Online Supplement to

"Estimation of time-varying covariance matrices for large datasets"

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This Supplement provides proofs of the results given in the text of the main paper. It is organised as follows: Section A provides proofs of the main results on exponential inequalities of Section 4 of the main paper. Section B provides proofs of Theorems 1-3 of the main paper. Section C contains auxiliary technical lemmas.

Formula numbering in this supplement includes the section number, e.g. $(A.1)$, and references to lemmas are signified as "Lemma A#", "Lemma B#", "Lemma C#", e.g. Lemma A1. Equation, lemma and theorem references to the main paper do not include section number and are signified as "Equation (#)", "Lemma #", "Theorem #", e.g. (1), Theorem 1.

In the proofs, C stands for a generic positive constant which may assume different values in different contexts, and we denote $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$.

A. Exponential inequalities. Proofs

This section contains the proofs of the results of Section 4 on Bernstein inequalities for (weighted) sums of random variables ξ_j that are dependent, unbounded and have thin- or heavy-tailed distributions.

We shall frequently refer to the α-mixing Assumption A and property (36) of (ξ_j) of Section 4 of the main paper. To denote that r.v.'s (ξ_j) have thin- or heavy-tailed distributions, we use respectively notation $(\xi_j) \in \mathcal{E}(s)$, $s > 0$ and $(\xi_j) \in \mathcal{H}(\theta)$, $\theta > 2$ of Section 4 of the main paper, see (37) and (38).

Merlevede, Peligrad and Rio (2009) in their Theorem 2 obtained a Bernstein type inequality for bounded α- mixing random variables. The following lemma is a minor auxiliary generalization of their result to a sequence of truncated random variables.

Lemma A1 *Let the sequence (ξ_k) of zero mean random variables satisfy Assumption A. Set $\xi_{D,k} := \xi_k I(|\xi_k| \le D)$ where $D > 0$. Suppose that*

$$m^* := \max_{k \ge 1} (E|\xi_k|^p)^{1/p} < \infty \quad \textit{for some } p > 2.$$

Then, there exist $0 < c < \infty$ *such that for all* $\zeta > 0$, $D > 0$ *and* $T \geq 2$,

$$P\Big(\big|\sum_{k=1}^{T}(\xi_{D,k} - E\xi_{D,k})\big| \geq \zeta\Big) \leq \exp\big(-\frac{c\zeta^2}{\bar{v}^2 T + D^2 + \zeta D \log^2 T}\big), \qquad (A.1)$$

with $\bar{v}^2 = m^*(1 + 24\sum_{j=1}^{\infty}\alpha_j^{1-2/p})$ *where* $c > 0$ *depends only on* c_* *in (36) of Assumption A.*

Proof of Lemma A1. By Theorem 14.1 of Davidson (1994), under Assumption A, the truncated process $(\xi_{D,t})$ is also α-mixing with mixing coefficients $\widetilde{\alpha}_k \leq \alpha_k$. Hence, the bound (2.3) of Theorem 2 in Merlevede *et al.* (2009) implies

$$P\Big(\big|\sum_{k=1}^{T}(\xi_{D,k} - E\xi_{D,k})\big| \geq \zeta\Big) \leq \exp\big(-\tfrac{c\zeta^2}{v_D^2 T + D^2 + \zeta D \log^2 T}\big),$$

with

$$v_D^2 = \sup_{i>0}\big(\mathrm{var}(\xi_{D,i}) + 2\sum_{j>i}|\mathrm{cov}(\xi_{D,i}, \xi_{D,j})|\big),$$

where c depends only on c_* in (36). We will show that $v_D^2 \leq \bar{v}^2$ which proves (A.1).

The conclusion (2.2) in Davydov (1968) applied with $p = q > 2$ gives

$$|\mathrm{cov}(\xi_{D,i}, \xi_{D,j})| \leq 12(E|\xi_{D,i}|^p)^{1/p}(E|\xi_{D,j}|^p)^{1/p}\widetilde{\alpha}_{|i-j|}^{1-2/p} \leq 12 m^* \alpha_{|i-j|}^{1-2/p}.$$

Observe that $\mathrm{var}(\xi_{D,i}) \leq E\xi_{D,i}^2 \leq (E|\xi_{D,i}|^p)^{2/p} \leq m^*$. Hence,

$$v_D^2 \leq m^*\big(1 + 24\sum_{j=1}^{\infty}\alpha_j^{1-2/p}\big) = \bar{v}^2 < \infty$$

which completes the proof of the lemma. □

The proof of Lemma 1 of the main paper combines the modified version, Lemma A1, of the exponential inequality for bounded random variables by Merlevede, Peligrad and Rio (2009), with a truncation argument employed in White and Wooldridge (1991).

Proof of Lemma 1. Without restriction of generality we prove the validity of (40), (41) for $\zeta \geq 1$. (The inequalities (40) and (41) can be extended to $0 < \zeta < 1$ by selecting large enough constant c_0.) Recall that $S_T = T^{-1/2}\sum_{k=1}^{T}(\xi_k - E\xi_k)$.

We start with (40). We need to prove that

$$P\big(|S_T| > \zeta\big) \leq f_T(2, \gamma, c, \zeta) = c_0\Big\{\exp\big(-c_1\zeta^2\big) + \exp\big(-c_2(\frac{\zeta\sqrt{T}}{\log^2 T})^{\gamma}\big)\Big\}, \qquad (A.2)$$

with $\gamma = s/(s+1)$ where positive constants c_0, c_1, c_2 do not depend on ζ, T. Denote by $D = D_{T,\zeta}$ the truncation constant depending on T, ζ which will be selected later. Write $\xi_k = w_k + v_k$ where $w_k = \xi_k I(|\xi_k| \leq D)$, $v_k = \xi_k I(|\xi_k| > D)$. Then,

$$\begin{aligned} S_T &= T^{-1/2}\sum_{k=1}^{T}(w_k - Ew_k) + T^{-1/2}\sum_{k=1}^{T}(v_k - Ev_k) \qquad (A.3)\\ &=: s_{T,1} + s_{T,2} \end{aligned}$$

and

$$P(|S_T| \geq \zeta) \leq P(|s_{T,1}| \geq \zeta/2) + P(|s_{T,2}| \geq \zeta/2).$$

Thus, to prove (A.2), it suffices to show that for some c, for all $\zeta \geq 1$, $T \geq 2$,

$$P\big(|s_{T,i}| \geq \zeta\big) \leq f_T(2, \gamma, c, \zeta), \quad i = 1, 2. \tag{A.4}$$

By Assumption A, $(\xi_j - E\xi_j)$ is an α-mixing process which mixing coefficients α_k satisfy (36). Hence, by Theorem 14.1 in Davidson (1994), $(w_j - Ew_j)$ and $(v_j - Ev_j)$ are α-mixing sequences and their respective mixing coefficients $\alpha_{w,k}$ and $\alpha_{v,k}$ satisfy

$$\alpha_{w,k} \leq \alpha_k, \quad \alpha_{v,k} \leq \alpha_k, \qquad k \geq 1. \tag{A.5}$$

Thus, by Lemma A1, for all $T \geq 2$ and $D > 0$,

$$P\big(|s_{T,1}| \geq \zeta\big) \leq \exp\big(- \frac{c_1 \zeta^2 T}{\bar{v}^2 T + D^2 + \zeta T^{1/2} D \log^2 T}\big) \tag{A.6}$$

where $c_1 > 0$ does not depend on T, D or ζ. Using, on the r.h.s. of (A.6), the inequality

$$-\frac{1}{|a| + |b| + |c|} \leq -\frac{1}{3\max(|a|, |b|, |c|)},$$

with $a = \bar{v}^2 T$, $b = D^2$, $c = \zeta T^{1/2} D \log^2 T$, we obtain

$$P\big(|s_{T,1}| \geq \zeta\big) \leq \exp\big(- c_1' \zeta^2\big) + \exp\big(- \frac{c_2' \zeta^2 T}{D^2}\big) + \exp\big(- \frac{c_2' \zeta T^{1/2}}{D \log^2 T}\big), \quad \zeta \geq 1 \tag{A.7}$$

with $c_1' = c_1/(3\bar{v}^2)$, $c_2' = c_1/3$. Setting

$$\Delta_T = \frac{T^{1/2}}{\log^2 T},$$

(A.7) becomes

$$\begin{aligned} &P\big(|s_{T,1}| \geq \zeta\big) \\ &\quad \leq \exp\big(- c_1' \zeta^2\big) + \exp\big(- c_2' \big(\frac{\zeta \Delta_T}{D}\big)^2 \log^4 T\big) + \exp\big(- \frac{c_2' \zeta \Delta_T}{D}\big). \end{aligned} \tag{A.8}$$

We select $D = D_{T,\zeta}$ such that $\zeta \Delta_T / D = D^s$. Then,

$$D = \big(\zeta \Delta_T\big)^{1/(s+1)}, \ D^s = \big(\zeta \Delta_T\big)^{s/(s+1)} \text{ and } \tfrac{\zeta \Delta_T}{D} = \big(\zeta \Delta_T\big)^{s/(s+1)}. \tag{A.9}$$

For $\zeta \geq 1$, $T \geq 2$ it holds $\zeta \Delta_T \geq \Delta_T \geq 1$. This together with (A.9) implies $\big(\zeta \Delta_T\big)/D \geq 1$. Notice that $\log^4 T \geq \log^4 2 =: v > 0$ for $T \geq 2$, and $v \in (0,\, 1)$. Hence,

$$\begin{aligned} &\big(\frac{\zeta \Delta_T}{D}\big)^2 \log^4 T \geq \big(\frac{\zeta \Delta_T}{D}\big) v, \\ &\frac{\zeta \Delta_T}{D} \geq v \frac{\zeta \Delta_T}{D} = v(\zeta \Delta_T)^{s/(s+1)} = v\big(\frac{\zeta \sqrt{T}}{\log^2 T}\big)^{s/(s+1)}. \end{aligned}$$

Applying these relations in (A.8), we obtain

$$\begin{aligned} P\big(|s_{T,1}| \ge \zeta\big) &\le \exp\big(-c_1'\zeta^2\big) + 2\exp\big(-\frac{c_2' v\zeta\Delta_T}{D}\big) \\ &\le \exp\big(-c_1'\zeta^2\big) + 2\exp\big(-c_2' v(\zeta\Delta_T)^{s/(s+1)}\big) \\ &\le 2\Big(\exp\big(-c_1'\zeta^2\big) + \exp\big(-c_2' v\big(\frac{\zeta\sqrt{T}}{\log^2 T}\big)^{s/(s+1)}\big)\Big) \\ &\le f_T(2,\gamma,c,\zeta). \end{aligned}$$

This proves (A.4) for $P(|s_{T,1}| \ge \zeta)$. Turning to $s_{T,2}$, by Markov inequality,

$$\begin{aligned} P\big(|s_{T,2}| \ge \zeta\big) &\le \zeta^{-2}T^{-1}E\Big(\sum_{k=1}^{T}(v_k - Ev_k)\Big)^2 \\ &\le \zeta^{-2}T^{-1}\sum_{j,k=1}^{T}\mathrm{cov}(v_j, v_k). \end{aligned} \tag{A.10}$$

Let $p, q > 1$, $1/p + 1/q < 1$. Assumption $(\xi_j) \in \mathcal{E}(s)$ implies $E|v_j|^p < \infty$, $E|v_j|^q < \infty$. Since $(v_j - Ev_j)$ is α-mixing sequence with the mixing coefficients $\alpha_{v,k} \le \alpha_k$, $k \ge 1$, then, by Conclusion 2.2 in Davydov (1968),

$$|\mathrm{cov}(v_j, v_k)| \le 12(E|v_j|^p)^{1/p}(E|v_j|^q)^{1/q}\alpha_{|j-k|}^{1-1/p-1/q},\ j \ne k. \tag{A.11}$$

In turn, for $j = k$, $\mathrm{var}(v_j) \le Ev_j^2$. Setting

$$V_p := \max_{j \ge 1}(E|v_j|^p)^{1/p},$$

we obtain

$$\begin{aligned} P\big(|s_{T,2}| \ge \zeta\big) &\le \zeta^{-2}T^{-1}\Big[\sum_{j=1}^{T}\mathrm{var}(v_j) + \sum_{j,k=1:\,k\ne j}^{T}\mathrm{cov}(v_j, v_k)\Big] \\ &\le \zeta^{-2}V_2^2 + \zeta^{-2}12V_pV_q\big(T^{-1}\sum_{j,k=1:k\ne j}^{T}\alpha_{|j-k|}^{e}\big) \end{aligned}$$

where $e := 1 - 1/p - 1/q > 0$. By (36),

$$T^{-1}\textstyle\sum_{j,k=1:j>k}^{T}\alpha_{|j-k|}^{e} = T^{-1}\sum_{s=1}^{T}\alpha_s^e(T-s) \le \sum_{s=1}^{\infty}\alpha_s^e < \infty.$$

This implies that with some C that does not depend on T or D, it holds that

$$P\big(|s_{T,2}| \ge \zeta\big) \le C\zeta^{-2}(V_2^2 + V_pV_q). \tag{A.12}$$

Set $p = q = 2 + \delta$ where $\delta > 0$ is a small number. Then, by (A.12),

$$P\big(|s_{T,2}| \ge \zeta\big) \le C\zeta^{-2}(V_2^2 + V_p^2) \le C\zeta^{-2}V_p^2 \tag{A.13}$$

because $V_2^2 = \max_j Ev_j^2 \le \max_j (E|v_j|^p)^{2/p} = V_p^2$. For $D > 0$, by (C.9) it holds that

$$E|v_j|^p = E[|\xi_j|^p I(|\xi_j| > D)] \le c_0' \exp(-c_1' D^s)$$

for some c_0', $c_1' > 0$ which do not depend on j and D. This implies

$$V_p^2 \le (c_0{}')^{2/p} \exp(-(2/p) c_1{}' D^s).$$

Thus, there exists $c_0 > 0$, $c_2 > 0$ such that for all $\zeta \ge 1$, $T \ge 2$, in view of (A.9),

$$\begin{aligned} P\big(|s_{T,2}| \ge \zeta\big) &\le C\zeta^{-2} \exp\big(-(2/p)c_2' D^s\big) \\ &\le c_0 \exp\big(-c_2(\zeta\Delta_T)^{s/(s+1)}\big) \\ &= c_0 \exp\Big(-c_2\big(\frac{\zeta\sqrt{T}}{\log^2 T}\big)^{s/(s+1)}\Big) \\ &\le f_T(2,\gamma,c,\zeta), \end{aligned}$$

which proves the bound (A.4) for $s_{T,2}$ and completes the proof of (A.2) and (40).

Proof of (41). Let $(\xi_j) \in \mathcal{H}(\theta)$. We need to prove that for any fixed $2 < \theta' < \theta$,

$$\begin{aligned} P(|S_T| > \zeta) &\le g_T(2,\theta',c,\zeta) \\ &= c_0\Big\{\exp(-c_1\zeta^2) + \zeta^{-\theta'} T^{-(\theta'/2-1)}\Big\}, \quad \zeta > 0, \ T \ge 2. \end{aligned} \tag{A.14}$$

Write $S_T = s_{T,1} + s_{T,2}$ as in (A.3). To verify (A.14), it remains to show that

$$P(|s_{T,i}| \ge \zeta) \le g_T(2,\theta',c,\zeta), \ \ i = 1,2 \quad \text{for some } c.$$

It suffices to consider the case $\zeta \ge 1$.

We start with the evaluation $P(|s_{T,1}| \ge \zeta)$. Set

$$D = \frac{a^{-1}\zeta\sqrt{T}}{\log^3(\zeta\sqrt{T})} \ge 1,$$

where $a > 0$ will be selected below. For $\zeta \ge 1$ it holds $\log(\zeta\sqrt{T}) \ge \log(\sqrt{T}) \ge \log(\sqrt{2}) =: b > 0$. Then, from (A.7) we obtain

$$\begin{aligned} P\big(|s_{T,1}| \ge \zeta\big) &\le \exp\big(-c_1'\zeta^2\big) + \exp\big(-c_2' a^2 \log^6(\zeta\sqrt{T})\big) + \exp\big(-c_2' a \log(\zeta\sqrt{T})\big) \\ &\le \exp\big(-c_1'\zeta^2\big) + \exp\big(-c_2' a^2 b^5 \log(\zeta\sqrt{T})\big) + \exp\big(-c_2' a \log(\zeta\sqrt{T})\big). \end{aligned}$$

Hence, selecting a such that $c_2' a^2 b^5 \ge \theta'$, $c_2' a \ge \theta'$, we obtain

$$P\big(|s_{T,1}| \ge \zeta\big) \le \exp\big(-c_1'\zeta^2\big) + 2(\zeta\sqrt{T})^{-\theta'}.$$

This proves the bound (A.14) for $P\big(|s_{T,1}| \geq \zeta\big)$.

Next we turn to $P(|s_{T,2}| \geq \zeta)$. By (A.13),

$$P(|s_{T,2}| \geq \zeta) \leq C\zeta^{-2} V_p^2$$

with $p = 2 + \delta$. According to (C.10), we can bound

$$E|v_j|^p = E[|\xi_j|^p I(|\xi_j| > D)] \leq c_0' D^{-(\theta - p)}$$

with some $c_0' > 0$ which does not depend on D and j. This implies

$$V_p^2 \leq (c_0')^{2/p} D^{-(\theta-p)(2/p)}.$$

Hence,

$$\begin{aligned} P(|s_{T,2}| \geq \zeta) &\leq C\zeta^{-2} D^{-(\theta-p)(2/p)} \\ &= C\zeta^{-2}(\zeta\sqrt{T})^{-(\theta'-2)} a_{T,\zeta}, \end{aligned} \tag{A.15}$$

where

$$a_{T,\zeta} := \frac{(\zeta\sqrt{T})^{\theta'-2}}{D^{(\theta-p)(2/p)}} = \frac{(a\log^3(\zeta\sqrt{T}))^{(\theta-p)(2/p)}}{(\zeta\sqrt{T})^{\gamma}}$$

and

$$\gamma = (\theta - p)(2/p) - (\theta' - 2) = \theta - \theta' - \theta(p-2)/p = \theta - \theta' - \theta\delta/p > 0$$

when $\theta > \theta'$, $p = 2 + \delta$ and $\delta > 0$ is selected sufficiently small. Since $\zeta\sqrt{T} \geq \sqrt{2}$ for $\zeta \geq 1$, $T \geq 2$, this implies that $\sup_{\zeta\geq 1,\, T\geq 2} a_{T,\zeta} \leq C' < \infty$. Thus, (A.15) implies

$$P(|s_{T,2}| \geq \zeta) \leq C\zeta^{-\theta'} T^{-(\theta'/2-1)} \leq g_T(2, \theta', c, \zeta)$$

which proves the bound (A.14) for $P(|s_{T,2}| \geq \zeta)$.

This completes the proof of (41) and the lemma. $\square$

We start the proof of Lemma 2 with the following technical lemma.

Lemma A2 *Let x_{tk}, $k, t \geq 1$ be random variables such that $E|x_{tk}| < \infty$ and a_{tk} and $v_{tk} > 0$ be real numbers such that*

$$\max_{n\geq 1} \max_{1\leq t\leq n} \sum_{1\leq k\leq n}^{n} |a_{tk}| v_{tk} < \infty. \tag{A.16}$$

Then there exists $\varepsilon > 0$ such that for all $\zeta > 0$, $t \geq 1$,

$$P\big(|\sum_{k=1}^{n} a_{tk} x_{tk}| \geq \zeta\big) \leq \varepsilon^{-1} \max_{1\leq k\leq n} E\Big[\frac{|x_{tk}|}{\zeta v_{tk}} I\big(\frac{|x_{tk}|}{\zeta v_{tk}} \geq \varepsilon\big)\Big]. \tag{A.17}$$

Proof of Lemma A2. By (A.16) there exists $\varepsilon > 0$ such that

$$\sum_{k=1}^{n} |a_{tk} v_{tk}| < 1/(2\varepsilon), \quad t \geq 1.$$

From

$$|x_{tk}/v_{tk}| = |x_{tk}/v_{tk}| \big(I(|x_{tk}/v_{tk}| \leq \varepsilon\zeta) + I(|x_{tk}/v_{tk}| > \varepsilon\zeta) \big) \leq \varepsilon\zeta + y_{tk},$$

where $y_{tk} := |x_{tk}/v_{tk}| I(|x_{tk}/v_{tk}| \geq \varepsilon\zeta)$, we obtain

$$\begin{aligned} |\sum_{k=1}^{n} a_{tk} x_{tk}| &\leq \sum_{k=1}^{n} |a_{tk} v_{tk}| \frac{|x_{tk}|}{v_{tk}} \leq \sum_{k=1}^{n} |a_{tk} v_{tk}| (\varepsilon\zeta) + \sum_{k=1}^{n} |a_{tk} v_{tk}| y_{tk} \\ &\leq \zeta/2 + \sum_{k=1}^{n} |a_{tk} v_{tk}| y_{tk}. \end{aligned}$$

Then, by Markov inequality,

$$\begin{aligned} P\big(|\textstyle\sum_{k=1}^{n} a_{tk} x_{tk}| \geq \zeta\big) &\leq P\big(\textstyle\sum_{k=1}^{n} |a_{tk} v_{tk}| y_{tk} \geq \zeta/2\big) \leq (\zeta/2)^{-1} \textstyle\sum_{k=1}^{n} |a_{tk} v_{tk}| E y_{tk} \\ &\leq (\zeta/2)^{-1} \big(\textstyle\sum_{k=1}^{n} |a_{tk} v_{tk}|\big) \max_{1 \leq k \leq n} E y_{tk} \\ &\leq (\zeta/2)^{-1} (2\varepsilon)^{-1} \max_{1 \leq k \leq n} E y_{tk} \end{aligned}$$

which proves (A.17). $\Box$

Proof of Lemma 2. Without restriction of generality assume that $\zeta \geq 1$. Notice that property (43) of $b_{H,k}$ implies

$$\max_{k=1,\ldots,T} b_{H,k} \Big(\frac{k \vee H}{H}\Big)^{1/2} \leq C, \quad \sum_{k=1}^{T-1} |b_{H,k} - b_{H,k+1}| \Big(\frac{k \vee H}{H}\Big)^{1/2} \leq C, \tag{A.18}$$

where $C < \infty$ does not depend on H, T.

Denote $\xi'_k := \xi_{t-k}$, $\xi''_k := \xi_{t+k}$ for $k \geq 0$. Write

$$\begin{aligned} S_{T,t} &= H^{-1/2} \textstyle\sum_{k=1}^{T} b_{H,|t-k|} (\xi_k - E\xi_k) \qquad \text{(A.19)} \\ &= H^{-1/2} \textstyle\sum_{k=1}^{t-1} b_{H,k} (\xi'_k - E\xi'_k) + H^{-1/2} \textstyle\sum_{k=0}^{T-t} b_{H,k} (\xi''_k - E\xi''_k) \\ &=: s^{(1)}_{T,t} + s^{(2)}_{T,t}. \end{aligned}$$

To prove (44) for $P(|S_{T,t}| \geq \zeta)$, it suffices to verify that for $\ell = 1, 2$,

$$P(|s^{(\ell)}_{T,t}| \geq \zeta) \leq f_H(2, \gamma, c, \zeta) \text{ if } (\xi_j) \in \mathcal{E}(s),\ s > 0. \tag{A.20}$$

To prove (45) for $P(|S_{T,t}| \geq \zeta)$, it suffices to show that for $\ell = 1, 2$,

$$P(|s^{(\ell)}_{T,t}| \geq \zeta) \leq g_H(2, \theta', c, \zeta) \text{ if } (\xi_j) \in \mathcal{H}(\theta), \quad \theta > 2. \tag{A.21}$$

We provide the proof for $s^{(1)}_{T,t}$. (For $s^{(2)}_{T,t}$ the proof is similar). Set

$$x_k = \sum_{i=1}^{k}(\xi'_i - E\xi'_i), \;\; y_k = k^{-1/2}x_k, \;\; \nu_k = \big(\tfrac{k\vee H}{k}\big)^{1/2} \;\text{ for } k = 1, ..., t-1.$$

Using summation by parts, we can write $s^{(1)}_{T,t}$ as

$$\begin{aligned} s^{(1)}_{T,t} &= H^{-1/2}\sum_{k=1}^{t-2}(b_{H,k} - b_{H,k+1})x_k + H^{-1/2}b_{H,t-1}x_{t-1} \\ &= \sum_{k=1}^{t-1} a_{tk}x_k, \end{aligned} \tag{A.22}$$

where

$$a_{tk} = H^{-1/2}(b_{H,k} - b_{H,k+1}) \;\text{ for }\; k = 1, ..., t-2, \quad a_{t,t-1} = H^{-1/2}b_{H,t-1}.$$

Subsequently, using notation y_k and ν_k introduced above, we can write

$$s^{(1)}_{T,t} = \sum_{k=1}^{t-1} a_{tk}\,(k \vee H)^{1/2}\,(y_k/\nu_k). \tag{A.23}$$

From (A.18) it follows

$$\sum_{k=1}^{t-1} a_{tk}\,(k \vee H)^{1/2} \leq C$$

where $C < \infty$ does not depend on t, H, T. Hence, by Lemma A2, there exists $\varepsilon > 0$ such that

$$p_{T,\zeta} = P(|s^{(1)}_{T,t}| \geq \zeta) \leq \varepsilon^{-1} \max_{1\leq k<t} E\big[\frac{|y_k|}{\zeta\nu_k} I(\frac{|y_k|}{\zeta\nu_k} \geq \varepsilon)\big]. \tag{A.24}$$

Notice that $\nu_k \geq 1$.

Proof of (44). Suppose that $(\xi_j) \in \mathcal{E}(s)$. Then, (40) of Lemma 1 implies

$$P(|y_k| \geq \zeta) \leq f_k(2, \gamma, c, \zeta), \;\; \zeta > 0, \; k \geq 2.$$

Therefore, by (C.11) of Lemma C2(ii),

$$E[|y_k|I(|y_k| \geq \varepsilon\zeta\nu_k)] \leq f_k\big(2, \gamma, c', \varepsilon\zeta\nu_k\big)$$

for some c' which does not depend on k. Thus, (A.24) implies

$$p_{T,\zeta} \leq C \max_{1\leq k<t} \nu_k^{-1} f_k\big(2, \gamma, c', \varepsilon\zeta\nu_k\big) \leq C \max_{1\leq k<t} f_k\big(2, \gamma, c', \varepsilon\zeta\nu_k\big). \tag{A.25}$$

For $k \geq H$, it holds that $\nu_k = 1$, and we have

$$\begin{aligned} f_k\big(2, \gamma, c, \varepsilon\zeta\nu_k\big) &= c_0\Big\{\exp\big(-c_1(\varepsilon\zeta)^2\big) + \exp\big(-c_2(\frac{\varepsilon\zeta\sqrt{k}}{\log^2 k})^{s/(s+1)}\big)\Big\} \\ &\leq f_H(2, \gamma, c, \zeta). \end{aligned}$$

For $1 \le k < H$, we have $\nu_k = (H/k)^{1/2} \ge 1$ and $\nu_k\sqrt{k} = \sqrt{H}$, which allows to conclude

$$\begin{aligned} f_k\big(2,\gamma,c,\varepsilon\zeta\nu_k\big) &= c_0\Big\{\exp(-c_1(\varepsilon\zeta\nu_k)^2) + \exp\big(-c_2(\frac{\varepsilon\zeta\nu_k\sqrt{k}}{\log^2 k})^{s/(s+1)}\big)\Big\} \\ &\le c_0\Big\{\exp\big(-c_1(\varepsilon\zeta)^2\big) + \exp\big(-c_2(\frac{\varepsilon\zeta\sqrt{H}}{\log^2 H})^{s/(s+1)}\big)\Big\} \\ &= f_H(2,\gamma,c,\zeta). \end{aligned}$$

Together with (A.24), this yields $p_{T,\zeta} \le f_H(2,\gamma,c,\zeta)$ which proves (A.20).

Proof of (45). Assume that $(\xi_j) \in \mathcal{H}(\theta)$ and let $\theta' \in (2,\theta)$. By (41) of Lemma 1,

$$P(|y_k| \ge \zeta) \le g_k(2,\theta',c,\zeta)$$

for $k \ge 2$, and by (C.12) of Lemma C2(iii),

$$E[|y_k| I(|y_k| \ge \varepsilon\zeta\nu_k)] \le \max(\varepsilon\zeta\nu_k,\, 1) g_k\big(2,\theta',c,\varepsilon\zeta\nu_k\big) \tag{A.26}$$

for some c which does not depend on k. Notice that $\zeta\nu_k \ge 1$. Then,

$$(\varepsilon\zeta\nu_k)^{-1}\max(\varepsilon\zeta\nu_k,\, 1) \le \max\big(1,(\varepsilon\zeta\nu_k)^{-1}\big) \le 1+\varepsilon^{-1}.$$

Thus, by (A.26) and (A.24),

$$p_{T,\zeta} \le C \max_{1\le k<t} g_k\big(2,\theta',c',\varepsilon\zeta\nu_k\big) \tag{A.27}$$

where C depends on ε. For $k \ge H$ we have $\nu_k = 1$, and therefore

$$\begin{aligned} g_k\big(2,\theta',c,\varepsilon\zeta\nu_k\big) &= g_k\big(2,\theta',c,\varepsilon\zeta\big) = c_0\Big\{\exp\big(-c_1(\varepsilon\zeta)^2\big) + (\varepsilon\zeta)^{-\theta'}k^{-(\theta'/2-1)}\Big\} \\ &\le g_H\big(2,\theta',c,\varepsilon\zeta\big). \end{aligned}$$

For $k \le H$, we have $\nu_k = (H/k)^{1/2} \ge 1$ and therefore

$$\begin{aligned} (\zeta\nu_k)^{-\theta'}k^{-(\theta'/2-1)} &= (\zeta(H/k)^{1/2})^{-\theta'}k^{-\theta'/2}k = (\zeta H^{1/2})^{-\theta'}k \\ &\le \zeta^{-\theta'}H^{-(\theta'/2-1)} \end{aligned}$$

which allows to conclude

$$\begin{aligned} g_k\big(2,\theta',c,\varepsilon\zeta\nu_k\big) &= c_0\Big\{\exp\big(-c_1(\varepsilon\zeta\nu_k)^2\big) + (\varepsilon\zeta\nu_k)^{-\theta'}k^{-(\theta'/2-1)}\Big\} \\ &\le c_0\Big\{\exp\big(-c_1(\varepsilon\zeta)^2\big) + (\varepsilon\zeta)^{-\theta'}H^{-(\theta'/2-1)}\Big\} \\ &= g_H\big(2,\theta',c,\varepsilon\zeta\big). \end{aligned}$$

Together with (A.27), this implies $p_{T,\zeta} \le g_H(2,\theta',c,\zeta)$ which proves (A.21). □

Proof of Lemma 3. (a) Write $\widetilde{S}_{T,t} = S_{T,t} + r_{T,t}$. Assumption $\zeta > 2|r_{T,t}|$ implies $\zeta - |r_{T,t}| \geq \zeta/2$. Therefore

$$P(|\widetilde{S}_{T,t}| \geq \zeta) \geq P(|S_{T,t}| \geq \zeta - |r_{T,t}|) \leq P(|S_{T,t}| \geq \zeta/2).$$

(b) If $|E\xi_k - E\xi_t| \leq C|k-t|/(t \vee k)$ for $k, t \geq 1$, then by (C.16) of Lemma C3,

$$|r_{T,t}| \leq CH^{-1/2} \textstyle\sum_{k=1}^{T} b_{H,|t-k|}(\frac{|t-k|}{t \vee k}) \leq c_1 \frac{H^{3/2}}{H \vee t}$$

for some $c_1 > 0$ which proves (46).
If $|E\xi_k - E\xi_t| \leq C|k-t|/T$ for $k, t = 1, ..., T$, then by (43),

$$\begin{aligned} |r_{T,t}| &\leq CH^{-1/2} \sum_{k=1}^{T} b_{H,|t-k|}(\frac{|t-k|}{T}) \\ &\leq C\Big(H^{-1} \sum_{k=1}^{T} b_{H,|t-k|}(\frac{|t-k|}{H})\Big) \frac{H^{3/2}}{T} \leq c_2 \frac{H^{3/2}}{T} \end{aligned} \tag{A.28}$$

for some $c_2 > 0$ which does not depend on t, H, T. This proves (47). □

Proof of Corollary 6.
Proof of (a). The bounds (44)-(45) together with definition of f_t, g_t in (39) imply

$$P(|S_{T,t}| \geq b) \to 0, \quad T \to \infty, \ b \to \infty.$$

Hence, $S_{T,t} = O_P(1)$ which proves (48).
Proof of (b). Assume that $(\xi_j) \in \mathcal{E}(s)$, $s > 0$. We will show that as $T \to \infty$, $b \to \infty$,

$$P\big(\max_{t=1,...,T} |S_{T,t}| > b\delta_{T,H}\big) = o_P(1), \quad \delta_{T,H} = (\log T)^{1/2} + \frac{(\log H)^2}{H^{1/2}} (\log T)^{1/\gamma} \tag{A.29}$$

with $\gamma = s/(s+1)$. Let $b > 0$. Then by (44) and (39),

$$\begin{aligned} &P\big(\max\nolimits_{t=1,...,T} |S_{T,t}| \geq b\delta_{T,H}\big) \leq \textstyle\sum_{t=1}^{T} P\big(|S_{T,t}| \geq b\delta_{T,H}\big) \\ &\leq \sum_{t=1}^{T} f_H(2, \gamma, c, b\delta_{T,H}) \\ &\leq Tc_0\Big\{\exp\big(-c_1(b\delta_{T,H})^2\big) + \exp\big(-c_2(\tfrac{(b\delta_{T,H})\sqrt{H}}{\log^2 H})^\gamma\big)\Big\} \\ &\leq Tc_0\Big\{\exp\big(-c_1 b^2 \log T\big) + \exp\big(-c_2 b^\gamma \log T\big)\Big\} \\ &\leq 2T^{-1} \to 0 \end{aligned}$$

for b such that $c_1 b^2 \geq 2$, $c_2 b^\gamma \geq 2$. This proves (A.29). Under assumption $cT^\delta \leq H \leq T$ it holds $\delta_{T,H} = O(\log^{1/2} T)$. Hence, (A.29) implies (50):

$$P\big(\max_{t=1,...,T} |S_{T,t}| > b \log^{1/2} T\big) \to 0, \quad T \to \infty, \ b \to \infty.$$

Next, assume that $(\xi_j) \in \mathcal{H}(\theta)$, $\theta > 2$. Let $\theta' \in (2, \theta)$. We will show that, as $T \to \infty$, $b \to \infty$,

$$P\big(\max_{t=1,\dots,T} |S_{T,t}| > b\delta_{T,H}\big) = o_P(1), \quad \delta_{T,H} = (\log T)^{1/2} + H^{1/2}\big(\frac{T}{H^{\theta'-1}}\big)^{1/\theta'}. \tag{A.30}$$

For $b > 0$, by (45) and definition of g_t, (39),

$$\begin{aligned} &P\big(\max_{t=1,\dots,T} |S_{T,t}| \geq b\delta_{T,H}\big) \\ &\quad \leq \sum_{t=1}^T P\big(|S_{T,t}| \geq b\delta_{T,H}\big) \leq \sum_{t=1}^T g_H(2, \theta', c, b\delta_{T,H}) \\ &\quad \leq Tc_0\Big\{\exp\big(-c_1(b\delta_{T,H})^2\big) + (b\delta_{T,H})^{-\theta'} H^{-(\theta'/2-1)}\Big\} \\ &\quad \leq c_0\{T\exp\big(-c_1 b^2 \log T\big) + b^{-\theta'}\big(\tfrac{T}{H^{\theta'/2-1}}\big)^{-1} T H^{-(\theta'/2-1)}\Big\} \\ &\quad \leq c_0\{T^{-1} + b^{-\theta'}\} \to 0 \end{aligned}$$

as $T \to \infty$ and $b \to \infty$. This proves (A.30). To prove that (A.30) implies (51), it suffices to show that for any $\varepsilon > 0$ there exists $2 < \theta' < \theta$ and $a > 0$ such that

$$\log^{1/2} T + (HT)^{1/\theta} H^{\varepsilon-1/2} \geq a\delta_{T,H}, \quad \delta_{T,H} = \log^{1/2} T + (TH)^{1/\theta'} H^{-1/2}. \tag{A.31}$$

Write

$$(HT)^{1/\theta} H^{\varepsilon-1/2} = (HT)^{1/\theta'} H^{-1/2} v_H, \quad v_H := (HT)^{1/\theta-1/\theta'} H^{\varepsilon}.$$

We will show that $v_H \geq a > 0$ for some $1 > a > 0$ which proves (A.31). By the assumption of the corollary, $cT^\delta \leq H \leq T$. Then,

$$v_H = \frac{H^\varepsilon}{(HT)^{1/\theta'-1/\theta}} \geq \frac{(cT^\delta)^\varepsilon}{T^{2(1/\theta'-1/\theta)}} = c^\varepsilon T^b, \;\; b := \delta\varepsilon - 2(1/\theta' - 1/\theta).$$

If $b \geq 0$, this implies $v_H \geq c^\varepsilon$. Clearly, $b \geq 0$ if θ' is selected sufficiently close to θ. □

Proof of Corollary 7. Let $0 \leq \nu \leq 1$. Write

$$v_{T,t} := H^{-1} \sum_{k=1}^T \widetilde{b}_{H,|t-k|} |\xi_k|, \quad \widetilde{b}_{H,|t-k|} := b_{H,|t-k|}(|t-k|/H)^\nu. \tag{A.32}$$

By (43), $\widetilde{b}_{H,|t-k|} \leq C(1 + (k/H)^{\nu-1})^{-1}$. It is easy to see that $\widetilde{b}_{H,k}$ satisfies (43) with parameter $\nu - 1$. Since under assumptions of corollary,

$$\max_{k\geq 1} E|\xi_k| \leq C < \infty, \qquad \max_{t\geq 1} H^{-1} \sum_{k=1}^T \widetilde{b}_{H,|t-k|} \leq C,$$

then

$$\max_{1\leq t\leq T} |v_{T,t}| \leq \max_{1\leq t\leq T} |v'_{T,t}| + C, \quad v'_{T,t} = H^{-1} \sum_{k=1}^T \widetilde{b}_{H,|t-k|}(|\xi_k| - E|\xi_k|).$$

Since (ξ_k) satisfies Assumption A, then by Theorem 14.1 in Davidson (1994), $(|\xi_k|)$ also satisfies Assumption A. To prove the claim (52)-(53) of the corollary, it remains to show that

$$\max_{1\le t\le T} |v'_{T,t}| = O_P(1). \tag{A.33}$$

Let $(\xi_j) \in \mathcal{E}(s)$, $s > 0$. Then, by (50) of Corollary 6 and assumption (49) on H,

$$\max_{1\le t\le T} |v'_{T,t}| = O(H^{-1/2}\log^{1/2} T) = o_P(1).$$

Let $(\xi_j) \in \mathcal{H}(\theta)$, $\theta > 2$. Then, by (51) of Corollary 6 and assumption (49), for any $\varepsilon > 0$,

$$\max_{1\le t\le T} |v'_{T,t}| = O_P\big(H^{-1/2}\log^{1/2} T + (TH)^{1/\theta}H^{\varepsilon-1}\big).$$

By assumption, $H \ge cT^{\delta}$ with $\delta > 1/(\theta-1)$ which implies that $(TH)^{1/\theta}H^{\varepsilon-1} = o(1)$ when ε is selected sufficiently small. This proves (A.33) and completes the proof of the corollary. □

Proof of Lemma 4. Without restriction of generality assume that $\zeta \ge 1$.

Proof of (58)-(59) for $P(|S^{(h)}_{T,t}| \ge \zeta)$. Denote

$$h'_k := h_{t-k},\ \xi'_k := \xi_{t-k},\ h''_k := h_{t+k},\ \xi''_k := \xi_{t+k} \text{ for } k \ge 0.$$

As in (A.19) write $S^{(h)}_{T,t}$ as

$$\begin{aligned} S^{(h)}_{T,t} &= H^{-1/2}\textstyle\sum_{k=1}^{T} b_{H,|t-k|}h_k(\xi_k - E\xi_k) \\ &= H^{-1/2}\textstyle\sum_{k=1}^{t-1} b_{H,k}h'_k\xi'_k + H^{-1/2}\sum_{k=0}^{T-t} b_{H,k}h''_k\xi''_k \\ &=: s^{(h)}_{T,t;1} + s^{(h)}_{T,t;2}. \end{aligned}$$

Proof of (58)-(59) for $S^{(h)}_{T,t}$ reduces to verification of these bounds for $s^{(h)}_{T,t;1}$ and $s^{(h)}_{T,t;2}$:

$$P(|s^{(h)}_{T,t;\ell}| \ge \zeta) \le \begin{cases} f_H\big(\gamma_1,\gamma_2,c,\zeta\wedge\zeta'\big) & \text{if } (\xi_j) \in \mathcal{E}(s), \quad \text{(A.34)} \\ g_H\big(\gamma_1,\theta',c,\zeta\wedge\zeta'\big) & \text{if } (\xi_j) \in \mathcal{H}(\theta),\ \theta > 2 \quad \text{(A.35)} \end{cases}$$

for $\ell = 1, 2$. We start with $s^{(h)}_{T,t;1}$. Denote

$$x_k = \sum_{i=1}^{k}(\xi'_i - E\xi'_i), \quad y_k = k^{-1/2}x_k, \quad y'_k = h'_k y_k, \quad \nu_k := \Big(\frac{k\vee H}{k}\Big)^{1/2}, \quad k \ge 1. \tag{A.36}$$

Then as in (A.22), summation by parts yields

$$\begin{aligned} s^{(h)}_{T,t;1} &= H^{-1/2}\textstyle\sum_{k=1}^{t-2}(b_{H,k}h'_k - b_{H,k+1}h'_{k+1})x_k + H^{-1/2}b_{H,t-1}h'_{t-1} \\ &= \big\{H^{-1/2}\textstyle\sum_{k=1}^{t-2}(b_{H,k} - b_{H,k+1})(h'_k x_k) + H^{-1/2}b_{H,t-1}(h'_{t-1}x_{t-1})\big\} \\ &\quad + H^{-1/2}\textstyle\sum_{k=1}^{t-2} b_{H,k}(h'_k - h'_{k+1})x_k \\ &=: s^{(1)}_{T,t;1} + s^{(2)}_{T,t;1}. \end{aligned} \tag{A.37}$$

Hence, it suffices to verify the bounds (A.34)-(A.35) for $s^{(1)}_{T,t;1}$ and $s^{(2)}_{T,t;1}$.

First, we evaluate $P(|s^{(1)}_{T,t;1}| \geq \zeta)$. The sum $s^{(1)}_{T,t;1}$ can be obtained from $s^{(1)}_{T,t}$ in (A.22) by replacing x_k by $h'_k x_k$. Therefore, the same argument as in the proof of (A.24) implies that there exists $\varepsilon > 0$ such that

$$P(|s^{(1)}_{T,t;1}| \geq \zeta) \leq \varepsilon^{-1} \max_{k=1,...,T} (\zeta\nu_k)^{-1} E[|y'_k| I(|y'_k| \geq \varepsilon\zeta\nu_k], \quad \zeta \geq 1, \;\; T \geq 2. \tag{A.38}$$

We now show that for all $\zeta > 0$, $k \geq 2$,

$$P\big(|y'_k| \geq \zeta\big) \leq f_k(\gamma_1, \gamma_2, c, \zeta) \quad \text{if} \quad (\xi_k) \in \mathcal{E}(s), \tag{A.39}$$

$$P\big(|y'_k| \geq \zeta\big) \leq g_k(\gamma_1, \theta', c, \zeta) \quad \text{if} \quad (\xi_j) \in \mathcal{H}(\theta), \tag{A.40}$$

with γ_1, γ_2 and θ' as in (58)-(59). Recall that $y'_k = h'_k y_k$ where $(h'_k) \in \mathcal{E}(\alpha)$ by assumption (56). Moreover, (40) and (41) imply that

$$P(|y_k| \geq \zeta) \leq f_k(2, \gamma, c, \zeta) \;\; \text{if } (\xi_k) \in \mathcal{E}(s),$$

$$P(|y_k| \geq \zeta) \leq g_k(2, \theta', c, \zeta) \;\; \text{if } (\xi_j) \in \mathcal{H}(\theta).$$

So, (A.39) and (A.40) follow from Lemma C1 (iii) and (iv), respectively.

As shown in the proof of (44) and (45), the relations (A.38)-(A.40) imply

$$P\big(|s^{(1)}_{T,t;1}| \geq \zeta\big) \leq f_H\big(\gamma_1, \gamma_2, c, \zeta\big) \quad \text{if} \quad (\xi_k) \in \mathcal{E}(s), \tag{A.41}$$

$$P\big(|s^{(1)}_{T,t;1}| \geq \zeta\big) \leq g_H\big(\gamma_1, \theta', c, \zeta\big) \quad \text{if} \quad (\xi_k) \in \mathcal{H}(\theta), \tag{A.42}$$

which verifies (A.34)-(A.35) for $s^{(1)}_{T,t;1}$. Next we show that setting $\zeta' = \zeta d_{Ht}$,

$$P\big(|s^{(2)}_{T,t;1}| \geq \zeta\big) \leq f_H\big(\gamma_1, \gamma_2, c, \zeta'\big) \quad \text{if} \quad (\xi_k) \in \mathcal{E}(s), \tag{A.43}$$

$$P\big(|s^{(2)}_{T,t;1}| \geq \zeta\big) \leq g_H\big(\gamma_1, \theta', c, \zeta'\big) \quad \text{if} \quad (\xi_k) \in \mathcal{H}(\theta). \tag{A.44}$$

Together with (A.37)-(A.40), the latter proves (A.34)-(A.35) for $s^{(h)}_{T,t;1}$.

We now prove (A.43)-(A.44). We have

$$P\big(|s^{(2)}_{T,t;1}| \geq \zeta\big) = P(d_{Ht}|s^{(2)}_{T,t;1}| \geq d_{Ht}\zeta). \tag{A.45}$$

In view of definition of h'_k, by assumptions (54)-(55),

$$h'_k - h'_{k+1} = h_{t-k} - h_{t-k-1} = \delta_{tk}^{-1/2}\xi_{tk}, \quad \text{for } k = 1, ..., t-2,$$

and $\delta_{tk} = t - k$ if (54) holds; $\delta_{tk} = T$ if (55) holds, while $(\xi_{tk}) \in \mathcal{E}(\alpha)$ by assumption (56). Then, with ν_k and y_k as in (A.36), setting $y''_k = \xi_{tk} x_k k^{-1/2} = \xi_{tk} y_k$, we can write

$$(h'_k - h'_{k+1})x_k = (\frac{k}{\delta_{tk}})^{1/2} \frac{\xi_{tk} x_k}{k^{1/2}} = (\frac{k \vee H}{\delta_{tk}})^{1/2} \frac{y''_k}{\nu_k}.$$

Hence,

$$d_{Ht}s^{(2)}_{T,t;1} = \sum_{k=1}^{t-2}\frac{d_{Ht}}{H^{1/2}}b_{H,k}(h'_k-h'_{k+1})\,x_k \tag{A.46}$$
$$= \sum_{k=1}^{t-2}\widetilde{a}_{tk}(\frac{y''_k}{\nu_k}), \quad \widetilde{a}_{tk}=\frac{d_{Ht}}{H^{1/2}}b_{H,k}(\frac{k\vee H}{\delta_{tk}})^{1/2}.$$

Next we show that for all t, H, T,

$$s_{Ht} := \sum_{k=1}^{t-2}|\widetilde{a}_{tk}| \le C < \infty. \tag{A.47}$$

Let (54) hold. Then, by definition, $d_{Ht}=(H\vee t)^{1/2}H^{-1}$, $\delta_{Ht}=|t-k|$, and by (C.17),

$$s_{Ht} = d_{Ht}\sum_{k=1}^{t-2}\frac{b_{H,k}}{H^{1/2}}(\frac{k\vee H}{t-k})^{1/2} \le (\frac{H}{H\vee t})^{-1/2}\big(\sum_{j=1}^{t-1}\frac{b_{H,|t-j|}}{H}(\frac{|t-j|\vee H}{j})^{1/2}\big) \le C, \quad t\ge 2.$$

Let (55) holds. Then, $d_{Ht}=T^{1/2}H^{-1}$, $\delta_{Ht}=T$, and by property (43) of $b_{H,k}$,

$$s_{Ht} = \sum_{k=1}^{t-2}\frac{b_{H,k}}{H}(\frac{k\vee H}{H})^{1/2} \le C, \quad t\ge 2,\ T\ge 2.$$

From (A.45)–(A.47) and Lemma A2 it follows that there exists $\varepsilon>0$ such that

$$P\big(|s^{(2)}_{T,t;1}|\ge\zeta\big) = P\big(|d_{Ht}s^{(2)}_{T,t;1}|\ge\zeta'\big) \le \varepsilon^{-1}\max_{1\le k\le t-2}E\Big[\frac{|y''_k|}{\zeta'\nu_k}I(\frac{|y''_k|}{\zeta'\nu_k}\ge\varepsilon)\Big]. \tag{A.48}$$

This bound is of the same type as (A.38) for $P(|s^{(1)}_{T,t;1}|\ge\zeta)$. Recall that $y''_k=\xi_{tk}y_k$ and by (56), variables ξ_{tk} have the property $(\xi_{tk})\in\mathcal{E}(\alpha)$. Hence, (A.48) implies (A.43)-(A.44) by the same argument as in the proof of (A.41)-(A.42) for $s^{(1)}_{T,t;1}$.

The proof of the bounds (A.34)-(A.35) for $s^{(h)}_{T,t;2}$ can be obtained using similar arguments as above for $s^{(h)}_{T,t;1}$. This completes the proof of (A.34)-(A.35) which imply (58)-(59) of Lemma 4 for $P\big(|S^{(h)}_{T,t}|\ge\zeta\big)$. □

Proof of Lemma 5. It suffices to verify (A.34)-(A.35) for $P\big(|\widetilde{S}^{(h)}_{T,t}|\ge\zeta\big)$. Write

$$\widetilde{S}^{(h)}_{T,t} = S^{(h)}_{T,t}+r_{T,t}, \quad r_{T,t} := H^{-1/2}\textstyle\sum_{k=1}^{T}b_{H,|t-k|}(h_kE\xi_k-h_tE\xi_t).$$

Since by Lemma 4, $P(|S^{(h)}_{T,t}|\ge\zeta)$ satisfies (58)-(59) and thus (A.34)-(A.35), to establish the corresponding bounds for $P\big(|\widetilde{S}^{(h)}_{T,t}|\ge\zeta\big)$, it suffices to show that $P(|r_{T,t}|\ge\zeta)$ satisfies (A.34)-(A.35) as well. We will prove that there exists $c_0>0$ and $c_1>0$ such that

$$P\big(|r_{T,t}|\ge\zeta\big) \le c_0\exp(-c_1\zeta^{\alpha}), \quad \zeta>0,\ T\ge 2. \tag{A.49}$$

Since $\alpha > \gamma_1 = 2\alpha/(2+\alpha)$, (A.49) together with definition (39) of f_t and g_t, implies (A.34)-(A.35) for $P\big(|r_{T,t}| \ge \zeta\big)$.

Proof of (A.49). Write

$$P\big(|r_{T,t}| \ge \zeta\big) = P\big(d_{Ht}|r_{T,t}| \ge \zeta'\big), \quad \zeta' = d_{Ht}\zeta.$$

Case 1. Suppose that $E|\xi_k - E\xi_t|$ and $|h_k - h_t|$ satisfy assumptions ((60), (54)). Then,

$$\begin{aligned} |h_k E\xi_k - h_t E\xi_t| &\le |h_k(E\xi_k - E\xi_t)| + |E\xi_t|\,|h_k - h_t| \\ &\le C(\frac{|t-k|}{t \vee k})^{1/2} z_k, \;\; z_k = |h_k| + |\xi_{tk}|. \end{aligned} \tag{A.50}$$

Under (54), by definition (57), $d_{Ht} = (t \vee H)^{1/2} H^{-1}$. Hence,

$$d_{Ht}|r_{T,t}| \le C \sum_{k=1}^{T} a_{tk} z_k, \quad a_{tk} = (\frac{t \vee H}{H})^{1/2} \frac{b_{H,|t-k|}}{H} (\frac{|t-k|}{t \vee k})^{1/2}.$$

Applying (C.16) with $\gamma = 1/2$, we get

$$\max_{t=1,\ldots,T} \sum_{k=1}^{T} a_{tk} \le C < \infty.$$

Hence, by Lemma A2, there exists $\varepsilon > 0$ such that

$$P\big(d_{Ht}|r_{T,t}| \ge \zeta'\big) \le \varepsilon^{-1} \max_{1 \le k \le T} E\Big[\frac{|z_k|}{\zeta'} I(\frac{|z_k|}{\zeta'} \ge \varepsilon)\Big]. \tag{A.51}$$

By assumption (56), $(z_k) \in \mathcal{E}(\alpha)$. Hence, by Lemma C2(i),

$$E\big[|z_k| I(|z_k| \ge \varepsilon\zeta')\big] \le c_0' \exp(-c_1' \zeta'^{\alpha})$$

which together with (A.51) implies

$$P\big(d_{Ht}|r_{T,t}| \ge \zeta'\big) \le c_0' \zeta'^{-1} \exp(-c_1' \zeta'^{\alpha}).$$

Therefore,

$$P\big(|r_{T,t}| \ge \zeta\big) \le c_0' \exp(-c_1' \zeta'^{\alpha}) \quad \text{for } \zeta' \ge 1. \tag{A.52}$$

This bound remains valid for $0 < \zeta' < 1$ if c_0' is selected such that $c_0' \exp(-c_1') \ge 1$. Then,

$$c_0' \exp(-c_1' \zeta'^{\alpha}) \ge c_0' \exp(-c_1') \ge 1 \;\text{ for } 0 < \zeta' < 1$$

and, thus, (A.52) holds. This proves (A.49).

Case 2. Suppose that $E|\xi_k - E\xi_t|$ and $|h_k - h_t|$ satisfy assumptions ((61), (55)). Then, instead of (A.50), we have the bound

$$|h_k E\xi_k - h_t E\xi_t| \le C(\frac{|t-k|}{T})^{1/2} z_k, \quad z_k = |h_k| + |\xi_{tk}|. \tag{A.53}$$

Under (55), by definition (57), $d_{Ht} = T^{1/2}H^{-1}$ for $t = 1, ..., T$. Hence,

$$d_{Ht}|r_{T,t}| \le C\sum_{k=1}^{T} a^*_{tk} z_k, \quad a^*_{tk} = (\frac{T}{H})^{1/2}\frac{b_{H,|t-k|}}{H}(\frac{|t-k|}{T})^{1/2} = \frac{b_{H,|t-k|}}{H}(\frac{|t-k|}{H})^{1/2}.$$

By the same argument as in (A.28) it follows that

$$\max_{t=1,\dots,T}\sum_{k=1}^{T} a^*_{tk} \le C < \infty.$$

Hence, as above, by Lemma A2, there exists $\varepsilon > 0$ such that (A.51) holds, and using the same argument as in Case 1, we obtain (A.49).

Thus, $P(|r_{T,t}| \ge \zeta)$ satisfies (A.49) which completes the proof of the lemma. □

Proof of Corollary 8. (a) Recall that $\zeta \wedge \zeta' = \zeta(1 \wedge d_{Ht})$. The bounds (58)-(59) together with definitions of f_t, g_t in (39) imply

$$P\big((1 \wedge d_{Ht})|S^{(h)}_{T,t}| \ge b\big) \to 0, \quad b \to \infty.$$

This proves (62):

$$S^{(h)}_{T,t} = O_P((1 \wedge d_{Ht})^{-1}) = O_P\big(1 + d^{-1}_{Ht}\big).$$

The same argument implies $\widetilde{S}^{(h)}_{T,t} = O_P\big(1 + d^{-1}_{Ht}\big)$, since by Lemma 5, $P(|\widetilde{S}^{(h)}_{T,t}| \ge \zeta)$ satisfies the same bounds (58)-(59).

(b) Under assumption (55), $d_{Ht} = T^{1/2}H^{-1}$. Set $z_{T,t} := (1 \wedge d_{Ht})\widetilde{S}^{(h)}_{T,t}$.

Assume that $(\xi_j) \in \mathcal{E}(s)$, $s > 0$. We will show that as $T \to \infty$, $b \to \infty$,

$$P\big(\max_{t=1,\dots,T}|z_{T,t}| > b\delta_{T,H}\big) = o_P(1), \quad \delta_{T,H} = (\log T)^{1/\gamma_1} + \frac{(\log H)^2}{H^{1/2}}(\log T)^{1/\gamma_2}, \tag{A.54}$$

where γ_1 and γ_2 are the same as in (58) of Lemma 4.

For $b > 0$, by (58), definition of f_t, (39), and equality $(\zeta \wedge \zeta')(1 \wedge d_{Ht})^{-1} = \zeta$,

$$\begin{aligned}
P(\max\nolimits_{t=1,\dots,T}|z_{T,t}| \ge b\delta_{T,H}) &\le \textstyle\sum_{t=1}^{T} P((1 \wedge d_{Ht})|S_{T,t}| \ge b\delta_{T,H})\\
&\le \textstyle\sum_{t=1}^{T} f_H(\gamma_1, \gamma_2, c, b\delta_{T,H})\\
&\le Tc_0\Big\{\exp\big(-c_1(b\delta_{T,H})^{\gamma_1}\big) + \exp\big(-c_2(\tfrac{(b\delta_{T,H})\sqrt{H}}{\log^2 H})^{\gamma_2}\big)\Big\}\\
&\le Tc_0\Big\{\exp\big(-c_1 b^{\gamma_1}\log T\big) + \exp\big(-c_2 b^{\gamma_2}\log T\big)\Big\}\\
&\le 2c_0T^{-1} \to 0
\end{aligned}$$

for b such that $c_1 b^{\gamma_1} \ge 2$, $c_2 b^{\gamma_2} \ge 2$. This proves (A.54). Since for $cT^\delta \le H \le T$ it holds $\delta_{T,H} = O\big(\log^{1/\gamma_1} T\big)$, (A.54) implies:

$$\max_{t=1,\dots,T} |z_{T,t}| = O_P(\delta_{T,H}) = O_P(\log^{1/\gamma_1} T).$$

This together with definition $z_{T,t} := (1 \wedge d_{Ht}) \widetilde{S}^{(h)}_{T,t}$, where $d_{Ht} = T^{1/2} H^{-1}$, and inequality $(1 \wedge T^{1/2} H^{-1})^{-1} \le 1 + HT^{-1/2}$, implies (63):

$$\max_{t=1,\dots,T} |\widetilde{S}^{(h)}_{T,t}| = O_P\big((1 \wedge T^{1/2}H^{-1})^{-1} \log^{1/\gamma_1} T\big) = O_P\big((1 + HT^{-1/2}) \log^{1/\gamma_1} T\big).$$

Next, consider the case $(\xi_j) \in \mathcal{H}(\theta)$, $\theta > 2$. First we show that for any $\theta' \in (2, \theta)$, as $T \to \infty$,

$$\max_{t=1,\dots,T} |z_{T,t}| = O_P(\delta_{T,H}), \quad \delta_{T,H} = (\log T)^{1/\gamma_1} + H^{1/2}\big(\frac{T}{H^{\theta'-1}}\big)^{1/\theta'}. \tag{A.55}$$

For $b > 0$, by (59), definition of g_t, (39) and equality $(\zeta \wedge \zeta')(1 \wedge d_{Ht})^{-1} = \zeta$, we obtain

$$\begin{aligned}
&P\big(\max_{t=1,\dots,T} |z_{T,t}| \ge b\delta_{T,H}\big)\\
&\quad \le \textstyle\sum_{t=1}^T P\big((1 \wedge d_{Ht})|S_{T,t}| \ge b\delta_{T,H}\big)\\
&\quad \le \textstyle\sum_{t=1}^T g_H(\gamma_1, \theta', c, b\delta_{T,H})\\
&\quad \le Tc_0\Big\{\exp\big(-c_1(b\delta_{T,H})^{\gamma_1}\big) + (b\delta_{T,H})^{-\theta'} H^{-(\theta'/2-1)}\Big\}\\
&\quad \le c_0\{T\exp\big(-c_1 b^{\gamma_1}\log T\big) + b^{-\theta'}(\tfrac{T}{H^{\theta'/2-1}})^{-1} T H^{-(\theta'/2-1)}\Big\}\\
&\quad \le c_0\{T^{-1} + b^{-\theta'}\} \to 0
\end{aligned}$$

as $T \to \infty$ and $b \to \infty$. This proves (A.55). The same argument as in the proof of (51) of Corollary 6 shows that validity of (A.55) for any $2 < \theta' < \theta$ implies that for any $\varepsilon > 0$,

$$\max_{t=1,\dots,T} |z_{T,t}| = O_P\big(\widetilde{\delta}_{T,H}\big), \quad \widetilde{\delta}_{T,H} = (\log T)^{1/\gamma_1} + H^{\varepsilon-1/2}(TH)^{1/\theta}. \tag{A.56}$$

In turn, since $d_{Ht} = T^{1/2}H^{-1}$, this yields

$$\begin{aligned}
\max_{t=1,\dots,T} |\widetilde{S}^{(h)}_{T,t}| &= O_P\big((1 \wedge +H^{-1}T^{1/2})^{-1}\widetilde{\delta}_{T,H}\big)\\
&= O_P\big((1 + HT^{-1/2})\{(\log T)^{1/\gamma_1} + H^{\varepsilon-1/2}(TH)^{1/\theta}\}\big)
\end{aligned}$$

which proves (64) and completes the proof of the corollary. □

Proof of Corollary 9. Denote

$$v_{T,t,\nu} = H^{-1} \sum_{k=1}^T b_{H,|t-k|} \big|\frac{t-k}{H}\big|^\nu |\xi_k|, \quad 0 \le \nu \le 1. \tag{A.57}$$

Proof of (67) and (69). Let (65) hold. Then,

$$|\Delta_{T,t}| \le CH^{-1}\sum_{k=1}^{T} b_{H,|t-k|}|\frac{t-k}{T}|\,|\xi_k| = C(H/T)v_{T,t,1}. \tag{A.58}$$

Since under assumptions of lemma, $\max_k E|\xi_k| < \infty$, together with (43) this implies

$$Ev_{T,t,1} \le C(\max_k E|\xi_k|)H^{-1}\sum_{k=1}^{T} b_{H,|t-k|}|\frac{t-k}{H}| = O(1).$$

Hence, $v_{T,t,1} = O_P(1)$ which together with (A.58) proves (67):

$$|\Delta_{T,t}| = C(H/T)O_P(1) = O_P(H/T).$$

Notice that by Corollary 7, under the assumptions of Corollary 9(b),

$$\max_{1\le s\le T}|v_{T,s,\nu}| = O_P(1), \quad 0 \le \nu \le 1. \tag{A.59}$$

This together with (A.58) proves (69):

$$\max_{1\le s\le T}|\Delta_{T,t}| \le C(H/T)\max_{1\le s\le T}|v_{T,s,1}| = O_P(H/T).$$

Proof of (68) and (70). Let (66) hold. Then

$$|\Delta_{T,t}| \le CH^{-1}\sum_{k=1}^{T} b_{H,|t-k|}(\frac{t-k}{T})^{1/2}|\nu_{tk}\xi_k|. \tag{A.60}$$

By (66), $(\nu_{tk}) \in \mathcal{E}(\alpha)$, $\alpha > 0$, while by assumption of corollary, $(\xi_j) \in \mathcal{E}(s)$, $s > 0$ or $(\xi_j) \in \mathcal{H}(\theta)$, $\theta > 2$. Thus, from Lemma C1 (i)-(ii) it follows $\max_{tk} E|\nu_{tk}\xi_k| < \infty$. Hence,

$$E|\Delta_{T,t}| \le C(H/T)^{1/2}(\max_{tk} E|\nu_{tk}\xi_k|)H^{-1}\sum_{k=1}^{T} b_{H,|t-k|}(\frac{t-k}{H})^{1/2} \le C(H/T)^{1/2},$$

where $C > 0$ does not depend on t, H, T. This proves (68), $\Delta_{T,t} = O_P\big((H/T)^{1/2}\big)$.
Next, by (A.60),

$$\max_{1\le t\le T}|\Delta_{T,t}| \le C(H/T)^{1/2}(\max_{1\le k,t\le T}|\nu_{tk}|)(\max_{1\le t\le T} v_{T,s,1/2}), \tag{A.61}$$

where $v_{T,s,1/2}$ is defined by (A.57). Since $(\nu_{tk}) \in \mathcal{E}(\alpha)$, (C.3) of Lemma C1 implies:

$$\max_{1\le k,t\le T}|\nu_{tk}| = O_P\big((\log T)^{1/\alpha}\big).$$

By (A.59),

$$\max_{1\le t\le T}|v_{T,s,1/2}| = O_P(1)$$

which together with (A.61) proves (70):

$$\max_{1\le t\le T}|\Delta_{T,t}| = O_P\big((H/T)^{1/2}(\log T)^{1/\alpha}\big).$$

This completes the proof of the corollary. □

B. Proofs of Theorems 1-3.

For convenience of the proof of Theorems 1-3, we include Lemma B1 which summarizes the key steps of the proof of Theorem 1, Bickel and Levina (2008) and adjusts them to our setting.

Recall notation of $p \times p$ covariance matrix $\mathbf{\Sigma}_t = [\sigma_{ij,t}]$, sample covariance estimator $\widehat{\mathbf{\Sigma}}_t = [\widehat{\sigma}_{ij,t}]$ of $\mathbf{\Sigma}_t$, (10), and the regularized sample covariance estimate defined in (11):

$$T_\lambda\big(\widehat{\mathbf{\Sigma}}_t\big) = \big(\widehat{\sigma}_{ij,t} I(|\widehat{\sigma}_{ij,t}| > \lambda)\big).$$

Denote

$$M = \max_{i,j=1,\dots,p} |\widehat{\sigma}_{ij,t} - \sigma_{ij,t}|, \quad N = \max_{i=1,\dots,p} \sum_{j=1}^{p} I\big(|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| > \lambda/2\big).$$

Recall the definition of the sparsity parameter n_p of covariance matrix $\mathbf{\Sigma}_t$ which is the maximum number of non-zero elements in a row of $\mathbf{\Sigma}_t$, see, e.g., (8).

Lemma B1 *(see Bickel and Levina (2008, proof of Theorem 1)). For any* $\lambda > 0$,

$$||T_\lambda(\widehat{\mathbf{\Sigma}}_t) - \mathbf{\Sigma}_t|| \;\; \le \;\; 2M\,N + Mn_p + 2\lambda n_p. \tag{B.1}$$

Moreover, if λ *is such that as* $T \to \infty$,

$$\max\nolimits_{i,j=1,\dots,p} P\big(|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| > \lambda/2\big) = o(p^{-2}), \tag{B.2}$$

then

$$\big\|T_\lambda\big(\widehat{\mathbf{\Sigma}}_t\big) - \mathbf{\Sigma}_t\big\| = O_P(n_p\lambda). \tag{B.3}$$

In addition, if (B.3) holds, $n_p\lambda = o(1)$ *and* $||\Sigma_t|| \ge c > 0$, *then*

$$\big\|T_\lambda\big(\widehat{\mathbf{\Sigma}}_t\big)^{-1} - \mathbf{\Sigma}_t^{-1}\big\| = O_P(n_p\lambda). \tag{B.4}$$

Proof. Verification of (B.1) follows closely the steps of the proof of Theorem 1, pp. 2582-2584 in Bickel and Levina (2008). For clarity, we include the details of the proof.

We have

$$T_\lambda(\widehat{\mathbf{\Sigma}}_t) - \mathbf{\Sigma}_t = [\delta_{ij,t}]_{i,j=1,\dots,p}, \quad \delta_{ij,t} = \widehat{\sigma}_{ij,t} I(|\widehat{\sigma}_{ij,t}| > \lambda) - \sigma_{ij,t}.$$

By the well-known property of the spectral norm of a symmetric matrix,

$$||T_\lambda(\widehat{\mathbf{\Sigma}}_t) - \mathbf{\Sigma}_t|| \le \max_{i=1,\dots,p} \big(\sum_{j=1}^{p} |\delta_{ij,t}|\big). \tag{B.5}$$

Write

$$
\begin{aligned}
\delta_{ij,t} &= \{\widehat{\sigma}_{ij,t} I(|\widehat{\sigma}_{ij,t}| > \lambda) - \sigma_{ij,t} I(|\sigma_{ij,t}| > \lambda/2)\} + \{-\sigma_{ij,t} I(|\sigma_{ij,t}| \leq \lambda/2)\} \\
&= \delta^{(1)}_{ij,t} + \delta^{(2)}_{ij,t}.
\end{aligned}
$$

Notice that

$$
|\delta^{(2)}_{ij,t}| \leq |\sigma_{ij,t}| I\big(|\sigma_{ij,t}| \leq \lambda/2\big) \leq (\lambda/2) I(|\sigma_{ij,t}| \neq 0).
$$

On the other hand,

$$
\begin{aligned}
\delta^{(1)}_{ij,t} &= \widehat{\sigma}_{ij,t}\big(I(|\widehat{\sigma}_{ij,t}| > \lambda) - I(|\sigma_{ij,t}| > \lambda/2)\big) + (\widehat{\sigma}_{ij,t} - \sigma_{ij,t}) I(|\sigma_{ij,t}| > \lambda/2) \\
&= \{\widehat{\sigma}_{ij,t} I(|\widehat{\sigma}_{ij,t}| > \lambda,\, |\sigma_{ij,t}| \leq \lambda/2)\} + \{-\widehat{\sigma}_{ij,t} I(|\widehat{\sigma}_{ij,t}| \leq \lambda,\, |\sigma_{ij,t}| > \lambda/2)\} \\
&\quad + \{(\widehat{\sigma}_{ij,t} - \sigma_{ij,t}) I(|\sigma_{ij,t}| > \lambda/2)\} = v^{(1)}_{ij,t} + v^{(2)}_{ij,t} + v^{(3)}_{ij,t}.
\end{aligned}
$$

Notice that for $|\widehat{\sigma}_{ij,t}| > \lambda$, $|\sigma_{ij,t}| \leq \lambda/2$ it holds

$$
\begin{aligned}
|\widehat{\sigma}_{ij,t}| &\leq 2(|\widehat{\sigma}_{ij,t}| - |\sigma_{ij,t}|) \leq 2|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}|, \\
|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| &\geq |\widehat{\sigma}_{ij,t}| - |\sigma_{ij,t}| > \lambda/2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|v^{(1)}_{ij,t}| &\leq 2|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| I\big(|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| > \lambda/2\big), \\
|v^{(2)}_{ij,t}| &\leq \lambda I(|\sigma_{ij,t}| \neq 0), \\
|v^{(3)}_{ij,t}| &\leq |\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| I(|\sigma_{ij,t}| \neq 0).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\delta_{ij,t}| &\leq |\delta^{(1)}_{ij,t}| + |\delta^{(2)}_{ij,t}| \leq |v^{(1)}_{ij,t}| + |v^{(2)}_{ij,t}| + |v^{(3)}_{ij,t}| + |\delta^{(2)}_{ij,t}| \\
&\leq 2|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| I\big(|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| > \lambda/2\big) + |\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| I\big(|\sigma_{ij,t}| \neq 0\big) + 2\lambda I\big(|\sigma_{ij,t}| \neq 0\big).
\end{aligned}
$$

Note that by definition of the sparsity parameter, $\max_{i=1,\ldots,p} \sum_{j=1}^{p} I(|\sigma_{ij,t}| \neq 0) = n_p$. Applying this in (B.5), we obtain

$$
\begin{aligned}
||T_\lambda(\widehat{\mathbf{\Sigma}}_t) - \mathbf{\Sigma}_t|| &\leq \max_{i=1,\ldots,p} \big(\sum_{j=1}^{p} |\delta_{ij,t}|\big) \\
&\leq 2\{\max_{i,j=1,\ldots,p} |\widehat{\sigma}_{ij,t} - \sigma_{ij,t}|\}\{\max_{i=1,\ldots,p} \sum_{j=1}^{p} I(|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| > \lambda/2)\} \\
&\quad + \{\max_{i,j=1,\ldots,p} |\widehat{\sigma}_{ij,t} - \sigma_{ij,t}|\}\{\max_i \sum_{j=1}^{p} I(|\sigma_{ij,t}| \neq 0)\} + 2\lambda\{\max_i \sum_{j=1}^{p} I(|\sigma_{ij,t}| \neq 0)\} \\
&\leq 2M\,N + M n_p + 2\lambda n_p
\end{aligned}
$$

which proves (B.1).

Proof of (B.3). If (B.2) holds then

$$P\big(M \geq \lambda/2\big) \leq \sum_{i,j=1}^{p} P\big(|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| > \lambda/2\big) \leq p^2 \max_{ij} P\big(|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| > \lambda/2\big) = o(1).$$

In turn,

$$P(N > 1) \leq P(N > 0) \leq P(M > \lambda/2) \to 0.$$

This shows that $M = O_P(\lambda)$ and $N = O_P(1)$ which together with (B.1) proves (B.3).

To prove (B.4), set $B := T_\lambda\big(\widehat{\Sigma}_t\big)$, $A := \Sigma_t$. By assumption, $\|A\| \geq c > 0$ and $n_p\lambda = o(1)$. By (B.3), $\|B - A\| = O_P(n_p\lambda) = o_P(1)$. Thus,

$$\|B\| \geq \|A + (B - A)\| \geq \|A\| - \|B - A\| \geq c - o_P(1) \geq c(1 + o_P(1)).$$

This implies $\|B^{-1}\| = O_P(1)$. Hence,

$$\begin{aligned} \|B^{-1} - A^{-1}\| &= \|A^{-1}(A - B)B^{-1}\| \leq \|A^{-1}\| \, \|A - B\| \, \|B^{-1}\| \\ &\leq c^{-1} O_P\big(n_p\lambda\big) O_P(1) = O_P\big(n_p\lambda\big) \end{aligned}$$

which proves (B.4). □

Proof of Theorem 1. Recall that

$$\lambda = \kappa (T^{-1} \log p)^{1/2} \tag{B.6}$$

has property $\lambda \to 0$ as $T \to \infty$ in view of (9). By assumption of the theorem, $(\boldsymbol{y_t})$ is a stationary sequence, the sample covariance matrix $\widehat{\boldsymbol{\Sigma}} = (\widehat{\sigma}_{ij})$ given by (5) is the estimate of $\boldsymbol{\Sigma} = (\sigma_{ij}) = \mathrm{var}(\boldsymbol{y_t})$ and σ_{ij} does not depend on t.

By Lemma B1, in view of definition (B.6), to show (B.3) and thus, the claim (6) of Theorem 1, it suffices to prove that for sufficiently large κ,

$$\max_{i,j=1,\dots,p} P\big(|\widehat{\sigma}_{ij} - \sigma_{ij}| > 2\lambda\big) = o(p^{-2}). \tag{B.7}$$

(Notice that (B.7) implies that (B.2) holds for sufficiently large κ which in turn proves (B.3).) Fix (i, j) and set $z_k = y_{ik} y_{jk}$. Because of stationarity assumption, $Ey_{ik} = Ey_{i1}$, $Ey_{jk} = Ey_{j1}$ and $Ez_k = \sigma_{ij}$ do not depend on k. Observe that

$$\sigma_{ij} = \mathrm{cov}(y_{ik}, y_{jk}) = Ez_k - Ey_{ik} Ey_{jk}.$$

Then we can write

$$\begin{aligned}
\widehat{\sigma}_{ij} - \sigma_{ij} &= T^{-1}\sum_{k=1}^{T} y_{ik}y_{jk} - \bar{y}_i\bar{y}_j - \sigma_{ij} \\
&= s_{T,ij} - \bar{y}_i\bar{y}_j + Ey_{i1}Ey_{j1}, \;\; s_{T,ij} := T^{-1}\sum_{k=1}^{T}(z_k - Ez_k), \quad \bar{y}_i = T^{-1}\sum_{k=1}^{T} y_{ik}.
\end{aligned}$$

Observe that

$$\bar{y}_i\bar{y}_j - Ey_{i1}Ey_{j1} = (\bar{y}_i - Ey_{i1})(\bar{y}_j - Ey_{j1}) + E[y_{i1}](\bar{y}_j - Ey_{j1}) + E[y_{j1}](\bar{y}_i - Ey_{i1}).$$

Both assumptions $(y_{ik}) \in \mathcal{E}(s)$ and $(y_{ik}) \in \mathcal{H}(\theta)$ imply that $m = \max_{i,k} E|y_{ik}| < \infty$. Therefore,

$$|\widehat{\sigma}_{ij} - \sigma_{ij}| \le |s_{T,ij}| + |\bar{y}_i - Ey_{i1}|\,|\bar{y}_j - Ey_{j1}| + m|\bar{y}_j - Ey_{j1}| + m|\bar{y}_i - Ey_{i1}|.$$

So, we obtain

$$\begin{aligned}
P\big(|\widehat{\sigma}_{ij} - \sigma_{ij}| > 4\lambda\big) &\le P\big(|s_{T,ij}| > \lambda\big) + P\big(|\bar{y}_i - Ey_{i1}|\,|\bar{y}_j - Ey_{j1}| > \lambda\big) \\
&\quad + P\big(m|\bar{y}_j - Ey_{j1}| > \lambda\big) + P\big(m|\bar{y}_i - Ey_{i1}| > \lambda\big).
\end{aligned} \tag{B.8}$$

Since $\lambda = o(1)$ as $T \to \infty$, then $\sqrt{\lambda} \ge \lambda$ for $\lambda \le 1$. Hence,

$$\begin{aligned}
&P\big(|\bar{y}_i - Ey_{i1}|\,|\bar{y}_j - Ey_{j1}| > \lambda\big) \\
&\le P\big(|\bar{y}_i - Ey_{i1}| > \sqrt{\lambda}\big) + P\big(|\bar{y}_j - Ey_{j1}| > \sqrt{\lambda}\big) \\
&\le P\big(|\bar{y}_i - Ey_{i1}| > \lambda\big) + P\big(|\bar{y}_j - Ey_{j1}| > \lambda\big).
\end{aligned} \tag{B.9}$$

Therefore, to prove (B.7), it suffices to show that uniformly in i, j, as $T \to \infty$,

$$\max_{i,j=1,\dots,p} P\big(|s_{T,ij}| > \lambda\big) = o(p^{-2}), \quad \max_{i=1,\dots,p} P\big(|\bar{y}_i - Ey_{i1}| > \lambda\big) = o(p^{-2}), \tag{B.10}$$

$$\max_{i=1,\dots,p} P\big(m|\bar{y}_i - Ey_{i1}| > \lambda\big) = o(p^{-2}), \tag{B.11}$$

when κ is selected sufficiently large. We will prove (B.10), while (B.11) can be shown using the same argument as in the proof of the second claim in (B.10).

Denote

$$S^*_{T,ij} = T^{1/2} s_{T,ij}, \quad S^*_{T,i} = T^{1/2}\bar{y}_i = T^{-1/2}\sum_{k=1}^{T}(y_{ik} - Ey_{ik}).$$

Then, with $\eta = T^{1/2}\lambda$,

$$\begin{aligned}
&P\big(|s_{T,ij}| > \lambda\big) \le P\big(|S^*_{T,ij}| > \eta\big), \\
&P\big(|\bar{y}_i - Ey_{i1}| > \lambda\big) \le P\big(|S^*_{T,i}| \ge \eta\big).
\end{aligned} \tag{B.12}$$

By Assumption M, the process $(\boldsymbol{y}_k - E\boldsymbol{y}_k)$ is α-mixing, and therefore processes $(z_k - Ez_k)$, $(y_{ik} - Ey_{ik})$ are also α-mixing with mixing coefficients satisfying (1).
(i) Let $(y_{ik}) \in \mathcal{E}(s)$. Then $(z_k) \in \mathcal{E}(s/2)$, and $Ez_k = \sigma_{ij}$ does not depend on k. Hence, (40) of Lemma 1 implies that with $\gamma = (s/2)(1 + s/2)$,

$$P\big(|S^*_{T,ij}| > \eta\big) \le f_T(2, \gamma, c, \eta), \qquad \text{(B.13)}$$
$$P\big(|S^*_{T,i}| > \eta\big) \le f_T(2, \gamma, c, \eta).$$

Notice that $\eta = \kappa(\log p)^{1/2}$. Then, by definition of f_t in (39),

$$\begin{aligned} f_T(2, \gamma, c, \eta) &= c_0\Big\{\exp(-c_1\eta^2) + \exp\big(-c_2(\frac{\eta T^{1/2}}{\log^2 T})^{\gamma}\Big\} \\ &= c_0\Big\{\exp(-c_1\kappa^2 \log T) + \exp\big(-c_2(\frac{\kappa(\log T)^{1/2}T^{1/2}}{\log^2 T})^{\gamma}\Big\} \\ &= o(p^{-2}) \end{aligned}$$

because $c_1\kappa^2 > 2$ when κ is chosen large enough, and under assumption (9), $T \ge cp^{\varepsilon}$,

$$\log p = o\big((\frac{(\log T)^{1/2}T^{1/2}}{\log^2 T})^{\gamma}\big).$$

This together with (B.13) and (B.12) proves (B.10).

(ii) Let $(y_{ik}) \in \mathcal{H}(\theta)$. Then, $(z_k) \in \mathcal{H}(\theta/2)$, and (41) of Lemma 1 implies

$$P\big(|S^*_{T,ij}| > \eta\big) \le g_T(2, \theta', c, \eta), \quad 2 < \theta' < \theta/2, \qquad \text{(B.14)}$$
$$P\big(|S^*_{T,i}| > \eta\big) \le g_T(2, \theta', c, \eta).$$

Recall that $\eta = \kappa(\log p)^{1/2}$. Then the function g_T given in (39) has property

$$\begin{aligned} g_T(2, \theta', c, \eta) &= c_0\Big\{\exp\big(-c_1\eta^2\big) + \eta^{-\theta}T^{-(\theta'/2-1)}\Big\} \\ &= c_0\Big\{\exp\big(-c_1\kappa^2 \log p\big) + (\kappa(\log p)^{1/2})^{-\theta'}T^{-(\theta'/2-1)}\Big\} \\ &= o(p^{-2}) \end{aligned}$$

because $c_1\kappa^2 > 2$ for large enough κ, and since under assumption (9) of the theorem,

$$p^2 = o(T^{\theta'/2-1}) \qquad \text{(B.15)}$$

if $\theta' \in (2, \theta/2)$ is selected close enough to $\theta/2$. Indeed, then $T \ge c_0p^{\varepsilon}$, $\varepsilon > 8/(\theta - 4)$ which implies $p^2 = o(T^{\theta'/2-1})$ if θ' is selected close enough to $\theta/2$.
This, together with (B.14) and (B.12) proves (B.10) which completes the proof of (6).

Property (7) follows using (B.4) of Lemma B1. □

Proof of Theorem 2. Recall that in Theorem 2

$$\lambda = \kappa(\log p)^{1/2} \max\big(H^{-1/2}, H/T\big). \tag{B.16}$$

By Lemma B1, to prove (B.3) which is equivalent to the claim (15) of theorem, it suffices to verify validity of (B.2) when parameter κ is selected sufficiently large. For notational simplicity, instead of (B.2) we will show that for sufficiently large κ,

$$\max_{i,j=1,...,p} P\big(|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| > 4\lambda\big) = o(p^{-2}). \tag{B.17}$$

Since κ can be arbitrary selected, (B.17) implies (B.2).

Recall that $\boldsymbol{y}_t = (y_{1t}, ..., y_{pt})'$. Set $z_k = y_{ik}y_{jk}$. Notice that

$$\sigma_{ij,k} = \mathrm{cov}(y_{ik}, y_{jk}) = Ez_k - Ey_{ik}Ey_{jk}.$$

Then,

$$\begin{aligned} \widehat{\sigma}_{ij,t} - \sigma_{ij,t} &= K_t^{-1} \textstyle\sum_{k=1}^{T} b_{H,|t-k|} y_{ik}y_{jk} - \bar{y}_{it}\bar{y}_{jt} - \sigma_{ij,t} \\ &= s_{T,ij,t} - \bar{y}_{it}\bar{y}_{jt} + Ey_{it}Ey_{jt}, \\ & \quad s_{T,ij,t} = K_t^{-1} \textstyle\sum_{k=1}^{T} b_{H,|t-k|}(z_k - Ez_t), \quad \bar{y}_{it} = K_t^{-1} \textstyle\sum_{k=1}^{T} b_{H,|t-k|} y_{ik}. \end{aligned} \tag{B.18}$$

Notice that

$$\bar{y}_{it}\bar{y}_{jt} - Ey_{it}Ey_{jt} = (\bar{y}_{it} - Ey_{it})(\bar{y}_{jt} - Ey_{jt}) + E[y_{it}](\bar{y}_{jt} - Ey_{jt}) + E[y_{jt}](\bar{y}_{it} - Ey_{it}).$$

Under assumption $(y_{ik}) \in \mathcal{E}(s)$ or $(y_{ik}) \in \mathcal{H}(\theta)$, $\max_{i,t} |Ey_{it}| \le m < \infty$. Hence,

$$|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| \le |s_{T,ij,t}| + |\bar{y}_{it} - Ey_{it}|\, |\bar{y}_{jt} - Ey_{jt}| + m|\bar{y}_{jt} - Ey_{jt}| + m|\bar{y}_{it} - Ey_{it}|.$$

Therefore,

$$\begin{aligned} P\big(|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| > 4\lambda\big) &\le P\big(|s_{T,ij,t}| > \lambda\big) + P\big(|\bar{y}_{it} - Ey_{it}|\, |\bar{y}_{jt} - Ey_{jt}| > \lambda\big) \\ &\quad + P\big(m|\bar{y}_{jt} - Ey_{jt}| > \lambda\big) + P\big(m|\bar{y}_{it} - Ey_{it}| > \lambda\big). \end{aligned} \tag{B.19}$$

Notice that $\lambda = o(1)$ as $T \to \infty$ by (14). Hence, $\sqrt{\lambda} \ge \lambda$ for $\lambda \le 1$. So,

$$\begin{aligned} & P\big(|\bar{y}_{it} - Ey_{it}|\, |\bar{y}_{jt} - Ey_{jt}| > \lambda\big) \\ &\quad \le P\big(|\bar{y}_{it} - Ey_{it}| > \sqrt{\lambda}\big) + P\big(|\bar{y}_{jt} - Ey_{jt}| > \sqrt{\lambda}\big) \\ &\quad \le P\big(|\bar{y}_{it} - Ey_{it}| > \lambda\big) + P\big(|\bar{y}_{jt} - Ey_{jt}| > \lambda\big). \end{aligned} \tag{B.20}$$

Therefore, to prove (B.17), it suffices to show that uniformly in i, j, as $T \to \infty$,

$$\max_{i,j=1,\dots,p} P\big(|s_{T,ij,t}| > \lambda\big) = o(p^{-2}), \quad \max_{i=1,\dots,p} P\big(|\bar{y}_{it} - Ey_{it}| > \lambda\big) = o(p^{-2}), \tag{B.21}$$

$$\max_{i=1,\dots,p} P\big(m|\bar{y}_{it} - Ey_{it}| > \lambda\big) = o(p^{-2}). \tag{B.22}$$

We will prove (B.21). ((B.22) can be shown using the same argument as in the proof of the second claim in (B.21)). Write

$$\begin{aligned} s_{T,ij,t} &= H^{1/2} K_t^{-1} \Big(H^{-1/2} \sum_{k=1}^{T} b_{H,|t-k|}(z_k - Ez_k) + H^{-1/2} \sum_{k=1}^{T} b_{H,|t-k|}(Ez_k - Ez_t) \Big) \\ &=: H^{1/2} K_t^{-1} (s^*_{T,ij,t} + r_{T,ij,t}), \\ \bar{y}_{it} - Ey_{it} &= H^{1/2} K_t^{-1} \Big(H^{-1/2} \sum_{k=1}^{T} b_{H,|t-k|}(y_{ik} - Ey_{ik}) + H^{-1/2} \sum_{k=1}^{T} b_{H,|t-k|}(Ey_{ik} - Ey_{it}) \Big) \\ &=: H^{1/2} K_t^{-1} (s^*_{T,i,t} + r_{T,i,t}). \end{aligned}$$

Observe that there exists $a_1, a_2 > 0$ such that for all $1 \le t \le T$, $T \ge 1$,

$$a_1 H \le K_t \le a_2 H.$$

Then

$$(K_t/H^{1/2})\lambda \ge a_1 H^{1/2} \lambda =: \eta. \tag{B.23}$$

Therefore

$$\begin{aligned} P\big(|s_{T,ij,t}| > \lambda\big) &\le P\big(|s^*_{T,ij,t} + r_{T,ij,t}| > \eta\big) \le P\big(|s^*_{T,ij,t}| > \eta - |r_{T,ij,t}|\big), \\ P\big(|\bar{y}_{it} - Ey_{it}| > \lambda\big) &\le P\big(|s^*_{T,i,t} + r_{T,i,t}| > \eta\big) \le P\big(|s^*_{T,i,t}| > \eta - |r_{T,i,t}|\big). \end{aligned}$$

First we show that, as $p \to \infty$,

$$|r_{T,ij,t}| \le \eta/2, \quad |r_{T,i,t}| \le \eta/2 \tag{B.24}$$

which implies

$$\begin{aligned} P\big(|s_{T,ij,t}| > \lambda\big) &\le P\big(|s^*_{T,ij,t}| \ge \eta/2\big), \\ P\big(|\bar{y}_{it} - Ey_{it}| > \lambda\big) &\le P\big(|s^*_{T,i,t}| \ge \eta/2\big). \end{aligned} \tag{B.25}$$

To verify (B.24), we use the equality $Ez_t = E[y_{it}y_{jt}] = \text{cov}(y_{it}, y_{jt}) + Ey_{it}Ey_{jt}$ which together with assumption (2) implies that uniformly in i, t, s,

$$\begin{aligned} |Ey_{it}| \le C, \quad |Ey_{it} - Ey_{is}| &\le C\frac{|t-s|}{t \vee s}, \\ |Ez_t| \le C, \quad |Ez_t - Ez_s| &\le C\frac{|t-s|}{t \vee s}. \end{aligned} \tag{B.26}$$

This together with (46) of Lemma 3 and the assumption of the theorem, $\delta T \le t \le T$, yields

$$
\begin{aligned}
|r_{T,ij,t}| &\le C_* \frac{H^{3/2}}{H \vee t} \le C_* \frac{H^{3/2}}{H \vee \delta T} \le C_* \frac{H^{3/2}}{\delta T}, \\
|r_{T,i,t}| &\le C_* \frac{H^{3/2}}{\delta T}
\end{aligned}
\tag{B.27}
$$

because $H = o(T)$ by the assumption (14). Since

$$
\lambda \sqrt{H} \quad \ge \quad \kappa (\log p)^{1/2} H^{3/2}/T,
$$

this implies

$$
\frac{\eta}{|r_{T,ij,t}|} \ge \frac{a_1 \kappa (\log p)^{1/2}}{C_*/\delta} > 2, \quad \frac{\eta}{|r_{T,i,t}|} > 2
$$

when $\kappa(\log p)^{1/2}$ is sufficiently large. This proves (B.24) and (B.25).

By Assumption M, the process $(\boldsymbol{x}_t - E\boldsymbol{x}_t)$ is α-mixing, and therefore $(z_t - Ez_t)$ is also α-mixing with mixing coefficients satisfying (1).

(i) Let $(\boldsymbol{y}_k) \in \mathcal{E}(s)$. Then, $(z_k) \in \mathcal{E}(s/2)$ and $(y_{ik}) \in \mathcal{E}(s/2)$. So, applying (44) of Lemma 2 we obtain

$$
\begin{aligned}
&P\big(|s^*_{T,ij,t}| > \eta/2\big) \le f_H(2, \gamma, c, \eta/2), \quad \gamma = (s/2)(1 + s/2), \\
&P\big(|s^*_{T,i,t}| > \eta/2\big) \le f_H(2, \gamma, c, \eta/2).
\end{aligned}
\tag{B.28}
$$

The function

$$
f_H(\gamma_1, \gamma_2, c, \zeta) \quad \le \quad c_0 \Big\{ \exp(-c_1 \zeta^{\gamma_1}) + \exp\big(- c_2 \big(\frac{\zeta H^{1/2}}{\log^2 H} \big)^{\gamma_2} \Big\}
\tag{B.29}
$$

given in (39) is non-increasing in ζ. By (B.23),

$$
\eta/2 \ge (a_1/2)\kappa (\log p)^{1/2}.
\tag{B.30}
$$

Thus,

$$
\begin{aligned}
&f_H(2, \gamma, c, \eta/2) \\
&\quad \le c_0 \Big\{ \exp\big(- c_1 (a_1/2)^2 \kappa^2 \log p \big) + \exp\big(- c_2 \big((a_1/2)\kappa \log^{1/2} p \frac{H^{1/2}}{\log^2 H} \big)^{\gamma} \big) \Big\} = o(p^{-2})
\end{aligned}
\tag{B.31}
$$

because $c_1(a_1/2)^2\kappa^2 > 2$ when κ is chosen large enough, and by (14), $H \ge c_0 p^{\varepsilon}$, which implies

$$
\log p = o\big(\big(\log^{1/2} p \frac{H^{1/2}}{\log^2 H} \big)^{\gamma} \big).
$$

This together with (B.28) and (B.25) proves (B.21).

(ii) Let $(\boldsymbol{y}_k) \in \mathcal{H}(\theta)$. Then, $(z_k) \in \mathcal{H}(\theta/2)$ and $(y_{ik}) \in \mathcal{H}(\theta/2)$, and using (45) of Lemma 2, we obtain

$$P\big(|s^*_{T,ij,t}| > \eta/2\big) \le g_H\big(2, \theta', c, \eta/2\big), \quad 2 < \theta' < \theta/2, \tag{B.32}$$
$$P\big(|s^*_{T,i,t}| > \eta/2\big) \le g_H\big(2, \theta', c, \eta/2\big).$$

The function

$$g_H(\gamma, \theta', c, \zeta) = c_0\Big\{\exp\big(-c_1\zeta^\gamma\big) + \zeta^{-\theta} t^{-(\theta'/2-1)}\Big\} \tag{B.33}$$

given in (39) is non-increasing in ζ. Again, using the bound $\eta/2 \ge (a_1/2)\kappa(\log p)^{1/2}$, we obtain

$$\begin{aligned} & g_H(2, \theta', c, \eta/2) \\ & \le c_0\Big\{\exp\big(-c_1(a_1/2)^2\kappa^2 \log p\big) + ((a_1/2)\kappa(\log p)^{1/2})^{-\theta'} H^{-(\theta'/2-1)}\Big\} \\ & = o(p^{-2}) \end{aligned} \tag{B.34}$$

because $c_1(a_1/2)^2\kappa^2 > 2$ for large enough κ and because $p^2 = o(H^{\theta'/2-1})$ under the assumption (14) of the theorem if $\theta' \in (2, \theta/2)$ is selected close enough to $\theta/2$, see the proof of (B.15). Clearly, (B.34), (B.32) and (B.25) prove (B.21).

This completes the proof of the claim (15) of theorem.

The claim (16) of the theorem is shown in (B.4) of Lemma B1.

The bandwidth $H_{opt} = T^{2/3}$ minimizes $\max\big(H^{-1/2}, (H/T)\big)$, so

$$\lambda = \kappa(\log p)^{1/2} \max\big(H^{-1/2}, (H/T)\big) \ge \lambda_{opt} = \kappa(\log p)^{1/2} T^{-1/3}$$

which proves the last claim of the theorem. □

Proof of Theorem 4. In this theorem,

$$\lambda = \kappa(\log p)^\nu \max\big(H^{-1/2}, (H/T)^{1/2}\big), \quad \nu = \frac{\alpha+4}{2\alpha}. \tag{B.35}$$

Notice that by (19), $\lambda = o(1)$. As in Theorem 2, to prove the main result (20) of this theorem, it suffices to verify (B.17), i.e. to show that uniformly in i, j, for sufficiently large κ it holds:

$$P\big(|\widehat{\sigma}_{ij,t} - \sigma_{ij,t}| > 4\lambda\big) = o(p^{-2}). \tag{B.36}$$

We will rewrite $\widehat{\sigma}_{ij,t} - \sigma_{ij,t}$ as follows. Observe that

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{x}_t = (y_{1k}, ..., y_{pk})', \quad \text{where } \; y_{ik} = \textstyle\sum_{u=1}^p h_{iu,k} x_{uk},$$
$$\boldsymbol{\Sigma}_t = \boldsymbol{H}_t \boldsymbol{\Sigma}_t^{(x)} \boldsymbol{H}_t' = (\sigma_{ij,t}), \quad \text{where } \; \sigma_{ij,t} = \textstyle\sum_{u,v=1}^p h_{iu,t} h_{ju,t} \sigma_{uv,t}^{(x)}$$

and $y_{ik}y_{jk} = \sum_{u,v=1}^{p} h_{iu,k}h_{jv,k}x_{uk}x_{vk}$. Since $\sigma_{uv,t}^{(x)} = E[x_{ut}x_{vt}] - Ex_{ut}Ex_{vt}$, then

$$\sigma_{ij,t} = \sum_{u,v=1}^{p} h_{iu,t}h_{ju,t}\big(E[x_{ut}x_{vt}] - E[x_{ut}]E[x_{vt}]\big).$$

So,

$$\begin{aligned}
\widehat{\sigma}_{ij,t} - \sigma_{ij,t} &= K_t^{-1}\textstyle\sum_{k=1}^{T} b_{H,|t-k|}y_{ik}y_{jk} - \bar{y}_{it}\bar{y}_{jt} - \sigma_{ij,t} \qquad (B.37)\\
&= \textstyle\sum_{u,v=1}^{p} \pi_{ij,uv,t}, \quad \text{where}\\
\pi_{ij,uv,t} &= \widetilde{s}_{ij,uv,t} - \bar{y}_{iu,t}\bar{y}_{jv,t} + (h_{iu,t}E[x_{ut}])(h_{jv,t}E[x_{vt}]),\\
&\quad \widetilde{s}_{ij,uv,t} = K_t^{-1}\textstyle\sum_{k=1}^{T} b_{H,|t-k|}\big(h_{iu,k}h_{jv,k}x_{uk}x_{vk} - h_{iu,t}h_{jv,t}E[x_{ut}x_{vt}]\big),\\
&\quad \bar{y}_{iu,t} = K_t^{-1}\textstyle\sum_{k=1}^{T} b_{H,|t-k|}h_{iu,k}x_{uk}.
\end{aligned}$$

By assumption of the theorem, the sparsity parameter n_H of $\boldsymbol{H}_t$ is finite and fixed, and does not depend on t, p, T. Therefore, for any fixed (i,j) the sum $\sum_{u,v=1}^{p}[...]$ in (B.37) includes no more than n_H^2 of non-zero terms. Without restriction of generality, assume that

$$\widehat{\sigma}_{ij,t} - \sigma_{ij,t} = \sum_{u,v=1}^{n_H} \pi_{ij,uv,t}.$$

Hence, to verify (B.36), it suffices to show that uniformly in i, j, u, v, for sufficiently large κ it holds:

$$P\big(|\pi_{ij,uv,t}| > 4\lambda'\big) = o(p^{-2}), \quad \lambda' = \lambda/n_H^2. \qquad (B.38)$$

Set $s_{iu,t} = \bar{y}_{iu,t} - h_{iu,t}E[x_{ut}]$, $\quad v_{iu,t} = h_{iu,t}E[x_{ut}]$. Then,

$$\pi_{ij,uv,t} = \widetilde{s}_{ij,uv,t} - (s_{iu,t}s_{jv,t} + v_{iu,t}s_{jv,t} + v_{jv,t}s_{iu,t}).$$

Thus, similarly to (B.19),

$$\begin{aligned}
\pi_{ij,uv,t} \;\le\;& P\big(|\widetilde{s}_{ij,uv,t}| > \lambda'\big) + P\big(|s_{iu,t}s_{jv,t}| > \lambda'\big) \qquad (B.39)\\
&+P\big(|v_{iu,t}s_{jv,t}| > \lambda'\big) + P\big(|v_{jv,t}s_{iu,t}| > \lambda'\big).
\end{aligned}$$

Since $\lambda \to 0$, assume that $\lambda \le 1$. Then, $\lambda' < 1$, and similarly to (B.20),

$$P\big(|s_{iu,t}s_{jv,t}| > \lambda'\big) \le P\big(|s_{iu,t}| > \lambda'\big) + P\big(|s_{jv,t}| > \lambda'\big).$$

Therefore, to prove (B.38), it suffices to show that uniformly in u, v, i, j, as $T \to \infty$, for sufficiently large κ it holds

$$\begin{aligned}
&P\big(|\widetilde{s}_{ij,uv,t}| > \lambda'\big) = o(p^{-2}), \quad P\big(|s_{jv,t}| > \lambda'\big) = o(p^{-2}), \qquad (B.40)\\
&P\big(|v_{iu,t}s_{jv,t}| > \lambda'\big) = o(p^{-2}).
\end{aligned}$$

Let i, j, u, v be fixed. Define $z_k := x_{uk}x_{vk}$, $\widetilde{h}_k := h_{iu,k}h_{jv,k}$. By Assumption M, the process $(\boldsymbol{x}_k - E\boldsymbol{x}_k)$ is α-mixing, and therefore the process $(z_k - Ez_k)$ is also α-mixing with mixing coefficients satisfying (1). Moreover, as in the proof of Theorem 2, Ez_k satisfies (B.26). By Assumption H, $(h_{iu,k})$ satisfies (18) with parameter α and $(\widetilde{h}_k)$ with parameter $\alpha/2$.

We can write

$$\widetilde{s}_{ij,uv,t} = K_t^{-1}H^{1/2}q_{ij,uv,t}, \quad q_{ij,uv,t} = H^{-1/2}\sum_{k=1}^{T} b_{H,|t-k|}\big(\widetilde{h}_k z_k - \widetilde{h}_t E z_t\big),$$
$$s_{iu,t} = K_t^{-1}H^{1/2}q_{iu,t}, \quad q_{iu,t} = H^{-1/2}\sum_{k=1}^{T} b_{H,|t-k|}\big(h_{iu,k}x_{uk} - h_{iu,t}Ex_{ut}\big).$$

This together with (B.23), setting $\eta = a_1 H^{1/2}\lambda'$, implies

$$\begin{aligned} P\big(|\widetilde{s}_{ij,uv,t}| > \lambda'\big) &\leq P\big(|q_{ij,uv,t}| > \eta\big), \\ P\big(|s_{iu,t}| > \lambda'\big) &\leq P\big(|q_{iu,t}| \geq \eta\big), \\ P\big(|v_{iu,t}s_{jv,t}| > \lambda'\big) &\leq P\big(|v_{iu,t}q_{jv,t}| \geq \eta\big). \end{aligned} \tag{B.41}$$

In addition, set $L = b(\log p)^{1/\alpha} > 1$, where $b > 0$ will be selected below. Then,

$$\begin{aligned} P\big(|v_{iu,t}q_{jv,t}| \geq \eta\big) &\leq P\big(|v_{iu,t}| \geq L\big) + P\big(L|q_{jv,t}| \geq \eta\big) \\ &= P\big(|v_{iu,t}| \geq L\big) + P\big(|q_{jv,t}| \geq L^{-1}\eta\big), \\ P\big(|q_{iu,t}| \geq \eta\big) &\leq P\big(|q_{iu,t}| \geq L^{-1}\eta\big). \end{aligned} \tag{B.42}$$

We will show that there exist sufficiently large $b > 0$ and $\kappa > 0$ such that

$$P\big(|v_{iu,t}| \geq L\big) = o(p^{-2}), \tag{B.43}$$
$$P\big(|q_{ij,uv,t}| \geq \eta\big) = o(p^{-2}), \quad P\big(|q_{jv,t}| \geq \eta/L\big) = o(p^{-2}) \tag{B.44}$$

which together with (B.42), (B.41) implies (B.40) which completes the proof of (B.36).

Proof of (B.43). By assumption, $(\boldsymbol{x}_t) \in \mathcal{E}(s)$ or $(\boldsymbol{x}_t) \in \mathcal{H}(\theta)$ which implies $\max_{i,t}|Ex_{it}| \leq m < \infty$. Therefore, $|v_{iu,t}| = |h_{iu,t}E[x_{ut}]| \leq m|h_{iu,t}|$. By Assumption H, $(h_{iu,t}) \in \mathcal{E}(\alpha)$. Therefore, $(v_{iu,t}) \in \mathcal{E}(\alpha)$ which implies that for some $c_0, c_1 \geq 0$,

$$P\big(|v_{iu,t}| \geq \zeta\big) \leq c_0 \exp(-c_1|\zeta|^{\alpha}), \quad \zeta > 0.$$

Using this bound with $\zeta = L = b(\log p)^{1/\alpha}$, we obtain

$$P\big(|v_{iu,t}| \geq L\big) \leq c_0 \exp(-c_1 b^{\alpha}\log p) = o(p^{-2})$$

when b is selected such that $c_1 b^{\alpha} > 2$. This proves (B.43).

Proof of (B.44).

(i) Let $(x_{ik}) \in \mathcal{E}(s)$. Recall that $q_{ij,uv,t}$ is a weighted sum of variables $\widetilde{h}_k z_k$, and by the assumptions of the theorem, $(h_{iu,k}) \in \mathcal{E}(\alpha)$. Thus, $(\widetilde{h}_k) \in \mathcal{E}(\alpha/2)$ and $(z_k) \in \mathcal{E}(s/2)$. On the other hand, $q_{iu,t}$ is a weighted sum of variables $h_{iu,k}x_{uk}$, where $(h_{iu,k}) \in \mathcal{E}(\alpha)$ and $(x_{uk}) \in \mathcal{E}(s)$. Hence, by the claim (58) of Lemma 5,

$$P\big(|q_{ij,uv,t}| > \eta\big) \le f_H\big(\gamma_1, \gamma_2, c, \eta(1 \wedge d_{Ht})\big), \qquad \text{(B.45)}$$
$$P\big(|q_{iu,t}| \ge \eta/L\big) \le f_H\big(\gamma_1', \gamma_2', c, (\eta/L)(1 \wedge d_{Ht})\big),$$

where

$$\gamma_1 = \frac{2(\alpha/2)}{\alpha/2+2} = \frac{2\alpha}{\alpha+4}, \quad \gamma_2 = \frac{(\alpha/2)(s/2)}{\alpha/2+s/2+1} = \frac{\alpha s}{2\alpha+2s+4}, \qquad \text{(B.46)}$$
$$\gamma_1' = \frac{2\alpha}{\alpha+2}, \quad \gamma_2' = \frac{\alpha s}{\alpha+s+1}.$$

By assumption of the theorem, $\delta T \le t \le T$. We will show below that

$$\eta(1 \wedge d_{Ht}) \ge a_\delta \kappa (\log p)^{1/\gamma_1}, \quad a_\delta = \delta^{1/2}(a_1/n_H^2), \qquad \text{(B.47)}$$
$$(\eta/L)(1 \wedge d_{Ht}) \ge a_\delta' \kappa (\log p)^{1/\gamma_1'}, \quad a_\delta' = b^{-1}\delta^{1/2}(a_1/n_H^2).$$

The function $f_H(\gamma_1, \gamma_2, c, \zeta)$, see (B.29), is non-increasing in ζ. So,

$$f_H\big(\gamma_1, \gamma_2, c, \eta(1 \wedge d_{Ht})\big) \le f_H(\gamma_1, \gamma_2, c, a_\delta \kappa (\log p)^{1/\gamma_1}),$$
$$f_H\big(\gamma_1', \gamma_2', c, (\eta/L)(1 \wedge d_{Ht})\big) \le f_H(\gamma_1', \gamma_2', c, a_\delta' \kappa (\log p)^{1/\gamma_1'}).$$

Notice that,

$$f_H(\gamma_1, \gamma_2, c, a_\delta \kappa (\log p)^{1/\gamma_1})$$
$$\le c_0 \Big\{ \exp\big(-c_1 (a_\delta \kappa)^{\gamma_1} \log p\big) + \exp\Big(-c_2 \big(a_\delta \kappa (\log p)^{1/\gamma_1} \frac{H^{1/2}}{\log^2 H}\big)^{\gamma_2}\Big) \Big\} = o(p^{-2})$$

because $c_1(a_\delta \kappa)^{\gamma_1} > 2$ when κ is selected sufficiently large, and because by the assumption (19), $H \ge c_0 p^{\varepsilon}$, which implies

$$\log p = o\Big(\big(\log^{1/\gamma_1} p \frac{H^{1/2}}{\log^2 H}\big)^{\gamma_2}\Big).$$

The same argument implies, that for sufficiently large κ,

$$f_H(\gamma_1', \gamma_2', c, a_\delta' \kappa (\log p)^{1/\gamma_1'}) = o(p^{-2}).$$

Together with (B.45) and (B.41) this proves (B.40).

Proof of (B.47). Notice that $\nu = (\alpha + 4)/(2\alpha)$ in (B.35) has property:

$$\nu\gamma_1 = 1, \quad (\nu - \alpha^{-1})\gamma_1' = 1. \tag{B.48}$$

By definition (57), $d_{Ht} = (t \vee H)^{1/2} H^{-1}$. By assumption, $\delta T \le t \le T$ and $H = o(T)$. Therefore,

$$\begin{aligned} d_{Ht} &\ge (\delta T \wedge H)^{1/2} H^{-1} \ge (\delta T)^{1/2} H^{-1}, \\ 1 \wedge d_{Ht} &\ge \delta^{1/2}(1 \wedge T^{1/2} H^{-1}). \end{aligned}$$

Since for any $e > 0$, $(1 \vee e)(1 \wedge e^{-1}) = 1$, we obtain

$$\begin{aligned} \eta(1 \wedge d_{Ht}) &= (a_1/n_H^2)\kappa(\log p)^\nu \{(H^{-1/2} \vee (H/T)^{1/2})H^{1/2}\}(1 \wedge d_{Ht}) \\ &\ge (a_1/n_H^2)\kappa(\log p)^\nu \delta^{1/2}(1 \vee HT^{-1/2})(1 \wedge T^{1/2}H^{-1}) \\ &= (a_1/n_H^2)\delta^{1/2}\kappa(\log p)^\nu = a_\delta \kappa (\log p)^\nu. \end{aligned}$$

Since by (B.48), $\nu = 1/\gamma_1$ this proves the first claim in (B.47).

On the other hand, $L^{-1} = b^{-1}(\log p)^{-1/\alpha}$, and therefore,

$$(\eta/L)(1 \wedge d_{Ht}) \ge b^{-1}(a_1/n_H^2)\delta^{1/2}\kappa(\log p)^{\nu - 1/\alpha} = a_\delta' \kappa (\log p)^{1/\gamma_1'}$$

by (B.48) which completes the proof of (B.47).

(ii) Let $(x_{it}) \in \mathcal{H}(\theta)$. Then $q_{ij,uv,t}$ is a weighted sum of variables $\widetilde{h}_k z_k$ where $(\widetilde{h}_k) \in \mathcal{E}(\alpha/2)$ and $(z_k) \in \mathcal{H}(\theta/2)$. In turn, $q_{iu,t}$ is a weighted sum of variables $h_{iu,k} x_{uk}$ where $(h_{iu,k}) \in \mathcal{E}(\alpha)$ and $(x_{uk}) \in \mathcal{H}(\theta)$. Thus, by the claim (59) of Lemma 5,

$$\begin{aligned} P\big(|q_{ij,uv,t}| > \eta\big) &\le g_H\big(\gamma_1, \theta', c, \eta(1 \wedge d_{Ht})\big), \quad \theta' \in (2, \theta/2), \\ P\big(|q_{iu,t}| \ge \eta/L\big) &\le g_H\big(\gamma_1', \theta', c, (\eta/L)(1 \wedge d_{Ht})\big), \end{aligned} \tag{B.49}$$

where γ_1 and γ_1' are the same as in (B.46). Since $g_H(\gamma_1, \gamma_2, c, \zeta)$, (B.33), is a non-increasing function in ζ, by (B.47) we can bound

$$\begin{aligned} &g_H\big(\gamma_1, \theta', c, \eta(1 \wedge d_{Ht})\big) \le g_H\big(\gamma_1, \theta', c, a_\delta\kappa(\log p)^{1/\gamma_1}\big), \\ &g_H\big(\gamma_1', \theta', c, (\eta/L)(1 \wedge d_{Ht})\big) \le g_H\big(\gamma_1', \theta', c, a_\delta'\kappa(\log p)^{1/\gamma_1'}\big). \end{aligned}$$

Notice that

$$\begin{aligned} &g_H\big(\gamma_1, \theta', c, a_\delta\kappa(\log p)^{1/\gamma_1}\big) \\ &\quad \le c_0\Big\{\exp\big(-c_1(a_\delta\kappa)^{\gamma_1}\log p\big) + \frac{1}{(a_\delta\kappa(\log p)^{1/\gamma_1})^{\theta'}}\frac{1}{H^{\theta'/2-1}}\Big\} = o(p^{-2}) \end{aligned}$$

when κ is selected such that $c_1(a_\delta\kappa)^{\gamma_1} > 2$, and $\theta' \in (2, \theta/2)$ is selected close enough to $\theta/2$, see the proof of (B.34). Similarly, it can be shown that for sufficiently large κ,

$$g_H\big(\gamma_1', \theta', c, a_\delta'\kappa(\log p)^{1/\gamma_1'}\big) = o(p^{-2}).$$

Together with (B.49) this implies (B.44). This completes the proof of the claim (20) of Theorem 4.

The claim (21) of the theorem is shown in (B.4) of Lemma B1.

The bandwidth $H_{opt} = T^{1/2}$ minimizes $\max\big(H^{-1/2}, (H/T)^{1/2}\big)$ which implies

$$\lambda = \kappa(\log p)^\nu \max\big(H^{-1/2}, (H/T)^{1/2}\big) \geq \lambda_{opt} = \kappa(\log p)^\nu T^{-1/4}$$

which proves the last claim of the theorem. □

C. Auxiliary results

This section contains auxiliary results used in the proofs.

Recall definition of functions f_t and g_t, (39).

Lemma C1 *(i) Let $x \in \mathcal{E}(\alpha)$, $y \in \mathcal{E}(\alpha')$ where $\alpha > 0$, $\alpha' > 0$. Then $xy \in \mathcal{E}(\widetilde{\alpha})$ where $\widetilde{\alpha} = \alpha\alpha'/(\alpha + \alpha')$.*

Moreover, $x + y \in \mathcal{E}(\min(\alpha, \alpha'))$ and $|z| \leq |x|$ implies $z \in \mathcal{E}(\alpha)$.

(ii) Let $x \in \mathcal{E}(\alpha)$, $y \in \mathcal{H}(\theta)$ where $\alpha > 0$, $\theta > 0$. Then $xy \in \mathcal{H}\big(\theta'\big)$ for any $0 < \theta' < \theta$.

(iii) Let $(x_t) \in \mathcal{E}(\alpha)$, $\alpha > 0$ and $P(|y_t| \geq \zeta) \leq f_t(\gamma_1, \gamma_2, c, \zeta)$, $\zeta > 0$, $t \geq 2$ with $\gamma_1, \gamma_2 > 0$. Then

$$P(|x_t y_t| \geq \zeta) \leq f_t(\widetilde{\gamma}_1, \widetilde{\gamma}_2, c', \zeta), \quad \zeta > 0, \quad t \geq 2, \tag{C.1}$$

where $\widetilde{\gamma}_1 = \alpha\gamma_1/(\alpha + \gamma_1)$, $\widetilde{\gamma}_2 = \alpha\gamma_2/(\alpha + \gamma_2)$ and c' does not depend on t, ζ.

(iv) Let $(x_t) \in \mathcal{E}(\alpha)$, $\alpha > 0$ and $P(|y_t| \geq \zeta) \leq g_t(\gamma, \theta, c, \zeta)$, $\zeta > 0$, $t \geq 2$ where $\gamma > 0$, $\theta > 2$. Then for any $\theta' \in (2,\, \theta)$,

$$P(|x_t y_t| \geq \zeta) \leq g_t(\widetilde{\gamma}, \theta', c', \zeta), \quad \zeta > 0, \quad t \geq 2, \tag{C.2}$$

where $\widetilde{\gamma} = \alpha\gamma/(\alpha + \gamma)$ and c' does not depend on t, ζ.

(v) If $(x_t) \in \mathcal{E}(\alpha)$, $(x_{tk}) \in \mathcal{E}(\alpha)$ for some $\alpha > 0$ then as $T \to \infty$,

$$\max_{1 \leq t \leq T} |x_t| = O_P\big((\log T)^{1/\alpha}\big), \qquad \max_{1 \leq t,k \leq T} |x_{tk}| = O_P\big((\log T)^{1/\alpha}\big). \tag{C.3}$$

Proof.

(i) Let $x \in \mathcal{E}(\alpha)$, $y \in \mathcal{E}(\alpha')$ where $\alpha > 0,\ \alpha' > 0$ and let $\widetilde{\alpha} = \alpha\alpha'/(\alpha + \alpha')$. Then for some $a > 0$,

$$E\exp(a|x|^{\alpha}) < \infty, \quad E\exp(a|y|^{\alpha'}) < \infty.$$

To prove (i), we will show that $E\exp(a|xy|^{\widetilde{\alpha}}) < \infty$.

Set $p = (\alpha + \alpha')/\alpha'$, $q = (\alpha + \alpha')/\alpha$. Then $p > 1, q > 1,\ 1/p + 1/q = 1$ and $\widetilde{\alpha}p = \alpha$, $\widetilde{\alpha}q = \alpha'$. Hence, for $k = 1, 2, ...$ by Hölder's inequality,

$$\begin{aligned} E|xy|^{\widetilde{\alpha}k} &= E[|x|^{\widetilde{\alpha}k}|y|^{\widetilde{\alpha}k}] \le (E|x|^{k\,\widetilde{\alpha}p})^{1/p}(E|y|^{k\,\widetilde{\alpha}q})^{1/q} = (E|x|^{k\,\alpha})^{1/p}(E|y|^{k\,\alpha'})^{1/q} \\ &\le (\max(E|x|^{k\,\alpha}, E|y|^{k\,\alpha'})^{1/p+1/q} = \max(E|x|^{k\,\alpha}, E|y|^{k\,\alpha'}) \\ &\le E|x|^{k\,\alpha} + E|y|^{k\,\alpha'}. \end{aligned}$$

Therefore,

$$\begin{aligned} E\exp(a|xy|^{\widetilde{\alpha}}) &\le \sum_{k=0}^{\infty} \frac{a^k E|xy|^{\widetilde{\alpha}k}}{k!} \le \sum_{k=0}^{\infty} \frac{a^k(E|x|^{k\,\alpha} + E|y|^{k\,\alpha'})}{k!} \\ &\le E\exp(a|x|^{\alpha}) + E\exp(a|y|^{\alpha'}) < \infty. \end{aligned}$$

(ii) Let $x \in \mathcal{E}(\alpha)$, $y \in \mathcal{H}(\theta)$ where $\alpha > 0,\ \theta > 0$. Then, for some $a > 0$,

$$E\exp(a|x|^{\alpha}) < \infty, \quad E|y|^{\theta} < \infty.$$

The latter implies that $E|x|^b < \infty$ for any $b > 0$.

Let $\theta' \in (0, \theta)$. To prove (ii), we will show that $E|xy|^{\theta'} < \infty$. Set $p = \theta/\theta'$ and let $q > 1$ be defined by equality $1/p + 1/q = 1$. Then, by Hölder inequality,

$$E|xy|^{\theta'} \le (E|x|^{\theta' q})^{1/q}(E|y|^{\theta' p})^{1/p} = (E|x|^{\theta' q})^{1/q}(E|y|^{\theta})^{1/p} < \infty.$$

This completes the proof of (ii).

Before proceeding to the proof of (iii)-(iv), we obtain the following two auxiliary results. First, consider the function

$$f(x) = x^{\alpha} + c(v/x)^{\alpha'}, \quad x > 0$$

where $\alpha > 0, \alpha' > 0$, $v > 0$, $c > 0$. It achieves its unique minimum at

$$x_0 = (c\alpha'/\alpha)^{1/(\alpha+\alpha')} v^{\alpha'/(\alpha+\alpha')}$$

because x_0 is a unique solution of equation $f'(x) = \alpha x^{\alpha-1} - c\alpha'(v/x)^{\alpha'} x^{-1} = 0$ and $f''(x_0) = x_0^{\alpha-2}\alpha(\alpha + \alpha') > 0$. Thus,

$$f(x) \ge f(x_0) = c' v^{\widetilde{\alpha}}, \quad x \ge 0 \tag{C.4}$$

where $\widetilde{\alpha} = \alpha\alpha'/(\alpha+\alpha')$ and $c' = (c\alpha'/\alpha)^{\alpha/(\alpha+\alpha')}(1+\alpha/\alpha')$.

Second, we obtain the upper bound for $P(|xy| \geq \zeta)$ for the product of r.v.'s x and y when $x \in \mathcal{E}(\alpha)$, $\alpha > 0$. Let $p, q > 1$, $1/p + 1/q = 1$. Then

$$\begin{aligned} P\big(|xy| \geq \zeta\big) &= \textstyle\sum_{k=0}^{\infty} P\big(\{|x| \in [k, k+1)\} \cap \{|xy| \geq \zeta\}\big) \\ &\leq \textstyle\sum_{k=0}^{\infty} P^{1/p}\big(|x| \in [k, k+1)\big) P^{1/q}\big(|y| \geq \zeta/(k+1)\big). \end{aligned}$$

Since $x \in \mathcal{E}(\alpha)$, then for $k \geq 0$,

$$P\big(|x| \in [k, k+1)\big) \leq P\big(|x| \geq k\big) \leq c_0' \exp(-2c_1' k^{\alpha}), \quad k \geq 0$$

for some $c_0' > 0$, $c_1' > 0$. Denote

$$g_{k\zeta} := \exp(-c_1' k^{\alpha}) P^{1/q}(|y| \geq \zeta/k).$$

Then,

$$\begin{aligned} P\big(|xy| \geq \zeta\big) &\leq C \textstyle\sum_{k=0}^{\infty} \exp\{-2c_1' k^{\alpha} + c_1'(k+1)^{\alpha}\} g_{k+1,\zeta} \\ &\leq C \max_{k \geq 1} g_{k\zeta} \textstyle\sum_{k=0}^{\infty} \{-2c_1' k^{\alpha} + c_1'(k+1)^{\alpha}\} \\ &\leq C \max_{k \geq 1} g_{k\zeta}. \end{aligned} \tag{C.5}$$

We use this result to evaluate $P(|xy| \geq \zeta)$ in parts (iii)-(iv) of the lemma.

(iii) Without restriction of generality, we assume that $\zeta \geq 1$. By (C.5),

$$P(|x_t y_t| \geq \zeta) \leq C \max_{k \geq 1} g_{k\zeta}. \tag{C.6}$$

Under assumptions of (iii), $g_{k\zeta} = \exp(-c_1' k^{\alpha}) f_t^{1/q}(2, \gamma, c, \zeta/k)$. To evaluate $f_t^{1/q}(2, \gamma, c, \zeta/k)$, denote $\zeta_t = \zeta\sqrt{t}/\log^2 t$. Using the definition of function f_t, (39), and inequality

$$(a+b)^{1/q} \leq a^{1/q} + b^{1/q}, \qquad a,\, b > 0, \tag{C.7}$$

we obtain

$$\begin{aligned} f_t^{1/q}(\gamma_1, \gamma_2, c, \zeta/k) &\leq C\Big(\exp\big(-c_1(\zeta/k)^{\gamma_1}\big) + \exp\big(-c_2(\zeta_t/k)^{\gamma_2}\big)\Big)^{1/q} \\ &\leq C\Big(\exp\big(-(c_1/q)(\zeta/k)^{\gamma_1}\big) + \exp\big(-(c_2/q)(\zeta_t/k)^{\gamma_2}\big)\Big). \end{aligned}$$

Hence, there exist constants c_1'', $c_2'' > 0$ such that

$$g_{k,\zeta} \leq C\{\exp(-c_1''(k^{\alpha} + (\zeta/k)^{\gamma_1})) + \exp(-c_2''(k^{\alpha} + (\zeta_t/k)^{\gamma_2}))\}.$$

Next, using (C.4) to bound $f(k) := k^\alpha + (\zeta/k)^{\gamma_1}$, $f(k) := k^\alpha + (\zeta_t/k)^{\gamma_2}$ from below, we obtain

$$g_{k,\zeta} \le c_0^*\big(\exp(-c_1^*\zeta^{\widetilde{\gamma}_1}) + \exp(-c_2^*\zeta_t^{\widetilde{\gamma}_2})\big) = f_t(\widetilde{\gamma}_1, \widetilde{\gamma}_2, c^*, \zeta), \quad k \ge 1,$$

with $\widetilde{\gamma}_1 = \alpha\gamma_1/(\alpha+\gamma_1)$, $\widetilde{\gamma}_2 = \alpha\gamma_2/(\alpha+\gamma_2)$. Thus, (C.6) implies

$$P\big(|x_t y_t| \ge \zeta\big) \le f_t(\widetilde{\gamma}_1, \widetilde{\gamma}_2, c^{''}, \zeta)$$

which proves (iii).

(iv) Let $\zeta \ge 1$. Under assumptions of (iv), (C.6) holds with

$$g_{k\zeta} = \exp(-c_1' k^\alpha) g_t^{1/q}(\gamma, \theta, c, \zeta/k).$$

Next we evaluate $g_t^{1/q}(\gamma, \theta, c, \zeta/k)$. Let $2 < \theta' < \theta$. Then, $\theta/\theta' > 1$ and $(\theta-2)/(\theta'-2) > 1$. Let $q > 1$ be such that $\min(\theta/\theta', (\theta-2)/(\theta'-2)) > q$. By (C.7) and definition of g_t, (39),

$$\begin{aligned} g_t^{1/q}(\gamma, \theta', c, \zeta/k) &\le C\Big(\exp\{-c_1(\zeta/k)^\gamma\} + (\zeta/k)^{-\theta} t^{-(\theta/2-1)}\Big)^{1/q} \\ &\le C\{\exp\{-(c_1/q)(\zeta/k)^\gamma\} + \zeta^{-\theta/q} t^{-(\theta/2-1)/q} k^{\theta/q}\}. \end{aligned} \qquad \text{(C.8)}$$

Definition of $q > 1$ implies $\theta/q > \theta'$ and $(\theta/2-1)/q > \theta'/2 - 1$. This together with (C.8) yields

$$g_t^{1/q}(\gamma, \theta', c, \zeta/k) \le C\big(\exp\{-(c_1/q)(\zeta/k)^\gamma\} + \zeta^{-\theta'} t^{-(\theta'/2-1)} k^{\theta/q}\big), \quad \zeta \ge 1, t \ge 1.$$

Hence,

$$\begin{aligned} \max_{k\ge1} g_{k\zeta} &\le C \max_{k\ge1} \exp\{-c_1''(k^\alpha + (\zeta/k)^\gamma)\} + C\zeta^{-\theta'} t^{-(\theta'/2-1)} \max_{k\ge1}\{\exp(-c_2' k^\alpha) k^{\theta/q}\} \\ &\le C\big(\max_{k\ge1} \exp\{-c_1''(k^\alpha + (\zeta/k)^\gamma)\} + \zeta^{-\theta'} t^{-(\theta'/2-1)}\big). \end{aligned}$$

Applying to $f(k) := k^\alpha + (\zeta/k)^\gamma$ the bound (C.4), we obtain

$$\max_{k\ge1} g_{k\zeta} \le c_0^*\Big(\exp(-c_1^*\zeta^{\widetilde{\gamma}}) + \zeta^{-\theta'} t^{-(\theta'/2-1)}\Big) = g_t(\widetilde{\gamma}, \theta', c^*, \zeta)$$

with $\widetilde{\gamma} = \alpha\gamma/(\alpha+\gamma)$. Then (C.6) implies $P(|x_t y_t| \ge \zeta) \le g_t(\widetilde{\gamma}, \theta', c^*, \zeta)$ which proves (iv).

(v) We need to show that, as $T \to \infty$, $b \to \infty$,

$$P\big(\max_{t=1,\dots,T} |x_t| \ge b(\log T)^{1/\alpha}\big) \to 0, \quad P\big(\max_{t,k=1,\dots,T} |x_{tk}| \ge b(\log T)^{1/\alpha}\big) \to 0.$$

By assumption, there exist $a > 0$ and $\alpha > 0$ such that

$$\max_{t\ge1} E\exp(a|x_t|^\alpha) < \infty, \quad \max_{t,k\ge1} E\exp(a|x_{tk}|^\alpha) < \infty.$$

Let b be such that $ab^\alpha \geq 2$. Then, as $T \to \infty$,

$$P\big(\max_{t=1,\dots,T} |x_t| \geq b(\log T)^{1/\alpha}\big) \leq \sum_{t=1}^{T} P\big(|x_t| \geq b(\log T)^{1/\alpha}\big)$$
$$\leq \sum_{t=1}^{T} \frac{E(\exp(a|x_t|^\alpha)}{\exp(ab^\alpha \log T)} \leq T^{-2} \sum_{t=1}^{T} C \to 0.$$

Similarly,

$$P\big(\max_{t,k=1,\dots,T} |x_{tk}| \geq b(\log T)^{1/\alpha}\big) \leq \sum_{t,k=1}^{T} P\big(|x_{tk}| \geq b(\log T)^{1/\alpha}\big)$$
$$\leq \sum_{t,k=1}^{T} \frac{E(\exp(a|x_{tk}|^\alpha)}{\exp(ab^\alpha \log T)} \leq T^{-2} \sum_{t,k=1}^{T} C \to 0.$$

This completes the proof of (v) and the lemma. □

Lemma C2 *Let $\gamma > 0$.*
(i) Let ξ be a zero mean random variable. Then for all $\zeta > 0$,

$$E\big(|\xi|^\gamma I(|\xi| > \zeta)\big) \leq \begin{cases} c_0 \exp(-c_1 \zeta^s) & \text{if } \xi \in \mathcal{E}(s),\ s > 0 \quad \text{(C.9)} \\ c_0 \zeta^{\gamma-\theta} & \text{if } \xi \in \mathcal{H}(\theta),\ \gamma < \theta \quad \text{(C.10)} \end{cases}$$

for some $c_0 > 0$, $c_1 > 0$ which do not depend on ζ.
(ii) Let s_t, $t \geq 1$ be zero mean random variables such that for some $\gamma_1 > 0$, $\gamma_2 > 0$ and c,

$$P\big(|s_t| \geq \zeta\big) \leq f_t(\gamma_1, \gamma_2, c, \zeta) \quad \text{for all } \zeta > 0,\ t \geq 2.$$

Then,

$$E\big[|s_t|^\gamma I(|s_t| > \zeta)\big] \leq f_t(\gamma_1, \gamma_2, c', \zeta),\ \ \zeta > 0,\ \ t \geq 2, \tag{C.11}$$

where c' does not depend on ζ, t.

(iii) Let s_t, $t \geq 1$ be zero mean random variables such that for some $\theta > 0$, $\gamma_1 > 0$ and c,

$$P\big(|s_t| \geq \zeta\big) \leq g_t(\gamma_1, \theta, c, \zeta) \quad \text{for all } \zeta > 0,\ t \geq 2.$$

Then, for $0 < \gamma < \theta$,

$$E\big[|s_t|^\gamma I(|s_t| > \zeta)\big] \leq \max(\zeta^\gamma, 1) g_t(\gamma_1, \theta, c', \zeta),\ \ \zeta > 0,\ \ t \geq 2, \tag{C.12}$$

where c' does not depend on ζ, t.

Proof. Without restriction of generality let $\zeta \geq 1$. Denote $F(x) = P(|\xi| \geq x)$. Then

$$E\big[|\xi|^{\gamma} I(|\xi| > \zeta)\big] = -\int_{\zeta}^{\infty} x^{\gamma} dF(x) = -\zeta^{\gamma} F(\zeta) + \int_{\zeta}^{\infty} x^{\gamma-1} F(x)dx. \tag{C.13}$$

(i) If $(\xi_k) \in \mathcal{E}(s)$, then $F(x) \leq c_0' \exp\big(-2c_1'|x|^s\big)$ for some $c_0', c_1' > 0$. Notice that $F(x) \geq F(\zeta)$, $x \geq \zeta$. Applying these bounds in (C.13), we obtain (C.9):

$$\begin{aligned} E\big[|\xi|^{\gamma} I(|\xi| > \zeta)\big] &\leq F^{1/2}(\zeta)\big\{\zeta^{\gamma} F^{1/2}(\zeta) + \textstyle\int_{1}^{\infty} x^{\gamma-1} F^{1/2}(x)dx\big\} \\ &\leq CF^{1/2}(\zeta) \leq C\exp(-c_1'\zeta^s). \end{aligned}$$

If $(\xi) \in \mathcal{H}(\theta)$, then $F(x) \leq c_0'|x|^{-\theta}$. Using this bound in (C.13), we obtain (C.10):

$$E[|\xi|^{\gamma} I(|\xi| > \zeta)] \leq C\big\{\zeta^{\gamma}|\zeta|^{-\theta} + \int_{\zeta}^{\infty} x^{\gamma-1} x^{-\theta} dx\big\} \leq C\zeta^{\gamma-\theta}.$$

(ii) Let again $\zeta \geq 1$. Denote $F_t^*(x) = P(|S_t| \geq x)$. Then as in (C.13),

$$\begin{aligned} E\big[|s_t|^{\gamma} I(|s_t| > \zeta)\big] &= -\int_{\zeta}^{\infty} x^{\gamma} dF_t^*(x) = -\zeta^{\gamma} F_t^*(\zeta) + \int_{\zeta}^{\infty} x^{\gamma-1} F_t^*(x)dx, \qquad \text{(C.14)} \\ E\big[|s_t|^{\gamma} I(|s_t| > \zeta)\big] &\leq F_t^{*1/2}(\zeta)\big\{\zeta^{\gamma} F_t^{*\,1/2}(\zeta) + \int_{1}^{\infty} x^{\gamma-1} F_t^{*\,1/2}(x)dx\big\}. \end{aligned}$$

By assumption, $P(|s_t| \geq \zeta) \leq f_t(\gamma_1, \theta, c, \zeta)$. Definition (39) of f_t implies that

$$f_t(\gamma_1, \gamma_2, c, \zeta) \leq c_0 \exp\big(-2c_1\zeta^{\min(\gamma_1,\gamma_2)}\big), \quad \zeta > 0,\ t \geq 2$$

for some $c_0, c_1 > 0$. Thus, by (C.14), for $\zeta \geq 1$

$$\begin{aligned} E\big[|s_t|^{\gamma} I(|s_t| > \zeta)\big] &\leq f_t^{1/2}(\gamma_1, \gamma_2, c, \zeta)\big(\zeta^{\gamma} f_t^{1/2}(\gamma_1, \gamma_2, c, \zeta) + \textstyle\int_{1}^{\infty} x^{\gamma-1} f_t^{1/2}(\gamma_1, \gamma_2, c, x)dx\big) \\ &\leq Cf_t^{1/2}(\gamma_1, \gamma_2, c, \zeta) \leq Cf_t(\gamma_1, \gamma_2, c', \zeta) \end{aligned}$$

for some c' in view of (C.7). This proves (C.11).

(iii) Let $\zeta \geq 1$. Since $P(|s_t| \geq \zeta) \leq g_t(\gamma_1, \theta, c, \zeta)$, (C.14) implies

$$E[|s_t|^{\gamma} I(|s_t| > \zeta)] \leq \zeta^{\gamma} g_t(\gamma_1, \theta, c, \zeta) + \int_{\zeta}^{\infty} x^{\gamma-1} g_t(\gamma_1, \theta, c, x)dx. \tag{C.15}$$

By definition (39), $g_t(\gamma_1, \theta, c, \zeta) \leq c_0\{\exp(-2c_1\zeta^{\gamma_1}) + \zeta^{-\theta} t^{-(\theta/2-1)}\}$ for some $c_0, c_1 > 0$. Thus,

$$\begin{aligned} &\int_{\zeta}^{\infty} x^{\gamma-1} g_t(\gamma_1, \theta, c, x)dx \\ &\qquad \leq C\Big(\exp(-c_1\zeta^{\gamma_1}) \int_{\zeta}^{\infty} x^{\gamma-1} \exp(-c_1 x^{\gamma_1})dx + \int_{\zeta}^{\infty} x^{\gamma-\theta-1} t^{-(\theta/2-1)} dx\Big) \\ &\qquad \leq C\zeta^{\gamma}\Big(\exp(-c_1\zeta^{\gamma_1}) + \zeta^{-\theta} t^{-(\theta/2-1)}\Big) = \zeta^{\gamma} g_t(\gamma_1, \theta, c', \zeta) \end{aligned}$$

for some c'. This together with (C.15) proves (C.12). $\Box$

Lemma C3 *Let $b_{H,k}$ satisfy (43) with $\nu > 3$ and $0 \le \gamma < 2$. Then for $1 \le t, H \le T$, $T \ge 1$,*

$$H^{-1}\sum_{k=1}^{T} b_{H,|t-k|}\Big(\frac{|t-k|}{t\vee k}\Big)^{\gamma} \le C\Big(\frac{H}{H\vee t}\Big)^{\gamma}, \tag{C.16}$$

$$H^{-1}\sum_{k=1}^{T} b_{H,|t-k|}\Big(\frac{|t-k|\vee H}{k}\Big)^{1/2} \le C\Big(\frac{H}{H\vee t}\Big)^{1/2}, \tag{C.17}$$

where $C > 0$ does not depend on t, T, H.

Proof. Notice that

$$\Big(\frac{H}{H\vee t}\Big)^{\gamma} = \Big(\Big(\frac{H}{t}\Big)^{\gamma}\wedge 1\Big) = \Big(\frac{H\wedge t}{t}\Big)^{\gamma}. \tag{C.18}$$

By (43), $b_{H,k} \le C(1+(k/H)^{\nu})^{-1}$ for $k \ge 0$ where $\nu \ge 3$. Therefore, for $0 \le \gamma < 2$,

$$H^{-1}\sum_{k=1}^{T} b_{H,k}(k/H)^{\gamma} \le C, \quad \max_{k\ge 1} b_{H,k}(k/H)^{\gamma} \le C, \tag{C.19}$$

where C does not depend on H, T.

Denote by $I_{\gamma,H}$ the l.h.s. of (C.16). Then, by (C.19), noting that $t \vee k \ge t$,

$$I_{\gamma,H} = H^{-1}\sum_{k=1}^{T} b_{H,|t-k|}\Big(\frac{|t-k|}{t\vee k}\Big)^{\gamma} = H^{-1}\sum_{k=1}^{T} b_{H,|t-k|}\Big(\frac{|t-k|}{H}\Big)^{\gamma}\Big(\frac{H}{t\vee k}\Big)^{\gamma} \le C\Big(\frac{H}{t}\Big)^{\gamma}.$$

On the other hand, since $|t-k|/(t\vee k) \le 1$, using (C.19) we obtain

$$I_{\gamma,H} \le H^{-1}\textstyle\sum_{k=1}^{T} b_{H,|t-k|} \le C$$

which together with (C.18) proves (C.16).

To prove (C.17), denote by I_H the l.h.s. of (C.17). Write

$$I_H = H^{-1}\sum_{k=t/2+1}^{T}[...] + H^{-1}\sum_{k=1}^{t/2}[...] =: I_{H;1} + I_{H;2}.$$

Then,

$$\begin{aligned} I_{H;1} &\le \Big[H^{-1}\sum_{k=t/2+1}^{T} b_{T,|t-k|}\Big(\frac{|t-k|\vee H}{H}\Big)^{1/2}\Big(\frac{H}{k}\Big)^{1/2}\Big] \\ &\le C\Big(\frac{H}{t}\Big)^{1/2}\Big[H^{-1}\sum_{k=1}^{T} b_{T,|t-k|}\Big(\frac{|t-k|\vee H}{H}\Big)^{1/2}\Big] \le C\Big(\frac{H}{t}\Big)^{1/2} \end{aligned}$$

by (C.19). On the other hand, for $1 \le k \le t/2$, it holds $|t-k| \ge t/2$. Then,

$$1 = (|t-k|/H)(H/|t-k|) \le 2(|t-k|/H)(H/t),$$

and

$$I_{H;2} \leq 2H^{-1} \sum_{k=1}^{t/2} \{b_{T,|t-k|} (\frac{|t-k| \vee H}{H})^{1/2} \frac{|t-k|}{H}\} (\frac{H}{t}) (\frac{H}{k})^{1/2}.$$

Then, using the second claim of (C.19), we obtain

$$I_{H;2} \leq C(\frac{H^{1/2}}{t}) \sum_{k=1}^{t/2} k^{-1/2} \leq C(\frac{H}{t})^{1/2}.$$

The bounds for $I_{H;1}$ and $I_{H;2}$ imply $I_H \leq C(H/t)^{1/2}$.

In view of (C.18), to prove (C.17), it remains to show that $I_H \leq C$. By (C.19),

$$\begin{aligned} I_H &\leq H^{-1} \textstyle\sum_{k=1}^{T} b_{T,|t-k|} (\frac{|t-k| \vee H}{H})^{1/2} (\frac{H}{k})^{1/2} \\ &\leq H^{-1} \textstyle\sum_{k=1}^{2H} b_{T,|t-k|} (\frac{|t-k| \vee H}{H})^{1/2} (\frac{H}{k})^{1/2} + 2H^{-1} \textstyle\sum_{k=2H}^{T} b_{T,|t-k|} (\frac{|t-k| \vee H}{H})^{1/2} \\ &\leq CH^{-1/2} \textstyle\sum_{k=1}^{2H} k^{-1/2} + CH^{-1} \textstyle\sum_{k=1}^{T} b_{T,|t-k|} (\frac{|t-k| \vee H}{H})^{1/2} \leq C, \end{aligned}$$

where $C < \infty$ does not depend on t, H and T. This proves $I_H \leq C$ and (C.17), and completes the proof of the lemma. □

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