# Online Supplement to 'Robust Tests for White Noise and Cross-Correlation'*

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## Preface

This Online Supplement comprises two separate documents (I and II). Supplement I provides proofs of all the results given in the main paper. Supplement II provides details of the full Monte Carlo experiment reported in the text of the main paper. The documents are arranged below in sequence.

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# Online Supplement I to 'Robust Tests for White Noise and Cross-Correlation'*

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## 1 Introduction

This Supplement I provides proofs of the results given in the text of the main paper. Equation references to the main paper are denoted with the affix M as (#M) and references to theorem and proposition numbers in the main paper are signified as "Theorem #M" and "Proposition #M". Details of the full Monte Carlo experiment are provided in Supplementary II. References used here are the same as those given in the main paper and are not listed.

## 2 Proofs of Results in the Main Paper

The univariate tests for the absence of autocorrelation for a time series $\{x_t\}$ in Section 2 form a special case of the bivariate tests given in Section 3 for the absence of cross-correlation between two series $\{x_t\}$ and $\{y_t\}$. Setting $y_t = x_t$ simplifies the assumptions of the bivariate tests. We demonstrate next how the results of Section 3 may be used to imply those of Section 2.

**Proof of Theorem 2.1M**. We show that under the assumptions of Theorem 2.1M, the bivariate series $\{x_t, y_t\}$ with $y_t = x_t$ satisfies the assumptions of Theorem 3.1M.

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First, in such a case, (22M) holds with $\mu_x = \mu_y$, $h_t = g_t$ and $\varepsilon_t = \eta_t$, while $q_n = \sum_{t=1}^n h_t^2 g_t^2 = \sum_{t=1}^n h_t^4$ and assumption (7M) on $h_t$ implies assumption (26M) on $(h_t, g_t)$.

Second, since in Theorem 2.1M, $\{\varepsilon_t\}$ is a stationary ergodic m.d. sequence with respect to some $\sigma$-field filtration $\mathcal{F}_t$ that includes the natural filtration then for any $k \geq 1$ the sequence $\omega_{tk} := \varepsilon_t \varepsilon_{t-k}$ is an m.d. sequence with respect to the same $\sigma$-field $\mathcal{F}_t$. Recall that $\mathbb{E}\varepsilon_t^4 < \infty$.

Third, we verify that the $\omega_{tk}$ satisfy Assumption A of Theorem 3.1M. Recall, that if $\{\varepsilon_t\}$ is a stationary ergodic sequence and $\phi(.)$ is a measurable function, then the sequence $z_t = \phi(\varepsilon_t, \varepsilon_{t+1}, ...)$ is also stationary and ergodic (Stout, 1974, Thm. 3.5.8). In addition, by Stout (1974, Cor. 3.5.2.), if $\{z_t\}$ is stationary and ergodic with $\mathbb{E}|z_1| < \infty$ then

$$\mathbb{E}|n^{-1} \sum_{t=1}^n z_t - \mathbb{E}z_1| \to 0, \quad n \to \infty. \tag{1}$$

This implies that for any $k, j \geq 0$, $\{\omega_{tk}\}$ and $\{\omega_{tk}\omega_{tj}\}$ are stationary ergodic sequences with $\mathbb{E}|\omega_{tk}\omega_{tj}| < \infty$, and $z_t = \omega_{tk}\omega_{tj} = \varepsilon_t^2 \varepsilon_{t-j}\varepsilon_{t-k}$ has property (1). This verifies Assumption A.

Fourth, $\{\varepsilon_t\}$ satisfies Assumption B of Theorem 3.1M since the $\varepsilon_t$ are uncorrelated variables.

Thus, all assumptions of Theorem 3.1 are satisfied and (28M) implies that

$$(\widetilde{t}_1, ..., \widetilde{t}_m) = (\widetilde{t}_{xx,1}, ..., \widetilde{t}_{xx,m}) \to_D \mathcal{N}(0, R_{xx}) \tag{2}$$

where $R_{xx} = (r_{xx,jk},\ j, k = 1, ..., m)$ is a matrix with elements $r_{xx,jk} = \mathrm{corr}(\varepsilon_1\varepsilon_{1-j},\ \varepsilon_1\varepsilon_{1-k})$. This proves (9M) and completes the proof of Theorem 2.1M. □

**Proof of Theorem 2.2M**. In the bivariate case $\{x_t,\ y_t\}$ with $y_t = x_t$ and $m_0 = 1$ all test statistics are the same as in the univariate case discussed in Theorem 2.2M. In addition, we showed above that under the assumptions of Theorem 2.1M the assumptions of Theorem 3.1M are satisfied. Hence Theorem 3.2M implies the results of Theorem 2.2M. □

To prove the results given in Section 3 we use the following theorem establishing the asymptotic normality of self-normalized sums of products $x_t y_{t-k}$, $t = 1, ..., n$ with lag $k \geq 0$ of the random variables

$$x_t = h_t \varepsilon_t, \quad y_t = g_t \eta_t \quad t = ... - 1, 0, 1, ... \tag{3}$$

where $\{\varepsilon_t\}$ and $\{\eta_t\}$ are stationary sequences, $\mathbb{E}\varepsilon_t = \mathbb{E}\eta_t = 0$, $\mathbb{E}\varepsilon_t^4 < \infty$, $\mathbb{E}\eta_t^4 < \infty$ and $h_t, g_t$ are positive real numbers. Define

$$\widetilde{t}^*_{xy,k} = \frac{\sum_{t=k+1}^n x_t y_{t-k}}{\left(\sum_{t=k+1}^n x_t^2 y_{t-k}^2\right)^{1/2}}, \quad \widetilde{t}_{xy,k} = \frac{\sum_{t=k+1}^n (x_t - \bar{x})(y_{t-k} - \bar{y})}{\left(\sum_{t=k+1}^n (x_t - \bar{x})^2 (y_{t-k} - \bar{y})^2\right)^{1/2}}. \tag{4}$$

Recall Assumptions A and B and assumption (26M) on $(h_t, g_t)$ from Section 3 which will be used in what follows.

The theorem below establishes the multivariate limit distribution of the vector $(\widetilde{t}^*_{xy,m_0}, ..., \widetilde{t}^*_{xy,m})$ when $\mathrm{corr}(x_t, y_{t-k}) = 0$ for $k = m_0, ..., m$ $(0 \le m_0 \le m)$.

**Theorem 2.1.** *Let $\{x_t,\, y_t\}$ be as in (3), and Assumption A and (26M) hold.*

*If for some $0 \le m_0 \le m$, $\{\varepsilon_t \eta_{t-k}\}$, $k = m_0, ..., m$ are m.d. sequences with respect to the same $\sigma$-field $\mathcal{F}_t$, then, as $n \to \infty$,*

$$(\widetilde{t}^*_{xy,m_0}, ..., \widetilde{t}^*_{xy,m}) \to_D \mathcal{N}(0, R_{xy}) \tag{5}$$

*where $R_{xy} = (r_{xy,jk},\, j, k = m_0, ..., m)$ is a matrix with elements*

$$r_{xy,jk} = \mathrm{corr}(\varepsilon_1 \eta_{1-j},\, \varepsilon_1 \eta_{1-k}).$$

*In particular, $\widetilde{t}^*_{xy,k} \to_D \mathcal{N}(0,1)$ for $k = m_0, ..., m$.*

**Corollary 2.1.** *Under the assumptions of Theorem 2.1 and Assumption B, as $n \to \infty$,*

$$\widetilde{t}_{xy,k} = \widetilde{t}^*_{xy,k} + o_p(1), \quad k = m_0, ..., m, \tag{6}$$

$$(\widetilde{t}_{xy,m_0}, ..., \widetilde{t}_{xy,m}) \to_D \mathcal{N}(0, R_{xy}). \tag{7}$$

**Proof of Theorem 2.1**. Denote

$$q_{njk} = \sum_{t=\max(j,k)+1}^n h_t^2 g_{t-j} g_{t-k}, \quad j, k \ge 0. \tag{8}$$

Write $\widetilde{t}^*_{xy,k}$ in (4) as a self-normalized sum

$$\widetilde{t}^*_{xy,k} = \frac{\sum_{t=k+1}^n \zeta_{tk}}{\left(\sum_{t=k+1}^n \zeta_{tk}^2\right)^{1/2}} \tag{9}$$

of random variables

$$\zeta_{tk} = (q_n \mathbb{E}[\varepsilon_1^2 \eta_{1-k}^2])^{-1/2} x_t y_{t-k}.$$

We will show that

$$\widetilde{t}^*_{xy,k} = \sum_{t=k+1}^n \zeta_{tk} + o_p(1) = s_{nk} + o_p(1), \quad s_{nk} := \sum_{t=k+1}^n \zeta_{tk}. \tag{10}$$

The latter follows from

$$\sum_{t=k+1}^n \zeta_{tk}^2 \to_p 1, \quad \sum_{t=k+1}^n \zeta_{tk} = O_p(1). \tag{11}$$

The first claim is shown in (43) below. Under the assumptions of the theorem, $\{\zeta_{tk}\}$ is an m.d. sequence. So, $\mathbb{E}\zeta_t = 0$, $\mathbb{E}[\zeta_{tk}\zeta_{sk}] = 0$ for $t \neq s$, and

$$\mathbb{E}\big(\sum_{t=k+1}^n \zeta_{tk}\big)^2 = \sum_{t=k+1}^n \mathbb{E}\zeta_{tk}^2 = q_n^{-1} \sum_{t=k+1}^n h_t^2 g_{t-k}^2 = q_{nkk}/q_n \to 1,$$

by Lemma 2.1. Hence $\sum_{t=k+1}^n \zeta_{tk} = O_p(1)$ which proves the second claim in (11) and completes verification of (10).

Hence, to prove (5), i.e. $(\widetilde{t}^*_{xy,m_0}, ..., \widetilde{t}^*_{xy,m}) \to_D \mathcal{N}(0, R_{xy})$, it suffices to show that

$$(s_{nm_0}, ..., s_{nm}) \to_D \mathcal{N}(0, R_{xy}). \tag{12}$$

By the Cramér-Wold device, the latter holds if for any real numbers $a_{m_0}, a_{m_0+1}, ..., a_m$,

$$S_n = \sum_{k=m_0}^m a_k s_{nk} \to \mathcal{N}(0, \sigma_m^2), \quad \sigma_m^2 = \sum_{j,k=m_0}^m a_j a_k r_{xy,jk}. \tag{13}$$

Using the definition of $s_{nk}$, we can write

$$S_n = \sum_{t=1}^n \widetilde{\zeta}_t, \quad \widetilde{\zeta}_t := \sum_{k=m_0}^m I(t \geq k+1) a_k (q_n \mathbb{E}[\varepsilon_1^2 \eta_{1-k}^2])^{-1/2} x_t y_{t-k}. \tag{14}$$

Under the assumptions of the theorem, $\{\widetilde{\zeta}_t\}$ is an m.d. sequence with finite variance $\mathbb{E}\widetilde{\zeta}_t^2 < \infty$. Hence, by Theorem 3.2 of Hall and Heyde (1980), to prove (13) it suffices to show that

$$(a) \sum_{t=1}^n \widetilde{\zeta}_t^2 \to_p \sigma_m^2, \quad (b) \max_{t=1,...,n} |\widetilde{\zeta}_t| \to_p 0, \quad (c)\ \mathbb{E}[\max_{t=1,...,n} \widetilde{\zeta}_t^2] = O(1). \tag{15}$$

Denote $\omega_{tk} = \varepsilon_t \eta_{t-k}$. We have

$$\sum_{t=k+1}^n \widetilde{\zeta}_t^2 = \sum_{j,k=m_0}^m a_j a_k (\mathbb{E}\omega_{1j}^2 \mathbb{E}\omega_{1k}^2)^{-1/2} \Big( q_n^{-1} \sum_{t=\max(j,k)+1}^n x_t^2 y_{t-k} y_{t-j} \Big).$$

Under the assumptions of the theorem, by (43) of Lemma 2.2 below,

$$q_n^{-1}\sum_{t=\max(j,k)+1}^n x_t^2 y_{t-j}y_{t-k}\to_p \mathbb{E}[\omega_{1j}\omega_{1k}].$$

Hence,

$$\sum_{t=k+1}^n \widetilde{\zeta}_t^2 \to \sum_{j,k=m_0}^m a_j a_k(\mathbb{E}\omega_{1j}^2\mathbb{E}\omega_{1k}^2)^{-1/2}\mathbb{E}[\omega_{1j}\omega_{1k}] = \sum_{j,k=m_0}^m a_j a_k r_{xy,jk} = \sigma_m^2$$

which proves (15)(a).

To prove (15)(b), notice that

$$|\widetilde{\zeta}_t| \le c_0\sum_{k=m_0}^m |\zeta_{tk}|I(t\ge k+1), \quad c_0 := \max_{k=m_0,\dots,m}|a_k|.$$

To show that $P(\max_{t=1,\dots,n}|\widetilde{\zeta}_t|\ge\varepsilon)\to 0$, it suffices to prove that for $k=m_0,\dots,m$ and for any $\varepsilon>0$,

$$P(\max_{t=k+1,\dots,n}|\zeta_{tk}|\ge\varepsilon)\to 0.$$

We have

$$P(\max_{t=k+1,\dots,n}|\zeta_{tk}|\ge\varepsilon) \le \sum_{t=k+1}^n P(|\zeta_{tk}|\ge\varepsilon)\le\varepsilon^{-2}\sum_{t=k+1}^n \mathbb{E}\zeta_{tk}^2 I(\zeta_{tk}^2\ge\varepsilon^2). \quad (16)$$

Write $\zeta_{tk}^2 = c_{tk}\omega_{tk}^2$, where $c_{tk}=(q_n\mathbb{E}\omega_{tk}^2)^{-1}h_t^2 g_{t-k}^2$. By (26M),

$$\max_{t=1,\dots,n} c_{tk}\le Cq_n^{-1}\max_{t=1,\dots,n}h_t^2\max_{t=1,\dots,n}g_t^2 =: \delta_n = o(1).$$

Therefore, $\zeta_{tk}^2\le\delta_n\omega_{tk}^2$. By the assumptions of the theorem, $\{\omega_{tk}^2\}$ is a stationary sequence such that $\mathbb{E}\omega_{1k}^2<\infty$. Hence,

$$\begin{aligned} P(\max_{t=k+1,\dots,n}|\zeta_{tk}|\ge\varepsilon) &\le \varepsilon^{-2}\sum_{t=k+1}^n c_{tk}\mathbb{E}\omega_{tk}^2 I(\omega_{tk}^2\ge\delta_n^{-1}\varepsilon^2) \quad (17)\\ &= \varepsilon^{-2}(\sum_{t=k+1}^n c_{tk})\mathbb{E}\omega_{1k}^2 I(\omega_{1k}^2\ge\delta_n^{-1}\varepsilon^2) = o(1)\end{aligned}$$

because $\sum_{t=k+1}^n c_{tk} = (\mathbb{E}\omega_{1k}^2)^{-1}q_{nkk}/q_n\to(\mathbb{E}\omega_{1k}^2)^{-1}$ by (38), and $\mathbb{E}\omega_{1k}^2 I(\omega_{1k}^2\ge\delta_n^{-1}\varepsilon^2)\to 0$ since $\mathbb{E}\omega_{1k}^2<\infty$ and $\delta_n\to 0$. This proves (b). (15)(c) can be shown using a similar argument. This proves (13) and (5) and completes the proof of the theorem. □

**Proof of Corollary 2.1**. Denote $\zeta^*_{tk} = (q_n \mathbb{E}\omega^2_{tk})^{-1/2}(x_t - \bar{x})(y_{t-k} - \bar{y})$. Then

$$\widetilde{t}_{xy,k} = \frac{\sum_{t=k+1}^{n} \zeta^*_{tk}}{\big(\sum_{t=k+1}^{n} \zeta^{*\,2}_{tk}\big)^{1/2}} = \frac{\sum_{t=k+1}^{n} \zeta_{tk} + R_{n1}}{\big(\sum_{t=k+1}^{n} \zeta^2_{tk} + R_{n2}\big)^{1/2}} \tag{18}$$

where $\zeta_{tk}$ is defined as in (9) and

$$R_{n1} = \sum_{t=k+1}^{n} (\zeta^*_{tk} - \zeta_{tk}), \quad R_{n2} = \sum_{t=k+1}^{n} (\zeta^{*\,2}_{tk} - \zeta^2_{tk}).$$

In (44) and (45) we show that

$$R_{n1} = o_p(1), \quad R_{n2} = o_p(1). \tag{19}$$

Together with (18) this implies

$$\widetilde{t}_{xy,k} = \frac{\sum_{t=k+1}^{n} \zeta_{tk} + o_p(1)}{\big(\sum_{t=k+1}^{n} \zeta^2_{tk} + o_p(1)\big)^{1/2}} = \frac{\sum_{t=k+1}^{n} \zeta_{tk}}{\big(\sum_{t=k+1}^{n} \zeta^2_{tk}\big)^{1/2}} + o_p(1) = \widetilde{t}^*_{xy,k} + o_p(1). \tag{20}$$

This verifies (6) and together with (5) proves (7). □

**Proof of Theorem 3.1M**. The claim of the theorem is shown in Corollary 2.1. □

**Proof of Corollary 3.1M**. Set $s_{xn} = \sum_{t=1}^{n}(x_t - \bar{x})^2$, $s_{yn} = \sum_{t=1}^{n}(y_t - \bar{y})^2$. By definition,

$$\widehat{c}_{xy,k} = \frac{\widetilde{t}_{xy,k}}{\widehat{\rho}_{xy,k}} = \frac{s^{1/2}_{xn} s^{1/2}_{yn}}{(\sum_{t=1}^{n} e^2_{xy,tk})^{1/2}}.$$

We will show that

$$s_{xn} = \mathbb{E}\varepsilon^2_1 (\sum_{t=1}^{n} h^2_t)(1 + o_p(1)), \quad s_{yn} = \mathbb{E}\eta^2_1 (\sum_{t=1}^{n} g^2_t)(1 + o_p(1)), \tag{21}$$

$$\sum_{t=1}^{n} e^2_{xy,tk} = \mathbb{E}[\varepsilon^2_1 \eta^2_{1-k}] q_n (1 + o_p(1)) \tag{22}$$

which implies (30M):

$$\widehat{c}_{xy,k} = \Big(\frac{\sum_{t=1}^{n} h^2_t \sum_{t=1}^{n} g^2_t}{\sum_{t=1}^{n} h^2_t g^2_t} \frac{\mathbb{E}\varepsilon^2_1 \mathbb{E}\eta^2_1}{\mathbb{E}[\varepsilon^2_1 \eta^2_{1-k}]}\Big)^{1/2}(1 + o_p(1)).$$

*Proof of (21).* We prove the claim for $s_{xn}$ (the claim for $s_{yn}$ follows using a similar argument). Without restriction of generality, assume that $\mathbb{E}x_t = 0$. Then, $x_t = h_t\varepsilon_t$,

$$s_{xn} = \sum_{t=1}^{n} x_t^2 - n\bar{x}^2 = \sum_{t=1}^{n} h_t^2\varepsilon_t^2 - n\bar{x}^2. \tag{23}$$

We have $\sum_{t=1}^{n}\mathbb{E}x_t^2 = \mathbb{E}\varepsilon_1^2\sum_{t=1}^{n}h_t^2$. We will show that

$$\mathrm{var}(\sum_{t=1}^{n} x_t^2) = o_p\big((\sum_{t=1}^{n} h_t^2)^2\big), \quad n\bar{x}^2 = o_p(\sum_{t=1}^{n} h_t^2) \tag{24}$$

which together with (23) proves (21) for $s_{xn}$.

Denote $\nu_{t-s} = \mathrm{cov}(\varepsilon_t^2, \varepsilon_s^2)$. Let $L > 0$. Write

$$\mathrm{var}(\sum_{t=1}^{n} x_t^2) = \sum_{t,s=1}^{n} h_t^2h_s^2\nu_{t-s} \le \sum_{t,s=1:|t-s|\ge L}^{n} h_t^2h_s^2|\nu_{t-s}| + \sum_{t,s=1:|t-s|\le L}^{n} h_t^2h_s^2|\nu_{t-s}| =: i_{1,n} + i_{2,n}.$$

By assumption of the corollary, $\nu_k \to 0$ as $k \to \infty$. Therefore, $\delta_L := \max_{k:|k|\ge L}|\nu_k| \to 0$ as $L \to \infty$. Then

$$i_{1,n} \le \delta_L\sum_{t,s=1}^{n} h_t^2h_s^2 = \delta_L(\sum_{t=1}^{n} h_t^2)^2.$$

On the other hand, for any fixed $L$,

$$i_{2,n} = \sum_{t,s=1:|t-s|\le L}^{n} h_t^2h_s^2|\nu_{t-s}| \le \nu_0(\max_{1\le j\le n} h_j^2)\sum_{t=1}^{n} h_t^2(\sum_{s:|t-s|\le L} 1) = o((\sum_{t=1}^{n} h_t^2)^2)$$

because $\max_{1\le s\le n} h_s^2 = o(\sum_{t=1}^{n} h_t^2)$ and $\nu_0 < E\varepsilon_0^4 < \infty$ by assumption of the corollary. This proves $i_{2,n} = o\big((\sum_{t=1}^{n} h_t^2)^2\big)$ which implies the first claim in (24).

To prove the second claim in (24), we use the bound $\mathbb{E}\bar{x}^2 \le Cn^{-2}\sum_{t=1}^{n} h_t^2$ established in (55), which yields $n\mathbb{E}\bar{x}^2 \le C\max_{t=1,\dots,n} h_t^2 = o(\sum_{t=1}^{n} h_t^2)$ by assumption of the corollary. Therefore, $n\bar{x}^2 = o_p(\sum_{t=1}^{n} h_t^2)$.

*Proof of (22).* Recall the notation $\zeta_{tk}^* = (q_n\mathbb{E}[\varepsilon_1^2\eta_{1-k}^2])^{-1/2}(x_t - \bar{x})(y_{t-k} - \bar{y})$ and $\zeta_{tk} = (q_n\mathbb{E}[\varepsilon_1^2\eta_{1-k}^2])^{-1/2}(x_t - \mathbb{E}x_t)(y_{t-k} - \mathbb{E}y_t)$ used in (18) and (10). Then

$$\sum_{t=k+1}^{n} e_{xy,tk}^2 = q_n\mathbb{E}[\varepsilon_1^2\eta_{1-k}^2]\,(\sum_{t=k+1}^{n} \zeta_{tk}^{*2}).$$

By (19) and (11), $\sum_{t=k+1}^{n}\zeta_{tk}^{*2} = \sum_{t=k+1}^{n}\zeta_{tk}^2 + o_p(1) = 1 + o_p(1)$, which proves (22) and completes the proof of the corollary. □

**Proof of Theorem 3.2M**. Suppose for simplicity that $\mu_x = \mu_y = 0$. First we show that $\widehat{r}_{xy,jk} \to_p r_{xy,jk}$.

Under the assumptions of the theorem, (43) and (45) of Lemma 2.2 imply that for $j, k = m_0, ..., m$,

$$q_n^{-1} \sum_{t=\max(j,k)+1}^{n} e_{xy,tj} e_{xy,tk} \to \mathbb{E}[(\varepsilon_1 \eta_{1-j})(\varepsilon_1 \eta_{1-k})].$$

This together with definition (33M) implies

$$\widehat{r}_{xy,jk} \to \frac{\mathbb{E}[(\varepsilon_1 \eta_{1-j})(\varepsilon_1 \eta_{1-k})]}{(\mathbb{E}(\varepsilon_1 \eta_{1-j})^2)^{1/2} (\mathbb{E}(\varepsilon_1 \eta_{1-k})^2)^{1/2}} = \mathrm{corr}(\varepsilon_1 \eta_{1-j}, \varepsilon_1 \eta_{1-k}) = r_{xy,jk}$$

because by assumption $\{\varepsilon_t \eta_{t-k}\}$ is an m.d. sequence and therefore $\mathbb{E}[\varepsilon_1 \eta_{1-k}] = 0$. Next we show that $\widehat{r}^*_{xy,jk} \to_p r_{xy,jk}$ for any $\lambda > 0$. Since $\widehat{r}_{xy,jk} \to_p r_{xy,jk}$, then

$$\widehat{r}^*_{xy,jk} = \widehat{r}_{xy,jk} I(|\tau_{xy,jk}| > \lambda) = (r_{xy,jk} + o_p(1)) I(|\tau_{xy,jk}| > \lambda). \tag{25}$$

If $r_{xy,jk} = 0$, then $|\widehat{r}^*_{xy,jk}| \leq |\widehat{r}_{xy,jk}| \to |r_{xy,jk}| = 0$.

Let $r_{xy,jk} \neq 0$. To show $\widehat{r}^*_{xy,jk} \to_p r_{xy,jk}$, in view of (25), it suffices to prove that $I(|\tau_{xy,jk}| > \lambda) \to_p 1$. To prove the latter we will show that

$$|\tau_{xy,jk}| \to_p \infty. \tag{26}$$

Write $\tau_{xy,jk} = A_n / B_n^{1/2}$ where

$$A_n = q_n^{-1} \textstyle\sum_{t=\max(j,k)+1}^{n} e_{xy,tj} e_{xy,tk}, \quad B_n = q_n^{-2} \textstyle\sum_{t=\max(j,k)+1}^{n} e_{xy,tj}^2 e_{xy,tk}^2.$$

We will prove (26) by showing that

$$A_n \to \mathbb{E}[\varepsilon_1^2 \eta_{1-j} \eta_{t-k}] = \mathrm{cov}(\varepsilon_1 \eta_{1-j}, \varepsilon_1 \eta_{1-k}) \neq 0, \quad B_n = o_p(1). \tag{27}$$

The results (43) and (45) of Lemma 2.2 imply the claim about $A_n$ in (27).

To evaluate $B_n$, denote $e'_{xy,tk} = x_t y_{t-k}$. Then

$$e_{xy,tj}^2 e_{xy,tk}^2 - e'^2_{xy,tj} e'^2_{xy,tk} = (e_{xy,tj}^2 - e'^2_{xy,tj})(e_{xy,tk}^2 - e'^2_{xy,tk}) + e'^2_{xy,tk}(e_{xy,tj}^2 - e'^2_{xy,tj}) + e'^2_{xy,tj}(e_{xy,tk}^2 - e'^2_{xy,tk}).$$

Hence, setting $v_{nk} = \sum_{t=\max(j,k)+1}^{n} |e_{xy,tk}^2 - e'^2_{xy,tk}|$, $q'_{nkk} = \sum_{t=\max(j,k)+1}^{n} e'^2_{xy,tk}$, we obtain

$$i_n := \sum_{t=\max(j,k)+1}^{n} |e^2_{xy,tj}e^2_{xy,tk} - e'^2_{xy,tj}e'^2_{xy,tk}| \le v_{nj}v_{nk} + q'_{nkk}v_{nj} + q'_{njj}v_{nk}.$$

Since $v_{nk} = o_p(q_n)$ by (45) of Lemma 2.2, and $q'_{nkk} = O_p(q_n)$ by (60), this implies $i_n = o_p(q_n^2)$. To prove $B_n = o_p(1)$, it remains to show that

$$i'_n := \textstyle\sum_{t=\max(j,k)+1}^{n} e'^2_{xy,tj}e'^2_{xy,tk} = o_p(q_n^2). \tag{28}$$

First observe that by (26M),

$$\tilde{\delta}_{nk} := \max_{t=k+1,\dots,n} h_t^2 g_{t-k}^2 \le \max_{t=1,\dots,n} h_t^2 \max_{t=1,\dots,n} g_t^2 = o(q_n).$$

Therefore, there exists $\nu_n \to \infty$ such that $\tilde{\delta}_{nk}\nu_n = o(q_n)$.

Recall that $e'^2_{xy,tk} = x_t^2 y_{t-k}^2 = h_t^2 g_{t-k}^2 \omega_{tk}^2$ where $\omega_{tk}^2 = \varepsilon_t^2 \eta_{t-k}^2$. Bound $\omega_{tk}^2$ as $\omega_{tk}^2 \le \omega_{tk}^2 I(\omega_{tk}^2 \ge \nu_n) + \nu_n$. Setting $z_{ntk} = \omega_{tk}^2 I(\omega_{tk}^2 \ge \nu_n)$, we obtain, $e'^2_{xy,tk} \le \nu_n\tilde{\delta}_{nk} + h_t^2 g_{t-k}^2 z_{ntk}$. Hence,

$$\begin{aligned}
i'_n &\le \nu_n\tilde{\delta}_{nk} \textstyle\sum_{t=\max(j,k)+1}^{n} e'^2_{xy,tj} + \sum_{t=\max(j,k)+1}^{n} h_t^2 g_{t-k}^2 z_{ntk} e'^2_{xy,tj} \\
&\le \nu_n\tilde{\delta}_{nk} q'_{njj} + (\textstyle\sum_{t=\max(j,k)+1}^{n} h_t^2 g_{t-k}^2 z_{ntk})(\sum_{t=\max(j,k)+1}^{n} e'^2_{xy,tj}) \\
&= \nu_n\tilde{\delta}_{nk} q'_{njj} + (\textstyle\sum_{t=\max(j,k)+1}^{n} h_t^2 g_{t-k}^2 z_{ntk}) q'_{njj} \\
&= o_p(q_n^2) + (\textstyle\sum_{t=\max(j,k)+1}^{n} h_t^2 g_{t-k}^2 z_{ntk}) O_p(q_n).
\end{aligned}$$

Notice that

$$\mathbb{E}(\sum_{t=\max(j,k)+1}^{n} h_t^2 g_{t-k}^2 z_{ntk}) = (\sum_{t=\max(j,k)+1}^{n} h_t^2 g_{t-k}^2)\mathbb{E}z_{n1k} = q_{nkk}\mathbb{E}[z_{n1k}] = o(q_n)$$

because $q_{nkk} = O(q_n)$ by Lemma 2.1 and $\mathbb{E}z_{n1k} = \mathbb{E}[\omega_{1k}^2 I(\omega_{1k}^2 \ge \nu_n)] \to 0$ because $\nu_n \to \infty$. This implies $i'_n = o_p(q_n^2)$ which proves (28), (27) and $\widehat{r}^*_{xy,jk} \to_p r_{xy,jk}$ and completes the proof of (36M).

Thus, $\widehat{R}_{xy} \to_p R_{xy}$ and $\widehat{R}^*_{xy} \to_p R_{xy}$. This together with (31M) proves (37M) which completes the proof of the theorem. □

**Proof of Lemma 3.1M**. Suppose that $a'\mathrm{Cov}(\eta)a = 0$ for some $a = (a_1, ..., a_m)'$ with $||a|| = 1$. Since the autocovariance sequence $\{\gamma_\eta(h) = \mathrm{cov}(\eta_j, \eta_{j-h})\}$ is non-negative definite and the spectral density $f_\eta(x)$ exists, $\mathrm{cov}(\eta_j, \eta_k) = \int_{-\pi}^{\pi} e^{i(j-k)x} f_\eta(x)dx$

by Herglotz's theorem. It follows that

$$a'\mathrm{Cov}(\eta)a \;=\; \sum_{j,k=1}^{m} a_j a_k \,\mathrm{cov}(\eta_j,\eta_k) = \int_{-\pi}^{\pi} |\sum_{j=1}^{m} a_j e^{ijx}|^2 f_\eta(x)dx = 0.$$

Therefore $|\sum_{j=1}^m a_j e^{ijx}|^2 f_\eta(x) = 0$ *a.e.* with respect to Lebesgue measure. Since $|\sum_{j=1}^m a_j e^{ijx}|^2$ can have at most a finite number of zeroes on $[-\pi,\pi]$, this implies $f_\eta(x) = 0$ *a.e.*. So, $\mathrm{var}(\eta_t) = \int_{-\pi}^{\pi} f_\eta(x)dx = 0$ which contradicts the assumption $\mathbb{E}\eta_t^2 > 0$. Since $\eta_t$ is a stationary series, the same argument implies that $\mathrm{Corr}(\eta)$ is positive definite.

Now let $a'\mathrm{Cov}(z)a = 0$ and $||a|| = 1$. By assumption, $z_j = \varepsilon_1\eta_{1-j}$, $\mathbb{E}z_j = 0$ and $\mathbb{E}z_j^2 < \infty$. Thus, $\mathrm{cov}(z_j,z_k) = \mathbb{E}[z_j z_k] = \mathbb{E}[\varepsilon_1^2\eta_{1-j}\eta_{1-k}]$, and

$$a'\mathrm{Cov}(z)a \;=\; \sum_{j,k=1}^{m} a_j a_k \,\mathrm{cov}(z_j,z_k) = \sum_{j,k=1}^{m} a_j a_k \mathbb{E}[z_j z_k] = \mathbb{E}[\varepsilon_1^2(\sum_{j=1}^{m} a_j\eta_{1-j})^2] = 0.$$

Since by assumption $\varepsilon_1 \neq 0$ *a.s.* this implies $\mathbb{E}(\sum_{j=1}^m a_j\eta_{1-j})^2 = 0$. So,

$$\mathbb{E}(\sum_{j=1}^{m} a_j\eta_{1-j})^2 = \sum_{j,k=1}^{m} a_j a_k \,\mathrm{cov}(\eta_j,\eta_k) = 0.$$

As shown above, this implies $\mathbb{E}\eta_j^2 = 0$ which leads to a contradiction.

Observe that $\mathbb{E}z_j^2 > 0$ for any $j \geq 1$. Otherwise, the equality $\mathbb{E}z_j^2 = \mathbb{E}[\varepsilon_1^2\eta_{1-j}^2] = 0$ and the assumption $\varepsilon_1 \neq 0$ *a.s.* would imply $\mathbb{E}\eta_{1-j}^2 = 0$ which leads to a contradiction. Then an argument similar to the above implies that $\mathrm{Corr}(z)$ is a positive definite matrix. $\square$

**Proof of Theorem 3.3M**. By assumption $\{\varepsilon_t\}$ is an m.d. sequence with respect to some $\sigma$-field $\mathcal{F}_t$ and $\{x_t\}$ and $\{y_t\}$ are mutually independent sequences. Therefore for $k = ..., -1, 0, 1, ...$ $\{\varepsilon_t\eta_{t-k}\}$ are m.d. sequences with respect to the $\sigma$-field $\mathcal{F}_{nt}^* = \mathcal{F}_t \cup \sigma(\eta_s, s = 1, ..., n)$, i.e. $\mathbb{E}[\varepsilon_t\eta_{t-k}|\mathcal{F}_{n,t-1}^*] = 0$. Moreover, for any negative or positive integers $j, k$,

$$\mathrm{corr}(\varepsilon_t\eta_{t-j},\, \varepsilon_t\eta_{t-k}) = \mathrm{corr}(\eta_j,\eta_k).$$

Clearly, Corollary 2.1 implies convergence $\widetilde{t}_{xy} = (\widetilde{t}_{xy,m_0}, ..., \widetilde{t}_{xy,m})' \to_D \mathcal{N}(0, R_y)$ in (38M) and together with Theorem 3.2M proves (39M).

The same argument as in the proof of (6) and (10) implies that

$$\widetilde{t}^*_{yx,k} = s_{yx,nk} + o_p(1), \quad s_{yx,nk} := \sum_{t=k+1}^n \zeta_{yx,nk} \tag{29}$$

where $\zeta_{yx,nk} = (q_n \mathbb{E}[\eta_1^2 \varepsilon_{1-k}^2])^{-1/2} y_t x_{t-k}$. Rewriting $s_{yx,nk}$ as

$$s_{yx,nk} = (q_n \mathbb{E}[\varepsilon_1^2 \eta_{1-k}^2])^{-1/2} \sum_{t=1}^{n-k} x_t y_{t+k} = s_{n(-k)}$$

we arrive at the sum as in (10) where $y_{t+k}$ appears with a negative lag. Since for any integer $k$, $\{\varepsilon_t \eta_{t+k}\}$ is an m.d. sequence with respect to $\sigma$-field $\mathcal{F}^*_{nt}$, the same argument as in the proof of Theorem 3.2M implies convergence $\widetilde{t}_{yx} = (\widetilde{t}_{yx,m_0}, ..., \widetilde{t}_{yx,m}) \to_D \mathcal{N}(0, R_y)$ in (38M), and the same argument as in the proof of Theorem 3.2M implies (40M). $\square$

**Proof of Proposition 3.1M**. Properties (6) and (10) do not require $\{\varepsilon_t\}$ or $\{\eta_t\}$ to be an m.d. sequence and hold under assumptions of the proposition. They imply (41M),

$$\widetilde{t}_{xy,k} = s_{nk} + o_p(1). \tag{30}$$

By definition (10), we have $s_{nk} = \sum_{t=k+1}^n \zeta_{tk}$, $\zeta_{tk} = (q_n \mathbb{E}[\varepsilon_1^2 \eta_{1-k}^2])^{-1/2} h_t g_{t-k} \omega_{tk}$, $\omega_{tk} = \varepsilon_t \eta_{t-k}$.

Clearly, $\mathbb{E} s_{nk} = 0$ since $\mathbb{E}\omega_{tk} = 0$. Moreover,

$$\gamma_{\omega,t-s} := \mathrm{cov}(\omega_{tk}, \omega_{sk}) = \mathrm{cov}(\varepsilon_t, \varepsilon_s)\mathrm{cov}(\eta_{t-k}, \eta_{s-k}) = \gamma_{\varepsilon,t-s}\gamma_{\eta,t-s}.$$

By mutual independence of $\{x_t\}$ and $\{y_t\}$,

$$r_j := \gamma_{\omega,j}/\mathbb{E}[\varepsilon_1^2]\mathbb{E}[\eta_1^2] = \mathrm{corr}(\varepsilon_1, \varepsilon_{1-j})\mathrm{corr}(\eta_1, \eta_{1-j}).$$

Moreover, under Assumption B, $\sum_j |r_j| < \infty$. Hence,

$$\begin{aligned} \mathbb{E} s_{nk}^2 &= \sum_{t,s=k+1}^n \mathbb{E}[\zeta_{tk}\zeta_{sk}] = q_n^{-1} \sum_{t,s=k+1}^n h_t g_{t-k} h_s g_{s-k} r_{t-s} \\ &= q_n^{-1} \sum_{t,s=k+1:|t-s|>L}^n [...] + q_n^{-1} \sum_{t,s=k+1:|t-s|\le L}^n [...] =: \nu_{n,1} + \nu_{n,2} \end{aligned} \tag{31}$$

where $L > 0$ is a fixed large number. Bound

$$\nu_{n,1} \le q_n^{-1} (\sum_{t=k+1}^n h_t^2 g_{t-k}^2)(\sum_{|j|>L} |r_j|) = q_n^{-1} q_{nkk} (\sum_{|j|>L} |r_j|). \tag{32}$$

By (38), $q_n^{-1} q_{nkk} \to 1$ as $n \to \infty$, while $\sum_{|j|>L} |r_j| \to 0$ as $L \to \infty$. Thus, $\nu_{n,1} \to 0$ as

$n, L \to \infty$.

Next we will show that for any fixed $L$, as $n \to \infty$,

$$\nu_{n,2} \to \sum_{|j|\le L} r_j. \tag{33}$$

Let $t - s = \ell$ where $\ell \ge 0$ is fixed. Then, $s = t - \ell$, and

$$q_n^{-1} \sum_{t,s=k+1:\, t-s=\ell}^{n} h_t g_{t-k} h_s g_{s-k} r_{t-s} = q_n^{-1} \Big( \sum_{t=k+1}^{n} h_t g_{t-k} h_{t-\ell} g_{t-\ell-k} \Big) r_\ell \to r_\ell$$

by the first claim in (39) below, which proves (33).

Since $\sum_{|j|\le L} r_j \to \sum_{j=-\infty}^{\infty} r_j = \sigma_{xy}^2$ as $L \to \infty$, this proves $\mathbb{E}s_{nk}^2 \to \sigma_{xy}^2$ as $n \to \infty$ which completes the proof of the proposition. $\Box$

**Proof of Theorem 3.4M**. Denote $\nu_n = \sum_{t=k+1}^{n} h_t g_{t-k} \omega_{tk}$. By (30),

$$\widetilde{t}_{xy,k} = s_{nk} + o_p(1) = (q_n \mathbb{E}[\omega_{1k}^2])^{-1/2} \nu_n + o_p(1).$$

We will show

$$\mathbb{E}\nu_n = \mathbb{E}[\varepsilon_1 \eta_{1-k}] (\textstyle\sum_{t=1}^{n} h_t g_t)(1 + o(1)), \quad \mathrm{var}(\nu_n) = O(q_n). \tag{34}$$

This implies $q_n^{-1/2} \nu_n = q_n^{-1/2} (\sum_{t=1}^{n} h_t g_t) \mathbb{E}[\varepsilon_1 \eta_{1-k}](1 + o_p(1))$ which proves (44M). Under assumption (43M), $\mathbb{E}\nu_n = (\sum_{t=k+1}^{n} h_t g_{t-k}) \mathbb{E}\omega_{1k} = \mathbb{E}\omega_{1k} (\sum_{t=1}^{n} h_t g_t)(1 + o(1))$. Under (42M), the same argument as in the proof $\mathrm{var}(s_{nk}) \to \sigma_{xy}^2$ in Proposition 3.1M implies that $\mathrm{var}(\nu_n) = O(q_n)$ which completes the proof of (34) and the theorem. $\Box$

**Proof of Theorem 4.1M**. We will verify (46M). (The proof of (47M) follows using a similar argument). Denote $\xi_{tk} = \varepsilon_t \varepsilon_{t-k}$, $|\xi|_{tk} = (|\varepsilon_t| - \mathbb{E}|\varepsilon_t|)(|\varepsilon_{t-k}| - \mathbb{E}|\varepsilon_{t-k}|)$. By Lemma 2.4,

$$\tau_{x,k} = \widetilde{\tau}_{x,k} + o_p(1), \quad \widetilde{\tau}_{x,k} := \frac{1}{\sigma_\varepsilon^2 (n-k)^{1/2}} \sum_{t=k+1}^{n} \xi_{tk}$$

$$\tau_{|x|,k} = \widetilde{\tau}_{|x|,k} + o_p(1), \quad \widetilde{\tau}_{|x|,k} := \frac{1}{\sigma_{|\varepsilon|}^2 (n-k)^{1/2}} \sum_{t=k+1}^{n} |\xi|_{tk}.$$

Hence, to prove (46M), it suffices to show that

$$(\widetilde{\tau}_{x,1}, \widetilde{\tau}_{|x|,1}, \ldots, \widetilde{\tau}_{x,m}, \widetilde{\tau}_{|x|,m}) \to_D \mathcal{N}(0, V_{x,|x|,2m}). \tag{35}$$

Observe that for any $k \geq 1$, $\{\xi_{tk}\}$, $\{|\xi|_{tk}\}$ are m.d. sequences with respect to the $\sigma$-field $\mathcal{F}_t = \sigma(\varepsilon_s, s \leq t)$. Moreover, $\mathbb{E}\xi_{tk}^2 = \mathbb{E}[\varepsilon_t^2 \varepsilon_{t-k}^2] = \sigma_\varepsilon^4$,

$\mathbb{E}|\xi|_{tk}^2 = (\mathbb{E}[(|\varepsilon_t| - \mathbb{E}|\varepsilon_t|)^2])^2 = \sigma_{|\varepsilon|}^4$, $\mathbb{E}[\xi_{tk}|\xi|_{tk}] = (\mathbb{E}[\varepsilon_t(|\varepsilon_t| - \mathbb{E}|\varepsilon_t|)])^2 = \mathrm{cov}^2(\varepsilon_1, |\varepsilon_1|)$. In addition, for $k > j \geq 1$, $\mathbb{E}[\xi_{tk}\xi_{tj}] = 0$, $\mathbb{E}[|\xi|_{tk}|\xi|_{tj}] = 0$, $\mathbb{E}[\xi_{tk}|\xi|_{tj}] = 0$. Finally,

$$n^{-1} \sum_{t=k+1}^n \xi_{tk}^2 \to \mathbb{E}\xi_{1k}^2 = \sigma_\varepsilon^4, \quad n^{-1} \sum_{t=k+1}^n |\xi|_{tk}^2 \to \mathbb{E}|\xi|_{1k}^2 = \sigma_{|\varepsilon|}^4, \tag{36}$$
$$n^{-1} = \sum_{t=k+1}^n \xi_{tk}|\xi|_{tk} \to \mathbb{E}[\xi_{tk}|\xi|_{tk}] = \mathrm{cov}^2(\varepsilon_1, |\varepsilon_1|).$$

Recall that an i.i.d. sequence $\{\varepsilon_t\}$ is ergodic. Therefore $\xi_{tk}^2$, $|\xi|_{tk}^2$ and $\xi_{tk}|\xi|_{tk}$ are also ergodic sequences since they are measurable functions $f(\varepsilon_t, \varepsilon_{t-k})$ of the ergodic sequence $\{\varepsilon_t\}$. The latter implies (36), see Remark 2.1. Hence, (35) follows using the same argument as in the proof of (12). $\square$

To prove Theorem 2.1 we use the following technical lemma. Recall the definition of $q_n$ and $q_{nkj}$ given in (26M) and (8). Set

$$q_{nijk} = \sum_{t=\max(i,j,k)+1}^n h_t h_{t-i} g_{t-j} g_{t-k}, \tag{37}$$
$$\Delta_{njk} = \sum_{t=\max(j,k)+2}^n |h_t^2 g_{t-j} g_{t-k} - h_{t-1}^2 g_{t-1-j} g_{t-1-k}|.$$

**Lemma 2.1.** *Let $h_t, g_t$, $t \geq 1$ satisfy (26M) and $k \geq i, j \geq 1$ be fixed. Then, as $n \to \infty$,*

$$q_n - q_{njk} = o(q_n), \quad q_{njk}/q_n \to 1, \tag{38}$$
$$q_n - q_{nijk} = o(q_n), \quad \Delta_{njk} = o(q_n). \tag{39}$$

**Proof of Lemma 2.1**. We have

$$|q_n - q_{nkj}| \leq \sum_{t=1}^k h_t^2 g_t^2 + \sum_{t=k+1}^n h_t^2 |g_t^2 - g_{t-j} g_{t-k}|. \tag{40}$$

By (26M), $\sum_{t=1}^k h_t^2 g_t^2 \leq k \max_{t=1,\dots,n} h_t^2 \max_{t=1,\dots,n} g_t^2 = o(q_n)$. Next we show that

$$\sum_{t=k+1}^n h_t^2 |g_t^2 - g_{t-j} g_{t-k}| = o(q_n). \tag{41}$$

Assumption (26M) implies that

$$\delta_n := (\sum_{t=1}^n h_t^4)^{1/2} (\sum_{t=2}^n (g_t - g_{t-1})^4)^{1/2} / (\sum_{t=1}^n h_t^2 g_t^2) \to 0.$$

We have $g_t^2 - g_{t-j}g_{t-k} = g_t(g_t - g_{t-j}) + g_t(g_t - g_{t-k}) + (g_{t-j} - g_t)(g_t - g_{t-k})$. Using the inequalities

$$|g_t(g_t - g_{t-k})| = |\delta_n^{1/4} g_t \delta_n^{-1/4}(g_t - g_{t-k})| \le \delta_n^{1/2} g_t^2 + \delta_n^{-1/2}(g_t - g_{t-k})^2,$$
$$|(g_{t-j} - g_t)(g_t - g_{t-k})| \le (g_{t-j} - g_t)^2 + (g_t - g_{t-k})^2,$$

we obtain $|g_t^2 - g_{t-j}g_{t-k}| \le 2\delta_n^{1/2} g_t^2 + (\delta_n^{-1/2} + 1)[(g_t - g_{t-j})^2 + (g_t - g_{t-k})^2]$. So,

$$\begin{aligned}\sum_{t=k+1}^{n} h_t^2 |g_t^2 - g_{t-j}g_{t-k}| &\le 2\delta_n^{1/2} \sum_{t=k+1}^{n} h_t^2 g_t^2 \\ &+ (\delta_n^{-1/2} + 1) \sum_{t=k+1}^{n} h_t^2[(g_t - g_{t-j})^2 + (g_t - g_{t-k})^2] \\ \le\ & 2\delta_n^{1/2} q_n + (\delta_n^{-1/2} + 1)(\sum_{t=k+1}^{n} h_t^4)^{1/2} \big[(\sum_{t=k+1}^{n} (g_t - g_{t-j})^4)^{1/2} + (\sum_{t=k+1}^{n} (g_t - g_{t-k})^4)^{1/2}\big].\end{aligned}$$

Using the inequality $(a_1 + .... + a_k)^4 \le k^3(a_1^4 + .... + a_k^4)$, we obtain

$$(g_t - g_{t-j})^4 = [(g_t - g_{t-1}) + (g_{t-1} - g_{t-2}) + ... + (g_{t-j+1} - g_{t-j})]^4 \le j^3[(g_t - g_{t-1})^4 + ... + (g_{t-j+1} - g_{t-j})^4].$$

Hence,

$$\sum_{t=k+1}^{n} (g_t - g_{t-j})^4 \le j^4 \sum_{t=2}^{n} (g_t - g_{t-1})^4$$

which together with definition of $\delta_n$ implies

$$\begin{aligned}\sum_{t=k+1}^{n} h_t^2 |g_t^2 - g_{t-j}g_{t-k}| &\le 2\delta_n^{1/2} q_n + (\delta_n^{-1/2} + 1)(j^2 + k^2)(\sum_{t=1}^{n} h_t^4)^{1/2}(\sum_{t=2}^{n} (g_t - g_{t-1})^4)^{1/2} \\ &= 2\delta_n^{1/2} q_n + (\delta_n^{-1/2} + 1)(j^2 + k^2)\delta_n q_n = o(q_n)\end{aligned}$$

since $\delta_n \to 0$.

This proves (41) and together with (40) proves the claim $q_n - q_{njk} = o(q_n)$ of (38). The latter implies $q_{njk}/q_n = 1 - (q_n - q_{njk})/q_n \to 1$.

Next we show the claim $q_{nijk} - q_n = o(q_n)$ in (39). We have $h_t h_{t-i} g_{t-j} g_{t-k} =$

$-h_t^2 g_{t-j}^2 + h_t^2 g_{t-j} g_{t-k} + h_t h_{t-i} g_{t-j}^2 + h_t(h_{t-i} - h_t) g_{t-j}(g_{t-k} - g_{t-j})$. Hence,

$$\begin{aligned} q_{nijk} &= \sum_{t=k+1}^{n} h_t h_{t-i} g_{t-j} g_{t-k} = q_n + (q_n - \sum_{t=k+1}^{n} h_t^2 g_{t-j}^2) + (\sum_{t=k+1}^{n} h_t^2 g_{t-j} g_{t-k} - q_n) \\ &\quad + (\sum_{t=k+1}^{n} h_t h_{t-i} g_{t-j}^2 - q_n) + (\sum_{t=k+1}^{n} h_t (h_{t-i} - h_t) g_{t-j}(g_{t-k} - g_{t-j})) \\ &=: q_n + v_{n,1} + v_{n,2} + v_{n,3} + v_{n,4} = q_n + o(q_n) + v_{n,4} \end{aligned}$$

because $v_{n,\ell} = o(q_n)$ for $\ell = 1, 2, 3$ by relation $q_{njk} - q_n = o(q_n)$ shown above in (38).

By the Hölder inequality,

$$\begin{aligned} |v_{n,4}| &\leq \Big( \sum_{t=k+1}^{n} h_t^4 \sum_{t=k+1}^{n} (h_{t-i} - h_t)^4 \sum_{t=k+1}^{n} g_{t-j}^4 \sum_{t=k+1}^{n} (g_{t-k} - g_{t-j})^4 \Big)^{1/4} \\ &\leq \Big( |k-j|^4 \sum_{t=1}^{n} h_t^4 \sum_{t=2}^{n} (g_t - g_{t-1})^4 \Big)^{1/4} \Big( i^4 \sum_{t=1}^{n} g_t^4 \sum_{t=2}^{n} (h_t - h_{t-1})^4 \Big)^{1/4} = o(q_n) \end{aligned}$$

by (26M) which proves $q_{nijk} - q_n = o(q_n)$.

We complete the proof of lemma by showing the last claim in (39), $\Delta_{njk} = o(q_n)$. Since $h_t^2 g_{t-j} g_{t-k} - h_{t-1}^2 g_{t-1-j} g_{t-1-k} = h_t^2 (g_{t-j} g_{t-k} - g_t^2) - h_{t-1}^2 (g_{t-1-j} g_{t-1-k} - g_{t-1}^2) + (h_t^2 g_t^2 - h_{t-1}^2 g_{t-1}^2)$, we can bound

$$\Delta_{njk} \leq \sum_{t=k+2}^{n} h_t^2 |g_{t-j} g_{t-k} - g_t^2| + \sum_{t=k+2}^{n} h_{t-1}^2 |g_{t-1-j} g_{t-1-k} - g_{t-1}^2| + \sum_{t=k+2}^{n} |h_t^2 g_t^2 - g_{t-1}^2 h_{t-1}^2|.$$

By (41) the first and the second sum is $o(q_n)$. We can bound the last sum by

$$\sum_{t=k+2}^{n} |h_t^2 g_t^2 - g_{t-1}^2 h_{t-1}^2| \leq \sum_{t=k+2}^{n} h_t^2 |g_t^2 - g_{t-1}^2| + \sum_{t=k+2}^{n} g_{t-1}^2 |h_t^2 - h_{t-1}^2|.$$

By (41) and under assumption (26M), the latter is $o(q_n)$ which completes the proof. □

Recall the notation $e'_{tk} = x_t y_{t-k} = h_t g_{t-k} \varepsilon_t \eta_{t-k}$ used in the proof of Theorem 3.2M and $e_{tk} = (x_t - \bar{x})(y_{t-k} - \bar{y})$ in (23M), respectively. We drop the subscript $xy$ in $e'_{tk}$ and $e_{tk}$ for simplicity in what follows.

**Lemma 2.2.** *Let $h_t, g_t$, $t \geq 1$ satisfy (26M). Assume that for some $k \geq j \geq 0$, $\{\varepsilon_t^2 \eta_{t-j} \eta_{t-k}\}$ is a stationary sequence, $\mathbb{E}|\varepsilon_1^2 \eta_{1-j} \eta_{1-k}| < \infty$, and as $n \to \infty$,*

$$\mathbb{E}\Big| (n^{-1} \textstyle\sum_{t=k+1}^{n} \varepsilon_t^2 \eta_{t-j} \eta_{t-k}) - \mathbb{E}[\varepsilon_1^2 \eta_{1-j} \eta_{1-k}] \Big| \to 0. \tag{42}$$

*Then,*

$$q_n^{-1} \sum_{t=k+1}^n e'_{tj} e'_{tk} \to_p \mathbb{E}[\varepsilon_1^2 \eta_{1-j} \eta_{1-k}]. \tag{43}$$

*In addition, if Assumption B holds, then*

$$q_n^{-1/2} \sum_{t=k+1}^n e_{tk} - q_n^{-1/2} \sum_{t=k+1}^n e'_{tk} \to_p 0, \tag{44}$$

$$q_n^{-1} \sum_{t=k+1}^n |e_{tj} e_{tk} - e'_{tj} e'_{tk}| \to_p 0. \tag{45}$$

*In particular, (42) holds if $\{\varepsilon_t^2 \eta_{t-j} \eta_{t-k}\}$ is an ergodic sequence.*

To prove Lemma 2.2, we shall use the following result.

**Lemma 2.3.** *(Dalla, Giraitis and Koul (2014), Lemma 10). Let $T_n = \sum_{t=1}^n c_{nt} V_t$, where $\{V_t\}$ is a stationary ergodic sequence, $\mathbb{E}|V_1| < \infty$, and $c_{nt}$ are real numbers such that for some $0 < \alpha_n < \infty$, $n \geq 1$,*

$$\sum_{t=1}^n |c_{nt}| = O(\alpha_n), \quad |c_{n1}| + \sum_{t=2}^n |c_{nt} - c_{n,t-1}| = o(\alpha_n). \tag{46}$$

*Then $\mathbb{E}|T_n - \mathbb{E}T_n| = o(\alpha_n)$.*

**Remark 2.1.** *The proof of Lemma 2.3 in Dalla, Giraitis and Koul (2014) uses the property*

$$\mathbb{E}\big|(n^{-1} \sum_{t=1}^n V_t) - \mathbb{E}V_1\big| \to 0, \tag{47}$$

*of ergodic sequence $\{V_t\}$, see Stout (1974, Cor. 3.5.2). Lemma 2.3 remains valid if the assumption of ergodicity of $\{V_t\}$ is replaced by assumption (47).*

*Proof of Lemma 2.2.* The proof is based on Lemma 2.3 and Remark 2.1. Denote the left hand side of (43) by $T_n$. Write

$$T_n = \sum_{t=k+1}^n c_{nt} V_t, \tag{48}$$

where stationary series $V_t = \varepsilon_t^2 \eta_{t-j} \eta_{t-k}$ satisfies (47) and $c_{nt} = q_n^{-1} h_t^2 g_{t-j} g_{t-k}$. Next we show that $c_{nt}$ satisfies (46) with $\alpha_n = 1$

In particular, we show that as $n \to \infty$,

$$\sum_{t=k+1}^n c_{nt} \to 1, \qquad c_{n,k+1} + \sum_{t=k+2}^n |c_{nt} - c_{n,t-1}| = o(1). \tag{49}$$

By (38), $\sum_{t=k+1}^{n} c_{nt} = q_n^{-1} q_{njk} \to 1$ which proves the first claim in (49). By definition (37) of $\Delta_{njk}$ and (39),

$$\sum_{t=k+2}^{n} |c_{nt} - c_{n,t-1}| \le q_n^{-1} \Delta_{njk} = o(1).$$

By definition of $c_{nt}$ and (26M),

$$|c_{n,t}| = q_n^{-1} h_t^2 |g_{t-j} g_{t-k}| \le q_n^{-1} \max_{t=1,\dots,n} h_t^2 \max_{t=1,\dots,n} g_t^2 = o(1).$$

This proves the second claim in (49).

Thus, by Lemma 2.3, $\mathbb{E}|T_n - \mathbb{E}T_n| \to 0$. Observe that

$$\mathbb{E}T_n = \sum_{t=k+1}^{n} c_{nt} \mathbb{E}V_t = \mathbb{E}[\varepsilon_1^2 \eta_{1-j} \eta_{1-k}] q_n^{-1} \sum_{t=k+1}^{n} h_t^2 g_{t-j} g_{t-k} = \mathbb{E}[\varepsilon_1^2 \eta_{1-j} \eta_{1-k}] q_n^{-1} q_{njk}$$

where $q_n^{-1} q_{njk} \to 1$ by (38). Hence, $T_n = \mathbb{E}T_n + o_p(1) = \mathbb{E}[\varepsilon_1^2 \eta_{1-j} \eta_{1-k}] + o_p(1)$ which proves (43).

*Proof of (44).* It suffices to show that

$$r_{n1} := \sum_{t=k+1}^{n} (e_{tk} - e'_{tk}) = o_p(q_n^{1/2}). \tag{50}$$

We have

$$e_{tk} - e'_{tk} = \bar{x}\bar{y} - \bar{y}x_t - \bar{x}y_{t-k}, \tag{51}$$
$$r_{n1} = \sum_{t=k+1}^{n} (\bar{x}\bar{y} - \bar{y}x_t - \bar{x}y_{t-k}) = (n-k)\bar{x}\bar{y} - 2n\bar{x}\bar{y} + \bar{y}\sum_{t=1}^{k} x_t + \bar{x}\sum_{t=n-k+1}^{n} y_t.$$

Hence,

$$|r_{n1}| \le 3n|\bar{x}\bar{y}| + |\bar{y}|\,|\sum_{t=1}^{k} x_t| + |\bar{x}|\,|\sum_{t=n-k+1}^{n} y_t|. \tag{52}$$

We will show below that for any fixed $k \ge 1$, as $n \to \infty$,

$$\bar{x} = o_p(n^{-1/2} q_n^{1/4}), \quad \sum_{t=1}^{k} x_t = o_p(q_n^{1/4}), \quad \sum_{t=1}^{n} x_t^2 = o_p(n q_n^{1/2}), \tag{53}$$
$$\bar{y} = o_p(n^{-1/2} q_n^{1/4}), \quad \sum_{t=n-k+1}^{n} y_t = o_p(q_n^{1/4}), \quad \sum_{t=1}^{n} y_t^2 = o_p(n q_n^{1/2}).$$

Together with (52) this implies $r_{n1} = o_p(q_n^{1/2})$ which verifies (50). Observe that

$$\mathbb{E}\bar{x}^2 \le Cn^{-2}\sum_{t=1}^n h_t^2, \quad \mathbb{E}\bar{y}^2 \le Cn^{-2}\sum_{t=1}^n g_t^2, \tag{54}$$
$$\mathbb{E}|\sum_{t=1}^k x_t| \le \mathbb{E}|\varepsilon_1|\sum_{t=1}^k |h_t|, \quad \mathbb{E}|\sum_{t=n-k+1}^n y_t| \le \mathbb{E}|\eta_1|\sum_{t=n-k+1}^n |g_t|.$$
$$\mathbb{E}\sum_{t=1}^n x_t^2 \le \mathbb{E}\varepsilon_1^2\sum_{t=1}^n h_t^2, \quad \mathbb{E}\sum_{t=1}^n y_t^2 \le \mathbb{E}\eta_1^2\sum_{t=1}^n g_t^2.$$

Indeed, by Assumption B, the stationary sequences $\{\varepsilon_t\}$ and $\{\eta_t\}$ have absolutely summable autocovariance functions $\gamma_{\varepsilon,k}$ and $\gamma_{\eta,k}$. Hence,

$$\begin{aligned}\mathbb{E}\bar{x}^2 &= n^{-2}\sum_{t,s=1}^n h_t h_s \mathrm{cov}(\varepsilon_t, \varepsilon_s) \le 2n^{-2}\sum_{t=1}^n h_t^2 \sum_{k=-\infty}^{\infty} |\mathrm{cov}(\varepsilon_t, \varepsilon_{t-k})| \\ &= Cn^{-2}\sum_{t=1}^n h_t^2, \;\; C = 2\sum_{k=-\infty}^{\infty}|\gamma_{\varepsilon,k}| < \infty,\end{aligned} \tag{55}$$

which proves the first claim in (54). The claim for $\mathbb{E}\bar{y}^2$ follows using similar arguments, and the remaining bounds in (54) are obvious.

Now we are ready to prove (53). By assumption (26M), $\max_{t=1,\dots,n} h_t^2 = o(q_n^{1/2})$ and $\max_{t=1,\dots,n} g_t^2 = o(q_n^{1/2})$. Hence from (54) we obtain $\mathbb{E}\bar{x}^2 = o(n^{-1}q_n^{1/2})$, $\mathbb{E}|\sum_{t=1}^k x_t| = o(q_n^{1/4})$ and $\mathbb{E}\sum_{t=1}^n x_t^2 = o(nq_n^{1/2})$ which implies the claimed orders for the sums involving the $x_t$'s in (53). The claimed orders for the sums involving the $y_t$'s follow using the same argument.

*Proof of (45).* It suffices to show that

$$r_{n2} := \sum_{t=k+1}^n |e_{tj}e_{tk} - e'_{tj}e'_{tk}| = o_p(q_n). \tag{56}$$

We have, $e_{tj}e_{tk} - e'_{tj}e'_{tk} = (e_{tj} - e'_{tj})(e_{tk} - e'_{tk}) + e'_{tk}(e_{tj} - e'_{tj}) + e'_{tj}(e_{tk} - e'_{tk})$. So, setting $D_{nk} = \sum_{t=k+1}^n (e_{tk} - e'_{tk})^2$,

$$|r_{n2}| \le D_{nj}^{1/2}D_{nk}^{1/2} + D_{nj}^{1/2}(\sum_{t=k+1}^n e'^2_{tk})^{1/2} + D_{nk}^{1/2}(\sum_{t=k+1}^n e'^2_{tj})^{1/2}. \tag{57}$$

We will show that

$$(a)\; D_{nk} = o_p(q_n), \quad (b)\; \sum_{t=k+1}^n e'^2_{tk} = O_p(q_n), \tag{58}$$

which together with (57) implies $r_{n2} = o_p(q_n)$ proving (56).

First we show (a). By (51),

$$(e_{tk}-e'_{tk})^2 = (\bar{x}\bar{y}-\bar{y}x_t-\bar{x}y_{t-k})^2 \le 3(\bar{x}\bar{y})^2+3(\bar{y}^2x_t^2)+3(\bar{x}^2y_{t-k}^2),$$
$$D_{nk} = \sum_{t=k+1}^{n}(e_{tk}-e'_{tk})^2 \le 3n(\bar{x}\bar{y})^2+3\bar{y}^2\sum_{t=k+1}^{n}x_t^2+3\bar{x}^2\sum_{t=k+1}^{n}y_{t-k}^2. \qquad (59)$$

In view of (53), $\bar{x}=o_p(n^{-1/2}q_n^{1/4})$, $\bar{y}=o_p(n^{-1/2}q_n^{1/4})$, $\sum_{t=k+1}^n x_t^2=o_p(nq_n^{1/2})$, and $\sum_{t=k+1}^n y_{t-k}^2=o_p(nq_n^{1/2})$. These orders in conjunction with (59) prove (a):

$$D_{nk}=o_p(q_n)+o_p(q_n^{1/2})n^{-1}o_p(nq_n^{1/2})=o_p(q_n).$$

(b) follows noting that by the definition of $e'_{tk}$,

$$\mathbb{E}[\sum_{t=k+1}^{n}e'^2_{tk}] = \sum_{t=k+1}^{n}h_t^2g_{t-k}^2\mathbb{E}[\varepsilon_t^2\eta_{t-k}^2] \le (\mathbb{E}\varepsilon_1^4\mathbb{E}\eta_1^4)^{1/2}\sum_{t=k+1}^{n}h_t^2g_{t-k}^2 \le Cq_{nkk}=O(q_n) \quad (60)$$

by (38) of Lemma 2.1. This completes the proof of (58) and of the lemma. □

**Lemma 2.4.** *Let $x_t=\mu+\varepsilon_t$, where $\{\varepsilon_t\}$ is an i.i.d. sequence with $\mathbb{E}\varepsilon_t=0$ and $\mathbb{E}\varepsilon_t^2<\infty$. Assume that $\mathbb{E}\varepsilon_t^4<\infty$ when $\widehat{\rho}_{x^2,k}$ is considered. Then for $k\ge 1$,*

$$\frac{n}{(n-k)^{1/2}}\widehat{\rho}_{x,k} = \frac{1}{\sigma_\varepsilon^2(n-k)^{1/2}}\sum_{t=k+1}^{n}\varepsilon_t\varepsilon_{t-k}+o_p(1), \qquad (61)$$
$$\frac{n}{(n-k)^{1/2}}\widehat{\rho}_{|x|,k} = \frac{1}{\sigma_{|\varepsilon|}^2(n-k)^{1/2}}\sum_{t=k+1}^{n}(|\varepsilon_t|-\mathbb{E}|\varepsilon_t|)(|\varepsilon_{t-k}|-\mathbb{E}|\varepsilon_{t-k}|)+o_p(1), \qquad (62)$$
$$\frac{n}{(n-k)^{1/2}}\widehat{\rho}_{x^2,k} = \frac{1}{\sigma_{\varepsilon^2}^2(n-k)^{1/2}}\sum_{t=k+1}^{n}(\varepsilon_t^2-\mathbb{E}\varepsilon_t^2)(\varepsilon_{t-k}^2-\mathbb{E}\varepsilon_{t-k}^2)+o_p(1), \qquad (63)$$

*where $\sigma_\varepsilon^2=\mathrm{var}(\varepsilon_1)$, $\sigma_{|\varepsilon|}^2=\mathrm{var}(|\varepsilon_1|)$ and $\sigma_{\varepsilon^2}^2=\mathrm{var}(\varepsilon_1^2)$.*

**Proof of Lemma 2.4**. Without loss of generality, assume that $\mu=0$. We prove (62). (The proof of (61) and (63) is simpler and follows using similar arguments).

Denote $z_t=|x_t-\bar{x}|-\mathbb{E}|x_t|$, $y_t=|x_t|-\mathbb{E}|x_t|$. Then, by (1M),

$$\frac{n}{(n-k)^{1/2}}\widehat{\rho}_{|x|,k} = \frac{(n-k)^{-1/2}\sum_{t=k+1}^{n}(z_t-\bar{z})(z_{t-k}-\bar{z})}{n^{-1}\sum_{t=1}^{n}(z_t-\bar{z})^2} \qquad (64)$$
$$= \frac{(n-k)^{-1/2}(\sum_{t=k+1}^{n}y_ty_{t-k}+Q_{n1})}{n^{-1}(\sum_{t=1}^{n}y_t^2+Q_{n2})}$$

where

$$Q_{n1} = \sum_{t=k+1}^{n} \big((z_t - \bar{z})(z_{t-k} - \bar{z}) - y_t y_{t-k}\big), \quad Q_{n2} = \sum_{t=1}^{n} \big((z_t - \bar{z})^2 - y_t^2\big).$$

We will show that

$$(a)\ Q_{n1} = o_p(n^{1/2}), \quad (b)\ Q_{n2} = o_p(n), \quad (c)\ n^{-1} \sum_{t=1}^{n} y_t^2 \to \mathrm{var}(|\varepsilon_1|). \tag{65}$$

Together with (64) this implies (62):

$$\frac{n}{(n-k)^{1/2}} \widehat{\rho}_{|x|,k} = \mathrm{var}(|\varepsilon_1|)^{-1} (n-k)^{-1/2} \sum_{t=k+1}^{n} y_t y_{t-k} + o_p(1). \tag{66}$$

*Proof of (65)(a).* To prove $Q_{n1} = o_p(n^{1/2})$, we write $Q_{n1} = q_{n1} + q_{n2}$ with

$$q_{n1} = \sum_{t=k+1}^{n} ((z_t - \bar{z})(z_{t-k} - \bar{z}) - z_t z_{t-k}), \quad q_{n2} = \sum_{t=k+1}^{n} (z_t z_{t-k} - y_t y_{t-k}). \tag{67}$$

We will show that

$$q_{n1} = o_p(n^{1/2}), \quad q_{n2} = o_p(n^{1/2}). \tag{68}$$

As in (51), we have

$$(z_t - \bar{z})(z_{t-k} - \bar{z}) - z_t z_{t-k} = \bar{z}^2 - \bar{z} z_t - \bar{z} z_{t-k}, \tag{69}$$

$$q_{n1} = \sum_{t=k+1}^{n} (\bar{z}^2 - \bar{z} z_t - \bar{z} z_{t-k}) = (n-k)\bar{z}^2 - 2n\bar{z}^2 + \bar{z} \sum_{t=1}^{k} z_t + \bar{z} \sum_{t=n-k+1}^{n} z_t.$$

Hence,

$$|q_{n1}| \le 3n|\bar{z}^2| + |\bar{z}|\, |\sum_{t=1}^{k} z_t| + |\bar{z}|\, |\sum_{t=n-k+1}^{n} z_t|. \tag{70}$$

Write

$$|\bar{z}| = n^{-1} |\sum_{t=1}^{b} z_t| \le n^{-1} |\sum_{t=1}^{n} (|x_t - \bar{x}| - |x_t|)| + n^{-1} |\sum_{t=1}^{n} (|x_t| - \mathbb{E}|x_t|)| \le |\bar{x}| + |\bar{y}| = O_p(n^{-1/2})$$

because $\big||x_t - \bar{x}| - |x_t|\big| \le |\bar{x}|$ and $|\bar{x}| = O_p(n^{-1/2})$, $|\bar{y}| = O_p(n^{-1/2})$. The latter holds because $\{x_t\}$ and $\{y_t\}$ are i.i.d random variables, $\mathbb{E}x_t = \mathbb{E}y_t = 0$, $\mathbb{E}x_t^2 < \infty$, $\mathbb{E}y_t^2 < \infty$. Since, $\mathbb{E}\sum_{t=1}^{k} |z_t| + \mathbb{E}\sum_{t=n-k+1}^{n} |z_t| \le 2k\mathbb{E}|z_1|$, this together with (70) implies $q_{n1} = O_p(1) = o_p(n^{1/2})$ which proves the first claim in (68).

To evaluate $q_{n2}$, write

$$\begin{aligned} z_t z_{t-k} - y_t y_{t-k} &= (|x_t - \bar{x}| - \mathbb{E}|x_t|)(|x_{t-k} - \bar{x}| - \mathbb{E}|x_{t-k}|) - (|x_t| - \mathbb{E}|x_t|)(|x_{t-k}| - \mathbb{E}|x_{t-k}|) \\ &= (|x_t - \bar{x}| - |x_t|)(|x_{t-k} - \bar{x}| - |x_{t-k}|) + (|x_t - \bar{x}| - |x_t|)y_{t-k} \\ &\quad +(|x_{t-k} - \bar{x}| - |x_{t-k}|)y_t. \end{aligned} \tag{71}$$

Hence,

$$\begin{aligned} q_{n2} &= \textstyle\sum_{t=k+1}^n (|x_t - \bar{x}| - |x_t|)(|x_{t-k} - \bar{x}| - |x_{t-k}|) + \sum_{t=k+1}^n (|x_t - \bar{x}| - |x_t|)y_{t-k} \\ &+ \textstyle\sum_{t=k+1}^n (|x_{t-k} - \bar{x}| - |x_{t-k}|)y_t =: q_{n2,1} + q_{n2,2} + q_{n2,3}. \end{aligned}$$

Since $\big||x_t - \bar{x}| - |x_t|\big| \le |\bar{x}|$, we have

$$|q_{n2,1}| \le n|\bar{x}|^2 = O_p(1).$$

Next we show $q_{n2,2} = o_p(n^{1/2})$. (The proof of $q_{n2,3} = o_p(n^{1/2})$ follows using a similar argument). Denote $\bar{x}_{(t-k)} = n^{-1}\sum_{j=1:j\ne t-k}^n x_j$, $\bar{x}_{(t-k,s-k)} = n^{-1}\sum_{j=1:j\ne t-k,s-k}^n x_j$. Hence

$$q_{n2,2} = \sum_{t=k+1}^n (|x_t - \bar{x}| - |x_t - \bar{x}_{(t-k)}|)y_{t-k} + \sum_{t=k+1}^n (|x_t - \bar{x}_{(t-k)}| - |x_t|)y_{t-k} =: v_n + v_n'.$$

We will show that

$$\mathbb{E}|v_n| = o(n^{1/2}), \quad \mathbb{E}v_n'^2 = o(n) \tag{72}$$

which proves $q_{n2,2} = o_p(n^{1/2})$.

Since $\big||x_t - \bar{x}| - |x_t - \bar{x}_{(t-k)}|\big| \le |\bar{x} - \bar{x}_{(t-k)}| = n^{-1}|x_{t-k}|$, we have

$$\mathbb{E}|v_n| \le \sum_{t=k+1}^n n^{-1}\mathbb{E}|x_{t-k}y_{t-k}| \le C$$

which implies $v_n = O_p(1)$. On the other hand,

$$\begin{aligned} \mathbb{E}v_n'^2 &= \mathbb{E}\sum_{t,s=k+1}^n (|x_t - \bar{x}_{(t-k)}| - |x_t|)y_{t-k}(|x_s - \bar{x}_{(s-k)}| - |x_s|)y_{s-k} \\ &= \mathbb{E}\sum_{t,s=k+1:|t-s|\le 2k}^n [...] + 2\mathbb{E}\sum_{t,s=k+1:t>s+2k}^n [...] =: S_{n1} + 2S_{n2}. \end{aligned}$$

To bound $S_{n1}$, notice that

$$\begin{aligned}
&\mathbb{E}\big|(|x_t-\bar{x}_{(t-k)}|-|x_t|)y_{t-k}(|x_s-\bar{x}_{(s-k)}|-|x_s|)y_{s-k}\big| \\
&\quad\le \mathbb{E}\big|\bar{x}_{(t-k)}y_{t-k}\bar{x}_{(s-k)}y_{s-k}\big| \le (\mathbb{E}[\bar{x}^2_{(t-k)}y^2_{t-k}]\mathbb{E}[\bar{x}^2_{(s-k)}y^2_{s-k}])^{1/2} \\
&\quad= \big(\mathbb{E}[\bar{x}^2_{(t-k)}]\mathbb{E}[y^2_{t-k}]\mathbb{E}[\bar{x}^2_{(s-k)}]\mathbb{E}[y^2_{s-k}]\big)^{1/2} \le Cn^{-1}
\end{aligned}$$

where $C$ does not depend on $t,s$. Hence, $|S_{n1}| \le Cn^{-1}\sum_{t,s=k+1:|t-s|\le 2k}^{n} 1 \le C = O(1)$.

To bound $S_{n2}$, write

$|x_t-\bar{x}_{(t-k)}|-|x_t| = (|x_t-\bar{x}_{(t-k)}|-|x_t-\bar{x}_{(t-k,s-k)}|)+(|x_t-\bar{x}_{(t-k,s-k)}|-|x_t|)$. Then,

$$\begin{aligned}
&(|x_t-\bar{x}_{(t-k)}|-|x_t|)y_{t-k}(|x_s-\bar{x}_{(s-k)}|-|x_s|)y_{s-k} \\
&\quad= (|x_t-\bar{x}_{(t-k)}|-|x_t-\bar{x}_{(t-k,s-k)}|)y_{t-k}(|x_s-\bar{x}_{(s-k)}|-|x_t-\bar{x}_{(t-k,s-k)}|)y_{s-k} \\
&\qquad+(|x_t-\bar{x}_{(t-k,s-k)}|-|x_t|)y_{t-k}(|x_s-\bar{x}_{(t-k,s-k)}|-|x_s|)y_{s-k} \\
&\qquad+(|x_t-\bar{x}_{(t-k)}|-|x_t-\bar{x}_{(t-k,s-k)}|)y_{t-k}(|x_s-\bar{x}_{(t-k,s-k)}|-|x_s|)y_{s-k} \\
&\qquad+(|x_t-\bar{x}_{(t-k,s-k)}|-|x_t|)y_{t-k}(|x_s-\bar{x}_{(s-k)}|-|x_s-\bar{x}_{(t-k,s-k)}|)y_{s-k} \\
&\quad=: g_{t1}+g_{t2}+g_{t3}+g_{t4}.
\end{aligned}$$

Observe that

$$|g_{t1}| \le |\bar{x}_{(t-k)}-\bar{x}_{(t-k,s-k)}||\bar{x}_{(s-k)}-\bar{x}_{(t-k,s-k)}||y_{t-k}y_{s-k}| \le (n^{-1}|x_{s-k}|)(n^{-1}|x_{t-k}|)|y_{t-k}y_{s-k}|.$$

Hence,

$$\mathbb{E}\sum_{t,s=k+1:t>s+2k}^{n}|g_{t1}| \le Cn^{-2}\sum_{t,s=k+1:t>s+2k}^{n} 1 \le C.$$

Recall that by assumption $\{x_t\}$ are i.i.d. random variables. Then for $t > s+2k$ in $g_{t2}$ and $g_{t3}$ only $y_{t-k} = |x_{t-k}| - \mathbb{E}|x_{t-k}|$ depends on $x_{t-k}$. Since $\mathbb{E}y_{t-k} = 0$, this implies $\mathbb{E}g_{t2} = 0$ and $\mathbb{E}g_{t3} = 0$. In $g_{t4}$ only $y_{s-k} = |x_{s-k}| - \mathbb{E}|x_{s-k}|$ depends on $x_{s-k}$. Hence, $\mathbb{E}g_{t4} = 0$.

This proves $S_{n2} = O(1)$ which completes the proof of (72) and proves (65)(a) for $Q_{n1}$.

*Proof of (65)(b).* Write

$$Q_{n2} = \sum_{t=1}^{n}\big((z_t-\bar{z})^2 - z_t^2\big) + \sum_{t=1}^{n}\big(z_t^2 - y_t^2\big).$$

By the first claim of (68), $\sum_{t=1}^{n}\big((z_t-\bar{z})^2 - z_t^2\big) = o_p(n)$. By (71),

$$z_t^2 - y_t^2 = (|x_t-\bar{x}|-|x_t|)^2 + 2(|x_t-\bar{x}|-|x_t|)y_t.$$

Hence, $|z_t^2 - y_t^2| \le \bar{x}^2 + 2|\bar{x}y_t|$. Then,

$$\begin{aligned}\mathbb{E}|\textstyle\sum_{t=1}^n \big(z_t^2 - y_t^2\big)| &\le \textstyle\sum_{t=1}^n \big(\mathbb{E}\bar{x}^2 + 2(\mathbb{E}\bar{x}^2)^{1/2}(\mathbb{E}y_t^2)^{1/2}\big) \\ &= n\mathbb{E}\bar{x}^2 + 2(\mathbb{E}\bar{x}^2)^{1/2}\textstyle\sum_{t=1}^n(\mathbb{E}y_t^2)^{1/2} \le Cn^{1/2} = o(n).\end{aligned}$$

This proves that $Q_{n2} = o_p(n)$.

*Proof of (65)(c).* Since an i.i.d. sequence $\{\varepsilon_t\}$ is also an ergodic sequence, then $y_t^2 = (|x_t| - \mathbb{E}|x_t|)^2 = (|\varepsilon_t| - \mathbb{E}|\varepsilon_t|)^2$ is a stationary ergodic sequence with $\mathbb{E}y_t^2 < \infty$. Thus, by Stout (1974, Cor. 3.5.2),

$$n^{-1}\sum_{t=1}^n y_t^2 \to \mathbb{E}y_1^2 = \mathrm{var}(|\varepsilon_1|). \qquad \square$$

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# Online Supplement II to ‘Robust Tests for White Noise and Cross-Correlation’*

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## 1 Introduction

This Supplement II provides details of the full Monte Carlo experiment reported in the text of the main paper. Equation references to the main paper are denoted with the affix M as (#M) and references to theorem and proposition numbers in the main paper are signified as “Theorem #M” and “Proposition #M”. Proofs of the theorems and propositions in the main paper are provided in Supplementary I. References used here are the same as those given in the main paper.

## 2 Monte Carlo study

We present here the full set of results for the Monte Carlo study on the finite sample performance of the standard and robust tests for zero serial correlation, cross-correlation and tests for the i.i.d. property. We evaluate the rejection frequency (in %) of the test statistics using 5,000 replications for sample sizes $n = 100, 300, 1000$. Here we present tables for $n = 300$. Results for the other sample sizes are available upon request. We set the significance level at $\alpha = 5\%$. For the univariate standard test $t_k, LB_m$ and the robust tests $\widetilde{t}_k, \widetilde{Q}_m$ for absence of serial correlation, results on size are reported for lags $k, m = 1, 2, ..., 40$ and on power for lags $k, m = 1, 2, ..., 20$ when $n = 100, 300$, while for $n = 1000$ on size for lags $k, m = 1, 4, ..., 118$ and on power for lags $k, m = 1, 4, ..., 58$. Same lags are used for the bivariate standard tests $t_{xy,k}, t_{yx,k}, HB_{xy,m}, HB_{yx,m}$ and the robust tests

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$\widetilde{t}_{xy,k}, \widetilde{t}_{yx,k}, \widetilde{Q}_{xy,m}, \widetilde{Q}_{yx,m}$, including the lag $k, m = 0$, and tests for the i.i.d. property $J_{x,|x|,k}$, $J_{x,x^2,k}$, $C_{x,|x|,m}$, $C_{x,x^2,m}$. We set the threshold $\lambda = 2.576$ in the robust cumulative statistics $\widetilde{Q}_m, \widetilde{Q}_{xy,m}, \widetilde{Q}_{yx,m}$. The models used in the univariate case for $\{x_t\}$ and in the bivariate case for $\{x_t, y_t\}$ are listed later on. First, we summarize our main findings.

**1** The standard tests for testing zero serial correlation perform well when the data are i.i.d., but may over-reject when the data are non-i.i.d. and this over-rejection increases with sample size. The robust tests achieve the right size. The power of the tests is similar and in a few cases the robust tests show some loss in power.

**2** All the tests for zero serial correlation, both standard and robust, produce spurious power when the data do not have constant mean. This spurious power increases with sample size. It is therefore advisable to examine whether the data have constant mean prior to applying the tests.

**3** The robust tests for testing zero cross-correlation (at individual and cumulative lags) achieve correct size when both series are uncorrelated with constant mean and either constant or time-varying variance. The robust test at an individual lag preserves the correct size when the leading series is uncorrelated and the lagged series is serially correlated, but the size of the cumulative test may become distorted.

The standard tests for testing zero cross-correlation (at individual and cumulative lags) perform well when both series are serially uncorrelated, stationary and mutually independent. But these tests over-reject when the series are mutually dependent or when both of them have time varying variance.

The powers of the standard and robust tests are similar. In a few cases the robust tests have some loss in power.

**4** All the tests for zero cross-correlation, standard and robust, produce spurious power when the two series do not have constant mean or both series have serial correlation. This spurious power increases with sample size for the first case and remains approximately constant in the second case. It is again advisable to examine whether each series has constant mean and no serial correlation prior to applying the tests.

**5** In testing for zero serial correlation or cross-correlation at a fixed lag $m$ using robust cumulative statistics, the need for thresholding increases when the sample size decreases. Thresholding is required even in the i.i.d. case at moderate lags $m$. The sensitivity of the robust test to the thresholding level depends on the nature of the departure for data from the i.i.d. assumption. The values $\lambda = 1.96, 2.576$ are good candidates for the threshold, with $\lambda = 2.576$ performing better at large lags.

**6** The tests for the i.i.d. property show satisfactory size and power.

**7** At individual lags the robust statistics for testing zero correlation or cross-correlation and the tests for the i.i.d. property have satisfactory size performance for all lags $k$ examined here.

## Tests for zero serial correlation

Tables 1-6 present testing results for zero serial correlation at individual lag $k$ based on the statistics $\widetilde{t}_k$ and $t_k$, and at cumulative lags $1, ..., m$ based on the statistics $\widetilde{Q}_m$ and $LB_m$. The results for size are given in Tables 1-2, for power in Tables 3-4 and for spurious power in Tables 5-6.

**Tables 1 and 2**. Models:

(a) $x_t = \varepsilon_t$, $\varepsilon_t \sim$ i.i.d. $N(0,1)$,

(b) $x_t = \varepsilon_t$, $\varepsilon_t \sim$ i.i.d. $t(6)$,

(c) $x_t = \varepsilon_t \varepsilon_{t-1}$, $\varepsilon_t \sim$ i.i.d. $N(0,1)$,

(d) $x_t = h_{1t}\varepsilon_t$, $h_{1t} = 1 + I(t/n > 0.5)$, $\varepsilon_t \sim$ i.i.d. $N(0,1)$,

(e) $x_t = h_{2t}\varepsilon_t$, $h_{2t} = t/n$, $\varepsilon_t \sim$ i.i.d. $N(0,1)$,

(f) $x_t = r_t$, $r_t = \sigma_t \varepsilon_t$, $\sigma_t^2 = 1 + 0.2r_{t-1}^2 + 0.7\sigma_{t-1}^2$, $\varepsilon_t \sim$ i.i.d. $N(0,1)$,

(g) $x_t = h_{1t} r_t$, $h_{1t} = 1 + I(t/n > 0.5)$, $r_t = \sigma_t \varepsilon_t$, $\sigma_t^2 = 1 + 0.2r_{t-1}^2 + 0.7\sigma_{t-1}^2$, $\varepsilon_t \sim$ i.i.d. $N(0,1)$.

In models (a) and (b), the data are i.i.d. Both the standard and robust tests, $t_k$, $LB_m$ and $\widetilde{t}_k$, $\widetilde{Q}_m$ have good size. However, the size of the standard test $t_k$ slightly drops when the lag $k$ increases because of standardization $\sqrt{n}$ in $t_k$ instead of $n/\sqrt{n-k}$ as is done in the cumulative $LB_m$ test. In model (c), the series is uncorrelated, but not independent. In models (d)-(g), the data have unconditional and/or conditional heteroskedasticity. In models (c)-(g) the robust tests $\widetilde{t}_k$, $\widetilde{Q}_m$ produce the appropriate size. On the other hand, the standard test $t_k$ overrejects at lag $k = 1$ in model (c) and at several lags in models (d)-(g), which is magnified in the cumulative test $LB_m$. Size performance is satisfactory at all $k$ for $\widetilde{t}_k$ and in the worst case up to $m \approx 31$ for $\widetilde{Q}_m$.

**Tables 3 and 4**. Models:

(a) $x_t = 0.2x_{t-1} + \varepsilon_t$,

(b) $x_t = \varepsilon_t + 0.2\varepsilon_{t-1}$,

(c) $x_t = r_{1,t}^2$, $r_{1,t} = \sigma_{1,t}\varepsilon_t$, $\sigma_{1,t}^2 = 1 + 0.2r_{1,t-1}^2$,

(d) $x_t = |r_{1,t}|$, $r_{1,t} = \sigma_{1,t}\varepsilon_t$, $\sigma_{1,t}^2 = 1 + 0.2r_{1,t-1}^2$,

(e) $x_t = r_{2,t}^2$, $r_{2,t} = \sigma_{2,t}\varepsilon_t$, $\sigma_{2,t}^2 = 1 + 0.2r_{2,t-1}^2 + 0.7\sigma_{2,t-1}^2$,

(f) $x_t = |r_{2,t}|$, $r_{2,t} = \sigma_{2,t}\varepsilon_t$, $\sigma_{2,t}^2 = 1 + 0.2r_{2,t-1}^2 + 0.7\sigma_{2,t-1}^2$,

(g) $x_t = |\varepsilon_t \varepsilon_{t-1}|$,

$\varepsilon_t \sim$ i.i.d. $N(0,1)$.

The tables below report the power of the tests for dependent stationary time series models (a)-(g). In (a)-(b), the data follow the AR(1) and MA(1) models. The standard $t_k$, $LB_m$ and robust $\widetilde{t}_k$, $\widetilde{Q}_m$ tests show similar power. In (c)-(f), the data are squared and absolute transformations of ARCH and GARCH series and we observe some loss in power for the robust statistics. In the last model (g), the series is correlated only at lag 1 and the standard and robust statistics have similar power properties.

**Tables 5 and 6**. Models:

(a) $x_t = m_{1t} + \varepsilon_t$, $m_{1t} = I(t/n > 0.5)$,

(b) $x_t = m_{2t} + \varepsilon_t$, $m_{2t} = I(0.25 < t/n \leq 0.75)$,

(c) $x_t = m_{3t} + \varepsilon_t$, $m_{3t} = 0.01t$,

(d) $x_t = m_{2t} + h_{1t}\varepsilon_t$, $m_{2t} = I(0.25 < t/n \leq 0.75)$, $h_{1t} = 1 + I(t/n > 0.5)$,

(e) $x_t = (h_{1t}\varepsilon_t)^2$, $h_{1t} = 1 + I(t/n > 0.5)$,

(f) $x_t = |h_{1t}\varepsilon_t|$, $h_{1t} = 1 + I(t/n > 0.5)$,

(g) $x_t = (m_{1t} + \varepsilon_t)^2$, $m_{1t} = I(t/n > 0.5)$,

$\varepsilon_t \sim$ i.i.d. $N(0,1)$.

In models (a)-(g), the data are independent over time but have non-constant mean. All tests over-reject and show spurious power. This is especially so in models (a)-(c) where data have either breaking or trending mean and constant variance. The effect is such that the cumulative tests reach 100% rejection frequency at some lags. The changes in variance seem to dampen this effect, as seen in model (d). Absolute values of a series with breaking variance produce higher rejection frequency compared to squared series, see models (e)-(f). When independent data with breaking mean are squared, as in model (g), the distortions of size are not as severe.

## Tests for zero cross-correlation

Tables 7-16 present testing results for zero serial cross-correlation at individual lag $k$ based on the statistics $\widetilde{t}_{xy,k}, \widetilde{t}_{yx,k}$ and $t_{xy,k}, t_{yx,k}$, and at cumulative lags $0, 1, ..., m$ based on the statistics $\widetilde{Q}_{xy,m}, \widetilde{Q}_{yx,m}$ and $HB_{xy,m}, HB_{yx,m}$. The results for size are given in Tables 7-12, for power in Tables 13-14 and for spurious power in Tables 15-16.

**Tables 7 and 8**. Models:

(a) $x_t = h_{1t}\varepsilon_t$, $y_t = h_{1t}\eta_t$, $h_{1t} = 1 + I(t/n > 0.5)$,

(b) $x_t = h_{1t}\varepsilon_t$, $y_t = h_{3t}\eta_t$, $h_{1t} = 1 + I(t/n > 0.5)$, $h_{3t} = 1 + 3I(t/n > 0.5)$,

(c) $x_t = r_{1t}$, $y_t = r_{2t}$, $r_{1t} = \sigma_{1t}\varepsilon_t$, $\sigma_{1t}^2 = 1 + 0.2r_{1,t-1}^2$, $r_{2t} = \sigma_{2t}\eta_t$, $\sigma_{2t}^2 = 1 + 0.2r_{2,t-1}^2 + 0.7\sigma_{2,t-1}^2$,

$\varepsilon_t, \eta_t \sim$ i.i.d. $N(0,1)$, with $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent.

In models (a)-(c), series are independent and thus with zero cross-correlation. In (a)-(b), both series have breaks in the variance. All the robust tests $\widetilde{t}_{xy,k}, \widetilde{t}_{yx,k}$ have the right size for all lags $k$, and so do the robust cumulative tests $\widetilde{Q}_{xy,m}, \widetilde{Q}_{yx,m}$ for all lags $m$. On the other hand, the standard tests $t_{xy,k}, t_{yx,k}$ show distortions in size at several lags, with the effect accumulating in the standard tests $HB_{xy,m}, HB_{yx,m}$. In (c), when the series follow stationary ARCH(1) and GARCH(1,1) models that are uncorrelated and mutually independent, all tests for cross-correlation, standard and modified, achieve the correct size. However, the standard tests $t_{xy,k}, t_{yx,k}$ become slightly undersized when the individual lag $k$ increases. This size distortion occurs because of the use of normalization $\sqrt{n}$ in $t_{xy,k}, t_{yx,k}$ instead of $n/\sqrt{n-k}$ as is done in the cumulative $HB_{xy,m}, HB_{yx,m}$ tests.

**Tables 9 and 10**. Models:

$$\text{(a) } x_t = \varepsilon_t,\ y_t = m_{1t} + h_{1t}\eta_t,\ m_{1t} = I(t/n > 0.5),\ h_{1t} = 1 + I(t/n > 0.5),$$

$$\text{(b) } x_t = h_{1t}\varepsilon_t,\ y_t = m_{1t} + \eta_t,\ m_{1t} = I(t/n > 0.5),\ h_{1t} = 1 + I(t/n > 0.5),$$

$$\text{(c) } x_t = \varepsilon_t,\ y_t = 0.7y_{t-1} + \eta_t,$$

$$\varepsilon_t, \eta_t \sim \text{i.i.d. } N(0,1), \text{ with } \{\varepsilon_t\} \text{ and } \{\eta_t\} \text{ mutually independent.}$$

In models (a)-(c), the two series are independent of each other. One of the two series, $x_t$, has no serial correlation, while the second series, $y_t$, has either a break in the mean or is autocorrelated. In all models, the standard $t_{xy,k}, t_{yx,k}$ and robust tests $\widetilde{t}_{xy,k}, \widetilde{t}_{yx,k}$ for the individual lag $k$ perform well at all lags. However, the cumulative versions of tests, standard $HB_{xy,m}, HB_{yx,m}$ and robust $\widetilde{Q}_{xy,m}, \widetilde{Q}_{yx,m}$, show distortions in size which increase in magnitude as the lag $m$ increases. In the simulation study, the statistics $\widetilde{Q}_{xy,m}, \widetilde{Q}_{yx,m}$ use respectively the matrices $\widehat{R}^*_{xy,m}, \widehat{R}^*_{yx,m}$, rather than using in both cases $\widehat{R}^*_{xy,m}$ as theory would suggest for model (c). Moreover, the correlation matrix $R_{xy,m} = (0.7^{|j-k|})_{j,k=0,1,\ldots,m}$ is not sparse and so poor performance of the $\widetilde{Q}_{yx,m}$ test is expected in this case.

**Tables 11 and 12**. Models:

$$\text{(a) } x_t = \varepsilon_t,\ y_t = |\varepsilon_t|\eta_t,$$

$$\text{(b) } x_t = \varepsilon_t,\ y_t = \varepsilon_t\varepsilon_{t-1},$$

$$\text{(c) } x_t = \varepsilon_t,\ y_t = \exp(z_t)\eta_t,\ z_t = 0.7z_{t-1} + \varepsilon_t,$$

$$\varepsilon_t, \eta_t \sim \text{i.i.d. } N(0,1), \text{ with } \{\varepsilon_t\} \text{ and } \{\eta_t\} \text{ mutually independent.}$$

In models (a)-(c), series $x_t$ and $y_t$ are series of uncorrelated random variables. They are not cross-correlated at any lag but they are not independent of each other. The size of the robust tests $\widetilde{t}_{xy,k}, \widetilde{t}_{yx,k}$ is satisfactory for all lags $k$ and for all lags $m$ for the cumulative tests $\widetilde{Q}_{xy,m}, \widetilde{Q}_{yx,m}$ albeit being a bit under-sized in model (c). The standard tests $t_{xy,k}, t_{yx,k}$ substantially over-reject at $k = 0$ in all models and also $t_{yx,k}$ over-rejects at $k = 1$ in model (b) and $k = 1, 2$ in model (c). As a consequence their cumulative versions $HB_{xy,m}, HB_{yx,m}$ show size distortions at several lags $m$.

**Tables 13 and 14**. Models:

(a) $x_t = r_{1t}$, $y_t = r_{2t}$, $r_{1t} = \sigma_{1t}\varepsilon_t$, $\sigma_{1t}^2 = 1 + 0.2r_{1,t-1}^2$, $r_{2t} = \sigma_{2t}\varepsilon_t$, $\sigma_{2t}^2 = 1 + 0.2r_{2,t-1}^2 + 0.7\sigma_{2,t-1}^2$,

(b) $x_t = h_{1t}\varepsilon_t$, $y_t = x_t + x_{t-1} + x_{t-2} + h_{1t}\eta_t$, $h_{1t} = 1 + I(t/n > 0.5)$,

(c) $x_t = h_{1t}\varepsilon_t$, $y_t = m_{1t} + x_t + x_{t-1} + x_{t-2} + h_{1t}\eta_t$, $m_{1t} = I(t/n > 0.5)$, $h_{1t} = 1 + I(t/n > 0.5)$,

$\varepsilon_t, \eta_t \sim$ i.i.d. $N(0,1)$, with $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent.

In models (a)-(c), series $x_t$ is serially uncorrelated, and the two series are cross-correlated. In (a), both $x_t$ and $y_t$ are series of uncorrelated variables (ARCH and GARCH). When the two series are only contemporaneously cross-correlated, as in model (a), we observe strong power for both standard $t_{xy,k}, t_{yx,k}$ and robust tests $\widetilde{t}_{xy,k}, \widetilde{t}_{yx,k}$ at lag $k = 0$, which is transmitted in all the cumulative tests. In models (b)-(c), series $y_t$ is autocorrelated, and $y_t$ depends also on $x_{t-1}$ and $x_{t-2}$. For individual lags $k = 0, 1, 2$, both $t_{yx,k}$ and $\widetilde{t}_{yx,k}$ exhibit strong power, which is further amplified by the cumulative tests $HB_{yx,m}$ and $\widetilde{Q}_{yx,m}$. For $k$ such that $x_t$ and $y_{t-k}$ (or $y_t$ and $x_{t-k}$) are not cross-correlated, the robust tests $\widetilde{t}_{xy,k}$, $\widetilde{t}_{yx,k}$ have correct size, while the standard tests $t_{xy,k}, t_{yx,k}$ show size distortions, in model (a), because the two series are not independent at those lags, and in models (b)-(c), because both series have breaks in unconditional variance.

**Tables 15 and 16**. Models:

(a) $x_t = m_{1t} + \varepsilon_t$, $y_t = m_{1t} + \eta_t$, $m_{1t} = I(t/n > 0.5)$,

(b) $x_t = m_{1t} + \varepsilon_t$, $y_t = m_{4t} + \eta_t$, $m_{1t} = I(t/n > 0.5)$, $m_{4t} = I(t/n > 0.25)$,

(c) $x_t = 0.7x_{t-1} + \varepsilon_t$, $y_t = 0.7y_{t-1} + \eta_t$,

$\varepsilon_t, \eta_t \sim$ i.i.d. $N(0,1)$, with $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent.

In models (a)-(c), the two series are mutually independent. They either both have a break in the mean or both are dependent AR(1) series. In spite of zero cross-correlation, all the tests, standard and robust, over-reject. When both series have a break in the mean, as in models (a)-(b), the over-rejection is stronger when the break is common. The spurious power is even more evident in the cumulative tests.

## Tests for i.i.d. property

Tables 17-20 report testing results for the i.i.d. property at individual lag $k$ based on the statistics $J_{x,|x|,k}$ and $J_{x,x^2,k}$, and at cumulative lags $1, ..., m$ based on the statistics $C_{x,|x|,m}$ and $C_{x,x^2,m}$. The results for size are given in Tables 17-18 and for power in Tables 19-20.

**Tables 17 and 18**. Models:

(a) $x_t = \varepsilon_t$, $\varepsilon_t \sim$ i.i.d. $N(0,1)$,

(b) $x_t = \varepsilon_t$, $\varepsilon_t \sim$ i.i.d. $t(6)$,

(c) $x_t = \varepsilon_t$, $\varepsilon_t \sim$ i.i.d. $\chi^2(3)$,

(d) $x_t = \exp(2\varepsilon_t)$, $\varepsilon_t \sim$ i.i.d. $N$(0,1).

In models (a)-(d) the data are i.i.d. with different distributions. Both tests $J_{x,|x|,k}$ and $J_{x,x^2,k}$ have good size for all individual lags $k$. When the data are highly skewed, as in model (d), the tests under-reject which is in line with theory. The cumulative tests $C_{x,|x|,m}$ and $C_{x,x^2,m}$ perform well for lags up to $m \approx 38$, observing though some distortions when the skewness is high. The size performance is overall better for the tests based on levels and absolute deviations from the sample mean, i.e., $J_{x,|x|,k}$ and $C_{x,|x|,m}$.

**Tables 19 and 20**. Models:

(a) $x_t = 0.2x_{t-1} + \varepsilon_t$,

(b) $x_t = r_t$, $r_t = \sigma_t\varepsilon_t$, $\sigma_t^2 = 1 + 0.2r_{t-1}^2$,

(c) $x_t = r_t$, $r_t = \sigma_t\varepsilon_t$, $\sigma_t^2 = 1 + 0.2r_{t-1}^2 + 0.7\sigma_{t-1}^2$,

(d) $x_t = \varepsilon_t\varepsilon_{t-1}$,

(e) $x_t = m_{1t} + \varepsilon_t$, $m_{1t} = I(t/n > 0.5)$,

(f) $x_t = h_{1t}\varepsilon_t$, $h_{1t} = 1 + I(t/n > 0.5)$,

(g) $x_t = h_{1t}y_t$, $h_{1t} = 1 + I(t/n > 0.5)$, $y_t = 0.2y_{t-1} + \varepsilon_t$,

(h) $x_t = m_{1t} + h_{1t}\varepsilon_t$, $m_{1t} = I(t/n > 0.5)$, $h_{1t} = 1 + I(t/n > 0.5)$,

$\varepsilon_t \sim$ i.i.d. $N$(0,1).

In models (a)-(h), the data are not i.i.d. They either have correlation, or non-constant mean or non-constant variance or both. The tests $J_{x,|x|,k}$ and $J_{x,x^2,k}$ for i.i.d. property at individual lags have satisfactory power and there is a boost in power, when the data have combined non-i.i.d features, as in models (g)-(h). The power of the cumulative tests $C_{x,|x|,m}$ and $C_{x,x^2,m}$ is magnified with increasing lag $m$. The power performance is overall better for the tests based on levels and absolute deviations from the sample mean, that is, $J_{x,|x|,k}$ and $C_{x,|x|,m}$.

## Effect of the threshold $\lambda$

Tables 21-22 are for the statistics $Q_m$ and $\widetilde{Q}_m$ for testing zero serial correlation at lags $1, ..., m$. Tables 23-24 are for the statistics $Q_{xy,m}$ and $\widetilde{Q}_{xy,m}$ for testing zero cross-correlation at lags $0, 1, ..., m$. Here, we write $\widetilde{Q}_m = \widetilde{Q}_m(\lambda)$ and $\widetilde{Q}_{xy,m} = \widetilde{Q}_{xy,m}(\lambda)$ and check the size of the tests for thresholds $\lambda = 1.645, 1.96, 2.576$ at different significance levels $\alpha = 10\%, 5\%, 1\%$. Recall that $Q_m = \widetilde{Q}_m(0)$ and $Q_{xy,m} = \widetilde{Q}_{xy,m}(0)$.

**Tables 21 and 24**. Models:

(a) $x_t = \varepsilon_t$,

(b) $x_t = h_{2t}\varepsilon_t$, $h_{2t} = t/n$,

(c) $x_t = \varepsilon_t$, $y_t = \eta_t$,

(d) $x_t = h_{1t}\varepsilon_t$, $y_t = h_{3t}\eta_t$, $h_{1t} = 1 + I(t/n > 0.5)$, $h_{3t} = 1 + 3I(t/n > 0.5)$,

$\varepsilon_t, \eta_t \sim$ i.i.d. $N(0,1)$, with $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent.

In models (a)-(d), the data are i.i.d. or independent with time varying variance. In all these cases, the univariate $Q_m$ and bivariate $Q_{xy,m}$ cumulative tests with no thresholding suffer size distortions that increase with increasing lag $m$. Size distortion is observed even for ideal i.i.d. normally distributed data, in models (a) and (c). It is more evident for heteroskedastic data, in models (b) and (d). On the other hand, when thresholding is applied, the size of the tests $\widetilde{Q}_m$ and $\widetilde{Q}_{xy,m}$ is satisfactory and not as sensitive to the value of the threshold $\lambda$ for all lags $m$ for i.i.d. data and up to $m \approx 30$ for heteroskedastic data. Overall, the thresholds $\lambda = 1.96, 2.576$ are good choices at all significance levels $\alpha = 10\%, 5\%, 1\%$, with $\lambda = 2.576$ giving better performance at large lags.

## Effect of the sample size $n$

### Tests for zero serial correlation

Here we summarize size and power properties of the standard test $t_k$ and robust test $\widetilde{t}_k$ at individual lag $k$ and cumulative Ljung-Box test $LB_m$ and robust cumulative test $\widetilde{Q}_m$ at lag $m$ for sample sizes $n = 100, 300, 1000$.

The robust $\widetilde{t}_k$ test is well sized for all models, sample sizes and lags $k$, while the size distortions of the standard $t_k$ test increase with sample size. The robust $\widetilde{Q}_m$ test is well-sized for all models, sample sizes at moderate lags $m$ and shows distortions after some lag that depends on the degree of the departure from i.i.d. (in the worst case, distortions start after lag $m \approx 17$ for $n = 100$, $m \approx 31$ for $n = 300$ and $m \approx 76$ for $n = 1000$). The size distortions of the standard $LB_m$ test increase with sample size. The power of all tests increases when lags $k, m$ are fixed and the sample size $n$ increases. When the departure from the null is weak, the power of all the tests is not as satisfactory for $n = 100$. At fixed lags $k, m$ spurious power increases with the sample size for all tests.

### Tests for zero cross-correlation

We now summarize size and power features of the standard tests $t_{xy,k}$, $t_{yx,k}$ and robust tests $\widetilde{t}_{xy,k}$, $\widetilde{t}_{yx,k}$ at individual lag $k$ and standard cumulative tests $HB_{xy,m}, HB_{yx,m}$ and robust cumulative tests $\widetilde{Q}_{xy,m}, \widetilde{Q}_{yx,m}$ at lag $m$ for sample sizes $n = 100, 300, 1000$.

We first focus on the ideal case when the series $x_t$ and $y_t$ have constant mean, are serially uncorrelated and mutually independent. If $x_t$ and $y_t$ are stationary, then both standard and robust tests are well-sized at all lags $k, m$ and for all sample sizes. If $x_t$ and $y_t$ are non-stationary, e.g. their unconditional variance is changing, the robust tests remain well-sized whereas the size of the standard tests may be severely distorted with the size distortion increasing slightly with sample size.

When one series has autocorrelation or non-constant mean, both standard and robust cumulative tests can be badly-sized even at low lags. Then, for fixed lag $m$, when $n$ increases the size of the standard tests $HB_{xy,m}, HB_{yx,m}$ tends to improve, while the size of the robust $\widetilde{Q}_{xy,m}, \widetilde{Q}_{yx,m}$ tests may improve or deteriorate depending on the specific model. Nevertheless, in this case the robust tests $\widetilde{t}_{xy,k}$, $\widetilde{t}_{yx,k}$ perform well at all lags $k$.

Next, we consider models of series $x_t$ and $y_t$ that are serially uncorrelated and have constant mean. In addition, $x_t$ and $y_t$ are uncorrelated but dependent on each other. Then the robust tests $\widetilde{t}_{xy,k}, \widetilde{t}_{yx,k}$ are well-sized for all sample sizes at all lags $k$ for all models. The robust $\widetilde{Q}_{xy,m}, \widetilde{Q}_{yx,m}$ tests are well sized for sample sizes $n = 300, 1000$ at all lags $m$ for all models; when $n = 100$ tests show size distortions after some large lag that depends the nature of the departure from i.i.d. (in the worst case, distortions start after lag $m \approx 20$ for $n = 100$). The size of the standard tests is distorted and remains approximately the same across sample sizes.

The power of all tests increases with sample size when lags $k, m$ is fixed. When the departure from the null is weak, the power of all tests is not as satisfactory for $n = 100$. Spurious power increases with sample size for all tests for fixed $k, m$ except the case when both series $x_t$ and $y_t$ are serially correlated: then spurious power remains approximately constant over $n$.

**Tests for the i.i.d. property**

The $J_{x,|x|,k}$ and $J_{x,x^2,k}$ tests are well-sized for all sample sizes and at all lags $k$ for all models. The small distortions due to skewness remain constant with increases in sample size. The $C_{x,|x|,m}$ and $C_{x,x^2,m}$ tests are well-sized for all sample sizes at moderate lags $m$ for all models and distortions due to skewness increase when sample size increases. Size performance is satisfactory for up to lag $m \approx 21$ for $n = 100$, $m \approx 38$ for $n = 300$ and $m \approx 67$ for $n = 1000$ (when considering the $C_{x,|x|,m}$ test) in the cases where the data are symmetrically distributed or not heavily skewed. For fixed lags $k, m$ power increases with sample size for all tests. When the departure from the null is weak, the power of all tests is not as satisfactory for $n = 100$.

**Effect of the threshold $\lambda$**

For fixed lag $m$, the need for thresholding in the univariate $\widetilde{Q}_m$ and bivariate $\widetilde{Q}_{xy,m}$ robust tests decreases with increases in sample size. For small (relatively to sample size) lags $m$ thresholding is hardly needed, but is essential for large (relative to sample size) lags. The values $\lambda = 1.96, 2.576$ are good candidates for the threshold, with $\lambda = 2.576$ performing better at relatively large lags $m$.

Table 1: Tests for zero serial correlation at lag $k$. Size of tests $\widetilde{t}_k$, $t_k$.

| | $x_t$ iid $N(0,1)$ | | $x_t$ iid $t(6)$ | | $x_t = \varepsilon_t \varepsilon_{t-1}$ $\varepsilon_t$ iid | | $x_t = h_{1t}\varepsilon_t$ $\varepsilon_t$ iid | | $x_t = h_{2t}\varepsilon_t$ $\varepsilon_t$ iid | | $x_t = r_t$ $r_t$ GARCH | | $x_t = h_{1t} r_t$ $r_t$ GARCH | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ |
| 1 | 4.60 | 4.48 | 4.42 | 4.46 | 4.70 | 23.04 | 4.28 | 8.02 | 4.76 | 13.40 | 4.56 | 12.58 | 4.22 | 16.36 |
| 2 | 5.14 | 4.96 | 5.24 | 4.92 | 4.34 | 4.32 | 5.46 | 9.22 | 5.12 | 14.22 | 4.70 | 12.34 | 5.10 | 16.20 |
| 3 | 5.30 | 4.98 | 5.06 | 4.80 | 5.00 | 5.06 | 4.70 | 8.64 | 4.60 | 13.78 | 4.36 | 10.88 | 4.84 | 15.32 |
| 4 | 4.76 | 4.80 | 4.66 | 4.84 | 4.18 | 4.54 | 4.92 | 8.90 | 4.70 | 13.12 | 4.72 | 10.20 | 5.04 | 13.80 |
| 5 | 5.02 | 4.72 | 4.86 | 4.60 | 4.48 | 4.36 | 4.72 | 8.08 | 4.92 | 12.96 | 4.76 | 9.26 | 4.86 | 12.72 |
| 6 | 4.74 | 4.86 | 4.74 | 4.64 | 5.22 | 5.06 | 4.74 | 8.48 | 4.70 | 13.02 | 4.72 | 8.38 | 4.28 | 11.82 |
| 7 | 4.66 | 4.46 | 4.42 | 4.10 | 4.64 | 4.28 | 5.04 | 8.32 | 5.18 | 13.02 | 4.74 | 8.14 | 4.90 | 11.76 |
| 8 | 4.66 | 4.40 | 4.64 | 4.20 | 5.08 | 4.72 | 4.84 | 7.94 | 5.10 | 12.34 | 4.40 | 7.42 | 4.58 | 11.40 |
| 9 | 4.78 | 4.44 | 4.78 | 4.28 | 5.08 | 4.58 | 4.52 | 7.58 | 4.46 | 12.12 | 4.88 | 7.16 | 5.12 | 10.10 |
| 10 | 5.08 | 4.68 | 4.90 | 4.52 | 4.68 | 4.32 | 4.46 | 8.32 | 4.86 | 12.62 | 4.96 | 6.68 | 4.50 | 9.70 |
| 11 | 4.94 | 4.56 | 4.96 | 4.44 | 4.92 | 4.24 | 4.90 | 7.98 | 4.76 | 11.66 | 4.82 | 6.06 | 4.66 | 8.58 |
| 12 | 4.62 | 4.30 | 4.48 | 4.00 | 4.68 | 4.26 | 4.50 | 7.56 | 4.60 | 11.64 | 4.30 | 5.68 | 4.52 | 8.58 |
| 13 | 5.02 | 4.58 | 4.90 | 4.26 | 4.92 | 4.38 | 5.56 | 8.50 | 5.80 | 12.86 | 4.80 | 5.74 | 5.26 | 9.28 |
| 14 | 5.26 | 4.62 | 5.08 | 4.44 | 5.46 | 4.98 | 5.34 | 8.06 | 5.04 | 11.88 | 5.24 | 5.74 | 5.14 | 8.86 |
| 15 | 4.80 | 4.38 | 4.84 | 4.36 | 4.82 | 4.02 | 4.92 | 7.40 | 4.84 | 11.28 | 4.52 | 4.80 | 4.38 | 7.82 |
| 16 | 5.46 | 4.88 | 5.58 | 4.74 | 4.98 | 4.24 | 5.12 | 7.86 | 4.72 | 11.26 | 5.20 | 5.40 | 5.00 | 8.50 |
| 17 | 4.66 | 4.14 | 4.74 | 3.98 | 4.40 | 3.84 | 4.62 | 7.54 | 5.02 | 11.30 | 4.76 | 4.70 | 5.02 | 7.80 |
| 18 | 5.00 | 4.22 | 4.84 | 4.26 | 4.92 | 3.76 | 5.08 | 7.48 | 4.74 | 11.48 | 4.86 | 4.84 | 4.76 | 7.72 |
| 19 | 4.82 | 4.10 | 4.80 | 4.34 | 4.60 | 4.04 | 4.94 | 7.38 | 5.52 | 11.24 | 4.98 | 4.68 | 5.02 | 7.30 |
| 20 | 5.32 | 4.44 | 5.48 | 4.66 | 4.76 | 4.04 | 5.28 | 7.96 | 5.04 | 11.42 | 5.18 | 4.72 | 5.04 | 7.58 |
| 21 | 4.56 | 3.58 | 4.72 | 3.94 | 4.96 | 4.04 | 4.72 | 7.14 | 5.32 | 11.24 | 4.36 | 4.06 | 4.76 | 7.06 |
| 22 | 4.86 | 4.04 | 4.78 | 3.94 | 4.34 | 4.16 | 5.04 | 7.18 | 4.94 | 10.76 | 4.58 | 3.94 | 4.66 | 6.62 |
| 23 | 5.20 | 4.42 | 5.26 | 4.68 | 4.20 | 3.62 | 5.14 | 7.76 | 5.06 | 10.46 | 4.98 | 4.24 | 4.98 | 7.12 |
| 24 | 5.04 | 4.34 | 5.22 | 4.66 | 4.10 | 3.68 | 5.32 | 7.40 | 4.96 | 9.96 | 5.24 | 4.42 | 5.16 | 6.42 |
| 25 | 4.96 | 4.08 | 4.92 | 3.96 | 4.42 | 3.92 | 5.00 | 7.14 | 5.48 | 11.02 | 5.26 | 4.50 | 5.06 | 6.86 |
| 26 | 5.20 | 4.12 | 5.08 | 4.00 | 5.02 | 3.90 | 4.64 | 6.56 | 5.04 | 10.10 | 4.88 | 3.84 | 4.92 | 6.40 |
| 27 | 5.14 | 4.02 | 5.04 | 3.82 | 4.48 | 3.62 | 5.00 | 6.96 | 4.84 | 9.78 | 5.00 | 3.82 | 4.86 | 6.66 |
| 28 | 5.04 | 3.92 | 4.90 | 3.86 | 4.34 | 3.70 | 4.90 | 6.80 | 5.00 | 9.90 | 4.96 | 3.74 | 4.66 | 6.14 |
| 29 | 5.22 | 4.16 | 5.06 | 4.28 | 4.60 | 3.68 | 5.20 | 7.26 | 5.28 | 9.94 | 5.50 | 4.04 | 5.18 | 6.52 |
| 30 | 5.12 | 3.96 | 5.14 | 3.82 | 4.90 | 3.66 | 4.84 | 6.90 | 4.68 | 8.98 | 5.14 | 3.76 | 4.94 | 5.66 |
| 31 | 5.42 | 3.94 | 5.44 | 3.92 | 4.70 | 3.34 | 4.94 | 6.38 | 4.96 | 9.36 | 5.46 | 3.94 | 4.90 | 5.66 |
| 32 | 4.82 | 3.44 | 4.92 | 3.36 | 4.62 | 3.56 | 5.44 | 6.60 | 5.08 | 9.08 | 5.24 | 3.24 | 5.28 | 6.08 |
| 33 | 4.62 | 3.56 | 4.60 | 3.36 | 4.84 | 3.80 | 4.80 | 6.28 | 4.70 | 8.96 | 4.90 | 3.20 | 4.70 | 5.40 |
| 34 | 5.24 | 3.98 | 5.16 | 3.74 | 4.60 | 3.34 | 5.42 | 7.54 | 5.50 | 10.06 | 5.38 | 3.76 | 5.68 | 6.50 |
| 35 | 5.16 | 3.66 | 4.94 | 3.48 | 4.30 | 3.38 | 5.12 | 6.20 | 4.58 | 8.28 | 5.10 | 3.26 | 4.92 | 5.46 |
| 36 | 5.00 | 3.24 | 4.98 | 3.38 | 5.00 | 3.96 | 4.96 | 6.38 | 5.38 | 9.42 | 4.88 | 3.24 | 5.18 | 5.64 |
| 37 | 5.46 | 4.04 | 5.16 | 3.86 | 4.56 | 3.30 | 5.40 | 6.38 | 5.10 | 8.40 | 5.00 | 3.14 | 5.12 | 5.30 |
| 38 | 5.44 | 3.76 | 5.50 | 3.70 | 4.52 | 3.58 | 5.24 | 6.18 | 4.68 | 8.30 | 5.14 | 3.40 | 5.04 | 5.42 |
| 39 | 4.94 | 3.68 | 5.02 | 3.64 | 5.20 | 3.50 | 4.94 | 6.06 | 4.82 | 7.84 | 5.44 | 3.26 | 5.08 | 5.14 |
| 40 | 5.20 | 3.38 | 4.96 | 3.60 | 5.02 | 3.24 | 5.22 | 5.86 | 5.02 | 7.82 | 5.08 | 2.90 | 5.60 | 5.12 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\varepsilon_t \sim$ i.i.d. $N(0,1)$, $h_{1t} = 1 + I(t/n > 0.5)$, $h_{2t} = t/n$, $r_t \sim$ GARCH(1,1), $\alpha = 0.2$, $\beta = 0.7$.

Table 2: Tests for zero serial correlation at lags 1, ..., $m$. Size of tests $\widetilde{Q}_m$, $LB_m$.

| | $x_t$ iid $N(0,1)$ | | $x_t$ iid $t(6)$ | | $x_t = \varepsilon_t\varepsilon_{t-1}$ $\varepsilon_t$ iid | | $x_t = h_{1t}\varepsilon_t$ $\varepsilon_t$ iid | | $x_t = h_{2t}\varepsilon_t$ $\varepsilon_t$ iid | | $x_t = r_t$ $r_t$ GARCH | | $x_t = h_{1t}r_t$ $r_t$ GARCH | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ |
| 1 | 4.60 | 4.68 | 4.42 | 4.52 | 4.70 | 23.42 | 4.28 | 8.24 | 4.76 | 13.56 | 4.56 | 12.76 | 4.22 | 16.52 |
| 2 | 4.76 | 4.64 | 4.84 | 5.04 | 3.84 | 18.94 | 4.54 | 10.90 | 4.54 | 18.44 | 4.46 | 15.88 | 4.38 | 22.14 |
| 3 | 4.60 | 4.62 | 4.66 | 4.66 | 3.98 | 16.78 | 4.40 | 11.42 | 4.46 | 21.28 | 4.40 | 18.48 | 4.18 | 26.08 |
| 4 | 4.68 | 4.74 | 4.32 | 4.90 | 4.20 | 15.24 | 4.38 | 12.70 | 4.36 | 24.92 | 4.72 | 18.98 | 4.08 | 29.58 |
| 5 | 4.40 | 4.58 | 4.30 | 4.62 | 4.12 | 14.20 | 4.34 | 13.60 | 4.86 | 27.34 | 4.80 | 20.02 | 4.24 | 30.92 |
| 6 | 4.54 | 4.88 | 4.50 | 4.70 | 4.14 | 13.72 | 4.64 | 14.34 | 4.76 | 29.26 | 4.84 | 20.36 | 4.76 | 32.34 |
| 7 | 4.64 | 4.86 | 4.40 | 4.62 | 4.40 | 13.08 | 4.98 | 15.68 | 4.94 | 31.52 | 4.86 | 20.54 | 4.88 | 33.58 |
| 8 | 4.44 | 4.82 | 4.20 | 4.34 | 4.66 | 12.82 | 4.76 | 16.32 | 5.22 | 33.98 | 4.88 | 20.80 | 4.98 | 34.48 |
| 9 | 4.74 | 4.78 | 4.32 | 4.52 | 4.28 | 12.18 | 4.82 | 17.14 | 4.88 | 35.70 | 5.10 | 21.82 | 4.90 | 34.84 |
| 10 | 4.64 | 4.92 | 4.42 | 4.66 | 4.72 | 12.06 | 4.70 | 17.60 | 4.68 | 38.02 | 5.18 | 21.50 | 4.94 | 35.42 |
| 11 | 4.66 | 5.00 | 4.22 | 4.82 | 4.70 | 11.54 | 4.68 | 18.74 | 4.82 | 39.12 | 4.86 | 21.54 | 5.20 | 36.32 |
| 12 | 4.72 | 5.18 | 4.52 | 4.90 | 4.96 | 11.52 | 4.94 | 19.26 | 4.98 | 40.80 | 5.28 | 21.02 | 4.94 | 36.20 |
| 13 | 4.66 | 5.12 | 4.52 | 4.82 | 4.66 | 11.34 | 4.96 | 20.58 | 5.34 | 42.08 | 5.12 | 20.82 | 4.84 | 37.00 |
| 14 | 4.76 | 5.06 | 4.48 | 4.84 | 4.86 | 11.10 | 5.02 | 21.08 | 5.58 | 43.80 | 5.24 | 20.98 | 5.18 | 37.74 |
| 15 | 5.04 | 5.54 | 4.70 | 4.94 | 5.02 | 10.82 | 5.26 | 21.06 | 5.64 | 45.00 | 5.42 | 20.52 | 5.30 | 37.46 |
| 16 | 5.22 | 5.58 | 4.82 | 5.36 | 5.24 | 10.54 | 5.34 | 21.96 | 5.52 | 46.58 | 5.54 | 20.52 | 5.30 | 38.32 |
| 17 | 5.02 | 5.52 | 4.80 | 5.34 | 5.32 | 10.32 | 5.26 | 22.92 | 5.52 | 47.64 | 5.58 | 20.64 | 5.48 | 38.24 |
| 18 | 5.12 | 5.52 | 4.82 | 5.34 | 5.08 | 10.12 | 5.32 | 23.38 | 5.72 | 48.60 | 5.44 | 20.00 | 5.50 | 37.90 |
| 19 | 5.26 | 5.58 | 4.92 | 5.12 | 5.00 | 9.64 | 5.34 | 24.18 | 5.96 | 49.76 | 5.68 | 19.28 | 5.46 | 38.24 |
| 20 | 5.04 | 5.50 | 4.90 | 5.20 | 5.04 | 9.62 | 5.38 | 25.18 | 5.92 | 50.86 | 5.48 | 19.42 | 5.40 | 38.74 |
| 21 | 5.08 | 5.62 | 5.02 | 5.16 | 5.32 | 9.48 | 5.56 | 25.62 | 6.04 | 52.02 | 5.56 | 19.14 | 5.54 | 38.88 |
| 22 | 5.32 | 5.60 | 4.98 | 5.32 | 5.10 | 9.40 | 5.64 | 25.82 | 6.00 | 53.60 | 5.86 | 18.96 | 5.82 | 39.06 |
| 23 | 5.18 | 5.88 | 5.20 | 5.38 | 5.06 | 9.36 | 5.46 | 26.66 | 6.04 | 54.52 | 5.86 | 18.72 | 6.02 | 38.86 |
| 24 | 5.16 | 5.76 | 5.30 | 5.56 | 4.94 | 9.40 | 5.46 | 26.84 | 6.14 | 55.68 | 5.72 | 18.54 | 5.94 | 39.48 |
| 25 | 5.64 | 6.16 | 5.36 | 5.74 | 4.88 | 9.10 | 5.54 | 27.58 | 6.32 | 56.74 | 6.08 | 18.64 | 6.02 | 39.42 |
| 26 | 5.64 | 6.14 | 5.18 | 5.66 | 5.08 | 8.86 | 5.72 | 27.80 | 6.42 | 57.38 | 5.86 | 18.14 | 6.02 | 39.34 |
| 27 | 5.58 | 6.10 | 5.58 | 5.60 | 5.04 | 8.84 | 6.02 | 28.48 | 6.60 | 58.22 | 5.92 | 18.14 | 6.30 | 39.58 |
| 28 | 5.56 | 6.08 | 5.36 | 5.72 | 5.02 | 8.66 | 5.80 | 29.10 | 6.60 | 58.98 | 5.90 | 18.08 | 6.16 | 39.32 |
| 29 | 5.74 | 6.28 | 5.64 | 5.74 | 5.04 | 8.60 | 5.78 | 29.56 | 6.80 | 59.86 | 5.76 | 17.96 | 6.52 | 39.74 |
| 30 | 5.68 | 6.58 | 5.64 | 5.96 | 5.06 | 8.58 | 5.96 | 30.22 | 6.80 | 60.22 | 5.94 | 17.92 | 6.56 | 40.14 |
| 31 | 5.68 | 6.44 | 5.78 | 6.12 | 4.80 | 8.48 | 5.88 | 30.84 | 6.94 | 60.44 | 6.00 | 17.86 | 6.64 | 40.22 |
| 32 | 5.66 | 6.40 | 5.86 | 6.06 | 4.76 | 8.40 | 6.16 | 31.10 | 7.12 | 60.68 | 5.88 | 17.74 | 6.66 | 40.38 |
| 33 | 5.70 | 6.52 | 5.90 | 6.20 | 4.68 | 8.36 | 6.30 | 31.90 | 7.02 | 61.72 | 5.90 | 17.40 | 6.54 | 40.22 |
| 34 | 5.88 | 6.70 | 5.76 | 6.32 | 4.94 | 8.36 | 6.30 | 32.22 | 7.28 | 62.48 | 6.04 | 17.40 | 6.76 | 40.52 |
| 35 | 5.88 | 6.82 | 5.80 | 6.30 | 4.88 | 8.58 | 6.38 | 32.86 | 7.26 | 62.80 | 6.38 | 17.14 | 6.56 | 40.88 |
| 36 | 5.70 | 6.72 | 5.76 | 6.44 | 5.24 | 8.68 | 6.52 | 33.38 | 7.34 | 63.74 | 6.40 | 17.08 | 6.52 | 41.04 |
| 37 | 5.74 | 6.80 | 5.82 | 6.44 | 5.08 | 8.52 | 6.60 | 33.64 | 7.36 | 64.06 | 6.36 | 16.96 | 6.42 | 40.98 |
| 38 | 6.00 | 6.96 | 5.80 | 6.78 | 5.24 | 8.44 | 6.76 | 33.80 | 7.30 | 64.68 | 6.48 | 16.96 | 6.50 | 41.16 |
| 39 | 6.22 | 7.04 | 6.06 | 6.78 | 5.30 | 8.44 | 6.74 | 34.42 | 7.50 | 65.44 | 6.60 | 16.94 | 6.60 | 41.12 |
| 40 | 6.06 | 7.08 | 6.16 | 7.02 | 5.40 | 8.44 | 6.90 | 34.82 | 7.60 | 65.60 | 6.48 | 16.88 | 6.68 | 41.18 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\varepsilon_t \sim$ i.i.d. $N(0,1)$, $h_{1t} = 1 + I(t/n > 0.5)$, $h_{2t} = t/n$, $r_t \sim$ GARCH(1,1), $\alpha = 0.2$, $\beta = 0.7$.

Table 3: Tests for zero serial correlation at lag $k$. Power of tests $\widetilde{t}_k$, $t_k$.

| | $x_t$ AR(1) $\phi=0.2$ | | $x_t$ MA(1) $\theta=0.2$ | | $x_t=r_{1t}^2$ $r_{1t}$ ARCH | | $x_t=\|r_{1t}\|$ $r_{1t}$ ARCH | | $x_t=r_{2t}^2$ $r_{2t}$ GARCH | | $x_t=\|r_{2t}\|$ $r_{2t}$ GARCH | | $x_t=\|\varepsilon_t\varepsilon_{t-1}\|$ $\varepsilon_t$ iid | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ |
| 1 | 92.28 | 92.40 | 91.02 | 91.12 | 46.08 | 73.86 | 60.18 | 70.40 | 60.66 | 83.34 | 77.68 | 84.06 | 99.94 | 100 |
| 2 | 10.88 | 10.52 | 6.24 | 6.32 | 6.14 | 10.94 | 6.82 | 8.68 | 50.66 | 75.20 | 68.56 | 76.30 | 12.46 | 9.56 |
| 3 | 6.08 | 5.66 | 5.90 | 5.60 | 7.50 | 5.30 | 5.72 | 5.90 | 42.66 | 65.58 | 59.30 | 67.66 | 9.94 | 7.76 |
| 4 | 5.40 | 5.28 | 5.28 | 5.28 | 7.26 | 4.74 | 5.56 | 5.42 | 34.14 | 57.06 | 50.76 | 59.04 | 10.16 | 7.24 |
| 5 | 5.82 | 5.74 | 5.66 | 5.56 | 7.50 | 4.22 | 5.36 | 5.24 | 27.56 | 48.16 | 41.98 | 50.34 | 8.94 | 6.96 |
| 6 | 5.92 | 5.36 | 5.66 | 5.34 | 7.38 | 4.12 | 5.32 | 4.72 | 21.70 | 40.28 | 35.00 | 41.88 | 9.64 | 6.66 |
| 7 | 5.50 | 5.14 | 5.40 | 5.02 | 8.08 | 3.98 | 5.32 | 4.80 | 18.24 | 34.72 | 30.16 | 36.76 | 9.82 | 6.96 |
| 8 | 5.40 | 5.04 | 5.30 | 4.96 | 7.78 | 4.08 | 5.88 | 5.44 | 14.88 | 28.12 | 25.42 | 30.88 | 10.34 | 7.30 |
| 9 | 5.84 | 5.60 | 5.84 | 5.48 | 7.90 | 4.32 | 6.14 | 5.32 | 12.60 | 24.10 | 21.68 | 26.76 | 9.96 | 7.94 |
| 10 | 5.96 | 5.62 | 5.88 | 5.52 | 7.40 | 3.62 | 5.32 | 4.80 | 11.18 | 19.76 | 18.20 | 21.86 | 9.46 | 6.84 |
| 11 | 6.24 | 5.90 | 6.14 | 5.64 | 8.26 | 4.40 | 6.08 | 4.86 | 9.94 | 18.00 | 16.26 | 19.96 | 9.90 | 6.62 |
| 12 | 5.66 | 5.10 | 5.44 | 5.02 | 8.28 | 4.26 | 6.10 | 5.34 | 9.34 | 13.96 | 13.66 | 16.54 | 10.52 | 7.04 |
| 13 | 5.42 | 5.00 | 5.46 | 4.90 | 8.36 | 4.00 | 5.66 | 5.10 | 9.24 | 12.92 | 13.10 | 15.28 | 10.02 | 7.22 |
| 14 | 5.90 | 5.24 | 5.92 | 5.10 | 7.44 | 3.66 | 5.88 | 5.02 | 7.80 | 9.72 | 10.90 | 12.46 | 10.06 | 6.56 |
| 15 | 5.54 | 4.94 | 5.50 | 4.90 | 8.00 | 3.96 | 5.84 | 5.08 | 8.48 | 9.74 | 10.58 | 12.04 | 9.90 | 6.82 |
| 16 | 6.34 | 5.76 | 6.22 | 5.64 | 7.74 | 4.30 | 5.94 | 4.66 | 8.90 | 8.70 | 9.98 | 10.66 | 9.98 | 7.16 |
| 17 | 5.50 | 4.84 | 5.44 | 4.68 | 7.96 | 3.80 | 6.06 | 4.72 | 9.98 | 7.90 | 10.30 | 10.62 | 9.76 | 6.82 |
| 18 | 5.66 | 4.94 | 5.64 | 5.00 | 7.58 | 3.86 | 5.64 | 4.68 | 8.80 | 7.72 | 9.04 | 9.42 | 9.76 | 7.02 |
| 19 | 6.04 | 5.18 | 5.84 | 5.06 | 8.16 | 3.80 | 6.42 | 5.10 | 10.16 | 6.52 | 10.16 | 9.48 | 10.46 | 7.08 |
| 20 | 6.32 | 5.46 | 6.16 | 5.24 | 7.78 | 3.52 | 6.02 | 4.48 | 9.74 | 5.80 | 9.18 | 8.30 | 10.84 | 6.92 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\varepsilon_t \sim$ i.i.d. $N(0,1)$, $r_{1t} \sim$ ARCH(1), $\alpha = 0.2$, $r_{2t} \sim$ GARCH(1,1), $\alpha = 0.2$, $\beta = 0.7$.

Table 4: Tests for zero serial correlation at lags $1, ..., m$. Power of tests $\widetilde{Q}_m$, $LB_m$.

| | $x_t$ AR(1) $\phi=0.2$ | | $x_t$ MA(1) $\theta=0.2$ | | $x_t=r_{1t}^2$ $r_{1t}$ ARCH | | $x_t=\|r_{1t}\|$ $r_{1t}$ ARCH | | $x_t=r_{2t}^2$ $r_{2t}$ GARCH | | $x_t=\|r_{2t}\|$ $r_{2t}$ GARCH | | $x_t=\|\varepsilon_t\varepsilon_{t-1}\|$ $\varepsilon_t$ iid | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ |
| 1 | 92.28 | 92.50 | 91.02 | 91.34 | 46.08 | 74.04 | 60.18 | 70.64 | 60.66 | 83.50 | 77.68 | 84.22 | 99.94 | 100 |
| 2 | 86.42 | 87.52 | 85.60 | 86.44 | 33.32 | 67.72 | 47.84 | 62.02 | 73.28 | 91.04 | 84.44 | 90.22 | 99.68 | 100 |
| 3 | 82.16 | 83.36 | 80.16 | 81.58 | 27.74 | 63.06 | 41.04 | 56.52 | 77.62 | 92.94 | 86.78 | 91.46 | 99.30 | 100 |
| 4 | 77.16 | 78.58 | 74.38 | 76.54 | 25.42 | 59.44 | 35.92 | 52.52 | 78.70 | 93.50 | 87.10 | 91.92 | 98.56 | 100 |
| 5 | 73.16 | 74.96 | 69.58 | 71.92 | 23.72 | 56.82 | 33.10 | 49.54 | 78.66 | 93.52 | 86.90 | 91.92 | 98.04 | 100 |
| 6 | 69.72 | 72.14 | 66.16 | 68.42 | 23.76 | 54.42 | 29.96 | 47.46 | 77.64 | 93.44 | 86.12 | 91.60 | 97.38 | 99.98 |
| 7 | 66.16 | 68.96 | 62.34 | 65.08 | 23.32 | 52.08 | 27.40 | 44.88 | 76.74 | 93.34 | 85.68 | 91.26 | 96.74 | 100 |
| 8 | 63.90 | 66.64 | 59.88 | 62.72 | 23.24 | 50.28 | 26.30 | 43.00 | 75.50 | 93.02 | 84.90 | 91.00 | 96.10 | 100 |
| 9 | 61.00 | 64.42 | 56.86 | 60.10 | 23.08 | 48.64 | 25.16 | 41.70 | 73.94 | 92.56 | 84.38 | 90.58 | 95.04 | 99.96 |
| 10 | 59.06 | 62.20 | 54.80 | 58.10 | 22.98 | 47.58 | 24.08 | 40.02 | 73.04 | 92.26 | 83.46 | 90.02 | 93.94 | 99.96 |
| 11 | 56.62 | 60.40 | 52.88 | 56.18 | 23.82 | 45.92 | 24.00 | 39.26 | 71.76 | 91.92 | 82.46 | 89.60 | 93.22 | 99.90 |
| 12 | 54.74 | 58.74 | 50.70 | 54.28 | 23.48 | 44.98 | 23.48 | 38.46 | 71.08 | 91.76 | 81.84 | 89.02 | 92.46 | 99.88 |
| 13 | 52.52 | 57.26 | 49.36 | 52.28 | 24.00 | 44.00 | 22.68 | 37.48 | 70.38 | 91.12 | 80.94 | 88.56 | 91.60 | 99.86 |
| 14 | 51.74 | 56.02 | 47.56 | 50.94 | 24.06 | 42.70 | 22.26 | 37.10 | 69.64 | 90.52 | 80.46 | 88.52 | 90.76 | 99.78 |
| 15 | 50.14 | 54.70 | 46.18 | 49.90 | 24.18 | 42.00 | 21.56 | 36.30 | 68.96 | 90.22 | 79.76 | 88.02 | 90.16 | 99.70 |
| 16 | 48.96 | 53.40 | 45.00 | 48.92 | 24.78 | 41.56 | 20.94 | 35.54 | 68.36 | 89.78 | 79.06 | 87.54 | 89.22 | 99.66 |
| 17 | 48.14 | 52.90 | 44.20 | 47.92 | 24.70 | 40.88 | 20.54 | 34.92 | 68.28 | 89.34 | 78.54 | 87.20 | 88.48 | 99.68 |
| 18 | 47.12 | 52.12 | 43.40 | 47.68 | 24.82 | 40.08 | 20.44 | 34.02 | 68.12 | 88.96 | 77.94 | 86.92 | 87.76 | 99.58 |
| 19 | 46.32 | 51.56 | 42.54 | 47.10 | 25.34 | 39.36 | 20.18 | 34.18 | 68.06 | 88.60 | 77.60 | 86.80 | 86.78 | 99.52 |
| 20 | 45.48 | 51.00 | 41.92 | 45.84 | 25.66 | 38.52 | 20.34 | 33.46 | 67.80 | 88.30 | 77.10 | 86.46 | 86.46 | 99.44 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\varepsilon_t \sim$ i.i.d. $N(0,1)$, $r_{1t} \sim$ ARCH(1), $\alpha = 0.2$, $r_{2t} \sim$ GARCH(1,1), $\alpha = 0.2$, $\beta = 0.7$.

Table 5: Tests for zero serial correlation at lag $k$. Spurious power of tests $\widetilde{t}_k$, $t_k$.

| | $x_t = m_{1t} + \varepsilon_t$ $\varepsilon_t$ iid | | $x_t = m_{2t} + \varepsilon_t$ $\varepsilon_t$ iid | | $x_t = m_{3t} + \varepsilon_t$ $\varepsilon_t$ iid | | $x_t = m_{2t} + h_{1t}\varepsilon_t$ $\varepsilon_t$ iid | | $x_t = (h_{1t}\varepsilon_t)^2$ $\varepsilon_t$ iid | | $x_t = \lvert h_{1t}\varepsilon_t \rvert$ $\varepsilon_t$ iid | | $x_t = (m_{1t} + \varepsilon_t)^2$ $\varepsilon_t$ iid | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ | $\widetilde{t}_k$ | $t_k$ |
| 1 | 91.92 | 91.22 | 92.12 | 91.52 | 100 | 100 | 27.66 | 34.98 | 35.36 | 45.94 | 62.46 | 67.36 | 10.42 | 18.02 |
| 2 | 91.18 | 90.44 | 91.04 | 90.38 | 100 | 100 | 28.28 | 35.36 | 35.30 | 47.40 | 62.50 | 67.28 | 10.42 | 18.26 |
| 3 | 92.10 | 91.24 | 90.72 | 89.92 | 100 | 100 | 26.72 | 33.30 | 34.74 | 46.10 | 62.06 | 66.90 | 10.22 | 16.96 |
| 4 | 90.86 | 90.24 | 89.16 | 88.60 | 100 | 100 | 27.38 | 34.56 | 35.26 | 46.06 | 62.04 | 66.40 | 11.12 | 17.42 |
| 5 | 90.20 | 89.58 | 89.06 | 87.96 | 100 | 100 | 26.84 | 33.26 | 33.52 | 45.26 | 60.66 | 65.60 | 10.48 | 17.30 |
| 6 | 90.20 | 88.96 | 87.60 | 86.42 | 100 | 100 | 25.02 | 31.10 | 33.02 | 44.28 | 59.50 | 64.60 | 9.98 | 16.88 |
| 7 | 89.62 | 88.94 | 87.46 | 86.02 | 100 | 100 | 24.08 | 29.80 | 32.00 | 43.92 | 59.02 | 63.80 | 9.80 | 16.48 |
| 8 | 89.00 | 88.06 | 85.76 | 84.16 | 100 | 100 | 23.70 | 29.18 | 31.88 | 43.04 | 59.08 | 62.78 | 10.58 | 16.36 |
| 9 | 88.28 | 86.92 | 83.76 | 82.46 | 100 | 100 | 23.06 | 28.14 | 30.66 | 42.56 | 57.76 | 62.16 | 9.02 | 15.28 |
| 10 | 87.80 | 86.18 | 83.44 | 81.68 | 100 | 100 | 22.38 | 27.50 | 30.30 | 42.70 | 56.62 | 61.52 | 9.72 | 15.30 |
| 11 | 87.00 | 85.46 | 81.70 | 80.16 | 100 | 100 | 22.14 | 27.86 | 28.64 | 39.72 | 54.88 | 58.48 | 8.86 | 14.90 |
| 12 | 87.10 | 85.02 | 80.24 | 78.28 | 100 | 100 | 21.02 | 26.14 | 28.66 | 39.64 | 54.62 | 58.52 | 9.12 | 15.36 |
| 13 | 85.98 | 84.28 | 78.88 | 76.66 | 100 | 100 | 21.30 | 25.82 | 28.76 | 39.76 | 53.98 | 58.42 | 9.02 | 15.22 |
| 14 | 85.32 | 83.04 | 77.84 | 75.30 | 100 | 100 | 19.66 | 24.26 | 26.98 | 38.46 | 53.04 | 57.54 | 8.16 | 14.42 |
| 15 | 85.02 | 83.10 | 76.78 | 74.18 | 100 | 100 | 18.84 | 23.90 | 27.34 | 39.34 | 53.42 | 57.42 | 7.82 | 14.26 |
| 16 | 83.86 | 82.02 | 73.66 | 71.24 | 100 | 100 | 20.20 | 24.22 | 25.62 | 36.86 | 51.12 | 54.90 | 8.30 | 13.90 |
| 17 | 83.92 | 81.54 | 72.22 | 69.52 | 100 | 100 | 17.70 | 21.64 | 25.52 | 36.34 | 51.06 | 54.44 | 8.36 | 13.66 |
| 18 | 81.76 | 79.38 | 69.50 | 66.72 | 100 | 100 | 17.24 | 20.80 | 24.50 | 35.56 | 49.72 | 52.92 | 8.28 | 13.66 |
| 19 | 82.52 | 80.08 | 69.04 | 65.76 | 100 | 100 | 16.74 | 20.18 | 25.54 | 35.70 | 48.84 | 52.24 | 7.50 | 12.66 |
| 20 | 81.18 | 78.22 | 66.88 | 63.32 | 100 | 100 | 16.72 | 20.40 | 24.28 | 34.60 | 48.20 | 51.82 | 8.14 | 13.02 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\varepsilon_t \sim$ i.i.d. $N(0,1)$, $m_{1t} = I(t/n > 0.5)$, $m_{2t} = I(0.25 < t/n \leq 0.75)$, $m_{3t} = 0.01t$, $h_{1t} = 1 + I(t/n > 0.5)$.

Table 6: Tests for zero serial correlation at lags $1, ..., m$. Spurious power of tests $\widetilde{Q}_m$, $LB_m$.

| | $x_t = m_{1t} + \varepsilon_t$ $\varepsilon_t$ iid | | $x_t = m_{2t} + \varepsilon_t$ $\varepsilon_t$ iid | | $x_t = m_{3t} + \varepsilon_t$ $\varepsilon_t$ iid | | $x_t = m_{2t} + h_{1t}\varepsilon_t$ $\varepsilon_t$ iid | | $x_t = (h_{1t}\varepsilon_t)^2$ $\varepsilon_t$ iid | | $x_t = \lvert h_{1t}\varepsilon_t \rvert$ $\varepsilon_t$ iid | | $x_t = (m_{1t} + \varepsilon_t)^2$ $\varepsilon_t$ iid | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ | $\widetilde{Q}_m$ | $LB_m$ |
| 1 | 91.92 | 91.40 | 92.12 | 91.60 | 100 | 100 | 27.66 | 35.26 | 35.36 | 46.24 | 62.46 | 67.66 | 10.42 | 18.32 |
| 2 | 97.88 | 97.74 | 97.98 | 97.70 | 100 | 100 | 37.92 | 48.62 | 49.94 | 65.10 | 79.20 | 84.22 | 13.08 | 24.18 |
| 3 | 99.30 | 99.22 | 98.98 | 98.92 | 100 | 100 | 44.76 | 56.64 | 61.92 | 76.58 | 87.02 | 91.18 | 14.78 | 28.70 |
| 4 | 99.68 | 99.66 | 99.42 | 99.42 | 100 | 100 | 50.36 | 63.04 | 70.14 | 83.72 | 91.34 | 94.68 | 17.36 | 32.90 |
| 5 | 99.72 | 99.70 | 99.72 | 99.76 | 100 | 100 | 54.82 | 68.48 | 75.20 | 88.28 | 94.28 | 96.84 | 19.42 | 36.74 |
| 6 | 99.84 | 99.84 | 99.82 | 99.84 | 100 | 100 | 57.82 | 71.58 | 79.06 | 91.46 | 95.56 | 97.82 | 20.36 | 39.78 |
| 7 | 99.90 | 99.92 | 99.84 | 99.86 | 100 | 100 | 60.64 | 74.90 | 82.06 | 93.96 | 96.68 | 98.60 | 22.44 | 41.64 |
| 8 | 99.88 | 99.92 | 99.94 | 99.94 | 100 | 100 | 63.02 | 77.40 | 84.60 | 95.18 | 97.20 | 98.88 | 24.08 | 44.78 |
| 9 | 99.86 | 99.92 | 99.92 | 99.96 | 100 | 100 | 64.68 | 79.30 | 86.36 | 96.16 | 97.88 | 99.20 | 25.58 | 46.82 |
| 10 | 99.86 | 99.94 | 99.92 | 99.98 | 100 | 100 | 66.14 | 80.88 | 87.88 | 97.16 | 98.20 | 99.38 | 27.02 | 48.98 |
| 11 | 99.84 | 99.96 | 99.84 | 99.98 | 100 | 100 | 68.12 | 82.12 | 88.88 | 97.50 | 98.58 | 99.56 | 28.00 | 50.56 |
| 12 | 99.60 | 99.98 | 99.64 | 99.98 | 100 | 100 | 68.98 | 82.90 | 90.10 | 97.98 | 98.64 | 99.62 | 28.96 | 51.36 |
| 13 | 99.40 | 99.98 | 99.50 | 99.98 | 100 | 100 | 70.26 | 84.08 | 90.84 | 98.36 | 98.92 | 99.66 | 29.44 | 52.70 |
| 14 | 98.96 | 100 | 99.06 | 99.98 | 99.70 | 100 | 70.92 | 84.90 | 91.44 | 98.66 | 99.00 | 99.74 | 30.76 | 54.02 |
| 15 | 98.46 | 100 | 98.56 | 99.98 | 99.70 | 100 | 70.90 | 85.70 | 92.36 | 98.88 | 99.10 | 99.74 | 31.14 | 55.36 |
| 16 | 98.34 | 100 | 98.20 | 100 | 99.58 | 100 | 71.92 | 86.16 | 92.90 | 98.96 | 99.18 | 99.78 | 31.48 | 56.52 |
| 17 | 97.56 | 100 | 98.02 | 100 | 99.36 | 100 | 72.30 | 86.88 | 93.24 | 99.02 | 99.20 | 99.80 | 32.06 | 57.28 |
| 18 | 96.64 | 100 | 97.72 | 100 | 99.24 | 100 | 72.20 | 87.08 | 93.70 | 99.02 | 99.28 | 99.84 | 32.96 | 58.02 |
| 19 | 96.54 | 100 | 96.90 | 99.98 | 99.14 | 100 | 72.78 | 87.60 | 93.86 | 99.12 | 99.34 | 99.86 | 33.50 | 58.76 |
| 20 | 95.98 | 100 | 96.26 | 100 | 99.00 | 100 | 72.92 | 87.98 | 94.20 | 99.30 | 99.36 | 99.86 | 33.98 | 59.40 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\varepsilon_t \sim$ i.i.d. $N(0,1)$, $m_{1t} = I(t/n > 0.5)$, $m_{2t} = I(0.25 < t/n \leq< 0.75)$, $m_{3t} = 0.01t$, $h_{1t} = 1 + I(t/n > 0.5)$.

Table 7: Tests for zero cross-correlation at lag $k$. $\{x_t\}$ and $\{y_t\}$ independent. Size of tests $\widetilde{t}_{xy,k}$, $\widetilde{t}_{yx,k}$, $t_{xy,k}$ and $t_{yx,k}$.

| | $x_t = h_{1t}\varepsilon_t$, $\varepsilon_t$ iid $y_t = h_{1t}\eta_t$, $\eta_t$ iid | | | | $x_t = h_{1t}\varepsilon_t$, $\varepsilon_t$ iid $y_t = h_{3t}\eta_t$, $\eta_t$ iid | | | | $x_t = r_{1t}$, $r_{1t}$ ARCH $y_t = r_{2t}$, $r_{2t}$ GARCH | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ |
| 0 | 4.50 | 4.50 | 9.22 | 9.22 | 4.64 | 4.64 | 11.00 | 11.00 | 5.08 | 5.08 | 5.26 | 5.26 |
| 1 | 4.48 | 5.46 | 8.42 | 10.18 | 4.38 | 5.54 | 10.08 | 12.02 | 5.04 | 5.20 | 4.88 | 5.18 |
| 2 | 5.30 | 5.02 | 10.10 | 8.96 | 5.26 | 4.96 | 11.88 | 11.12 | 5.30 | 5.06 | 5.02 | 5.02 |
| 3 | 5.30 | 5.00 | 9.34 | 9.68 | 5.30 | 5.20 | 11.04 | 12.18 | 4.52 | 5.02 | 4.56 | 4.98 |
| 4 | 5.10 | 5.34 | 8.90 | 9.50 | 5.00 | 5.30 | 10.68 | 11.20 | 4.52 | 5.20 | 4.36 | 5.10 |
| 5 | 5.26 | 5.12 | 9.20 | 9.22 | 5.24 | 5.28 | 11.42 | 11.10 | 4.70 | 4.84 | 4.76 | 4.70 |
| 6 | 5.26 | 4.60 | 8.98 | 8.00 | 5.30 | 4.70 | 10.66 | 9.82 | 5.32 | 4.64 | 4.98 | 4.66 |
| 7 | 4.86 | 4.78 | 8.48 | 8.34 | 4.92 | 4.96 | 10.24 | 10.44 | 4.92 | 5.34 | 4.70 | 4.90 |
| 8 | 4.52 | 4.68 | 8.14 | 8.22 | 4.58 | 4.46 | 10.04 | 10.04 | 4.96 | 5.24 | 4.64 | 5.02 |
| 9 | 4.40 | 5.04 | 7.98 | 8.44 | 4.64 | 4.76 | 9.80 | 9.96 | 4.82 | 5.02 | 4.58 | 4.66 |
| 10 | 5.10 | 4.74 | 8.36 | 7.96 | 5.12 | 4.60 | 10.04 | 9.70 | 4.80 | 4.50 | 4.66 | 4.56 |
| 11 | 4.62 | 5.12 | 8.38 | 8.80 | 4.68 | 5.44 | 9.96 | 10.84 | 4.50 | 5.08 | 4.02 | 4.72 |
| 12 | 4.84 | 4.80 | 8.24 | 8.18 | 4.74 | 4.66 | 9.74 | 9.98 | 4.80 | 4.90 | 4.54 | 4.54 |
| 13 | 5.04 | 4.62 | 7.88 | 8.38 | 4.86 | 4.64 | 9.66 | 10.44 | 4.98 | 5.10 | 4.72 | 4.44 |
| 14 | 4.82 | 5.12 | 7.60 | 8.24 | 4.88 | 5.04 | 9.40 | 10.64 | 5.22 | 5.18 | 4.36 | 4.36 |
| 15 | 4.36 | 4.70 | 7.58 | 8.16 | 4.38 | 4.60 | 9.40 | 10.16 | 4.56 | 5.06 | 4.16 | 4.88 |
| 16 | 5.42 | 4.84 | 8.26 | 7.82 | 5.44 | 4.70 | 9.64 | 9.94 | 5.54 | 4.86 | 5.02 | 4.24 |
| 17 | 5.04 | 5.00 | 8.12 | 8.08 | 4.96 | 5.14 | 9.08 | 10.06 | 5.02 | 4.56 | 4.36 | 3.96 |
| 18 | 4.86 | 4.80 | 7.86 | 7.86 | 5.00 | 4.92 | 9.40 | 9.76 | 4.54 | 5.26 | 4.20 | 4.54 |
| 19 | 5.06 | 5.00 | 7.68 | 7.74 | 4.80 | 4.94 | 8.84 | 9.62 | 5.10 | 4.78 | 4.54 | 4.24 |
| 20 | 4.76 | 5.30 | 7.38 | 7.78 | 4.54 | 5.08 | 8.42 | 9.90 | 5.08 | 5.18 | 4.46 | 4.42 |
| 21 | 5.02 | 4.66 | 7.70 | 7.60 | 5.00 | 4.96 | 8.86 | 9.62 | 4.74 | 4.80 | 4.22 | 3.88 |
| 22 | 5.02 | 5.30 | 7.68 | 8.12 | 5.16 | 5.28 | 9.42 | 9.72 | 4.76 | 4.88 | 4.14 | 4.18 |
| 23 | 4.88 | 5.48 | 7.26 | 8.34 | 4.68 | 5.54 | 8.84 | 10.20 | 4.80 | 5.40 | 4.38 | 4.50 |
| 24 | 4.64 | 5.50 | 7.10 | 8.16 | 4.84 | 5.46 | 8.30 | 9.98 | 4.38 | 5.34 | 3.74 | 4.28 |
| 25 | 4.98 | 5.08 | 6.78 | 7.40 | 4.94 | 5.10 | 8.04 | 9.44 | 4.86 | 4.60 | 3.80 | 3.78 |
| 26 | 5.42 | 5.78 | 7.86 | 8.18 | 5.36 | 5.82 | 8.96 | 9.84 | 4.66 | 5.10 | 3.92 | 4.18 |
| 27 | 5.16 | 4.90 | 7.08 | 7.02 | 4.94 | 5.10 | 8.12 | 8.96 | 4.96 | 4.56 | 4.02 | 3.54 |
| 28 | 5.08 | 5.20 | 6.80 | 6.64 | 4.70 | 5.18 | 7.90 | 8.34 | 4.84 | 5.64 | 4.06 | 4.36 |
| 29 | 5.52 | 5.28 | 7.30 | 7.24 | 5.30 | 5.06 | 8.30 | 9.32 | 5.62 | 5.36 | 4.50 | 4.20 |
| 30 | 4.54 | 4.56 | 6.46 | 6.28 | 4.58 | 4.50 | 7.36 | 8.50 | 4.66 | 4.80 | 3.82 | 3.72 |
| 31 | 4.20 | 5.06 | 6.20 | 7.00 | 4.40 | 4.90 | 7.12 | 8.60 | 4.38 | 4.90 | 3.22 | 3.80 |
| 32 | 4.82 | 5.04 | 6.42 | 6.80 | 4.72 | 4.76 | 7.48 | 8.34 | 5.20 | 5.22 | 3.92 | 4.30 |
| 33 | 4.52 | 4.46 | 6.56 | 5.98 | 4.64 | 4.64 | 7.44 | 7.52 | 4.52 | 4.74 | 3.32 | 3.76 |
| 34 | 4.92 | 5.04 | 6.44 | 6.92 | 4.68 | 5.28 | 7.32 | 8.76 | 5.12 | 5.04 | 3.92 | 4.00 |
| 35 | 4.60 | 4.72 | 6.24 | 6.08 | 4.68 | 4.66 | 7.16 | 7.62 | 4.74 | 5.02 | 3.90 | 3.78 |
| 36 | 4.80 | 4.68 | 6.20 | 6.08 | 4.94 | 4.72 | 7.18 | 8.08 | 5.14 | 5.10 | 3.92 | 3.72 |
| 37 | 5.50 | 5.06 | 6.44 | 6.20 | 5.38 | 5.08 | 7.22 | 8.16 | 4.70 | 4.72 | 3.46 | 3.52 |
| 38 | 4.80 | 4.46 | 6.02 | 5.38 | 4.70 | 4.64 | 6.70 | 7.00 | 4.92 | 4.62 | 3.44 | 3.42 |
| 39 | 5.18 | 4.60 | 6.06 | 5.74 | 5.12 | 4.60 | 6.66 | 7.42 | 5.42 | 5.04 | 3.84 | 3.64 |
| 40 | 5.04 | 4.98 | 5.92 | 5.88 | 5.06 | 5.06 | 6.70 | 7.80 | 4.42 | 5.30 | 3.22 | 3.92 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $h_{1t} = 1 + I(t/n > 0.5)$, $h_{3t} = 1 + 3I(t/n > 0.5)$, $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$, $r_{1t} \sim$ ARCH(1), $\alpha = 0.2$, $r_{2t} \sim$ GARCH(1,1), $\alpha = 0.2$, $\beta = 0.7$, $\{r_{1t}\}$ and $\{r_{2t}\}$ mutually independent.

Table 8: Tests for zero cross-correlation at lags $0, 1, ..., m$. $\{x_t\}$ and $\{y_t\}$ independent. Size of tests $\widetilde{Q}_{xy,m}$, $\widetilde{Q}_{yx,m}$, $HB_{xy,m}$ and $HB_{yx,m}$.

| | $x_t = h_{1t}\varepsilon_t$, $\varepsilon_t$ iid $y_t = h_{1t}\eta_t$, $\eta_t$ iid | | | | $x_t = h_{1t}\varepsilon_t$, $\varepsilon_t$ iid $y_t = h_{3t}\eta_t$, $\eta_t$ iid | | | | $x_t = r_{1t}$, $r_{1t}$ ARCH $y_t = r_{2t}$, $r_{2t}$ GARCH | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ |
| 0 | 4.50 | 4.50 | 9.22 | 9.22 | 4.64 | 4.64 | 11.00 | 11.00 | 5.08 | 5.08 | 5.26 | 5.26 |
| 1 | 4.40 | 4.72 | 10.40 | 11.16 | 4.56 | 4.46 | 13.12 | 14.38 | 5.10 | 4.90 | 5.32 | 4.96 |
| 2 | 4.72 | 4.54 | 12.22 | 12.78 | 4.46 | 4.52 | 16.30 | 16.46 | 5.10 | 4.80 | 5.30 | 4.92 |
| 3 | 4.82 | 4.88 | 14.02 | 13.86 | 4.84 | 4.74 | 18.28 | 18.34 | 5.30 | 5.00 | 5.40 | 5.16 |
| 4 | 4.62 | 4.92 | 14.92 | 14.88 | 4.50 | 4.76 | 20.16 | 20.58 | 4.58 | 5.12 | 4.86 | 5.30 |
| 5 | 4.44 | 4.78 | 15.94 | 15.72 | 4.58 | 4.66 | 21.62 | 22.50 | 4.84 | 5.20 | 5.16 | 5.12 |
| 6 | 4.78 | 4.30 | 17.04 | 16.38 | 4.82 | 4.38 | 23.84 | 23.96 | 4.74 | 5.16 | 5.14 | 5.10 |
| 7 | 4.92 | 4.44 | 17.70 | 17.80 | 5.00 | 4.54 | 25.12 | 25.62 | 4.78 | 4.84 | 4.90 | 5.10 |
| 8 | 4.82 | 4.54 | 18.74 | 19.04 | 4.78 | 4.60 | 26.96 | 26.74 | 4.64 | 4.96 | 5.02 | 5.14 |
| 9 | 5.00 | 4.72 | 19.22 | 19.72 | 4.80 | 4.58 | 27.54 | 28.88 | 4.66 | 4.80 | 4.88 | 5.32 |
| 10 | 5.10 | 4.28 | 19.90 | 19.68 | 4.84 | 4.40 | 29.10 | 29.82 | 4.70 | 4.86 | 4.76 | 5.14 |
| 11 | 4.82 | 4.42 | 20.84 | 20.98 | 4.70 | 4.70 | 30.30 | 31.08 | 4.42 | 4.90 | 4.48 | 5.26 |
| 12 | 4.76 | 4.50 | 21.10 | 21.72 | 4.88 | 4.38 | 31.28 | 32.18 | 4.32 | 4.60 | 4.88 | 5.16 |
| 13 | 4.80 | 4.46 | 21.90 | 22.70 | 4.94 | 4.22 | 31.98 | 33.80 | 4.40 | 4.48 | 4.98 | 5.14 |
| 14 | 4.68 | 4.28 | 22.70 | 23.08 | 4.62 | 4.40 | 33.06 | 35.22 | 4.56 | 4.46 | 4.96 | 5.14 |
| 15 | 4.68 | 4.44 | 23.30 | 23.98 | 4.44 | 4.22 | 34.02 | 35.86 | 4.62 | 4.52 | 4.92 | 5.14 |
| 16 | 4.70 | 4.56 | 25.10 | 24.90 | 4.48 | 4.48 | 34.84 | 38.02 | 4.66 | 4.36 | 5.28 | 5.26 |
| 17 | 4.42 | 4.56 | 24.98 | 25.64 | 4.38 | 4.38 | 36.46 | 39.36 | 4.74 | 4.56 | 5.20 | 4.94 |
| 18 | 4.66 | 4.46 | 25.56 | 26.24 | 4.40 | 4.58 | 37.20 | 40.12 | 4.72 | 4.62 | 5.36 | 5.08 |
| 19 | 4.56 | 4.42 | 26.22 | 26.92 | 4.34 | 4.42 | 38.26 | 40.90 | 4.86 | 4.48 | 5.50 | 5.20 |
| 20 | 4.40 | 4.14 | 26.78 | 27.58 | 4.26 | 4.44 | 38.72 | 41.90 | 4.78 | 4.56 | 5.58 | 5.46 |
| 21 | 4.32 | 4.38 | 27.28 | 28.26 | 4.12 | 4.38 | 40.12 | 42.50 | 4.80 | 4.54 | 5.56 | 5.32 |
| 22 | 4.06 | 4.48 | 27.64 | 28.60 | 3.92 | 4.42 | 41.24 | 44.32 | 4.62 | 4.82 | 5.40 | 5.20 |
| 23 | 4.06 | 4.38 | 28.68 | 29.62 | 3.86 | 4.28 | 41.82 | 45.40 | 4.66 | 4.92 | 5.26 | 5.52 |
| 24 | 4.20 | 4.20 | 29.40 | 30.10 | 4.08 | 4.22 | 42.86 | 46.80 | 4.68 | 4.70 | 5.36 | 5.48 |
| 25 | 4.10 | 4.00 | 30.02 | 30.68 | 4.08 | 4.28 | 43.30 | 47.84 | 4.76 | 4.94 | 5.36 | 5.52 |
| 26 | 4.16 | 4.18 | 30.66 | 31.66 | 4.08 | 4.06 | 44.72 | 49.04 | 4.28 | 4.70 | 5.20 | 5.32 |
| 27 | 4.08 | 4.34 | 31.28 | 32.50 | 4.02 | 4.44 | 44.78 | 49.88 | 4.22 | 4.54 | 5.18 | 5.44 |
| 28 | 4.08 | 4.36 | 31.68 | 32.78 | 4.18 | 4.48 | 45.86 | 50.20 | 4.50 | 4.70 | 5.46 | 5.44 |
| 29 | 3.92 | 4.52 | 32.04 | 32.86 | 4.12 | 4.48 | 46.86 | 51.24 | 4.58 | 4.58 | 5.42 | 5.72 |
| 30 | 3.98 | 4.82 | 32.32 | 34.18 | 4.24 | 4.52 | 47.42 | 51.86 | 4.34 | 4.58 | 5.32 | 5.84 |
| 31 | 3.96 | 4.48 | 32.56 | 34.58 | 4.08 | 4.34 | 47.78 | 52.56 | 4.28 | 4.60 | 4.90 | 5.70 |
| 32 | 4.18 | 4.54 | 32.90 | 34.88 | 4.22 | 4.44 | 48.12 | 53.08 | 4.54 | 4.42 | 5.14 | 5.80 |
| 33 | 4.24 | 4.34 | 33.00 | 34.96 | 4.08 | 4.10 | 48.84 | 53.82 | 4.34 | 4.46 | 5.20 | 5.74 |
| 34 | 4.20 | 4.52 | 33.44 | 35.70 | 4.08 | 4.48 | 49.56 | 54.74 | 4.44 | 4.34 | 4.98 | 5.70 |
| 35 | 4.32 | 4.62 | 34.08 | 36.24 | 4.26 | 4.58 | 49.66 | 55.60 | 4.64 | 4.68 | 5.12 | 5.78 |
| 36 | 4.52 | 4.52 | 34.32 | 36.20 | 4.32 | 4.50 | 50.34 | 56.54 | 4.76 | 4.82 | 5.30 | 5.68 |
| 37 | 4.48 | 4.58 | 34.62 | 36.42 | 4.30 | 4.48 | 50.62 | 56.76 | 4.60 | 4.56 | 5.18 | 5.80 |
| 38 | 4.42 | 4.56 | 35.12 | 36.94 | 4.40 | 4.58 | 51.22 | 57.52 | 4.56 | 4.48 | 5.30 | 5.78 |
| 39 | 4.50 | 4.48 | 35.80 | 37.32 | 4.70 | 4.50 | 51.24 | 57.82 | 4.64 | 4.72 | 5.20 | 5.74 |
| 40 | 4.36 | 4.54 | 35.80 | 37.88 | 4.64 | 4.68 | 51.88 | 58.48 | 4.38 | 4.80 | 5.48 | 5.90 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $h_{1t} = 1 + I(t/n > 0.5)$, $h_{3t} = 1 + 3I(t/n > 0.5)$, $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$, $r_{1t} \sim$ ARCH(1), $\alpha = 0.2$, $r_{2t} \sim$ GARCH(1,1), $\alpha = 0.2$, $\beta = 0.7$, $\{r_{1t}\}$ and $\{r_{2t}\}$ mutually independent.

Table 9: Tests for zero cross-correlation at lag $k$. $\{x_t\}$ and $\{y_t\}$ independent. Size of tests $\widetilde{t}_{xy,k}$, $\widetilde{t}_{yx,k}$, $t_{xy,k}$ and $t_{yx,k}$.

| | $x_t = \varepsilon_t$, $\varepsilon_t$ iid $y_t = m_{1t} + h_{1t}\eta_t$, $\eta_t$ iid | | | | $x_t = h_{1t}\varepsilon_t$, $\varepsilon_t$ iid $y_t = m_{1t} + \eta_t$, $\eta_t$ iid | | | | $x_t = \varepsilon_t$, $\varepsilon_t$ iid $y_t = 0.7y_{t-1} + \eta_t$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ |
| 0 | 4.84 | 4.84 | 4.96 | 4.96 | 4.86 | 4.86 | 4.86 | 4.86 | 5.06 | 5.06 | 4.92 | 4.92 |
| 1 | 4.94 | 4.88 | 4.84 | 5.16 | 4.74 | 5.10 | 4.84 | 5.12 | 4.84 | 4.88 | 5.06 | 4.72 |
| 2 | 5.42 | 5.62 | 5.46 | 5.74 | 5.32 | 5.40 | 5.32 | 5.50 | 5.10 | 5.10 | 4.96 | 4.92 |
| 3 | 5.04 | 5.30 | 4.66 | 5.54 | 4.84 | 5.34 | 4.86 | 5.12 | 4.42 | 5.28 | 4.30 | 5.16 |
| 4 | 4.72 | 5.40 | 4.78 | 5.42 | 4.92 | 4.86 | 5.06 | 4.80 | 4.64 | 5.06 | 4.20 | 5.12 |
| 5 | 5.22 | 4.98 | 4.72 | 5.00 | 5.06 | 5.14 | 5.00 | 4.62 | 5.32 | 5.12 | 5.30 | 4.84 |
| 6 | 5.48 | 4.32 | 5.24 | 4.12 | 5.46 | 4.44 | 5.54 | 3.92 | 4.80 | 4.72 | 4.54 | 4.40 |
| 7 | 4.90 | 5.40 | 4.54 | 5.30 | 4.52 | 5.24 | 4.54 | 4.76 | 5.06 | 4.94 | 4.86 | 4.70 |
| 8 | 4.74 | 4.76 | 4.38 | 4.62 | 4.96 | 4.82 | 5.14 | 4.48 | 4.82 | 4.74 | 4.54 | 4.38 |
| 9 | 4.42 | 5.68 | 4.10 | 5.16 | 4.58 | 5.42 | 4.62 | 5.14 | 4.92 | 5.44 | 4.54 | 5.36 |
| 10 | 4.98 | 4.98 | 4.64 | 4.88 | 5.18 | 4.62 | 5.28 | 4.32 | 4.92 | 4.72 | 4.66 | 4.50 |
| 11 | 4.60 | 5.04 | 3.80 | 4.72 | 4.90 | 4.78 | 4.86 | 4.10 | 5.02 | 4.62 | 4.80 | 4.20 |
| 12 | 5.04 | 4.96 | 4.14 | 4.64 | 5.16 | 4.70 | 5.06 | 4.10 | 5.10 | 5.02 | 4.68 | 4.44 |
| 13 | 5.20 | 5.04 | 4.38 | 4.72 | 5.12 | 5.04 | 4.82 | 4.34 | 4.96 | 4.94 | 4.26 | 4.48 |
| 14 | 5.24 | 5.16 | 4.54 | 4.80 | 5.28 | 4.80 | 5.18 | 4.04 | 4.76 | 4.98 | 4.30 | 4.58 |
| 15 | 4.98 | 5.18 | 4.34 | 4.94 | 5.02 | 4.82 | 4.86 | 4.08 | 5.26 | 5.12 | 4.90 | 4.68 |
| 16 | 6.04 | 4.90 | 5.02 | 4.50 | 6.14 | 5.04 | 5.80 | 4.06 | 5.08 | 4.58 | 4.34 | 4.02 |
| 17 | 5.10 | 5.16 | 4.38 | 4.84 | 5.44 | 5.06 | 5.14 | 4.20 | 5.18 | 4.86 | 4.32 | 4.12 |
| 18 | 4.90 | 4.92 | 3.76 | 4.66 | 4.90 | 4.52 | 4.92 | 3.58 | 5.08 | 4.88 | 4.54 | 4.28 |
| 19 | 5.18 | 4.50 | 4.16 | 4.38 | 5.08 | 4.36 | 5.06 | 3.40 | 5.00 | 4.64 | 4.20 | 4.00 |
| 20 | 4.98 | 5.58 | 4.14 | 5.26 | 5.18 | 5.48 | 4.94 | 4.34 | 4.40 | 5.40 | 3.86 | 4.58 |
| 21 | 4.92 | 4.54 | 3.62 | 4.28 | 5.42 | 4.72 | 4.92 | 3.50 | 4.26 | 5.22 | 3.66 | 4.28 |
| 22 | 5.02 | 5.16 | 3.94 | 5.28 | 5.06 | 4.84 | 4.86 | 4.08 | 4.64 | 5.32 | 3.84 | 4.46 |
| 23 | 4.56 | 5.42 | 3.44 | 5.20 | 5.04 | 5.34 | 4.58 | 3.60 | 4.68 | 5.44 | 4.14 | 4.66 |
| 24 | 4.62 | 5.36 | 3.54 | 5.00 | 5.10 | 5.04 | 5.00 | 3.72 | 4.54 | 5.08 | 3.54 | 4.02 |
| 25 | 5.06 | 4.98 | 3.70 | 4.90 | 5.36 | 4.84 | 5.14 | 3.50 | 4.96 | 4.90 | 4.16 | 4.06 |
| 26 | 5.06 | 5.40 | 3.68 | 5.12 | 5.20 | 5.16 | 4.84 | 3.50 | 5.30 | 4.82 | 4.38 | 3.98 |
| 27 | 4.92 | 4.94 | 3.64 | 4.58 | 5.34 | 4.94 | 4.88 | 3.42 | 4.72 | 4.36 | 3.80 | 3.68 |
| 28 | 4.80 | 5.06 | 3.48 | 4.60 | 5.30 | 5.12 | 5.20 | 3.52 | 5.12 | 4.96 | 3.80 | 3.92 |
| 29 | 5.18 | 5.24 | 3.44 | 4.54 | 5.18 | 5.40 | 4.94 | 3.54 | 5.26 | 4.94 | 4.26 | 3.90 |
| 30 | 5.38 | 4.50 | 3.56 | 4.16 | 5.64 | 4.10 | 5.22 | 2.68 | 4.78 | 4.74 | 3.76 | 3.82 |
| 31 | 4.36 | 5.10 | 2.84 | 4.68 | 4.72 | 4.66 | 4.32 | 2.84 | 5.82 | 4.94 | 4.24 | 3.62 |
| 32 | 4.48 | 4.96 | 3.02 | 4.54 | 5.06 | 4.78 | 4.40 | 3.12 | 5.12 | 4.98 | 3.66 | 3.68 |
| 33 | 4.66 | 4.40 | 3.10 | 3.88 | 4.88 | 4.32 | 4.64 | 2.58 | 4.98 | 5.04 | 3.86 | 3.74 |
| 34 | 4.84 | 4.96 | 3.04 | 4.66 | 4.98 | 4.60 | 4.50 | 2.96 | 5.22 | 4.96 | 4.02 | 3.58 |
| 35 | 4.96 | 5.00 | 3.32 | 4.68 | 5.16 | 4.56 | 4.76 | 2.58 | 5.28 | 5.02 | 3.84 | 3.94 |
| 36 | 5.18 | 4.46 | 2.94 | 4.04 | 5.36 | 4.22 | 4.62 | 2.50 | 5.02 | 4.96 | 3.72 | 3.58 |
| 37 | 4.68 | 4.76 | 2.92 | 4.24 | 5.06 | 4.38 | 4.58 | 2.26 | 5.18 | 4.42 | 3.90 | 3.22 |
| 38 | 4.70 | 4.72 | 2.68 | 4.10 | 5.44 | 4.64 | 4.72 | 2.64 | 4.90 | 4.54 | 3.72 | 2.96 |
| 39 | 5.38 | 4.86 | 3.02 | 3.98 | 5.84 | 4.46 | 5.40 | 2.56 | 5.04 | 4.38 | 3.72 | 3.34 |
| 40 | 4.72 | 5.14 | 3.04 | 4.52 | 4.66 | 4.66 | 4.28 | 2.48 | 4.46 | 4.72 | 3.06 | 3.46 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$, $m_{1t} = I(t/n > 0.5)$, $h_{1t} = 1 + I(t/n > 0.5)$.

Table 10: Tests for zero cross-correlation at lags $0, 1, ..., m$. $\{x_t\}$ and $\{y_t\}$ independent. Size of tests $\widetilde{Q}_{xy,m}$, $\widetilde{Q}_{yx,m}$, $HB_{xy,m}$ and $HB_{yx,m}$.

| | $x_t = \varepsilon_t$, $\varepsilon_t$ iid $y_t = m_{1t} + h_{1t}\eta_t$, $\eta_t$ iid | | | | $x_t = h_{1t}\varepsilon_t$, $\varepsilon_t$ iid $y_t = m_{1t} + \eta_t$, $\eta_t$ iid | | | | $x_t = \varepsilon_t$, $\varepsilon_t$ iid $y_t = 0.7y_{t-1} + \eta_t$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ |
| 0 | 4.84 | 4.84 | 4.96 | 4.96 | 4.86 | 4.86 | 4.86 | 4.86 | 5.06 | 5.06 | 4.92 | 4.92 |
| 1 | 4.84 | 4.98 | 5.16 | 4.86 | 5.20 | 4.74 | 5.28 | 5.04 | 4.84 | 6.34 | 6.76 | 6.48 |
| 2 | 5.24 | 5.04 | 5.20 | 5.14 | 5.50 | 5.54 | 5.74 | 5.44 | 4.96 | 7.50 | 8.36 | 7.74 |
| 3 | 5.32 | 4.98 | 5.72 | 5.30 | 5.80 | 5.54 | 6.14 | 5.74 | 6.50 | 8.82 | 8.80 | 9.04 |
| 4 | 5.14 | 5.20 | 5.14 | 5.34 | 6.08 | 5.88 | 6.12 | 5.98 | 8.40 | 9.66 | 8.84 | 9.78 |
| 5 | 5.14 | 5.74 | 5.22 | 5.68 | 6.52 | 6.50 | 6.58 | 6.36 | 11.80 | 10.16 | 9.46 | 10.38 |
| 6 | 4.94 | 5.50 | 5.18 | 5.50 | 6.62 | 6.34 | 6.84 | 6.50 | 14.90 | 10.52 | 9.78 | 10.58 |
| 7 | 5.22 | 5.14 | 5.26 | 5.76 | 7.28 | 6.18 | 7.30 | 6.40 | 18.70 | 10.78 | 10.04 | 10.82 |
| 8 | 5.08 | 5.16 | 5.10 | 5.74 | 8.14 | 6.40 | 7.30 | 6.58 | 21.54 | 11.46 | 9.90 | 11.44 |
| 9 | 4.82 | 5.48 | 5.28 | 5.90 | 8.66 | 6.64 | 7.60 | 6.72 | 23.30 | 11.12 | 9.88 | 11.18 |
| 10 | 4.92 | 5.12 | 5.54 | 5.74 | 10.12 | 6.94 | 7.98 | 6.74 | 23.38 | 11.32 | 10.36 | 11.58 |
| 11 | 5.22 | 5.34 | 5.24 | 6.00 | 11.22 | 7.18 | 8.22 | 6.96 | 22.68 | 11.64 | 10.92 | 11.68 |
| 12 | 5.04 | 5.52 | 5.06 | 6.40 | 12.04 | 7.30 | 8.48 | 7.48 | 22.52 | 11.66 | 11.02 | 11.90 |
| 13 | 5.20 | 5.58 | 5.40 | 6.16 | 13.40 | 7.54 | 8.54 | 7.64 | 23.06 | 11.78 | 11.18 | 11.90 |
| 14 | 5.24 | 5.30 | 5.24 | 6.22 | 14.84 | 7.44 | 8.62 | 7.90 | 22.36 | 11.62 | 11.50 | 11.80 |
| 15 | 5.40 | 5.56 | 4.98 | 6.64 | 16.02 | 7.64 | 8.68 | 7.72 | 22.34 | 11.98 | 11.80 | 12.18 |
| 16 | 5.42 | 5.56 | 5.40 | 6.60 | 16.50 | 7.80 | 9.56 | 8.00 | 22.60 | 12.00 | 11.86 | 12.12 |
| 17 | 5.48 | 5.72 | 5.34 | 6.64 | 17.10 | 7.96 | 9.40 | 7.88 | 23.14 | 12.26 | 12.02 | 12.24 |
| 18 | 5.72 | 5.70 | 5.30 | 6.78 | 17.24 | 8.18 | 9.46 | 8.14 | 23.22 | 12.58 | 12.40 | 12.62 |
| 19 | 5.76 | 5.58 | 5.16 | 6.92 | 18.44 | 8.28 | 9.74 | 8.10 | 23.70 | 12.54 | 12.24 | 12.82 |
| 20 | 5.82 | 5.74 | 5.26 | 7.14 | 19.12 | 8.60 | 9.88 | 7.90 | 23.70 | 12.58 | 11.98 | 12.76 |
| 21 | 6.28 | 5.94 | 5.06 | 7.26 | 20.08 | 8.60 | 10.02 | 8.16 | 23.46 | 12.44 | 12.08 | 12.80 |
| 22 | 6.32 | 6.08 | 5.08 | 7.48 | 19.58 | 8.64 | 10.10 | 8.34 | 24.10 | 12.82 | 12.34 | 12.96 |
| 23 | 6.34 | 5.84 | 5.00 | 7.38 | 19.68 | 8.90 | 10.26 | 8.36 | 23.58 | 13.08 | 12.08 | 13.34 |
| 24 | 6.10 | 5.98 | 5.26 | 7.44 | 19.38 | 8.82 | 10.68 | 8.28 | 24.40 | 13.38 | 12.12 | 13.34 |
| 25 | 6.40 | 6.30 | 5.26 | 7.74 | 19.22 | 8.76 | 10.92 | 8.12 | 24.42 | 13.24 | 12.44 | 13.50 |
| 26 | 6.56 | 6.22 | 5.20 | 7.80 | 20.50 | 8.84 | 10.96 | 8.22 | 24.90 | 13.16 | 12.60 | 13.72 |
| 27 | 6.74 | 6.10 | 5.18 | 7.76 | 20.56 | 9.02 | 10.86 | 8.20 | 25.22 | 13.20 | 12.70 | 13.78 |
| 28 | 6.76 | 6.40 | 5.00 | 8.18 | 20.56 | 8.88 | 11.18 | 8.20 | 24.10 | 12.86 | 12.74 | 13.50 |
| 29 | 6.96 | 6.46 | 4.96 | 8.22 | 20.00 | 8.96 | 11.56 | 8.44 | 23.76 | 13.12 | 12.64 | 13.74 |
| 30 | 7.36 | 6.54 | 5.00 | 8.24 | 20.22 | 9.04 | 11.90 | 8.22 | 23.48 | 13.10 | 12.60 | 13.36 |
| 31 | 7.82 | 6.30 | 5.04 | 8.22 | 20.32 | 9.04 | 11.66 | 8.32 | 23.64 | 13.04 | 13.08 | 13.34 |
| 32 | 7.90 | 6.44 | 5.28 | 8.44 | 20.64 | 9.42 | 12.06 | 8.12 | 23.66 | 13.52 | 13.16 | 13.36 |
| 33 | 7.84 | 6.58 | 5.12 | 8.70 | 21.24 | 9.48 | 12.22 | 8.34 | 23.62 | 13.54 | 13.30 | 13.68 |
| 34 | 8.14 | 6.70 | 5.02 | 9.00 | 21.38 | 9.66 | 12.34 | 8.42 | 23.74 | 13.68 | 13.60 | 13.76 |
| 35 | 8.22 | 6.96 | 5.18 | 9.12 | 20.98 | 9.76 | 12.50 | 8.38 | 23.62 | 13.78 | 13.80 | 13.70 |
| 36 | 8.40 | 6.92 | 5.42 | 9.34 | 21.16 | 9.64 | 12.74 | 8.46 | 23.12 | 13.54 | 13.72 | 13.74 |
| 37 | 8.84 | 6.94 | 5.26 | 9.34 | 20.96 | 9.50 | 12.88 | 8.40 | 23.34 | 13.46 | 13.70 | 13.64 |
| 38 | 8.88 | 6.82 | 5.08 | 9.24 | 20.98 | 9.64 | 13.32 | 8.28 | 22.98 | 13.22 | 13.62 | 13.74 |
| 39 | 8.82 | 6.76 | 5.04 | 9.72 | 21.22 | 9.54 | 13.54 | 8.34 | 22.42 | 13.24 | 13.80 | 13.50 |
| 40 | 8.88 | 6.92 | 4.98 | 9.94 | 21.90 | 9.76 | 13.54 | 8.24 | 22.52 | 12.90 | 13.96 | 13.34 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$, $m_{1t} = I(t/n > 0.5)$, $h_{1t} = 1 + I(t/n > 0.5)$.

Table 11: Tests for zero cross-correlation at lag $k$. $\{x_t\}$ and $\{y_t\}$ uncorrelated (not independent). Size of tests $\widetilde{t}_{xy,k}$, $\widetilde{t}_{yx,k}$, $t_{xy,k}$ and $t_{yx,k}$.

| | $x_t = \varepsilon_t$, $\varepsilon_t$ iid $y_t = \lvert\varepsilon_t\rvert\eta_t$, $\eta_t$ iid | | | | $x_t = \varepsilon_t$, $\varepsilon_t$ iid $y_t = \varepsilon_t\varepsilon_{t-1}$ | | | | $x_t = \varepsilon_t$, $\varepsilon_t$ iid $y_t = \exp(z_t)\eta_t$, $z_t = 0.7z_{t-1} + \varepsilon_t$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ |
| 0 | 5.14 | 5.14 | 24.68 | 24.68 | 4.62 | 4.62 | 24.36 | 24.36 | 3.38 | 3.38 | 29.92 | 29.92 |
| 1 | 4.88 | 4.92 | 4.90 | 5.02 | 4.84 | 5.02 | 4.60 | 25.74 | 3.88 | 3.86 | 4.56 | 16.44 |
| 2 | 4.90 | 4.96 | 5.02 | 5.08 | 4.72 | 4.84 | 4.64 | 4.66 | 4.02 | 4.04 | 4.26 | 9.80 |
| 3 | 4.28 | 5.34 | 4.24 | 5.22 | 5.14 | 5.30 | 5.10 | 4.92 | 3.68 | 3.50 | 4.80 | 6.14 |
| 4 | 4.42 | 5.12 | 4.54 | 5.06 | 5.28 | 4.80 | 5.12 | 4.68 | 4.10 | 3.38 | 4.86 | 5.50 |
| 5 | 4.82 | 4.72 | 4.82 | 4.94 | 5.22 | 4.88 | 5.10 | 4.58 | 3.66 | 3.92 | 5.06 | 5.16 |
| 6 | 4.36 | 4.70 | 4.50 | 4.62 | 5.18 | 5.04 | 4.66 | 4.74 | 3.60 | 3.34 | 4.50 | 4.88 |
| 7 | 4.68 | 5.48 | 4.36 | 4.90 | 5.34 | 4.80 | 4.88 | 4.62 | 4.04 | 3.66 | 4.94 | 4.42 |
| 8 | 5.00 | 4.80 | 4.24 | 4.56 | 4.58 | 4.90 | 4.40 | 4.62 | 4.04 | 4.00 | 4.84 | 4.82 |
| 9 | 4.70 | 5.48 | 4.38 | 4.88 | 4.56 | 5.12 | 4.18 | 4.80 | 3.84 | 4.08 | 4.38 | 4.34 |
| 10 | 4.92 | 4.94 | 4.72 | 4.76 | 4.60 | 4.84 | 4.34 | 4.58 | 3.78 | 4.18 | 4.56 | 4.98 |
| 11 | 4.86 | 5.08 | 4.20 | 4.52 | 4.96 | 4.86 | 4.90 | 4.50 | 3.80 | 4.34 | 4.20 | 4.58 |
| 12 | 4.86 | 5.46 | 4.58 | 5.14 | 5.04 | 4.78 | 4.80 | 4.32 | 4.04 | 3.52 | 4.64 | 4.56 |
| 13 | 4.42 | 5.12 | 3.96 | 4.84 | 4.88 | 5.10 | 4.30 | 4.48 | 3.50 | 4.36 | 3.74 | 4.70 |
| 14 | 4.90 | 4.90 | 4.76 | 4.60 | 4.52 | 4.60 | 4.10 | 4.58 | 3.42 | 3.66 | 4.00 | 4.54 |
| 15 | 4.96 | 4.98 | 4.30 | 4.52 | 4.88 | 4.52 | 4.46 | 4.10 | 3.66 | 4.06 | 3.76 | 4.54 |
| 16 | 5.34 | 4.92 | 4.76 | 4.62 | 5.34 | 5.00 | 4.44 | 4.72 | 3.86 | 3.86 | 4.44 | 4.62 |
| 17 | 5.32 | 4.74 | 4.96 | 4.26 | 4.70 | 4.54 | 4.30 | 4.02 | 4.32 | 3.60 | 5.14 | 4.10 |
| 18 | 4.80 | 4.40 | 4.16 | 3.98 | 4.60 | 5.04 | 3.98 | 4.24 | 4.10 | 3.80 | 4.38 | 4.00 |
| 19 | 5.08 | 5.22 | 4.46 | 4.48 | 5.22 | 5.00 | 4.28 | 4.44 | 4.06 | 4.00 | 4.44 | 4.08 |
| 20 | 4.96 | 4.90 | 4.36 | 4.54 | 5.40 | 5.20 | 4.56 | 4.14 | 4.02 | 4.20 | 4.44 | 4.26 |
| 21 | 5.02 | 4.82 | 4.30 | 4.36 | 4.38 | 5.24 | 3.74 | 3.84 | 3.88 | 3.72 | 4.30 | 4.32 |
| 22 | 4.46 | 5.08 | 3.76 | 4.46 | 4.78 | 4.90 | 4.00 | 3.88 | 3.34 | 4.08 | 3.70 | 4.00 |
| 23 | 4.60 | 5.06 | 3.94 | 4.38 | 5.42 | 4.66 | 4.28 | 4.00 | 4.06 | 4.28 | 4.44 | 4.62 |
| 24 | 4.76 | 4.92 | 4.02 | 4.28 | 4.94 | 4.74 | 3.84 | 3.76 | 3.86 | 4.22 | 3.94 | 4.50 |
| 25 | 5.28 | 5.00 | 4.40 | 4.30 | 5.24 | 4.82 | 4.08 | 4.02 | 3.92 | 3.90 | 4.38 | 3.76 |
| 26 | 4.96 | 5.10 | 4.28 | 4.10 | 4.62 | 4.58 | 3.70 | 3.66 | 3.80 | 4.14 | 3.98 | 3.72 |
| 27 | 5.04 | 4.60 | 4.10 | 3.76 | 4.96 | 5.02 | 3.84 | 3.82 | 3.56 | 3.86 | 3.86 | 3.98 |
| 28 | 5.02 | 4.90 | 4.00 | 3.90 | 5.30 | 4.96 | 3.88 | 3.94 | 3.30 | 4.20 | 3.44 | 3.70 |
| 29 | 5.36 | 5.18 | 4.08 | 4.02 | 4.52 | 4.78 | 3.64 | 3.88 | 4.02 | 3.58 | 3.66 | 3.96 |
| 30 | 4.48 | 4.30 | 3.48 | 3.54 | 5.04 | 4.64 | 3.94 | 3.52 | 4.00 | 3.62 | 4.16 | 3.98 |
| 31 | 4.34 | 5.14 | 3.44 | 3.80 | 5.00 | 4.70 | 3.72 | 3.56 | 3.80 | 3.66 | 3.84 | 3.60 |
| 32 | 5.44 | 5.34 | 3.80 | 3.88 | 4.86 | 4.68 | 3.66 | 3.44 | 4.40 | 3.84 | 3.80 | 4.16 |
| 33 | 4.44 | 4.96 | 3.42 | 4.06 | 4.84 | 4.82 | 3.26 | 3.42 | 3.90 | 3.48 | 3.80 | 4.06 |
| 34 | 5.18 | 5.38 | 4.20 | 4.04 | 4.68 | 4.74 | 3.48 | 3.66 | 3.62 | 3.70 | 3.56 | 4.00 |
| 35 | 3.98 | 4.82 | 3.36 | 3.84 | 4.92 | 4.66 | 3.96 | 3.84 | 4.14 | 3.98 | 3.90 | 3.80 |
| 36 | 5.06 | 4.48 | 3.64 | 3.30 | 4.62 | 4.80 | 3.52 | 3.70 | 3.94 | 4.22 | 4.04 | 4.02 |
| 37 | 5.04 | 4.40 | 3.76 | 3.46 | 4.78 | 4.50 | 3.34 | 3.12 | 3.62 | 3.18 | 3.54 | 3.16 |
| 38 | 5.36 | 4.92 | 3.72 | 3.76 | 4.96 | 5.06 | 3.50 | 3.54 | 3.22 | 3.66 | 3.84 | 4.10 |
| 39 | 5.30 | 4.70 | 3.84 | 3.54 | 4.76 | 4.70 | 3.44 | 3.40 | 4.14 | 3.52 | 3.98 | 3.84 |
| 40 | 4.58 | 4.76 | 3.24 | 3.56 | 4.62 | 5.18 | 3.30 | 3.94 | 4.16 | 4.30 | 3.90 | 3.76 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$.

Table 12: Tests for zero cross-correlation at lags $0, 1, ..., m$. $\{x_t\}$ and $\{y_t\}$ uncorrelated (not independent). Size of tests $\widetilde{Q}_{xy,m}$, $\widetilde{Q}_{yx,m}$, $HB_{xy,m}$ and $HB_{yx,m}$.

| | $x_t = \varepsilon_t$, $\varepsilon_t$ iid $y_t = \|\varepsilon_t\|\eta_t$, $\eta_t$ iid | | | | $x_t = \varepsilon_t$, $\varepsilon_t$ iid $y_t = \varepsilon_t \varepsilon_{t-1}$ | | | | $x_t = \varepsilon_t$, $\varepsilon_t$ iid $y_t = \exp(z_t)\eta_t$, $z_t = 0.7z_{t-1} + \varepsilon_t$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ |
| 0 | 5.14 | 5.14 | 24.68 | 24.68 | 4.62 | 4.62 | 24.36 | 24.36 | 3.38 | 3.38 | 29.92 | 29.92 |
| 1 | 4.58 | 5.10 | 20.46 | 20.68 | 4.42 | 4.90 | 19.74 | 34.50 | 3.62 | 5.22 | 23.98 | 30.74 |
| 2 | 4.86 | 4.62 | 19.30 | 18.94 | 4.76 | 4.92 | 17.54 | 30.44 | 3.30 | 5.22 | 20.92 | 29.74 |
| 3 | 4.58 | 4.84 | 17.78 | 17.08 | 4.42 | 4.96 | 16.16 | 27.80 | 2.94 | 5.10 | 18.34 | 28.26 |
| 4 | 4.32 | 4.64 | 16.16 | 15.76 | 4.68 | 5.02 | 15.44 | 25.92 | 3.00 | 5.14 | 17.04 | 27.38 |
| 5 | 4.48 | 4.76 | 14.98 | 15.06 | 4.94 | 4.82 | 15.06 | 24.48 | 2.88 | 5.02 | 16.64 | 26.48 |
| 6 | 4.50 | 4.82 | 13.98 | 14.74 | 4.54 | 4.90 | 14.00 | 22.88 | 2.84 | 4.60 | 15.94 | 24.98 |
| 7 | 4.70 | 4.54 | 13.68 | 13.98 | 4.74 | 4.78 | 13.46 | 21.88 | 2.90 | 4.64 | 15.12 | 24.34 |
| 8 | 4.28 | 4.60 | 12.78 | 13.26 | 4.56 | 4.80 | 13.14 | 20.72 | 2.76 | 4.40 | 14.80 | 22.96 |
| 9 | 4.38 | 4.62 | 12.68 | 13.20 | 4.46 | 4.86 | 12.54 | 20.44 | 2.76 | 4.04 | 14.08 | 22.28 |
| 10 | 4.04 | 4.48 | 12.00 | 12.80 | 4.50 | 4.68 | 11.86 | 19.30 | 2.76 | 4.16 | 13.46 | 21.78 |
| 11 | 4.38 | 4.42 | 11.60 | 12.48 | 4.36 | 4.92 | 11.30 | 19.36 | 2.78 | 4.28 | 13.14 | 21.06 |
| 12 | 4.26 | 4.58 | 11.58 | 12.44 | 4.60 | 4.94 | 11.28 | 19.14 | 2.74 | 4.30 | 13.06 | 20.10 |
| 13 | 4.34 | 4.32 | 10.96 | 11.32 | 4.60 | 5.06 | 11.24 | 18.14 | 2.94 | 4.32 | 12.30 | 19.68 |
| 14 | 4.30 | 4.36 | 10.90 | 11.32 | 4.34 | 4.94 | 10.56 | 17.60 | 2.86 | 4.26 | 12.02 | 19.82 |
| 15 | 4.46 | 4.32 | 10.86 | 11.20 | 3.96 | 4.94 | 10.38 | 17.02 | 2.84 | 4.38 | 11.72 | 19.44 |
| 16 | 4.72 | 4.22 | 10.84 | 10.82 | 4.12 | 5.02 | 10.32 | 16.90 | 2.82 | 4.48 | 11.48 | 19.36 |
| 17 | 4.80 | 4.34 | 10.62 | 10.54 | 3.98 | 4.82 | 9.90 | 16.78 | 2.76 | 4.18 | 11.60 | 18.80 |
| 18 | 5.02 | 4.56 | 10.40 | 10.44 | 4.08 | 4.68 | 9.58 | 16.52 | 2.86 | 4.08 | 11.18 | 18.10 |
| 19 | 4.74 | 4.62 | 10.42 | 10.20 | 4.14 | 4.64 | 9.44 | 16.40 | 3.02 | 4.08 | 11.08 | 17.74 |
| 20 | 4.84 | 4.62 | 10.42 | 10.20 | 4.22 | 4.68 | 9.10 | 16.00 | 3.22 | 4.00 | 10.90 | 17.32 |
| 21 | 4.94 | 4.62 | 10.24 | 10.26 | 4.24 | 4.88 | 9.28 | 15.90 | 3.14 | 3.82 | 10.44 | 17.18 |
| 22 | 4.70 | 4.48 | 9.98 | 9.98 | 4.26 | 5.02 | 9.08 | 15.62 | 3.10 | 3.96 | 10.52 | 16.80 |
| 23 | 4.66 | 4.40 | 9.52 | 9.84 | 4.12 | 4.92 | 8.56 | 15.30 | 3.22 | 3.98 | 10.56 | 16.64 |
| 24 | 4.78 | 4.86 | 9.54 | 9.64 | 4.16 | 4.84 | 8.54 | 14.98 | 3.24 | 4.00 | 10.46 | 16.50 |
| 25 | 4.80 | 4.76 | 9.06 | 9.66 | 4.30 | 4.92 | 8.66 | 14.48 | 3.44 | 3.80 | 10.34 | 16.28 |
| 26 | 4.42 | 4.62 | 9.14 | 9.60 | 4.26 | 4.98 | 8.48 | 14.28 | 3.02 | 3.78 | 9.78 | 16.24 |
| 27 | 4.64 | 4.82 | 9.60 | 9.52 | 4.22 | 4.90 | 8.46 | 13.96 | 3.12 | 3.62 | 9.74 | 16.24 |
| 28 | 4.80 | 4.72 | 9.24 | 9.48 | 4.32 | 5.20 | 8.42 | 13.94 | 3.06 | 3.62 | 9.52 | 16.36 |
| 29 | 4.74 | 4.72 | 9.28 | 9.06 | 4.46 | 5.18 | 8.28 | 13.78 | 3.04 | 3.76 | 9.46 | 16.02 |
| 30 | 4.70 | 4.58 | 9.26 | 9.12 | 4.26 | 5.16 | 8.52 | 13.62 | 2.94 | 3.58 | 9.74 | 15.52 |
| 31 | 4.76 | 4.82 | 8.76 | 9.28 | 4.38 | 5.06 | 8.52 | 13.10 | 2.90 | 3.64 | 9.44 | 15.68 |
| 32 | 4.68 | 4.68 | 8.86 | 9.26 | 4.34 | 4.92 | 8.26 | 13.14 | 2.98 | 3.58 | 9.52 | 15.56 |
| 33 | 4.34 | 4.62 | 8.60 | 8.76 | 4.42 | 4.68 | 8.36 | 12.82 | 2.96 | 3.58 | 9.56 | 15.34 |
| 34 | 4.40 | 4.64 | 8.58 | 9.00 | 4.46 | 5.02 | 8.08 | 12.50 | 2.96 | 3.66 | 9.54 | 15.18 |
| 35 | 4.42 | 4.64 | 8.42 | 8.62 | 4.44 | 4.96 | 8.12 | 12.78 | 2.76 | 3.96 | 9.80 | 15.34 |
| 36 | 4.48 | 4.62 | 8.60 | 8.66 | 4.34 | 5.00 | 7.94 | 12.62 | 2.92 | 3.98 | 9.54 | 15.16 |
| 37 | 4.34 | 4.50 | 8.52 | 8.50 | 4.48 | 5.00 | 8.18 | 12.44 | 3.04 | 3.78 | 9.78 | 15.28 |
| 38 | 4.40 | 4.36 | 8.34 | 8.48 | 4.40 | 5.00 | 8.06 | 12.36 | 3.24 | 3.66 | 9.66 | 15.00 |
| 39 | 4.38 | 4.52 | 8.24 | 8.42 | 4.38 | 5.14 | 7.98 | 12.22 | 3.22 | 3.70 | 9.98 | 15.06 |
| 40 | 4.46 | 4.62 | 8.02 | 8.38 | 4.60 | 5.20 | 7.66 | 12.06 | 3.02 | 3.78 | 9.84 | 14.98 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$.

Table 13: Tests for zero cross-correlation at lag $k$. Power of tests $\widetilde{t}_{xy,k}$, $\widetilde{t}_{yx,k}$, $t_{xy,k}$ and $t_{yx,k}$.

| | $x_t = r_{1t}$, $r_{1t} = \sigma_{1t}\varepsilon_t$ ARCH $y_t = r_{2t}$, $r_{2t} = \sigma_{2t}\varepsilon_t$ GARCH | | | | $x_t = h_{1t}\varepsilon_t$ $y_t = x_t + x_{t-1} + x_{t-2} + h_{1t}\eta_t$ | | | | $x_t = h_{1t}\varepsilon_t$ $y_t = m_{1t} + x_t + x_{t-1} + x_{t-2} + h_{1t}\eta_t$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ |
| 0 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| 1 | 4.46 | 4.48 | 9.06 | 9.74 | 4.98 | 100 | 6.74 | 100 | 4.84 | 100 | 6.42 | 100 |
| 2 | 4.92 | 4.82 | 5.50 | 9.58 | 4.92 | 100 | 6.94 | 100 | 5.00 | 100 | 6.80 | 100 |
| 3 | 4.94 | 4.82 | 4.46 | 8.86 | 4.90 | 5.02 | 6.60 | 6.80 | 4.78 | 4.66 | 6.70 | 6.30 |
| 4 | 4.88 | 4.66 | 4.72 | 8.40 | 4.52 | 5.34 | 6.48 | 6.98 | 4.80 | 4.86 | 6.18 | 6.76 |
| 5 | 5.10 | 4.86 | 4.56 | 7.66 | 4.74 | 4.88 | 6.24 | 6.32 | 4.78 | 5.10 | 6.52 | 6.60 |
| 6 | 4.66 | 4.76 | 4.36 | 7.44 | 4.58 | 4.88 | 5.68 | 6.54 | 4.64 | 5.16 | 6.00 | 6.38 |
| 7 | 4.62 | 4.76 | 4.34 | 7.08 | 5.08 | 5.12 | 6.58 | 6.44 | 5.18 | 4.90 | 6.46 | 6.28 |
| 8 | 4.74 | 4.32 | 4.12 | 6.20 | 4.70 | 5.16 | 6.22 | 6.36 | 4.64 | 4.98 | 6.22 | 6.56 |
| 9 | 5.00 | 4.82 | 4.38 | 6.72 | 4.98 | 5.10 | 6.58 | 6.44 | 5.14 | 4.96 | 6.48 | 6.08 |
| 10 | 5.20 | 5.04 | 4.28 | 6.28 | 4.56 | 4.84 | 5.98 | 6.34 | 4.74 | 4.98 | 5.98 | 6.26 |
| 11 | 4.64 | 4.70 | 4.06 | 5.76 | 4.66 | 4.98 | 6.06 | 5.96 | 4.60 | 5.14 | 6.02 | 6.16 |
| 12 | 4.72 | 4.72 | 3.64 | 5.34 | 5.10 | 4.72 | 7.00 | 5.64 | 5.24 | 4.66 | 6.84 | 5.46 |
| 13 | 5.28 | 5.02 | 4.42 | 5.50 | 5.02 | 4.70 | 6.28 | 5.68 | 5.08 | 4.70 | 6.22 | 5.52 |
| 14 | 5.42 | 5.36 | 4.56 | 5.60 | 5.10 | 5.40 | 6.34 | 6.00 | 5.38 | 4.86 | 6.68 | 5.64 |
| 15 | 4.80 | 4.82 | 3.86 | 4.88 | 5.28 | 5.10 | 6.58 | 5.90 | 5.30 | 4.98 | 6.40 | 5.86 |
| 16 | 5.08 | 5.48 | 4.34 | 5.78 | 4.82 | 5.02 | 5.72 | 5.90 | 5.12 | 4.94 | 6.10 | 5.54 |
| 17 | 4.84 | 4.58 | 3.88 | 4.60 | 5.18 | 5.18 | 6.12 | 6.06 | 5.06 | 5.20 | 6.38 | 5.86 |
| 18 | 5.04 | 4.86 | 4.18 | 4.74 | 5.02 | 4.50 | 6.00 | 5.20 | 5.12 | 4.48 | 6.14 | 4.94 |
| 19 | 4.96 | 4.92 | 3.84 | 4.66 | 5.12 | 4.74 | 6.20 | 5.44 | 5.28 | 4.86 | 6.04 | 5.50 |
| 20 | 5.50 | 5.26 | 4.40 | 4.84 | 4.58 | 4.82 | 5.50 | 5.52 | 4.50 | 4.70 | 5.62 | 5.28 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$, $r_{1t} \sim$ ARCH(1), $\alpha = 0.2$, $r_{2t} \sim$ GARCH(1,1), $\alpha = 0.2$, $\beta = 0.7$, $m_{1t} = I(t/n > 0.5)$, $h_{1t} = 1 + I(t/n > 0.5)$.

Table 14: Tests for zero cross-correlation at lags $0, 1, ..., m$. Power of tests $\widetilde{Q}_{xy,m}$, $\widetilde{Q}_{yx,m}$, $HB_{xy,m}$, $HB_{yx,m}$.

| | $x_t = r_{1t}$, $r_{1t} = \sigma_{1t}\varepsilon_t$ ARCH $y_t = r_{2t}$, $r_{2t} = \sigma_{2t}\varepsilon_t$ GARCH | | | | $x_t = h_{1t}\varepsilon_t$ $y_t = x_t + x_{t-1} + x_{t-2} + h_{1t}\eta_t$ | | | | $x_t = h_{1t}\varepsilon_t$ $y_t = m_{1t} + x_t + x_{t-1} + x_{t-2} + h_{1t}\eta_t$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ |
| 0 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| 1 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| 2 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| 3 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 |
| 4 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 100 | 99.94 | 100 | 100 | 100 |
| 5 | 100 | 100 | 100 | 100 | 99.98 | 100 | 99.98 | 100 | 99.90 | 100 | 99.98 | 100 |
| 6 | 100 | 100 | 100 | 100 | 99.92 | 100 | 99.96 | 100 | 99.88 | 100 | 99.96 | 100 |
| 7 | 100 | 100 | 100 | 100 | 99.90 | 100 | 99.94 | 100 | 99.82 | 100 | 99.94 | 100 |
| 8 | 100 | 100 | 100 | 100 | 99.86 | 100 | 99.94 | 100 | 99.84 | 100 | 99.94 | 100 |
| 9 | 100 | 100 | 100 | 100 | 99.74 | 100 | 99.94 | 100 | 99.62 | 100 | 99.94 | 100 |
| 10 | 100 | 100 | 100 | 100 | 99.68 | 100 | 99.94 | 100 | 99.50 | 100 | 99.94 | 100 |
| 11 | 100 | 100 | 100 | 100 | 99.64 | 100 | 99.94 | 100 | 99.38 | 100 | 99.94 | 100 |
| 12 | 100 | 100 | 100 | 100 | 99.60 | 100 | 99.94 | 100 | 99.20 | 100 | 99.94 | 100 |
| 13 | 100 | 100 | 100 | 100 | 99.52 | 100 | 99.90 | 100 | 99.22 | 100 | 99.90 | 100 |
| 14 | 100 | 100 | 100 | 100 | 99.44 | 100 | 99.92 | 100 | 98.98 | 100 | 99.90 | 100 |
| 15 | 100 | 100 | 100 | 100 | 99.32 | 100 | 99.92 | 100 | 98.68 | 100 | 99.90 | 100 |
| 16 | 100 | 100 | 100 | 100 | 99.20 | 100 | 99.92 | 100 | 98.64 | 100 | 99.90 | 100 |
| 17 | 100 | 100 | 100 | 100 | 99.10 | 100 | 99.92 | 100 | 98.44 | 100 | 99.86 | 100 |
| 18 | 100 | 100 | 100 | 100 | 98.92 | 100 | 99.88 | 100 | 98.08 | 100 | 99.84 | 100 |
| 19 | 100 | 100 | 100 | 100 | 98.62 | 100 | 99.88 | 100 | 97.82 | 100 | 99.80 | 100 |
| 20 | 100 | 100 | 100 | 100 | 98.28 | 100 | 99.88 | 100 | 97.30 | 100 | 99.78 | 100 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$, $r_{1t} \sim$ ARCH(1), $\alpha = 0.2$, $r_{2t} \sim$ GARCH(1,1), $\alpha = 0.2$, $\beta = 0.7$, $m_{1t} = I(t/n > 0.5)$, $h_{1t} = 1 + I(t/n > 0.5)$.

Table 15: Tests for zero cross-correlation at lag $k$. Spurious power of tests $\widetilde{t}_{xy,k}$, $\widetilde{t}_{yx,k}$, $t_{xy,k}$ and $t_{yx,k}$.

| | $x_t = m_{1t} + \varepsilon_t$, $\varepsilon_t$ iid $y_t = m_{1t} + \eta_t$, $\eta_t$ iid | | | | $x_t = m_{1t} + \varepsilon_t$, $\varepsilon_t$ iid $y_t = m_{4t} + \eta_t$, $\eta_t$ iid | | | | $x_t = 0.7x_{t-1} + \varepsilon_t$, $\varepsilon_t$ iid $y_t = 0.7y_{t-1} + \eta_t$, $\eta_t$ iid | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ | $\widetilde{t}_{xy,k}$ | $\widetilde{t}_{yx,k}$ | $t_{xy,k}$ | $t_{yx,k}$ |
| 0 | 94.94 | 94.94 | 94.40 | 94.40 | 43.16 | 43.16 | 43.18 | 43.18 | 24.98 | 24.98 | 24.80 | 24.80 |
| 1 | 94.56 | 94.20 | 94.06 | 93.92 | 43.20 | 42.48 | 42.88 | 41.80 | 25.44 | 24.94 | 25.12 | 24.68 |
| 2 | 93.34 | 93.76 | 92.60 | 93.22 | 42.76 | 41.14 | 42.26 | 40.66 | 25.56 | 25.56 | 25.22 | 25.36 |
| 3 | 94.04 | 93.86 | 93.36 | 93.06 | 43.82 | 40.52 | 43.08 | 39.78 | 25.08 | 25.50 | 24.48 | 25.34 |
| 4 | 93.28 | 93.84 | 92.72 | 93.16 | 43.46 | 39.78 | 42.94 | 38.74 | 24.32 | 25.48 | 23.98 | 24.84 |
| 5 | 92.70 | 93.36 | 91.94 | 92.18 | 44.58 | 39.32 | 43.92 | 38.28 | 25.12 | 24.28 | 24.64 | 23.80 |
| 6 | 92.30 | 93.36 | 91.60 | 92.62 | 45.30 | 36.52 | 44.24 | 35.54 | 25.06 | 24.50 | 24.30 | 23.78 |
| 7 | 91.78 | 92.72 | 90.84 | 91.86 | 44.68 | 37.28 | 42.92 | 35.96 | 24.76 | 25.22 | 24.24 | 24.62 |
| 8 | 91.58 | 92.14 | 90.34 | 91.46 | 45.88 | 36.52 | 44.40 | 35.22 | 25.00 | 25.16 | 24.26 | 24.76 |
| 9 | 91.02 | 90.76 | 89.90 | 89.72 | 45.92 | 34.64 | 44.38 | 33.06 | 24.06 | 25.80 | 23.56 | 25.02 |
| 10 | 90.42 | 91.34 | 89.04 | 90.22 | 46.56 | 32.72 | 44.96 | 30.90 | 24.72 | 25.62 | 23.78 | 24.24 |
| 11 | 90.20 | 90.64 | 89.06 | 89.50 | 46.94 | 34.04 | 45.54 | 32.20 | 25.12 | 24.56 | 23.86 | 23.56 |
| 12 | 88.98 | 89.90 | 87.48 | 88.28 | 47.00 | 31.56 | 45.42 | 29.68 | 25.40 | 25.08 | 24.54 | 24.30 |
| 13 | 88.16 | 89.16 | 86.52 | 87.54 | 47.88 | 30.18 | 46.04 | 28.50 | 25.28 | 25.30 | 24.16 | 24.10 |
| 14 | 87.84 | 88.98 | 86.24 | 87.54 | 48.78 | 29.34 | 46.04 | 27.16 | 25.00 | 24.78 | 23.46 | 23.64 |
| 15 | 86.84 | 87.44 | 85.20 | 85.60 | 47.54 | 27.52 | 45.02 | 25.34 | 25.18 | 25.18 | 23.58 | 24.10 |
| 16 | 86.00 | 86.50 | 84.16 | 84.78 | 49.18 | 27.50 | 46.82 | 25.30 | 25.40 | 24.56 | 23.92 | 23.60 |
| 17 | 85.12 | 86.90 | 83.22 | 84.54 | 48.58 | 25.78 | 45.98 | 23.16 | 25.14 | 24.20 | 23.58 | 23.00 |
| 18 | 84.14 | 85.94 | 82.26 | 83.60 | 50.38 | 24.84 | 47.80 | 22.56 | 24.86 | 24.54 | 23.18 | 23.14 |
| 19 | 83.90 | 85.22 | 81.54 | 82.48 | 49.02 | 25.90 | 46.34 | 22.96 | 24.44 | 25.66 | 22.86 | 23.78 |
| 20 | 82.54 | 84.26 | 80.46 | 81.34 | 49.04 | 24.90 | 46.16 | 21.70 | 24.54 | 25.68 | 22.76 | 23.86 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$, $m_{1t} = I(t/n > 0.5)$, $m_{4t} = I(t/n > 0.25)$.

Table 16: Tests for zero cross-correlation at lags $0, 1, ..., m$. Spurious power of tests $\widetilde{Q}_{xy,m}$, $\widetilde{Q}_{yx,m}$, $HB_{xy,m}$ and $HB_{yx,m}$.

| | $x_t = m_{1t} + \varepsilon_t$, $\varepsilon_t$ iid $y_t = m_{1t} + \eta_t$, $\eta_t$ iid | | | | $x_t = m_{1t} + \varepsilon_t$, $\varepsilon_t$ iid $y_t = m_{4t} + \eta_t$, $\eta_t$ iid | | | | $x_t = 0.7x_{t-1} + \varepsilon_t$, $\varepsilon_t$ iid $y_t = 0.7y_{t-1} + \eta_t$, $\eta_t$ iid | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ | $\widetilde{Q}_{xy,m}$ | $\widetilde{Q}_{yx,m}$ | $HB_{xy,m}$ | $HB_{yx,m}$ |
| 0 | 94.94 | 94.94 | 94.40 | 94.40 | 43.16 | 43.16 | 43.18 | 43.18 | 24.98 | 24.98 | 24.80 | 24.80 |
| 1 | 99.04 | 99.04 | 99.08 | 99.02 | 56.80 | 54.76 | 58.18 | 57.44 | 20.62 | 20.60 | 30.90 | 31.08 |
| 2 | 99.70 | 99.78 | 99.72 | 99.82 | 64.60 | 61.50 | 66.62 | 65.98 | 19.16 | 19.56 | 34.56 | 34.88 |
| 3 | 99.88 | 99.92 | 99.86 | 99.90 | 70.72 | 65.28 | 73.52 | 71.30 | 18.76 | 19.50 | 37.78 | 38.26 |
| 4 | 99.92 | 99.96 | 99.92 | 99.98 | 74.82 | 67.86 | 77.58 | 75.42 | 19.50 | 20.28 | 40.88 | 40.98 |
| 5 | 99.98 | 99.98 | 100 | 100 | 78.20 | 69.54 | 81.04 | 77.90 | 20.70 | 21.34 | 43.60 | 43.52 |
| 6 | 100 | 99.98 | 100 | 100 | 80.80 | 71.00 | 83.70 | 80.10 | 21.74 | 22.42 | 46.00 | 46.34 |
| 7 | 100 | 99.98 | 100 | 100 | 82.64 | 71.62 | 85.86 | 81.60 | 23.18 | 23.82 | 48.70 | 48.56 |
| 8 | 99.98 | 99.98 | 100 | 100 | 84.22 | 72.52 | 87.48 | 82.76 | 23.82 | 24.82 | 50.98 | 51.12 |
| 9 | 99.94 | 99.96 | 100 | 100 | 85.72 | 73.62 | 88.68 | 83.58 | 24.56 | 25.26 | 53.30 | 53.14 |
| 10 | 99.72 | 99.78 | 100 | 100 | 86.50 | 74.34 | 89.54 | 84.38 | 24.96 | 25.40 | 55.28 | 55.06 |
| 11 | 99.62 | 99.66 | 100 | 100 | 87.44 | 75.10 | 90.26 | 85.14 | 24.66 | 25.66 | 57.12 | 57.16 |
| 12 | 99.44 | 99.24 | 100 | 100 | 87.88 | 74.68 | 91.36 | 85.56 | 25.06 | 25.18 | 58.64 | 59.04 |
| 13 | 98.88 | 98.70 | 100 | 100 | 88.40 | 74.68 | 92.20 | 85.92 | 25.14 | 25.32 | 60.50 | 61.12 |
| 14 | 98.46 | 98.24 | 100 | 100 | 89.10 | 74.50 | 92.84 | 85.90 | 24.86 | 24.64 | 61.76 | 62.98 |
| 15 | 97.78 | 98.04 | 100 | 100 | 89.32 | 72.62 | 93.60 | 86.06 | 24.80 | 24.70 | 63.56 | 64.08 |
| 16 | 97.22 | 97.20 | 100 | 100 | 89.84 | 72.20 | 94.12 | 86.32 | 24.44 | 24.20 | 65.14 | 65.46 |
| 17 | 96.62 | 96.28 | 100 | 100 | 90.46 | 70.62 | 94.60 | 86.58 | 24.30 | 24.78 | 66.36 | 66.76 |
| 18 | 96.04 | 95.72 | 100 | 100 | 90.58 | 69.60 | 95.22 | 86.60 | 24.94 | 24.28 | 67.56 | 67.96 |
| 19 | 95.34 | 94.68 | 100 | 100 | 90.88 | 68.78 | 95.62 | 86.78 | 24.82 | 24.50 | 68.78 | 69.38 |
| 20 | 94.80 | 94.74 | 100 | 100 | 91.06 | 68.16 | 95.92 | 86.60 | 24.30 | 25.32 | 70.10 | 70.74 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$, $m_{1t} = I(t/n > 0.5)$, $m_{4t} = I(t/n > 0.25)$.

Table 17: Tests for i.i.d. property at lag $k$. Size of tests $J_{x,|x|,k}$, $J_{x,x^2,k}$.

| | $x_t$ iid $N$(0,1) | | $x_t$ iid $t$(6) | | $x_t$ iid $\chi^2$(3) | | $x_t = \exp(2\varepsilon_t)$ $\varepsilon_t$ iid $N$(0,1) | |
|---|---|---|---|---|---|---|---|---|
| $k$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ |
| 1 | 4.54 | 4.30 | 4.74 | 3.98 | 5.06 | 4.70 | 3.58 | 2.64 |
| 2 | 4.86 | 4.76 | 4.82 | 4.64 | 4.54 | 4.68 | 3.28 | 2.24 |
| 3 | 4.50 | 4.14 | 4.46 | 4.32 | 4.90 | 4.80 | 3.58 | 2.60 |
| 4 | 4.78 | 4.94 | 4.82 | 4.68 | 4.60 | 4.48 | 3.58 | 2.62 |
| 5 | 4.46 | 4.24 | 4.20 | 4.24 | 4.42 | 4.50 | 3.82 | 2.86 |
| 6 | 4.50 | 4.46 | 4.12 | 4.44 | 4.60 | 4.68 | 3.46 | 2.56 |
| 7 | 4.48 | 4.12 | 4.28 | 3.94 | 4.34 | 4.30 | 4.00 | 2.92 |
| 8 | 4.32 | 4.16 | 4.20 | 3.94 | 4.24 | 4.46 | 3.62 | 2.46 |
| 9 | 4.60 | 4.48 | 4.40 | 4.46 | 5.00 | 4.74 | 3.40 | 2.54 |
| 10 | 4.60 | 4.36 | 4.74 | 4.44 | 4.58 | 4.76 | 3.64 | 2.48 |
| 11 | 5.30 | 4.82 | 5.02 | 4.46 | 4.90 | 5.00 | 3.50 | 2.38 |
| 12 | 4.50 | 4.14 | 4.46 | 4.18 | 4.94 | 4.72 | 3.48 | 2.48 |
| 13 | 5.04 | 4.66 | 4.56 | 3.98 | 5.40 | 4.78 | 3.26 | 2.26 |
| 14 | 4.64 | 4.32 | 4.30 | 4.16 | 4.68 | 4.80 | 3.34 | 2.22 |
| 15 | 4.64 | 4.22 | 4.58 | 4.14 | 4.70 | 4.76 | 4.04 | 2.92 |
| 16 | 5.04 | 4.68 | 4.96 | 4.36 | 5.02 | 5.12 | 4.06 | 2.90 |
| 17 | 5.00 | 4.56 | 4.58 | 3.92 | 4.52 | 4.24 | 3.54 | 2.64 |
| 18 | 4.84 | 4.50 | 4.70 | 4.30 | 4.76 | 5.20 | 3.50 | 2.52 |
| 19 | 5.12 | 4.72 | 4.74 | 4.60 | 4.70 | 4.88 | 3.94 | 3.04 |
| 20 | 4.86 | 4.50 | 4.70 | 4.50 | 5.08 | 4.92 | 3.70 | 2.56 |
| 21 | 5.02 | 4.14 | 4.70 | 4.38 | 4.80 | 4.60 | 3.52 | 2.64 |
| 22 | 5.08 | 4.78 | 4.80 | 4.46 | 5.22 | 5.08 | 3.46 | 2.60 |
| 23 | 4.86 | 5.08 | 4.72 | 4.88 | 5.02 | 4.68 | 3.40 | 2.42 |
| 24 | 5.04 | 4.88 | 5.08 | 4.78 | 4.74 | 4.60 | 3.68 | 2.74 |
| 25 | 5.18 | 5.22 | 5.30 | 5.06 | 4.56 | 4.36 | 3.60 | 2.62 |
| 26 | 4.92 | 4.94 | 4.76 | 4.16 | 5.26 | 5.56 | 3.68 | 2.46 |
| 27 | 4.68 | 4.20 | 4.38 | 4.54 | 4.66 | 4.80 | 3.32 | 2.62 |
| 28 | 4.58 | 4.84 | 4.64 | 4.32 | 4.84 | 4.72 | 3.26 | 2.26 |
| 29 | 4.96 | 4.78 | 4.90 | 4.92 | 5.00 | 4.92 | 3.60 | 2.60 |
| 30 | 4.60 | 4.68 | 4.46 | 4.32 | 4.34 | 4.34 | 3.64 | 2.70 |
| 31 | 4.86 | 4.98 | 4.90 | 4.84 | 4.60 | 4.54 | 3.48 | 2.40 |
| 32 | 4.68 | 4.48 | 4.60 | 4.32 | 5.50 | 5.28 | 3.16 | 2.48 |
| 33 | 4.40 | 4.12 | 4.36 | 4.16 | 4.78 | 4.48 | 3.30 | 2.28 |
| 34 | 5.06 | 4.76 | 4.56 | 4.54 | 5.54 | 5.58 | 3.30 | 2.34 |
| 35 | 4.92 | 4.54 | 4.80 | 4.26 | 4.64 | 4.90 | 3.22 | 2.32 |
| 36 | 4.86 | 4.50 | 4.48 | 3.98 | 4.76 | 5.18 | 3.50 | 2.60 |
| 37 | 5.02 | 4.86 | 4.68 | 4.46 | 5.32 | 4.88 | 3.40 | 2.44 |
| 38 | 4.84 | 4.58 | 4.62 | 4.68 | 4.90 | 4.64 | 3.28 | 2.66 |
| 39 | 4.86 | 4.78 | 4.92 | 4.84 | 4.82 | 5.06 | 3.94 | 2.92 |
| 40 | 5.10 | 4.80 | 4.98 | 4.66 | 4.56 | 4.44 | 3.46 | 2.54 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$.

Table 18: Tests for i.i.d. property at lags $1, ..., m$. Size of tests $C_{x,|x|,m}$, $C_{x,x^2,m}$.

| | $x_t$ iid $N(0,1)$ | | $x_t$ iid $t(6)$ | | $x_t$ iid $\chi^2(3)$ | | $x_t = \exp(2\varepsilon_t)$ $\varepsilon_t$ iid $N(0,1)$ | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ |
| 1 | 4.54 | 4.30 | 4.74 | 3.98 | 5.06 | 4.70 | 3.58 | 2.64 |
| 2 | 4.88 | 4.48 | 4.66 | 4.52 | 4.96 | 5.30 | 4.72 | 3.42 |
| 3 | 4.66 | 4.66 | 4.76 | 4.90 | 5.30 | 5.98 | 6.18 | 4.52 |
| 4 | 4.84 | 4.76 | 4.56 | 5.38 | 5.22 | 6.44 | 6.82 | 5.08 |
| 5 | 5.02 | 4.68 | 4.50 | 5.56 | 5.42 | 6.74 | 7.76 | 5.96 |
| 6 | 4.78 | 4.84 | 4.50 | 5.66 | 5.82 | 7.10 | 8.18 | 6.12 |
| 7 | 4.88 | 4.52 | 4.50 | 5.64 | 5.66 | 6.96 | 8.60 | 6.44 |
| 8 | 4.94 | 4.66 | 4.56 | 5.78 | 5.62 | 7.12 | 8.86 | 6.86 |
| 9 | 4.68 | 4.78 | 4.46 | 5.70 | 5.68 | 7.16 | 9.08 | 7.08 |
| 10 | 5.04 | 4.90 | 4.76 | 5.70 | 5.86 | 7.46 | 9.10 | 7.06 |
| 11 | 5.40 | 4.92 | 4.86 | 5.48 | 5.56 | 7.46 | 9.40 | 7.34 |
| 12 | 5.54 | 4.90 | 4.94 | 5.44 | 5.46 | 7.38 | 9.74 | 7.52 |
| 13 | 5.62 | 4.94 | 4.98 | 5.32 | 5.52 | 7.46 | 9.70 | 7.48 |
| 14 | 5.36 | 4.78 | 4.80 | 5.24 | 5.46 | 7.14 | 9.64 | 7.50 |
| 15 | 5.44 | 4.72 | 4.94 | 5.22 | 5.54 | 6.94 | 9.82 | 7.88 |
| 16 | 5.48 | 4.78 | 4.74 | 5.10 | 5.38 | 7.16 | 9.96 | 7.80 |
| 17 | 5.54 | 4.62 | 4.86 | 5.08 | 5.68 | 6.92 | 9.96 | 7.84 |
| 18 | 5.50 | 4.82 | 4.76 | 4.82 | 5.76 | 6.76 | 9.84 | 7.84 |
| 19 | 5.74 | 4.70 | 5.00 | 4.88 | 5.76 | 6.82 | 9.88 | 7.76 |
| 20 | 5.70 | 4.86 | 5.04 | 4.92 | 5.56 | 7.02 | 9.78 | 7.82 |
| 21 | 5.86 | 4.86 | 5.06 | 4.82 | 5.84 | 6.74 | 9.70 | 7.96 |
| 22 | 5.30 | 4.80 | 4.86 | 4.80 | 5.70 | 6.76 | 9.50 | 7.98 |
| 23 | 5.58 | 4.98 | 4.78 | 4.78 | 6.06 | 6.86 | 9.68 | 7.84 |
| 24 | 5.82 | 5.18 | 5.12 | 4.96 | 6.08 | 7.02 | 9.76 | 7.90 |
| 25 | 5.88 | 5.24 | 5.08 | 4.92 | 6.28 | 6.82 | 9.76 | 7.72 |
| 26 | 5.82 | 5.54 | 5.16 | 4.86 | 6.32 | 6.84 | 9.46 | 7.60 |
| 27 | 6.06 | 5.26 | 5.30 | 4.86 | 6.20 | 6.80 | 9.46 | 7.62 |
| 28 | 6.10 | 5.28 | 5.48 | 5.16 | 6.34 | 7.00 | 9.32 | 7.32 |
| 29 | 6.28 | 5.50 | 5.66 | 5.28 | 6.60 | 6.96 | 9.30 | 7.20 |
| 30 | 6.02 | 5.38 | 5.58 | 5.20 | 6.62 | 6.88 | 9.26 | 7.14 |
| 31 | 6.36 | 5.56 | 5.84 | 5.32 | 6.74 | 6.62 | 9.28 | 7.08 |
| 32 | 6.64 | 5.80 | 5.90 | 5.48 | 6.60 | 6.60 | 9.20 | 7.14 |
| 33 | 6.28 | 5.72 | 5.78 | 5.20 | 6.60 | 6.72 | 9.10 | 7.10 |
| 34 | 6.42 | 5.88 | 5.72 | 5.28 | 6.64 | 6.62 | 8.88 | 6.98 |
| 35 | 6.44 | 5.72 | 5.72 | 5.08 | 6.92 | 6.56 | 8.84 | 7.18 |
| 36 | 6.34 | 5.62 | 5.90 | 5.22 | 6.64 | 6.66 | 8.50 | 7.08 |
| 37 | 6.58 | 5.64 | 5.86 | 5.04 | 6.70 | 6.54 | 8.58 | 6.88 |
| 38 | 6.74 | 5.66 | 6.02 | 4.94 | 6.90 | 6.56 | 8.70 | 6.94 |
| 39 | 6.92 | 6.04 | 6.12 | 5.12 | 7.16 | 6.62 | 8.72 | 6.76 |
| 40 | 6.98 | 6.00 | 6.28 | 5.10 | 7.08 | 6.64 | 8.60 | 6.82 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$.

Table 19: Tests for i.i.d. property at lag $k$. Power of tests $J_{x,|x|,k}$, $J_{x,x^2,k}$.

| | $x_t$ AR(1) $\phi = 0.2$ | | $x_t$ ARCH(1) $\alpha = 0.2$ | | $x_t$ GARCH(1,1) $\alpha = 0.2$, $\beta = 0.7$ | | $x_t = \varepsilon_t \varepsilon_{t-1}$ $\varepsilon_t$ iid $N$(0,1) | |
|---|---|---|---|---|---|---|---|---|
| $k$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ |
| 1 | 86.50 | 86.52 | 63.42 | 68.64 | 80.12 | 79.48 | 99.98 | 89.64 |
| 2 | 8.58 | 8.20 | 8.68 | 10.36 | 71.84 | 70.50 | 7.44 | 4.50 |
| 3 | 5.10 | 4.84 | 5.54 | 5.22 | 61.84 | 60.64 | 6.34 | 4.72 |
| 4 | 5.54 | 5.46 | 5.12 | 4.98 | 53.90 | 52.44 | 6.06 | 4.84 |
| 5 | 5.28 | 5.46 | 4.80 | 4.54 | 45.84 | 44.00 | 6.04 | 5.12 |
| 6 | 4.96 | 4.92 | 4.58 | 4.46 | 38.30 | 36.94 | 6.24 | 4.84 |
| 7 | 5.08 | 4.80 | 4.62 | 4.38 | 33.06 | 31.48 | 6.18 | 4.42 |
| 8 | 5.32 | 4.60 | 5.14 | 4.00 | 27.66 | 26.02 | 6.94 | 5.12 |
| 9 | 5.22 | 5.12 | 5.04 | 4.72 | 24.14 | 22.22 | 7.18 | 5.16 |
| 10 | 4.80 | 5.10 | 4.94 | 4.90 | 20.76 | 19.34 | 6.32 | 4.98 |
| 11 | 5.74 | 5.48 | 5.30 | 4.80 | 18.30 | 16.74 | 6.06 | 4.54 |
| 12 | 5.30 | 4.76 | 5.26 | 4.72 | 15.48 | 13.64 | 6.40 | 4.66 |
| 13 | 5.64 | 5.28 | 5.22 | 4.78 | 14.58 | 12.92 | 6.48 | 4.70 |
| 14 | 5.42 | 4.92 | 4.84 | 3.86 | 12.16 | 10.40 | 6.38 | 4.94 |
| 15 | 4.94 | 5.32 | 5.04 | 4.68 | 11.36 | 9.86 | 6.28 | 4.90 |
| 16 | 5.66 | 5.28 | 5.66 | 4.94 | 10.74 | 9.18 | 6.56 | 5.22 |
| 17 | 5.62 | 4.86 | 5.10 | 4.40 | 10.30 | 8.12 | 6.30 | 4.60 |
| 18 | 5.16 | 4.74 | 5.14 | 4.66 | 9.34 | 7.92 | 6.02 | 4.44 |
| 19 | 5.60 | 5.54 | 5.50 | 4.72 | 9.42 | 7.34 | 6.20 | 4.74 |
| 20 | 5.14 | 5.40 | 5.28 | 4.66 | 8.06 | 6.56 | 6.44 | 4.80 |
| | $x_t = m_{1t} + \varepsilon_t$ $\varepsilon_t$ iid $N$(0,1) | | $x_t = h_{1t}\varepsilon_t$ $\varepsilon_t$ iid $N$(0,1) | | $x_t = h_{1t} y_t$ $y_t$ AR(1), $\phi = 0.2$ | | $x_t = m_{1t} + h_{1t}\varepsilon_t$ $\varepsilon_t$ iid $N$(0,1) | |
| $k$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ | $J_{x,\|x\|,k}$ | $J_{x,x^2,k}$ |
| 1 | 86.06 | 85.64 | 59.22 | 41.00 | 95.58 | 91.86 | 61.22 | 51.08 |
| 2 | 85.70 | 85.40 | 58.84 | 41.74 | 61.34 | 45.30 | 61.00 | 51.94 |
| 3 | 85.32 | 85.06 | 59.44 | 41.02 | 59.00 | 40.98 | 58.68 | 49.18 |
| 4 | 85.26 | 85.04 | 58.88 | 41.72 | 58.20 | 42.20 | 60.12 | 51.20 |
| 5 | 83.98 | 83.80 | 57.74 | 40.08 | 58.02 | 41.04 | 58.80 | 49.32 |
| 6 | 83.36 | 83.12 | 55.76 | 39.34 | 56.96 | 40.30 | 57.42 | 48.56 |
| 7 | 82.64 | 82.26 | 55.78 | 39.82 | 55.28 | 39.14 | 56.38 | 46.40 |
| 8 | 82.76 | 82.32 | 55.64 | 38.64 | 56.96 | 39.88 | 55.22 | 46.22 |
| 9 | 80.96 | 80.58 | 55.24 | 37.58 | 54.54 | 38.74 | 53.74 | 44.80 |
| 10 | 79.80 | 79.96 | 53.76 | 36.80 | 53.82 | 37.84 | 52.32 | 44.00 |
| 11 | 79.54 | 79.10 | 52.28 | 36.22 | 53.24 | 36.66 | 51.00 | 43.42 |
| 12 | 79.42 | 79.12 | 51.38 | 35.86 | 51.72 | 36.34 | 49.86 | 41.44 |
| 13 | 77.94 | 78.02 | 52.36 | 36.12 | 52.58 | 36.98 | 49.40 | 41.94 |
| 14 | 76.78 | 76.54 | 50.58 | 35.10 | 50.56 | 36.26 | 48.90 | 40.48 |
| 15 | 76.44 | 76.30 | 50.92 | 35.58 | 51.16 | 35.52 | 49.12 | 41.08 |
| 16 | 75.12 | 74.72 | 49.00 | 34.44 | 49.86 | 35.38 | 47.58 | 40.06 |
| 17 | 74.92 | 74.30 | 48.08 | 33.82 | 47.98 | 34.22 | 45.24 | 38.06 |
| 18 | 72.94 | 72.70 | 47.36 | 33.28 | 47.98 | 33.84 | 45.08 | 37.40 |
| 19 | 72.88 | 72.88 | 46.98 | 32.70 | 47.68 | 33.28 | 44.50 | 36.58 |
| 20 | 71.44 | 71.00 | 46.76 | 33.06 | 46.96 | 34.10 | 43.48 | 35.70 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $m_{1t} = I(t/n > 0.5)$, $h_{1t} = 1 + I(t/n > 0.5)$.

Table 20: Tests for i.i.d. property at lag $k$. Power of tests $C_{x,|x|,m}$, $C_{x,x^2,m}$.

| | $x_t$ AR(1) $\phi = 0.2$ | | $x_t$ ARCH(1) $\alpha = 0.2$ | | $x_t$ GARCH(1,1) $\alpha = 0.2$, $\beta = 0.7$ | | $x_t = \varepsilon_t \varepsilon_{t-1}$ $\varepsilon_t$ iid $N(0,1)$ | |
|---|---|---|---|---|---|---|---|---|
| $m$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ |
| 1 | 86.50 | 86.52 | 63.42 | 68.64 | 80.12 | 79.48 | 99.98 | 89.64 |
| 2 | 78.48 | 78.64 | 54.30 | 61.02 | 86.70 | 87.96 | 99.98 | 83.66 |
| 3 | 72.32 | 72.16 | 48.50 | 56.10 | 88.46 | 90.08 | 99.96 | 79.46 |
| 4 | 66.48 | 66.14 | 43.88 | 51.98 | 88.96 | 90.86 | 99.98 | 76.32 |
| 5 | 62.18 | 61.34 | 41.34 | 48.42 | 89.46 | 91.28 | 99.94 | 73.82 |
| 6 | 57.90 | 57.10 | 38.54 | 46.18 | 89.24 | 91.46 | 99.88 | 71.12 |
| 7 | 54.76 | 54.70 | 36.22 | 43.46 | 88.68 | 91.04 | 99.74 | 67.80 |
| 8 | 52.04 | 51.68 | 34.28 | 41.44 | 88.24 | 90.52 | 99.62 | 66.02 |
| 9 | 50.42 | 49.94 | 33.04 | 40.06 | 87.68 | 90.14 | 99.42 | 64.50 |
| 10 | 48.22 | 47.92 | 31.96 | 38.70 | 87.04 | 89.42 | 99.16 | 62.50 |
| 11 | 46.30 | 45.74 | 31.12 | 37.88 | 86.42 | 88.96 | 98.94 | 60.98 |
| 12 | 44.64 | 43.92 | 30.48 | 36.90 | 85.64 | 88.74 | 98.68 | 59.48 |
| 13 | 43.58 | 42.34 | 29.34 | 36.22 | 84.90 | 87.92 | 98.62 | 58.06 |
| 14 | 42.20 | 41.14 | 28.72 | 35.48 | 84.56 | 87.22 | 98.22 | 57.02 |
| 15 | 41.64 | 40.24 | 27.62 | 34.40 | 84.32 | 86.94 | 98.02 | 55.68 |
| 16 | 40.50 | 38.90 | 26.86 | 33.76 | 83.80 | 86.26 | 97.88 | 54.50 |
| 17 | 39.78 | 38.12 | 26.14 | 33.28 | 83.12 | 85.64 | 97.68 | 53.42 |
| 18 | 39.06 | 37.74 | 25.52 | 32.04 | 82.82 | 84.92 | 97.10 | 52.16 |
| 19 | 38.20 | 36.58 | 25.46 | 31.68 | 82.04 | 84.72 | 96.74 | 50.90 |
| 20 | 37.58 | 36.20 | 25.30 | 31.00 | 81.84 | 84.12 | 96.20 | 50.30 |
| | $x_t = m_{1t} + \varepsilon_t$ $\varepsilon_t$ iid $N(0,1)$ | | $x_t = h_{1t}\varepsilon_t$ $\varepsilon_t$ iid $N(0,1)$ | | $x_t = h_{1t} y_t$ $y_t$ AR(1), $\phi = 0.2$ | | $x_t = m_{1t} + h_{1t}\varepsilon_t$ $\varepsilon_t$ iid $N(0,1)$ | |
| $m$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ | $C_{x,\|x\|,m}$ | $C_{x,x^2,m}$ |
| 1 | 86.06 | 85.64 | 59.22 | 41.00 | 95.58 | 91.86 | 61.22 | 51.08 |
| 2 | 95.58 | 95.26 | 77.86 | 58.76 | 96.90 | 93.28 | 79.38 | 70.78 |
| 3 | 98.36 | 98.14 | 86.76 | 70.20 | 97.84 | 95.00 | 88.30 | 81.36 |
| 4 | 99.10 | 98.82 | 91.16 | 77.86 | 98.44 | 96.18 | 93.36 | 87.88 |
| 5 | 99.44 | 99.28 | 94.56 | 83.82 | 99.02 | 96.90 | 96.14 | 91.94 |
| 6 | 99.64 | 99.58 | 96.02 | 87.32 | 99.34 | 97.44 | 97.32 | 94.08 |
| 7 | 99.76 | 99.76 | 97.34 | 90.52 | 99.48 | 98.04 | 98.00 | 96.02 |
| 8 | 99.78 | 99.78 | 97.62 | 92.40 | 99.52 | 98.44 | 98.56 | 96.86 |
| 9 | 99.86 | 99.84 | 98.22 | 93.64 | 99.62 | 98.44 | 98.92 | 97.64 |
| 10 | 99.86 | 99.86 | 98.78 | 94.78 | 99.66 | 98.88 | 99.16 | 98.22 |
| 11 | 99.92 | 99.88 | 99.02 | 95.74 | 99.64 | 98.82 | 99.36 | 98.62 |
| 12 | 99.94 | 99.90 | 99.18 | 96.24 | 99.74 | 99.04 | 99.40 | 98.92 |
| 13 | 99.94 | 99.90 | 99.38 | 96.70 | 99.76 | 99.12 | 99.50 | 99.14 |
| 14 | 99.94 | 99.92 | 99.48 | 97.34 | 99.80 | 99.34 | 99.70 | 99.28 |
| 15 | 99.96 | 99.94 | 99.52 | 97.78 | 99.84 | 99.54 | 99.72 | 99.44 |
| 16 | 99.94 | 99.94 | 99.58 | 98.08 | 99.84 | 99.66 | 99.76 | 99.52 |
| 17 | 99.94 | 99.92 | 99.56 | 98.30 | 99.82 | 99.56 | 99.78 | 99.62 |
| 18 | 99.94 | 99.92 | 99.62 | 98.42 | 99.86 | 99.66 | 99.78 | 99.62 |
| 19 | 99.92 | 99.94 | 99.68 | 98.60 | 99.82 | 99.58 | 99.76 | 99.66 |
| 20 | 99.94 | 99.92 | 99.68 | 98.66 | 99.84 | 99.62 | 99.82 | 99.76 |

Rejection frequencies (in %) at the 5% significance level, $n = 300$. In models: $m_{1t} = I(t/n > 0.5)$, $h_{1t} = 1 + I(t/n > 0.5)$.

Table 21: Tests for zero serial correlation at lags $1, ..., m$. Size of tests $Q_m$, $\widetilde{Q}_m$ with different thresholds $\lambda$.

| | $\alpha = 10\%$ | | | | $\alpha = 5\%$ | | | | $\alpha = 1\%$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\widetilde{Q}_m = \widetilde{Q}_m(\lambda)$ | | | | $\widetilde{Q}_m = \widetilde{Q}_m(\lambda)$ | | | | $\widetilde{Q}_m = \widetilde{Q}_m(\lambda)$ | | |
| $m$ | $Q_m$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ | $Q_m$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ | $Q_m$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ |
| 1 | 9.74 | 9.74 | 9.74 | 9.74 | 4.60 | 4.60 | 4.60 | 4.60 | 0.80 | 0.80 | 0.80 | 0.80 |
| 2 | 9.92 | 9.68 | 9.72 | 9.76 | 4.84 | 4.84 | 4.78 | 4.76 | 0.76 | 0.70 | 0.66 | 0.68 |
| 3 | 9.82 | 9.64 | 9.72 | 9.78 | 4.28 | 4.52 | 4.66 | 4.60 | 0.88 | 0.82 | 0.80 | 0.86 |
| 4 | 9.80 | 9.44 | 9.56 | 9.36 | 4.38 | 4.52 | 4.58 | 4.68 | 0.82 | 0.86 | 0.80 | 0.94 |
| 5 | 9.28 | 9.46 | 9.44 | 9.50 | 4.58 | 4.26 | 4.34 | 4.40 | 1.00 | 1.06 | 1.02 | 1.06 |
| 6 | 9.32 | 9.18 | 9.52 | 9.50 | 4.36 | 4.34 | 4.40 | 4.54 | 0.90 | 0.94 | 1.04 | 1.16 |
| 7 | 8.76 | 8.82 | 9.16 | 9.48 | 4.12 | 4.42 | 4.52 | 4.64 | 0.76 | 0.86 | 0.82 | 1.08 |
| 8 | 8.10 | 8.82 | 8.94 | 9.26 | 4.12 | 4.20 | 4.26 | 4.44 | 0.72 | 0.72 | 0.76 | 0.86 |
| 9 | 8.42 | 8.60 | 8.82 | 9.06 | 3.72 | 4.22 | 4.26 | 4.74 | 0.72 | 0.84 | 0.84 | 0.96 |
| 10 | 8.28 | 8.68 | 8.86 | 9.24 | 3.80 | 4.24 | 4.28 | 4.64 | 0.82 | 0.94 | 0.90 | 1.10 |
| 11 | 8.10 | 8.90 | 8.92 | 9.08 | 3.72 | 4.10 | 4.30 | 4.66 | 0.64 | 0.90 | 0.98 | 1.10 |
| 12 | 8.08 | 9.00 | 8.90 | 9.40 | 3.88 | 4.06 | 4.28 | 4.72 | 0.66 | 0.90 | 1.02 | 1.12 |
| 13 | 7.90 | 8.72 | 9.14 | 9.48 | 3.92 | 4.14 | 4.20 | 4.66 | 0.56 | 0.86 | 0.94 | 1.04 |
| 14 | 8.04 | 8.72 | 9.24 | 9.76 | 3.88 | 4.40 | 4.60 | 4.76 | 0.50 | 0.84 | 1.00 | 1.22 |
| 15 | 8.08 | 8.76 | 9.26 | 9.54 | 3.86 | 4.40 | 4.60 | 5.04 | 0.50 | 0.76 | 0.88 | 1.14 |
| 16 | 8.22 | 9.28 | 9.56 | 9.88 | 3.48 | 4.44 | 4.66 | 5.22 | 0.44 | 0.76 | 0.90 | 1.14 |
| 17 | 7.68 | 8.74 | 9.12 | 9.84 | 3.32 | 4.40 | 4.60 | 5.02 | 0.46 | 0.80 | 0.94 | 1.22 |
| 18 | 7.18 | 8.60 | 8.68 | 9.70 | 3.18 | 4.24 | 4.64 | 5.12 | 0.42 | 0.76 | 0.88 | 1.24 |
| 19 | 7.18 | 8.46 | 8.70 | 9.70 | 3.24 | 4.42 | 4.72 | 5.26 | 0.44 | 0.94 | 0.98 | 1.30 |
| 20 | 7.24 | 8.30 | 8.68 | 9.50 | 2.90 | 4.38 | 4.50 | 5.04 | 0.38 | 0.86 | 1.00 | 1.28 |
| 21 | 7.08 | 8.52 | 8.80 | 9.66 | 2.92 | 4.40 | 4.76 | 5.08 | 0.44 | 0.80 | 0.96 | 1.26 |
| 22 | 6.82 | 8.60 | 9.02 | 9.72 | 2.94 | 4.14 | 4.68 | 5.32 | 0.36 | 0.80 | 0.92 | 1.20 |
| 23 | 7.06 | 8.58 | 9.24 | 9.98 | 2.88 | 4.64 | 4.44 | 5.18 | 0.30 | 0.98 | 0.96 | 1.24 |
| 24 | 7.00 | 8.86 | 9.42 | 10.24 | 2.92 | 4.52 | 4.76 | 5.16 | 0.28 | 0.84 | 1.02 | 1.46 |
| 25 | 6.84 | 9.08 | 9.32 | 10.28 | 2.74 | 4.66 | 4.96 | 5.64 | 0.16 | 0.86 | 1.02 | 1.40 |
| 26 | 6.52 | 9.06 | 8.98 | 10.22 | 2.54 | 4.60 | 4.74 | 5.64 | 0.18 | 0.82 | 1.04 | 1.46 |
| 27 | 6.16 | 8.96 | 9.16 | 9.96 | 2.62 | 4.64 | 4.72 | 5.58 | 0.24 | 0.86 | 1.10 | 1.36 |
| 28 | 6.12 | 8.96 | 9.22 | 9.88 | 2.28 | 4.68 | 4.76 | 5.56 | 0.20 | 0.80 | 1.12 | 1.46 |
| 29 | 5.92 | 9.32 | 9.22 | 10.36 | 2.44 | 4.74 | 4.66 | 5.74 | 0.16 | 0.84 | 1.20 | 1.52 |
| 30 | 5.68 | 9.36 | 9.88 | 10.50 | 2.40 | 4.92 | 4.62 | 5.68 | 0.20 | 0.98 | 1.10 | 1.46 |
| 31 | 5.84 | 9.56 | 9.62 | 10.68 | 2.28 | 5.02 | 4.78 | 5.68 | 0.20 | 1.10 | 1.16 | 1.52 |
| 32 | 5.74 | 9.54 | 9.56 | 10.88 | 2.08 | 5.04 | 4.62 | 5.66 | 0.16 | 1.16 | 1.16 | 1.48 |
| 33 | 5.66 | 9.86 | 9.52 | 10.72 | 1.84 | 5.04 | 4.60 | 5.70 | 0.18 | 1.14 | 1.20 | 1.52 |
| 34 | 5.60 | 10.02 | 9.72 | 11.14 | 2.00 | 5.16 | 5.00 | 5.88 | 0.16 | 1.24 | 1.16 | 1.54 |
| 35 | 5.46 | 9.90 | 9.56 | 11.10 | 1.98 | 5.24 | 4.86 | 5.88 | 0.14 | 1.20 | 1.06 | 1.56 |
| 36 | 5.08 | 10.06 | 9.62 | 10.96 | 2.08 | 5.46 | 5.06 | 5.70 | 0.20 | 1.42 | 1.20 | 1.46 |
| 37 | 5.12 | 10.22 | 9.36 | 11.02 | 2.10 | 5.60 | 5.06 | 5.74 | 0.18 | 1.42 | 1.22 | 1.58 |
| 38 | 5.24 | 10.62 | 9.52 | 10.96 | 2.00 | 5.72 | 5.20 | 6.00 | 0.10 | 1.50 | 1.22 | 1.68 |
| 39 | 4.90 | 10.70 | 9.70 | 11.00 | 1.88 | 5.82 | 5.10 | 6.22 | 0.10 | 1.44 | 1.40 | 1.56 |
| 40 | 5.20 | 11.34 | 9.94 | 10.88 | 1.82 | 6.06 | 5.04 | 6.06 | 0.08 | 1.84 | 1.30 | 1.66 |

Rejection frequencies (in %) at the 10%, 5% and 1% significance level, $n = 300$. Model: $x_t = \varepsilon_t$, $\varepsilon_t \sim$ i.i.d. $N(0,1)$.

Table 22: Tests for zero serial correlation at lags $1, ..., m$. Size of tests $Q_m$, $\widetilde{Q}_m$ with different thresholds $\lambda$.

| | $\alpha = 10\%$ | | | | $\alpha = 5\%$ | | | | $\alpha = 1\%$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\widetilde{Q}_m = \widetilde{Q}_m(\lambda)$ | | | | $\widetilde{Q}_m = \widetilde{Q}_m(\lambda)$ | | | | $\widetilde{Q}_m = \widetilde{Q}_m(\lambda)$ | | |
| $m$ | $Q_m$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ | $Q_m$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ | $Q_m$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ |
| 1 | 9.66 | 9.66 | 9.66 | 9.66 | 4.76 | 4.76 | 4.76 | 4.76 | 0.76 | 0.76 | 0.76 | 0.76 |
| 2 | 9.66 | 9.78 | 9.74 | 9.80 | 4.44 | 4.44 | 4.46 | 4.54 | 0.64 | 0.72 | 0.76 | 0.90 |
| 3 | 9.52 | 9.42 | 9.36 | 9.60 | 4.38 | 4.52 | 4.46 | 4.46 | 0.64 | 0.64 | 0.70 | 0.80 |
| 4 | 9.26 | 9.34 | 9.64 | 9.54 | 3.88 | 4.12 | 4.26 | 4.36 | 0.48 | 0.62 | 0.64 | 0.76 |
| 5 | 8.38 | 9.18 | 9.34 | 9.60 | 3.82 | 4.04 | 4.40 | 4.86 | 0.48 | 0.64 | 0.76 | 0.80 |
| 6 | 7.90 | 8.46 | 8.76 | 9.26 | 3.64 | 3.88 | 4.36 | 4.76 | 0.54 | 0.70 | 0.96 | 1.08 |
| 7 | 7.84 | 8.30 | 9.04 | 9.48 | 3.36 | 4.04 | 4.36 | 4.94 | 0.50 | 0.64 | 0.90 | 1.14 |
| 8 | 7.54 | 8.62 | 9.08 | 9.58 | 3.30 | 4.28 | 4.52 | 5.22 | 0.52 | 0.76 | 0.98 | 1.18 |
| 9 | 6.90 | 8.56 | 9.00 | 9.58 | 3.06 | 4.08 | 4.34 | 4.88 | 0.46 | 0.86 | 0.98 | 1.16 |
| 10 | 6.72 | 8.78 | 9.06 | 9.62 | 2.72 | 3.80 | 4.10 | 4.68 | 0.36 | 0.64 | 1.00 | 1.16 |
| 11 | 6.16 | 8.56 | 8.94 | 9.64 | 2.58 | 3.70 | 4.32 | 4.82 | 0.28 | 0.74 | 0.96 | 1.14 |
| 12 | 5.84 | 8.34 | 8.54 | 9.56 | 2.48 | 3.78 | 4.20 | 4.98 | 0.18 | 0.64 | 0.90 | 1.18 |
| 13 | 5.88 | 8.40 | 8.92 | 9.90 | 2.58 | 3.94 | 4.52 | 5.34 | 0.18 | 0.72 | 1.04 | 1.28 |
| 14 | 5.82 | 9.26 | 8.96 | 10.06 | 2.28 | 4.42 | 4.68 | 5.58 | 0.26 | 0.94 | 1.16 | 1.56 |
| 15 | 5.50 | 9.22 | 9.14 | 10.34 | 2.02 | 4.22 | 4.58 | 5.64 | 0.28 | 1.04 | 1.02 | 1.50 |
| 16 | 5.54 | 9.38 | 9.32 | 10.68 | 1.84 | 4.36 | 4.76 | 5.52 | 0.20 | 1.12 | 1.02 | 1.54 |
| 17 | 5.06 | 9.38 | 9.14 | 10.52 | 1.68 | 4.84 | 4.90 | 5.52 | 0.14 | 1.28 | 1.10 | 1.62 |
| 18 | 4.74 | 9.26 | 9.54 | 10.60 | 1.70 | 4.76 | 4.64 | 5.72 | 0.12 | 1.38 | 1.22 | 1.50 |
| 19 | 4.48 | 9.60 | 9.58 | 10.38 | 1.56 | 5.06 | 4.96 | 5.96 | 0.20 | 1.58 | 1.14 | 1.68 |
| 20 | 4.36 | 10.02 | 9.72 | 10.38 | 1.58 | 5.04 | 4.94 | 5.92 | 0.18 | 1.76 | 1.14 | 1.68 |
| 21 | 3.96 | 10.36 | 9.74 | 10.52 | 1.38 | 5.74 | 5.02 | 6.04 | 0.12 | 2.02 | 1.26 | 1.78 |
| 22 | 3.78 | 10.54 | 10.20 | 10.80 | 1.32 | 5.94 | 5.26 | 6.00 | 0.10 | 2.38 | 1.30 | 1.88 |
| 23 | 3.58 | 10.96 | 10.06 | 10.98 | 1.22 | 6.84 | 5.20 | 6.04 | 0.10 | 2.66 | 1.26 | 2.00 |
| 24 | 3.70 | 11.66 | 10.26 | 11.42 | 1.04 | 7.22 | 5.58 | 6.14 | 0.12 | 3.16 | 1.50 | 2.04 |
| 25 | 3.44 | 12.66 | 10.62 | 11.62 | 1.08 | 7.94 | 5.74 | 6.32 | 0.02 | 3.64 | 1.68 | 2.12 |
| 26 | 3.28 | 13.16 | 10.70 | 11.42 | 1.06 | 8.30 | 5.86 | 6.42 | 0.02 | 3.98 | 1.80 | 2.10 |
| 27 | 3.08 | 13.84 | 11.00 | 11.52 | 0.92 | 8.86 | 6.00 | 6.60 | 0.04 | 4.30 | 1.80 | 2.00 |
| 28 | 2.90 | 14.82 | 11.04 | 11.78 | 0.90 | 9.64 | 6.06 | 6.60 | 0.04 | 4.80 | 1.98 | 2.16 |
| 29 | 2.80 | 15.52 | 11.42 | 11.58 | 0.94 | 10.44 | 6.52 | 6.80 | 0.02 | 5.36 | 2.10 | 2.24 |
| 30 | 2.70 | 15.82 | 11.24 | 11.38 | 0.92 | 11.02 | 6.34 | 6.80 | 0.02 | 6.00 | 2.18 | 2.20 |
| 31 | 2.42 | 16.64 | 11.32 | 11.60 | 0.80 | 11.84 | 6.52 | 6.94 | 0.04 | 6.36 | 2.40 | 2.40 |
| 32 | 2.46 | 17.52 | 11.70 | 11.72 | 0.74 | 12.36 | 6.96 | 7.12 | 0.04 | 6.80 | 2.60 | 2.28 |
| 33 | 2.32 | 17.76 | 11.68 | 11.86 | 0.76 | 12.50 | 7.28 | 7.02 | 0.06 | 6.66 | 2.52 | 2.46 |
| 34 | 2.36 | 18.94 | 12.34 | 11.90 | 0.62 | 13.52 | 7.62 | 7.28 | 0.04 | 7.38 | 2.68 | 2.38 |
| 35 | 2.22 | 19.14 | 12.36 | 11.78 | 0.58 | 14.18 | 7.48 | 7.26 | 0.06 | 8.12 | 2.70 | 2.40 |
| 36 | 2.02 | 19.66 | 12.70 | 12.16 | 0.56 | 14.66 | 7.48 | 7.34 | 0.02 | 8.80 | 2.82 | 2.46 |
| 37 | 2.10 | 20.54 | 13.22 | 12.08 | 0.50 | 15.70 | 7.96 | 7.36 | 0.02 | 9.64 | 3.22 | 2.62 |
| 38 | 1.94 | 21.38 | 13.60 | 12.08 | 0.50 | 16.44 | 8.12 | 7.30 | 0.04 | 10.10 | 3.18 | 2.58 |
| 39 | 1.78 | 22.74 | 13.72 | 12.32 | 0.44 | 17.10 | 8.56 | 7.50 | 0.04 | 10.94 | 3.78 | 2.54 |
| 40 | 1.62 | 23.46 | 14.02 | 12.48 | 0.46 | 17.96 | 8.96 | 7.60 | 0.04 | 11.48 | 4.12 | 2.50 |

Rejection frequencies (in %) at the 10%, 5% and 1% significance level, $n = 300$. Model: $x_t = h_{2t}\varepsilon_t$, $h_{2t} = t/n$, $\varepsilon_t \sim$ i.i.d. $N(0,1)$.

Table 23: Tests for zero cross-correlation at lags 0, ..., $m$. Size of tests $Q_{xy,m}$, $\widetilde{Q}_{xy,m}$ with different thresholds $\lambda$.

| | $\alpha = 10\%$ | | | | $\alpha = 5\%$ | | | | $\alpha = 1\%$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\widetilde{Q}_{xy,m} = \widetilde{Q}_{xy,m}(\lambda)$ | | | | $\widetilde{Q}_{xy,m} = \widetilde{Q}_{xy,m}(\lambda)$ | | | | $\widetilde{Q}_{xy,m} = \widetilde{Q}_{xy,m}(\lambda)$ | | |
| $m$ | $Q_{xy,m}$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ | $Q_{xy,m}$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ | $Q_{xy,m}$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ |
| 0 | 10.06 | 10.06 | 10.06 | 10.06 | 4.92 | 4.92 | 4.92 | 4.92 | 0.84 | 0.84 | 0.84 | 0.84 |
| 1 | 9.80 | 9.96 | 9.88 | 9.90 | 4.98 | 5.14 | 5.16 | 5.22 | 1.00 | 1.08 | 1.08 | 1.08 |
| 2 | 10.24 | 10.00 | 10.00 | 10.08 | 5.38 | 5.52 | 5.44 | 5.48 | 1.18 | 1.08 | 1.10 | 1.12 |
| 3 | 10.20 | 10.22 | 10.18 | 10.26 | 5.50 | 5.44 | 5.34 | 5.40 | 0.94 | 0.94 | 0.94 | 0.94 |
| 4 | 9.86 | 9.64 | 9.62 | 9.90 | 5.08 | 5.08 | 5.12 | 5.12 | 0.72 | 0.70 | 0.76 | 0.70 |
| 5 | 9.86 | 9.92 | 10.02 | 10.38 | 4.62 | 4.96 | 4.94 | 5.06 | 0.78 | 0.76 | 0.82 | 0.88 |
| 6 | 10.04 | 10.30 | 10.22 | 10.40 | 4.88 | 5.08 | 4.86 | 5.02 | 0.78 | 0.86 | 0.90 | 0.82 |
| 7 | 10.08 | 9.98 | 10.16 | 10.18 | 4.72 | 4.96 | 4.76 | 4.72 | 0.58 | 0.84 | 0.90 | 0.90 |
| 8 | 9.44 | 9.74 | 9.98 | 10.08 | 4.48 | 4.64 | 4.52 | 4.70 | 0.54 | 0.82 | 0.86 | 0.92 |
| 9 | 9.32 | 9.52 | 9.42 | 9.38 | 4.10 | 4.44 | 4.80 | 4.78 | 0.56 | 0.78 | 0.78 | 0.78 |
| 10 | 9.12 | 9.58 | 9.34 | 9.44 | 4.06 | 4.48 | 4.38 | 4.48 | 0.64 | 0.84 | 0.90 | 0.92 |
| 11 | 8.80 | 9.08 | 9.04 | 9.24 | 3.76 | 4.20 | 4.40 | 4.32 | 0.54 | 0.78 | 0.90 | 0.92 |
| 12 | 8.62 | 9.14 | 9.22 | 9.40 | 3.66 | 4.24 | 4.28 | 4.66 | 0.48 | 0.72 | 0.80 | 0.84 |
| 13 | 8.60 | 9.22 | 9.34 | 9.32 | 3.70 | 4.36 | 4.48 | 4.62 | 0.42 | 0.58 | 0.64 | 0.82 |
| 14 | 8.54 | 9.28 | 9.24 | 9.28 | 3.56 | 4.46 | 4.40 | 4.62 | 0.42 | 0.58 | 0.62 | 0.80 |
| 15 | 8.36 | 8.92 | 9.22 | 9.42 | 3.44 | 4.40 | 4.38 | 4.56 | 0.42 | 0.72 | 0.70 | 0.78 |
| 16 | 8.56 | 9.22 | 9.38 | 9.66 | 3.68 | 4.40 | 4.38 | 4.60 | 0.36 | 0.68 | 0.78 | 0.88 |
| 17 | 8.38 | 9.36 | 9.12 | 9.30 | 3.56 | 4.46 | 4.60 | 4.92 | 0.46 | 0.68 | 0.64 | 0.88 |
| 18 | 8.36 | 9.56 | 9.16 | 9.42 | 3.68 | 4.78 | 4.90 | 5.00 | 0.52 | 0.78 | 0.74 | 0.86 |
| 19 | 8.48 | 9.92 | 9.76 | 9.70 | 3.42 | 4.64 | 4.80 | 5.08 | 0.44 | 0.82 | 0.72 | 0.92 |
| 20 | 8.54 | 10.28 | 9.82 | 10.30 | 3.46 | 4.90 | 4.94 | 5.16 | 0.40 | 0.76 | 0.62 | 0.80 |
| 21 | 8.22 | 10.24 | 9.92 | 10.26 | 3.56 | 4.86 | 5.00 | 5.02 | 0.50 | 0.82 | 0.64 | 0.98 |
| 22 | 7.88 | 10.36 | 9.78 | 10.10 | 3.50 | 4.90 | 4.82 | 4.88 | 0.48 | 0.78 | 0.72 | 0.94 |
| 23 | 7.86 | 9.92 | 9.78 | 9.84 | 3.48 | 4.90 | 4.68 | 4.78 | 0.38 | 0.82 | 0.82 | 0.90 |
| 24 | 7.90 | 10.12 | 9.42 | 9.52 | 3.36 | 4.84 | 4.38 | 4.60 | 0.36 | 0.94 | 0.76 | 0.96 |
| 25 | 7.52 | 9.76 | 9.34 | 9.34 | 3.30 | 4.72 | 4.54 | 4.74 | 0.38 | 1.00 | 0.76 | 0.90 |
| 26 | 7.32 | 9.90 | 9.34 | 9.34 | 3.02 | 4.84 | 4.16 | 4.80 | 0.32 | 0.96 | 0.88 | 0.94 |
| 27 | 7.28 | 9.74 | 9.24 | 9.22 | 2.90 | 4.80 | 4.38 | 4.74 | 0.30 | 0.92 | 0.88 | 1.04 |
| 28 | 7.56 | 10.44 | 9.50 | 9.50 | 2.90 | 5.00 | 4.36 | 4.68 | 0.20 | 0.92 | 0.82 | 0.82 |
| 29 | 7.54 | 10.48 | 9.64 | 9.62 | 2.88 | 5.16 | 4.56 | 4.58 | 0.36 | 0.90 | 0.72 | 0.82 |
| 30 | 7.30 | 10.52 | 9.22 | 9.22 | 2.74 | 5.28 | 4.46 | 4.32 | 0.38 | 0.84 | 0.76 | 0.82 |
| 31 | 7.30 | 10.84 | 9.38 | 9.44 | 2.72 | 5.40 | 4.34 | 4.54 | 0.36 | 0.86 | 0.72 | 0.82 |
| 32 | 7.36 | 10.98 | 9.68 | 9.54 | 2.70 | 5.56 | 4.30 | 4.42 | 0.34 | 0.88 | 0.80 | 0.76 |
| 33 | 7.28 | 11.16 | 9.46 | 9.72 | 2.72 | 5.86 | 4.40 | 4.44 | 0.30 | 1.00 | 0.82 | 0.74 |
| 34 | 7.28 | 11.60 | 9.72 | 9.36 | 2.58 | 5.90 | 4.42 | 4.58 | 0.32 | 0.98 | 0.94 | 0.80 |
| 35 | 7.10 | 11.70 | 9.68 | 9.20 | 2.64 | 5.98 | 4.42 | 4.72 | 0.28 | 1.12 | 0.86 | 0.84 |
| 36 | 7.10 | 11.76 | 9.98 | 9.46 | 2.72 | 5.98 | 4.48 | 4.86 | 0.26 | 0.98 | 0.88 | 0.80 |
| 37 | 7.06 | 12.66 | 9.84 | 9.38 | 2.78 | 6.16 | 4.50 | 4.84 | 0.30 | 1.06 | 0.82 | 0.84 |
| 38 | 7.18 | 13.00 | 9.92 | 9.70 | 2.86 | 6.56 | 4.58 | 4.84 | 0.30 | 1.26 | 0.80 | 0.76 |
| 39 | 7.14 | 13.42 | 10.24 | 9.62 | 2.84 | 6.90 | 4.72 | 4.74 | 0.30 | 1.38 | 0.78 | 0.84 |
| 40 | 7.08 | 13.96 | 10.14 | 9.38 | 2.78 | 6.88 | 4.86 | 4.62 | 0.28 | 1.48 | 0.82 | 0.74 |

Rejection frequencies (in %) at the 10%, 5% and 1% significance level, $n = 300$. Model: $x_t = \varepsilon_t$, $y_t = \eta_t$, $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$.

Table 24: Tests for zero cross-correlation at lags 0, ..., $m$. Size of tests $Q_{xy,m}$, $\widetilde{Q}_{xy,m}$ with different thresholds $\lambda$.

| | $\alpha = 10\%$ | | | | $\alpha = 5\%$ | | | | $\alpha = 1\%$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\widetilde{Q}_{xy,m} = \widetilde{Q}_{xy,m}(\lambda)$ | | | | $\widetilde{Q}_{xy,m} = \widetilde{Q}_{xy,m}(\lambda)$ | | | | $\widetilde{Q}_{xy,m} = \widetilde{Q}_{xy,m}(\lambda)$ | | |
| $m$ | $Q_{xy,m}$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ | $Q_{xy,m}$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ | $Q_{xy,m}$ | $\lambda = 1.645$ | $\lambda = 1.96$ | $\lambda = 2.576$ |
| 0 | 9.56 | 9.56 | 9.56 | 9.56 | 4.64 | 4.64 | 4.64 | 4.64 | 0.70 | 0.70 | 0.70 | 0.70 |
| 1 | 9.22 | 9.44 | 9.52 | 9.52 | 4.40 | 4.56 | 4.58 | 4.56 | 0.72 | 0.76 | 0.76 | 0.72 |
| 2 | 9.82 | 9.94 | 9.94 | 9.90 | 4.32 | 4.38 | 4.38 | 4.46 | 0.54 | 0.70 | 0.72 | 0.74 |
| 3 | 9.86 | 10.14 | 10.16 | 10.24 | 4.54 | 5.02 | 4.82 | 4.84 | 0.60 | 0.72 | 0.78 | 0.78 |
| 4 | 9.80 | 10.14 | 10.14 | 10.30 | 4.10 | 4.54 | 4.54 | 4.50 | 0.58 | 0.62 | 0.70 | 0.68 |
| 5 | 8.96 | 9.96 | 9.80 | 9.80 | 3.86 | 4.36 | 4.40 | 4.58 | 0.60 | 0.60 | 0.68 | 0.74 |
| 6 | 8.84 | 9.62 | 9.72 | 9.84 | 4.02 | 4.54 | 4.54 | 4.82 | 0.54 | 0.62 | 0.64 | 0.72 |
| 7 | 8.92 | 9.74 | 9.48 | 9.64 | 3.96 | 4.70 | 4.88 | 5.00 | 0.34 | 0.52 | 0.64 | 0.74 |
| 8 | 8.88 | 9.86 | 9.90 | 9.82 | 3.82 | 4.52 | 4.70 | 4.78 | 0.34 | 0.64 | 0.64 | 0.64 |
| 9 | 8.30 | 9.68 | 9.64 | 9.92 | 3.82 | 4.76 | 4.90 | 4.80 | 0.32 | 0.66 | 0.68 | 0.70 |
| 10 | 8.46 | 9.80 | 9.74 | 9.68 | 3.60 | 4.54 | 4.70 | 4.84 | 0.36 | 0.60 | 0.66 | 0.60 |
| 11 | 8.18 | 9.74 | 9.72 | 9.70 | 3.30 | 4.48 | 4.66 | 4.70 | 0.26 | 0.60 | 0.52 | 0.56 |
| 12 | 7.82 | 9.14 | 9.04 | 9.32 | 3.00 | 4.44 | 4.68 | 4.88 | 0.28 | 0.52 | 0.50 | 0.62 |
| 13 | 7.76 | 9.70 | 9.62 | 9.54 | 2.98 | 4.48 | 4.52 | 4.94 | 0.36 | 0.58 | 0.64 | 0.64 |
| 14 | 7.44 | 9.28 | 9.66 | 9.74 | 2.78 | 4.30 | 4.24 | 4.62 | 0.30 | 0.70 | 0.80 | 0.68 |
| 15 | 7.32 | 9.56 | 9.12 | 9.54 | 2.72 | 4.26 | 4.16 | 4.44 | 0.20 | 0.78 | 0.82 | 0.76 |
| 16 | 6.82 | 9.92 | 9.56 | 9.84 | 2.74 | 4.26 | 4.46 | 4.48 | 0.16 | 0.82 | 0.74 | 0.80 |
| 17 | 7.14 | 9.90 | 9.56 | 9.96 | 2.76 | 4.44 | 4.34 | 4.38 | 0.20 | 0.76 | 0.74 | 0.82 |
| 18 | 7.18 | 10.28 | 9.48 | 9.76 | 2.54 | 4.56 | 4.26 | 4.40 | 0.16 | 0.74 | 0.64 | 0.72 |
| 19 | 7.16 | 10.22 | 9.68 | 9.44 | 2.36 | 4.86 | 4.20 | 4.34 | 0.14 | 0.84 | 0.80 | 0.84 |
| 20 | 6.94 | 10.62 | 9.78 | 9.50 | 2.34 | 4.94 | 4.32 | 4.26 | 0.14 | 0.92 | 0.90 | 0.82 |
| 21 | 6.62 | 10.66 | 9.64 | 9.54 | 2.32 | 5.40 | 4.32 | 4.12 | 0.14 | 1.14 | 0.94 | 0.80 |
| 22 | 6.58 | 11.08 | 10.08 | 9.26 | 2.28 | 5.68 | 4.38 | 3.92 | 0.04 | 1.16 | 0.96 | 0.72 |
| 23 | 6.24 | 11.48 | 10.10 | 9.46 | 2.18 | 5.72 | 4.26 | 3.86 | 0.10 | 1.24 | 1.00 | 0.72 |
| 24 | 5.94 | 11.76 | 10.02 | 9.08 | 2.08 | 6.40 | 4.58 | 4.08 | 0.06 | 1.52 | 0.96 | 0.72 |
| 25 | 6.00 | 12.70 | 10.08 | 9.22 | 1.86 | 6.78 | 4.28 | 4.08 | 0.02 | 1.64 | 0.94 | 0.72 |
| 26 | 5.70 | 13.44 | 9.88 | 9.02 | 1.92 | 7.42 | 4.46 | 4.08 | 0.02 | 2.28 | 0.96 | 0.62 |
| 27 | 5.58 | 14.30 | 10.04 | 9.10 | 1.78 | 7.84 | 4.48 | 4.02 | 0.04 | 2.50 | 0.94 | 0.56 |
| 28 | 5.34 | 14.66 | 10.02 | 9.38 | 1.68 | 8.70 | 4.70 | 4.18 | 0.02 | 2.90 | 0.94 | 0.58 |
| 29 | 5.22 | 15.74 | 10.30 | 9.16 | 1.46 | 9.26 | 4.74 | 4.12 | 0.00 | 3.44 | 1.14 | 0.68 |
| 30 | 5.16 | 16.72 | 10.94 | 9.04 | 1.58 | 9.92 | 5.02 | 4.24 | 0.04 | 3.96 | 1.20 | 0.70 |
| 31 | 4.94 | 17.88 | 10.80 | 9.06 | 1.62 | 10.68 | 5.24 | 4.08 | 0.04 | 4.40 | 1.36 | 0.78 |
| 32 | 5.12 | 18.94 | 10.96 | 9.42 | 1.48 | 12.14 | 5.70 | 4.22 | 0.02 | 5.12 | 1.36 | 0.72 |
| 33 | 4.86 | 19.98 | 11.26 | 9.18 | 1.26 | 13.10 | 5.78 | 4.08 | 0.02 | 5.84 | 1.46 | 0.76 |
| 34 | 4.90 | 21.04 | 11.80 | 9.26 | 1.32 | 14.04 | 6.12 | 4.08 | 0.00 | 6.34 | 1.74 | 0.82 |
| 35 | 4.66 | 22.32 | 12.28 | 9.44 | 1.22 | 15.30 | 6.34 | 4.26 | 0.00 | 7.12 | 1.80 | 0.80 |
| 36 | 4.52 | 23.76 | 12.80 | 9.26 | 1.36 | 16.40 | 6.58 | 4.32 | 0.00 | 8.08 | 1.92 | 0.80 |
| 37 | 4.30 | 24.62 | 13.02 | 9.04 | 1.42 | 16.96 | 7.04 | 4.30 | 0.02 | 8.82 | 2.14 | 0.82 |
| 38 | 4.24 | 26.12 | 13.34 | 8.80 | 1.12 | 18.58 | 7.36 | 4.40 | 0.02 | 10.16 | 2.26 | 0.68 |
| 39 | 4.24 | 26.78 | 13.94 | 9.22 | 1.16 | 19.16 | 7.58 | 4.70 | 0.04 | 10.70 | 2.38 | 0.72 |
| 40 | 4.16 | 27.14 | 14.38 | 9.02 | 1.08 | 20.06 | 8.14 | 4.64 | 0.04 | 11.80 | 2.70 | 0.76 |

Rejection frequencies (in %) at the 10%, 5% and 1% significance level, $n = 300$. Model: $x_t = h_{1t}\varepsilon_t$, $y_t = h_{3t}\eta_t$, $h_{1t} = 1 + I(t/n > 0.5)$, $h_{3t} = 1 + 3I(t/n > 0.5)$, $\{\varepsilon_t\}$ and $\{\eta_t\}$ mutually independent i.i.d. $N(0,1)$.