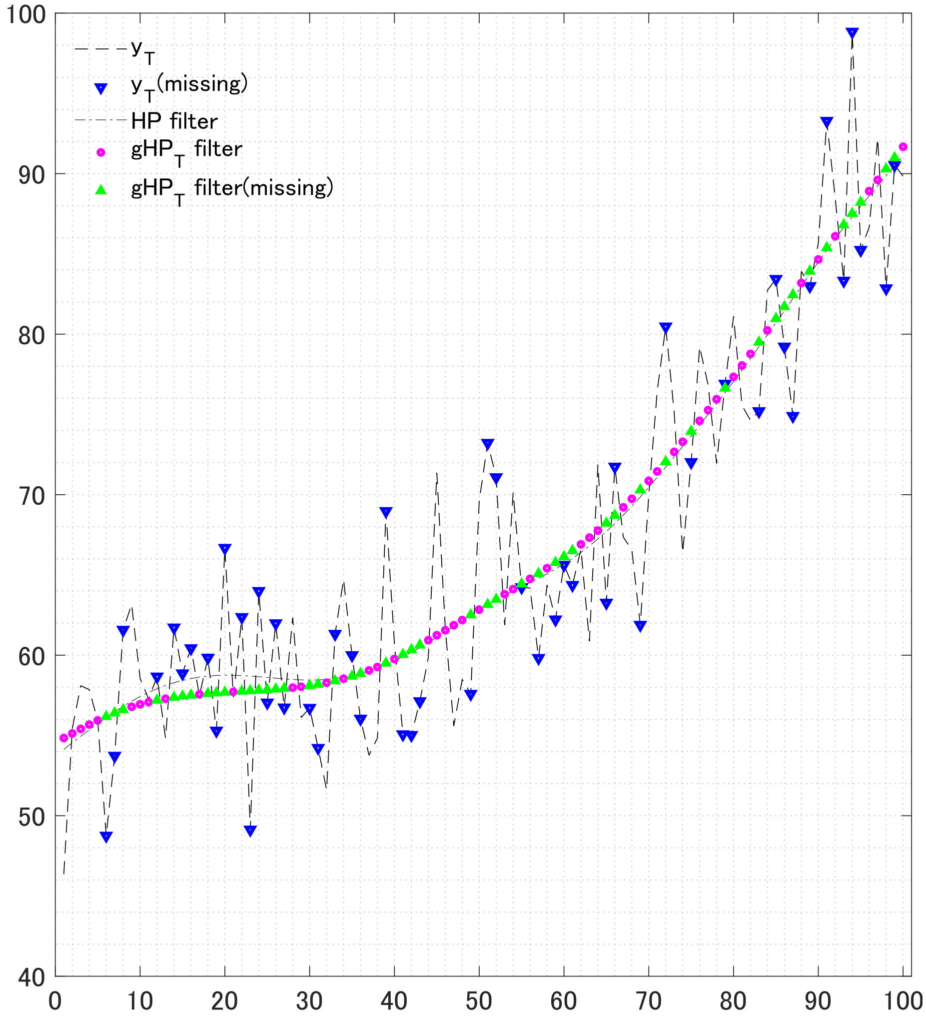

Figure B.10: For the explanation of $\mathtt{y_T}$ and $\mathtt{y_T(missing)}$, see Figure B.6. `HP filter` denotes $\widehat{\boldsymbol{x}}$ in (16), which is estimated with $\lambda = 1600$ from not only available observations but also missing observations. $\mathtt{gHP_T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 1600$ and $\mathtt{gHP_T}$ `filter(missing)` denotes $\boldsymbol{S}_\perp\widehat{\boldsymbol{x}}_T$ in (13) estimated with $\lambda_T = 1600$.

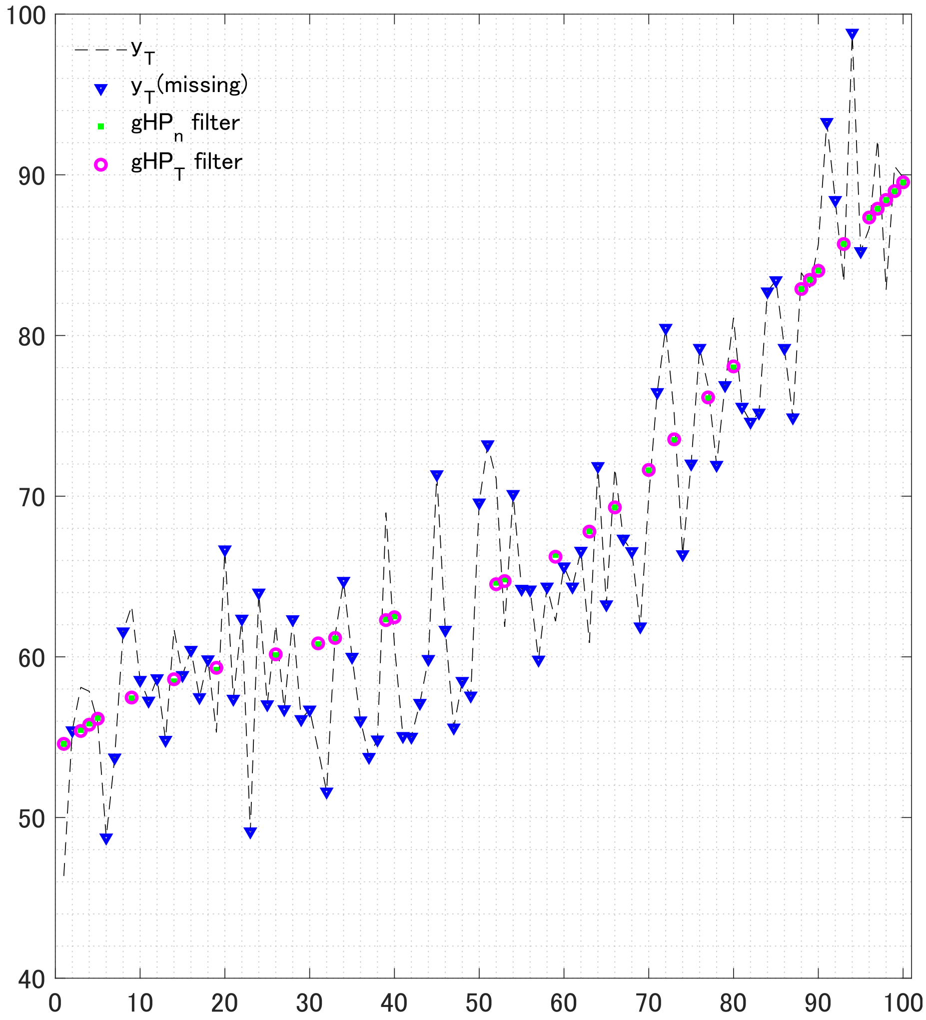

Figure B.11: For the explanation of $\mathtt{y_T}$, see Figure 5 in Yamada (2021). $\mathtt{y_T}$`(missing)` denotes $70(= 100 - 30)$ missing observations selected randomly from $\{y_2, \ldots, y_{T-1}\}$. `gHP`$_\mathtt{T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 1600$. `gHP`$_\mathtt{T}$ `filter` denotes $\widehat{\boldsymbol{x}}_n$ in (8) estimated with $\lambda_n = 464.34$, which is specified so that $\|\boldsymbol{y}_n - \widehat{\boldsymbol{x}}_n\|^2 = \|\boldsymbol{y}_n - \boldsymbol{S}\widehat{\boldsymbol{x}}_T\|^2$.

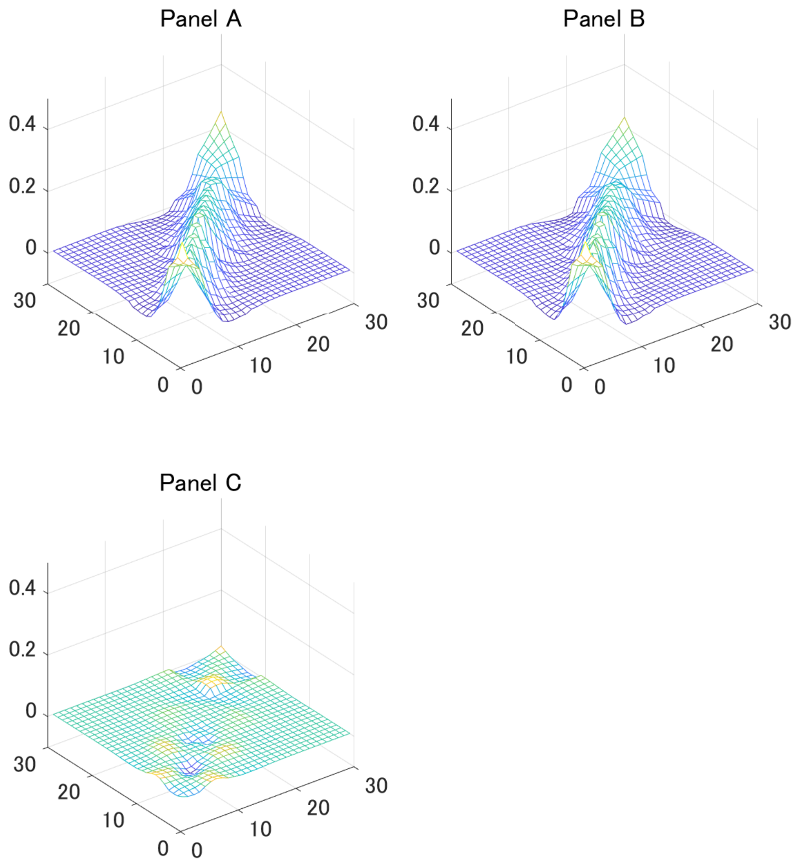

Figure B.12: Panel A (resp. Panel B) plots the smoother matrix corresponding to $\widehat{\boldsymbol{x}}_n$ (resp. $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$) in Figure B.11. Panel C plots their difference.

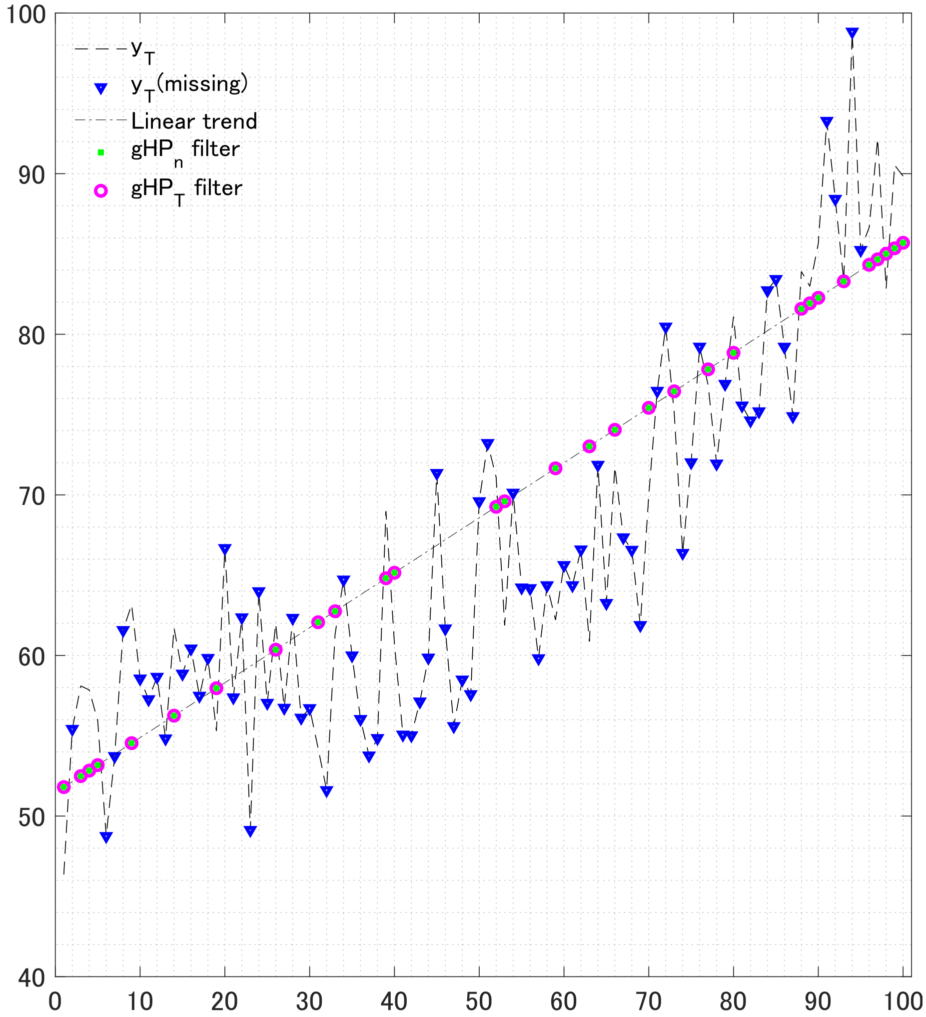

Figure B.13: For the explanation of $\mathtt{y_T}$ and $\mathtt{y_T(missing)}$, see Figure B.11. $\mathtt{gHP_n}$ `filter` denotes $\widehat{\boldsymbol{x}}_n$ in (8) estimated with $\lambda_n = 10^8$ and $\mathtt{gHP_T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 10^8$. `Linear trend` denotes $\boldsymbol{P}\boldsymbol{y}_n [= \boldsymbol{\Pi}_n(\boldsymbol{\Pi}_n'\boldsymbol{\Pi}_n)^{-1}\boldsymbol{\Pi}_n'\boldsymbol{y}_n]$.

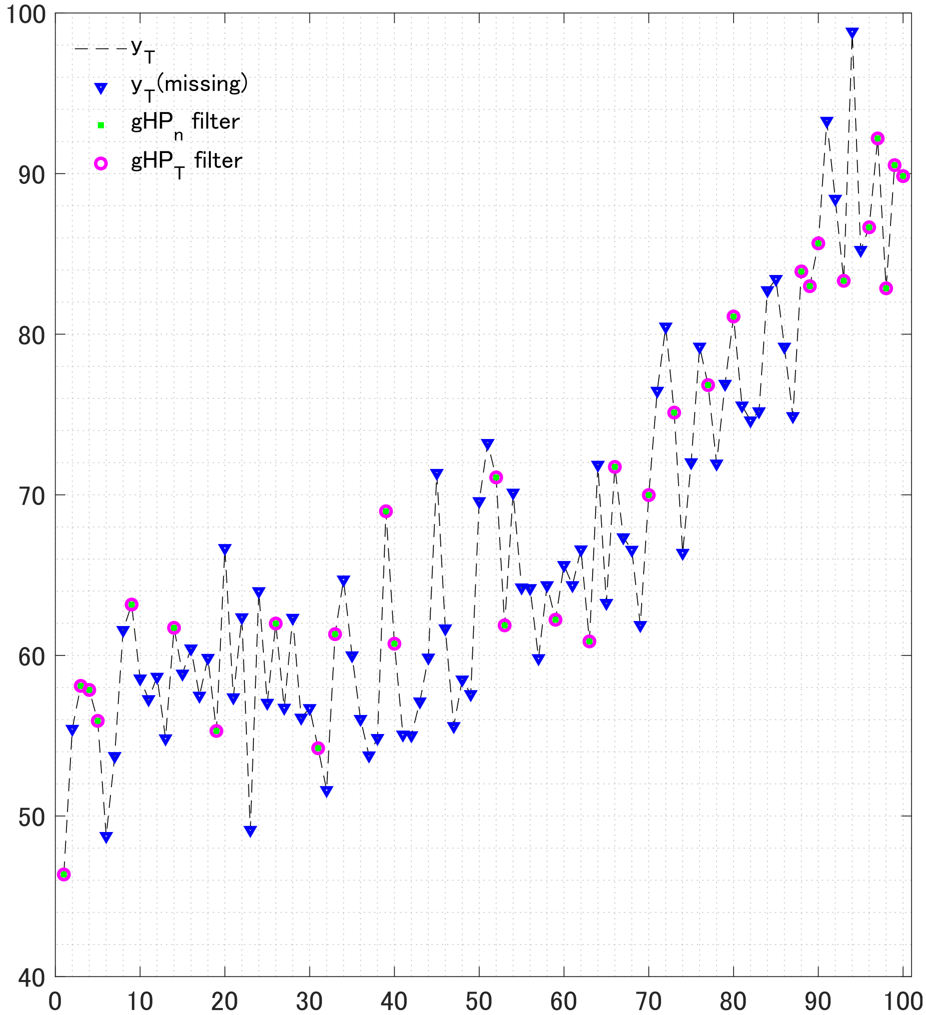

Figure B.14: For the explanation of $\mathtt{y_T}$ and $\mathtt{y_T(missing)}$, see Figure B.11. $\mathtt{gHP_n}$ `filter` denotes $\widehat{\boldsymbol{x}}_n$ in (8) estimated with $\lambda_n = 10^{-4}$ and $\mathtt{gHP_T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 10^{-4}$.

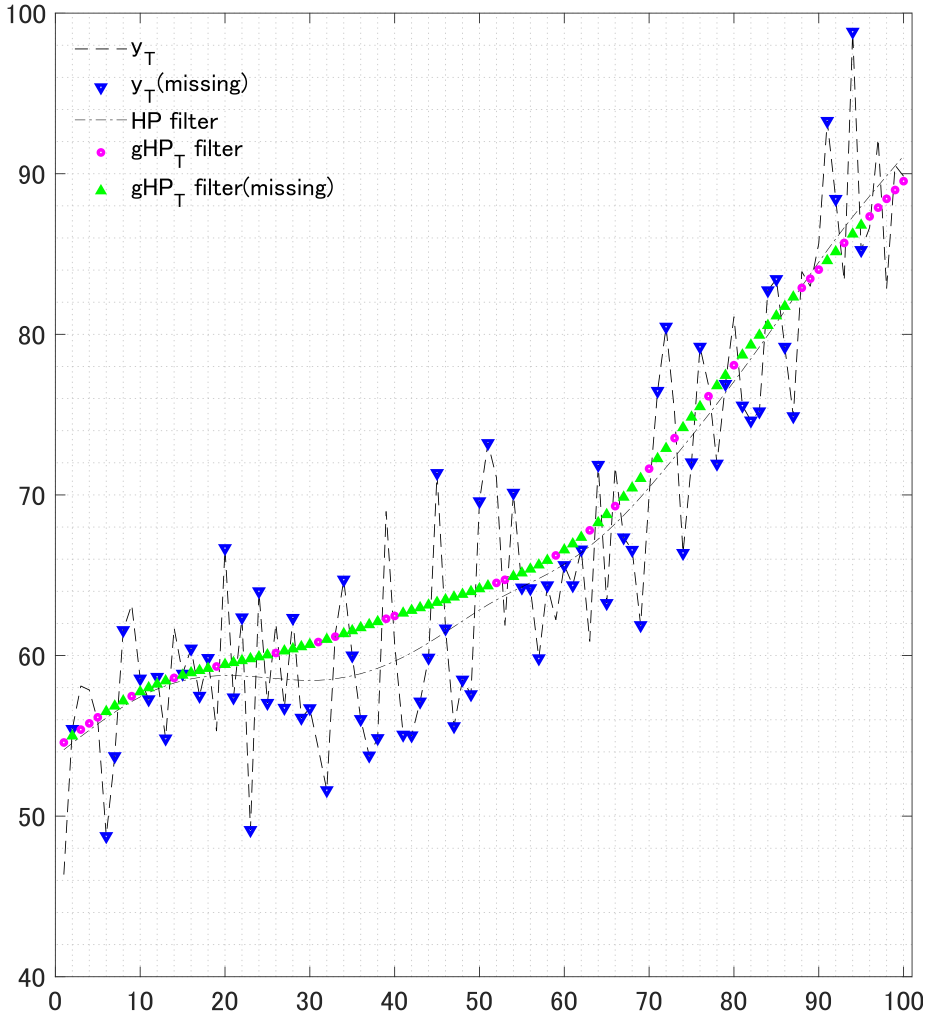

Figure B.15: For the explanation of $\mathtt{y_T}$ and $\mathtt{y_T(missing)}$, see Figure B.11. `HP filter` denotes $\widehat{\boldsymbol{x}}$ in (16), which is estimated with $\lambda = 1600$ from not only available observations but also missing observations. $\mathtt{gHP_T}$ `filter` denotes $\boldsymbol{S}\widehat{\boldsymbol{x}}_T$ in (12) estimated with $\lambda_T = 1600$ and $\mathtt{gHP_T}$ `filter(missing)` denotes $\boldsymbol{S}_{\perp}\widehat{\boldsymbol{x}}_T$ in (13) estimated with $\lambda_T = 1600$.

References

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2. Grant, M. and S. Boyd, 2008, Graph implementations for nonsmooth convex programs, in *Recent Advances in Learning and Control*, edited by V. Blondel, S. Boyd, and H. Kimura, 95–110, Springer, London.

3. Yamada, H., 2021, Trend extraction from economic time series with missing observations by generalized Hodrick–Prescott filters, *Econometric Theory*. [doi will be inserted here by typesetter].