

Supplement to the Paper: Unit Root Test with High-Frequency Data¹

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This paper provides a supplement to the article titled “Unit Root Test with High-Frequency Data”. It offers detailed proof for some of the Lemmas and Remarks.

1. ASYMPTOTICS OF THE DF STATISTIC

The null hypothesis of a unit root is specified as

$$y_{i\Delta} = y_{(i-1)\Delta} + \sigma\sqrt{\Delta}\varepsilon_{i\Delta}, \tag{S1}$$

with initial value y_0 , where σ is a constant and $\varepsilon_{i\Delta} \stackrel{i.i.d.}{\sim} \mathcal{N}(0, 1)$. The alternative hypothesis is

$$y_{i\Delta} = \alpha_0 + \beta_0 y_{(i-1)\Delta} + \lambda_0 \varepsilon_{i\Delta}, \tag{S2}$$

where $\alpha_0 = \mu(1 - e^{\theta\Delta})$ with μ and θ being constant, $\beta_0 = e^{\theta\Delta}$ and $\lambda_0^2 = \frac{\sigma^2}{2\theta}(e^{2\theta\Delta} - 1)$. The regression model is specified as follows:

$$y_{i\Delta} = \alpha + \beta y_{(i-1)\Delta} + v_{i\Delta}, \tag{S3}$$

where $v_{i\Delta}$ is the error term. The Dickey-Fuller statistic is

$$DF = (\hat{\beta} - 1) \left[\frac{T \sum_{j=1}^T y_{(j-1)\Delta}^2 - \left(\sum_{j=1}^T y_{(j-1)\Delta} \right)^2}{\sum_{j=1}^T \left(y_{j\Delta} - \hat{\alpha} - \hat{\beta} y_{(j-1)\Delta} \right)^2} \right]^{1/2}, \tag{S4}$$

where $\hat{\alpha}$ and $\hat{\beta}$ represent the OLS estimates of α and β .

Proof of Lemma 2.1. (a) Under model (S1),

$$y_{T\Delta} = \sigma\sqrt{\Delta} \sum_{j=1}^T \varepsilon_{j\Delta} + y_0 = \sigma N^{1/2} \left(T^{-1/2} \sum_{j=1}^T \varepsilon_{j\Delta} \right) + y_0 \implies \sigma N^{1/2} \left(w_1 + \frac{y_0}{\sigma N^{1/2}} \right) \equiv \sigma N^{1/2} \Psi_1,$$

given $T^{-1/2} \sum_{j=1}^T \varepsilon_{j\Delta} \implies w_1$.

(b) The quantity

$$\begin{aligned} T^{-1} \sum_{j=1}^T y_{j\Delta} &= T^{-1} \sum_{j=1}^T \left[\sigma\sqrt{\Delta} \sum_{i=1}^j \varepsilon_{i\Delta} + y_0 \right] \\ &= \sigma N^{1/2} \left[\frac{1}{T} \sum_{j=1}^T \left(T^{-1/2} \sum_{i=1}^j \varepsilon_{i\Delta} \right) \right] + T^{-1} \sum_{j=1}^T y_0 \\ &\implies \sigma N^{1/2} \left(\int_0^1 w_s ds + \frac{y_0}{\sigma N^{1/2}} \right) \equiv \sigma N^{1/2} \Psi_2. \end{aligned}$$

(c) The quantity

$$T^{-1} \sum_{j=1}^T y_{j\Delta}^2 = T^{-1} \sum_{j=1}^T \left[\sigma\sqrt{\Delta} \sum_{i=1}^j \varepsilon_{i\Delta} + y_0 \right]^2$$

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$$\begin{aligned}
&= \sigma^2 \Delta T \left[\frac{1}{T} \sum_{j=1}^T \left(T^{-1/2} \sum_{i=1}^j \varepsilon_{i\Delta} \right)^2 \right] + y_0^2 + 2y_0 \sigma \sqrt{\Delta} T^{1/2} \left[\frac{1}{T} \sum_{j=1}^T \left(T^{-1/2} \sum_{i=1}^j \varepsilon_{i\Delta} \right) \right] \\
&\implies \sigma^2 N \left[\int_0^1 w_s^2 ds + \frac{y_0^2}{\sigma^2 N} + 2 \frac{y_0}{\sigma N^{1/2}} \int_0^1 w_s ds \right] \equiv \sigma^2 N \Psi_3.
\end{aligned}$$

(d) Note that by squaring equation (S1), subtracting $y_{(j-1)\Delta}^2$ from both sides, summing over $j = 1, \dots, T$, re-organizing the equation, and multiplying by $T^{-1/2}$ we get

$$\begin{aligned}
T^{-1/2} \sum_{j=1}^T y_{(j-1)\Delta} \varepsilon_{j\Delta} &= \frac{1}{2\sigma N^{1/2}} \left[\sum_{j=1}^T (y_{j\Delta}^2 - y_{(j-1)\Delta}^2) - \sigma^2 \Delta \sum_{j=1}^T \varepsilon_{j\Delta}^2 \right] \\
&= \frac{1}{2\sigma N^{1/2}} \left[y_{T\Delta}^2 - y_0^2 - \sigma^2 N \left(T^{-1} \sum_{j=1}^T \varepsilon_{j\Delta}^2 \right) \right] \\
&\implies \frac{1}{2} \sigma N^{1/2} \left(w_1^2 + 2 \frac{y_0}{\sigma N^{1/2}} w_1 - 1 \right) \equiv \sigma N^{1/2} \Psi_4
\end{aligned}$$

since $y_{T\Delta}^2 \implies \Psi_1$ and $T_n^{-1} \sum_{j=1}^n \varepsilon_j^2 \rightarrow 1$. □

Proof of Lemma 2.2. (a) Use (S2) to obtain

$$\begin{aligned}
y_{T\Delta} &= \alpha_0 \frac{1 - e^{T\theta\Delta}}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^T e^{(T-j)\theta\Delta} \varepsilon_{j\Delta} + e^{T\theta\Delta} y_0 \\
&= \alpha_0 \frac{1 - e^c}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^T e^{\frac{T-j}{T}c} \varepsilon_{j\Delta} + e^c y_0 \\
&= \mu (1 - e^c) + \Delta^{1/2} T^{1/2} \frac{\lambda_0}{\Delta^{1/2}} \left(T^{-1/2} \sum_{j=1}^T e^{\frac{T-j}{T}c} \varepsilon_{j\Delta} \right) + e^c y_0 \\
&\implies \sigma N^{1/2} \left[\frac{\mu}{\sigma N^{1/2}} (1 - e^c) + J_c(1) + e^c \frac{y_0}{\sigma N^{1/2}} \right] \equiv \sigma N^{1/2} \Xi_1
\end{aligned}$$

since

$$T^{-1/2} \sum_{j=1}^T e^{\frac{T-j}{T}c} \varepsilon_{j\Delta} \implies J_c(1) = \int_0^1 e^{c(1-s)} dw_s$$

from Lemma 1(a) of Phillips (1987a), and

$$\frac{\lambda_0^2}{\Delta} = \frac{\sigma^2}{2\theta} \frac{(e^{2\theta\Delta} - 1)}{\Delta} \rightarrow \sigma^2 \text{ as } \Delta \rightarrow 0.$$

(b) We have

$$\begin{aligned}
T^{-1} \sum_{i=1}^T y_{i\Delta} &= T^{-1} \sum_{i=1}^T \left[\alpha_0 \frac{1 - e^{i\theta\Delta}}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right] \\
&= T^{-1} \frac{\alpha_0}{1 - e^{\theta\Delta}} \left(T - \sum_{i=1}^T e^{i\theta\Delta} \right) + T^{-1} \lambda_0 \sum_{i=1}^T \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + T^{-1} y_0 \sum_{i=1}^T e^{i\theta\Delta} \\
&= \mu \left(1 - \frac{\Delta}{N} e^{\theta\Delta} \frac{1 - e^c}{1 - e^{\theta\Delta}} \right) + N^{1/2} \frac{\lambda_0}{\Delta^{1/2}} \left(T^{-3/2} \sum_{i=1}^T \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \right) + y_0 \frac{\Delta}{N} e^{\theta\Delta} \frac{1 - e^c}{1 - e^{\theta\Delta}} \\
&\implies \sigma N^{1/2} \left[\frac{\mu}{\sigma N^{1/2}} + \int_0^1 J_c(r) dr + \frac{y_0 - \mu}{\sigma N^{1/2}} \frac{e^c - 1}{c} \right] \equiv \sigma N^{1/2} \Xi_2
\end{aligned}$$

since

$$T^{-3/2} \sum_{i=1}^T \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \implies \int_0^1 J_c(r) dr$$

using Lemma 1(b) of Phillips (1987b), and

$$\frac{\Delta}{N} e^{\theta\Delta} \frac{1 - e^c}{1 - e^{\theta\Delta}} = \frac{e^{\theta\Delta}}{N} (1 - e^c) \frac{\Delta}{1 - e^{\theta\Delta}} \rightarrow \frac{e^c - 1}{c}.$$

(c) Consider

$$\begin{aligned}
T^{-1} \sum_{i=1}^T y_{i\Delta}^2 &= T^{-1} \sum_{i=1}^T \left[\alpha_0 \frac{1 - e^{i\theta\Delta}}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right]^2 \\
&= T^{-1} \sum_{i=1}^T \left[\left(\alpha_0 \frac{1 - e^{i\theta\Delta}}{1 - e^{\theta\Delta}} \right)^2 + \left(\lambda_0 \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \right)^2 + e^{2i\theta\Delta} y_0^2 \right. \\
&\quad \left. + 2\alpha_0 \frac{1 - e^{i\theta\Delta}}{1 - e^{\theta\Delta}} \lambda_0 \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + 2\alpha_0 \frac{1 - e^{i\theta\Delta}}{1 - e^{\theta\Delta}} e^{i\theta\Delta} y_0 + 2\lambda_0 \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} e^{i\theta\Delta} y_0 \right] \\
&= \mu^2 + 2\mu (y_0 - \mu) T^{-1} \sum_{i=1}^T e^{i\theta\Delta} + (y_0 - \mu)^2 T^{-1} \sum_{i=1}^T e^{2i\theta\Delta} \\
&\quad + \lambda_0^2 T^{-1} \sum_{i=1}^T \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \right)^2 + 2\mu \lambda_0 T^{-1} \sum_{i=1}^T \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \\
&\quad - 2\mu \lambda_0 T^{-1} \sum_{i=1}^T e^{i\theta\Delta} \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + 2\lambda_0 y_0 T^{-1} \sum_{i=1}^T e^{i\theta\Delta} \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \\
&\Rightarrow \sigma^2 N \left[\frac{\mu^2}{\sigma^2 N} + 2 \frac{\mu (y_0 - \mu)}{\sigma^2 N} \frac{e^c - 1}{c} + \left(\frac{y_0 - \mu}{\sigma N^{1/2}} \right)^2 \frac{e^{2c} - 1}{2c} \right. \\
&\quad \left. + \int_0^1 J_c(r)^2 dr + 2 \frac{\mu}{\sigma N^{1/2}} \int_0^1 J_c(r) dr + 2 \frac{y_0 - \mu}{\sigma N^{1/2}} \int_0^1 e^{cr} J_c(r) dr \right] \\
&\equiv \sigma^2 N \Xi_3
\end{aligned}$$

since

$$\begin{aligned}
T^{-3/2} \sum_{i=1}^T \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \right)^2 &= T^{-3/2} \sum_{i=1}^T \left(\sum_{j=1}^i e^{\frac{i-j}{T}c} \varepsilon_{j\Delta} \right)^2 \Rightarrow \int_0^1 J_c(r) dr, \\
T^{-2} \sum_{i=1}^T \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \right)^2 &= T^{-1} \sum_{i=1}^T \left(T^{-1/2} \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \right)^2 \Rightarrow \int_0^1 J_c(r)^2 dr, \\
T^{-3/2} \sum_{i=1}^T e^{i\theta\Delta} \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} &= T^{-3/2} \sum_{i=1}^T e^{\frac{i}{T}c} \sum_{j=1}^i e^{\frac{i-j}{T}c} \varepsilon_{j\Delta} \Rightarrow \int_0^1 e^{cr} J_c(r) dr.
\end{aligned}$$

(d) By squaring (S2), subtracting $y_{(i-1)\Delta}^2$ from both sides, summing over $t = 1, \dots, T$, re-organizing the equation, and multiplying by $T^{-1/2}$ we get

$$\begin{aligned}
T^{-1/2} \sum_{i=1}^T y_{(i-1)\Delta} \varepsilon_{i\Delta} &= \frac{T^{-1/2}}{2e^{\theta\Delta} \lambda_0} \left[\sum_{i=1}^T (y_{i\Delta}^2 - y_{(i-1)\Delta}^2) - T \alpha_0^2 - (e^{2\theta\Delta} - 1) \sum_{i=1}^T y_{(i-1)\Delta}^2 \right. \\
&\quad \left. - \lambda_0^2 \sum_{i=1}^T \varepsilon_{i\Delta}^2 - 2\alpha_0 \lambda_0 \sum_{i=1}^T \varepsilon_{i\Delta} - 2\alpha_0 e^{\theta\Delta} \sum_{i=1}^T y_{(i-1)\Delta} \right] \\
&= \frac{1}{2e^{\theta\Delta} \sigma N^{1/2}} \left[y_{T\Delta}^2 - y_0^2 - T \alpha_0^2 - N \frac{e^{2\theta\Delta} - 1}{\Delta} \left(T^{-1} \sum_{i=1}^T y_{(i-1)\Delta}^2 \right) \right. \\
&\quad \left. - N \frac{\lambda_0^2}{\Delta} \left(T^{-1} \sum_{i=1}^T \varepsilon_{i\Delta}^2 \right) - 2\alpha_0 N^{1/2} \frac{\lambda_0}{\Delta^{1/2}} \left(T^{-1/2} \sum_{i=1}^T \varepsilon_{i\Delta} \right) - 2N e^{\theta\Delta} \frac{\alpha_0}{\Delta} \left(T^{-1} \sum_{i=1}^T y_{(i-1)\Delta} \right) \right] \\
&\Rightarrow \frac{\sigma N^{1/2}}{2} \left(\Xi_1^2 - \frac{y_0^2}{\sigma^2 N} - 2c \Xi_3 - 1 + 2c \frac{\mu}{\sigma N^{1/2}} \Xi_2 \right) \equiv \sigma N^{1/2} \Xi_4.
\end{aligned}$$

□

Proof of Remark 2.2. The DF statistic can be rewritten as

$$\begin{aligned} DF &= \left[\left(\hat{\beta} - \beta_0 \right) + (\beta_0 - 1) \right] \left[\frac{T \sum_{i=1}^T y_{(i-1)\Delta}^2 - \left(\sum_{i=1}^T y_{(i-1)\Delta} \right)^2}{\sum_{i=1}^T \left(y_{i\Delta} - \hat{\alpha} - \hat{\beta} y_{(i-1)\Delta} \right)^2} \right]^{1/2} \\ &= t_{\hat{\beta}} + (\beta_0 - 1) \left[\frac{T \sum_{i=1}^T y_{(i-1)\Delta}^2 - \left(\sum_{i=1}^T y_{(i-1)\Delta} \right)^2}{\sum_{i=1}^T \left(y_{i\Delta} - \hat{\alpha} - \hat{\beta} y_{(i-1)\Delta} \right)^2} \right]^{1/2}. \end{aligned}$$

(1) When $\theta > 0$, from Wang and Yu (2016), under the DGP of (S2) and the double asymptotic scheme,

$$\begin{aligned} t_{\hat{\beta}} &\implies \mathcal{N}(0, 1), \\ \frac{e^{2\theta\Delta} - 1}{e^{2\theta N}} \sum_{i=1}^T y_{(i-1)\Delta}^2 &\implies (y_0 - \mu + \sigma Z_1)^2, \\ \frac{e^{\theta\Delta} - 1}{e^{\theta N}} \sum_{i=1}^T y_{i\Delta} &\implies \sigma Z_1 + y_0 - \mu, \\ \frac{1}{N} \sum_{i=1}^T \left(y_{i\Delta} - \hat{\alpha} - \hat{\beta} y_{(i-1)\Delta} \right)^2 &\rightarrow \sigma^2, \end{aligned}$$

with $Z_1 \sim \mathcal{N}(0, \frac{1}{2\theta})$. Therefore,

$$\begin{aligned} (\beta_0 - 1) \frac{\left[T \sum_{i=1}^T y_{(i-1)\Delta}^2 - \left(\sum_{i=1}^T y_{(i-1)\Delta} \right)^2 \right]^{1/2}}{\left[\sum_{i=1}^T \left(y_{i\Delta} - \hat{\alpha} - \hat{\beta} y_{(i-1)\Delta} \right)^2 \right]^{1/2}} &= \left(e^{\theta\Delta} - 1 \right) \left[\frac{T \frac{e^{2\theta N}}{e^{2\theta\Delta} - 1} (y_0 - \mu + \sigma Z_1)^2}{\sigma^2 N} \right]^{1/2} [1 + o_p(1)] \\ &\sim e^{\theta N} \sqrt{\frac{\theta}{2}} \frac{1}{\sigma} |y_0 - \mu + \sigma Z_1| \rightarrow +\infty. \end{aligned}$$

It follows that

$$DF = (\beta_0 - 1) \frac{\left[T \sum_{i=1}^T y_{(i-1)\Delta}^2 - \left(\sum_{i=1}^T y_{(i-1)\Delta} \right)^2 \right]^{1/2}}{\left[\sum_{i=1}^T \left(y_{i\Delta} - \hat{\alpha} - \hat{\beta} y_{(i-1)\Delta} \right)^2 \right]^{1/2}} [1 + o_p(1)] \sim e^{\theta N} \sqrt{\frac{\theta}{2}} \frac{1}{\sigma} |y_0 - \mu + \sigma Z_1| \rightarrow +\infty.$$

(2) Consider the case of $\theta < 0$. From Wang and Yu (2016), assuming $E|\varepsilon_{1\Delta}|^{2+\delta} < \infty$ for some $\delta > 0$, under the DGP of (S2) and the double asymptotic scheme,

$$\begin{aligned} t_{\hat{\beta}} &\implies \mathcal{N}(0, 1), \\ \frac{1}{T} \sum_{i=1}^T y_{(i-1)\Delta}^2 &\implies -\frac{1}{2\theta} + \frac{\mu^2}{\sigma^2}, \\ \frac{1}{T} \sum_{i=1}^T y_{i\Delta} &\implies \frac{\mu}{\sigma}, \\ \frac{1}{N} \sum_{i=1}^T \left(y_{i\Delta} - \hat{\alpha} - \hat{\beta} y_{(i-1)\Delta} \right)^2 &\rightarrow \sigma^2. \end{aligned}$$

Therefore,

$$\begin{aligned} (\beta_0 - 1) \frac{\left[T \sum_{i=1}^T y_{(i-1)\Delta}^2 - \left(\sum_{i=1}^T y_{(i-1)\Delta} \right)^2 \right]^{1/2}}{\left[\sum_{i=1}^T \left(y_{i\Delta} - \hat{\alpha} - \hat{\beta} y_{(i-1)\Delta} \right)^2 \right]^{1/2}} &= \frac{T}{\sigma N^{1/2}} \left(e^{\theta\Delta} - 1 \right) \left[\frac{1}{T} \sum_{i=1}^T y_{(i-1)\Delta}^2 - \left(\frac{1}{T} \sum_{i=1}^T y_{i\Delta} \right)^2 \right]^{1/2} [1 + o_p(1)] \\ &\sim N^{1/2} \frac{\theta}{\sigma} \left(-\frac{1}{2\theta} \right)^{1/2} \rightarrow -\infty. \end{aligned}$$

It follows that when $\theta < 0$

$$DF = (\beta_0 - 1) \frac{\left[T \sum_{i=1}^T y_{(i-1)\Delta}^2 - \left(\sum_{i=1}^T y_{(i-1)\Delta} \right)^2 \right]^{1/2}}{\left[\sum_{i=1}^T \left(y_{i\Delta} - \hat{\alpha} - \hat{\beta} y_{(i-1)\Delta} \right)^2 \right]^{1/2}} [1 + o_p(1)] \sim N^{1/2} \frac{\theta}{\sigma} \left(-\frac{1}{2\theta} \right)^{1/2} \rightarrow -\infty.$$

□

2. ASYMPTOTICS FOR MODELS WITH JUMPS

The null hypothesis of a unit root with jumps is

$$y_{i\Delta} = \sum_{k=1}^K \phi_k I_{i\Delta}^k + y_{(i-1)\Delta} + \sigma \sqrt{\Delta} \varepsilon_{i\Delta}, \quad (\text{S5})$$

and the alternative is

$$y_{i\Delta} = \alpha_0 + \sum_{k=1}^K \phi_k I_{i\Delta}^k + \beta_0 y_{(i-1)\Delta} + \lambda_0 \varepsilon_{i\Delta}, \quad (\text{S6})$$

where $\sum_{k=1}^K \phi_k I_{i\Delta}^k$ is the jump component, defined in the main paper.

Proof of Lemma 3.1. The null hypothesis (S5) can be rewritten as

$$y_{i\Delta} = \sum_{j=1}^i \sum_{k=1}^K \phi_k I_{j\Delta}^k + \sigma \sqrt{\Delta} \sum_{j=1}^i \varepsilon_{j\Delta} + y_0. \quad (\text{S7})$$

(a) We have

$$y_{T\Delta} = \sum_{j=1}^T \sum_{k=1}^K \phi_k I_{j\Delta}^k + \sigma \sqrt{\Delta} \sum_{j=1}^T \varepsilon_{j\Delta} + y_0 \implies \sigma N^{1/2} \left[\sum_{k=1}^K \frac{\phi_k}{\sigma N^{1/2}} + \Psi_1 \right] \equiv \sigma N^{1/2} \tilde{\Psi}_1$$

since

$$\sum_{j=1}^T \sum_{k=1}^K \phi_k I_{j\Delta}^k = \sum_{k=1}^K \phi_k \sum_{j=1}^T I_{j\Delta}^k = \sum_{k=1}^K \phi_k$$

and from the proof of Lemma 2.1

$$\sigma \sqrt{\Delta} \sum_{j=1}^T \varepsilon_{j\Delta} + y_0 \implies \sigma N^{1/2} \Psi_1.$$

(b) The quantity

$$\begin{aligned} T^{-1} \sum_{i=1}^T y_{i\Delta} &= T^{-1} \sum_{i=1}^T \sum_{j=1}^i \sum_{k=1}^K \phi_k I_{j\Delta}^k + T^{-1} \sum_{i=1}^T \left(\sigma \sqrt{\Delta} \sum_{j=1}^i \varepsilon_{j\Delta} + y_0 \right) \\ &\implies \sigma N^{1/2} \left[\sum_{k=1}^K \frac{\phi_k}{\sigma N^{1/2}} (1 - r_k) + \Psi_2 \right] \equiv \sigma N^{1/2} \tilde{\Psi}_2 \end{aligned}$$

since

$$\begin{aligned} T^{-1} \sum_{i=1}^T \sum_{j=1}^i \sum_{k=1}^K \phi_k I_{j\Delta}^k &= T^{-1} \sum_{k=1}^K \phi_k \sum_{i=1}^T \sum_{j=1}^i I_{j\Delta}^k \\ &= T^{-1} \sum_{k=1}^K \phi_k \left[I_{1\Delta}^k + (I_{1\Delta}^k + I_{2\Delta}^k) + \cdots + (I_{1\Delta}^k + I_{2\Delta}^k + \cdots + I_{T\Delta}^k) \right] \\ &= T^{-1} \sum_{k=1}^K \phi_k (T - \tau_k) = \sum_{k=1}^K \phi_k (1 - r_k) \end{aligned}$$

and from the proof of Lemma 2.1,

$$T^{-1} \sum_{i=1}^T \left(\sigma \sqrt{\Delta} \sum_{j=1}^i \varepsilon_{j\Delta} + y_0 \right) \implies \sigma N^{1/2} \Psi_2.$$

(c) The quantity

$$\begin{aligned} T^{-1} \sum_{i=1}^T y_{i\Delta}^2 &= T^{-1} \sum_{i=1}^T \left[\sum_{j=1}^i \sum_{k=1}^K \phi_k I_{j\Delta}^k + \sigma \sqrt{\Delta} \sum_{j=1}^i \varepsilon_{j\Delta} + y_0 \right]^2 \\ &= T^{-1} \sum_{i=1}^T \left(\sum_{j=1}^i \sum_{k=1}^K \phi_k I_{j\Delta}^k \right)^2 + T^{-1} \sum_{i=1}^T \left(\sigma \sqrt{\Delta} \sum_{j=1}^i \varepsilon_{j\Delta} + y_0 \right)^2 \\ &\quad + 2T^{-1} \sum_{i=1}^T \sum_{j=1}^i \sum_{k=1}^K \phi_k I_{j\Delta}^k \left(\sigma \sqrt{\Delta} \sum_{j=1}^i \varepsilon_{j\Delta} + y_0 \right). \end{aligned}$$

The first term

$$\begin{aligned} T^{-1} \sum_{i=1}^T \left(\sum_{j=1}^i \sum_{k=1}^K \phi_k I_{j\Delta}^k \right)^2 &= T^{-1} \sum_{i=1}^T \left(\sum_{k=1}^K \phi_k \sum_{j=1}^i I_{j\Delta}^k \right)^2 \\ &= T^{-1} \left[\sum_{k=1}^{K-1} (\tau_{k+1} - \tau_k) \left(\sum_{j=1}^k \phi_j \right)^2 + (T - \tau_K) \left(\sum_{j=1}^K \phi_j \right)^2 \right] \\ &= \begin{cases} (1 - r_1) \phi_1^2 & \text{if } K = 1 \\ \sum_{k=1}^{K-1} (r_{k+1} - r_k) \left(\sum_{j=1}^k \phi_j \right)^2 + (1 - r_K) \left(\sum_{j=1}^K \phi_j \right)^2 & \text{if } K > 1. \end{cases} \end{aligned}$$

The second term, from the proof of Lemma 2.1,

$$T^{-1} \sum_{j=1}^T \left[\sigma \sqrt{\Delta} \sum_{i=1}^j \varepsilon_{i\Delta} + y_0 \right]^2 \implies \sigma^2 N \Psi_3.$$

The third term

$$\begin{aligned} &2T^{-1} \sum_{i=1}^T \sum_{j=1}^i \sum_{k=1}^K \phi_k I_{j\Delta}^k \left(\sigma \sqrt{\Delta} \sum_{j=1}^i \varepsilon_{j\Delta} + y_0 \right) \\ &= 2\sigma N^{1/2} \left(T^{-3/2} \sum_{k=1}^K \phi_k \sum_{i=1}^T \sum_{j=1}^i I_{j\Delta}^k \sum_{j=1}^i \varepsilon_{j\Delta} \right) + 2y_0 \left(T^{-1} \sum_{i=1}^T \sum_{j=1}^i \sum_{k=1}^K \phi_k I_{j\Delta}^k \right) \\ &\implies 2\sigma N^{1/2} \sum_{k=1}^K \phi_k \int_{r_k}^1 w_s ds + 2y_0 \sum_{k=1}^K \phi_k (1 - r_k) \end{aligned}$$

since

$$\begin{aligned} &T^{-3/2} \sum_{i=1}^T \left(\sum_{k=1}^K \phi_k \sum_{j=1}^i I_{j\Delta}^k \right) \left(\sum_{j=1}^i \varepsilon_{j\Delta} \right) \\ &= T^{-3/2} \sum_{k=1}^K \phi_k \sum_{i=1}^T \left(\sum_{j=1}^i I_{j\Delta}^k \sum_{j=1}^i \varepsilon_{j\Delta} \right) \\ &= T_n^{-3/2} \sum_{k=1}^K \phi_k \left[(T - \tau_k) \sum_{i=1}^{\tau_k} \varepsilon_{i\Delta} + \sum_{i=\tau_k+1}^T (T - i + 1) \varepsilon_{i\Delta} \right] \\ &= \sum_{k=1}^K \phi_k \left[\frac{T - \tau_k}{T} \left(T^{-1/2} \sum_{i=1}^{\tau_k} \varepsilon_{i\Delta} \right) + T_n^{-3/2} \sum_{i=\tau_k+1}^{T_n} (T - i) \varepsilon_{i\Delta} \right] \\ &= \sum_{k=1}^K \phi_k \left[\frac{T - \tau_k}{T} \left(T^{-1/2} \sum_{i=1}^{\tau_k} \varepsilon_{i\Delta} \right) + T^{-1/2} \sum_{i=\tau_k+1}^T \varepsilon_{i\Delta} - T^{-3/2} \sum_{i=\tau_k+1}^T i \varepsilon_{i\Delta} \right] \\ &\implies \sum_{k=1}^K \phi_k \left[(1 - r_k) w_{r_k} + (w_1 - w_{r_k}) - \left(w_1 - r_k w_{r_k} - \int_{r_k}^1 w_s ds \right) \right] \\ &= \sum_{k=1}^K \phi_k \int_{r_k}^1 w_s ds. \end{aligned}$$

Therefore,

$$T^{-1} \sum_{i=1}^T y_{i\Delta}^2 \Rightarrow \sigma^2 N \left[\Psi_3 + \Delta_1 + (1 - r_K) \left(\sum_{j=1}^K \frac{\phi_j}{\sigma N^{1/2}} \right)^2 + 2 \sum_{k=1}^K \frac{\phi_k}{\sigma N^{1/2}} \int_{r_k}^1 w_s ds + 2 \frac{y_0}{\sigma N^{1/2}} \sum_{k=1}^K \frac{\phi_k}{\sigma N^{1/2}} (1 - r_k) \right] \equiv \sigma^2 N \tilde{\Psi}_3$$

with

$$\Delta_1 = \begin{cases} (1 - r_1) \frac{\phi_1^2}{\sigma^2 N} & \text{if } K = 1 \\ \sum_{k=1}^{K-1} (r_{k+1} - r_k) \left(\sum_{j=1}^k \frac{\phi_j}{\sigma N^{1/2}} \right)^2 + (1 - r_K) \left(\sum_{j=1}^K \frac{\phi_j}{\sigma N^{1/2}} \right)^2 & \text{if } K > 1. \end{cases}$$

(d) By squaring (S5), subtracting $y_{(i-1)\Delta}^2$ from both sides, summing over $i = 1, \dots, T$, re-organizing the equation, and multiplying by $T^{-1/2}$, we get

$$T^{-1/2} \sum_{i=1}^T y_{(i-1)\Delta} \varepsilon_{i\Delta} = \frac{T^{-1/2}}{2\sigma\sqrt{\Delta}} \left[y_{T\Delta}^2 - y_0^2 - \sigma^2 \Delta \sum_{i=1}^T \varepsilon_{i\Delta}^2 - \sum_{i=1}^T \left(\sum_{k=1}^K \phi_k I_{i\Delta}^k \right)^2 - 2 \sum_{i=1}^T \sum_{k=1}^K \phi_k I_{i\Delta}^k \left(y_{(i-1)\Delta} + \sigma\sqrt{\Delta} \varepsilon_{(i-1)\Delta} \right) \right].$$

We have $y_{T\Delta} \Rightarrow \tilde{\Psi}_1$, $\sigma^2 \Delta \sum_{i=1}^T \varepsilon_{i\Delta}^2 \rightarrow \sigma^2 N$, $\sum_{i=1}^T \left(\sum_{k=1}^K \phi_k I_{i\Delta}^k \right)^2 = \sum_{k=1}^K \phi_k^2$ and

$$\begin{aligned} & 2 \sum_{i=1}^T \left(\sum_{k=1}^K \phi_k I_{i\Delta}^k \right) \left(y_{(i-1)\Delta} + \sigma\sqrt{\Delta} \varepsilon_{(i-1)\Delta} \right) \\ &= 2 \sum_{k=1}^K \phi_k \sum_{i=1}^T I_{i\Delta}^k y_{(i-1)\Delta} + \sigma\sqrt{\Delta} \sum_{k=1}^K \phi_k \sum_{i=1}^T I_{i\Delta}^k \varepsilon_{(i-1)\Delta} \\ &= 2 \sum_{k=1}^K \phi_k y_{(\tau_{k-1})\Delta} + \sigma\sqrt{\Delta} \sum_{k=1}^K \phi_k \varepsilon_{(\tau_{k-1})\Delta} \Rightarrow 2\sigma N^{1/2} \sum_{k=1}^K \phi_k \left(w_{r_k} + \frac{y_0}{\sigma N^{1/2}} \right) \end{aligned}$$

since $\sigma\sqrt{\Delta} \sum_{k=1}^K \phi_k \varepsilon_{(\tau_{k-1})\Delta} \rightarrow 0$ and

$$y_{(\tau_{k-1})\Delta} = \sigma\sqrt{\Delta} \sum_{j=1}^{\tau_{k-1}} \varepsilon_j + y_0 + \sum_{k=1}^K \phi_k \sum_{j=1}^{\tau_{k-1}} I_j^k \Rightarrow N^{1/2} \left(w_{r_k} + \frac{y_0}{\sigma N^{1/2}} + \Delta_2 \right),$$

where $\Delta_2 = 0$ if $K = 0$ and $\Delta_2 = \sum_{j=1}^{k-1} \frac{\phi_j}{\sigma N^{1/2}}$ if $K > 1$. Therefore,

$$\begin{aligned} T^{-1/2} \sum_{i=1}^T y_{(i-1)\Delta} \varepsilon_{i\Delta} &\Rightarrow \frac{\sigma N^{1/2}}{2} \left[\tilde{\Psi}_1 - \frac{y_0^2}{\sigma^2 N} - 1 - \sum_{k=1}^K \frac{\phi_k^2}{\sigma^2 N} - 2 \sum_{k=1}^K \frac{\phi_k}{\sigma N^{1/2}} \left(w_{r_k} + \frac{y_0}{\sigma N^{1/2}} + \Delta_2 \right) \right] \\ &\equiv \sigma N^{1/2} \tilde{\Psi}_4. \end{aligned}$$

□

Proof of Lemma 3.2. (a) The alternative model can be rewritten as

$$y_{i\Delta} = \alpha_0 \frac{1 - e^{i\theta\Delta}}{1 - e^{\theta\Delta}} + \sum_{j=1}^i e^{(i-j)\theta\Delta} \left(\lambda_0 \varepsilon_{j\Delta} + \sum_{k=1}^K \phi_k I_{j\Delta}^k \right) + e^{i\theta\Delta} y_0. \tag{2.1}$$

It follows that

$$y_{T\Delta} = \alpha_0 \frac{1 - e^{T\theta\Delta}}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^T e^{(T-j)\theta\Delta} \varepsilon_{j\Delta} + e^{T\theta\Delta} y_0 + \sum_{j=1}^T e^{(T-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k.$$

From the proof of Lemma 2.2,

$$\alpha_0 \frac{1 - e^{T\theta\Delta}}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^T e^{(T-j)\theta\Delta} \varepsilon_{j\Delta} + e^{T\theta\Delta} y_0 \Rightarrow \sigma N^{1/2} \Xi_1.$$

The last term

$$\sum_{j=1}^T e^{(T-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k = \sum_{k=1}^K \phi_k \sum_{j=1}^T e^{\frac{T-j}{T}c} I_{j\Delta}^k = \sum_{k=1}^K \phi_k e^{(1-r_k)c}.$$

Therefore,

$$y_{T\Delta} \implies \sigma N^{1/2} \left[\Xi_1 + \sum_{k=1}^K \frac{\phi_k}{\sigma N^{1/2}} e^{(1-r_k)c} \right] \equiv \sigma N^{1/2} \tilde{\Xi}_1.$$

(b) The quantity

$$\begin{aligned} T^{-1} \sum_{i=1}^T y_{i\Delta} &= T^{-1} \sum_{i=1}^T \left[\alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^i e^{(T-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 + \sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right] \\ &= T^{-1} \sum_{i=1}^T \left[\alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^i e^{(T-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right] + T^{-1} \sum_{i=1}^T \sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k. \end{aligned}$$

From the proof of Lemma 2.2,

$$T^{-1} \sum_{i=1}^T \left[\alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^i e^{(T-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right] \implies \sigma N^{1/2} \Xi_2.$$

Furthermore,

$$\begin{aligned} T^{-1} \sum_{i=1}^T \sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k &= T^{-1} \sum_{k=1}^K \phi_k \sum_{i=1}^T \sum_{j=1}^i e^{(i-j)\theta\Delta} I_{j\Delta}^k \\ &= T^{-1} \sum_{k=1}^K \phi_k \sum_{i=1}^T \frac{1-e^{(T-i+1)\theta\Delta}}{1-e^{\theta\Delta}} I_{i\Delta}^k \\ &= T^{-1} \sum_{k=1}^K \phi_k \frac{1-e^{(T-\tau_k+1)\theta\Delta}}{1-e^{\theta\Delta}} \\ &\rightarrow \frac{1}{c} \sum_{k=1}^K \phi_k \left[e^{(1-r_k)c} - 1 \right]. \end{aligned}$$

Therefore,

$$T^{-1} \sum_{i=1}^T y_{i\Delta} \implies \sigma N^{1/2} \left\{ \Xi_2 + \frac{1}{c} \sum_{k=1}^K \frac{\phi_k}{\sigma N^{1/2}} \left[e^{(1-r_k)c} - 1 \right] \right\} \equiv \sigma N^{1/2} \tilde{\Xi}_2.$$

(c) The quantity

$$\begin{aligned} T^{-1} \sum_{i=1}^T y_{i\Delta}^2 &= T^{-1} \sum_{i=1}^T \left[\alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 + \sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right]^2 \\ &= T^{-1} \sum_{i=1}^T \left(\alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right)^2 + T^{-1} \sum_{i=1}^T \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right)^2 \\ &\quad + 2T^{-1} \sum_{i=1}^T \left(\alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} \right) \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right) \\ &\quad + 2T^{-1} \sum_{i=1}^T \lambda_0 \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right) \\ &\quad + 2T^{-1} \sum_{i=1}^T e^{i\theta\Delta} y_0 \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right). \end{aligned}$$

From the proof of Lemma 2.2,

$$T^{-1} \sum_{i=1}^T \left(\alpha_0 \frac{1-e^{i\theta\Delta}}{1-e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} + e^{i\theta\Delta} y_0 \right)^2 \implies \sigma^2 N \Xi_3.$$

The second term

$$T^{-1} \sum_{i=1}^T \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right)^2 = T^{-1} \sum_{i=1}^T \left(\sum_{k=1}^K \phi_k \sum_{j=1}^i e^{(i-j)\theta\Delta} I_{j\Delta}^k \right)^2.$$

If $K = 1$, we have

$$T^{-1} \sum_{i=1}^T \left(\sum_{k=1}^K \phi_k \sum_{j=1}^i e^{(i-j)\theta\Delta} I_{j\Delta}^k \right)^2 = T^{-1} \sum_{i=\tau_1}^T \phi_1^2 e^{2(i-\tau_1)\theta\Delta} = \frac{\Delta}{N} \phi_1^2 \frac{e^{2c(1-r_1)} - 1}{e^{2\theta\Delta} - 1} \rightarrow \phi_1^2 \frac{1}{2c} \left[e^{2c(1-r_1)} - 1 \right].$$

If $K > 1$,

$$\begin{aligned} & T^{-1} \sum_{i=1}^T \left(\sum_{k=1}^K \phi_k \sum_{j=1}^i e^{(i-j)\theta\Delta} I_{j\Delta}^k \right)^2 \\ &= T^{-1} \sum_{k=1}^{K-1} \sum_{i=\tau_k}^{\tau_{k+1}-1} \left(\sum_{j=1}^k \phi_j e^{(i-\tau_j)\theta\Delta} \right)^2 + T^{-1} \sum_{i=\tau_K}^T \left(\sum_{j=1}^K \phi_j e^{(i-\tau_j)\theta\Delta} \right)^2 \\ &= T^{-1} \sum_{k=1}^{K-1} \sum_{i=\tau_k}^{\tau_{k+1}-1} e^{2i\theta\Delta} \left(\sum_{j=1}^k \phi_j e^{-\tau_j\theta\Delta} \right)^2 + T^{-1} \sum_{i=\tau_K}^T e^{2i\theta\Delta} \left(\sum_{j=1}^K \phi_j e^{-\tau_j\theta\Delta} \right)^2 \\ &= T^{-1} \sum_{k=1}^{K-1} \left(\sum_{j=1}^k \phi_j e^{-\tau_j\theta\Delta} \right)^2 \left(\sum_{i=\tau_k}^{\tau_{k+1}-1} e^{2i\theta\Delta} \right) + T^{-1} \left(\sum_{j=1}^K \phi_j e^{-\tau_j\theta\Delta} \right)^2 \sum_{i=\tau_K}^T e^{2i\theta\Delta} \\ &= T^{-1} \sum_{k=1}^{K-1} \left(\sum_{j=1}^k \phi_j e^{-\tau_j\theta\Delta} \right)^2 \left[e^{2r_k\theta} \frac{1 - e^{2\theta(r_{k+1}-r_k)}}{1 - e^{2\theta\Delta}} \right] + T^{-1} \left(\sum_{j=1}^K \phi_j e^{-\tau_j\theta\Delta} \right)^2 \left[e^{2r_K\theta} \frac{1 - e^{2\theta(1-r_K)}}{1 - e^{2\theta\Delta}} \right] \\ &\rightarrow \frac{1}{2c} \left[\sum_{k=1}^{K-1} \left(\sum_{j=1}^k \phi_j e^{-r_j\theta} \right)^2 e^{2r_k\theta} \left[e^{2\theta(r_{k+1}-r_k)} - 1 \right] + \left(\sum_{j=1}^K \phi_j e^{-r_j\theta} \right)^2 e^{2r_K\theta} \left[e^{2\theta(1-r_K)} - 1 \right] \right]. \end{aligned}$$

The third term

$$\begin{aligned} & 2T^{-1} \sum_{i=1}^T \left(\alpha_0 \frac{1 - e^{i\theta\Delta}}{1 - e^{\theta\Delta}} \right) \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right) \\ &= 2 \frac{\alpha_0}{1 - e^{\theta\Delta}} T^{-1} \sum_{i=1}^T \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right) - 2 \frac{\alpha_0}{1 - e^{\theta\Delta}} T^{-1} \sum_{i=1}^T e^{i\theta\Delta} \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right) \\ &= 2\mu T^{-1} \sum_{k=1}^K \phi_k \sum_{i=1}^T \sum_{j=1}^i e^{(i-j)\theta\Delta} I_{j\Delta}^k - 2\mu T^{-1} \sum_{k=1}^K \phi_k \sum_{i=1}^T \sum_{j=1}^i e^{i\theta\Delta} e^{(i-j)\theta\Delta} I_{j\Delta}^k \\ &= 2\mu T^{-1} \sum_{k=1}^K \phi_k e^{-\tau_k\theta\Delta} \sum_{i=\tau_k}^T e^{i\theta\Delta} - 2\mu T^{-1} \sum_{k=1}^K \phi_k e^{-\tau_k\theta\Delta} \sum_{i=\tau_k}^T e^{2i\theta\Delta} \\ &= 2\mu T^{-1} \sum_{k=1}^K \phi_k e^{-\tau_k\theta\Delta} \left[\sum_{i=\tau_k}^T e^{i\theta\Delta} - \sum_{i=\tau_k}^T e^{2i\theta\Delta} \right] \\ &= \frac{\mu}{c} \sum_{k=1}^K \phi_k \left[2e^{c(1-r_k)} - 2 - e^{c(2-r_k)} + e^{r_k c} \right]. \end{aligned}$$

The fourth term

$$\begin{aligned} & 2T^{-1} \sum_{i=1}^T \lambda_0 \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \right) \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right) \\ &= 2\lambda_0 T^{-1} \sum_{k=1}^K \phi_k \sum_{i=1}^T \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \right) \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} I_{j\Delta}^k \right) \\ &= 2\lambda_0 T^{1/2} \sum_{k=1}^K \phi_k \left[\frac{1}{T} \sum_{i=\tau_k}^T e^{(i-\tau_k)\theta\Delta} \left(T^{-1/2} \sum_{j=1}^i e^{(i-j)\theta\Delta} \varepsilon_{j\Delta} \right) \right] \\ &\Rightarrow 2\sigma N^{1/2} \sum_{k=1}^K \phi_k \int_{r_k}^1 e^{c(r-r_k)} J_c(r) dr. \end{aligned}$$

The fifth term

$$\begin{aligned}
2T^{-1} \sum_{i=1}^T e^{i\theta\Delta} y_0 \left(\sum_{j=1}^i e^{(i-j)\theta\Delta} \sum_{k=1}^K \phi_k I_{j\Delta}^k \right) &= 2y_0 T^{-1} \sum_{k=1}^K \phi_k \sum_{i=1}^T \left(\sum_{j=1}^i e^{i\theta\Delta} e^{(i-j)\theta\Delta} I_{j\Delta}^k \right) \\
&= 2y_0 T^{-1} \sum_{k=1}^K \phi_k \sum_{i=\tau_k}^T e^{(2i-\tau_k)\theta\Delta} \\
&= 2y_0 \sum_{k=1}^K \phi_k \left(e^{-\tau_k\theta\Delta} T^{-1} \sum_{i=\tau_k}^T e^{2i\theta\Delta} \right) \\
&= \frac{y_0}{c} \sum_{k=1}^K \phi_k e^{r_k c} \left[e^{2c(1-r_k)} - 1 \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
T^{-1} \sum_{i=1}^T y_{(i-1)\Delta}^2 &\implies \sigma^2 N \Xi_3 + \frac{1}{2c} \sum_{k=1}^K \phi_k^2 \left[e^{2(1-r_k)c} - 1 \right] + \frac{\mu}{c} \sum_{k=1}^K \phi_k \left[2e^{c(1-r_k)} - 2 - e^{c(2-r_k)} + e^{r_k c} \right] \\
&\quad + 2\sigma N^{1/2} \sum_{k=1}^K \phi_k \int_{r_k}^1 e^{c(r-r_k)} J_c(r) dr + \frac{y_0}{c} \sum_{k=1}^K \phi_k e^{r_k c} \left[e^{2c(1-r_k)} - 1 \right] \\
&= \sigma^2 N \left\{ \Xi_3 + \Delta_3 + \frac{1}{c} \frac{\mu}{\sigma N^{1/2}} \sum_{k=1}^K \frac{\phi_k}{\sigma N^{1/2}} \left[2e^{c(1-r_k)} - 2 - e^{c(2-r_k)} + e^{r_k c} \right] \right. \\
&\quad \left. + 2 \sum_{k=1}^K \frac{\phi_k}{\sigma N^{1/2}} \int_{r_k}^1 e^{c(r-r_k)} J_c(r) dr + \frac{1}{c} \frac{y_0}{\sigma N^{1/2}} \sum_{k=1}^K \frac{\phi_k}{\sigma N^{1/2}} e^{r_k c} \left[e^{2c(1-r_k)} - 1 \right] \right\} \\
&\equiv \sigma^2 N \tilde{\Xi}_3,
\end{aligned}$$

where

$$\Delta_3 = \begin{cases} \frac{\phi_1^2}{\sigma^2 N} \frac{1}{2c} \left[e^{2c(1-r_1)} - 1 \right] & \text{if } K = 1 \\ \frac{1}{2c} \left[\sum_{k=1}^{K-1} \left(\sum_{j=1}^k \frac{\phi_j}{\sigma N^{1/2}} e^{-r_j \theta} \right)^2 e^{2r_k \theta} \left[e^{2\theta(r_{k+1}-r_k)} - 1 \right] + \left(\sum_{j=1}^K \frac{\phi_j}{\sigma N^{1/2}} e^{-r_j \theta} \right)^2 e^{2r_k \theta} \left[e^{2\theta(1-r_k)} - 1 \right] \right] & \text{if } K > 1 \end{cases}.$$

(d) By squaring (S6), subtracting $y_{(i-1)\Delta}^2$ from both sides, summing over $i = 1, \dots, T$, re-organizing the equation, and multiplying T^{-1} such that

$$\begin{aligned}
&T^{-1/2} \sum_{i=1}^T y_{(i-1)\Delta} \varepsilon_{i\Delta} \\
&= \frac{T^{-1/2}}{2e^{\theta\Delta} \lambda_0} \left[\sum_{i=1}^T (y_{i\Delta}^2 - y_{(i-1)\Delta}^2) - T\alpha_0^2 - (e^{2\theta\Delta} - 1) \sum_{i=1}^T y_{(i-1)\Delta}^2 - \lambda_0^2 \sum_{i=1}^T \varepsilon_{i\Delta}^2 - 2\alpha_0 e^{\theta\Delta} \sum_{i=1}^T y_{(i-1)\Delta} \right. \\
&\quad \left. - 2\alpha_0 \lambda_0 \sum_{i=1}^T \varepsilon_{i\Delta} - \sum_{i=1}^T \left(\sum_{k=1}^K \phi_k I_{i\Delta}^k \right)^2 - 2\alpha_0 \sum_{i=1}^T \sum_{k=1}^K \phi_k I_{i\Delta}^k - 2e^{\theta\Delta} \sum_{i=1}^T y_{(i-1)\Delta} \sum_{k=1}^K \phi_k I_{i\Delta}^k - 2\lambda_0 \sum_{i=1}^T \left(\sum_{k=1}^K \phi_k I_{i\Delta}^k \right) \varepsilon_{i\Delta} \right].
\end{aligned}$$

From (a), (b) and (c),

$$\begin{aligned}
&\frac{T^{-1/2}}{2e^{\theta\Delta} \lambda_0} \left[\sum_{i=1}^T (y_{i\Delta}^2 - y_{(i-1)\Delta}^2) - T\alpha_0^2 - (e^{2\theta\Delta} - 1) \sum_{i=1}^T y_{(i-1)\Delta}^2 - \lambda_0^2 \sum_{i=1}^T \varepsilon_{i\Delta}^2 - 2\alpha_0 e^{\theta\Delta} \sum_{i=1}^T y_{(i-1)\Delta} - 2\alpha_0 \lambda_0 \sum_{i=1}^T \varepsilon_{i\Delta} \right] \\
&\implies \frac{\sigma N^{1/2}}{2} \left[\tilde{\Xi}_1 - \frac{y_0^2}{\sigma^2 N} - 2c\tilde{\Xi}_3 - 1 + 2c \frac{\mu}{\sigma N^{1/2}} \tilde{\Xi}_2 \right].
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{i=1}^T \left(\sum_{k=1}^K \phi_k I_{i\Delta}^k \right)^2 &= \sum_{k=1}^K \phi_k^2 \sum_{i=1}^T I_{i\Delta}^k = \sum_{k=1}^K \phi_k^2; \\
2\alpha_0 \sum_{i=1}^T \sum_{k=1}^K \phi_k I_{i\Delta}^k &= 2\mu (1 - e^{\theta\Delta}) \sum_{k=1}^K \phi_k \sum_{i=1}^T I_{i\Delta}^k = 2\mu (1 - e^{\theta\Delta}) \sum_{k=1}^K \phi_k \rightarrow 0; \\
2\lambda_0 \sum_{i=1}^T \left(\sum_{k=1}^K \phi_k I_{i\Delta}^k \right) \varepsilon_{i\Delta} &= 2\lambda_0 \sum_{k=1}^K \phi_k \sum_{i=1}^T I_{i\Delta}^k \varepsilon_{i\Delta} = 2\lambda_0 \sum_{k=1}^K \phi_k \varepsilon_{\tau_k \Delta} \rightarrow 0;
\end{aligned}$$

and

$$\begin{aligned}
 & 2e^{\theta\Delta} \sum_{i=1}^T y_{(i-1)\Delta} \sum_{k=1}^K \phi_k I_{i\Delta}^k \\
 &= 2e^{\theta\Delta} \sum_{k=1}^K \phi_k \sum_{i=1}^T y_{(i-1)\Delta} I_{i\Delta}^k = 2e^{\theta\Delta} \sum_{k=1}^K \phi_k y_{(\tau_k-1)\Delta} \\
 &= 2e^{\theta\Delta} \sum_{k=1}^K \phi_k \left[\alpha_0 \frac{1 - e^{(\tau_k-1)\theta\Delta}}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{\tau_k-1} e^{(\tau_k-1-j)\theta\Delta} \varepsilon_{j\Delta} + e^{(\tau_k-1)\theta\Delta} y_0 + \sum_{j=1}^{\tau_k-1} e^{(\tau_k-1-j)\theta\Delta} \sum_{s=1}^K \phi_s I_{j\Delta}^s \right] \\
 &\implies 2\sigma^2 N \sum_{k=1}^K \frac{\phi_k}{N^{1/2}\sigma} \left[\frac{\mu}{N^{1/2}\sigma} (1 - e^{r_k c}) + J_c(r_k) + e^{r_k c} \frac{y_0}{N^{1/2}\sigma} + \Delta_4 \right], \tag{2.2}
 \end{aligned}$$

where $\Delta_4 = 0$ if $K = 1$ and $\Delta_4 = \sum_{j=1}^{k-1} e^{(r_k-r_j)\theta} \frac{\phi_j}{N^{1/2}\sigma}$ if $K > 1$. This is because

$$\alpha_0 \frac{1 - e^{(\tau_k-1)\theta\Delta}}{1 - e^{\theta\Delta}} + \lambda_0 \sum_{j=1}^{\tau_k-1} e^{(\tau_k-1-j)\theta\Delta} \varepsilon_{j\Delta} + e^{(\tau_k-1)\theta\Delta} y_0 \implies N^{1/2}\sigma \left[\frac{\mu}{N^{1/2}\sigma} (1 - e^{r_k c}) + J_c(r_k) + e^{r_k c} \frac{y_0}{N^{1/2}\sigma} \right]$$

and

$$\sum_{j=1}^{\tau_k-1} e^{(\tau_k-1-j)\theta\Delta} \sum_{s=1}^K \phi_s I_{j\Delta}^s = \begin{cases} \sum_{s=1}^K \phi_s e^{(\tau_k-1)\theta\Delta} \sum_{j=1}^{\tau_k-1} e^{-j\theta\Delta} I_{j\Delta}^s = 0 & \text{if } K = 1 \\ \sum_{j=1}^{k-1} e^{(\tau_k-1-\tau_j)\theta\Delta} \phi_j \rightarrow N^{1/2}\sigma \sum_{j=1}^{k-1} e^{(r_k-r_j)\theta} \frac{\phi_j}{N^{1/2}\sigma} & \text{if } K > 1 \end{cases}.$$

Therefore,

$$\begin{aligned}
 T^{-1/2} \sum_{i=1}^T y_{(i-1)\Delta} \varepsilon_{i\Delta} &= \frac{\sigma N^{1/2}}{2} \left\{ \tilde{\Xi}_1^2 - \frac{y_0^2}{\sigma^2 N} - 2c \tilde{\Xi}_3 - 1 + 2c \frac{\mu}{\sigma N^{1/2}} \tilde{\Xi}_2 - \sum_{k=1}^K \frac{\phi_k^2}{\sigma^2 N} \right. \\
 &\quad \left. - 2 \sum_{k=1}^K \frac{\phi_k}{N^{1/2}\sigma} \left[\frac{\mu}{N^{1/2}\sigma} (1 - e^{r_k c}) + J_c(r_k) + e^{r_k c} \frac{y_0}{N^{1/2}\sigma} + \Delta_4 \right] \right\} \\
 &= \sigma N^{1/2} \tilde{\Xi}_4.
 \end{aligned}$$

□

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