Supplement to "Generalized Laplace Inference in Multiple Change-Points Models"

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Abstract

This supplemental material is structured as follows. Section A contains the Mathematical Appendix which includes all proofs of the results in the paper. Section B includes further simulation results comparing the GL-LN method to the GL estimators proposed in Casini and Perron (2020).

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A Mathematical Appendix

The mathematical appendix is structured as follows. Section A.2 presents some preliminary lemmas which will be used in the sequel. The proofs of the theoretical results in the paper are in Section A.3-A.5.

A.1 Additional Notation

The $(i,\, j)$ element of A is denoted by $A^{(i,j)}$. For a matrix A, the orthogonal projection matrices P_A, M_A are defined as $P_A = A\left(A'A\right)^{-1}A'$ and $M_A = I - P_A$, respectively. Also, for a projection matrix P, $\|PA\| \leq \|A\|$. We denote the d-dimensional identity matrix by I_d. When the context is clear we omit the subscript notation in the projection matrices. We denote the $i \times j$ upper-left (resp., lower-right) sub-block of A as $[A]_{\{i\times j,\cdot\}}$ (resp., $[A]_{\{\cdot,i\times j\}}$). Note that the norm of A is equal to the square root of the maximum eigenvalue of $A'A$, and thus, $\|A\| \leq \left[\mathrm{tr}\left(A'A\right)\right]^{1/2}$. For a sequence of matrices $\{A_T\}$, we write $A_T = o_{\mathbb{P}}(1)$ if each of its elements is $o_{\mathbb{P}}(1)$ and likewise for $O_{\mathbb{P}}(1)$. For a random variable ξ and a number $r \geq 1$, $\|\xi\|_r = \left(\mathbb{E}\|\xi\|^r\right)^{1/r}$. K is a generic constant that may vary from line to line; we may sometime write K_r to emphasize the dependence of K on a number r. For two scalars a and b, $a \wedge b = \inf\{a,\, b\}$. We may use $\sum_k$ when the limits of the summation are clear from the context. Unless otherwise sated $\mathbf{A}^c$ denotes the complementary set of $\mathbf{A}$.

A.2 Preliminary Lemmas

We first present results related to the extremum criterion function $Q_T\left(\delta\left(T_b\right),\, T_b\right)$ under the following assumption (Assumptions 3.1-3.2 are not needed in this section).

Assumption A.1. *We consider model (2.3) with Assumptions 2.1-2.4 and 3.3-3.5.*

Lemma A.1. *The following inequalities hold $\mathbb{P}$-a.s.:*

$$\left(Z_0'MZ_0\right) - \left(Z_0'MZ_2\right)\left(Z_2'MZ_2\right)^{-1}\left(Z_2'MZ_0\right) \geq D'\left(X_\Delta'X_\Delta\right)\left(X_2'X_2\right)^{-1}\left(X_0'X_0\right)D, \qquad T_b < T_b^0 \tag{A.1}$$

$$\left(Z_0'MZ_0\right) - \left(Z_0'MZ_2\right)\left(Z_2'MZ_2\right)^{-1}\left(Z_2'MZ_0\right) \geq D'\left(X_\Delta'X_\Delta\right)\left(X'X - X_2'X_2\right)^{-1}\left(X'X - X_0'X_0\right)D, \qquad T_b \geq T_b^0 \tag{A.2}$$

Proof. See Lemma A.1 in Bai (1997). □

Recall that $Q_T\left(\delta\left(T_b\right),\, T_b\right) = \delta\left(T_b\right)\left(Z_2'MZ_2\right)\delta\left(T_b\right)$. We decompose $Q_T\left(\delta\left(T_b\right),\, T_b\right) - Q_T\left(\delta\left(T_b^0\right),\, T_b^0\right)$ into a "deterministic" and a "stochastic" component. It follows by definition that,

$$\delta\left(T_b\right) = \left(Z_2'MZ_2\right)^{-1}\left(Z_2'MY\right) = \left(Z_2'MZ_2\right)^{-1}\left(Z_2'MZ_0\right)\delta_T + \left(Z_2'MZ_2\right)^{-1}Z_2Me,$$

and

$$\delta\left(T_b^0\right) = \left(Z_0'MZ_0\right)^{-1}\left(Z_0'MY\right) = \delta_T + \left(Z_0'MZ_0\right)^{-1}\left(Z_0'Me\right).$$

Therefore

$$Q_T\left(\delta\left(T_b\right),\, T_b\right) - Q_T\left(\delta\left(T_b^0\right),\, T_b^0\right) = \delta\left(T_b\right)'\left(Z_2'MZ_2\right)\delta\left(T_b\right) - \delta\left(T_b^0\right)'\left(Z_0'MZ_0\right)\delta\left(T_b^0\right) \tag{A.3}$$

$$\triangleq g_d\left(\delta_T,\, T_b\right) + g_e\left(\delta_T,\, T_b\right), \tag{A.4}$$

where

$$g_d\left(\delta_T,\, T_b\right) = \delta_T'\left\{\left(Z_0'MZ_2\right)\left(Z_2'MZ_2\right)^{-1}\left(Z_2'MZ_0\right) - Z_0'MZ_0\right\}\delta_T, \tag{A.5}$$

and

$$g_e(\delta_T, T_b) = 2\delta_T'(Z_0'MZ_2)(Z_2'MZ_2)^{-1}Z_2Me - 2\delta_T'(Z_0'Me) \tag{A.6}$$
$$+ e'MZ_2(Z_2'MZ_2)^{-1}Z_2Me - e'MZ_0(Z_0'MZ_0)^{-1}Z_0'Me. \tag{A.7}$$

(A.5) constitutes the deterministic component and $g_e(\delta_T, T_b)$ the stochastic one. Denote

$$X_\Delta \triangleq X_2 - X_0 = \left(0, \ldots, 0, x_{T_b+1}, \ldots, x_{T_b^0}, 0, \ldots,\right)', \qquad \text{for } T_b < T_b^0$$
$$X_\Delta \triangleq -(X_2 - X_0) = \left(0, \ldots, 0, x_{T_b^0+1}, \ldots, x_{T_b}, 0, \ldots,\right)', \qquad \text{for } T_b > T_b^0$$

whereas $X_\Delta \triangleq 0$ when $T_b = T_b^0$. Observe that $X_2 = X_0 + X_\Delta \text{sign}(T_b^0 - T_b)$. When the sign is immaterial, we simply write $X_2 = X_0 + X_\Delta$. Next, let $Z_\Delta = X_\Delta D$, and define

$$\overline{g}_d(\delta_T, T_b) \triangleq -\frac{g_d(\delta_T, T_b)}{|T_b - T_b^0|}. \tag{A.8}$$

We arbitrarily define $\overline{g}_d(\delta^0, T_b) = \delta_T'\delta_T$ when $T_b = T_b^0$. Observe that $\overline{g}_d(\delta_T, T_b)$ is non-negative because the matrix inside the braces in (A.5) is negative semidefinite. (A.3) can be written as

$$Q_T(\delta(T_b), T_b) - Q_T\left(\delta\left(T_b^0\right), T_b^0\right) = -\left|T_b - T_b^0\right| \overline{g}_d(\delta_T, T_b) + g_e(\delta_T, T_b), \qquad \text{for all } T_b. \tag{A.9}$$

We use the notation $u = T\|\delta_T\|^2(\lambda_b - \lambda_0)$ and $T_b = T\lambda_b$. For $\eta > 0$, let $B_{T,\eta} \triangleq \{T_b : |T_b - T_b^0| \leq T\eta\}$, $B_{T,K} \triangleq \left\{T_b : |T_b - T_b^0| \leq K/\|\delta_T\|^2\right\}$ and $B_{T,K}^c \triangleq \left\{T_b : T\eta \geq |T_b - T_b^0| > K/\|\delta_T\|^2\right\}$, with $K > 0$. Note that $B_{T,\eta} = B_{T,K} \cup B_{T,K}^c$. Further, let $B_{T,\eta}^c \triangleq \{T_b : |T_b - T_b^0| > T\eta\}$.

Lemma A.2. *Under Assumption A.1,* $Q_T(\delta(T_b), T_b) - Q_T(\delta(T_b^0), T_b^0) = -\delta_T' Z_\Delta' Z_\Delta \delta_T + 2\text{sgn}(T_b^0 - T_b)\delta_T' Z_\Delta' e + o_{\mathbb{P}}(1)$ *uniformly on* $B_{T,K}$ *for* K *large enough.*

Proof. It follows from Lemma A.5 in Bai (1997). □

Lemma A.3. *Under Assumption A.1, for* $T_b = T_b^0 + \left\lfloor u/\|\delta_T\|^2\right\rfloor$*, we have* $\delta_T' Z_\Delta' Z_\Delta \delta_T = \delta_T' \sum_{t=T_b+1}^{T_b^0} z_t z_t' \delta_T = |u|(\delta^0)'\overline{V}\delta^0 + o_{\mathbb{P}}(1)$*, where* $\overline{V} = V_1$ *if* $u \leq 0$ *and* $\overline{V} = V_2$ *if* $u > 0$.

Proof. It follows from basic arguments (cf. Assumptions 3.4-3.5). □

Lemma A.4. *Under Assumption A.1, for any* $\epsilon > 0$ *there exists a* $C < \infty$ *and a positive sequence* $\{\nu_T\}$*, with* $\nu_T \to \infty$ *as* $T \to \infty$*, such that*

$$\liminf_{T\to\infty} \mathbb{P}\left[\sup_{K \leq |u| \leq \eta T\|\delta_T\|^2} Q_T(\delta(T_b), T_b) - Q_T\left(\delta\left(T_b^0\right), T_b^0\right) < -C\nu_T\right] \geq 1 - \epsilon,$$

for all sufficiently large K *and a sufficiently small* $\eta > 0$.

Proof. Note that on $\left\{K \leq |u| \leq \eta T\|\delta_T\|^2\right\}$ we have $K/\|\delta_T\|^2 \leq |T_b - T_b^0| \leq \eta T$. In view of (A.8), the statement $Q_T(\delta(T_b), T_b) - Q_T(\delta(T_b^0), T_b^0) < -C\nu_T$ follows from showing that as $T \to \infty$,

$$\mathbb{P}\left(\sup_{T_b \in B_{K,T}^c} g_e(\delta_T, T_b) \geq \inf_{T_b \in B_{K,T}^c} \left|T_b - T_b^0\right|^\kappa \overline{g}_d(\delta_T, T_b)\right) < \epsilon,$$

where $\kappa \in (1/2,\, 1)$. Suppose $T_b < T_b^0$. We show that

$$\mathbb{P}\left(\sup_{T_b \in B_{K,T}^c} \frac{\|\delta_T\|}{K} g_e(\delta_T,\, T_b) \geq \frac{1}{\|\delta_T\|^{2\kappa-1}} \left(\frac{1}{K}\right)^{1-\kappa} \inf_{T_b \in B_{K,T}^c} \overline{g}_d(\delta_T,\, T_b)\right) < \epsilon. \tag{A.10}$$

Lemma A.5-(ii) stated below implies that $\inf_{T_b \in B_{T,K}^c} \overline{g}_d(\delta_T,\, T_b)$ is bounded away from zero as $T \to \infty$ for large K and small η. Next, we show that

$$\sup_{T_b \in B_{K,T}^c} K^{-1} \|\delta_T\| g_e(\delta_T,\, T_b) = o_{\mathbb{P}}(1). \tag{A.11}$$

Consider the first term of (A.6),

$$\begin{aligned} 2\delta_T' (Z_0' M Z_2)(Z_2' M Z_2)^{-1} Z_2 M e &= 2\delta_T' (Z_0' M Z_2/T)(Z_2' M Z_2/T)^{-1} Z_2 M e \\ &= 2C \|\delta_T\| O_{\mathbb{P}}(1) O_{\mathbb{P}}(1) O_{\mathbb{P}}\left(T^{1/2}\right) = C O_{\mathbb{P}}\left(\|\delta_T\| T^{1/2}\right). \end{aligned}$$

When multiplied by $\|\delta_T\|/K$, this term is $O_{\mathbb{P}}\left(\|\delta_T\|^2 T^{1/2}/K\right)$ which goes to zero for large K. The second term in (A.6), when multiplied by $\|\delta_T\|/K$, is

$$2K^{-1} \|\delta_T\| \delta_T' (Z_0' M e) = K^{-1} \|\delta_T\| O_{\mathbb{P}}\left(\|\delta_T\| T^{1/2}\right) = K^{-1} O_{\mathbb{P}}\left(\|\delta_T\|^2 T^{1/2}\right),$$

which converges to zero using the same argument as for the first term. Consider now the first term of (A.7), $T^{-1/2} e' M Z_2 (Z_2' M Z_2/T)^{-1} T^{-1/2} Z_2 M e = O_{\mathbb{P}}(1)$. A similar argument can be used for the second term which is also $O_{\mathbb{P}}(1)$. The latter two terms multiplied by $\|\delta_T\|/K$ is $O_{\mathbb{P}}(\|\delta_T\|/K) = o_{\mathbb{P}}(1)$. This proves (A.11) and thus (A.10). To conclude the proof, note that $\kappa \in (1/2,\, 1)$ implies $\|\delta_T\|^{-(2\kappa-1)} \to \infty$, so that we can choose $\nu_T = \left(\|\delta_T\|^2/K\right)^{-(1-\kappa)}$. □

Lemma A.5. *Let* $\widetilde{g}_d \triangleq \inf_{|T_b - T_b^0| > K\|\delta_T\|^{-2}} \overline{g}_d(\delta_T,\, T_b)$. *Under Assumption A.1,*

(i) for any $\epsilon > 0$ *there exists some* $C > 0$ *such that* $\liminf_{T\to\infty} \mathbb{P}\left(\widetilde{g}_d > C\|\delta_T\|^2\right) \leq 1 - \epsilon$*;*

(ii) with $B_{T,K}^c = \left\{T_b : T\eta \geq |T_b - T_b^0| \geq K/\|\delta_T\|^2\right\}$, *for any* $\epsilon > 0$ *there exists a* $C > 0$ *such that* $\liminf_{T\to\infty} \mathbb{P}\left(\inf_{T_b \in B_{T,K}^c} \overline{g}_d(\delta_T,\, T_b) > C\right) \leq 1 - \epsilon$.

Proof. Part (i) was proved in Lemma A.2 of Bai (1997). As for part (ii), by Lemma A.1,

$$\overline{g}_d\left(\delta^0,\, T_b\right) \geq \delta_T D' \frac{X_\Delta' X_\Delta}{T_b^0 - T_b} (X_2' X_2)^{-1} (X_0' X_0) D\delta_T \geq \lambda_{J,T_b},$$

where λ_{J,T_b} is the minimum eigenvalue of $D' J(T_b) D$, with $J(T_b) \triangleq \|\delta_T\|^2 (T_b^0 - T_b)^{-1} X_\Delta' X_\Delta (X_2' X_2)^{-1} (X_0' X_0)$. It is sufficient to show that, for $T_b \in B_{T,K}^c$, λ_{J,T_b} is bounded away from zero with large probability for large K and small η. We have $\left\|J(T_b)^{-1}\right\| \leq \left\|\left[\|\delta_T\|^2 (T_b^0 - T_b)^{-1} X_\Delta' X_\Delta\right]^{-1}\right\| \left\|(X_2' X_2)(X_0' X_0)^{-1}\right\|$ and by Assumption 2.3-2.4 $\left\|(X_2' X_2)(X_0' X_0)^{-1}\right\| \leq \|X' X\| \left\|(X_0' X_0)^{-1}\right\|$ is bounded. Next, note that $(T_b^0 - T_b)^{-1} X_\Delta' X_\Delta = (T_b^0 - T_b)^{-1} \sum_{t=T_b+1}^{T_b^0} x_t x_t'$ is larger than $(T\eta)^{-1} \sum_{t=T_b^0 - \lfloor K/\|\delta_T\|^2 \rfloor}^{T_b^0} x_t x_t'$ on $B_{T,K}^c$, and for all K, $\left(\|\delta_T\|^2/K\right) \sum_{t=T_b^0 - \lfloor K/\|\delta_T\|^2 \rfloor}^{T_b^0} x_t x_t'$ is positive definite with large probability as $T \to \infty$ by Assumption 2.3. Now, $(K/T\eta)\left(\|\delta_T\|^2/K\right) \sum_{t=T_b^0 - \lfloor K/\|\delta_T\|^2 \rfloor}^{T_b^0} x_t x_t' = O_{\mathbb{P}}(1)$, by choosing sufficiently large K and small η. Thus, $\left\|\left[\|\delta_T\|^2 (T_b^0 - T_b)^{-1} X_\Delta' X_\Delta\right]^{-1}\right\|$ is bounded with large probability for such large

K and small η, which in turn implies that $\|J(T_b)^{-1}\|$ is bounded. Since D has full column rank, λ_{J,T_b} is bounded away from zero for sufficiently large K and small η. □

Lemma A.6. *Under Assumption A.1, for any $\epsilon > 0$ there exists a $C > 0$ such that*

$$\liminf_{T\to\infty} \mathbb{P}\left[\sup_{|u|\geq T\|\delta_T\|^2\eta} Q_T(\delta(T_b), T_b) - Q_T\left(\delta\left(T_b^0\right), T_b^0\right) < -C\nu_T\right] \geq 1-\epsilon,$$

for every $\eta > 0$, where $\nu_T \to \infty$.

Proof. Fix any $\eta > 0$. Note that on $\left\{|u| \geq T\|\delta_T\|^2\eta\right\}$ we have $|T_b - T_b^0| \geq T\eta$. We proceed in a similar manner to Lemma A.4. Let $B_{T,\eta}^c \triangleq \{T_b : |T_b - T_b^0| \geq T\eta\}$ and recall (A.8). First, as in Lemma A.5-(i), we have $\inf_{T_b\in B_{T,\eta}^c} \overline{g}_d(\delta_T, T_b) \geq C\|\delta_T\|^2$ with large probability for some $C > 0$. Noting that $T\eta \inf_{T_b\in B_{T,\eta}^c} \overline{g}_d(\delta_T, T_b)$ diverges at rate $\tau_T = T\|\delta_T\|^2$, the claim follows if we can show that $g_e(\delta_T, T_b) = O_{\mathbb{P}}(\tau_T^{\varpi})$, with $0 \leq \varpi < 1$ uniformly on $B_{T,\eta}^c$. This is shown in Lemma A.7 below, which suggests setting $\varpi \in (1/2, 1)$. Then, choose $\nu_T = \left(T\|\delta_T\|^2\right)^{1-\varpi}$. □

Lemma A.7. *Under Assumption A.1, uniformly on $B_{T,\eta}^c$, $|g_e(\delta_T, T_b)| = O_{\mathbb{P}}\left(\|\delta_T\| T^{1/2}\log T\right)$.*

Proof. We show that $T^{-1}|g_e(\delta^0, T_b)| = O_{\mathbb{P}}\left(\|\delta_T\| T^{-1/2}\log T\right)$ uniformly on $B_{T,\eta}^c$. Note that

$$\sup_{T_b\in B_{T,\eta}^c} |g_e(\delta_T, T_b)| \leq \sup_{q\leq T_b\leq T-q} |g_e(\delta_T, T_b)|,$$

and recall that $q = \dim(z_t)$ is needed for identification. Observe that

$$\sup_{q\leq T_b\leq T-q} \left\|\left(Z_2'MZ_2\right)^{-1/2} Z_2'Me\right\| = O_{\mathbb{P}}(\log T), \tag{A.12}$$

by the law of iterated logarithms [cf. Billingsley (1995), Ch. 1, Theorem 9.5]. Next,

$$\sup_{q\leq T_b\leq T-q} T^{-1/2}\left(Z_0'MZ_2\right)\left(Z_2'MZ_2\right)^{-1/2} = O_{\mathbb{P}}(1), \tag{A.13}$$

which can be proved using the inequality $(Z_0'MZ_2)(Z_2'MZ_2)(Z_0'MZ_2) \leq Z_0'MZ_0 = O_{\mathbb{P}}(T)$ (valid for all T_b). Thus, by (A.12) and (A.13), the first term on the right-hand side of (A.6) multiplied by T^{-1} is such that

$$\sup_{q\leq T_b\leq T-q} 2\delta_T' T^{-1}\left(Z_0'MZ_2\right)\left(Z_2'MZ_2\right)^{-1} Z_2'Me = O_{\mathbb{P}}\left(\|\delta_T\| T^{-1/2}\log T\right). \tag{A.14}$$

The second term on the right-hand side of (A.6) is $2\delta_T' Z_0'Me = O_{\mathbb{P}}\left(\|\delta_T\| T^{1/2}\right)$. Using (A.12), and dividing by T, the first term of (A.7) is $O_{\mathbb{P}}\left((\log T)^2/T\right)$ while the last term is $O_{\mathbb{P}}(T^{-1})$. When divided by T, they are of order $O_{\mathbb{P}}\left((\log T)^2/T\right)$ and $O_{\mathbb{P}}(T^{-1})$, respectively. Therefore, $|g_e(T_b, \delta^0)| = O_{\mathbb{P}}\left(\|\delta_T\| T^{1/2}\log T\right)$, uniformly on $B_{T,\eta}^c$. □

A.3 Proofs of Results in Section 3

We denote by $\boldsymbol{P}$ the class of polynomial functions $p : \mathbb{R} \to \mathbb{R}$. Let $\mathbf{U}_T \triangleq \{u \in \mathbb{R} : \lambda_b^0 + u/\psi_T \in \Gamma^0\}$, $\Gamma_{T,\psi} \triangleq \{u \in \mathbb{R} : |u| \leq \psi_T\}$, $\Gamma_{T,\psi}^c \triangleq \mathbb{R} - \Gamma_{T,\psi}$, and $\widetilde{\mathbf{U}}_T^c \triangleq \mathbf{U}_T - \Gamma_{T,\psi}$. For $u \in \mathbb{R}$, let $R_{T,v}(u) \triangleq Q_{T,v}(u) -$

$\Lambda^0(u)$ and $\overline{G}_{T,v}(u) \triangleq \sup_{\widetilde{v}\in\mathbf{V}} \widetilde{G}_{T,v}(u,\, \widetilde{v})$. The generic constant $0 < C < \infty$ used below may change from line to line. Finally, let $\widetilde{\gamma}_T \triangleq \gamma_T / T \|\delta_T\|^2$.

A.3.1 Proof of Proposition 3.1

We begin with the proof for the case of a fixed shift.

Lemma A.8. *Under Assumption 2.1-2.4, 3.1-3.3 (except that $\delta_T = \delta^0$) and 3.6-(i), $\widehat{\lambda}_b^{\mathrm{GL}} = \lambda_b^0 + o_{\mathbb{P}}(1)$.*

Proof. Let $\overline{S}_T(\delta(\lambda_b),\, \lambda_b) \triangleq Q_T(\delta(\lambda_b),\, \lambda_b) - Q_T(\delta(\lambda_b^0),\, \lambda_b^0)$. From (A.9),

$$\overline{S}_T\left(\widehat{\delta}(\lambda_b),\, \lambda_b\right) = -\left|T_b - T_b^0\right| \overline{g}_d\left(\delta^0,\, T_b\right) + g_e\left(\delta^0,\, T_b\right),$$

where $g_e(\delta^0,\, T_b)$ and $\overline{g}_d(\delta^0,\, T_b)$ are defined in (A.6)-(A.8). By Lemma A.24 in Bai (1997), $\liminf_{T\to\infty} \overline{g}_d(\delta^0,\, T_b) > 0$ and $T^{-1}\sup_{T_b} |g_e(\delta^0,\, T_b)| = O_{\mathbb{P}}\left(T^{-1/2}\log T\right)$. Thus, for any $B > 0$ if $\left|\widehat{\lambda}_b^{\mathrm{GL}} - \lambda_b^0\right| > B$ we have that,

$$-\overline{S}_T\left(\widehat{\delta}(\lambda_b),\, \lambda_b\right) \to \infty \,\text{at rate}\, TB. \tag{A.15}$$

Let $p_T(u) \triangleq p_{1,T}(u)/\overline{p}_T$ with $p_{1,T}(u) = \exp\left(Q_T(\delta(u),\, u)\right)$ and $\overline{p}_T \triangleq \int_{\mathbf{U}_T} p_{1,T}(w)\, dw$. By definition, $\widehat{\lambda}_b^{\mathrm{GL}}$ is the minimum of the function $\int_{\Gamma^0} l(s-u)\, p_{1,T}(u)\, \pi(u)\, du$ with $s \in \Gamma^0$. Using a change in variables,

$$\begin{aligned}&\int_{\Gamma^0} l(s-u)\, p_{1,T}(u)\, \pi(u)\, du \\ &\qquad = T^{-1}\overline{p}_T \int_{\mathbf{U}_T} l\left(T\left(s-\lambda_b^0\right) - u\right) p_T\left(\lambda_b^0 + T^{-1}u\right) \pi\left(\lambda_b^0 + T^{-1}u\right) du,\end{aligned}$$

where $\mathbf{U}_T \triangleq \left\{u \in \mathbb{R} :\, \lambda_b^0 + T^{-1}u \in \Gamma^0\right\}$. Thus, $\lambda_{\delta,T} \triangleq T\left(\widehat{\lambda}_b^{\mathrm{GL}} - \lambda_b^0\right)$ is the minimum of the function,

$$\mathcal{S}_T(s) \triangleq \int_{\mathbf{U}_T} l(s-u)\, \frac{p_T\left(\lambda_b^0 + T^{-1}u\right) \pi\left(\lambda_b^0 + T^{-1}u\right)}{\int_{\mathbf{U}_T} p_T\left(\lambda_b^0 + T^{-1}w\right) \pi\left(\lambda_b^0 + T^{-1}w\right) dw} du,$$

where the optimization is over $\mathbf{U}_T$. We shall show that for any $B > 0$,

$$\mathbb{P}\left[\left|\widehat{\lambda}_b^{\mathrm{GL}} - \lambda_b^0\right| > B\right] \le \mathbb{P}\left[\inf_{|s|>TB} \mathcal{S}_T(s) \le \mathcal{S}_T(0)\right] \to 0. \tag{A.16}$$

By assumption the prior is bounded and so we can proceed to the proof for the case $\pi(u) = 1$ for all u. By the properties of the family $\boldsymbol{L}$ of loss functions, we can find $\overline{u}_1,\, \overline{u}_2 \in \mathbb{R}$, with $0 < \overline{u}_1 < \overline{u}_2$ such that as T increases,

$$\bar{l}_{1,T} \triangleq \sup\left\{l(u) :\, u \in \Gamma_{1,T}\right\} < \bar{l}_{2,T} \triangleq \inf\left\{l(u) :\, u \in \Gamma_{2,T}\right\},$$

where $\Gamma_{1,T} \triangleq \mathbf{U}_T \cap (|u| \le \overline{u}_1)$ and $\Gamma_{2,T} \triangleq \mathbf{U}_T \cap (|u| > \overline{u}_2)$. With this notation,

$$\mathcal{S}_T(0) \le \bar{l}_{1,T} \int_{\Gamma_{1,T}} p_T(u)\, du + \int_{\mathbf{U}_T \cap (|u| > \overline{u}_1)} l(u)\, p_T(u)\, du.$$

If $l \in \boldsymbol{L}$ then for a sufficiently large T the following relationship holds: $l(u) - \inf_{|v|>TB/2} l(v) \le 0$, $|u| \le (TB/2)^{\vartheta}$ for some $\vartheta > 0$. It also follows that for large T we have $TB > 2\overline{u}_2$ and $(TB/2)^{\vartheta} > \overline{u}_2$. Let

$\Gamma_{T,B} \triangleq \{u : (|u| > TB/2) \cap \mathbf{U}_T\}$. Then, whenever $|s| > TB$ and $|u| \leq TB/2$, we have,

$$|u - s| > TB/2 > \overline{u}_2 \qquad \text{and} \qquad \inf_{u \in \Gamma_{T,B}} l(u) \geq \bar{l}_{2,T}. \tag{A.17}$$

With this notation,

$$\begin{aligned}\inf_{|s|>TB} \mathcal{S}_T(s) &\geq \inf_{u \in \Gamma_{T,B}} l_T(u) \int_{(|w| \leq TB/2) \cap \mathbf{U}_T} p_T(w)\, dw \\ &\geq \bar{l}_{2,T} \int_{(|w| \leq TB/2) \cap \mathbf{U}_T} p_T(w)\, dw,\end{aligned}$$

from which it follows that

$$\begin{aligned}\mathcal{S}_T(0) - \inf_{|s|>TB} \mathcal{S}_T(s) \leq& -\varpi \int_{\Gamma_{1,T}} p_T(u)\, du \\ &+ \int_{\mathbf{U}_T \cap \left((TB/2)^{\vartheta} \geq |u| \geq \overline{u}_1\right)} \left(l(u) - \inf_{|s|>TB/2} l_T(s)\right) p_T(u)\, du \\ &+ \int_{\mathbf{U}_T \cap \left(|u| > (TB/2)^{\vartheta}\right)} l(u)\, p_T(u)\, du,\end{aligned}$$

where $\varpi \triangleq \bar{l}_{2,T} - \bar{l}_{1,T}$. The last inequality can be manipulated further using (A.17),

$$\begin{aligned}\mathcal{S}_T(0) - \inf_{|s|>TB} \mathcal{S}_T(s) \leq& -\varpi \int_{\Gamma_{1,T}} p_T(u)\, du \\ &+ \int_{\mathbf{U}_T \cap \left(|u| > (TB/2)^{\vartheta}\right)} l_T(u)\, p_T(u)\, du.\end{aligned} \tag{A.18}$$

Since $l \in \boldsymbol{L}$, we have $l(u) \leq |u|^a$, $a > 0$ when u is large enough. Thus, given (A.15), the second term of (A.18) converges to zero. Since $\int_{\Gamma_{1,T}} p_T(u)\, du > 0$ the first term of (A.18) is negative which then leads to $\mathcal{S}_T(0) - \inf_{|s|>TB} \mathcal{S}_T(s) < 0$ or $\mathcal{S}_T(0) < \inf_{|s|>TB} \mathcal{S}_T(s)$. Thus, we have (A.16). □

Lemma A.9. *Under Assumption 2.1-2.4, 3.1-3.3 and 3.6-(i), for $l \in \boldsymbol{L}$ and any $B > 0$ and $\varepsilon > 0$, we have for all large T, $\mathbb{P}\left[\left|\widehat{\lambda}_b^{\mathrm{GL}} - \lambda_b^0\right| > B\right] < \varepsilon$.*

Proof. The structure of the proof is similar to that of Lemma A.8. By Proposition 1 in Bai (1997), eq. (A.15) holds with $O_{\mathbb{P}}\left(T \|\delta_T\|^2\right)$ in place of $O_{\mathbb{P}}(TB)$, $B > 0$. One can then follow the same steps as in the previous lemma to yield the result. □

Lemma A.10. *Under Assumption 2.1-2.4, 3.1-3.3 and 3.6-(i), for $l \in \boldsymbol{L}$ and for every $\varepsilon > 0$ there exists a $B < \infty$ such that for all large T, $\mathbb{P}\left[Tv_T^2 \left|\widehat{\lambda}_b^{\mathrm{GL}} - \lambda_b^0\right| > B\right] < \varepsilon$.*

Proof. See Lemma A.29 which proves a stronger result needed for Theorem 3.2. □

Parts (i) and (ii) of Proposition 3.1 follow from Lemma A.9 and Lemma A.10, respectively.

A.3.2 Proof of Theorem 3.1

We start with the following lemmas.

Lemma A.11. *For any $a \in \mathbb{R}$, $|c| \leq 1$, and integer $i \geq 0$, $\left|\exp(ca) - \sum_{j=0}^{i} (ca)^j / j!\right| \leq |c|^{i+1} \exp(|a|)$.*

Proof. The proof is immediate and the same as the one in Jun, Pinkse, and Wan (2015). Using simple manipulations,

$$\left|\exp(ca)-\sum_{j=0}^{i}(ca)^j/j!\right|\leq\left|\sum_{j=i+1}^{\infty}\frac{(ca)^j}{j!}\right|\leq|c|^{i+1}\left|\sum_{j=i+1}^{\infty}\frac{(a)^j}{j!}\right|\leq|c|^{i+1}\exp(|a|).$$

□

Lemma A.12. *$\widetilde{G}_{T,v}(u,\widetilde{v})\Rightarrow\mathscr{W}(u)$ in $\mathbb{D}_b(\mathbf{C}\times\mathbf{V})$, where $\mathbf{C}\subset\mathbb{R}$ and $\mathbf{V}\subset\mathbb{R}^{p+2q}$ are both compact sets, and*

$$\mathscr{W}(u)\triangleq\begin{cases}2\left((\delta^0)'\Sigma_1\delta^0\right)^{1/2}W_1(-u), & \text{if } u<0\\ 2\left((\delta^0)'\Sigma_2\delta^0\right)^{1/2}W_2(u), & \text{if } u\geq 0.\end{cases}$$

Proof. Consider $u<0$. According to the expansion of the criterion function given in Lemma A.2, for any $(u,\widetilde{v})\in\mathbf{C}\times\mathbf{V}$, $\widetilde{G}_{T,v}(u,\widetilde{v})$ satisfies $2\text{sgn}\left(T_b^0-T_b(u)\right)\delta_T'Z_\Delta'e+o_{\mathbb{P}}(1)$. Then, $\delta_T'Z_\Delta'e=(\delta^0)'v_T\sum_{t=\lfloor u/v_T^2\rfloor}^{T_b^0}z_te_t$ $\Rightarrow(\delta^0)'\mathscr{G}_1(-u)$, where $\mathscr{G}_1$ is a multivariate Gaussian process. In particular, $(\delta^0)'\mathscr{G}_1(-u)$ is equivalent in law to $\left((\delta^0)'\Sigma_1\delta^0\right)^{1/2}W_1(-u)$, where $W_1(\cdot)$ is a standard Wiener process on $[0,\infty)$. Similarly, for $u\geq0$, $\delta_T'Z_\Delta'e\Rightarrow\left((\delta^0)'\Sigma_2\delta^0\right)^{1/2}W_2(u)$, where $W_2(\cdot)$ is another standard Wiener process on $[0,\infty)$ which is independent of W_1. Hence, $\widetilde{G}_{T,v}(u,\widetilde{v})\Rightarrow\mathscr{W}(u)$ in $\mathbb{D}_b(\mathbf{C}\times\mathbf{V})$. □

Lemma A.13. *Fix any $a>0$ and let $\varpi\in(1/2,1]$. (i) For any $\nu>0$ and any $\varepsilon>0$,*

$$\limsup_{T\to\infty}\mathbb{P}\left[\sup_{u\in\Gamma_{T,\psi}^c}\left\{\overline{G}_{T,v}(u)-a\left\|\delta^0\right\|^2|u|^{\varpi}\right\}>\nu\right]<\varepsilon.$$

(ii) For $\widetilde{u}\in\mathbb{R}_+$ let $\widetilde{\Gamma}\triangleq\{u\in\mathbb{R}:\,|u|>\widetilde{u}\}$. Then, for every $\epsilon>0$,

$$\lim_{\widetilde{u}\to\infty}\lim_{T\to\infty}\mathbb{P}\left[\sup_{u\in\widetilde{\Gamma}}\left\{\overline{G}_{T,v}(u)-a\left\|\delta^0\right\|^2|u|^{\varpi}\right\}>\epsilon\right]=0.$$

Proof. We begin with part (i). Upon using Lemma A.12 and the continuous mapping theorem, with any nonnegative integer i,

$$\begin{aligned}\limsup_{T\to\infty}\mathbb{P}\left[\sup_{u\in\Gamma_{T,\psi}^c}\left\{\overline{G}_{T,v}(u)-a\left\|\delta^0\right\|^2|u|^{\varpi}\right\}>\nu\right]&\leq\lim_{T\to\infty}\mathbb{P}\left[\sup_{|u|>\overline{u}}\left\{\overline{G}_{T,v}(u)-a\left\|\delta^0\right\||u|^{\varpi}\right\}>\nu\right]\\&\leq\lim_{T\to\infty}\mathbb{P}\left[\sup_{|u|\geq i}\left\{\overline{G}_{T,v}(u)>a\left\|\delta^0\right\||u|^{\varpi}\right\}>\nu\right]\\&\leq\mathbb{P}\left[\sup_{|u|\geq i}\left\{|\mathscr{W}(u)|-a\left\|\delta^0\right\||u|^{\varpi}\right\}>\nu\right]\\&\leq\sum_{r=i+1}^{\infty}\mathbb{P}\left[\sup_{r-1\leq|u|<r}\left\{|\mathscr{W}(u)|-a\left\|\delta^0\right\||u|^{\varpi}\right\}>\nu\right].\end{aligned}$$

Then,

$$\begin{aligned}
&\sum_{r=i+1}^{\infty}\mathbb{P}\left[\sup_{r-1\leq|u|<r}\frac{1}{\sqrt{r}}\left|\mathscr{W}\left(u\right)\right|>\inf_{r-1<|u|<r}a\frac{1}{\sqrt{r}}\left\|\delta^{0}\right\|\left|u\right|^{\varpi}\right]\\
&\quad=\sum_{r=i+1}^{\infty}\mathbb{P}\left[\sup_{1-1/r\leq|u|/r\leq1}\left|\mathscr{W}\left(u/r\right)\right|>\inf_{1-1/r<|u|/r\leq1}a\left(\frac{r}{r}\right)^{\varpi-1/2}\frac{|u|^{\varpi}}{\sqrt{r}}\left\|\delta^{0}\right\|\right]\\
&\quad=\sum_{r=i+1}^{\infty}\mathbb{P}\left[\sup_{1-1/r<s\leq1}\left|\mathscr{W}\left(s\right)\right|>\inf_{c<s\leq1}ar^{\varpi-1/2}s^{\varpi}\left\|\delta^{0}\right\|\right]\\
&\quad=\sum_{r=i+1}^{\infty}\mathbb{P}\left[\sup_{s\leq1}\left|\mathscr{W}\left(s\right)\right|>r^{\varpi-1/2}c^{\varpi}C\left\|\delta^{0}\right\|\right],
\end{aligned}\tag{A.19}$$

where $0<c\leq1$. By Markov's inequality,

$$\sum_{r=i+1}^{\infty}\mathbb{P}\left[\sup_{c<s\leq1}\left|\mathscr{W}\left(s\right)\right|^{4}>C^{4}\left\|\delta^{0}\right\|^{4}r^{4(\varpi-1/2)}c^{4\varpi}\right]\leq\frac{C}{\left\|\delta^{0}\right\|^{4}}\frac{\mathbb{E}\left(\sup_{s\leq1}\left|\mathscr{W}\left(s\right)\right|^{4}\right)}{c^{4\varpi}}\sum_{r=i+1}^{\infty}r^{-(4\varpi-2)}.\tag{A.20}$$

By Proposition A.2.4 in van der Vaart and Wellner (1996), $\mathbb{E}(\sup_{s\leq1}|\mathscr{W}(s)|^{4})\leq C\mathbb{E}\left(\sup_{s\leq1}|\mathscr{W}(s)|\right)^{4}$ for some $C<\infty$, which is finite by Corollary 2.2.8 in van der Vaart and Wellner (1996). Choose K (thus $\overline{u}$) large enough such that the right-hand side in (A.20) can be made arbitrarily smaller than $\varepsilon>0$. The proof of the second part is similar and omitted. □

Lemma A.14. *Fix any $a>0$. For any $\varepsilon>0$ there exists a $C<\infty$ such that*

$$\mathbb{P}\left[\sup_{u\in\mathbb{R}}\left\{\overline{G}_{T,v}\left(u\right)-a\left\|\delta^{0}\right\|^{2}|u|\right\}>C\right]<\varepsilon,\qquad\text{for all }T.$$

Proof. For any finite T, $\overline{G}_{T,v}(u)\in\mathbb{D}_{b}$ by definition. As for the limiting case, fix any $0<\overline{u}<\infty$,

$$\begin{aligned}
\limsup_{T\rightarrow\infty}\mathbb{P}\left[\sup_{u\in\mathbb{R}}\left\{\overline{G}_{T,v}\left(u\right)-a\left\|\delta^{0}\right\|^{2}|u|\right\}>C\right]&\leq\limsup_{T\rightarrow\infty}\mathbb{P}\left[\sup_{|u|\leq\overline{u}}\overline{G}_{T,v}\left(u\right)>C\right]\\
&\quad+\limsup_{T\rightarrow\infty}\mathbb{P}\left[\sup_{|u|>\overline{u}}\overline{G}_{T,v}\left(u\right)>a\left\|\delta^{0}\right\|^{2}\overline{u}\right].
\end{aligned}$$

The second term converges to zero letting $\overline{u}\rightarrow\infty$ from Lemma A.13-(ii). For the first term, let $C\rightarrow\infty$, use the continuous mapping theorem and Lemma A.12 to deduce that it converges to zero by the properties of $\mathscr{W}\in\mathbb{D}_{b}$. □

Lemma A.15. *Let*

$$\begin{aligned}
A_{1}\left(u,\,\widetilde{v}\right)&=u^{m}\pi_{T,v}\left(u\right)\exp\left(\widetilde{\gamma}_{T}\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right),\\
A_{2}\left(u,\,\widetilde{v}\right)&=u^{m}\pi^{0}\exp\left(\widetilde{\gamma}_{T}\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)-\Lambda_{0}\left(u\right)\right).
\end{aligned}\tag{A.21}$$

For $m\geq0$,

$$\liminf_{T\rightarrow\infty}\mathbb{P}\left[\sup_{\widetilde{v}\in\mathbf{V}}\left|\int_{\Gamma_{T,\psi}^{c}}\left(A_{1}\left(u,\,\widetilde{v}\right)-A_{2}\left(u,\,\widetilde{v}\right)\right)\right|<\epsilon\right]\geq1-\epsilon.$$

Proof. We consider each integrand $A_i(u, \widetilde{v})$ $(i = 1, 2)$ separately on $\Gamma^c_{T,\psi}$. Let us consider A_1 first. Lemma A.4 yields that whenever $\widetilde{\gamma}_T \to \kappa_\gamma < \infty$, $A_1(u, \widetilde{v}) \leq C_1 \exp(-C_2 \nu_T)$ where $0 < C_1, C_2 < \infty$ and ν_T is a divergent sequence. Note that the number C_1 follows from Assumption 3.2 (cf. $\pi(\cdot) < \infty$). The argument for $A_2(u, \widetilde{v})$ relies on Lemma A.13-(i), which shows that $G_{T,v}(u, \widetilde{v})$ is always less than $C|u|^{\varpi}$ uniformly on $\Gamma^c_{T,\psi}$, with $C > 0$ and $\varpi \in (1/2, 1)$. Thus, $A_2(u, \widetilde{v}) = o_{\mathbb{P}}(1)$ uniformly on $\mathbf{V}$. □

Let $\Gamma_{T,K} \triangleq \{u \in \mathbb{R} : |u| < K, K > 0\}$, and $\Gamma_{T,\eta} \triangleq \{u \in \mathbb{R} : K \leq |u| \leq \eta\psi_T, K, \eta > 0\}$.

Lemma A.16. *For any polynomial function $p \in \boldsymbol{P}$ and any $C < \infty$, let*

$$D_T \triangleq \sup_{\widetilde{v} \in \mathbf{W}} \int_{\Gamma_{T,K}} |p(u)| \exp\left\{C\widetilde{G}_{T,v}(u, \widetilde{v})\right\} |\exp(R_{T,v}(u)) - 1| \exp\left(-\Lambda^0(u)\right) du = o_{\mathbb{P}}(1).$$

Proof. Let $0 < \epsilon < 1$. We shall use Lemma A.11 with $i = 0$, $a = R_{T,v}(u)/c$, and $c = \epsilon$ to deduce that $D_T = O_{\mathbb{P}}(\epsilon)$ and then let $\epsilon \to 0$. Note that

$$\epsilon^{-1} D_T \leq C \int_{\Gamma_{T,K}} |p(u)| \exp\left(C\overline{G}_{T,v}(u, \widetilde{v}) + \left|\epsilon^{-1} R_{T,v}(u)\right| - \Lambda^0(u)\right) du.$$

By definition, $K \geq u = \|\delta_T\|^2 (T_b - T^0_b)$ on $\Gamma_{T,K}$. By Lemmas A.2-A.3, on $\Gamma_{T,K}$ we have $R_{T,v}(u) = O_{\mathbb{P}}\left(\|\delta_T\|^2\right)$ for each u. Thus, for large enough T, the right-hand side above is $O_{\mathbb{P}}(1)$ and does not depend on ϵ. Thus, $D_T = \epsilon O_{\mathbb{P}}(1)$. The claim of the lemma follows by letting ϵ approach zero. □

Lemma A.17. *For $p \in \boldsymbol{P}$,*

$$D_{2,T} \triangleq \sup_{\widetilde{v} \in \mathbf{V}} \int_{\Gamma_{T,\eta}} |p(u)| \exp\left\{\widetilde{\gamma}_T \widetilde{G}_{T,v}(u, \widetilde{v})\right\} \exp\left(-\Lambda^0(u)\right) \left|\pi_{T,v}(u) - \pi^0\right| du = o_{\mathbb{P}}(1).$$

Proof. By the differentiability of $\pi(\cdot)$ at λ^0_b (cf. Assumption 3.2), for any $u \in \mathbb{R}$ $|\pi_{T,v}(u) - \pi^0| \leq \left|\pi\left(\lambda^0_{b,T}(v)\right) - \pi^0\right| + C\psi_T^{-1}|u|$, with $C > 0$. The first term on the right-hand side is $o(1)$ and does not depend on u. Recalling that $\overline{G}_{T,v}(u, \widetilde{v}) = \sup_{\widetilde{v} \in \mathbf{V}} \left|\widetilde{G}_{T,v}(u, \widetilde{v})\right|$,

$$D_{2,T} \leq K\left[o(1) \int_{\Gamma_{T,\eta}} d_T(u) du + \psi_T^{-1} \int_{\Gamma_{T,\eta}} |u| d_T(u) du\right] \leq K\left[o(1) O_{\mathbb{P}}(1) + \psi_T^{-1} O_{\mathbb{P}}(1)\right],$$

where $d_T(u) \triangleq |p(u)| \exp\left\{\widetilde{\gamma}_T \overline{G}_{T,v}(u, \widetilde{v})\right\} |\exp(-\Lambda^0(u))|$ and the $O_{\mathbb{P}}(1)$ terms follows from Lemma A.14 and $\widetilde{\gamma}_T \to \kappa_\gamma < \infty$. Since $\psi_T \to \infty$, we have $D_{2,T} = o_{\mathbb{P}}(1)$. □

Lemma A.18. *For any $p \in \boldsymbol{P}$ and constants $C_1, C_2 > 0$, $\int_{\Gamma^c_{T,\psi}} |p(u)| \exp\left(C_1 \overline{G}_T(u) - C_2|u|\right) du = o_{\mathbb{P}}(1)$.*

Proof. It follows from Lemma A.13. □

Lemma A.19. *For $p \in \boldsymbol{P}$ and constants $a_1, a_2, a_3 \geq 0$, with $a_2 + a_3 > 0$, let*

$$D_{3,T} \triangleq \int_{\widetilde{\mathbf{U}}^c_T} |p(u)| \exp\left(\widetilde{\gamma}_T \left\{a_1 \overline{G}_{T,v}(u) + a_2 Q_{T,v}(u) - a_3 \Lambda^0(u)\right\}\right) du = o_{\mathbb{P}}(1).$$

Proof. It follows from Lemma A.6. □

Lemma A.20. *For any integer $m \geq 0$,*

$$\sup_{\widetilde{v}\in\mathbf{V}}\left|\int_{\mathbb{R}} u^m \exp\left(\widetilde{\gamma}_T \widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)\right)\left[\pi_{T,v}\left(u\right)\exp\left(Q_{T,v}\left(u\right)\right)-\pi^0\exp\left(-\varLambda^0\left(u\right)\right)\right]du\right|$$
$$=\sup_{\widetilde{v}\in\mathbf{V}}\left|\int_{\mathbb{R}}\left(A_1\left(u,\,\widetilde{v}\right)-A_2\left(u,\,\widetilde{v}\right)\right)du\right|$$
$$=o_{\mathbb{P}}\left(1\right).$$

Proof. By Assumption 3.2, $A_1\left(u,\,\widetilde{v}\right)=0$ for $u\in\Gamma_{T,\psi}^c-\widetilde{\mathbf{U}}_T^c$. Then, omitting arguments, we can write,

$$\sup\left|\int_{\mathbb{R}}\left(A_1-A_2\right)\right|\leq\sup\left|\int_{\Gamma_{T,\psi}}\left(A_1-A_2\right)\right|+\sup\left|\int_{\Gamma_{T,\psi}^c}A_2\right|+\sup\left|\int_{\widetilde{\mathbf{U}}_T^c}A_1\right|. \tag{A.22}$$

The first right-hand side term above converges in probability to zero by Lemmas A.16-A.17. The second and the last term are each $o_{\mathbb{P}}\left(1\right)$ by, receptively, Lemma A.18 and Lemma A.19. □

We are now in a position to conclude the proof of Theorem 3.1.

Proof. Let $\mathbf{V}\subset\mathbb{R}^{p+2q}$ be a compact set. From (3.11),

$$\psi_T\left(\widehat{\lambda}_b^{\mathrm{GL},*}\left(\widetilde{v},\,v\right)-\lambda_{b,T}^0\left(v\right)\right)=\frac{\int_{\mathbb{R}}u\exp\left(\widetilde{\gamma}_T\left[\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right]\right)\pi_{T,v}\left(u\right)du}{\int_{\mathbb{R}}\exp\left(\widetilde{\gamma}_T\left[\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right]\right)\pi_{T,v}\left(u\right)du}.$$

For a large enough T, by Lemma A.20 the right-hand is uniformly in $\widetilde{v}\in\mathbf{V}$ equal to

$$\frac{\int_{\mathbb{R}}u\exp\left(\widetilde{\gamma}_T\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)\right)\exp\left(-\varLambda^0\left(u\right)\right)du}{\int_{\mathbb{R}}\exp\left(\widetilde{\gamma}_T\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)\right)\exp\left(-\varLambda^0\left(u\right)\right)du}+o_{\mathbb{P}}\left(1\right).$$

The first term is integrable with large probability by Lemmas A.13-A.14. Thus, by Lemma A.12 and the continuous mapping theorem, we have for each $v\in\mathbf{V}$,

$$T\left\Vert\delta_T\right\Vert^2\left(\widehat{\lambda}_b^{\mathrm{GL},*}\left(\widetilde{v},\,v\right)-\lambda_{b,T}^0\left(v\right)\right)\Rightarrow\frac{\int_{\mathbb{R}}u\exp\left(\mathscr{W}\left(u\right)\right)\exp\left(-\varLambda^0\left(u\right)\right)du}{\int_{\mathbb{R}}\exp\left(\mathscr{W}\left(u\right)\right)\exp\left(-\varLambda^0\left(u\right)\right)du}. \tag{A.23}$$

Note that $\partial_\theta Q_T^0\left(\theta,\,\cdot\right)$ is monotonic and bounded for all $\theta\in\mathbf{S}$. The argument of Theorem 4.1 in Jurečková (1977) can be used in (A.23) to achieve uniformity in v. □

A.3.3 Proof of Proposition 3.2

We first need to introduce further notation. For a scalar $\overline{u}>0$ define $\Gamma_{\overline{u}}\triangleq\left\{u:\in\mathbb{R}:\,\left|u\right|\leq\overline{u}\right\}.$ Note that $\widetilde{\gamma}_T^{-1}=o\left(1\right)$. We shall be concerned with the asymptotic properties of the following statistic:

$$\xi_T\left(\widetilde{v}\right)=\frac{\int_{\Gamma_{\overline{u}}}u\exp\left(\widetilde{\gamma}_T\left(\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right)\right)\pi_{T,v}\left(u\right)du}{\int_{\Gamma_{\overline{u}}}\exp\left(\widetilde{\gamma}_T\left(\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right)\right)\pi_{T,v}\left(u\right)du}.$$

Furthermore, for every $\widetilde{v}\in\mathbf{V}$, let $\xi_0\left(\widetilde{v}\right)=\arg\max_{u\in\Gamma_{\overline{u}}}\mathscr{V}\left(u\right)$. It turns out that $\xi_0\left(\widetilde{v}\right)$ is flat in $\widetilde{v}$ and thus we write $\xi_0=\xi_0\left(\widetilde{v}\right)$. Finally, recall that $u=T\left\Vert\delta_T\right\Vert^2\left(\lambda_b-\lambda_{b,T}^0\left(v\right)\right).$

Lemma A.21. *Let $\Gamma^c_{T,\overline{u}} = \mathbf{U}_T - \Gamma_{\overline{u}}$. Then for any $\epsilon > 0$ and $m = 0,\, 1$,*

$$\lim_{\overline{u}\to\infty}\lim_{T\to\infty}\mathbb{P}\left(\frac{\sup_{\widetilde{v}\in\mathbf{V}}\int_{\Gamma^c_{T,\overline{u}}}|u|^m\exp\left(\widetilde{\gamma}_T\left(\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right)\right)\pi_{T,v}\left(u\right)du}{\sup_{\widetilde{v}\in\mathbf{V}}\int_{\mathbb{R}}\exp\left(\widetilde{\gamma}_T\left(\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right)\right)\pi_{T,v}\left(u\right)du}>\epsilon\right)=0.$$

Proof. Let J_1 and J_2 denote the numerator and denominator, respectively, in the display of the lemma. Then,

$$\mathbb{P}\left(J_1/J_2>\epsilon\right)\leq\mathbb{P}\left(J_2\leq\exp\left(-\overline{a}\widetilde{\gamma}_T\right)\right)+\mathbb{P}\left(J_1>\epsilon\exp\left(-\overline{a}\widetilde{\gamma}_T\right)\right), \tag{A.24}$$

for any constant $\overline{a} > 0$. Let us consider the second term term in (A.24). For an arbitrary $a > 0$, let $\mathbf{H}\left(\overline{u},\, a\right) = \left\{u\in\Gamma^c_{T,\overline{u}} : \sup_{\widetilde{v}\in\mathbf{V}}\left|\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)\right|\leq a\left|u\right|\right\}$. Let $\overline{\lambda} = 2\sup_{\lambda_b\in\varGamma^0}\left|\lambda_b\right|$. Note that $\overline{\lambda} < 2$ and $\sup_{u\in\mathbf{H}(\overline{u},\,a)}|u| \leq \overline{\lambda}T\left\|\delta_T\right\|^2$. By Assumption 2.3 and 2.4, and Lemma A.6, $Q_{T,v}\left(u\right) \leq -\min(\varLambda^0\left(u\right)/2,$ $\eta\overline{\lambda}\left\|\delta_T\right\|^2 T)$ uniformly for all large T where $\eta > 0$. Thus,

$$\begin{aligned}
&\sup_{u\in\mathbf{H}(\overline{u},\,a)}\sup_{\widetilde{v}\in\mathbf{V}}\exp\left(\widetilde{\gamma}_T\left[\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right]\right) &\text{(A.25)}\\
&\quad\leq\sup_{u\in\mathbf{H}(\overline{u},\,a)}\sup_{\widetilde{v}\in\mathbf{V}}\exp\left(\widetilde{\gamma}_T\left[a\left|u\right|-\varLambda^0\left(u\right)/4+\left[\varLambda^0\left(u\right)/2+Q_{T,v}\left(u\right)\right]\right]\right)\\
&\quad\leq\sup_{u\in\mathbf{H}(\overline{u},\,a)}\exp\left(\widetilde{\gamma}_T\left[a\left|u\right|-\varLambda^0\left(u\right)-\min\left(\varLambda^0\left(u\right)/4,\,\varLambda^0\left(u\right)/4+\eta\left\|\delta_T\right\|^2T\right)\right]\right)\\
&\quad\leq\sup_{u\in\mathbf{H}(\overline{u},\,c)}\exp\left(\widetilde{\gamma}_T\left[a\left|u\right|-C_2\left|u\right|\right]\right)+\exp\left(\gamma_T\left[a\overline{\lambda}-\eta C\right]\right)\\
&\quad\leq\sup_{u\in\mathbf{H}(\overline{u},\,c)}\exp\left(\gamma_T\left[a-C_2\right]\right)+\exp\left(\gamma_T\left[a\overline{\lambda}-\eta C\right]\right)=o\left(\exp\left(-\gamma_T\overline{a}_1\right)\right),
\end{aligned}$$

when $a > 0$ is chosen sufficiently small and for some $\overline{a}_1 > 0$. Furthermore, by Lemma A.13-(ii) below with $\varpi = 1$,

$$\lim_{\overline{u}\to\infty}\lim_{T\to\infty}\mathbb{P}\left(u\in\left\{\Gamma^c_{T,\overline{u}}-\mathbf{H}\left(\overline{u},\,c\right)\right\}\right)\leq\lim_{\overline{u}\to\infty}\lim_{T\to\infty}\mathbb{P}\left(\sup_{|u|>\overline{u}}\frac{\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)}{|u|}>a\right)=0. \tag{A.26}$$

By combining (A.25)-(A.26), $\mathbb{P}\left(J_1>\epsilon\exp\left(-\overline{a}\widetilde{\gamma}_T\right)\right)\to 0$ as $T\to\infty$. Next, we consider the first right-hand side term in (A.24). Recall the definition of λ_+ from Assumption 3.5 and let $0 < b \leq \overline{a}/4\lambda_+$. Note that for $G_{T,v}\left(b\right) \triangleq \sup_{|u|\leq b}\sup_{\widetilde{v}\in\mathbf{V}}\left|\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)\right|$,

$$\mathbb{P}\left(J_2\leq\exp\left(-\overline{a}\widetilde{\gamma}_T\right)\right)\leq\mathbb{P}\left(G_{T,v}\left(b\right)\leq\overline{a},\,J_2\leq\exp\left(-\overline{a}\widetilde{\gamma}_T\right)\right)+\mathbb{P}\left(G_{T,v}\left(b\right)>\overline{a}\right). \tag{A.27}$$

Under Assumption 3.2 and the second part of Assumption 3.5, using the definition of b,

$$\begin{aligned}
\mathbb{P}\left(G_{T,v}\left(b\right)\leq\overline{a},\,J_2\leq\exp\left(-\overline{a}\widetilde{\gamma}_T\right)\right)&\leq\mathbb{P}\left(C_\pi\int_{|u|\leq b}\exp\left(\widetilde{\gamma}_T\left(-\overline{a}/2-\lambda_+b\right)\right)du\leq\exp\left(-\overline{a}\widetilde{\gamma}_T\right)\right)\\
&\leq\mathbb{P}\left(C_\pi b\exp\left(\overline{a}\widetilde{\gamma}_T/2\right)\leq 1\right)\to 0,
\end{aligned}$$

as $T\to\infty$. We shall use the uniform convergence in Lemma A.12 for the second right-hand side term in (A.27) to deduce that (recall that $\overline{a}$ was chosen sufficiently small and $b\leq\overline{a}/4\lambda_+$),

$$\lim_{b\to 0}\lim_{T\to\infty}\mathbb{P}\left(G_{T,v}\left(b\right)>\overline{a}\right)\leq\lim_{b\to 0}\mathbb{P}\left(\sup_{|u|\leq b}\left|\mathscr{W}\left(u\right)\right|>\overline{a}\right)=0.$$

□

Lemma A.22. *As $T \to \infty$, $\xi_T(\widetilde{v}) \Rightarrow \xi_0$ in $\mathbb{D}_b(\mathbf{V})$.*

Proof. Let $\mathbf{B} = \Gamma_{\overline{u}} \times \mathbf{V}$. For any fixed $\overline{u}$, Lemma A.12 and the result $\sup_{(u,\widetilde{v})\in\mathbf{B}} |Q_{T,v}(u) - \Lambda^0(u)| = o_{\mathbb{P}}(1)$ (cf. Lemma A.3), imply that $\overline{Q}_T \Rightarrow \mathscr{V}$ in $\mathbb{D}_b(\mathbf{B})$. By the Skorokhod representation theorem [cf. Theorem 6.4 in Billingsley (1999)] we can find a probability space $\left(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}}\right)$ on which there exist processes $\widetilde{Q}_T(u, \widetilde{v})$ and $\widetilde{\mathscr{V}}(u)$ which have the same law as $\overline{Q}_T(u, \widetilde{v})$ and $\mathscr{V}(u)$, respectively, and with the property that

$$\sup_{(u,\widetilde{v})\in\mathbf{B}} \left|\widetilde{Q}_T(u, \widetilde{v}) - \widetilde{\mathscr{V}}(u)\right| \to 0 \qquad \widetilde{\mathbb{P}} - \text{a.s.} \tag{A.28}$$

Let

$$\widetilde{\xi}_T(\widetilde{v}) \triangleq \frac{\int_{\Gamma_{\overline{u}}} u \exp\left(\widetilde{\gamma}_T \widetilde{Q}_{T,v}(u, \widetilde{v})\right) \pi_{T,v}(u)\, du}{\int_{\Gamma_{\overline{u}}} \exp\left(\widetilde{\gamma}_T \widetilde{Q}_{T,v}(u, \widetilde{v})\right) \pi_{T,v}(u)\, du},$$

and $\widetilde{\xi}_0 \triangleq \arg\max_{u\in\Gamma_{\overline{u}}} \widetilde{\mathscr{V}}(u)$. We shall rely on (A.28) to establish that

$$\sup_{\widetilde{v}\in\mathbf{V}} \left|\widetilde{\xi}_T(\widetilde{v}) - \widetilde{\xi}_0\right| \to 0 \qquad \widetilde{\mathbb{P}} - \text{a.s.} \tag{A.29}$$

Let us indicate any pair of sample paths of $\widetilde{Q}_T(u, \widetilde{v})$ and $\widetilde{\mathscr{V}}$, for which (A.28) holds with a superscript ω, by $\widetilde{Q}^{\omega}_{T,v}$ and $\widetilde{\mathscr{V}}^{\omega}$, respectively. For arbitrary sets $\mathbf{S}_1, \mathbf{S}_2 \subset \mathbf{B}$, let $\widetilde{\rho}(\mathbf{S}_1, \mathbf{S}_2) \triangleq \text{Leb}(\mathbf{S}_1 - \mathbf{S}_2) + \text{Leb}(\mathbf{S}_2 - \mathbf{S}_1)$ where $\text{Leb}(\mathbf{A})$ is the Lebesgue measure of the set $\mathbf{A}$. Further, for an arbitrary scalar $c > 0$ and function $\Upsilon : \mathbf{B} \to \mathbb{R}$, define $\mathbf{S}(\Upsilon, c) \triangleq \left\{(u, \widetilde{v}) \in \mathbf{B} : \left|\Upsilon(u, \widetilde{v}) - \widetilde{\mathscr{V}}_{\text{M}}\right| \leq c\right\}$ where $\widetilde{\mathscr{V}}_{\text{M}} \triangleq \max_{u\in\Gamma_{\overline{u}}} \widetilde{\mathscr{V}}^{\omega}(u)$. The first step is to show that

$$\widetilde{\rho}\left(\mathbf{S}\left(\widetilde{Q}^{\omega}_{T,v}, c\right), \mathbf{S}\left(\widetilde{\mathscr{V}}^{\omega}, c\right)\right) = o(1). \tag{A.30}$$

Let $\mathbf{S}_{1,T}(c) = \mathbf{S}\left(\widetilde{Q}^{\omega}_{T,v}, c\right) - \mathbf{S}\left(\widetilde{\mathscr{V}}^{\omega}, c\right)$ and $\mathbf{S}_{2,T}(c) = \mathbf{S}\left(\widetilde{\mathscr{V}}^{\omega}, c\right) - \mathbf{S}\left(\widetilde{Q}^{\omega}_{T,v}, c\right)$. We first establish that $\text{Leb}(\mathbf{S}_{2,T}(c)) = o(1)$. For an arbitrary $\overline{c} > 0$, define the set $\widetilde{\mathbf{S}}_T(\overline{c}) \triangleq \{(u, \widetilde{v}) \in \mathbf{B} : \left|\widetilde{Q}^{\omega}_{T,v}(u, \widetilde{v}) - \widetilde{\mathscr{V}}^{\omega}(u)\right| \leq \overline{c}\}$ and its complement (relative to $\mathbf{B}$) $\widetilde{\mathbf{S}}^c_T(\overline{c}) \triangleq \left\{(u, \widetilde{v}) \in \mathbf{B} : \left|\widetilde{Q}^{\omega}_{T,v}(u, \widetilde{v}) - \widetilde{\mathscr{V}}^{\omega}(u)\right| > \overline{c}\right\}$. We have

$$\begin{aligned}\text{Leb}(\mathbf{S}_{2,T}(c)) &= \text{Leb}\left(\mathbf{S}_{2,T}(c) \cap \widetilde{\mathbf{S}}_T(\overline{c})\right) + \text{Leb}\left(\mathbf{S}_{2,T}(c) \cap \widetilde{\mathbf{S}}^c_T(\overline{c})\right) \\ &\leq \text{Leb}\left(\mathbf{S}_{2,T}(c) \cap \widetilde{\mathbf{S}}_T(\overline{c})\right) + \text{Leb}\left(\widetilde{\mathbf{S}}^c_T(\overline{c})\right).\end{aligned}$$

Note that $\text{Leb}\left(\widetilde{\mathbf{S}}^c_T(\overline{c})\right) = o(1)$ since the path ω satisfies (A.28). Furthermore, $\mathbf{S}_{2,T}(c) \cap \widetilde{\mathbf{S}}_T(\overline{c}) \subset \mathbf{C}_T(c, \overline{c})$ where $\mathbf{C}_T(c, \overline{c}) \triangleq \left\{(u, \widetilde{v}) \in \mathbf{B} : c \leq \left|\widetilde{Q}^{\omega}_{T,v}(u, \widetilde{v}) - \widetilde{\mathscr{V}}_{\text{M}}\right| \leq c + \overline{c}\right\}$. In view of (A.28),

$$\begin{aligned}\lim_{\overline{c}\downarrow 0} \lim_{T\to\infty} \text{Leb}(\mathbf{C}_T(c, \overline{c})) &= \lim_{\overline{c}\downarrow 0} \text{Leb}\left\{(u, \widetilde{v}) \in \mathbf{B} : c \leq \left|\widetilde{\mathscr{V}}^{\omega}(u) - \widetilde{\mathscr{V}}_{\text{M}}\right| \leq c + \overline{c}\right\} \\ &= \text{Leb}\left\{(u, \widetilde{v}) \in \mathbf{B} : \left|\widetilde{\mathscr{V}}^{\omega}(u) - \widetilde{\mathscr{V}}_{\text{M}}\right| = c\right\} = 0,\end{aligned}$$

by the path properties of $\widetilde{\mathscr{V}}^{\omega}$. Since $\text{Leb}(\mathbf{S}_{1,T}(c)) = o(1)$ can be proven in a similar fashion, (A.30) holds.

For $m=0,\,1$, $C_1<\infty$ and by Assumption 3.2 we know there exists some $C_2<\infty$ such that

$$\sup_{\widetilde{v}\in\mathbf{V}}\int_{\mathbf{S}^c\left(\widetilde{Q}^{\omega}_{T,v}(u,\widetilde{v}),c\right)}|u|^m\exp\left(\widetilde{\gamma}_T\left(\widetilde{Q}^{\omega}_{T,v}\left(u,\,\widetilde{v}\right)-\mathscr{V}_{\mathrm{M}}\right)\right)\pi_{T,v}\left(u\right)du\leq C_1\exp\left(-c\widetilde{\gamma}_T\right)C_2\int_{\Gamma_{\overline{u}}}|u|^m\,du=o\left(1\right),$$

since $\{u\leq\overline{u}\}$ on $\Gamma_{\overline{u}}$ and recalling that $\widetilde{\gamma}_T\rightarrow\infty$. This gives an upper bound to the same function where u replaces $|u|$. Then,

$$\sup_{\widetilde{v}\in\mathbf{V}}\frac{\int_{\Gamma_{\overline{u}}}u\exp\left(\widetilde{\gamma}_T\widetilde{Q}^{\omega}_{T,v}\left(u,\,\widetilde{v}\right)\right)\pi_{T,v}\left(u\right)du}{\int_{\Gamma_{\overline{u}}}\exp\left(\widetilde{\gamma}_T\widetilde{Q}^{\omega}_{T,v}\left(u,\,\widetilde{v}\right)\right)\pi_{T,v}\left(u\right)du}\leq\operatorname{ess\,sup}\mathbf{S}\left(\widetilde{Q}^{\omega}_{T,v},\,c\right)+o\left(1\right).$$

By (A.28) we deduce $\operatorname{ess\,sup}\mathbf{S}\left(\widetilde{Q}^{\omega}_{T,v},\,c\right)+o\left(1\right)=\operatorname{ess\,sup}\mathbf{S}\left(\widetilde{\mathscr{V}}^{\omega},\,c\right)+o\left(1\right)$. The same argument yields

$$\inf_{\widetilde{v}\in\mathbf{V}}\frac{\int_{\Gamma_{\overline{u}}}u\exp\left(\widetilde{\gamma}_T\widetilde{Q}^{\omega}_{T,v}\left(u,\,\widetilde{v}\right)\right)\pi_{T,v}\left(u\right)du}{\int_{\Gamma_{\overline{u}}}\exp\left(\widetilde{\gamma}_T\widetilde{Q}^{\omega}_{T,v}\left(u,\,\widetilde{v}\right)\right)\pi_{T,v}\left(u\right)du}\geq\operatorname{ess\,inf}\mathbf{S}\left(\widetilde{\mathscr{V}}^{\omega},\,c\right)+o\left(1\right).$$

Since almost every path ω of the Gaussian process $\widetilde{\mathscr{V}}$ achieves its maximum at a unique point on compact sets [cf. Bai (1997) and Lemma 2.6 in Kim and Pollard (1990)], we have

$$\lim_{c\downarrow0}\operatorname{ess\,inf}\mathbf{S}\left(\widetilde{\mathscr{V}}^{\omega},\,c\right)=\lim_{c\downarrow0}\operatorname{ess\,sup}\mathbf{S}\left(\widetilde{\mathscr{V}}^{\omega},\,c\right)=\arg\max_{u\in\Gamma_{\overline{u}}}\widetilde{\mathscr{V}}^{\omega}\left(u\right).$$

Hence, we have proved (A.29) which by the dominated convergence theorem then implies the weak convergence of $\widetilde{\xi}_T$ toward $\widetilde{\xi}_0$. Since the law of $\widetilde{\xi}_T$ ($\widetilde{\xi}_0$) under $\widetilde{\mathbb{P}}$ is the same as the law of ξ_T (ξ_0) under $\mathbb{P}$, the claim of the Lemma follows. □

We are now in a position to conclude the proof of Proposition 3.2. For a set $\mathbf{T}\subset\mathbb{R}$ and $m=0,\,1$ we define $J_m\left(\mathbf{T}\right)\triangleq\int_{\mathbf{T}}u^m\exp\left(\widetilde{\gamma}_T\left(\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right)\right)\pi_{T,v}\left(u\right)du$. Hence, with this notation equation (3.11) can be rewritten as $T\left\Vert\delta_T\right\Vert^2\left(\widehat{\lambda}^{\mathrm{GL},*}_b\left(\widetilde{v},\,v\right)-\lambda^0_{b,T}\left(v\right)\right)=J_1\left(\mathbb{R}\right)/J_0\left(\mathbb{R}\right)$. Applying simple manipulations, we obtain,

$$J_1\left(\mathbb{R}\right)/J_0\left(\mathbb{R}\right)=\frac{J_1\left(\Gamma_{\overline{u}}\right)+J_1\left(\Gamma^c_{\overline{u},T}\right)}{J_0\left(\Gamma_{\overline{u}}\right)+J_0\left(\Gamma^c_{\overline{u},T}\right)}=\frac{J_1\left(\Gamma_{\overline{u}}\right)}{J_0\left(\Gamma_{\overline{u}}\right)}\left[1-\frac{J_0\left(\Gamma^c_{\overline{u},T}\right)}{J_0\left(\mathbb{R}\right)}\right]+\frac{J_1\left(\Gamma^c_{\overline{u},T}\right)}{J_0\left(\mathbb{R}\right)}.\qquad\text{(A.31)}$$

By Lemma A.21, $J_m\left(\Gamma^c_{\overline{u},T}\right)/J_0\left(\mathbb{R}\right)=o_{\mathbb{P}}\left(1\right)$ ($m=0,\,1$) uniformly in $\widetilde{v}\in\mathbf{V}$. By Lemma A.22, with $\xi_T\left(\widetilde{v}\right)=J_1\left(\Gamma_{\overline{u}}\right)/J_0\left(\Gamma_{\overline{u}}\right)$, the first right-hand side term in (A.31) converges weakly to $\arg\max_{u\in\mathbb{R}}\mathscr{V}\left(u\right)$ in $\mathbb{D}_b\left(\mathbf{V}\right)$.

A.3.4 Proof of Corollary 3.1

The proof involves a simple change in variable. We refer to Proposition 3 in Bai (1997).

A.3.5 Proof of Theorem 3.2

We begin by introducing some notation. Since $l\in\boldsymbol{L}$, for all real numbers B sufficiently large and ϑ sufficiently small the following relationship holds

$$\inf_{|u|>B}l\left(u\right)-\sup_{|u|\leq B^{\vartheta}}l\left(u\right)\geq0.\qquad\text{(A.32)}$$

Let $\zeta_{T,v}(u,\, \widetilde{v}) = \exp\left(G_{T,v}(u,\, \widetilde{v}) - \varLambda^0(u)\right)$, $\Gamma_T \triangleq \left\{u \in \mathbb{R} :\, \lambda_b \in \varGamma^0\right\}$ and

$$\Gamma_M = \{u \in \mathbb{R} :\, M \leq |u| < M+1\} \cap \Gamma_T,$$

and define

$$J_{1,M} \triangleq \int_{\Gamma_M} \zeta_{T,v}(u,\, \widetilde{v})\, \pi_{T,v}(u)\, du, \qquad\qquad J_2 \triangleq \int_{\Gamma_T} \zeta_{T,v}(u,\, \widetilde{v})\, \pi_{T,v}(u)\, du. \tag{A.33}$$

In some steps in the proof we shall be working with elements of the following families of functions. A function $f_T : \mathbb{R} \to \mathbb{R}$ is said to belong to the family $\boldsymbol{F}$ if it satisfies the following properties: (1) For fixed T, $f_T(x)$ increases monotocically to infinity with $x \in [0,\, \infty)$; (2) For any $b < \infty$, $x^b \exp(-f_T(x)) \to 0$ as both T and x diverge to infinity.

Proof. The random variable $T\, \|\delta_T\|^2 \left(\widehat{\lambda}_b^{\mathrm{GL}} - \lambda_0\right) = \widetilde{\tau}_T$ is a minimizer of the function

$$\Psi_{l,T}(s) = \int_{\Gamma_T} l(s-u)\, \frac{\exp\left(\widetilde{G}_{T,v}(u,\, \widetilde{v}) + Q_{T,v}(u)\right) \pi_{T,v}(u)}{\int_{\Gamma_T} \exp\left(\widetilde{G}_{T,v}(w,\, \widetilde{v}) + Q_{T,v}(u)\right) \pi_{T,v}(w)\, dw} du.$$

Observe that Lemmas A.16-A.20 apply to any polynomial $p \in \boldsymbol{P}$; therefore, they are still valid for $l \in \boldsymbol{L}$. We then have that the asymptotic behavior of $\Psi_{l,T}(s)$ only matters when u (and thus s) varies on $\Gamma_K = \{u \in \mathbb{R} :\, u \leq K\}$. By Lemmas A.27-A.28, for any $\vartheta > 0$, there exists a $\overline{T}$ such that for all $T > \overline{T}$,

$$\mathbb{E}\left[\int_{\Gamma_K} \frac{\exp\left(\widetilde{G}_{T,v}(u,\, \widetilde{v}) + Q_{T,v}(u)\right)}{\int_{\Gamma_T} \exp\left(\widetilde{G}_{T,v}(w,\, \widetilde{v}) + Q_{T,v}(w)\right) dw} du\right] \leq \frac{c_\vartheta}{K^\vartheta}. \tag{A.34}$$

Therefore, for all $T > \overline{T}$,

$$\Psi_{l,T}(s) = \frac{\int_{|u| \leq K} l(s-u) \exp\left(\widetilde{G}_{T,v}(u,\, \widetilde{v}) + Q_{T,v}(u)\right) du}{\int_{|w| \leq K} \exp\left(\widetilde{G}_{T,v}(w,\, \widetilde{v}) + Q_{T,v}(w)\right) dw} + o_{\mathbb{P}}(1), \tag{A.35}$$

where the $o_{\mathbb{P}}(1)$ term is uniform in $T > \overline{T}$ as K increases to infinity. By Assumption 3.2, $\left|\pi_{T,v}(u) - \pi^0\right| \leq \left|\pi\left(\lambda_{b,T}^0(v)\right) - \pi^0\right| + C\psi_T^{-1}|u|$, with $C > 0$. On $\{|u| \leq K\}$, the first term on the right-hand side is $o(1)$ and does not depend on u. The second term is negligible when T is large. Thus, without loss of generality we set $\pi_{T,v}(u) = 1$ for all u in what follows.

Next, we show the convergence of the marginal distributions of the estimate $\Psi_{l,T}(s)$ to the marginals of the random function $\Psi_l(s)$, where the region of integration in the definition of both the numerator and denominator of $\Psi_{l,T}(s)$ and $\Psi_l(s)$ is restricted to $\{|u| \leq K\}$ only, in view of (A.35). For a finite integer n, choose arbitrary real numbers a_j $(j = 0, \ldots, n)$ and introduce the following estimate:

$$\sum_{j=1}^{n} a_j \int_{|u| \leq K} l(s_j - u)\, \zeta_{T,v}(u,\, \widetilde{v})\, du + a_0 \int_{|u| \leq K} l(s_0 - u)\, \zeta_{T,v}(u,\, \widetilde{v})\, du. \tag{A.36}$$

By Lemmas A.24 and A.30, we can invoke Theorem I.A.22 in Ibragimov and Has'minskiĭ (1981) which gives the convergence in distribution of the estimate in (A.36) towards the distribution of the following random variable:

$$\sum_{j=1}^{n} a_j \int_{|u| \leq K} l(s_j - u) \exp(\mathscr{V}(u))\, du + a_0 \int_{|u| \leq K} l(s_0 - u) \exp(\mathscr{V}(u))\, du.$$

By the Cramer-Wold Theorem [cf. Theorem 29.4 in Billingsley (1995)] this suffices for the convergence in distribution of the vector

$$\int_{|u|\leq K} l\left(s_i-u\right)\zeta_{T,v}\left(u,\,\widetilde{v}\right)du,\ldots,\int_{|u|\leq K} l\left(s_n-u\right)\zeta_{T,v}\left(u,\,\widetilde{v}\right)du,\qquad \int_{|u|\leq K} l\left(s_0-u\right)\zeta_{T,v}\left(u,\,\widetilde{v}\right)du,$$

to the distribution of the vector

$$\int_{|u|\leq K} l\left(s_i-u\right)\exp\left(\mathscr{V}\left(u\right)\right)du,\ldots,\int_{|u|\leq K} l\left(s_n-u\right)\exp\left(\mathscr{V}\left(u\right)\right)du,\qquad \int_{|u|\leq K} l\left(s_0-u\right)\exp\left(\mathscr{V}\left(u\right)\right)du.$$

As a consequence, for any $K_1,\,K_2<\infty$, the marginal distributions of

$$\frac{\int_{|u|\leq K_1} l\left(s-u\right)\exp\left(\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right)du}{\int_{|w|\leq K_2}\exp\left(\widetilde{G}_{T,v}\left(w,\,\widetilde{v}\right)+Q_{T,v}\left(w\right)\right)dw},$$

converge to the marginals of $\int_{|u|\leq K_1} l\left(s-u\right)\exp\left(\mathscr{V}\left(u\right)\right)du/\left(\int_{|w|\leq K_2}\exp\left(\mathscr{V}\left(w\right)\right)dw\right)$. The same convergence result extends to the distribution of

$$\int_{M\leq|u|<M+1}\frac{\exp\left(\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right)}{\int_{|w|\leq K_2}\exp\left(\widetilde{G}_{T,v}\left(w,\,\widetilde{v}\right)+Q_{T,v}\left(w\right)\right)dw}du,$$

towards the distribution of $\int_{M\leq|u|<M+1}(\exp\left(\mathscr{V}\left(u\right)\right)du/\int_{|w|\leq K_2}\exp\left(\mathscr{V}\left(w\right)\right)dw)$. By choosing $K_2>M+1$ we deduce

$$\mathbb{E}\left[\int_{M\leq|u|<M+1}\frac{\exp\left(\mathscr{V}\left(u\right)\right)}{\int_{\mathbb{R}}\exp\left(\mathscr{V}\left(w\right)\right)dw}du\right]\leq\lim_{T\to\infty}\mathbb{E}\left[\int_{\Gamma_M}\frac{\exp\left(\widetilde{G}_{T,v}\left(u,\,\widetilde{v}\right)+Q_{T,v}\left(u\right)\right)}{\int_{|w|\leq K_2}\exp\left(\widetilde{G}_{T,v}\left(w,\,\widetilde{v}\right)+Q_{T,v}\left(w\right)\right)dw}du\right]\leq c_{\vartheta}M^{-\vartheta},$$

in view of (A.34). This leads to

$$\Psi_l\left(s\right)=\int_{|u|\leq K} l\left(s-u\right)\frac{\exp\left(\mathscr{V}\left(u\right)\right)du}{\int_{|w|\leq K}\exp\left(\mathscr{V}\left(w\right)\right)dw}+o_{\mathbb{P}}\left(1\right),\tag{A.37}$$

where the $o_{\mathbb{P}}\left(1\right)$ term is uniform as K increases to infinity. We then have the convergence of the finite-dimensional distributions of $\Psi_{l,T}\left(s\right)$ toward $\Psi_l\left(s\right)$. Next, we need to prove the tightness of the sequence $\{\Psi_{l,T}\left(s\right),\,T\geq1\}$. More specifically, we shall show that the family of distributions on the space of continuous functions $\mathbb{C}_b\left(K\right)$ generated by the contractions of $\Psi_{l,T}\left(s\right)$ on $\{|s|\leq K\}$ are dense. For any $l\in\boldsymbol{L}$ the inequality $l\left(u\right)\leq2^r\left(1+|u|^2\right)^r$ holds for some r. Let

$$\Upsilon_K\left(\varpi\right)\triangleq\int_{\mathbb{R}}\sup_{|s|\leq K,\,|y|\leq\varpi}\left|l\left(s+y-u\right)-l\left(s-u\right)\right|\left(1+|u|^2\right)^{-r-1}du.$$

Fix $K<\infty$. We show $\lim_{\varpi\downarrow0}\Upsilon_K\left(\varpi\right)=0$. Note that for any $\kappa>0$, we can choose an M such that

$$\int_{|u|>M}\sup_{|s|\leq K,\,|y|\leq\varpi}\left|l\left(s+y-u\right)-l\left(s-u\right)\right|\left(1+|u|^2\right)^{-r-1}du<\kappa.$$

We now use Lusin's Theorem [cf. Section 3.3 in Royden and Fitzpatrick (2010)]. Since $l\left(\cdot\right)$ is measurable, there exists a continuous function $g\left(u\right)$ in the interval $\{u\in\mathbb{R}:\,|u|\leq K+2M\}$ which agrees with $l\left(u\right)$ except on a set whose measure does not exceed $\kappa\left(2\overline{L}\right)^{-1}$, where $\overline{L}$ is the upper bound of $l\left(\cdot\right)$ on

$\{u \in \mathbb{R} : |u| \leq K + 2M\}$. Denote the modulus of continuity of $g(\cdot)$ by $w_g(\varpi)$. Without loss of generality assume $|g(u)| \leq \overline{L}$ for all u satisfying $|u| \leq K + 2M$. Then,

$$\begin{aligned}
&\int_{|u|>M} \sup_{|s|\leq K,\, |y|\leq \varpi} |l(s+y-u) - l(s-u)| \left(1+|u|^2\right)^{-r-1} du \\
&\qquad \leq \int_{\mathbb{R}} \sup_{|s|\leq K,\, |y|\leq \varpi} |l(s+y-u) - l(s-u)| \left(1+|u|^2\right)^{-r-1} du \\
&\qquad \leq w_g(\varpi) \int_{\mathbb{R}} \sup_{|s|\leq K,\, |y|\leq \varpi} \left(1+|u|^2\right)^{-r-k} du + 2\overline{L}\text{Leb}\left\{u \in \mathbb{R} : |u| \leq K + 2M,\, l \neq g\right\},
\end{aligned}$$

and $\overline{L} \leq Cw_g(\varpi) + \kappa$ for some C. Hence, $\Upsilon_K(\varpi) \leq Cw_g(\varpi) + 2\kappa$ since κ can be chosen arbitrarily small and (for each fixed κ) $w_g(\varpi) \to 0$ as $\varpi \downarrow 0$ by definition. By Assumption 3.7, there exists a number $C < \infty$ such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{|s|\leq K,\, |y|\leq \varpi} |\Psi_{l,T}(s+y) - \Psi_{l,T}(s)|\right] \\
&\quad \leq \int_{\mathbb{R}} \sup_{|s|\leq K,\, |y|\leq \varpi} |l(s+y-u) - l(s-u)| \,\mathbb{E}\left(\frac{\exp\left(\widetilde{G}_{T,v}(u, \widetilde{v}) + Q_{T,v}(u)\right)}{\int_{\mathbf{U}_T} \exp\left(\widetilde{G}_{T,v}(w, \widetilde{v}) + Q_{T,v}(w)\right) dw}\right) du \\
&\quad \leq C\Upsilon_K(\varpi).
\end{aligned}$$

Markov's inequality together with the above bound establish that the family of distributions generated by the contractions of $\Psi_{T,l}$ is dense in $\mathbb{C}_b(K)$. Since the finite-dimensional convergence in distribution was demonstrated above, we can deduce the weak convergence $\Psi_{l,T} \Rightarrow \Psi_l$ in $\mathbb{D}_b(\mathbf{V})$ uniformly in $\lambda_b^0 \in \mathbf{K}$. Finally, we examine the oscillations of the minimum points of the sample criterion $\Psi_{l,T}$. Consider an open bounded interval $\mathbf{A}$ that satisfies $\mathbb{P}\{\xi_l^0 \in b(\mathbf{A})\} = 0$, where $b(\mathbf{A})$ denotes the boundary of the set $\mathbf{A}$. Choose a real number K sufficiently large such that $\mathbf{A} \subset \{s : |s| \leq K\}$ and define for $|s| \leq K$ the functionals $H_{\mathbf{A}}(\Psi) = \inf_{s\in\mathbf{A}} \Psi_l(s)$ and $H_{\mathbf{A}^c}(\Psi) = \inf_{s\in\mathbf{A}^c} \Psi_l(s)$. Let $\mathbf{M}_T$ denote the set of minimum points of $\Psi_{l,T}$. We have

$$\begin{aligned}
\mathbb{P}[\mathbf{M}_T \subset \mathbf{A}] &= \mathbb{P}[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi),\, \mathbf{M}_T \subset \{s : |s| \leq K\}] \\
&\geq \mathbb{P}[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)] - \mathbb{P}[\mathbf{M}_T \not\subseteq \{s : |s| \leq K\}].
\end{aligned}$$

Therefore,

$$\liminf_{T\to\infty} \mathbb{P}[\mathbf{M}_T \subset \mathbf{A}] \geq \mathbb{P}[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)] - \sup_T \mathbb{P}[\mathbf{M}_T \not\subseteq \{s : |s| \leq K\}],$$

and $\limsup_{T\to\infty} \mathbb{P}[\mathbf{M}_T \subset \mathbf{A}] \leq \mathbb{P}[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)]$. Moreover, the minimum of the population criterion $\Psi_l(\cdot)$ satisfies $\mathbb{P}[\xi_l^0 \in \mathbf{A}] \leq \mathbb{P}[H_{\mathbf{A}}(\Psi) < H_{\mathbf{A}^c}(\Psi)]$ and $\mathbb{P}[\xi_l^0 \in \mathbf{A}] + \mathbb{P}[|\xi_l^0| > K] \geq \mathbb{P}[H_{\mathbf{A}}(\Psi) \leq H_{\mathbf{A}^c}(\Psi)]$. Lemma A.29 shall be used to deduce that the following relationship holds,

$$\limsup_{T\to\infty} \mathbb{E}\left[l\left(T\|\delta_T\|^2\left(\widehat{\lambda}_b^{\text{GL}} - \lambda_b^0\right)\right)\right] < \infty,$$

for any loss function $l \in \boldsymbol{L}$. Hence, the set $\mathbf{M}_T$ of absolute minimum points of the function $\Psi_{l,T}(s)$ are uniformly stochastically bounded for all T large enough: $\lim_{K\to\infty} \mathbb{P}[\mathbf{M}_T \not\subseteq \{s : |s| \leq K\}] = 0$. The latter result together with the uniqueness assumption (cf. Assumption 3.5) yield

$$\lim_{K\to\infty}\left\{\sup_{T}\mathbb{P}\left[\mathbf{M}_T\not\subseteq\{s:\,|s|\leq K\}\right]+\mathbb{P}\left[\left|\xi_l^0\right|>K\right]\right\}=0.$$

Hence, we have

$$\lim_{T\to\infty}\mathbb{P}\left[\mathbf{M}_T\subset\mathbf{A}\right]=\mathbb{P}\left[\xi_l^0\in\mathbf{A}\right]. \tag{A.38}$$

The last step involves showing that the length of the set $\mathbf{M}_T$ approaches zero in probability as $T\to\infty$. Let $\mathbf{A}_d$ denote an interval in $\mathbb{R}$ centered at the origin and of length $d<\infty$. Equation (A.38) guarantees that $\lim_{d\to\infty}\sup_{T\to\infty}\mathbb{P}\left[\mathbf{M}_T\not\subseteq\mathbf{A}_d\right]=0$. Choose any $\epsilon>0$ and divide $\mathbf{A}_d$ into admissible subintervals whose lengths do not exceed $\epsilon/2$. Then,

$$\mathbb{P}\left[\sup_{s_i,s_j\in\mathbf{M}_T}|s_i-s_j|>\epsilon\right]\leq\mathbb{P}\left[\mathbf{M}_T\not\subseteq\mathbf{A}_d\right]+(1+2d/\epsilon)\sup\mathbb{P}\left[H_{\mathbf{A}}\left(\Psi_{l,T}\right)=H_{\mathbf{A}^c}\left(\Psi_{l,T}\right)\right],$$

where the term $1+2d/\epsilon$ is an upper bound on the admissible number of subintervals and the supremum in the second term is over all possible open bounded subintervals $\mathbf{A}\subset\mathbf{A}_d$. The weak convergence result implies $\mathbb{P}\left[H_{\mathbf{A}}\left(\Psi_{l,T}\right)=H_{\mathbf{A}^c}\left(\Psi_{l,T}\right)\right]\to\mathbb{P}\left[H_{\mathbf{A}}\left(\Psi_l\right)=H_{\mathbf{A}^c}\left(\Psi_l\right)\right]$ as $T\to\infty$. Since $\mathbb{P}\left[H_{\mathbf{A}}\left(\Psi_l\right)=H_{\mathbf{A}^c}\left(\Psi_l\right)\right]=0$ and $\mathbb{P}\left[\mathbf{M}_T\not\subseteq\mathbf{A}_d\right]\to0$ for large d, then $\mathbb{P}\left[\sup_{s_i,s_j\in\mathbf{M}_T}|s_i-s_j|>\epsilon\right]=o(1)$. Since $\epsilon>0$ can be chosen arbitrary small we deduce that the distribution of $T\left\|\delta_T\right\|^2\left(\widehat{\lambda}_b^{\mathrm{GL}}-\lambda_b^0\right)$ converges to the distribution of ξ_l^0. □

Lemma A.23. *Let $u_1,\,u_2\in\mathbb{R}$ be of the same sign with $0<|u_1|<|u_2|$. For any integer $r>0$ and some constants c_r and C_r which depend on r only, we have uniformly in $\widetilde{v}\in\mathbf{V}$,*

$$\mathbb{E}\left[\left(\zeta_{T,v}^{1/2r}\left(u_2,\,\widetilde{v}\right)-\zeta_{T,v}^{1/2r}\left(u_1,\,\widetilde{v}\right)\right)^{2r}\right]\leq c_r\left|\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_i\right)\delta^0\right|^r\leq C_r\left|u_2-u_1\right|^r,$$

where Σ_i is defined in Assumption 3.5 and $i=1$ if $u_1<0$ and $i=2$ if $u_1>0$.

Proof. The proof is given for the case $u_2>u_1>0$. The other case is similar and thus omitted. We follow closely the proof of Lemma III.5.2 in Ibragimov and Has'minskiĭ (1981). Let $\mathcal{V}\left(u_i\right)=\exp\left(\mathscr{V}\left(u_i\right)\right)$, $i=1,\,2$. We have $\mathbb{E}\left[\left(\mathcal{V}^{1/2r}\left(u_2\right)-\mathcal{V}^{1/2r}\left(u_1\right)\right)^{2r}\right]=\sum_{j=0}^{2r}\binom{2r}{j}(-1)^j\,\mathbb{E}_{u_1}\left[\mathcal{V}_{u_1}^{j/2r}\left(u_2\right)\right]$, where $\mathcal{V}_{u_1}\left(u_2\right)\triangleq\exp\left(\mathscr{V}\left(u_2\right)-\mathscr{V}\left(u_1\right)\right)$. Using the Gaussian property of $\mathscr{V}(u)$, for each $u\in\mathbb{R}$, we have

$$\mathbb{E}_{u_1}\left[\mathcal{V}^{j/2r}\left(u_2\right)\right]=\exp\left(\frac{1}{2}\left(\frac{j}{2r}\right)^24\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0-\frac{j}{2r}\left|\varLambda^0\left(u_2\right)-\varLambda^0\left(u_1\right)\right|\right). \tag{A.39}$$

Then, $\mathbb{E}\left[\left(\mathcal{V}^{1/2r}\left(u_2\right)-\mathcal{V}^{1/2r}\left(u_1\right)\right)^{2r}\right]=\sum_{j=0}^{2r}\binom{2r}{j}(-1)^j\,d^{j/2r}$ with

$$d\triangleq\exp\left(\frac{j}{2r}2\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0-\left|\varLambda^0\left(u_2\right)-\varLambda^0\left(u_1\right)\right|\right).$$

Let $B\triangleq2\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0-\left|\varLambda^0\left(u_2\right)-\varLambda^0\left(u_1\right)\right|$. There are different cases to be considered:
(1) $B<0$. Note that

$$d=\exp\left(\frac{j}{2r}2\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0-\left|\left(\delta^0\right)'\left(\left|u_2-u_1\right|\Sigma_2\right)\delta^0\right|+B\right)$$

$$= \exp\left(-\frac{2r-j}{r}\left(\delta^0\right)'\left(|u_2-u_1|\,\Sigma_2\right)\delta^0\right)e^B,$$

which then results in

$$\mathbb{E}\left[\left(\mathcal{V}^{1/2r}\left(u_2\right)-\mathcal{V}^{1/2r}\left(u_1\right)\right)^{2r}\right]\leq p_r\left(a\right), \tag{A.40}$$

where $p_r\left(a\right)\triangleq\sum_{j=0}^{2r}\binom{2r}{j}\left(-1\right)^j a^{(2r-j)}$ and $a=e^{B/2r}\exp\left(-r^{-1}\left(\delta^0\right)'\left(|u_2-u_1|\,\Sigma_2\right)\delta^0\right)$.
(2) $2\left(\delta^0\right)'\left(|u_2-u_1|\,\Sigma_2\right)\delta^0 = \left|\Lambda^0\left(u_2\right)-\Lambda^0\left(u_1\right)\right|$. This case is the same as the previous one but with $a=\exp\left(-r^{-1}\left(\delta^0\right)'\left(|u_2-u_1|\,\Sigma_2\right)\delta^0\right)$.
(3) $B>0$. Upon simple manipulations, $\mathbb{E}\left[\left(\mathcal{V}^{1/2r}\left(u_2\right)-\mathcal{V}^{1/2r}\left(u_1\right)\right)^{2r}\right]\leq p_r\left(a\right)$, where

$$p_r\left(a\right)=e^{-B/2r}\sum_{j=0}^{2r}\binom{2r}{j}\left(-1\right)^j a^{(2r-j)},$$

with $a=\exp\left(-r^{-1}\left(\delta^0\right)'\left(|u_2-u_1|\,\Sigma_2\right)\delta^0\right)$. We can thus proceed with the same proof for all the above cases. Let us consider the first case. We show that at the point $a=1$, the polynomial $p_r\left(a\right)$ admits a root of multiplicity r. This can be established by verifying the equalities $p_r\left(1\right)=p_r^{(1)}\left(1\right)=\cdots=p_r^{(r-1)}\left(1\right)=0$. One then recognizes that $p_r^{(i)}\left(a\right)$ is a linear combination of summations $\mathcal{S}_k$ $(k=0,\,1,\ldots,\,2i)$ given by $\mathcal{S}_k=e^B\sum_{j=0}^{2r}\binom{2r}{j}j^k$. Thus, one only needs to verify that $\mathcal{S}_k=0$ for $k=0,\,1,\ldots,\,2r-2$. This follows because the expression for $\mathcal{S}_k$ is found by applying the operator $e^Ba\left(d/da\right)$ to the function $\left(1-a^2\right)^{2r}$ and evaluating it at $a=1$. Consequently, $\mathcal{S}_k=0$ for $k=0,\,1,\ldots,\,2r-1$. Using this result into (A.40) we find, with $\widetilde{p}_r\left(a\right)$ being a polynomial of degree r^2-r,

$$\mathbb{E}\left[\left(\mathscr{V}^{1/2r}\left(u_2\right)-\mathscr{V}^{1/2r}\left(u_1\right)\right)^{2r}\right]=\left(1-a\right)^r\widetilde{p}_r\left(a\right)\leq\left(r^{-1}\left(\delta^0\right)'\left(|u_2-u_1|\,\Sigma_2\right)\delta^0\right)^r\widetilde{p}_r\left(a\right), \tag{A.41}$$

where the last inequality follows from $1-e^{-c}\leq c$, for $c>0$. Next, let $\overline{\zeta}_{T,v}^{1/2r}\left(u_2,\,u_1\right)=\zeta_{T,v}^{1/2r}\left(u_2\right)-\zeta_{T,v}^{1/2r}\left(u_1\right)$. By Lemmas A.3 and A.12, the continuous mapping theorem and (A.41), $\lim_{T\to\infty}\mathbb{E}\left[\overline{\zeta}_{T,v}^{1/2r}\left(u_2,\,u_1\right)\right]\leq\left(1-a\right)^r\widetilde{p}_r\left(a\right)$, uniformly in $\widetilde{v}\in\mathbf{V}$. Noting that $j\leq 2r$, we can set $C_r=\max_{0\leq a\leq 1}e^B\widetilde{p}_r\left(a\right)/r^r$ to prove the lemma. □

Lemma A.24. *For $u_1,\,u_2\in\mathbb{R}$ being of the same sign and satisfying $0<|u_1|<|u_2|<K<\infty$. Then, for all T sufficiently large, we have*

$$\mathbb{E}\left[\left(\zeta_{T,v}^{1/4}\left(u_2,\,\widetilde{v}\right)-\zeta_{T,v}^{1/4}\left(u_1,\,\widetilde{v}\right)\right)^4\right]\leq C_1\left|u_2-u_1\right|^2, \tag{A.42}$$

where $0<C_1<\infty$. Furthermore, for the constant C_1 from Lemma A.23, we have

$$\mathbb{P}\left[\zeta_{T,v}\left(u,\,\widetilde{v}\right)>\exp\left(-3C_1\left|u\right|/2\right)\right]\leq\exp\left(-C_1\left|u\right|/4\right). \tag{A.43}$$

Both relationships are valid uniformly in $\widetilde{v}\in\mathbf{V}$.

Proof. Suppose $u>0$. The relationship in (A.42) follows from Lemma A.23 with $r=2$. By Markov's inequality and Lemma A.23,

$$\mathbb{P}\left[\zeta_{T,v}\left(u,\,\widetilde{v}\right)>\exp\left(-3C_1\left|u\right|/2\right)\right]\leq\exp\left(3C_1\left|u\right|/4\right)\mathbb{E}\left[\zeta_{T,v}^{1/2}\left(u,\,\widetilde{v}\right)\right]$$

$$\leq \exp\left(3C_1 |u|/4 - \left(\delta^0\right)' (|u| \Sigma_2) \delta^0\right) \leq \exp(-C_1 |u|/4).$$

□

Lemma A.25. *Under the conditions of Lemma A.24, for any $\vartheta > 0$ there exists a finite real number c_ϑ and a $\overline{T}$ such that for all $T > \overline{T}$,* $\sup_{\widetilde{v}\in\mathbf{V}} \mathbb{P}\left[\sup_{|u|>M} \zeta_{T,v}(u, \widetilde{v}) > M^{-\vartheta}\right] \leq c_\vartheta M^{-\vartheta}$.

Proof. It can be shown using Lemmas A.23-A.24. □

Lemma A.26. *For every sufficiently small $\epsilon \leq \overline{\epsilon}$, where $\overline{\epsilon}$ depends on the smoothness of $\pi(\cdot)$, there exists $0 < C < \infty$ such that*

$$\mathbb{P}\left[\int_0^\epsilon \zeta_{T,v}(u, \widetilde{v}) \pi\left(\lambda_b^0 + u/\psi_T\right) du < \epsilon\pi\left(\lambda_b^0\right)\right] < C\epsilon^{1/2}. \tag{A.44}$$

Proof. Since $\mathbb{E}(\zeta_{T,v}(0, \widetilde{v})) = 1$ and $\mathbb{E}(\zeta_{T,v}(u, \widetilde{v})) \leq 1$ for sufficiently large T, we have

$$\mathbb{E}|\zeta_{T,v}(u, \widetilde{v}) - \zeta_{T,v}(0, \widetilde{v})| \leq \left(\mathbb{E}\left|\zeta_{T,v}^{1/2}(u, \widetilde{v}) + \zeta_{T,v}^{1/2}(0, \widetilde{v})\right|^2 \mathbb{E}\left|\zeta_{T,v}^{1/2}(u, \widetilde{v}) - \zeta_{T,v}^{1/2}(0, \widetilde{v})\right|^2\right)^{1/2} \leq C|u|^{1/2}, \tag{A.45}$$

by Lemma A.23 with $r = 1$. By Assumption 3.2, $|\pi_{T,v}(u) - \pi^0| \leq \left|\pi\left(\lambda_{b,T}^0(v)\right) - \pi^0\right| + C\psi_T^{-1}|u|$, with $C > 0$. The first term on the right-hand side is $o(1)$ (and independent of u) while the second is asymptotically negligible for small u. Thus, for a sufficiently small $\overline{\epsilon} > 0$,

$$\int_0^\epsilon \zeta_{T,v}(u, \widetilde{v}) \pi_{T,v}(u) du > \frac{\pi^0}{2} \int_0^\epsilon \zeta_{T,v}(u, \widetilde{v}) du.$$

Next, using $\zeta_{T,v}(0, \widetilde{v}) = 1$,

$$\begin{aligned}\mathbb{P}\left[\int_0^\epsilon \zeta_{T,v}(u, \widetilde{v}) \pi_{T,v}(u) du < \epsilon/2\right] &\leq \mathbb{P}\left[\int_0^\epsilon (\zeta_{T,v}(u, \widetilde{v}) - \zeta_{T,v}(0, \widetilde{v})) du < -\epsilon/2\right] \\ &\leq \mathbb{P}\left[\int_0^\epsilon |\zeta_{T,v}(u, \widetilde{v}) - \zeta_{T,v}(0, \widetilde{v})| du > \epsilon/2\right],\end{aligned}$$

and by Markov's inequality together with (A.45) the last expression is less than or equal to

$$(2/\epsilon) \int_0^\epsilon \mathbb{E}|\zeta_{T,v}(u, \widetilde{v}) - \zeta_{T,v}(0, \widetilde{v})| du < 2C\epsilon^{1/2}.$$

□

Lemma A.27. *For $f_T \in \boldsymbol{F}$, and M sufficiently large, there exist constants $c, C > 0$ such that*

$$\mathbb{P}[J_{1,M} > \exp(-cf_T(M))] \leq C\left(1 + M^C\right) \exp(-cf_T(M)), \tag{A.46}$$

uniformly in $\widetilde{v} \in \mathbf{V}$.

Proof. In view of the smotheness property of $\pi(\cdot)$, without loss of generality we consider the case of the uniform prior (i.e., $\pi_{T,v}(u) = 1$ for all u). We begin by dividing the open interval $\{u : M \leq |u| < M + 1\}$ into I disjoint segments denoting the i-th one by Π_i. For each segment Π_i choose a point u_i and define $J_{1,M}^{\Pi} \triangleq \sup_{\widetilde{v}\in\mathbf{V}} \sum_{i\in I} \zeta_{T,v}(u_i, \widetilde{v}) \text{Leb}(\Pi_i) = \sup_{\widetilde{v}\in\mathbf{V}} \sum_{i\in I} \int_{\Pi_i} \zeta_{T,v}(u_i, \widetilde{v}) du$. Then,

$$\mathbb{P}\left[J_{1,M}^{\Pi} > (1/4)\exp(-cf_T(M))\right] \leq \mathbb{P}\left[\max_{i\in I} \sup_{\widetilde{v}\in\mathbf{V}} \zeta_{T,v}^{1/2}(u_i, \widetilde{v}) (\text{Leb}(\Gamma_M))^{1/2} > (1/2)\exp(-f_T(M)/2)\right]$$

$$\leq \sum_{i\in I} \mathbb{P}\left[\zeta_{T,v}^{1/2}(u_i, \widetilde{v}) > (1/2)(\text{Leb}(\Gamma_M))^{-1/2}\exp(-f_T(M)/2)\right]$$
$$\leq 2I(\text{Leb}(\Gamma_M))^{1/2}\exp(-f_T(M)/12), \tag{A.47}$$

where the last inequality follows from applying Lemma A.24 to each summand. Upon using the inequality $\exp(-f_T(M)/2) < 1/2$ (which is valid for sufficiently large M), we have

$$\mathbb{P}[J_{1,M} > \exp(-f_T(M)/2)] \leq \mathbb{P}\left[\left|J_{1,M} - J_{1,M}^{\Pi}\right| > (1/2)\exp(-f_T(M)/2)\right] + \mathbb{P}\left[J_{1,M}^{\Pi} > \exp(-f_T(M))\right].$$

Focusing on the first term,

$$\begin{aligned}\mathbb{E}\left[J_{1,M} - J_{1,M}^{\Pi}\right] &\leq \sum_{i\in I}\int_{\Pi_i}\mathbb{E}\left|\zeta_{T,v}^{1/2}(u, \widetilde{v}) - \zeta_{T,v}^{1/2}(u_i, \widetilde{v})\right| du\\ &\leq \sum_{i\in I}\int_{\Pi_i}\left(\mathbb{E}\left|\zeta_{T,v}^{1/2}(u, \widetilde{v}) + \zeta_{T,v}^{1/2}(u_i, \widetilde{v})\right|\mathbb{E}\left|\zeta_{T,v}^{1/2}(u, \widetilde{v}) - \zeta_{T,v}^{1/2}(u_i, \widetilde{v})\right|\right)^{1/2} du\\ &\leq C(1+M)^C\sum_{i\in I}\int_{\Pi_i}|u_i - u|^{1/2}\,du,\end{aligned}$$

where for the last inequality we have used Lemma A.24 since we can always choose the partition of the segments such that each Π_i contains either positive or negative u_i. Since each summand on the right-hand side above is less than $C(MI^{-1})^{3/2}$ there exist numbers C_1 and C_2 such that

$$\mathbb{E}\left[J_{1,M} - J_{1,M}^{\Pi}\right] \leq C_1\left(1 + M^{C_2}\right)I^{-1/2}. \tag{A.48}$$

Using (A.47) and (A.48) we have

$$\mathbb{P}[J_{1,M} > \exp(-f_T(M)/2)] \leq C_1\left(1 + M^{C_2}\right)I^{-1/2} + 2I(\text{Leb}(\Gamma_M))^{1/2}\exp(-f_T(M)/12).$$

The relationship in the last display leads to the claim of the lemma if we choose I satisfying $1 \leq I^{3/2}\exp(-f_T(M)/4) \leq 2$. □

Lemma A.28. *For $f_T \in \boldsymbol{F}$, and M sufficiently large, there exist constants c, $C > 0$ such that*

$$\mathbb{E}[J_{1,M}/J_2] \leq C\left(1 + M^C\right)\exp(-cf_T(M)), \tag{A.49}$$

uniformly in $\widetilde{v} \in \mathbf{V}$.

Proof. Note that $J_{1,M}/J_2 \leq 1$. Thus, for any $\epsilon > 0$,

$$\mathbb{E}[J_{1,M}/J_2] \leq \mathbb{P}[J_{1,M} > \exp(-cf_T(M)/2)] + (4/\epsilon)\exp(-cf_T(M)) + \mathbb{P}\left[\int_{\Gamma_T}\zeta_{T,v}(u, \widetilde{v})\,du < \epsilon/4\right].$$

By Lemma A.27, the first term is bounded by $C\left(1 + M^C\right)\exp(-cf_T(M)/4)$ while for the last term we can use (A.44) to deduce

$$\mathbb{E}[J_{1,M}/J_2] \leq C\left(1 + M^C\right)\exp(-cf_T(M)) + (4/\epsilon)\exp(-cf_T(M)) + C\epsilon^{1/2}.$$

Finally, choose $\epsilon = \exp((-2c/3)f_T(M))$ to complete the proof of the lemma. □

Lemma A.29. *For $l \in \boldsymbol{L}$ and any $\vartheta > 0$, $\lim_{B\to\infty}\lim_{T\to\infty}B^{\vartheta}\mathbb{P}\left[\psi_T\left(\widehat{\lambda}_b^{\text{GL}} - \lambda_b^0\right) > B\right] = 0.$*

Proof. Let $p_T(u) \triangleq p_{1,T}(u)/\overline{p}_T$ where $p_{1,T}(u) = \exp\left(\widetilde{G}_{T,v}(u, \widetilde{v}) + Q_{T,v}(u)\right)$ and $\overline{p}_T \triangleq \int_{\mathbf{U}_T} p_{1,T}(w)\,dw$. By definition, $\widehat{\lambda}_b^{\mathrm{GL}}$ is the minimum of the function $\int_{\Gamma^0} l\left(T\|\delta_T\|^2(s-u)\right)p_{1,T}(u)\,\pi_{T,v}(u)\,du$ with $s \in \Gamma^0$. Upon using a change in variables,

$$\begin{aligned}
&\int_{\Gamma^0} l\left(T\|\delta_T\|^2(s-u)\right)p_{1,T}(u)\,\pi_{T,v}(u)\,du \\
&\quad= \left(T\|\delta_T\|^2\right)^{-1}\overline{p}_T\int_{\mathbf{U}_T} l\left(T\|\delta_T\|^2\left(s-\lambda_b^0\right)-u\right)p_T\left(\lambda_{b,T}^0(v)+\left(T\|\delta_T\|^2\right)^{-1}u\right) \\
&\qquad\times \pi_{T,v}\left(\lambda_{b,T}^0(v)+\left(T\|\delta_T\|^2\right)^{-1}u\right)du.
\end{aligned}$$

Thus, $\lambda_{\delta,T} \triangleq T\|\delta_T\|^2\left(\widehat{\lambda}_b^{\mathrm{GL}}-\lambda_b^0\right)$ is the minimum of the function

$$\mathcal{S}_T(s) \triangleq \int_{\mathbf{U}_T} l(s-u)\,\frac{p_T\left(\lambda_b^0+\left(T\|\delta_T\|^2\right)^{-1}u\right)\pi_{T,v}\left(\lambda_b^0+\left(T\|\delta_T\|^2\right)^{-1}u\right)}{\int_{\mathbf{U}_T} p_T\left(\lambda_b^0+\left(T\|\delta_T\|^2\right)^{-1}w\right)\pi_{T,v}\left(\lambda_b^0+\left(T\|\delta_T\|^2\right)^{-1}w\right)dw}\,du,$$

where the optimization is over $\mathbf{U}_T$. The random function $\mathcal{S}_T(\cdot)$ converges with probability one in view of Lemmas A.27-A.28 together with the properties of the loss function l [cf. (A.35) and the discussion surrounding it]. Therefore, we shall show that the random function $\mathcal{S}_T(s)$ is strictly larger than $\mathcal{S}_T(0)$ on $\{|s|>B\}$ with high probability as $T\to\infty$. This reflects that

$$\mathbb{P}\left[\left|T\|\delta_T\|^2\left(\widehat{\lambda}_b^{\mathrm{GL}}-\lambda_b^0\right)\right|>B\right] \leq \mathbb{P}\left[\inf_{|s|>B}\mathcal{S}_T(s)\leq\mathcal{S}_T(0)\right]. \tag{A.50}$$

We present the proof for the case $\pi_{T,v}(u)=1$ for all u. The general case follows with no additional difficulties due to the assumptions satisfied by the prior $\pi(\cdot)$. By the properties of the family $\boldsymbol{L}$ of loss functions, we can find $\overline{u}_1,\,\overline{u}_2\in\mathbb{R}$, with $0<\overline{u}_1<\overline{u}_2$ such that as T increases,

$$\bar{l}_{1,T}\triangleq\sup\left\{l(u):\ u\in\Gamma_{1,T}\right\}<\bar{l}_{2,T}\triangleq\inf\left\{l(u):\ u\in\Gamma_{2,T}\right\},$$

where $\Gamma_{1,T}\triangleq\mathbf{U}_T\cap(|u|\leq\overline{u}_1)$ and $\Gamma_{2,T}\triangleq\mathbf{U}_T\cap(|u|>\overline{u}_2)$. With this notation,

$$\mathcal{S}_T(0)\leq\bar{l}_{1,T}\int_{\Gamma_{1,T}}p_T(u)\,du+\int_{\mathbf{U}_T\cap(|u|>\overline{u}_1)}l(u)\,p_T(u)\,du.$$

Furthermore, if $l\in\boldsymbol{L}$, then for sufficiently large B the following relationships hold: (i) $l(u)-\inf_{|v|>B/2}l(v)\leq 0$; (ii) $|u|\leq(B/2)^{\vartheta}$, $\vartheta>0$. We shall assume that B is chosen so that $B>2\overline{u}_2$ and $(B/2)^{\vartheta}>\overline{u}_2$ hold. Let $\Gamma_{T,B}\triangleq\{u:\ (|u|>B/2)\cap\mathbf{U}_T\}$. Then, whenever $|s|>B$ and $|u|\leq B/2$, we have,

$$|u-s|>B/2>\overline{u}_2 \qquad\text{and}\qquad \inf_{u\in\Gamma_{T,B}}l(u)\geq\bar{l}_{2,T}. \tag{A.51}$$

With this notation,

$$\begin{aligned}
\inf_{|s|>B}\mathcal{S}_T(s) &\geq \inf_{u\in\Gamma_{T,B}}l_T(u)\int_{(|w|\leq B/2)\cap\mathbf{U}_T}p_T(w)\,dw \\
&\geq \bar{l}_{2,T}\int_{(|w|\leq B/2)\cap\mathbf{U}_T}p_T(w)\,dw,
\end{aligned}$$

from which it follows that

$$\mathcal{S}_T(0) - \inf_{|s|>B} \mathcal{S}_T(s) \leq -\varpi \int_{\Gamma_{1,T}} p_T(u)\, du + \int_{\mathbf{U}_T \cap \left((B/2)^{\vartheta} \geq |u| \geq \overline{u}_1\right)} \left(l(u) - \inf_{|s|>B/2} l_T(s) \right) p_T(u)\, du$$
$$+ \int_{\mathbf{U}_T \cap \left(|u| > (B/2)^{\vartheta}\right)} l(u)\, p_T(u)\, du,$$

where $\varpi \triangleq \bar{l}_{2,T} - \bar{l}_{1,T}$. The last inequality can be manipulated further using (A.51), so that

$$\mathcal{S}_T(0) - \inf_{|s|>B} \mathcal{S}_T(s) \leq -\varpi \int_{\Gamma_{1,T}} p_T(u)\, du + \int_{\mathbf{U}_T \cap \left(|u| > (B/2)^{\vartheta}\right)} l_T(u)\, p_T(u)\, du. \tag{A.52}$$

Let $B_{\vartheta} \triangleq (B/2)^{\vartheta}$ and fix an arbitrary number $\overline{a} > 0$. For the first term of (A.52), Lemma A.26 implies that for sufficiently large T, we have

$$\mathbb{P}\left[\int_{\Gamma_{1,T}} p_T(u)\, du < 2\left(\varpi B^{\overline{a}}\right)^{-1} \right] \leq c\left(\varpi B^{\overline{a}}\right)^{-1/2}, \tag{A.53}$$

where $0 < c < \infty$. Next, let us consider the second term of (A.52). We show that for large enough T, an arbitrary number $\overline{a} > 0$,

$$\mathbb{P}\left[\int_{\mathbf{U}_T \cap \{|u| > B_{\vartheta}\}} l(u)\, p_T(u)\, du > B^{-\overline{a}} \right] \leq c B^{-\overline{a}}. \tag{A.54}$$

Since $l \in \boldsymbol{L}$, we have $l(u) \leq |u|^{a}$, $a > 0$ when u is large enough. Choosing B large leads to

$$\mathbb{E}\left[\int_{\mathbf{U}_T \cap \{|u| > B_{\vartheta}\}} l(u)\, p_T(u)\, du \right] \leq \sum_{i=0}^{\infty} (B_{\vartheta} + i + 1)^{a}\, \mathbb{E}\left(J_{1, B_{\vartheta}+i}/J_2\right),$$

where $J_{1,B_{\vartheta}+i}$, J_2 are defined as in (A.33). By Lemma A.28,

$$\mathbb{E}\left(J_{1, B_{\vartheta}+i}/J_2\right) \leq c\left(1 + (B_{\vartheta} + i)^{a}\right) \exp\left(-b f_T(B_{\vartheta} + i)\right),$$

where $f_T \in \boldsymbol{F}$ and thus for some b, $0 < c < \infty$,

$$\mathbb{E}\left[\int_{\mathbf{U}_T \cap \{|u| > B_{\vartheta}\}} l(u)\, p_T(u)\, du \right] \leq c \int_{B_{\vartheta}}^{\infty} (1 + v^{a}) \exp\left(-b f_T(v)\right) dv \leq c \exp\left(-b f_T(B_{\vartheta})\right).$$

By property (ii) of the function f_T in the class $\boldsymbol{F}$, for any $d \in \mathbb{R}$, $\lim_{v\to\infty} \lim_{T\to\infty} v^{d} e^{-b f_T(v)} = 0$. Thus, we know that for T large enough and some $0 < c < \infty$,

$$\mathbb{E}\left[\int_{\mathbf{U}_T \cap \{|u| > B_{\vartheta}\}} l(u)\, p_T(u)\, du \right] \leq c B^{-2\overline{a}},$$

from which we deduce (A.54) after applying Markov's inequality. Therefore, for sufficiently large T and large B, combining equation (A.50), and (A.53)-(A.54), we have

$$\mathbb{P}\left[T \left\| \delta_T \right\|^2 \left(\widehat{\lambda}_b^{\mathrm{GL}} - \lambda_b^0 \right) > B \right]$$
$$\leq \mathbb{P}\left[-\varpi \int_{\Gamma_{1,T}} p_T(u)\, du + \int_{\mathbf{U}_T \cap \{|u| > B_{\vartheta}\}} l_T(u)\, p_T(u)\, du \leq 0 \right]$$

$$\leq \mathbb{P}\left[\int_{\Gamma_{1,T}} p_T(u)\, du < 2\left(\varpi B^{\overline{a}}\right)^{-1}\right] + \mathbb{P}\left[\int_{\mathbf{U}_T \cap \{|u| > B_\vartheta\}} l(u)\, p_T(u)\, du > B^{-\overline{a}}\right]$$
$$\leq c\left(B^{-\overline{a}/2} + B^{-\overline{a}}\right),$$

which can be made arbitrarily small choosing B large enough. □

Lemma A.30. *As $T \to \infty$, the marginal distributions of $\zeta_{T,v}(u, \widetilde{v})$ converge to the marginal distributions of $\exp(\mathscr{V}(u))$.*

Proof. The results follow from Lemma A.3, Lemma A.12 and the continuous mapping theorem. □

A.4 Proofs of Section 4

A.4.1 Proof of Proposition 4.1

The preliminary lemmas below consider the Gaussian process $\mathscr{W}$ on the positive half-line with $s > 0$. The case $s \leq 0$ is similar and omitted. The generic constant $C > 0$ used in the proofs of this section may change from line to line.

Lemma A.31. *For $\varpi > 3/4$, we have $\lim_{T\to\infty} \limsup_{|s|\to\infty} \left|\widehat{\mathscr{W}}_T(s)\right| / |s|^{\varpi} = 0$, $\mathbb{P}$-a.s.*

Proof. For any $\epsilon > 0$, if we can show that

$$\sum_{i=1}^{\infty} \mathbb{P}\left[\sup_{i-1\leq |s| < i} \left|\widehat{\mathscr{W}}_T(s)\right| / |s|^{\varpi} > \epsilon\right] < \infty, \tag{A.55}$$

then by the Borel-Cantelli lemma, $\mathbb{P}\left[\limsup_{|s|\to\infty} \left|\widehat{\mathscr{W}}_T(s)\right| / |s|^{\varpi} > \epsilon\right] = 0$. Proceeding as in the proof of Lemma A.13,

$$\mathbb{P}\left[\sup_{i-1\leq |s| < i} \left|\widehat{\mathscr{W}}_T(s)\right| / |s|^{\varpi} > \epsilon\right] \leq \mathbb{P}\left[\sup_{|s|\leq 1} \left|\widehat{\mathscr{W}}_T(s)\right| > \epsilon i^{\varpi - 1/2}\right]$$
$$\leq \frac{1}{\epsilon^4}\mathbb{E}\left[\mathbb{E}\left(\sup_{|s|\leq 1}\left(\widehat{\mathscr{W}}_T(s)\right)^4 \mid \widehat{\Sigma}_T\right)\right]\frac{1}{i^{4\varpi - 2}}.$$

The series $\sum_{i=1}^{\infty} i^{-p}$ is a Riemann's zeta function and satisfies $\sum_{i=1}^{\infty} i^{-p} < \infty$ if $p > 1$. Then,

$$\sum_{i=1}^{\infty} \mathbb{P}\left[\sup_{i-1\leq |s| < i} \left|\widehat{\mathscr{W}}_T(s)\right| / |s|^{\varpi} > \epsilon\right] \leq \left(C/\epsilon^4\right)\mathbb{E}\left[\mathbb{E}\left(\sup_{|s|\leq 1}\left(\widehat{\mathscr{W}}_T(s)\right)^4 \mid \widehat{\Sigma}_T\right)\right]$$
$$\leq \left(C/\epsilon^4\right)\mathbb{E}\left[\mathbb{E}\left(\sup_{|s|\leq 1}\widehat{\mathscr{W}}_T(s) \mid \widehat{\Sigma}_T\right)\right]^4, \tag{A.56}$$

where $C > 0$ and the last inequality follows from Proposition A.2.4 in van der Vaart and Wellner (1996). The process $\widehat{\mathscr{W}}_T$, conditional on $\widehat{\Sigma}_T$, is sub-Gaussian with respect to the semimetric $d_{VW}^2(t, s) = \widehat{\Sigma}_T(t, t) + \widehat{\Sigma}_T(s, s)$, which by invoking Assumption 4.1-(ii,iii) is bounded by

$$\widehat{\Sigma}_T(t - s, t - s) \leq |t - s| \sup_{|s|=1} \widehat{\Sigma}_T(s, s).$$

Theorem 2.2.8 in van der Vaart and Wellner (1996) then implies

$$\mathbb{E}\left(\sup_{|s|\leq 1}\widehat{\mathscr{W}}_T(s) \mid \widehat{\Sigma}_T\right) \leq C \sup_{|s|=1} \widehat{\Sigma}_T^{1/2}(s, s).$$

The above inequality can be used into the right-hand side of (A.56) to deduce that the latter is bounded by $C\mathbb{E}\left(\sup_{|s|=1}\widehat{\Sigma}_T^2(s,\,s)\right)$. By Assumption 4.1-(iv) $C\mathbb{E}\left(\sup_{|s|=1}\widehat{\Sigma}_T^2(s,\,s)\right)<\infty$, and the proof is concluded. □

Lemma A.32. $\left\{\widehat{\mathscr{W}_T}\right\}$ *converges weakly toward* $\mathscr{W}$ *on compact subsets of* $\mathbb{D}_b$.

Proof. By the definition of $\widehat{\mathscr{W}_T}(\cdot)$, we have the finite-dimensional convergence in distribution of $\widehat{\mathscr{W}_T}$ toward $\mathscr{W}$. Hence, it remains to show the (asymptotic) stochastic equicontinuity of the sequence of processes $\left\{\widehat{\mathscr{W}_T},\,T\geq 1\right\}$. Let $\mathbf{C}\subset\mathbb{R}_+$ be any compact set. Fix any $\eta>0$ and $\epsilon>0$. We show that for any positive sequence $\{d_T\}$, with $d_T\downarrow 0$, and for every $t,\,s\in\mathbf{C}$,

$$\limsup_{T\to\infty}\mathbb{P}\left(\sup_{|t-s|<d_T}\left|\widehat{\mathscr{W}_T}(t)-\widehat{\mathscr{W}_T}(s)\right|>\eta\right)<\epsilon. \tag{A.57}$$

By Markov's inequality, $\mathbb{P}\left(\sup_{|t-s|<d_T}\left|\widehat{\mathscr{W}_T}(t)-\widehat{\mathscr{W}_T}(s)\right|>\eta\right)\leq\mathbb{E}\left(\sup_{|t-s|<d_T}\left|\widehat{\mathscr{W}_T}(t)-\widehat{\mathscr{W}_T}(s)\right|\right)/\eta$. Let $\widehat{\Upsilon}_T(t,\,s)$ denote the covariance matrix of $\left(\widehat{\mathscr{W}_T}(t)\,,\,\widehat{\mathscr{W}_T}(s)\right)'$ and $\mathcal{N}$ be a two-dimensional standard normal vector. Letting $\imath\triangleq\begin{bmatrix}1 & -1\end{bmatrix}'$, we have

$$\begin{aligned}\left[\mathbb{E}\sup_{|t-s|<d_T}\left|\widehat{\mathscr{W}_T}(t)-\widehat{\mathscr{W}_T}(s)\right|\right]^2&=\left[\mathbb{E}\sup_{|t-s|<d_T}\left|\imath'\widehat{\Upsilon}_T^{1/2}(t,\,s)\,\mathcal{N}\right|\right]^2\leq\mathbb{E}\left[\sup_{|t-s|<d_T}\imath'\widehat{\Upsilon}_T(t,\,s)\,\imath\right]\\&=\mathbb{E}\left[\sup_{|t-s|<d_T}\widehat{\Sigma}_T(t-s,\,t-s)\right]\\&\leq d_T\mathbb{E}\left[\sup_{|s|=1}\widehat{\Sigma}_T(s,\,s)\right],\end{aligned}$$

and so $\mathbb{E}\left[\sup_{|t-s|<d_T}\widehat{\Sigma}_T(t-s,\,t-s)\right]\leq 2d_T\mathbb{E}\left[\sup_{|s|=1}\widehat{\Sigma}_T(s,\,s)\right]$ where we have used Assumption 4.1-(iii) in the last step. As $d_T\downarrow 0$ the right-hand side goes to zero since $\mathbb{E}\left[\sup_{|s|=1}\widehat{\Sigma}_T(s,\,s)\right]=O(1)$ by Assumption 4.1-(iv). □

Lemma A.33. *Fix* $0<a<\infty$. *For any* $p\in\boldsymbol{P}$ *and for any positive sequence* $\{a_T\}$ *satisfying* $a_T\overset{\mathbb{P}}{\to}a$,

$$\int_{\mathbb{R}}|p(s)|\exp\left(\widehat{\mathscr{W}_T}(s)\right)\exp\left(-a_T|s|\right)ds\overset{d}{\to}\int_{\mathbb{R}}|p(s)|\exp\left(\mathscr{W}(s)\right)\exp\left(-a|s|\right)ds.$$

Proof. Let $\mathbf{B}_+$ be a compact subset of $\mathbb{R}_+/\{0\}$. Let

$$\boldsymbol{G}=\left\{(W,\,a_T)\in\mathbb{D}_b(\mathbb{R},\,\mathscr{B},\,\mathbb{P})\times\mathbf{B}_+:\,\limsup_{|s|\to\infty}|W(s)|\,/\,|s|^{\varpi}=0,\,\varpi>3/4,\,a_T=a+o_{\mathbb{P}}(1)\right\},$$

and denote by $f:\boldsymbol{G}\to\mathbb{R}$ the functional given by $f(\boldsymbol{G})=\int|p(s)|\exp\left(W(s)\right)\exp\left(-a_T|s|\right)ds$. In view of the continuity of $f(\cdot)$ and $a_T\overset{\mathbb{P}}{\to}a$, the claim of the lemma follows by Lemmas A.31-A.32 and the continuous mapping theorem. □

We are now in a position to conclude the proof of Proposition 4.1. Suppose $\gamma_T=CT\left\|\widehat{\delta}_T\right\|^2$ for some $C>0$. Under mean-squared loss function, $\widehat{\xi}_T$ admits a closed form:

$$\widehat{\xi}_T=\frac{\int u\exp\left(\widehat{\mathscr{W}_T}(u)-\widehat{\Lambda}_T(u)\right)du}{\int\exp\left(\widehat{\mathscr{W}_T}(u)-\widehat{\Lambda}_T(u)\right)du}.$$

By Lemma A.33, we deduce that $\widehat{\xi}_T$ converges in law to the distribution stated in (3.12). For general loss functions, a result corresponding to Lemma A.33 can be shown to hold since $l(\cdot)$ is assumed to be continuous.

A.5 Proofs of Section 5

Rewrite the GL estimator $\widehat{\boldsymbol{\lambda}}_b^{\mathrm{GL}}$ as the minimizer of

$$\mathcal{R}_{l,T} \triangleq \int_{\Gamma^0} l\left(s-\boldsymbol{\lambda}_b\right) \frac{\exp\left(-Q_T\left(\delta\left(\boldsymbol{\lambda}_b\right),\, \boldsymbol{\lambda}_b\right)\right)\pi\left(\boldsymbol{\lambda}_b\right)}{\int_{\Gamma^0}\exp\left(-Q_T\left(\delta\left(\boldsymbol{\lambda}_b\right),\, \boldsymbol{\lambda}_b\right)\right)\pi\left(\boldsymbol{\lambda}_b\right)d\boldsymbol{\lambda}_b} d\boldsymbol{\lambda}_b. \tag{A.58}$$

We show with the following lemma that, for each i, $\widehat{\lambda}_i^{\mathrm{GL}} \xrightarrow{\mathbb{P}} \lambda_i^0$ no matter whether the magnitude of the shifts is fixed or not. Then, the proof of Theorem 3.2 can be repeated for each $i = 1, \ldots, m$ separately. We begin with the proof for the case of fixed shifts.

Lemma A.34. *Under Assumption 5.1-5.2, except that $\Delta_{T,i} = \Delta_i^0$ for all i, for $l \in \boldsymbol{L}$ and any $B > 0$ and $\varepsilon > 0$, we have for all large T, $\mathbb{P}\left[\left|\widehat{\lambda}_i^{\mathrm{GL}} - \lambda_i^0\right| > B\right] < \varepsilon$ for each i.*

Proof. Let $S_T\left(\delta\left(\boldsymbol{\lambda}_b\right),\, \boldsymbol{\lambda}_b\right) \triangleq Q_T\left(\delta\left(\boldsymbol{\lambda}_b\right),\, \boldsymbol{\lambda}_b\right) - Q_T\left(\delta\left(\boldsymbol{\lambda}_b^0\right),\, \boldsymbol{\lambda}_b^0\right)$. Without loss of generality, we assume there are only three change-points and provide a proof by contradiction for the consistency result. In particular, we suppose that all but the second change-point are consistently estimated. That is, consider the case $T_2 < T_2^0$ and for some finite $C > 0$ assume that $|\lambda_2 - \lambda_2^0| > C$. $Q_T\left(\delta\left(\boldsymbol{\lambda}_b\right),\, \boldsymbol{\lambda}_b\right)$ can be decomposed as,

$$Q_T\left(\delta\left(\boldsymbol{\lambda}_b\right),\, \boldsymbol{\lambda}_b\right) = \sum_{t=1}^T e_t^2 + \sum_{t=1}^T d_t^2 - 2\sum_{t=1}^T e_t d_t,$$

where $d_t = w_t'\left(\widehat{\phi} - \phi^0\right) + z_t'\left(\widehat{\delta}_k - \delta_j^0\right)$, for $t \in \left[\widehat{T}_{k-1}+1,\, \widehat{T}_k\right] \cap \left[T_{j-1}^0+1,\, T_j^0\right]$ $(k,\, j = 1, \ldots, m+1)$ where $\widehat{\phi}$ and $\widehat{\delta}_k$ are asymptotically equivalent to the corresponding least-squares estimates. Bai and Perron (1998) showed that

$$T^{-1}\sum_{t=1}^T d_t^2 \xrightarrow{\mathbb{P}} K > 0 \qquad \text{and} \qquad T^{-1}\sum_{t=1}^T e_t d_t = o_{\mathbb{P}}(1).$$

Note that $Q_T\left(\delta\left(\boldsymbol{\lambda}_b^0\right),\, \boldsymbol{\lambda}_b^0\right) = S_T\left(T_1^0,\, T_2^0,\, T_3^0\right)$, where $S_T\left(T_1^0,\, T_2^0,\, T_3^0\right)$ denotes the sum of squared residuals evaluated at $\left(T_1^0,\, T_2^0,\, T_3^0\right)$. Since $T^{-1}S_T\left(T_1^0,\, T_2^0,\, T_3^0\right)$ is asymptotically equivalent to $T^{-1}\sum_{t=1}^T e_t^2$, this implies that $T^{-1}S_T\left(\delta\left(\boldsymbol{\lambda}_b\right),\, \boldsymbol{\lambda}_b\right) > 0$ for all large T. For some finite $K > 0$, this implies

$$S_T\left(\delta\left(\boldsymbol{\lambda}_b\right),\, \boldsymbol{\lambda}_b\right) \geq TK. \tag{A.59}$$

Let $\mathbf{U}_T \triangleq \left\{u \in \mathbb{R} :\, \boldsymbol{\lambda}_b^0 + T^{-1}u \in \Gamma^0\right\}$. Define $p_T(u) \triangleq p_{1,T}(u)/\overline{p}_T$ where $p_{1,T}(u) = \exp\left(-Q_T\left(\delta(u),\, u\right)\right)$ and $\overline{p}_T \triangleq \int_{\mathbf{U}_T} p_{1,T}(w)\,dw$. By definition, $\widehat{\boldsymbol{\lambda}}_b^{\mathrm{GL}}$ is the minimum of the function $\int_{\Gamma^0} l(s-u)\,p_{1,T}(u)\,\pi(u)\,du$ with $s \in \Gamma^0$. Upon using a change in variables,

$$\begin{aligned}&\int_{\Gamma^0} l(s-u)\,p_{1,T}(u)\,\pi(u)\,du \\ &\quad = T^{-1}\overline{p}_T \int_{\mathbf{U}_T} l\left(T\left(s-\boldsymbol{\lambda}_b^0\right) - u\right) p_T\left(\boldsymbol{\lambda}_b^0 + T^{-1}u\right)\pi\left(\boldsymbol{\lambda}_b^0 + T^{-1}u\right)du.\end{aligned}$$

Thus, $\boldsymbol{\lambda}_{\delta,T} \triangleq T\left(\widehat{\boldsymbol{\lambda}}_b^{\mathrm{GL}} - \boldsymbol{\lambda}_b^0\right)$ is the minimum of the function,

$$\mathcal{S}_T(s) \triangleq \int_{\mathbf{U}_T} l(s-u) \frac{p_T\left(\boldsymbol{\lambda}_b^0 + T^{-1}u\right)\pi\left(\boldsymbol{\lambda}_b^0 + T^{-1}u\right)}{\int_{\mathbf{U}_T} p_T\left(\boldsymbol{\lambda}_b^0 + T^{-1}w\right)\pi\left(\boldsymbol{\lambda}_b^0 + T^{-1}w\right)dw} du,$$

where the optimization is over $\mathbf{U}_T$. As in the proof of Lemma A.8, we exploit the following relationship,

$$\mathbb{P}\left[\left|\widehat{\boldsymbol{\lambda}}_b^{\mathrm{GL}} - \boldsymbol{\lambda}_b^0\right| > B\right] \leq \mathbb{P}\left[\inf_{|s|>TB} \mathcal{S}_T(s) \leq \mathcal{S}_T(0)\right]. \tag{A.60}$$

Thus, we need to show that the random function $\mathcal{S}_T(s)$ is strictly larger than $\mathcal{S}_T(0)$ on $\{|s| > TB\}$ with high probability as $T \to \infty$. The same steps as in Lemma A.8 lead to,

$$\begin{aligned} &\mathcal{S}_T(0) - \inf_{|s|>TB} \mathcal{S}_T(s) \qquad\qquad \text{(A.61)} \\ &\leq -\varpi \int_{\Gamma_{1,T}} p_T(u)\,du + \int_{\mathbf{U}_T \cap \left(|u|>(TB/2)^{\vartheta}\right)} l_T(u)\,p_T(u)\,du. \end{aligned}$$

We can use the relationship (A.59) in place of (A.15) in Lemma A.8 to show that the second term above converges to zero. The first term is negative using the same argument as in Lemma A.8. Thus, $\mathcal{S}_T(0) - \inf_{|s|>TB} \mathcal{S}_T(s) < 0$. This gives a contradiction to the fact that $\widehat{\boldsymbol{\lambda}}_b^{\mathrm{GL}}$ minimizes $\int_{\Gamma^0} l(s-u)\,p_{1,T}(u)\,\pi(u)\,du$. Hence, each change-point is consistently estimated. □

Lemma A.35. *Under Assumption 5.1-5.2, for $l \in \boldsymbol{L}$ and any $B > 0$ and $\varepsilon > 0$, we have for all large T, $\mathbb{P}\left[\left|\widehat{\lambda}_i^{\mathrm{GL}} - \lambda_i^0\right| > B\right] < \varepsilon$ for each i.*

Proof. The structure of the proof is similar to that of Lemma A.34. The difference consists on the fact that now $T^{-1}\sum_{t=1}^T d_t^2 \xrightarrow{\mathbb{P}} 0$ even when a break is not consistently estimated. However, Bai and Perron (1998) showed that $T^{-1}\sum_{t=1}^T d_t^2 > 2T^{-1}\sum_{t=1}^T e_t d_t$ and thus one can proceed as in the aforementioned proof to complete the proof. □

Lemma A.36. *Under Assumption 5.1-5.2, for $l \in \boldsymbol{L}$ and for every $\varepsilon > 0$ there exists a $B < \infty$ such that for all large T, $\mathbb{P}\left[Tv_T^2\left|\widehat{\lambda}_i^{\mathrm{GL}} - \lambda_i^0\right| > B\right] < \varepsilon$ for each i.*

Proof. Let $S_T(\delta(\boldsymbol{\lambda}_b), \boldsymbol{\lambda}_b) \triangleq Q_T(\delta(\boldsymbol{\lambda}_b), \boldsymbol{\lambda}_b) - Q_T\left(\delta\left(\boldsymbol{\lambda}_b^0\right), \boldsymbol{\lambda}_b^0\right)$. Without loss of generality, we assume there are only three change-points and provide an explicit proof only for λ_2^0. We use the same notation as in Bai and Perron (1998), pp. 69-70. Note that their results concerning the estimates of the regression parameters can be used in our context because once we have the consistency of the fractional change-points the estimates of the regression parameters are asymptotically equivalent to the corresponding least-squares estimates. For each $\epsilon > 0$, let $V_\epsilon = \left\{(T_1, T_2, T_3);\, \left|\widehat{T}_i - T_i^0\right| \leq \epsilon T,\, i = 1 \leq i \leq 3\right\}$. By the consistency result, for each $\epsilon > 0$ and T large, we have $\left|\widehat{T}_i - T_i^0\right| \leq \epsilon T$, where $\widehat{T}_i = \widehat{T}_i^{\mathrm{GL}} = T\widehat{\lambda}_i^{\mathrm{GL}}$. Hence, $\mathbb{P}\left(\left\{\widehat{T}_1, \widehat{T}_2, \widehat{T}_3\right\} \in V_\epsilon\right) \to 1$ with high probability. Therefore we only need to examine the behavior of $S_T(\delta(\boldsymbol{\lambda}_b), \boldsymbol{\lambda}_b)$ for those T_i that are close to the true break dates such that $\left|T_i - T_i^0\right| < \epsilon T$ for all i. By symmetry, we can, without loss of generality, consider the case $T_2 < T_2^0$. For $C > 0$, define

$$V_\epsilon^*(C) = \left\{(T_1, T_2, T_3);\, \left|\widehat{T}_i - T_i^0\right| < \epsilon T,\, 1 \leq i \leq 3,\, T_2 - T_2^0 < -C/v_T^2\right\}.$$

Define the sum of squared residuals evaluated at (T_1, T_2, T_3) by $S_T(T_1, T_2, T_3)$. Let $SSR_1 = S_T(T_1, T_2, T_3)$, $SSR_2 = S_T(T_1, T_2^0, T_3)$ and $SSR_3 = S_T(T_1, T_2, T_2^0, T_3)$. We have omitted the dependence on δ. With this notation, we have $S_T(\delta(\boldsymbol{\lambda}_b), \boldsymbol{\lambda}_b) = S_T(T_1, T_2, T_3) - S_T(T_1^0, T_2^0, T_3^0)$ which can be decomposed as

$$\begin{aligned} & S_T(\delta(\boldsymbol{\lambda}_b), \boldsymbol{\lambda}_b) \\ & \quad = [(SSR_1 - SSR_3) - (SSR_2 - SSR_3)] + \left(SSR_2 - S_T\left(T_1^0, T_2^0, T_3^0\right)\right). \end{aligned} \tag{A.62}$$

In their Proposition 4-(ii), Bai and Perron (1998) showed that the first term on the right-hand side above satisfies the following: for every $\varepsilon > 0$, there exists $B > 0$ and $\epsilon > 0$ such that for large T,

$$\mathbb{P}\left[\min\left\{\left[S_T(T_1, T_2, T_3) - S_T\left(T_1, T_2^0, T_3\right)\right] / \left(T_2^0 - T_2\right)\right\} \leq 0\right] < \varepsilon,$$

where the minimum is taken over $V_\epsilon^*(C)$. The second term of (A.62) divided by $T_2^0 - T_2$ can be shown to be negligible for $\{T_1, T_2, T_3\} \in V_\epsilon^*(C)$ and C large enough because on $V_\epsilon^*(C)$ the consistency result guarantees that $\widehat{\lambda}_i$ can be made arbitrarily close to λ_i^0. This leads to a result similar to (A.59) where T is replaced by v_T^{-2}. Then one can continue with the same argument used in the second part of the proof of Lemma A.34. □

A.6 Proofs of Section 6

A.6.1 Proof of Proposition 6.1

Let

$$p_{1,T}\left(y|\, \lambda_b^0 + \psi_T^{-1} u\right) \triangleq \exp\left(\left(\widetilde{G}_{T,0}(u, 0) + Q_{T,0}(u)\right)/2\right),$$

where $\widetilde{G}_{T,0}(u, 0)$ and $Q_{T,0}(u)$ were defined in equation (3.7). Let $p_1(y|\,\lambda_b) \triangleq \exp\left(\left(L^2(\lambda_b) - L^2(\lambda_0)\right)/2\right)$ where $L(\lambda_b) = (T_b(T - T_b))^{1/2}\left(\overline{Y}_{T_b}^* - \overline{Y}_{T_b}\right)$ with $\overline{Y}_{T_b} = T_b^{-1}\sum_{t=1}^{T_b} y_t$ and $\overline{Y}_{T_b}^* = (T - T_b)^{-1}\sum_{t=T_b+1}^{T} y_t$. Following Bhattacharya (1994) we use a prior $\breve{\pi}(\cdot)$ on the random variable $\overline{\lambda}_b$. The posterior distribution of $\overline{\lambda}_b = \lambda_b$ is given by $p(\lambda_b|\, y) = h(\lambda_b) / \int_0^1 h(s)\, ds$ where $h(\lambda_b) = p_1(y|\,\lambda_b)\, \breve{\pi}(\lambda_b)$. The total variation distance between two probability measures ν_1 and ν_2 defined on some probability space $S \in \mathbb{R}$ is denoted as $|\nu_1 - \nu_2|_{\text{TV}} \triangleq \int_S |\nu_1(u) - \nu_2(u)|\, du$. Given the local parameter $\lambda_b = \lambda_b^0 + \left(T v_T^2\right)^{-1} u$ with $u \in [-M, M]$ for a given $M > 0$, the posterior for u is equal to $p^*(u|\, y) = \left(T v_T^2\right)^{-1} p\left(\left(T v_T^2\right)^{-1} u + \lambda_b^0 |\, y\right)$ while the quasi-posterior is given by $p_T^*(u|\, y) = \left(T v_T^2\right)^{-1} p_T\left(\left(T v_T^2\right)^{-1} u + \lambda_b^0 |\, y\right)$.

Lemma A.37. *Let Assumption 3.2-3.3 and 3.6-(i) hold and $\breve{\pi}(\cdot)$ satisfy Assumption 3.2. Then,*

$$\left|p_T^*\left(T v_T^2\left(\overline{\lambda}_b - \lambda_b^0\right) |\, y\right) - p^*\left(T v_T^2\left(\overline{\lambda}_b - \lambda_b^0\right) |\, y\right)\right|_{\text{TV}} \xrightarrow{\mathbb{P}} 0.$$

Proof. By Assumption 3.2, $\pi(\cdot)$ and $\breve{\pi}(\cdot)$ are bounded, and

$$\sup_{|u| \leq M}\left|\pi\left(\left(T v_T^2\right)^{-1} u + \lambda_b^0\right) - \pi\left(\lambda_b^0\right)\right| \xrightarrow{\mathbb{P}} 0,$$
$$\sup_{|u| \leq M}\left|\breve{\pi}\left(\left(T v_T^2\right)^{-1} u + \lambda_b^0\right) - \breve{\pi}\left(\lambda_b^0\right)\right| \xrightarrow{\mathbb{P}} 0.$$

Since $\pi(\cdot)$ $[\breve{\pi}(\cdot)]$ appears in both the numerator and denominator of $p_T^*(\cdot|\, y)$ $[p^*(\cdot|\, y)]$, it cancels from that expression asymptotically. Turning to the Laplace estimator, the results of Section 3 (see Lemmas A.2 and A.4) imply that for $u \leq 0$, using $Q(\delta(\lambda_b), \lambda_b)/2$ in place of $Q(\delta(\lambda_b), \lambda_b)$,

$$\exp\left(\left(\widetilde{G}_{T,0}(u, 0) + Q_{T,0}(u)\right)/2\right) \tag{A.63}$$

$$= \exp\left(\delta_T \sum_{t=0}^{v_T^{-2}|u|} e_{T_b^0-t} - |u|\,\delta_0^2/2\right)(1+A_T),$$

where $A_T = o_{\mathbb{P}}(1)$ is uniform in the region $u \leq \eta T v_T^2$ for small $\eta > 0$. By symmetry, the case $u > 0$ results in the same relationship as (A.63) with $e_{T_b^0-t}$ replaced by $e_{T_b^0+t}$. The results in the proof of Theorem 1 in Bai (1994) combined with the arguments referenced for the derivation of (A.63) suggest that for $u \leq 0$,

$$\exp\left(\left(L^2\left(\left(Tv_T^2\right)^{-1} u + \lambda_b^0\right) - L^2\left(\lambda_b^0\right)\right)/2\right) \tag{A.64}$$

$$= \exp\left(\delta_T \sum_{t=0}^{v_T^{-2}|u|} e_{T_b^0-t} - |u|\,\delta_0^2/2\right)(1+B_T),$$

where $B_T = o_{\mathbb{P}}(1)$ is uniform in the region $u \leq \eta T v_T^2$ for small $\eta > 0$. By symmetry, the case $u > 0$ results in the same relationship as (A.64) with $e_{T_b^0-t}$ replaced by $e_{T_b^0+t}$. By Lemma A.6 and the results in Bai (1994), $p_T(u|\,y)$ and $p(u|\,y)$ are negligible uniformly in u for $u > \eta T v_T^2$ for every η. Thus, (A.63)-(A.64) yield,

$$\left|p_T^*\left(Tv_T^2\left(\overline{\lambda}_b - \lambda_b^0\right),\, y\right) - p^*\left(Tv_T^2\left(\overline{\lambda}_b - \lambda_b^0\right),\, y\right)\right|_{\text{TV}} \leq |A_T| + |B_T| \xrightarrow{\mathbb{P}} 0.$$

□

Continuing with the proof of Proposition 6.1, we begin with part (i). Note that $\varphi(\lambda_b,\, y)$ is defined by

$$\int (1-\varphi(\lambda_b,\, y))\, p_T(y|\,\lambda_b)\, d\mathit{\Pi}(\lambda_b) \geq 1-\alpha$$

for all y, where $\mathit{\Pi}(\cdot)$ is a probability measure on $\mathit{\Gamma}^0$ such that $\mathit{\Pi}(\lambda_b) = \pi(\lambda_b)\, d\lambda_b$. The fact that $|1-\varphi(\lambda_b,\, y)| \leq 1$ and Lemma A.37 lead to,

$$\int (1-\varphi(\lambda_b,\, y))\, p_T(y|\,\lambda_b)\, d\mathit{\Pi}(\lambda_b) \tag{A.65}$$

$$= \int (1-\varphi(\lambda_b,\, y))\, p(y|\,\lambda_b)\, d\mathit{\Pi}(\lambda_b) + o_{\mathbb{P}}(1).$$

Given that Definition 4.1 of the GL confidence interval involves an inequality that explicitly allows for conservativeness, (A.65) implies the following relationship,

$$\int \varphi(\lambda_b,\, y)\, p_T(y|\,\lambda_b)\, d\mathit{\Pi}(\lambda_b) = \int \varphi(\lambda_b,\, y)\, p(y|\,\lambda_b)\, d\mathit{\Pi}(\lambda_b) + \varepsilon_T$$

$$\leq \alpha \int p(y|\,\lambda_b)\, d\mathit{\Pi}(\lambda_b),$$

where $\varepsilon_T = \int \varphi(\lambda_b,\, y)\left(p_T(y|\,\lambda_b) - p(y|\,\lambda_b)\right) d\mathit{\Pi}(\lambda_b)$. Rearranging, we have,

$$\int (\alpha - \varphi(\lambda_b,\, y))\, p(y|\,\lambda_b)\, d\mathit{\Pi}(\lambda_b) - \varepsilon_T \geq 0,$$

for all y. Now multiply both sides by $\widetilde{b}(y) \geq 0$ and integrating with respect to $\zeta(y)$ yields,

$$\int\int (\alpha - \varphi(\lambda_b, y)) \widetilde{b}(y) p(y|\lambda_b) d\zeta(y) d\Pi(\lambda_b) - \varepsilon_T \int \widetilde{b}(y) d\zeta(y) \geq 0,$$

or

$$(1-\alpha) \int L_\alpha\left(\varphi, \widetilde{b}, \lambda_b\right) d\Pi(\lambda_b) - \varepsilon_T \int \widetilde{b}(y) d\zeta(y) \geq 0.$$

Taking the limit as $T \to \infty$,

$$(1-\alpha) \int L_\alpha\left(\varphi, \widetilde{b}, \lambda_b\right) d\Pi(\lambda_b) \geq 0.$$

The latter implies that $L_\alpha\left(\varphi, \widetilde{b}, \lambda_b\right) \geq 0$ for some λ_b. Thus, φ is bet-proof at level $1-\alpha$.

We now prove part (ii). We use a proof by contradiction. If $\int \varphi'(\lambda_b, y) d\lambda_b \geq \int \varphi(\lambda_b, y) d\lambda_b$ for all $y \in \mathcal{Y}$ and $\int \varphi'(\lambda_b, y) d\lambda_b > \int \varphi(\lambda_b, y) d\lambda_b$ for all $y \in \mathcal{Y}_0$ with $\zeta(\mathcal{Y}_0) > 0$, then we show that $\int \varphi'(\lambda_b, y) p(y|\lambda_b) d\zeta(y) > \alpha$ for some $\lambda_b \in \Gamma^0$. By Lemma A.37 and (6.1) holding with equality,

$$\begin{aligned}\int \varphi(\lambda_b, y) p_T(y|\lambda_b) d\Pi(\lambda_b) &= \alpha \int p_T(y|\lambda_b) d\Pi(\lambda_b) \\ &= \alpha \int p(y|\lambda_b) d\Pi(\lambda_b) + o_{\mathbb{P}}(1).\end{aligned}$$

Integrating both sides with respect to $\zeta(y)$ yields,

$$\int \left(\int \varphi(\lambda_b, y) p(y|\lambda_b) d\zeta(y) \right) d\Pi(\lambda_b) = \alpha + o_{\mathbb{P}}(1). \tag{A.66}$$

By Assumption (3.2), $\pi(\lambda_b) > 0$ for all $\lambda_b \in \Gamma^0$. Taking the limit as $T \to \infty$ of both sides of (A.66) yields $\int \left(\int \varphi(\lambda_b, y) p(y|\lambda_b) d\zeta(y) \right) d\Pi(\lambda_b) = \alpha$. The latter holds only if $\int \varphi(\lambda_b, y) p(y|\lambda_b) d\zeta(y) = \alpha$ for all $\lambda_b \in \Gamma^0$. This means that φ is similar. The definition of HPD confidence set $\varphi(\lambda_b, y)$ implies that for ζ-almost all y, if $\int \varphi(\lambda_b, y) d\lambda_b = \int \varphi'(\lambda_b, y) d\lambda_b$ then $\int \varphi(\lambda_b, y) p_T(\lambda_b|y) d\lambda_b \leq \int \varphi'(\lambda_b, y) p_T(\lambda_b|y) d\lambda_b$. The latter relationship and Lemma A.37 imply that,

$$\int \varphi(\lambda_b, y) p(y|\lambda_b) d\Pi(\lambda_b) \leq \int \varphi'(\lambda_b, y) p(y|\lambda_b) d\Pi(\lambda_b),$$

for all $y \in \mathcal{Y}$ and

$$\int \varphi(\lambda_b, y) p(y|\lambda_b) d\Pi(\lambda_b) < \int \varphi'(\lambda_b, y) p(y|\lambda_b) d\Pi(\lambda_b),$$

for all $y \in \mathcal{Y}_0$. Integrating both sides with respect to ζ yields

$$\begin{aligned}&\int \left(\int \varphi(\lambda_b, y) p(y|\lambda_b) d\zeta(y) \right) d\Pi(\lambda_b) \\ &\quad < \int \left(\int \varphi'(\lambda_b, y) p(y|\lambda_b) d\zeta(y) \right) d\Pi(\lambda_b),\end{aligned}$$

or

$$\int \left(\int (\varphi(\lambda_b, y) - \varphi'(\lambda_b, y)) p(y|\lambda_b) d\zeta(y) \right) d\Pi(\lambda_b) < 0.$$

Since $\varphi(\lambda_b, y)$ is similar, there exists a λ_b such that $\int \varphi'(\lambda_b, y) p(y|\lambda_b) d\zeta(y) > \alpha$. Thus, φ' is not of level $1 - \alpha$. □

B Comparison to Casini and Perron (2020)

In this section we compare the GL-LN method to the GL estimators/confidence intervals proposed in Casini and Perron (2020). Table 1-2 report the results. We have considered a data-generating mechanism with higher serial dependence in the errors. In terms of the empirical performance of the estimators, Table 1 shows that overall the estimator that does better is $\widehat{\lambda}_b^{\mathrm{GL-LN}}$. $\widehat{\lambda}_b^{\mathrm{GL-CR-Iter}}$ is the one that does best when $\lambda_b^0 = 0.5$ but it does worse in relative terms when the break is in the tails. The performance of $\widehat{\lambda}_b^{\mathrm{GL-LN}}$ is in general superior to $\widehat{\lambda}_b^{\mathrm{GL-CR}}$ especially for medium to large breaks both in terms of MAE and RMSE. From other simulations (not reported), we conclude that GL-LN does in general better for moderate to large breaks. $\widehat{\lambda}_b^{\mathrm{GL-CR-Iter}}$ is the one that does best when the break is in the middle but its precision deteriorates as the break moves to the tails. In addition, $\widehat{\lambda}_b^{\mathrm{GL-LN}}$ is valid for models with multiple breaks and models with trending regressors that are not covered in Casini and Perron (2020). So overall we believe that the estimators $\widehat{\lambda}_b^{\mathrm{GL-LN}}$, $\widehat{\lambda}_b^{\mathrm{GL-CR}}$ and $\widehat{\lambda}_b^{\mathrm{GL-CR-Iter}}$ can be seen as complementary.

Turning to the finite-sample performance of the confidence intervals, Table 2 clearly shows that when there is higher serial dependence in the errors, the method that dominates is GL-LN. The gain in terms of coverage accuracy and lengths can be substantial relative to the GL-CR and GL-CR-Iter. When the serial dependence in the errors is low (not reported), the difference in performance of the three confidence intervals becomes smaller.

Overall, we find that both estimation and confidence intervals based on GL-LN perform well relative to the continuous record counterparts, where major gains appear to occur when there is high serial correlation in the errors.

Table 1: Small-sample accuracy of the estimates of the break point T_b^0

		MAE	Std	RMSE	$Q_{0.25}$	$Q_{0.75}$	MAE	Std	RMSE	$Q_{0.25}$	$Q_{0.75}$
				$\lambda_0 = 0.3$					$\lambda_0 = 0.5$		
$\delta^0 = 0.3$	OLS	26.84	28.12	33.00	21	76	23.02	26.86	26.76	25	75
	GL-LN	13.63	14.07	17.25	27	56	10.84	13.03	14.40	35	65
	GL-CR	12.79	13.13	18.46	29	57	11.84	13.17	13.12	35	65
	GL-CR-Iter	14.47	10.29	20.21	28	58	8.76	10.01	10.24	41	59
	GL-Uni	21.78	21.73	27.71	28	66	17.84	20.90	20.98	32	68
$\delta^0 = 0.4$	OLS	23.62	26.99	30.23	21	70	21.23	25.43	25.44	25	75
	GL-LN	11.53	13.66	15.44	27	51	10.11	12.15	13.37	37	63
	GL-CR	16.36	13.86	21.49	29	61	11.56	11.97	12.25	36	64
	GL-CR-Iter	17.19	10.81	20.35	28	57	8.30	9.95	10.01	43	57
	GL-Uni	20.18	21.25	26.30	28	64	16.53	19.97	19.98	34	64
$\delta^0 = 0.6$	OLS	19.80	24.62	26.25	21	57	17.34	22.39	22.34	37	65
	GL-LN	8.86	11.63	12.77	29	42	8.05	10.29	11.18	41	59
	GL-CR	12.84	13.66	18.23	30	56	9.96	11.93	11.99	38	58
	GL-CR-Iter	14.85	11.52	17.56	29	52	7.26	9.20	9.22	44	55
	GL-Uni	16.04	20.05	22.77	26	56	13.85	17.81	17.94	38	60
$\delta^0 = 1$	OLS	11.69	18.43	19.26	27	40	9.38	14.40	14.40	46	54
	GL-LN	5.63	9.56	9.57	27	31	5.40	8.21	8.59	49	51
	GL-CR	6.82	10.85	12.81	27	38	6.96	9.43	9.52	44	53
	GL-CR-Iter	10.67	7.54	13.02	30	39	4.44	6.71	6.85	47	53
	GL-Uni	9.44	14.60	15.15	27	37	8.17	12.34	12.34	45	54

The model is $y_t = \delta_1^0 + \delta^0 \mathbf{1}_{\{t > \lfloor T\lambda_0 \rfloor\}} + e_t$, $e_t = 0.6e_{t-1} + u_t$, $u_t \sim i.i.d. \mathscr{N}(0, 0.49)$, $T = 100$.

Table 2: Small-sample coverage rates and lengths of the confidence sets

		$\delta^0 = 0.4$		$\delta^0 = 0.8$		$\delta^0 = 1.6$	
		Cov.	Lgth.	Cov.	Lgth.	Cov.	Lgth.
$\lambda_0 = 0.5$	OLS-CR	0.910	67.57	0.911	68.87	0.945	42.30
	Bai (1997)	0.808	67.57	0.811	50.22	0.894	20.74
	GL-LN	0.925	57.43	0.965	37.35	0.985	9.30
	GL-CR	0.885	60.05	0.884	52.63	0.926	32.61
	GL-CR-Iter	0.911	76.72	0.911	69.06	0.944	42.20
$\lambda_0 = 0.35$	OLS-CR	0.927	75.58	0.910	66.20	0.944	39.15
	Bai (1997)	0.838	66.86	0.821	49.34	0.893	20.77
	GL-LN	0.965	54.57	0.974	32.88	0.984	9.39
	GL-CR	0.898	57.32	0.888	50.29	0.924	29.06
	GL-CR-Iter	0.930	75.87	0.913	66.13	0.944	38.71
$\lambda_0 = 0.2$	OLS-CR	0.910	75.24	0.917	64.17	0.953	34.26
	Bai (1997)	0.808	67.03	0.852	50.40	0.937	21.76
	GL-LN	0.921	57.96	0.962	39.63	0.969	10.86
	GL-CR	0.912	56.87	0.909	48.68	0.932	23.91
	GL-CR-Iter	0.894	75.15	0.923	64.14	0.953	34.06

The model is $y_t = \delta_1^0 + \delta^0 \mathbf{1}_{\{t > \lfloor T\lambda_0 \rfloor\}} + e_t$, $e_t = 0.6e_{t-1} + u_t$, $u_t \sim i.i.d. \mathscr{N}(0, 0.49)$, $T = 100$.

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