Supplement to "Estimation of Volatility Functions in Jump Diffusions Using Truncated Bipower Increments"

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This supplement contains two sections, Sections C and D. Section C provides the proofs of Lemmas A.1-A.4 in the Mathematical Appendix; Section D includes some preliminary lemmas, Lemmas D.1-D.13 that are useful for the proofs in Section C.

C Proofs of Lemmas A.1-A.4

Proof of Lemma A.1. By the occupation time formula and change of variables, we have

$$\begin{aligned}\frac{1}{h}\int_0^T f_{x,h}(X_t)[g(X_t)-g(x)]dt &= \int f(u)[g(x+hu)-g(x)]\ell(T,x+hu)du\\ &= g'(x)h\int(\iota f)(u)\ell(T,x+hu)du + \frac{\imath_2(f)}{2}g''(x)h^2\ell(T,x)\\ &\quad + o_p(h^2\ell(T,x)).\end{aligned} \tag{C.1}$$

For the first term in (C.1), we may write $\int(\iota f)(u)\ell(T,x+hu)du$ as

$$\begin{aligned}&\int(\iota f)(u)\sigma^{-2}(x+hu)\ell[T,x+hu]du\\ &= \sigma^{-2}(x)\int(\iota f)(u)\ell[T,x+hu]du + (\sigma^{-2})'(x)\imath_2(f)h\ell[T,x] + o_p(h\ell(T,x))\end{aligned} \tag{C.2}$$

by Taylor expansion. Then, we are left to analyze the first term in (C.2).

Let $u>0$, and $\varphi(u,v) = 1\{0\le (v-x)/h < u\}$ and $\Phi(u,v)=\int_{-\infty}^v \varphi(u,w)dw$. By the Bouleau-Yor formula (see, e.g., Theorem 78 in Chapter IV of Protter (2005)),

$$\begin{aligned}\ell[T,x+hu]-\ell[T,x] &= 2\int_0^T \varphi(u,X_t)dX_t^c\\ &\quad + 2\int_0^T\int_{\mathbb{R}}\left[\Phi(u,X_{t-}+z\tau(X_{t-}))-\Phi(u,X_{t-})\right]\Lambda(dt,dz)+O_{a.s.}(hu),\end{aligned}$$

from which, together with $\imath_1(f)=0$ and Fubini's theorem for stochastic integrals (see, e.g., Theorem 64 in Chapter IV of Protter (2005)), we may readily deduce that

$$\begin{aligned}\int(\iota f)(u)\ell[T,x+hu]du &= \int(\iota f)(u)\left(\ell[T,x+hu]-\ell[T,x]\right)du \\ &= 2\left(A_T+B_T+C_T+D_T\right)+O_{a.s.}(h), \end{aligned} \tag{C.3}$$

with

$$\begin{aligned} A_T+B_T &= \int_0^T(\iota f)_1\left(\frac{X_t-x}{h}\right)\left(\sigma(X_t)dW_t+\mu(X_t)dt\right), \\ C_T+D_T &= \int_0^T\int_{\mathbb{R}}\int_{X_{t-}}^{X_{t-}+z\tau(X_{t-})}(\iota f)_1\left(\frac{v-x}{h}\right)dv\left(\Gamma(dt,dz)+\lambda(dz)dt\right). \end{aligned}$$

Using similar arguments, we may show that (C.3) also holds for $u<0$.

By Lemma A.1 in PW, we have

$$B_T=\imath_2(f)\mu(x)h\ell(T,x)(1+o_p(1)), \tag{C.4}$$

noting that $\imath(f_1)=\imath_1(f)$ for f defined on $[-1,1]$, and therefore, $\imath((\iota f)_1)=\imath_1(\iota f)=\imath_2(f)$.

Next, it follows from the occupation time formula and changing the order of integrals that

$$D_T=h\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lambda_1\left(\frac{x-u+hv}{\tau(u)}\right)(\iota f)_1(v)dv\ell(T,u)du=\imath_2(f)h\xi(T,x)(1+o_p(1)) \tag{C.5}$$

due to Lemma A.2 in PW. The stated result then follows from (C.1)-(C.5). □

Proof of Lemma A.2. We may write

$$|\Delta_iW||\Delta_{i+1}W|-\omega\delta=|\Delta_iW|\left(|\Delta_{i+1}W|-\sqrt{\omega\delta}\right)+\sqrt{\omega\delta}\left(|\Delta_iW|-\sqrt{\omega\delta}\right), \tag{C.6}$$

from which we have $N_T=U_T+R_T$, where

$$\begin{aligned} U_T &= \frac{1}{\sqrt{\delta h}}\sum_{i=2}^{n-1}\left[\left(f_{x,h}g\right)(X_{(i-2)\delta})|\Delta_{i-1}W|+(f_{x,h}g)(X_{(i-1)\delta})\sqrt{\omega\delta}\right]\left(|\Delta_iW|-\sqrt{\omega\delta}\right) \\ R_T &= \frac{1}{\sqrt{\delta h}}\left[\left(f_{x,h}g\right)(X_{(n-2)\delta})|\Delta_{n-1}W|\left(|\Delta_nW|-\sqrt{\omega\delta}\right)+\left(f_{x,h}g\right)(X_0)\sqrt{\omega\delta}\left(|\Delta_1W|-\sqrt{\omega\delta}\right)\right]. \end{aligned}$$

It is easy to show that R_T is asymptotically negligible, and therefore, we have $N_T=U_T(1+o_p(1))$.

For each $T,\delta,h>0$, let $V^{T,\delta,h}=\left(V_1^{T,\delta,h},V_2^{T,\delta,h}\right)$, with $V_j^{T,\delta,h}=\left(V_{j,t}^{T,\delta,h}\right)_{t\geq0}$ for $j=1,2$ as processes indexed by t and $V_{j,t}^{T,\delta,h}=\sum_{i=2}^{\lfloor Tt/\delta\rfloor}\zeta_{j,i}^{T,\delta,h}$, where

$$\begin{aligned} \zeta_{1,i}^{T,\delta,h} &= \frac{1}{\sqrt{h}}\left(\chi_{x,h}\varphi\right)(X_{(i-1)\delta})\Delta_iW \\ \zeta_{2,i}^{T,\delta,h} &= \frac{1}{\sqrt{\delta h}}\left[\left(f_{x,h}g\right)(X_{(i-2)\delta})|\Delta_{i-1}W|+(f_{x,h}g)(X_{(i-1)\delta})\sqrt{\omega\delta}\right]\left(|\Delta_iW|-\sqrt{\omega\delta}\right). \end{aligned}$$

For the stated result in Lemma A.2, it then suffices to show that

$$\left(V_{1,1}^{T,\delta,h}, V_{2,1}^{T,\delta,h}\right) =_d \ell(T,x)^{1/2} Z(1+o_p(1)) \tag{C.7}$$

as $\delta, h \to 0$, and T either fixed or $T \to \infty$.

Case 1. T is fixed. In this case, we show that for any $0 < t \leq 1$,

$$\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \mathbb{E}_{(i-1)\delta}\left(\zeta_i^{T,\delta,h}\right) = 0, \tag{C.8}$$

$$\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \mathbb{E}_{(i-1)\delta}\left[\left(\zeta_i^{T,\delta,h}\right)\left(\zeta_i^{T,\delta,h}\right)^\top\right] \to_p \ell(tT,x)\Sigma, \tag{C.9}$$

$$\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \mathbb{E}_{(i-1)\delta}\left(\|\zeta_i^{T,\delta,h}\|^4\right) \to_p 0, \tag{C.10}$$

$$\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \mathbb{E}_{(i-1)\delta}\left(\zeta_i^{T,\delta,h} \Delta_i H\right) \to_p 0 \tag{C.11}$$

for H being W or any bounded martingale orthogonal to W, where $\zeta_i^{T,\delta,h} = \big(\zeta_{1,i}^{T,\delta,h}, \zeta_{2,i}^{T,\delta,h}\big)$. Then, by Lemma 3.7 in Jacod (2012), the process $V^{T,\delta,h}$ converges stably in law (as $\delta, h \to 0$) to a continuous process defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and which, conditionally on $\mathcal{F}$, is a bivariate centered Gaussian process, with conditional variance process given by the right hand side of (C.9). Then, (C.7) follows with $t = 1$.

First, (C.8) clearly holds. For (C.9), by Lemmas A.9 and A.14 in PW,

$$\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \mathbb{E}_{(i-1)\delta}\left(\zeta_{1,i}^{T,\delta,h}\right)^2 = \frac{\delta}{h}\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \left(\chi_{x,h}\varphi\right)^2\left(X_{(i-1)\delta}\right) \to_p \imath(\chi^2)\varphi^2(x)\ell(tT,x) \tag{C.12}$$

under $\delta = o_p(h^2)$. Moreover, by Lemma D.2 and $\mathbb{E}_{(i-1)\delta}\big(|\Delta_i W| - \sqrt{\omega\delta}\big)^2 = (1-\omega)\delta$, we have

$$\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \mathbb{E}_{(i-1)\delta}\left(\zeta_{2,i}^{T,\delta,h}\right)^2 \to_p c(\pi)\imath(f^2)g^2(x)\ell(tT,x), \tag{C.13}$$

which, together with (C.12), implies that (C.9) holds, noting that $\mathbb{E}_{(i-1)\delta}\big(\zeta_{1,i}^{T,\delta,h}\zeta_{2,i}^{T,\delta,h}\big) = 0$.

For (C.10), by analogous arguments as (C.12), we have

$$\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \mathbb{E}_{(i-1)\delta}\left(\zeta_{1,i}^{T,\delta,h}\right)^4 \leq \frac{c\delta^2}{h^2}\sum_{i=2}^{\lfloor Tt/\delta \rfloor} (\chi_{x,h}\varphi)^4(X_{(i-1)\delta}) = O_p\left(\frac{\delta\ell(Tt,x)}{h}\right) = o_p(1),$$

and

$$\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \mathbb{E}_{(i-1)\delta}\left(\zeta_{2,i}^{T,\delta,h}\right)^4 \leq \frac{c}{h^2}\sum_{i=2}^{\lfloor Tt/\delta \rfloor}\left[(f_{x,h}g)^4(X_{(i-2)\delta})(\Delta_{i-1}W)^4 + (f_{x,h}g)^4(X_{(i-1)\delta})\delta^2\right]$$
$$\leq_p \frac{c\delta^2}{h^2}\sum_{i=2}^{\lfloor Tt/\delta \rfloor}\left[(f_{x,h}g)^4(X_{(i-2)\delta}) + (f_{x,h}g)^4(X_{(i-1)\delta})\right] = o_p(1).$$

where the second relation "$\leq_p$" holds by Lenglart domination property. Therefore, (C.10) follows.

For (C.11), it suffices to show that for $j = 1, 2$,

$$\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \mathbb{E}_{(i-1)\delta}\left(\zeta_{j,i}^{T,\delta,h}\Delta_i H\right) \to_p 0. \tag{C.14}$$

For $j = 1$, (C.14) holds for H being a bounded martingale orthogonal to W since $\mathbb{E}_{(i-1)\delta}(\Delta_i W \Delta_i H) = 0$ for $2 \leq i \leq n$. For $H = W$, it holds that

$$\sum_{i=2}^{\lfloor Tt/\delta \rfloor} \mathbb{E}_{(i-1)\delta}\left(\zeta_{1,i}^{T,\delta,h}\Delta_i H\right) = \frac{\delta}{\sqrt{h}}\sum_{i=2}^{\lfloor Tt/\delta \rfloor}(\chi_{x,h}\varphi)(X_{(i-1)\delta}) = O_p\left(\sqrt{h}\ell(tT, x)\right) = o_p(1).$$

For $j = 2$, (C.14) holds for $H = W$ since $\mathbb{E}_{(i-1)\delta}\big[(|\Delta_i W| - \sqrt{\omega\delta})\Delta_i W\big] = 0$ for $2 \leq i \leq n$. Moreover, using analogous arguments as in the proof of Lemma 3.18 in Jacod (2012), we may readily show that for H being any bounded martingale orthogonal to W, $\mathbb{E}_{(i-1)\delta}\big(\zeta_{2,i}^{T,\delta,h}\Delta_i H\big) = 0$ for $2 \leq i \leq n$, which completes the proof of (C.11).

Case 2. $T \to \infty$. By Equation (14) in Kanaya (2016), it holds that $\limsup_{\delta\to 0}\sup_{s,t\in[0,\infty),|t-s|\in[0,\delta]} |W_t - W_s| = 2\sqrt{\delta\log(1/\delta)}$ almost surely as $\delta \to 0$, as the global modulus of continuity of Brownian motion. Then we readily have

$$\max_{i\geq 1}\left|\zeta_{1,i}^{T,\delta,h}\right| \leq \frac{1}{\sqrt{h}}\|\chi_{x,h}\varphi\|_\infty\left(\max_{i\geq 1}|\Delta_i W|\right) \to 0$$
$$\max_{i\geq 1}\left|\zeta_{2,i}^{T,\delta,h}\right| \leq \frac{1}{\sqrt{\delta h}}\|f_{x,h}g\|_\infty\left(\max_{i\geq 1}|\Delta_i W| + \sqrt{\omega\delta}\right)^2 \to 0 \tag{C.15}$$

almost surely under $\delta = o(h^2)$.

Next, for each $T > 0$, let $(\ell_t^T)_{t\geq 0}$ be a process given by $\ell_t^T = \ell(Tt, x)/\kappa_T$. Noting that we write $\kappa(T)$ as κ_T for simplicity, and $\kappa(\cdot)$ is as in Assumption 2.1 (g). Using similar arguments as Lemma D.2, we may readily deduce that for predictable quadratic variation processes $\langle V_j^{T,\delta,h}\rangle$ with $j = 1, 2$, it holds that for each $t > 0$,

$$\sup_{0<s\leq t}\left|\kappa_T^{-1}\langle V_{j,s}^{T,\delta,h}\rangle - a_j(x)\ell_s^T\right| \to_p 0, \tag{C.16}$$

where $a_1(x) = \imath(\chi^2)\varphi^2(x)$ and $a_2(x) = c(\pi)\imath(f^2)g^2(x)$.

Moreover, it follows from Lemma D.2 in PW that

$$\ell^T \to_{st} m(x)L \tag{C.17}$$

as $T \to \infty$ on an extended probability space $\big(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\big)$ of $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$, on which L and $\mathcal{F}$ are independent, where "$\to_{st}$" denotes stable convergence in law, and $L = (L_t)_{t\geq 0}$ denotes a Mittag-Leffler process of index $\rho \in (0,1]$ as in Assumption 2.1 (g). Together with (C.16), we have

$$\Big(\ell^T, \kappa_T^{-1}\langle V_1^{T,\delta,h}\rangle, \kappa_T^{-1}\langle V_2^{T,\delta,h}\rangle\Big) \to_{st} (m(x)L, (a_1 m)(x)L, (a_2 m)(x)L) \tag{C.18}$$

as $T \to \infty$ on the extended probability space $\big(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\big)$. Moreover, note that $\mathbb{E}_{(i-1)\delta}\big[\Delta_i W\big(|\Delta_i W| - \sqrt{\omega\delta}\big)\big] = 0$, which implies that the predictable quadratic covariation between $V_1^{T,\delta,h}$ and $V_2^{T,\delta,h}$ is zero. It then follows from (C.15), (C.18), Theorem 5.5 in Ueltzhöfer (2013), and (3.5) in Höpfner et al. (1990) that

$$\begin{aligned}
&\Big(\ell^T, \kappa_T^{-1}\langle V_1^{T,\delta,h}\rangle, \kappa_T^{-1}\langle V_2^{T,\delta,h}\rangle, \kappa_T^{-1/2} V_1^{T,\delta,h}, \kappa_T^{-1/2} V_2^{T,\delta,h}\Big) \\
&\to_{st} \Big(m(x)L, (a_1 m)(x)L, (a_2 m)(x)L, \sqrt{(a_1 m)(x)} B_1 \circ L, \sqrt{(a_2 m)(x)} B_2 \circ L\Big),
\end{aligned} \tag{C.19}$$

as $h, \delta \to 0$ and $T \to \infty$ on an extended probability space $\big(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}\big)$, on which L, B and $\mathcal{F}$ are mutually independent, with $B = (B_1, B_2)$ as a two dimensional Brownian motion. Therefore, it follows from (C.16) and (C.19) that (C.7) also holds in the case of $T \to \infty$, which completes the proof. □

Proof of Lemma A.3. We first define

$$\begin{aligned}
&Z_{i1} = |\Delta_i X||\Delta_{i+1} X_1^c|,\ Z_{i2} = |\Delta_i X||\Delta_{i+1} X^d|,\ Z_{i3} = |\sigma|(X_{(i-1)\delta})|\Delta_{i+1} W||\Delta_i X_1^c|, \\
&Z_{i4} = |\sigma|(X_{(i-1)\delta})|\Delta_{i+1} W||\Delta_i X^d|,\ Z_{i5} = |\Delta_i X||\Delta_{i+1} W||\sigma(X_{i\delta}) - \sigma(X_{(i-1)\delta})|, \\
&Z_{i6} = |\Delta_i X|\Big|\int_{i\delta}^{(i+1)\delta} (\sigma(X_t) - \sigma(X_{i\delta}))dW_t\Big|, \\
&Z_{i7} = |\sigma|(X_{(i-1)\delta})|\Delta_{i+1} W|\Big|\int_{(i-1)\delta}^{i\delta} (\sigma(X_t) - \sigma(X_{(i-1)\delta}))dW_t\Big| \\
&Z_{i8} = \sigma^2(X_{(i-1)\delta})|\Delta_i W||\Delta_{i+1} W|\left(1 - 1\{|\Delta_i X| \leq \delta^\beta, |\Delta_{i+1} X| \leq \delta^\beta\}\right),
\end{aligned} \tag{C.20}$$

so that we may readily write

$$\begin{aligned}
&\Big||\Delta_i X||\Delta_{i+1} X| 1\{|\Delta_i X| \leq \delta^\beta, |\Delta_{i+1} X| \leq \delta^\beta\} - \sigma^2(X_{(i-1)\delta})|\Delta_i W||\Delta_{i+1} W|\Big| \\
&\leq \big||\Delta_i X||\Delta_{i+1} X| - \sigma^2(X_{(i-1)\delta})|\Delta_i W||\Delta_{i+1} W|\big| 1\{|\Delta_i X| \leq \delta^\beta, |\Delta_{i+1} X| \leq \delta^\beta\} \\
&\quad + \sigma^2(X_{(i-1)\delta})|\Delta_i W||\Delta_{i+1} W|\left(1 - 1\{|\Delta_i X| \leq \delta^\beta, |\Delta_{i+1} X| \leq \delta^\beta\}\right) \\
&\leq \big||\Delta_i X||\Delta_{i+1} X| - \sigma^2(X_{(i-1)\delta})|\Delta_i W||\Delta_{i+1} W|\big| + Z_{i8} \leq \sum_{j=1}^{8} Z_{ij},
\end{aligned} \tag{C.21}$$

from which it follows that $\left|R_T(K,\sigma^2)\right| \le \sum_{j=1}^{8} R_j$, where

$$R_j = \frac{\pi}{2h}\sum_{i=1}^{n-1} K_{x,h}(X_{(i-1)\delta})Z_{ij}$$

for $j = 1, 2, \ldots, 8$. However, it follows from Lemmas D.6, D.7, D.12, D.8, D.10, D.11 and D.13 that $R_j = O_p(\delta^{1/2}\ell(T,x))$ for $j = 1, 2, \ldots, 8$, from which the stated result follows. □

Proof of Lemma A.4. Firstly, by applying Lemma A.16 in PW with $f = K$ and $g = \sigma^2$, we have

$$B_T(K) = O_p\left(\delta T^{2pq}\ell(T,x) + \delta h^{-1/2}T^{pq}\ell(T,x)^{1/2}\right) = o_p(h^2\ell(T,x)) \tag{C.22}$$

under $\delta = o(h^3 \wedge T^{-6pq})$. Secondly, note that $C_T(K) = -M_T(K,\sigma^2) - R_T(K,\sigma^2)$ with $M_T(K,\sigma^2)$ and $R_T(K,\sigma^2)$ defined as in Section 3.1 and that, as shown in the proof of Theorem 3.1, $M_T(K,\sigma^2) = O_p\big(\sqrt{\delta\ell(T,x)/h}\big)$ and $R_T(K,\sigma^2) = O_p\big(\delta^{1/2}\ell(T,x)\big)$, from which it follows that

$$C_T(K) = O_p\left(\sqrt{\frac{\delta\ell(T,x)}{h}}\right) + O_p\left(\delta^{1/2}\ell(T,x)\right). \tag{C.23}$$

For $A_T(K)$, we write $A_T(K) = F_T + G_T$, where

$$\begin{aligned}
F_T &= \frac{2}{h}\sum_{i=1}^{n} K_{x,h}(X_{(i-1)\delta})\int_{(i-1)\delta}^{i\delta}(X_t - X_{(i-1)\delta})\mu(X_t)dt \\
G_T &= \frac{2}{h}\sum_{i=1}^{n} K_{x,h}(X_{(i-1)\delta})\int_{(i-1)\delta}^{i\delta}(X_{t-} - X_{(i-1)\delta})\left(\sigma(X_t)dW_t + \int_{\mathbb{R}} z\tau(X_{t-})\Gamma(dt,dz)\right).
\end{aligned}$$

Using similar arguments as $B_T(K)$ in (C.22), we have

$$F_T = O_p\left(\delta T^{2pq}\ell(T,x) + \delta h^{-1/2}T^{pq}\ell(T,x)^{1/2}\right) = o_p(h^2\ell(T,x)) \tag{C.24}$$

under $\delta = o(h^3 \wedge T^{-6pq})$, noting that we may write $(X_t - X_{(i-1)\delta})\mu(X_t) = \iota\mu(X_t) - \iota\mu(X_{(i-1)\delta}) - \left[x + h\left((X_{(i-1)\delta} - x)/h\right)\right](\mu(X_t) - \mu(X_{(i-1)\delta}))$. Moreover, for G_T, we have

$$\begin{aligned}
\langle G\rangle_T &\le T(\sigma^2+\tau^2)\frac{4}{h^2}\sum_{i=1}^{n} K_{x,h}^2(X_{(i-1)\delta})\int_{(i-1)\delta}^{i\delta}(X_t - X_{(i-1)\delta})^2 dt \\
&\le_p T(\sigma^2+\tau^2)\frac{4}{h^2}\sum_{i=1}^{n} K_{x,h}^2(X_{(i-1)\delta})\int_{(i-1)\delta}^{i\delta}\int_{(i-1)\delta}^{t}(\sigma^2+\tau^2)(X_s)dsdt \\
&\le T^2(\sigma^2+\tau^2)\left(\frac{2\delta^2}{h^2}\sum_{i=1}^{n} K_{x,h}^2(X_{(i-1)\delta})\right) = O_p\left(\delta T^{2pq}h^{-1}\ell(T,x)\right),
\end{aligned}$$

which implies that

$$G_T = O_p\left(\delta^{1/2}T^{pq}h^{-1/2}\ell(T,x)^{1/2}\right) = o_p\left(h^{1/2}\ell(T,x)^{1/2}\right). \tag{C.25}$$

under $\delta = o(h^3 \wedge T^{-6pq})$. Then, the stated result follows from (C.22)-(C.25). □

D Lemmas D.1-D.13 and their proofs

This section contains some preliminary lemmas, Lemmas D.1-D.13 and their proofs. Lemmas D.1 and D.2 are useful for the proof of Lemma A.2; and Lemmas D.3-D.13 are used in the proof of Lemma A.3.

Lemma D.1. *Let (i) f and g be twice continuously differentiable, (ii) Assumptions 2.1, 2.3 and 2.5 (b) hold, (iii) η satisfies Assumption 2.5 (a) for $\eta = \tau^2$, and (iv) $\delta = o(h^2 \wedge T^{-pq})$. Then*

$$\frac{\delta}{h}\sum_{i=2}^{n}\left(f_{x,h}g\right)\left(X_{(i-2)\delta}\right)\left(f_{x,h}g\right)\left(X_{(i-1)\delta}\right) = \imath(f^2)g^2(x)\ell(T,x)(1+o_p(1)).$$

Proof. We write

$$\frac{\delta}{h}\sum_{i=2}^{n}\left(f_{x,h}g\right)\left(X_{(i-2)\delta}\right)\left(f_{x,h}g\right)\left(X_{(i-1)\delta}\right) = A_T + B_T, \tag{D.1}$$

where

$$\begin{aligned}
A_T &= \frac{\delta}{h}\sum_{i=2}^{n}\left(f_{x,h}g\right)^2\left(X_{(i-2)\delta}\right) \\
B_T &= \frac{\delta}{h}\sum_{i=2}^{n}\left(f_{x,h}g\right)\left(X_{(i-2)\delta}\right)\left[\left(f_{x,h}g\right)\left(X_{(i-1)\delta}\right) - \left(f_{x,h}g\right)\left(X_{(i-2)\delta}\right)\right],
\end{aligned}$$

which will be considered in the sequel. For A_T, using similar arguments as (C.12), we have

$$A_T = \imath(f^2)g^2(x)\ell(T,x)(1+o_p(1)) \tag{D.2}$$

under $\delta = o(h^2)$. For B_T, by Itô's formula, we may write $B_T = P_T + Q_T + R_T$, where

$$\begin{aligned}
P_T &= \frac{\delta}{h}\sum_{i=2}^{n}(f_{x,h}g)(X_{(i-2)\delta})\int_{(i-2)\delta}^{(i-1)\delta}\left[(f_{x,h}g)'\mu + (f_{x,h}g)''\sigma^2/2\right](X_t)dt \\
Q_T &= \frac{\delta}{h}\sum_{i=2}^{n}(f_{x,h}g)(X_{(i-2)\delta})\int_{(i-2)\delta}^{(i-1)\delta}\left[(f_{x,h}g)'\sigma\right](X_t)dW_t \\
R_T &= \frac{\delta}{h}\sum_{i=2}^{n}(f_{x,h}g)(X_{(i-2)\delta})\int_{(i-2)\delta}^{(i-1)\delta}\int_{\mathbb{R}}\left[(f_{x,h}g)(X_{t-}+z\tau(X_{t-})) - (f_{x,h}g)(X_{t-})\right]\Lambda(dt,dz).
\end{aligned}$$

For P_T, we have

$$\begin{aligned}
|P_T| &\leq \delta T\left((f_{x,h}g)'\mu + \frac{1}{2}(f_{x,h}g)''\sigma^2\right)\left(\sup_{|x-y|\leq h}|g|(y)\right)\left(\frac{\delta}{h}\sum_{i=2}^{n}f_{x,h}(X_{(i-2)\delta})\right) \\
&= O_p\left(\delta h^{-2}\ell(T,x)\right) = o_p(\ell(T,x)),
\end{aligned}$$

noting that $T\left((f_{x,h}g)'\mu+(f_{x,h}g)''\sigma^2/2\right) = O(h^{-2})$, since f has support $[-1,1]$ and μ,σ^2,g,g',g'' are locally bounded. For R_T, we may easily deduce from Lenglart domination property that

$$\begin{aligned}
|R_T| &\le \delta T\left((f_{x,h}g)'\right) T(\tau)\left(\sup_{|x-y|\le h}|g|(y)\right)\left(\frac{1}{h}\sum_{i=2}^{n} f_{x,h}(X_{(i-2)\delta})\int_{(i-2)\delta}^{(i-1)\delta}\int_{\mathbb{R}}|z|\Lambda(dt,dz)\right)\\
&\le_p \delta T\left((f_{x,h}g)'\right) T(\tau)\left(\sup_{|x-y|\le h}|g|(y)\right)\left(\frac{\imath(|\iota|\lambda)\delta}{h}\sum_{i=2}^{n} f_{x,h}(X_{(i-2)\delta})\right)\\
&= O_p\left(\delta h^{-1}T^{pq/2}\ell(T,x)\right) = o_p(\ell(T,x))
\end{aligned}$$

under $\delta = o(h^2\wedge T^{-pq})$.

Finally, we have

$$\begin{aligned}
[Q]_T &= \frac{\delta^2}{h^2}\sum_{i=2}^{n}\left(f_{x,h}g\right)^2(X_{(i-2)\delta})\int_{(i-2)\delta}^{(i-1)\delta}\left[(f_{x,h}g)'\sigma\right]^2(X_t)dt\\
&\le \frac{\delta^2}{h}T\left(\left[(f_{x,h}g)'\sigma\right]^2\right)\left(\sup_{|x-y|\le h} g^2(y)\right)\left(\frac{\delta}{h}\sum_{i=2}^{n} f_{x,h}^2(X_{(i-2)\delta})\right) = O_p\left(\delta^2h^{-3}\ell(T,x)\right),
\end{aligned}$$

from which it follows that $Q_T = O_p\left(\delta h^{-3/2}\ell(T,x)^{1/2}\right) = o_p(\ell(T,x))$. Therefore, $B_T = o_p(\ell(T,x))$, from which, together with (D.1) and (D.2), the stated result follows. □

Lemma D.2. *Let the conditions in Lemma D.1 hold. Then*

$$\frac{1}{h}\sum_{i=2}^{n-1}\left[\left(f_{x,h}g\right)(X_{(i-2)\delta})|\Delta_{i-1}W| + \left(f_{x,h}g\right)(X_{(i-1)\delta})\sqrt{\omega\delta}\right]^2 = (1+3\omega)\imath(f^2)g^2(x)\ell(T,x)(1+o_p(1)). \tag{D.3}$$

Proof. We may readily write the left hand side of (D.3) as $A_T+B_T+C_T+D_T$, where

$$\begin{aligned}
A_T &= \frac{\delta}{h}\sum_{i=2}^{n-1}\left[\left(f_{x,h}g\right)^2(X_{(i-2)\delta})+\omega\left(f_{x,h}g\right)^2(X_{(i-1)\delta})\right]+\frac{2\omega\delta}{h}\sum_{i=2}^{n-1}(f_{x,h}g)(X_{(i-2)\delta})(f_{x,h}g)(X_{(i-1)\delta})\\
B_T &= \frac{2}{h}\sum_{i=2}^{n-1}\left(f_{x,h}g\right)^2(X_{(i-2)\delta})\int_{(i-2)\delta}^{(i-1)\delta}\int_{(i-2)\delta}^{t} dW_s dW_t\\
C_T &= \frac{2\sqrt{\omega\delta}}{h}\sum_{i=2}^{n-1}\left(f_{x,h}g\right)^2(X_{(i-2)\delta})\left(|\Delta_{i-1}W|-\sqrt{\omega\delta}\right)\\
D_T &= \frac{2\sqrt{\omega\delta}}{h}\sum_{i=2}^{n-1}\left(f_{x,h}g\right)(X_{(i-2)\delta})\left[\left(f_{x,h}g\right)(X_{(i-1)\delta})-\left(f_{x,h}g\right)(X_{(i-2)\delta})\right]\left(|\Delta_{i-1}W|-\sqrt{\omega\delta}\right).
\end{aligned}$$

Note that $\mathbb{E}|\Delta_{i-1}W| = \sqrt{\omega\delta}$ and $(\Delta_{i-1}W)^2 = \delta + 2\int_{(i-2)\delta}^{(i-1)\delta}\int_{(i-2)\delta}^{t} dW_s dW_t$.

Using similar arguments as (C.12), together with Lemma D.1, we may readily deduce that

$$A_T = (1+3\omega)\imath(f^2)g^2(x)\ell(T,x)(1+o_p(1)).$$

In the sequel, we show that

$$B_T, C_T, D_T = o_p(\ell(T,x)),$$

which then completes the proof. For B_T, it holds that

$$\begin{aligned}[B]_T &= \frac{4}{h^2}\sum_{i=2}^{n-1}(f_{x,h}g)^4\left(X_{(i-2)\delta}\right)\int_{(i-2)\delta}^{(i-1)\delta}\left(\int_{(i-2)\delta}^{t}dW_s\right)^2 dt \\ &\le_p \frac{4\delta}{h}\left(\frac{\delta}{h}\sum_{i=2}^{n-1}(f_{x,h}g)^4\left(X_{(i-2)\delta}\right)\right) = O_p(\delta h^{-1}\ell(T,x)),\end{aligned}$$

which implies that $B_T = O_p\left((\delta h^{-1}\ell(T,x))^{1/2}\right) = o_p(\ell(T,x))$ under $\delta = o(h)$. Similarly, we have $C_T = O_p\left((\delta h^{-1}\ell(T,x))^{1/2}\right) = o_p(\ell(T,x))$. Next, we use Cauchy-Schwarz inequality and deduce that $|D_T| \le c\sqrt{P_T Q_T}$, where

$$\begin{aligned}P_T &= \frac{1}{h}\sum_{i=2}^{n-1}(f_{x,h}g)\left(X_{(i-2)\delta}\right)\left(|\Delta_{i-1}W| - \sqrt{\omega\delta}\right)^2 \\ Q_T &= \frac{\delta}{h}\sum_{i=2}^{n-1}(f_{x,h}g)\left(X_{(i-2)\delta}\right)\left[(f_{x,h}g)\left(X_{(i-1)\delta}\right) - (f_{x,h}g)\left(X_{(i-2)\delta}\right)\right]^2.\end{aligned}$$

However, similarly C_T, we may easily show that $P_T = o_p(\ell(T,x))$. Moreover, we may write

$$\begin{aligned}\left[(f_{x,h}g)\left(X_{(i-1)\delta}\right) - (f_{x,h}g)\left(X_{(i-2)\delta}\right)\right]^2 &= \left[(f_{x,h}g)^2\left(X_{(i-1)\delta}\right) - (f_{x,h}g)^2\left(X_{(i-2)\delta}\right)\right] \\ &\quad - 2(f_{x,h}g)\left(X_{(i-2)\delta}\right)\left[(f_{x,h}g)\left(X_{(i-1)\delta}\right) - (f_{x,h}g)\left(X_{(i-2)\delta}\right)\right],\end{aligned}$$

and show, as for B_T in the proof of Lemma D.1, that $Q_T = o_p(\ell(T,x))$, which implies that $D_T = o_p(\ell(T,x))$. The proof is therefore complete. □

Lemma D.3. *Let (i) g be twice continuously differentiable on $\mathcal{D}$, (ii) Assumptions 2.1, 2.3 and 2.5 (b) hold, (iii) η satisfies Assumption 2.5 (a) for $\eta = \mu, \sigma^2, \tau^2, g'$, and (iv) $\delta = o(h^2 \wedge T^{-4pq})$. Then*

$$\frac{1}{h}\sum_{i=1}^{n} f_{x,h}(X_{(i-1)\delta})\int_{i\delta}^{(i+1)\delta} g(X_t)dt = \imath(f)g(x)\ell(T,x)(1+o_p(1)).$$

Proof. We write

$$\frac{1}{h}\sum_{i=1}^{n} f_{x,h}(X_{(i-1)\delta})\int_{i\delta}^{(i+1)\delta} g(X_t)dt = A_T + B_T + C_T,$$

where

$$A_T = \frac{\delta}{h}\sum_{i=1}^{n}\left(f_{x,h}g\right)\left(X_{(i-1)\delta}\right)$$
$$B_T = \frac{\delta}{h}\sum_{i=1}^{n} f_{x,h}(X_{(i-1)\delta})\left[g(X_{i\delta}) - g(X_{(i-1)\delta})\right]$$
$$C_T = \frac{1}{h}\sum_{i=1}^{n} f_{x,h}(X_{(i-1)\delta})\int_{i\delta}^{(i+1)\delta}\left[g(X_t) - g(X_{i\delta})\right]dt.$$

By similar arguments as (C.12), we have

$$A_T = \imath(f)g(x)\ell(T,x)(1 + o_p(1)) \tag{D.4}$$

under $\delta = o(h^2)$. For B_T, we may write

$$|B_T| \le T(g')\frac{\delta}{h}\sum_{i=1}^{n} f_{x,h}(X_{(i-1)\delta})\left(|\Delta_i X^c| + |\Delta_i X^d|\right) = P_T + Q_T,$$

for which we have

$$P_T \le T(g')\max_{1\le i\le n}|\Delta_i X^c|\left(\frac{\delta}{h}\sum_{i=1}^{n} f_{x,h}(X_{(i-1)\delta})\right) = O_p\left(\delta^{1/2}T^{3pq/2}\ell(T,x)\sqrt{\log(T/\delta)}\right),$$

under $\delta = o(h^2)$, due to the modulus of continuity of diffusion, and

$$\begin{aligned} Q_T &\le T(g'\tau)\frac{\delta}{h}\sum_{i=1}^{n} f_{x,h}(X_{(i-1)\delta})\int_{(i-1)\delta}^{i\delta}\int_{\mathbb{R}}|z|\Lambda(dt,dz) \\ &\le_p \imath(|\iota|\lambda)\delta T(g'\tau)\left(\frac{\delta}{h}\sum_{i=1}^{n} f_{x,h}(X_{(i-1)\delta})\right) = O_p\left(\delta T^{3pq/2}\ell(T,x)\right), \end{aligned}$$

by Lenglart domination property. Therefore, it follows that

$$B_T = O_p\left(\delta^{1/2}T^{3pq/2}\ell(T,x)\sqrt{\log(T/\delta)}\right) = o_p(\ell(T,x)). \tag{D.5}$$

under $\delta = o(T^{-4pq})$. We may also similarly deduce that $C_T = o_p(\ell(T,x))$, which completes the proof, together with (D.4) and (D.5). □

Lemma D.4. *Let (i) σ be twice continuously differentiable on $\mathcal{D}$, (ii) Assumptions 2.1, 2.3 and 2.5 (b) hold, (iii) η satisfies Assumption 2.5 (a) for $\eta = \mu, \sigma^2, \tau^2, \sigma^{2\prime}, \sigma^{2\prime\prime}$, and (iv) $\delta = o(h^2 \wedge T^{-2pq})$. Then*

$$\frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X| = O_p(\delta^{1/2}\ell(T,x)).$$

Proof. Note that

$$\frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X| \leq A_T + B_T + C_T, \tag{D.6}$$

where A_T, B_T and C_T are defined as the left hand side in (D.6) with $|\Delta_i X|$ replaced by $|\Delta_i X_1^c|$, $|\Delta_i X_2^c|$ and $|\Delta_i X^d|$, respectively. For A_T, we have

$$A_T \leq \delta T(\mu)\left(\frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\right) = O_p(\delta T^{pq}\ell(T,x)) = o_p(\delta^{1/2}\ell(T,x)) \tag{D.7}$$

under $\delta = o(h^2 \wedge T^{-2pq})$. For B_T, we use Itô's formula to have $B_T \leq P_T + Q_T + R_T + S_T$, where

$$\begin{aligned}
P_T &= \frac{\delta}{h}\sum_{i=1}^{n-1}(f_{x,h}|\sigma|)(X_{(i-1)\delta})|\Delta_i W| \\
Q_T &= \frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\left|\int_{(i-1)\delta}^{i\delta}\int_{(i-1)\delta}^{t}(\mu\sigma' + \sigma^2\sigma''/2)(X_s)dsdt\right| \\
R_T &= \frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\left|\int_{(i-1)\delta}^{i\delta}\int_{(i-1)\delta}^{t}\sigma\sigma'(X_s)dW_s dt\right| \\
S_T &= \frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\left|\int_{(i-1)\delta}^{i\delta}\int_{(i-1)\delta}^{t}\int_{\mathbb{R}}\left(\sigma(X_{s-} + z\tau(X_{s-})) - \sigma(X_{s-})\right)\Lambda(ds,dz)dt\right|,
\end{aligned}$$

which will be considered subsequently.

It follows from Lenglart domination property that

$$P_T \leq_p \sqrt{\frac{\pi\delta}{2}}\left(\sup_{|x-y|\leq h}|\sigma(y)|\right)\left(\frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\right) = O_p(\delta^{1/2}\ell(T,x)).$$

Also, we have

$$Q_T \leq \delta^2 T(\mu\sigma' + \sigma^2\sigma''/2)\left(\frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\right) = O_p\left(\delta^2 T^{3pq/2}\ell(T,x)\right),$$

and

$$\begin{aligned}
R_T &\leq \delta \sup_{t\in[0,T]}\sup_{s\in[0,\delta]}\left|\int_t^{t+s}(\sigma\sigma')(X_r)dW_r\right|\left(\frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\right) \\
&\leq_p \delta^{3/2}T(\sigma\sigma')\sqrt{\log(T/\delta)}\ell(T,x) = O_p\left(\delta^{3/2}T^{pq}\ell(T,x)\sqrt{\log(T/\delta)}\right),
\end{aligned}$$

due to the modulus of continuity of diffusion. Moreover, we may change the order of integrals to have

$$S_T = \frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\left|\int_{(i-1)\delta}^{i\delta}\int_{\mathbb{R}}(i\delta - t)\left[\sigma(X_{t-} + z\tau(X_{t-})) - \sigma(X_{t-})\right]\Lambda(dt,dz)\right|$$
$$\leq \delta^2 T(\sigma'\tau)\left(\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\int_{(i-1)\delta}^{i\delta}\int_{\mathbb{R}}|z|\Lambda(dt,dz)\right) = O_p\left(\delta^2 T^{pq}\ell(T,x)\right),$$

as for Q_T in the proof of Lemma D.3. Consequently, we have

$$B_T = O_p(\delta^{1/2}\ell(T,x)) \tag{D.8}$$

under $\delta = o(T^{-2pq})$.

Finally, we may deduce that $C_T = O_p(\delta T^{pq/2}\ell(T,x)) = o_p(\delta^{1/2}\ell(T,x))$ similarly S_T above, from which, together with (D.6), (D.7) and (D.8), the stated result follows. □

Lemma D.5. *Let (i) g be twice continuously differentiable on $\mathcal{D}$, (ii) Assumptions 2.1, 2.3, 2.5 (b) and 2.7 hold, (iii) η satisfies Assumption 2.5 (a) for $\eta = \mu, \sigma^2, \tau^2, \sigma^{2\prime}, \sigma^{2\prime\prime}, \tau^{2\prime}, \tau^{2\prime\prime}, g, g', g''$, and (iv) $\delta = o(h^2 \wedge T^{-6pq})$. Then*

$$\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|g(X_{i\delta}) - g(X_{(i-1)\delta})\right| = O_p(\ell(T,x)).$$

Proof. We write

$$\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|g(X_{i\delta}) - g(X_{(i-1)\delta})\right| \leq A_T + B_T + C_T, \tag{D.9}$$

where

$$A_T = \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{(i-1)\delta}^{i\delta}(\mu g' + \sigma^2 g''/2)(X_t)dt\right|$$
$$B_T = \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{(i-1)\delta}^{i\delta}(\sigma g')(X_t)dW_t\right|$$
$$C_T = \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{(i-1)\delta}^{i\delta}\int_{\mathbb{R}}\left(g(X_{t-} + z\tau(X_{t-})) - g(X_{t-})\right)\Lambda(dt,dz)\right|.$$

For A_T, it follows from Lemma D.4 that

$$A_T \leq T(\mu g' + \sigma^2 g''/2)\frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X| = O_p\left(\delta^{1/2}T^{2pq}\ell(T,x)\right) = o_p(\ell(T,x)) \tag{D.10}$$

under $\delta = o(h^2 \wedge T^{-4pq})$.

For B_T, we show that

$$B_T \leq P_T + Q_T = O_p(\ell(T,x)), \tag{D.11}$$

where P_T and Q_T are defined in the same way as B_T with $|\Delta_i X|$ replaced by $|\Delta_i X_1^c|$ and $|\Delta_i X_2^c + \Delta_i X^d|$ respectively. We have

$$\begin{aligned} P_T &\leq_p \delta^{1/2} T(\mu) T(\sigma g') \sqrt{\log(T/\delta)} \left(\frac{\delta}{h} \sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta}) \right) \\ &= O_p\left(\delta^{1/2} T^{5pq/2} \ell(T,x) \sqrt{\log(T/\delta)} \right) = o_p(\ell(T,x)) \end{aligned}$$

under $\delta = o(T^{-6pq})$, due to the modulus of continuity of diffusion. Moreover, we may use Cauchy-Schwarz inequality to have $Q_T \leq \sqrt{U_T V_T}$, where

$$\begin{aligned} U_T &= \frac{1}{h} \sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta}) \left(\int_{(i-1)\delta}^{i\delta} \sigma(X_t) dW_t + \int_{(i-1)\delta}^{i\delta} \int_{\mathbb{R}} z\tau(X_{t-}) \Lambda(dt, dz) \right)^2 \\ V_T &= \frac{1}{h} \sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta}) \left(\int_{(i-1)\delta}^{i\delta} (\sigma g')(X_t) dW_t \right)^2. \end{aligned}$$

However, it follows from Lenglart domination property that

$$U_T \leq_p \frac{1}{h} \sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta}) \int_{(i-1)\delta}^{i\delta} \left(\sigma^2(X_t) + \imath_2(\lambda) \tau^2(X_t) \right) dt = O_p(\ell(T,x))$$

due to Lemmas A.9, A.14 and A.16 in PW under $\delta = o\left(h^2 \wedge T^{-6pq}\right)$. Similarly, we may deduce that $V_T = O_p(\ell(T,x))$, and therefore, $Q_T = O_p(\ell(T,x))$. Consequently, (D.11) follows.

Finally, for C_T, we show that

$$C_T \leq M_T + N_T = O_p(\ell(T,x)), \tag{D.12}$$

where M_T and N_T are defined in the same way as C_T with $|\Delta_i X|$ replaced by $|\Delta_i X^c|$ and $|\Delta_i X^d|$ respectively. For M_T, similarly Q_T in the proof of Lemma D.3, we may deduce that

$$\begin{aligned} M_T &\leq T(\tau g') \left(\max_{1 \leq i \leq n} |\Delta_i X^c| \right) \left(\frac{1}{h} \sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta}) \int_{(i-1)\delta}^{i\delta} \int_{\mathbb{R}} |z| \Lambda(dt, dz) \right) \\ &= O_p\left(\delta^{1/2} T^{2pq} \ell(T,x) \sqrt{\log(T/\delta)} \right) = o_p(\ell(T,x)), \end{aligned}$$

under $\delta = o(T^{-5pq})$, due to the modulus of continuity of diffusion. Therefore, it suffices to show that

$$N_T = O_p(\ell(T,x)), \tag{D.13}$$

from which, together with (D.9), (D.10), (D.11) and (D.12), the stated result follows immediately.

To establish (D.13), we write $X_t = X_{t-} + z\tau(X_{t-})$, and let

$$E_{it} = \left\{ |X_t - X_{(i-1)\delta}| \leq \delta^\beta \right\} \cap \left\{ |X_{t-} - X_{(i-1)\delta}| \leq \delta^\beta \right\}$$

and define F_T and G_T similarly as N_T with $g(X_t) - g(X_{t-})$ replaced by $\big(g(X_t) - g(X_{t-})\big) 1(E_{it})$

and $\big(g(X_t)-g(X_{t-})\big)1(E_{it}^c)$ respectively, so that $N_T=F_T+G_T$. It follows that

$$\begin{aligned}
F_T &\le \left(\sup_{|x-y|\le h+\delta^\beta}|g'(y)|\right)\frac{1}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})\left|\Delta_iX^d\right|\left|\int_{(i-1)\delta}^{i\delta}\int_{\mathbb{R}}|z|\tau(X_{t-})\Lambda(dt,dz)\right|\\
&\le \left(\sup_{|x-y|\le h+\delta^\beta}|g'(y)|\right)\frac{1}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})\left(\int_{(i-1)\delta}^{i\delta}\int_{\mathbb{R}}|z|\tau(X_{t-})\Lambda(dt,dz)\right)^2\\
&\le_p \frac{\imath_2(\lambda)}{h}\sum_{i=1}^{n}f_{x,h}(X_{(i-1)\delta})\int_{(i-1)\delta}^{i\delta}\tau^2(X_t)dt=O_p(\ell(T,x)),
\end{aligned}$$

due to similar arguments as U_T above. Moreover, we may apply Cauchy-Schwartz inequality to deduce that

$$\begin{aligned}
G_T^2 \le &\left[\frac{1}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})\left(\Delta_iX^d\right)^2\right]\\
&\times\left[\frac{1}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})\left(\int_{(i-1)\delta}^{i\delta}\int_{\mathbb{R}}(g(X_t)-g(X_{t-}))1(E_{it}^c)\Lambda(dt,dz)\right)^2\right].
\end{aligned}$$

We may easily show that the first term in parenthesis is of order $O_p(\ell(T,x))$ similarly as above. Moreover, since $\sup_{t\in[0,T]}|g(X_t)-g(X_{t-})|\le T(g'\tau)|z|$, we may bound the second term in parenthesis by

$$\begin{aligned}
&T\big(g'^2\tau^2\big)\frac{1}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})\left[\int_{(i-1)\delta}^{i\delta}\int_{\mathbb{R}}|z|1(E_{it}^c)\Lambda(dt,dz)\right]^2\\
&\le_p T\big(g'^2\tau^2\big)\frac{\imath_2(\lambda)}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})\int_{(i-1)\delta}^{i\delta}\mathbb{P}\left(E_{it}^c\middle|\mathcal{F}_{(i-1)\delta}\right)dt\\
&\le 2\imath_2(\lambda)T\big(g'^2\tau^2\big)\left(\sup_{|x-y|\le h}\sup_{0<t\le\delta}\mathbb{P}_y\left(|X_t-y|>\delta^\beta\right)\right)\left(\frac{\delta}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})\right)\\
&=O_p\left(\delta^{1-\alpha\beta-\varepsilon}T^{3pq}\ell(T,x)\right)=o_p(\ell(T,x)),
\end{aligned}$$

where the first inequality in probability follows from Lenglart domination property, the second inequality holds since f has support $[-1,1]$, the third equality follows from Lemmas A.9, A.13 and A.14 in PW for any $\varepsilon>0$, and the fourth equality holds if we choose $\varepsilon<1/2-\alpha\beta$ given $\delta=o(T^{-6pq})$. Therefore, we have $G_T=o_p(\ell(T,x))$, which implies (D.13), as was to be shown. □

Lemma D.6. *Let the conditions in Lemma D.5 hold with g replaced by μ. Then*

$$\frac{1}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})|\Delta_iX||\Delta_{i+1}X_1^c|=O_p(\delta^{1/2}\ell(T,x)).$$

Proof. We may apply Itô's formula to have

$$\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X||\Delta_{i+1}X_1^c| \le A_T + B_T + C_T + D_T + E_T,$$

where

$$\begin{aligned}
A_T &= \frac{\delta}{h}\sum_{i=1}^{n-1} (f_{x,h}|\mu|)\,(X_{(i-1)\delta})|\Delta_i X| \\
B_T &= \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t}(\mu\mu' + \sigma^2\mu''/2)(X_s)dsdt\right| \\
C_T &= \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t}(\sigma\mu')(X_s)dW_s dt\right| \\
D_T &= \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t}\int_{\mathbb{R}}(\mu(X_{s-} + z\tau(X_{s-})) - \mu(X_{s-}))\Lambda(ds,dz)dt\right| \\
E_T &= \frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\mu(X_{i\delta}) - \mu(X_{(i-1)\delta})\right|.
\end{aligned}$$

We have $A_T = O_p(\delta^{1/2}\ell(T,x))$ by Lemma D.4, and also by changing the order of integrals $B_T, C_T, D_T = o_p(\delta^{1/2}\ell(T,x))$ as in the proof of Lemma D.5. Moreover, it follows from Lemma D.5 that $E_T = O_p(\delta\ell(T,x))$, which completes the proof. □

Lemma D.7. *Let the conditions in Lemma D.5 hold with g replaced by τ. Then*

$$\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X||\Delta_{i+1}X^d| = O_p(\delta^{1/2}\ell(T,x)).$$

Proof. It follows from Lenglart domination property that

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X||\Delta_{i+1}X^d| \\
&\le \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left(\int_{i\delta}^{(i+1)\delta}\tau(X_{t-})\int_{\mathbb{R}}|z|\Lambda(dt,dz)\right) \\
&\le_p \frac{\imath(|\iota|\lambda)}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left(\int_{i\delta}^{(i+1)\delta}\tau(X_t)dt\right) = O_p(\delta^{1/2}\ell(T,x)),
\end{aligned}$$

similarly as in the proof of Lemma D.5. □

Lemma D.8. *Let the conditions in Lemma D.5 hold with g replaced by σ. Then*

$$\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X||\Delta_{i+1}W|\left|\sigma(X_{i\delta})-\sigma(X_{(i-1)\delta})\right| = O_p(\delta^{1/2}\ell(T,x)).$$

Proof. Due to Lenglart domination property, we have

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X||\Delta_{i+1}W|\left|\sigma(X_{i\delta})-\sigma(X_{(i-1)\delta})\right| \\
&\leq_p \sqrt{\frac{2\delta}{\pi}}\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\sigma(X_{i\delta})-\sigma(X_{(i-1)\delta})\right| = O_p(\delta^{1/2}\ell(T,x)),
\end{aligned}$$

due to Lemma D.5. □

Lemma D.9. *Let (i) σ and g be twice continuously differentiable on $\mathcal{D}$, (ii) Assumptions 2.1, 2.3 and 2.5 (b) hold, (iii) η satisfies Assumption 2.5 (a) for $\eta = \mu, \sigma^2, \tau^2, \sigma^{2\prime}, \sigma^{2\prime\prime}, g, g'$, and (iv) $\delta = o\left(h^2 \wedge T^{-6pq}\right)$. Then*

$$\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t} g(X_s)dW_s dW_t\right| = O_p(\delta^{1/2}\ell(T,x)).$$

Proof. We write

$$\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t} g(X_s)dW_s dW_t\right| = A_T + B_T, \tag{D.14}$$

where A_T and B_T are defined in the same way as the left hand side in (D.14) with $|\Delta_i X|$ replaced by $|\Delta_i X_1^c + \Delta_i X^d|$ and $|\Delta_i X_2^c|$ respectively.

Using the modulus of continuity of diffusion, we have

$$\begin{aligned}
A_T &\leq \left(\max_{1\leq i\leq n}\left|\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t} g(X_s)dW_s dW_t\right|\right)\left(\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X_1^c + \Delta_i X^d|\right) \\
&\leq_p \delta T(g)\sqrt{\log(T/\delta)}\left[\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\left(\delta T(\mu) + T(\tau)\int_{(i-1)\delta}^{i\delta}\int_{\mathbb{R}}|z|\Lambda(dt,dz)\right)\right] \\
&= O_p\left(\delta T^{2pq}\ell(T,x)\sqrt{\log(T/\delta)}\right),
\end{aligned} \tag{D.15}$$

due to Lenglart domination property. Moreover, it follows from Cauchy-Schwarz inequality that

$B_T \le \sqrt{P_T Q_T}$, where

$$P_T = \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})(\Delta_i X_2^c)^2$$
$$Q_T = \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\left(\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t} g(X_s)dW_s dW_t\right)^2,$$

for which we have

$$P_T \le_p \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\int_{(i-1)\delta}^{i\delta}\sigma^2(X_t)dt = O_p(\ell(T,x)) \tag{D.16}$$

by similar arguments as U_T in the proof of Lemma D.5, and similarly,

$$\begin{aligned} Q_T &\le_p \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\int_{i\delta}^{(i+1)\delta}\left(\int_{i\delta}^{t} g(X_s)dW_s\right)^2 dt \\ &\le_p \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t} g^2(X_s)dsdt = O_p(\delta\ell(T,x)) \end{aligned} \tag{D.17}$$

by Lemma D.3. The stated result follows immediately from (D.14), (D.15) (D.16) and (D.17) under $\delta = o(T^{-6pq})$. □

Lemma D.10. *Let (i) σ be twice continuously differentiable on $\mathcal{D}$, (ii) Assumptions 2.1, 2.3 and 2.5 (b) hold, (iii) η satisfies Assumption 2.5 (a) for $\eta = \mu, \sigma^2, \tau^2, \sigma^{2\prime}, \sigma^{2\prime\prime}$, and (iv) $\delta = o(h^2 \wedge T^{-6pq})$. Then*

$$\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{i\delta}^{(i+1)\delta}(\sigma(X_t)-\sigma(X_{i\delta}))dW_t\right| = O_p(\delta^{1/2}\ell(T,x)).$$

Proof. We use Itô's formula to have

$$\frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{i\delta}^{(i+1)\delta}(\sigma(X_t)-\sigma(X_{i\delta}))dW_t\right| \le A_T + B_T + C_T, \tag{D.18}$$

where

$$A_T = \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t}(\mu\sigma' + \sigma^2\sigma''/2)(X_s)dsdW_t\right|$$
$$B_T = \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t}(\sigma\sigma')(X_s)dW_s dW_t\right|$$
$$C_T = \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\left|\int_{i\delta}^{(i+1)\delta}\int_{i\delta}^{t}\int_{\mathbb{R}}(\sigma(X_{s-}+z\tau(X_{s-}))-\sigma(X_{s-}))\Lambda(ds,dz)dW_t\right|.$$

As shown in Lemma D.9, we have $B_T = O_p(\delta^{1/2}\ell(T,x))$. By changing the order of integrals and using the modulus of continuity of diffusion, and subsequently applying Lemma D.4, we may also easily deduce that

$$\begin{aligned} A_T &= \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X| \left| \int_{i\delta}^{(i+1)\delta}\int_t^{(i+1)\delta} dW_s(\mu\sigma' + \sigma^2\sigma''/2)(X_t)dt \right| \\ &= O_p\left(\delta T^{3pq/2}\ell(T,x)\sqrt{\log(T/\delta)}\right). \end{aligned}$$

Similarly, by the modulus of continuity of diffusion and Lenglart domination property,

$$\begin{aligned} C_T &= \frac{1}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X| \left| \int_{i\delta}^{(i+1)\delta}\int_{\mathbb{R}}\int_t^{(i+1)\delta} dW_s(\sigma(X_{t-}+z\tau(X_{t-})) - \sigma(X_{t-}))\Lambda(dt,dz) \right| \\ &\leq_p \imath(|\iota|\lambda)\delta^{1/2}T(\sigma'\tau)\sqrt{\log(T/\delta)}\left(\frac{\delta}{h}\sum_{i=1}^{n-1} f_{x,h}(X_{(i-1)\delta})|\Delta_i X|\right) = O_p\left(\delta T^{pq}\ell(T,x)\sqrt{\log(T/\delta)}\right). \end{aligned}$$

The stated result therefore follows under $\delta = o(T^{-6pq})$. ☐

Lemma D.11. *Let the conditions in Lemma D.10 hold. Then*

$$\frac{1}{h}\sum_{i=1}^{n-1}\left(f_{x,h}|\sigma|\right)(X_{(i-1)\delta})|\Delta_{i+1}W| \left| \int_{(i-1)\delta}^{i\delta} (\sigma(X_t) - \sigma(X_{(i-1)\delta}))dW_t \right| = O_p(\delta^{1/2}\ell(T,x)).$$

Proof. The proof is almost identical to that of Lemma D.10, and therefore omitted. ☐

Lemma D.12. *Let (i) μ and τ be twice continuously differentiable on $\mathcal{D}$, (ii) Assumptions 2.1, 2.3 and 2.5 (b) hold, (iii) η satisfies Assumption 2.5 (a) for $\eta = \mu, \sigma^2, \tau^2, \mu', \tau^{2\prime}$, and (iv) $\delta = o(h^2 \wedge T^{-4pq})$. Then*

$$\begin{aligned} &\frac{1}{h}\sum_{i=1}^{n-1}(f_{x,h}|\sigma|)(X_{(i-1)\delta})|\Delta_{i+1}W||\Delta_i X_1^c| = O_p(\delta^{1/2}\ell(T,x)), \\ &\frac{1}{h}\sum_{i=1}^{n-1}(f_{x,h}|\sigma|)(X_{(i-1)\delta})|\Delta_{i+1}W||\Delta_i X^d| = O_p(\delta^{1/2}\ell(T,x)). \end{aligned}$$

Proof. The stated results follow readily from Lenglart domination property. We have

$$\frac{1}{h}\sum_{i=1}^{n-1}(f_{x,h}|\sigma|)(X_{(i-1)\delta})|\Delta_{i+1}W||\Delta_i X_1^c| \leq_p \frac{\delta^{1/2}}{h}\sum_{i=1}^{n-1}(f_{x,h}|\sigma|)(X_{(i-1)\delta})|\Delta_i X_1^c|,$$

from which the first part readily follows due to Lemma D.3, and Lemmas A.9 and A.14 in PW. The second part also follows immediately from

$$\frac{1}{h}\sum_{i=1}^{n-1}(f_{x,h}|\sigma|)(X_{(i-1)\delta})|\Delta_{i+1}W||\Delta_i X^d| \leq_p \imath(|\iota|\lambda)\frac{\delta^{1/2}}{h}\sum_{i=1}^{n-1}\left(f_{x,h}|\sigma|\right)(X_{(i-1)\delta})\int_{(i-1)\delta}^{i\delta}\tau(X_t)dt,$$

and Lemma D.3. □

Lemma D.13. *Let Assumptions 2.1, 2.3, 2.7 and 3.1 hold. Then*

$$\frac{1}{h}\sum_{i=1}^{n-1}(f_{x,h}\sigma^2)(X_{(i-1)\delta})|\Delta_i W||\Delta_{i+1}W|\left(1\left\{|\Delta_i X|\leq\delta^\beta,|\Delta_{i+1}X|\leq\delta^\beta\right\}-1\right)=O_p(\delta^{1/2}\ell(T,x)).$$

Proof. We may write

$$\begin{aligned}\Big|1\{|\Delta_i X|\leq\delta^\beta,|\Delta_{i+1}X|\leq\delta^\beta\}-1\Big| &\leq 1\{|\Delta_i X|>\delta^\beta\}+1\{|\Delta_{i+1}X|>\delta^\beta\}\\ &\leq 2\times 1\{|\Delta_i X|>\delta^\beta/2\}+1\{|X_{(i+1)\delta}-X_{(i-1)\delta}|>\delta^\beta/2\},\end{aligned} \tag{D.19}$$

noting that

$$\begin{aligned}1\{|\Delta_{i+1}X|>\delta^\beta\} &\leq 1\{|X_{(i+1)\delta}-X_{(i-1)\delta}|>\delta^\beta-|\Delta_i X|\}\\ &\leq 1\{|\Delta_i X|>\delta^\beta/2\}+1\{|X_{(i+1)\delta}-X_{(i-1)\delta}|>\delta^\beta/2\}.\end{aligned}$$

It then follows from (D.19) that

$$\begin{aligned}&\left|\frac{1}{h}\sum_{i=1}^{n-1}(f_{x,h}\sigma^2)(X_{(i-1)\delta})|\Delta_i W||\Delta_{i+1}W|\left(1\{|\Delta_i X|\leq\delta^\beta,|\Delta_{i+1}X|\leq\delta^\beta\}-1\right)\right|\\ &\leq\left(\max_{1\leq i\leq n}|\Delta_i W|^2\right)\left(\sup_{|y-x|\leq h}\sigma^2(y)\right)\left[\frac{1}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})\left|1\{|\Delta_i X|\leq\delta^\beta,|\Delta_{i+1}X|\leq\delta^\beta\}-1\right|\right]\\ &\leq_p \delta\log(T/\delta)\left(A_T+B_T\right),\end{aligned} \tag{D.20}$$

where the second relation "$\leq_p$" follows from the modulus of continuity of Brownian motion, and the local boundedness of σ^2, with

$$\begin{aligned}A_T &= \frac{2}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})1\{|\Delta_i X|>\delta^\beta/2\},\\ B_T &= \frac{1}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})1\{|X_{(i+1)\delta}-X_{(i-1)\delta}|>\delta^\beta/2\}.\end{aligned}$$

For A_T, we have

$$\begin{aligned}A_T &\leq_p \frac{1}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})\mathbb{E}_{(i-1)\delta}\{|\Delta_i X|>\delta^\beta/2\}\\ &\leq \delta^{-1}\left(\sup_{|y-x|\leq h}\sup_{0<t\leq\delta}\mathbb{P}_y\left(|X_t-y|>\delta^\beta/2\right)\right)\left(\frac{\delta}{h}\sum_{i=1}^{n-1}f_{x,h}(X_{(i-1)\delta})\right)\\ &= O_p\left(\delta^{-\alpha\beta-\varepsilon}\ell(T,x)\right)\end{aligned} \tag{D.21}$$

for any $\varepsilon>0$, where the third equality follows from Lemma A.13 in PW. Similarly, we have

$B_T = O_p\left(\delta^{-\alpha\beta-\varepsilon}\ell(T,x)\right)$ for any $\varepsilon > 0$, which, together with (D.19)-(D.21), completes the proof by choosing $0 < \varepsilon < 1/2 - \alpha\beta$. □

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