Online Supplement

Finite-sample size control of IVX-based tests in predictive regressions

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I Additional proofs

Proof of Lemma 1

Use the Phillips-Solo decomposition (see Phillips and Solo, 1992) of v_t to write $v_t := \psi\nu_t + \Delta\tilde{v}_t$ where $\tilde{v}_t$ is a linear process in ν_t with absolutely summable coefficients (and as such uniformly L_r-bounded itself), leading to

$$x_t = \psi \sum_{j=0}^{t-1} \rho^j \nu_{t-j} + \tilde{v}_t - \rho^{t-1}\tilde{v}_1 + (\rho - 1)\sum_{j=1}^{t-1} \rho^{j-1}\tilde{v}_{t-j}.$$

Hence, Minkowski's norm inequality gives

$$\|x_t\|_r \leq \psi \left\| \sum_{j=0}^{t-1} \rho^j \nu_{t-j} \right\|_r + \|\tilde{v}_t\|_r + \rho^{t-1}\|\tilde{v}_1\|_r + |\rho - 1| \sum_{j=1}^{t-1} \rho^{j-1} \|\tilde{v}_{t-j}\|_r$$

such that $\|x_t\|_r \leq \psi \left\| \sum_{j=0}^{t-1} \rho^j \nu_{t-j} \right\|_r + C$. For $r = 4$ we have with the iid property of ν_t

$$\mathrm{E}\left(x_t^4\right) = \sum_{i=0}^{t-1}\sum_{j=0}^{t-1}\sum_{k=0}^{t-1}\sum_{l=0}^{t-1} \rho^i\rho^j\rho^k\rho^l\, \mathrm{E}\left(\nu_{t-i}\nu_{t-j}\nu_{t-k}\nu_{t-l}\right) \leq CT^2$$

and $T^{-1/2}x_t$ is thus uniformly L_4-bounded as required. A similar reasoning leads to the second part of the result.

Proof of Lemma 3

For computing the limit of $\mathrm{E}\left(B_T\right)$, note that

$$\begin{aligned} \Delta\tilde{x}_{t-1} &= (\rho - 1)\,\tilde{x}_{t-2} + \nu_{t-1} \\ &= (\rho - 1)\sum_{j=1}^{t-2} \rho^{j-1}\nu_{t-j-1} + \nu_{t-1} \end{aligned}$$

and thus

$$\tilde{z}_{t-1} = (1 - \varrho L)_+^{-1}\,\Delta\tilde{x}_{t-1}$$

$$
\begin{aligned}
&= (\rho-1)(1-\varrho L)_+^{-1}\sum_{j=1}^{t-2}\rho^{j-1}\nu_{t-j-1}+(1-\varrho L)_+^{-1}\nu_{t-1} \\
&= (\rho-1)(1-\varrho L)_+^{-1}\sum_{j=1}^{t-2}\rho^{j-1}\nu_{t-j-1}+\sum_{j=0}^{t-2}\varrho^{j}\nu_{t-j-1} \\
&= \sum_{j=1}^{t-1}\frac{\varrho^{t-1-j}(1-\varrho)-\rho^{t-1-j}(1-\rho)}{\rho-\varrho}\nu_j \equiv \sum_{j=1}^{t-1}c_{j,t-1}\nu_j
\end{aligned}
$$

by defining $c_{t-1,t-1}=1$. We may then focus on $\mathrm{E}\left(\sum_{t=2}^{T}\tilde{z}_{t-1}u_t\cdot\sum_{t=2}^{T}\tilde{z}_{t-1}^2u_t^2\right)=S_0$,

$$
\begin{aligned}
S_0 &= \mathrm{E}\left(\sum_{t=2}^{T}\tilde{z}_{t-1}^3u_t^3\right)+\mathrm{E}\left(\sum_{t=2}^{T}\sum_{s=2}^{t-1}\tilde{z}_{t-1}u_t\tilde{z}_{s-1}^2u_s^2\right)+\mathrm{E}\left(\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\tilde{z}_{t-1}u_t\tilde{z}_{s-1}^2u_s^2\right) \\
&= \mathrm{E}\left(\sum_{t=2}^{T}\tilde{z}_{t-1}^3u_t^3\right)+\mathrm{E}\left(\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\tilde{z}_{t-1}u_t\tilde{z}_{s-1}^2u_s^2\right),
\end{aligned}
$$

since $\mathrm{E}\left(\sum_{t=2}^{T}\tilde{z}_{t-1}u_t\right)=0$.

Let $S_0=S_{0,1}+S_{0,2}$, with $S_{0,1}=\mathrm{E}\left(\sum_{t=2}^{T}\tilde{z}_{t-1}^3u_t^3\right)$ and $S_{0,2}=\mathrm{E}\left(\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\tilde{z}_{t-1}u_t\tilde{z}_{s-1}^2u_s^2\right)$. We work out $S_{0,2}$ first. Recall that

$$
\tilde{z}_{t-1}=\sum_{j=1}^{t-1}c_{j,t-1}\upsilon_j \text{ with } c_{j,t-1}=\frac{\varrho^{t-1-j}(1-\varrho)-\rho^{t-1-j}(1-\rho)}{\rho-\varrho}.
$$

Using the independence of shocks we obtain

$$
\begin{aligned}
\mathrm{E}\left(\tilde{z}_{t-1}u_t\tilde{z}_{s-1}^2\right) &= \mathrm{E}\left(\left(\sum_{j=1}^{t-1}c_{j,t-1}\upsilon_j\right)u_t\left(\sum_{j=1}^{t-1}c_{j,s-1}\upsilon_j+\sum_{j=t}^{s-1}c_{j,s-1}\upsilon_j\right)^2\right) \\
&= 2\,\mathrm{E}\left(\left(\sum_{j=1}^{t-1}c_{j,t-1}\upsilon_j\right)u_t\left(\sum_{j=1}^{t-1}c_{j,s-1}\upsilon_j\right)\left(\sum_{j=t}^{s-1}c_{j,s-1}\upsilon_j\right)\right) \\
&= 2\sigma_{u\upsilon}^2\left(\frac{t}{T}\right)c_{t,s-1}\sum_{j=1}^{t-1}c_{j,t-1}c_{j,s-1}\sigma_\upsilon^2\left(\frac{j}{T}\right)
\end{aligned}
$$

which implies

$$
\mathrm{E}\left(\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\tilde{z}_{t-1}u_t\tilde{z}_{s-1}^2u_s^2\right)=2\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\sigma_u^2\left(\frac{s}{T}\right)\sigma_{u\upsilon}\left(\frac{t}{T}\right)c_{t,s-1}c_{j,t-1}c_{j,s-1}\sigma_\upsilon^2\left(\frac{j}{T}\right).
$$

Now since $c_{j,t-1}=\frac{\varrho^{t-1-j}(1-\varrho)-\rho^{t-1-j}(1-\rho)}{\rho-\varrho}=\frac{1-\varrho}{\rho-\varrho}\left(\varrho^{t-1-j}-\frac{1-\rho}{1-\varrho}\rho^{t-1-j}\right)$ we have

$$
\begin{aligned}
c_{t,s-1}c_{j,t-1}c_{j,s-1} &= \varrho^{2s-3-2j}-\frac{1-\rho}{1-\varrho}\varrho^{s-2-j}\rho^{s-1-j}-\frac{1-\rho}{1-\varrho}\varrho^{2s-2-t-j}\rho^{t-1-j} \\
&\quad+\left(\frac{1-\rho}{1-\varrho}\right)^2\varrho^{s-1-t}\rho^{t+s-2-2j}-\frac{1-\rho}{1-\varrho}\varrho^{t+s-2-2j}\rho^{s-1-t}
\end{aligned} \tag{S.1}
$$

$$
\begin{aligned}
&+\left(\frac{1-\rho}{1-\varrho}\right)^2 \rho^{2s-2-t-j}\varrho^{t-1-j} \\
&+\left(\frac{1-\rho}{1-\varrho}\right)^2 \rho^{s-2-j}\varrho^{s-1-j} - \left(\frac{1-\rho}{1-\varrho}\right)^3 \rho^{2s-2j-3} \\
= &\sum_{k=1}^{8} \alpha_k, \text{say},
\end{aligned}
$$

where we have dropped the term $\left(\frac{1-\varrho}{\rho-\varrho}\right)^3 = 1 + O\left(T^{\eta-1}\right)$. Further, those terms that have a factor of $\left(\frac{1-\rho}{1-\varrho}\right)^2$ and the last term with $\left(\frac{1-\rho}{1-\varrho}\right)^3$ are clearly dominated by the other terms and hence vanish. In the following we will, therefore, look only at the terms that are associated with α_1, α_2, α_3 and α_5.

We start by focusing on the first term stemming from $c_{t,s-1}c_{j,t-1}c_{j,s-1}$, namely $\varrho^{2s-2j-3}$.

$$
S_{0,2}^{(\alpha_1)} = \sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sum_{s=t+1}^{T}\sum_{j=1}^{t-1} \sigma_u^2\left(\frac{s}{T}\right) \varrho^{2s-2j-3} \sigma_v^2\left(\frac{j}{T}\right).
$$

Note here that $1 < j < t < s \leq T$ hence

$$
\begin{aligned}
\left|\sigma_u^2\left(\frac{s}{T}\right) - \sigma_u^2\left(\frac{t}{T}\right)\right| &< C\frac{|s-t|}{T} < C\frac{|s-j|}{T}, \\
\left|\sigma_v^2\left(\frac{j}{T}\right) - \sigma_v^2\left(\frac{t}{T}\right)\right| &< C\frac{|t-j|}{T} < C\frac{|s-j|}{T},
\end{aligned}
$$

and

$$
\left|\sigma_u^2\left(\frac{s}{T}\right) - \sigma_u^2\left(\frac{t}{T}\right)\right|\left|\sigma_v^2\left(\frac{j}{T}\right) - \sigma_v^2\left(\frac{t}{T}\right)\right| < C\frac{(s-j)^2}{T^2}.
$$

Now as

$$
\begin{aligned}
S_{0,2}^{(\alpha_1)} = &\sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sum_{s=t+1}^{T}\sum_{j=1}^{t-1} \left(\sigma_u^2\left(\frac{s}{T}\right) - \sigma_u^2\left(\frac{t}{T}\right) + \sigma_u^2\left(\frac{t}{T}\right)\right) \varrho^{2s-2j-3} \left(\sigma_v^2\left(\frac{j}{T}\right) - \sigma_v^2\left(\frac{t}{T}\right) + \sigma_v^2\left(\frac{t}{T}\right)\right) \\
= &\sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sum_{s=t+1}^{T}\sum_{j=1}^{t-1} \left(\sigma_u^2\left(\frac{s}{T}\right) - \sigma_u^2\left(\frac{t}{T}\right)\right)\left(\sigma_v^2\left(\frac{j}{T}\right) - \sigma_v^2\left(\frac{t}{T}\right)\right) \varrho^{2s-2j-3} \\
&+\sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sum_{s=t+1}^{T}\sum_{j=1}^{t-1} \left(\sigma_u^2\left(\frac{s}{T}\right) - \sigma_u^2\left(\frac{t}{T}\right)\right) \sigma_v^2\left(\frac{t}{T}\right) \varrho^{2s-2j-3} \\
&+\sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sum_{s=t+1}^{T}\sum_{j=1}^{t-1} \sigma_u^2\left(\frac{t}{T}\right) \varrho^{2s-2j-3} \left(\sigma_v^2\left(\frac{j}{T}\right) - \sigma_v^2\left(\frac{t}{T}\right)\right) \\
&+\sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sum_{s=t+1}^{T}\sum_{j=1}^{t-1} \sigma_u^2\left(\frac{t}{T}\right) \varrho^{2s-2j-3} \left(\sigma_v^2\left(\frac{t}{T}\right)\right),
\end{aligned}
$$

we may write

$$
\left|S_{0,2}^{(\alpha_1)} - \sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) \sum_{s=t+1}^{T}\sum_{j=1}^{t-1} \varrho^{2s-2j-3}\right|
$$

$$
\begin{aligned}
&\leq \sum_{t=2}^{T-1}\sigma_{uv}\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\left|\sigma_u^2\left(\frac{s}{T}\right)-\sigma_u^2\left(\frac{t}{T}\right)\right|\left|\sigma_v^2\left(\frac{j}{T}\right)-\sigma_v^2\left(\frac{t}{T}\right)\right|\varrho^{2s-2j-3}\\
&\quad+\sum_{t=2}^{T-1}\sigma_{uv}\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\left|\sigma_u^2\left(\frac{s}{T}\right)-\sigma_u^2\left(\frac{t}{T}\right)\right|\sigma_v^2\left(\frac{t}{T}\right)\varrho^{2s-2j-3}\\
&\quad+\sum_{t=2}^{T-1}\sigma_{uv}\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\sigma_u^2\left(\frac{t}{T}\right)\left|\sigma_v^2\left(\frac{j}{T}\right)-\sigma_v^2\left(\frac{t}{T}\right)\right|\varrho^{2s-2j-3}\\
&\leq \frac{C}{T^2}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-j)^2\varrho^{2s-2j-3}+\frac{C}{T}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-t)\varrho^{2s-2j-3}+\frac{C}{T}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{2s-2j-3}(j-t).
\end{aligned}
$$

Now note that

$$
\begin{aligned}
\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-j)^2\varrho^{2s-2j} &= -\frac{2\varrho^4\left(4+7\varrho^2+\varrho^4\right)\left(1-\varrho^{2T}\right)}{\left(1-\varrho^2\right)^5}+\frac{2\varrho^4\left(2+\varrho^2\right)+\varrho^{2+2T}\left(1+12\varrho^2+5\varrho^4\right)}{\left(1-\varrho^2\right)^4}T\\
&\quad+\frac{2\varrho^{2+2T}\left(1+2\varrho^2\right)}{\left(1-\varrho^2\right)^3}T^2+\frac{\varrho^{2+2T}}{\left(1-\varrho^2\right)^2}T^3\\
&= O\left(T^{4\eta+1}\right),
\end{aligned}
$$

and in the same way we can show that

$$
\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-t)\varrho^{2s-2j-3}=O\left(T^{3\eta+1}\right)=\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{2s-2j-3}(j-t).
$$

Therefore

$$
\begin{aligned}
&\left|S_{0,2}^{(\alpha_1)}-\sum_{t=2}^{T-1}\sigma_{uv}\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{2s-2j-3}\right|\\
&\leq \frac{C}{T^2}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-j)^2\varrho^{2s-2j-3}+\frac{C}{T}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-t)\varrho^{2s-2j-3}+\frac{C}{T}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{2s-2j-3}(j-t)\\
&= O\left(T^{4\eta-1}\right)+O\left(T^{3\eta}\right)=O\left(T^{3\eta}\right),\text{ since }3\eta>4\eta-1.
\end{aligned}
$$

The latter in turn implies

$$
T^{1/2-\eta/2}\frac{1}{T^{1/2+\eta/2}}\frac{1}{T^{1+\eta}}\times\left|S_{0,2}^{(\alpha_1)}-\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{2s-2j-3}\right|=O\left(T^{\eta-1}\right).
$$

We will now show that

$$
T^{1/2-\eta/2}\frac{1}{T^{1/2+\eta/2}}\frac{1}{T^{1+\eta}}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{2s-2j-3}=\frac{1}{4a^2}\int_0^1\sigma_{uv}(x)\,\sigma_u^2(x)\,\sigma_v^2(x)\,dx+O\left(T^{\eta
$$

First observe that

$$
\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{2s-2j}=\frac{\varrho^4-\varrho^{2t+2}+\varrho^{2T+2}-\varrho^{2T-2t+4}}{\left(1-\varrho^2\right)^2}.
$$

Hence we have

$$
\begin{aligned}
& \frac{1}{T^{1/2+\eta/2}} \frac{1}{T^{1+\eta}} \sum_{t=2}^{T-1} \sigma_{uv}^2\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) \sum_{s=t+1}^{T} \sum_{j=1}^{t-1} \varrho^{2s-2j} \\
= & \frac{1}{T^{3/2+3\eta/2}} \frac{\varrho^4}{(1-\varrho)^2(1+\varrho)^2} \sum_{t=2}^{T-1} \sigma_{uv}^2\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) \\
& -\frac{1}{T^{3/2+3\eta/2}} \frac{1}{(1-\varrho^2)^2} \sum_{t=2}^{T-1} \sigma_{uv}^2\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) \varrho^{2t+2} \\
& +\frac{1}{T^{3/2+3\eta/2}} \frac{\varrho^{2T+2}}{(1-\varrho^2)^2} \sum_{t=2}^{T-1} \sigma_{uv}^2\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) \\
& -\frac{1}{T^{3/2+3\eta/2}} \frac{1}{(1-\varrho^2)^2} \sum_{t=2}^{T-1} \sigma_{uv}^2\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) \varrho^{2T-2t+4} \\
= & \frac{1}{T^{3/2+3\eta/2}} \frac{(1-aT^{-\eta})^4}{(aT^{-\eta})^2(2+aT^{-\eta})^2} \sum_{t=2}^{T-1} \sigma_{uv}^2\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) + R_1^{(\alpha_1)} + R_2^{(\alpha_1)} + R_3^{(\alpha_1)}.
\end{aligned}
$$

For $R_1^{(\alpha_1)}$ note that

$$
\begin{aligned}
\left|R_1^{(\alpha_1)}\right| & = \frac{1}{T^{3/2+3\eta/2}} \frac{1}{(1-\varrho^2)^2} \sum_{t=2}^{T-1} \sigma_{uv}^2\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) \varrho^{2t+2} \\
& \leq \frac{C}{T^{3/2-\eta/2}} \sum_{t=2}^{T-1} \varrho^{2t} = O\left(T^{-3/2+3\eta/2}\right).
\end{aligned}
$$

$R_2^{(\alpha_1)} = \frac{1}{T^{3/2+3\eta/2}} \frac{\varrho^{2T+2}}{(1-\varrho^2)^2} \sum_{t=2}^{T-1} \sigma_{uv}^2\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right)$ is clearly dominated by $R_1^{(\alpha_1)}$ and for $R_3^{(\alpha_1)}$ we have

$$
\begin{aligned}
\left|R_3^{(\alpha_1)}\right| & = \frac{1}{T^{3/2+3\eta/2}} \frac{1}{(1-\varrho^2)^2} \sum_{t=2}^{T-1} \sigma_{uv}^2\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) \varrho^{2T-2t+4} \\
& \leq \frac{C}{T^{3/2-\eta/2}} \sum_{t=2}^{T-1} \varrho^{2T-2t} = O\left(T^{-3/2+3\eta/2}\right).
\end{aligned}
$$

Further, note that $T^{1/2-\eta/2}\left|R_1^{(\alpha_1)} + R_2^{(\alpha_1)} + R_3^{(\alpha_1)}\right| = O\left(T^{-1+\eta}\right)$, hence

$$
\begin{aligned}
& T^{1/2-\eta/2} \frac{1}{T^{1/2+\eta/2}} \frac{1}{T^{1+\eta}} \sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) \sum_{s=t+1}^{T} \sum_{j=1}^{t-1} \varrho^{2s-2j-3} \\
= & T^{1/2-\eta/2} \frac{1}{T^{1/2+\eta/2}} \frac{1}{T^{1+\eta}} \sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) \frac{\varrho^4 - \varrho^{2t+2} + \varrho^{2T+2} - \varrho^{2T-2t+4}}{(1-\varrho^2)^2} \\
= & T^{1/2-\eta/2} \frac{1}{T^{1/2+3\eta/2}} \frac{\varrho}{(1-\varrho^2)^2} \frac{1}{T} \sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) + O\left(T^{\eta-1}\right) \\
= & \frac{1}{a^2} \frac{(1-aT^{-\eta})}{(2-aT^{-\eta})^2} \frac{1}{T} \sum_{t=2}^{T-1} \sigma_{uv}\left(\frac{t}{T}\right) \sigma_u^2\left(\frac{t}{T}\right) \sigma_v^2\left(\frac{t}{T}\right) + O\left(T^{\eta-1}\right)
\end{aligned}
$$

$$\rightarrow \quad \frac{1}{4a^2}\int_0^1 \sigma_{uv}(x)\,\sigma_u^2(x)\,\sigma_v^2(x)\,dx.$$

Putting all these arguments together we have

$$T^{1/2-\eta/2}\frac{1}{T^{1/2+\eta/2}}\frac{1}{T^{1+\eta}}S_{0,2}^{(\alpha_1)} \rightarrow \frac{1}{4a^2}\int_0^1 \sigma_{uv}(x)\,\sigma_u^2(x)\,\sigma_v^2(x)\,dx. \tag{S.2}$$

We now turn to the second term stemming from $c_{t,s-1}c_{j,t-1}c_{j,s-1}$ and given as α_2 below (S.1), leading to the analysis of

$$S_{0,2}^{(\alpha_2)} = -\frac{1-\rho}{1-\varrho}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\sigma_u^2\left(\frac{s}{T}\right)\sigma_{uv}\left(\frac{t}{T}\right)\varrho^{s-2-j}\rho^{s-1-j}\sigma_v^2\left(\frac{j}{T}\right).$$

Following the same lines of arguments as for $S_{0,2}^{(\alpha_1)}$ and noting that

$$\begin{aligned}
\frac{C}{T}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-t)\,\varrho^{s-2-j}\rho^{s-1-j} &= O\left(T^{3\eta}\right)\\
\frac{C}{T}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(j-t)\,\varrho^{s-2-j}\rho^{s-1-j} &= O\left(T^{3\eta}\right)
\end{aligned}$$

and as

$$\frac{C}{T^2}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-j)^2\,\varrho^{s-2-j}\rho^{s-1-j} = O\left(T^{5\eta-2}\right)$$

we have that

$$\begin{aligned}
&T^{1/2-\eta/2}\frac{1}{T^{1/2+\eta/2}}\frac{1}{T^{1+\eta}}\cdot\frac{1}{\varrho^2\rho}\times\\
&\left|\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\sigma_u^2\left(\frac{s}{T}\right)(\varrho\rho)^{s-j}\sigma_v^2\left(\frac{j}{T}\right)-\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(\varrho\rho)^{s-j}\right|
\end{aligned}$$

is of order $T^{1/2-\eta/2}O\left(T^{-\frac{3}{2}\eta-\frac{3}{2}}\right)O\left(T^{4\eta-1}\right)=O\left(T^{\eta-1}\right)$.

Next observe that

$$\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{s-2-j}\rho^{s-1-j} = \frac{\rho-\rho^t\varrho^{t-1}+\varrho^{T-1}\rho^T-\varrho^{T-t}\rho^{T-t+1}}{(1-\rho\varrho)^2}.$$

Therefore

$$\begin{aligned}
&\frac{1}{T^{1/2+\eta/2}}\frac{1}{T^{1+\eta}}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{s-2-j}\rho^{s-1-j}\\
&= \frac{1}{T^{3/2+3\eta/2}}\frac{\rho}{(1-\rho\varrho)^2}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{T^{3/2+3\eta/2}}\frac{1}{(1-\rho\varrho)^2}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\rho^t\varrho^{t-1} \\
& +\frac{1}{T^{3/2+3\eta/2}}\frac{\varrho^{T-1}\rho^T}{(1-\rho\varrho)^2}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right) \\
& -\frac{1}{T^{3/2+3\eta/2}}\frac{1}{(1-\rho\varrho)^2}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\varrho^{T-t}\rho^{T-t+1} \\
= \; & \frac{1}{T^{3/2+3\eta/2}}\frac{\rho}{(1-\rho\varrho)^2}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)+R_1^{(\alpha_2)}+R_2^{(\alpha_2)}+R_3^{(\alpha_2)},
\end{aligned}
$$

for which again one can elementarily show that $R_1^{(\alpha_2)}$, $R_2^{(\alpha_2)}$ and $R_3^{(\alpha_2)}$ are $o\left(T^{\eta/2-1/2}\right)$ and since $\frac{\rho}{(1-\rho\varrho)^2}=O\left(T^{2\eta}\right)$, we have

$$
\frac{T^{1/2-\eta/2}}{T^{3/2+3\eta/2}}\frac{\rho}{(1-\rho\varrho)^2}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\to\frac{1}{a^2}\int_0^1\sigma_{uv}(x)\,\sigma_u^2(x)\,\sigma_v^2(x)\,dx.
$$

Since $\frac{1-\rho}{1-\varrho}=O\left(T^{\eta-1}\right)$, we obtain

$$
T^{1/2-\eta/2}S_{0,2}^{(\alpha_2)}=O\left(T^{\eta-1}\right). \tag{S.3}
$$

We now turn to the third term stemming from $c_{t,s-1}c_{j,t-1}c_{j,s-1}$ and given as α_3 below (S.1), $\alpha_3=\frac{1-\rho}{1-\varrho}\varrho^{2s-2-t-j}\rho^{t-1-j}$, i.e. we analyze

$$
S_{0,2}^{(\alpha_3)}=-\frac{1-\rho}{1-\varrho}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\sigma_u^2\left(\frac{s}{T}\right)\sigma_{uv}\left(\frac{t}{T}\right)\varrho^{2s-2-t-j}\rho^{t-1-j}\sigma_v^2\left(\frac{j}{T}\right).
$$

Here note that

$$
\begin{aligned}
\frac{C}{T}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-t)\,\varrho^{2s-2-t-j}\rho^{t-1-j} &= O\left(T^{3\eta}\right)\\
\frac{C}{T}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(j-t)\,\varrho^{2s-2-t-j}\rho^{t-1-j} &= O\left(T^{3\eta}\right)
\end{aligned}
$$

and as

$$
\frac{C}{T^2}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-j)^2\,\varrho^{2s-2-t-j}\rho^{t-1-j}=O\left(T^{5\eta-2}\right)
$$

we have

$$
\begin{aligned}
& T^{1/2-\eta/2}\frac{1}{T^{1/2+\eta/2}}\frac{1}{T^{1+\eta}}\times \\
& \left|\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\sigma_u^2\left(\frac{s}{T}\right)\varrho^{2s-2-t-j}\rho^{t-1-j}\sigma_v^2\left(\frac{j}{T}\right)-\right.
\end{aligned}
$$

$$\left.\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{2s-2-t-j}\rho^{t-1-j}\right|$$
$$= T^{1/2-\eta/2}O\left(T^{-\frac{3}{2}\eta-\frac{3}{2}}\right)O\left(T^{3\eta}\right)=O\left(T^{\eta-1}\right).$$

Next observe that

$$\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{2s-2-t-j}\rho^{t-1-j}=\frac{\left(1-\varrho^{2T-2t}\right)\left(\varrho\rho-\varrho^t\rho^t\right)}{\left(1-\varrho^2\right)\rho\left(1-\varrho\rho\right)}.$$

Therefore

$$\begin{aligned}
&\frac{1}{T^{1/2+\eta/2}}\frac{1}{T^{1+\eta}}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{2s-2-t-j}\rho^{t-1-j}\\
=\;&\frac{1}{T^{3/2+3\eta/2}}\frac{\varrho}{\left(1-\varrho^2\right)\rho\left(1-\varrho\rho\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\\
&-\frac{1}{T^{3/2+3\eta/2}}\frac{1}{\left(1-\varrho^2\right)\rho\left(1-\varrho\rho\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\rho^t\varrho^t\\
&+\frac{1}{T^{3/2+3\eta/2}}\frac{1}{\left(1-\varrho^2\right)\rho\left(1-\varrho\rho\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\varrho^{2T-2t}\rho^t\varrho^t\\
&-\frac{1}{T^{3/2+3\eta/2}}\frac{\varrho}{\left(1-\varrho^2\right)\left(1-\varrho\rho\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\varrho^{2T-2t}\\
=\;&\frac{1}{T^{3/2+3\eta/2}}\frac{\varrho}{\left(1-\varrho^2\right)\left(1-\varrho\rho\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)+R_1^{(\alpha_3)}+R_2^{(\alpha_3)}+R_3^{(\alpha_3)},
\end{aligned}$$

for which again one can elementarily show that $R_1^{(\alpha_3)}$, $R_2^{(\alpha_3)}$ and $R_3^{(\alpha_3)}$ are $o\left(T^{\eta/2-1/2}\right)$ and since $\frac{\varrho}{(1-\varrho^2)\rho(1-\varrho\rho)}=O\left(T^{2\eta}\right)$, we have

$$\frac{T^{1/2-\eta/2}}{T^{3/2+3\eta/2}}\frac{\varrho}{\left(1-\varrho^2\right)\left(1-\rho^2\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\to\frac{1}{a^2}\int_0^1\sigma_{uv}\left(x\right)\sigma_u^2\left(x\right)\sigma_v^2\left(x\right)dx.$$

Now as $\frac{1-\rho}{1-\varrho}=O\left(T^{\eta-1}\right)$ we obtain

$$T^{1/2-\eta/2}S_{0,2}^{(\alpha_3)}=O\left(T^{\eta-1}\right). \tag{S.4}$$

Finally we turn to the α_5 given under (S.1): $\frac{1-\rho}{1-\varrho}\varrho^{t+s-2-2j}\rho^{s-1-t}$

$$S_{0,2}^{(\alpha_5)}=-\frac{1-\rho}{1-\varrho}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\sigma_u^2\left(\frac{s}{T}\right)\sigma_{uv}\left(\frac{t}{T}\right)\varrho^{t+s-2-2j}\rho^{s-1-t}\sigma_v^2\left(\frac{j}{T}\right).$$

Here note that

$$\frac{C}{T}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\left(s-t\right)\varrho^{t+s-2-2j}\rho^{s-1-t}=O\left(T^{3\eta}\right)=\frac{C}{T}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\left(j-t\right)\varrho^{t+s-2-2j}\rho^{s-1-t},$$

and as $\frac{C}{T^2}\sum_{t=2}^{T-1}\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}(s-j)^2\varrho^{t+s-2-2j}\rho^{s-1-t}=O\left(T^{5\eta-2}\right)$ we have

$$\begin{aligned}
& T^{1/2-\eta/2}\frac{1}{T^{1/2+\eta/2}}\frac{1}{T^{1+\eta}}\times \\
& \left|\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\sigma_u^2\left(\frac{s}{T}\right)\varrho^{t+s-2-2j}\rho^{s-1-t}\sigma_v^2\left(\frac{j}{T}\right)\right. \\
& \left.-\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{t+s-2-2j}\rho^{s-1-t}\right| \\
= & T^{1/2-\eta/2}O\left(T^{-\frac{3}{2}\eta-\frac{3}{2}}\right)O\left(T^{3\eta}\right)=O\left(T^{\eta-1}\right).
\end{aligned}$$

Next observe that

$$\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{t+s-2-2j}\rho^{s-1-t}=\frac{\left(\varrho-\varrho^{2t-1}\right)\left(1-\varrho^{T-t}\rho^{T-t}\right)}{\left(1-\varrho^2\right)\left(1-\varrho\rho\right)}.$$

Therefore

$$\begin{aligned}
& \frac{1}{T^{1/2+\eta/2}}\frac{1}{T^{1+\eta}}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\sum_{s=t+1}^{T}\sum_{j=1}^{t-1}\varrho^{t+s-2-2j}\rho^{s-1-t} \\
= & \frac{1}{T^{3/2+3\eta/2}}\frac{\varrho}{\left(1-\varrho^2\right)\left(1-\varrho\rho\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right) \\
& -\frac{1}{T^{3/2+3\eta/2}}\frac{1}{\left(1-\varrho^2\right)\left(1-\varrho\rho\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\rho^{2t-1} \\
& +\frac{1}{T^{3/2+3\eta/2}}\frac{1}{\left(1-\varrho^2\right)\left(1-\varrho\rho\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\varrho^{T+t-1}\rho^{T-t} \\
& -\frac{1}{T^{3/2+3\eta/2}}\frac{\varrho}{\left(1-\varrho^2\right)\left(1-\varrho\rho\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\varrho^{T-t}\rho^{T-t} \\
= & \frac{1}{T^{3/2+3\eta/2}}\frac{\varrho}{\left(1-\varrho^2\right)\left(1-\varrho\rho\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)+R_1^{(\alpha_5)}+R_2^{(\alpha_5)}+R_3^{(\alpha_5)},
\end{aligned}$$

for which again one can elementarily show that $R_1^{(\alpha_5)}$, $R_2^{(\alpha_5)}$ and $R_3^{(\alpha_5)}$ are $o(1)$ and since $\frac{\varrho}{(1-\varrho^2)(1-\varrho\rho)}=O\left(T^{2\eta}\right)$, we have

$$\frac{T^{1/2-\eta/2}}{T^{3/2+3\eta/2}}\frac{\varrho}{\left(1-\varrho^2\right)\left(1-\rho^2\right)}\sum_{t=2}^{T-1}\sigma_{uv}^2\left(\frac{t}{T}\right)\sigma_u^2\left(\frac{t}{T}\right)\sigma_v^2\left(\frac{t}{T}\right)\to\frac{1}{a^2}\int_0^1\sigma_{uv}(x)\,\sigma_u^2(x)\,\sigma_v^2(x)\,dx.$$

Now as $\frac{1-\rho}{1-\varrho}=O\left(T^{\eta-1}\right)$ we obtain

$$T^{1/2-\eta/2}S_{0,2}^{(\alpha_5)}=O\left(T^{\eta-1}\right). \tag{S.5}$$

Putting (S.2)-(S.5) together we obtain

$$T^{1/2-\eta/2}E\left(S_{0,2}\right)\to\frac{1}{4a^2}\int_0^1\sigma_{uv}\left(x\right)\sigma_u^2\left(x\right)\sigma_v^2\left(x\right)dx.$$

We may now turn to $S_{0,1}$. Assuming Lipschitz continuity for the third moments we have

$$\begin{aligned}S_{0,1} &= \sum_{t=2}^{T}\sum_{j=1}^{t-1}\sum_{k=1}^{t-1}\sum_{l=1}^{t-1}c_{j,t-1}c_{k,t-1}c_{l,t-1}\,\mathrm{E}\left(v_jv_kv_l\right)\mathrm{E}\left(u_t^3\right)\\ &= \sum_{t=2}^{T}\sum_{j=1}^{t-1}c_{j,t-1}^3\,\mathrm{E}\left(v_j^3\right)\mathrm{E}\left(u_t^3\right).\end{aligned}$$

Now again as for the analysis of $S_{0,2}$ we have to look at all the terms stemming from the expansion of $c_{j,t-1}^3$. We start by considering the effect of the first term, namely ϱ^{3t-3j}.

$$\begin{aligned}&\sum_{t=2}^{T}\sum_{j=1}^{t-1}\varrho^{3t-3j}\,\mathrm{E}\left(v_j^3\right)\mathrm{E}\left(u_t^3\right)\\ =& \sum_{t=2}^{T}\sigma_u^3\left(\frac{t}{T}\right)\sum_{j=1}^{t-1}\varrho^{3t-3j}\sigma_v^3\left(\frac{j}{T}\right)\\ =& \sum_{t=2}^{T}\sigma_u^3\left(\frac{t}{T}\right)\sum_{j=1}^{t-1}\varrho^{3t-3j}\left(\sigma_v^3\left(\frac{j}{T}\right)-\sigma_v^3\left(\frac{t}{T}\right)+\sigma_v^3\left(\frac{t}{T}\right)\right)\\ =& \sum_{t=2}^{T}\sigma_u^3\left(\frac{t}{T}\right)\sum_{j=1}^{t-1}\varrho^{3t-3j}\left(\sigma_v^3\left(\frac{j}{T}\right)-\sigma_v^3\left(\frac{t}{T}\right)\right)+\sum_{t=2}^{T}\sigma_u^3\left(\frac{t}{T}\right)\sum_{j=1}^{t-1}\varrho^{3t-3j}\sigma_v^3\left(\frac{t}{T}\right)\end{aligned}$$

which implies that

$$\begin{aligned}&\left|\sum_{t=2}^{T}\sum_{j=1}^{t-1}\varrho^{3t-3j}\,\mathrm{E}\left(v_j^3\right)\mathrm{E}\left(u_t^3\right)-\sum_{t=2}^{T}\sigma_u^3\left(\frac{t}{T}\right)\sigma_v^3\left(\frac{t}{T}\right)\sum_{j=1}^{t-1}\varrho^{3t-3j}\right|\\ \leq& \sum_{t=2}^{T}\sigma_u^3\left(\frac{t}{T}\right)\sum_{j=1}^{t-1}\varrho^{3t-3j}\left|\sigma_v^3\left(\frac{j}{T}\right)-\sigma_v^3\left(\frac{t}{T}\right)\right|\\ \leq& \frac{C}{T}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\varrho^{3t-3j}\left(t-j\right)=O\left(T^{3\eta}\right)\end{aligned}$$

On the other hand $\sum_{j=1}^{t-1}\varrho^{3t-3j}=\frac{\varrho^3-\varrho^{3t}}{1-\varrho^3}$, hence

$$\begin{aligned}&T^{1/2-\eta/2}\times\frac{1}{T^{1/2+\eta/2}}\frac{1}{T^{1+\eta}}\sum_{t=2}^{T}\sum_{j=1}^{t-1}\varrho^{3t-3j}\,\mathrm{E}\left(v_j^3\right)\mathrm{E}\left(u_t^3\right)\\ =& T^{-2\eta}\frac{1}{T}\sum_{t=2}^{T}\sigma_u^3\left(\frac{t}{T}\right)\sigma_v^3\left(\frac{t}{T}\right)\sum_{j=1}^{t-1}\varrho^{3t-3j}+O\left(T^{\eta-1}\right)\\ =& T^{-2\eta}\frac{1}{T}\sum_{t=2}^{T}\sigma_u^3\left(\frac{t}{T}\right)\sigma_v^3\left(\frac{t}{T}\right)\frac{\varrho^3-\varrho^{3t}}{1-\varrho^3}+O\left(T^{\eta-1}\right)\end{aligned}$$

$$= \frac{\varrho^3}{1-\varrho^3} T^{-2\eta} \frac{1}{T} \sum_{t=2}^{T} \sigma_u^3\left(\frac{t}{T}\right) \sigma_v^3\left(\frac{t}{T}\right) - \frac{1}{1-\varrho^3} T^{-2\eta} \frac{1}{T} \sum_{t=2}^{T} \sigma_u^3\left(\frac{t}{T}\right) \sigma_v^3\left(\frac{t}{T}\right) \varrho^{3t} + O\left(T^{\eta-1}\right)$$

First note that $\frac{\varrho^3}{1-\varrho^3} T^{-2\eta} \frac{1}{T} \sum_{t=2}^{T} \sigma_u^3\left(\frac{t}{T}\right) \sigma_v^3\left(\frac{t}{T}\right) = O\left(T^{-\eta}\right)$. Further

$$\left|\sum_{t=2}^{T} \sigma_u^3\left(\frac{t}{T}\right) \sigma_v^3\left(\frac{t}{T}\right) \varrho^{3t}\right| \leq C \left|\sum_{t=2}^{T} \varrho^{3t}\right| = O\left(T^{\eta}\right).$$

Therefore $T^{1/2-\eta/2} \times \frac{1}{T^{1/2+\eta/2}} \frac{1}{T^{1+\eta}} \sum_{t=2}^{T} \sum_{j=1}^{t-1} \varrho^{3t-3j} \operatorname{E}\left(v_j^3\right) \operatorname{E}\left(u_t^3\right) = o(1)$. The other terms stemming from expanding $c_{j,t-1}^3$ can be shown, in a similar way, to vanish.

Proof of Proposition 2

We prove first that, the null $\beta = 0$, the following holds:

$$t_{vx}^{rec} = \frac{\sum_{t=2}^{T}\left(\tilde{z}_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)\left(u_t - \frac{\sum_{j=t}^{T} u_j}{T-t+1}\right)}{\sqrt{\sum_{t=2}^{T}\left(\tilde{z}_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2 u_t^2}} + O_p\left(T^{-\eta/2}\right),$$

where $\tilde{z}_{t-1} = (1-\varrho L)_+^{-1} \Delta\tilde{x}_{t-1}$ with $\tilde{x}_t = \sum_{j=0}^{t-2} \rho^j \nu_{t-1-j}$.

Using arguments like those given in the proof of Lemma 2, we conclude that

$$\frac{\sum_{t=2}^{T}\left(z_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1} z_j\right)\left(u_t - \frac{\sum_{j=t}^{T} u_j}{T-t+1}\right)}{\sqrt{\sum_{t=2}^{T}\left(z_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1} z_j\right)^2 \widehat{u}_t^2}} = \frac{\sum_{t=2}^{T}\left(z_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1} z_j\right)\left(u_t - \frac{\sum_{j=t}^{T} u_j}{T-t+1}\right)}{\sqrt{\sum_{t=2}^{T}\left(z_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1} z_j\right)^2 u_t^2}} + O_p\left(T^{-1/2}\right).$$

Using $z_{t-1} = \psi\tilde{z}_{t-1} + r_{t-1}$ from Lemma 2, we have that

$$\begin{aligned}
&\frac{1}{T^{1+\eta}} \sum_{t=2}^{T}\left(z_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1} z_j\right)^2 u_t^2 = \\
&\quad = \frac{1}{T^{1+\eta}} \sum_{t=2}^{T}\left(\psi\tilde{z}_{t-1} + r_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1}\left(\psi\tilde{z}_j + r_j\right)\right)^2 u_t^2 \\
&\quad = \frac{\psi^2}{T^{1+\eta}} \sum_{t=2}^{T}\left(\tilde{z}_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2 u_t^2 + \frac{1}{T^{1+\eta}} \sum_{t=2}^{T}\left(r_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1} r_j\right)^2 u_t^2 \\
&\qquad + \frac{2\psi}{T^{1+\eta}} \sum_{t=2}^{T}\left(\tilde{z}_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)\left(r_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1} r_j\right) u_t^2. \\
&\quad = \frac{\psi^2}{T^{1+\eta}} \sum_{t=2}^{T}\left(\tilde{z}_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2 u_t^2 + A_{1T} + A_{2T}.
\end{aligned}$$

Examine A_{1T} first,

$$A_{1T} = \frac{1}{T^{1+\eta}}\sum_{t=2}^{T} r_{t-1}^2 u_t^2 + \frac{1}{T^{1+\eta}}\sum_{t=2}^{T}\left(\frac{1}{t-1}\sum_{j=1}^{t-1} r_j\right)^2 u_t^2 - \frac{2}{T^{1+\eta}}\sum_{t=2}^{T}\frac{r_{t-1}}{t-1}\sum_{j=1}^{t-1} r_j u_t^2.$$

Thanks to the independence of r_{t-1} and u_t and since r_t is uniformly L_4-bounded, we have

$$\mathrm{E}\left(\left|\frac{1}{T^{1+\eta}}\sum_{t=2}^{T} r_{t-1}^2 u_t^2\right|\right) \leq \frac{C}{T^{1+\eta}}\sum_{t=2}^{T}\sqrt{\mathrm{E}\left(r_{t-1}^4\right)}\,\mathrm{E}\left(u_t^2\right) = O\left(T^{-\eta}\right).$$

For the second term of A_{1T}, note that

$$\begin{aligned}\mathrm{E}\left(\frac{1}{T^{1+\eta}}\sum_{t=2}^{T}\left(\frac{1}{t-1}\sum_{j=1}^{t-1} r_j\right)^2 u_t^2\right) &= \frac{\sigma_u^2}{T^{1+\eta}}\sum_{t=2}^{T}\frac{1}{(t-1)^2}\,\mathrm{E}\left(\left(\sum_{j=1}^{t-1} r_j\right)^2\right)\\ &\leq \frac{C}{T^{\eta}},\end{aligned}$$

while, for the last term, we have

$$\begin{aligned}\frac{2}{T^{1+\eta}}\,\mathrm{E}\left(\left|\sum_{t=2}^{T}\frac{r_{t-1}}{t-1}\sum_{j=1}^{t-1} r_j u_t^2\right|\right) &\leq \frac{2\sigma_u^2}{T^{1+\eta}}\sum_{t=2}^{T}\frac{1}{t-1}\sqrt{\mathrm{E}\left(r_{t-1}^2\right)}\sqrt{\mathrm{E}\left(\left(\sum_{j=1}^{t-1} r_j\right)^2\right)}\\ &= O\left(T^{-\eta}\right).\end{aligned}$$

Moving on to A_{2T}, write

$$\begin{aligned}\mathrm{E}(|A_{2T}|) \leq\ & \frac{2\psi}{T^{1+\eta}}\,\mathrm{E}\left(\left|\sum_{t=2}^{T}\tilde{z}_{t-1} r_{t-1} u_t^2\right|\right) + \frac{2\psi}{T^{1+\eta}}\,\mathrm{E}\left(\left|\sum_{t=2}^{T}\frac{\tilde{z}_{t-1} u_t^2}{t-1}\sum_{j=1}^{t-1} r_j\right|\right)\\ & + \frac{2\psi}{T^{1+\eta}}\,\mathrm{E}\left(\left|\sum_{t=2}^{T}\frac{r_{t-1} u_t^2}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right|\right) + \frac{2\psi}{T^{1+\eta}}\,\mathrm{E}\left(\left|\sum_{t=2}^{T}\frac{1}{(t-1)^2}\sum_{j=1}^{t-1}\tilde{z}_j\sum_{j=1}^{t-1} r_j u_t^2\right|\right)\\ \leq\ & \frac{2\psi}{T^{1+\eta}}\sum_{t=2}^{T}\sqrt{\mathrm{E}\left(\tilde{z}_{t-1}^2\right)\mathrm{E}\left(r_{t-1}^2\right)}\,\mathrm{E}(u_t^2) + \frac{2\psi}{T^{1+\eta}}\sum_{t=2}^{T}\sqrt{\mathrm{E}\left(\tilde{z}_{t-1}^2\right)\mathrm{E}(u_t^4)}\sqrt{\frac{\mathrm{E}\left(\left(\sum_{j=1}^{t-1} r_j\right)^2\right)}{(t-1)^2}}\\ & + \frac{2\psi}{T^{1+\eta}}\sum_{t=2}^{T}\sqrt{\mathrm{E}(r_{t-1}^2)\,\mathrm{E}(u_t^4)}\sqrt{\frac{\mathrm{E}\left(\left(\sum_{j=1}^{t-1}\tilde{z}_j\right)^2\right)}{(t-1)^2}}\\ & + \frac{2\psi}{T^{1+\eta}}\sum_{t=2}^{T}\sqrt{\frac{\mathrm{E}\left(\left(\sum_{j=1}^{t-1}\tilde{z}_j\right)^2\right)}{(t-1)^2}\frac{\mathrm{E}\left(\left(\sum_{j=1}^{t-1} r_j\right)^2\right)}{(t-1)^2}\,\mathrm{E}(u_t^4)}\\ =\ & O\left(T^{-\eta/2}\right)\end{aligned}$$

where we have everywhere used the L_4 boundedness and also $\mathrm{Var}\left(\sum_{j=1}^{t-1}\tilde{z}_j\right) \leq CtT^{2\eta}$ which can easily be established using the expansion for $\tilde{z}_{t-1}$ given in the proof of Proposition 1. Therefore

$$\frac{1}{T^{1+\eta}}\sum_{t=2}^{T}\left(z_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}z_j\right)^2 u_t^2=\frac{\psi^2}{T^{1+\eta}}\sum_{t=2}^{T}\left(\tilde{z}_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2 u_t^2+O_p\left(T^{-\eta/2}\right). \quad (S.6)$$

Furthermore

$$\begin{aligned}
&\frac{1}{T^{1/2+\eta/2}}\sum_{t=2}^{T}\left(z_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}z_j\right)\left(u_t-\frac{\sum_{j=t}^{T}u_j}{T-t+1}\right)\\
&\quad=\frac{1}{T^{1/2+\eta/2}}\sum_{t=2}^{T}\left(\psi\tilde{z}_{t-1}+r_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}\left(\psi\tilde{z}_j+r_j\right)\right)\left(u_t-\frac{\sum_{j=t}^{T}u_j}{T-t+1}\right)\\
&\quad=\frac{1}{T^{1/2+\eta/2}}\psi\sum_{t=2}^{T}\left(\tilde{z}_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)\left(u_t-\frac{\sum_{j=t}^{T}u_j}{T-t+1}\right)\\
&\qquad+\frac{1}{T^{1/2+\eta/2}}\sum_{t=2}^{T}\left(r_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}r_j\right)\left(u_t-\frac{\sum_{j=t}^{T}u_j}{T-t+1}\right).
\end{aligned}$$

We now re-arrange the sum terms to exploit the serial independence of u_t, leading after some algebra exploiting the L_r boundedness of r_t to

$$\mathrm{Var}\left(\frac{1}{T^{1/2+\eta/2}}\sum_{t=2}^{T}\left(r_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}r_j\right)\left(u_t-\frac{\sum_{j=t}^{T}u_j}{T-t+1}\right)\right)=O\left(T^{-\eta}\right)$$

and therefore

$$\begin{aligned}
&\frac{1}{T^{1/2+\eta/2}}\sum_{t=2}^{T}\left(z_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}z_j\right)\left(u_t-\frac{\sum_{j=t}^{T}u_j}{T-t+1}\right)\\
&\quad=\frac{1}{T^{1/2+\eta/2}}\psi\sum_{t=2}^{T}\left(\tilde{z}_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)\left(u_t-\frac{\sum_{j=t}^{T}u_j}{T-t+1}\right)+O_p\left(T^{-\eta/2}\right). \quad (S.7)
\end{aligned}$$

The desired representation follows from equations (S.6) and (S.7).

Write now

$$\begin{aligned}
\frac{\sum_{t=2}^{T}\left(\tilde{z}_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)\left(u_t-\frac{\sum_{j=t}^{T}u_j}{T-t+1}\right)}{\sqrt{\sum_{t=2}^{T}\left(\tilde{z}_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2 u_t^2}} &= \frac{\sum_{t=2}^{T}\tilde{z}_{t-1}u_t}{\sqrt{D_T}}-\frac{\sum_{t=2}^{T}\frac{1}{T-t+1}\tilde{z}_{t-1}\sum_{j=t}^{T}u_j}{\sqrt{D_T}}\\
&\quad-\frac{\sum_{t=2}^{T}\frac{1}{t-1}u_t\sum_{j=1}^{t-1}\tilde{z}_j}{\sqrt{D_T}}+\frac{\sum_{t=2}^{T}\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\frac{1}{T-t+1}\sum_{j=t}^{T}u_j}{\sqrt{D_T}}\\
&= \frac{\sum_{t=2}^{T}\tilde{z}_{t-1}u_t}{\sqrt{D_T}}+\frac{C_{1,T}}{\sqrt{D_T}}+\frac{C_{2,T}}{\sqrt{D_T}}+\frac{C_{3,T}}{\sqrt{D_T}}.
\end{aligned}$$

First we look at the variance of $C_{1,T} = \sum_{t=2}^{T} \frac{1}{T-t+1} \tilde{z}_{t-1} \sum_{j=t}^{T} u_j$:

$$\begin{aligned}
\mathrm{Var}\left(C_{1,T}\right) &= \mathrm{Var}\left(\sum_{j=2}^{T} u_j \sum_{t=2}^{j} \frac{\tilde{z}_{t-1}}{T-t+1}\right) \\
&\leq \max_{s\in[0,1]} \sigma_u^2(s) \sum_{j=2}^{T} \mathrm{Var}\left(\sum_{t=2}^{j} \frac{\tilde{z}_{t-1}}{T-t+1}\right) \\
&= \max_{s\in[0,1]} \sigma_u^2(s) \sum_{j=2}^{T} \mathrm{Var}\left(\sum_{t=2}^{j} \sum_{k=1}^{t-1} \frac{c_{k,t-1}}{T-t+1} \nu_k\right)
\end{aligned}$$

with $c_{k,t}$ from the proof of Proposition 1. Hence

$$\begin{aligned}
\mathrm{Var}\left(C_{1,T}\right) &\leq \max_{s\in[0,1]} \sigma_u^2(s) \sum_{j=2}^{T} \mathrm{Var}\left(\sum_{k=1}^{j-1} \nu_k \sum_{t=k+1}^{j} \frac{c_{k,t-1}}{T-t+1}\right) \\
&= \max_{s\in[0,1]} \sigma_u^2(s) \max_{s\in[0,1]} \sigma_\nu^2(s) \sum_{j=2}^{T} \sum_{k=1}^{j-1} \left(\sum_{t=k+1}^{j} \frac{c_{k,t-1}}{T-t+1}\right)^2 \\
&= O\left(\left(\frac{1-\varrho}{\rho-\varrho}\right)^2 \sum_{j=2}^{T} \sum_{k=1}^{j-1} \left(\sum_{t=k+1}^{j} \frac{\varrho^{t-1-k}}{T-t+1}\right)^2\right).
\end{aligned}$$

With

$$\begin{aligned}
\sum_{j=2}^{T} \sum_{k=1}^{j-1} \left(\sum_{t=k+1}^{j} \frac{\varrho^{t-1-k}}{T-t+1}\right)^2 &\leq \sum_{j=2}^{T} \sum_{k=1}^{j-1} \left(\sum_{t=1}^{T} \frac{\varrho^{t-1-k}}{T-t+1}\right)^2 \\
&= \sum_{j=2}^{T} \sum_{k=1}^{j-1} \varrho^{-k} \left(\sum_{t=1}^{T} \frac{\varrho^{t-1}}{T-t+1}\right)^2 \\
&= \sum_{j=2}^{T} \sum_{k=1}^{j-1} \varrho^{-k} \left(\varrho^{T} \sum_{t=1}^{T-1} \frac{\varrho^{-t}}{t} + \frac{1}{T}\right)^2 \\
&= \sum_{j=2}^{T} \sum_{k=1}^{j-1} \varrho^{-k} \left(\varrho^{T} \frac{1}{T} \sum_{t=1}^{T-1} \frac{\left(\varrho^{T}\right)^{-t/T}}{t/T} + \frac{1}{T}\right)^2 \\
&= \sum_{j=2}^{T} \sum_{k=1}^{j-1} \varrho^{-k} \left(\varrho^{T} O\left(\int_{1/T}^{1-1/T} \frac{\left(\varrho^{-T}\right)^{x}}{x}\right) + \frac{1}{T}\right)^2,
\end{aligned}$$

and, with $\mathrm{Ei}\left(x\right) = \int_x^\infty \frac{\exp(-z)}{z} \mathrm{d}z$,

$$\begin{aligned}
\int_{1/T}^{1-1/T} \frac{\left(\varrho^{-T}\right)^{x}}{x} \mathrm{d}x &= \mathrm{Ei}\left(\left(1-\frac{1}{T}\right)\log\left(\varrho^{-T}\right)\right) - \mathrm{Ei}\left(\frac{1}{T}\log\left(\varrho^{-T}\right)\right) \\
&= O\left(\log T\right),
\end{aligned}$$

leading to

$$\begin{aligned}
&\sum_{j=2}^{T}\sum_{k=1}^{j-1}\varrho^{-k}\left(\varrho^{T}O\left(\int_{1/T}^{1-1/T}\frac{\left(\varrho^{-T}\right)^{x}}{x}\right)+\frac{1}{T}\right)^{2}\\
=\;& O\left(\log^{2}T\right)\sum_{j=2}^{T}\sum_{k=1}^{j-1}\varrho^{2T-k}\\
=\;& O\left(\log^{2}T\right)\sum_{j=2}^{T}\frac{\varrho^{2T}-\varrho^{2T-j+1}}{\varrho-1}\\
=\;& O\left(\log^{2}T\right)\left(\frac{(T-1)\,\varrho^{2T}}{\varrho-1}+\frac{\varrho}{\varrho-1}\frac{\varrho^{2T-1}-\varrho^{T}}{\varrho-1}\right)\\
=\;& O\left(Te^{-aT^{1-\eta}}\log^{2}T\right),
\end{aligned}$$

which in turn implies that

$$T^{-1/2-\eta}C_{1,T}=o\left(1\right).$$

Now we turn to the variance of $C_{2,T}=\sum_{t=2}^{T}\frac{1}{t-1}u_{t}\sum_{j=1}^{t-1}\tilde{z}_{j}$:

$$\begin{aligned}
\operatorname{Var}\left(C_{2,T}\right) &= \sum_{t=2}^{T}\operatorname{Var}\left(\sum_{j=1}^{t-1}\frac{1}{t-1}u_{t}\tilde{z}_{j}\right)\\
&\leq \max_{s\in[0,1]}\sigma_{u}^{2}(s)\sum_{t=2}^{T}\frac{1}{(t-1)^{2}}\operatorname{E}\left[\left(\sum_{j=1}^{t-1}\sum_{k=1}^{j}c_{k,j}\nu_{k}\right)^{2}\right]\\
&\leq \max_{s\in[0,1]}\sigma_{u}^{2}(s)\max_{s\in[0,1]}\sigma_{\nu}^{2}(s)\sum_{t=2}^{T}\frac{1}{(t-1)^{2}}\sum_{k=1}^{t-1}\left(\sum_{j=k}^{t-1}c_{k,j}\right)^{2}
\end{aligned}$$

like above. Since $\sum_{j=k}^{t-1}c_{k,j}=\frac{\rho^{t-k}-\varrho^{t-k}}{\rho-\varrho}$, we obtain

$$\begin{aligned}
\sum_{k=1}^{t-1}\left(\sum_{j=k}^{t-1}c_{k,j}\right)^{2} &= \sum_{k=1}^{t-1}\left(\frac{\rho^{t-k}-\varrho^{t-k}}{\rho-\varrho}\right)^{2}\\
&= \frac{1}{(\rho-\varrho)^{2}}\sum_{k=1}^{t-1}\left(\rho^{2t-2k}+\varrho^{2t-2k}-2\left(\rho\varrho\right)^{t-k}\right)\\
&= \frac{1}{(\rho-\varrho)^{2}}\left(\frac{\rho^{2}-\rho^{2t}}{1-\rho^{2}}+\frac{\varrho^{2}-\varrho^{2t}}{1-\varrho^{2}}-2\frac{(\varrho\rho)^{2}-(\rho\varrho)^{2t}}{1-(\rho\varrho)^{2}}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{t=2}^{T}\frac{1}{(t-1)^{2}}\frac{\rho^{2-2t}}{(\rho-\varrho)^{2}} &= \frac{\rho^{2}}{(\rho-\varrho)^{2}}\frac{1}{T^{2}}\sum_{t=1}^{T-1}\frac{\left(\rho^{-2T}\right)^{t/T}}{(t/T)^{2}}\\
&= O\left(T^{2\eta-1}\int_{1/T}^{1-1/T}\frac{\left(\rho^{-2T}\right)^{x}}{x^{2}}dx\right)\\
&= O\left(T^{2\eta}\right),
\end{aligned}$$

where

$$\begin{aligned}\int_{1/T}^{1-1/T} \frac{\left(\rho^{-2T}\right)^x}{x^2} dx &= \log\left(\rho^{-2T}\right) \mathrm{Ei}\left(x \log\left(\rho^{-2T}\right)\right) - \frac{\rho^{-2Tx}}{x}|_{1/T}^{1-1/T} \\ &= \log\left(\rho^{-2T}\right) \mathrm{Ei}\left(\left(1-\frac{1}{T}\right)\log\left(\rho^{-2T}\right)\right) - \frac{T\rho^{-2(T-1)}}{T-1} \\ &\quad - \log\left(\rho^{-2T}\right) \mathrm{Ei}\left(\frac{1}{T}\log\left(\rho^{-2T}\right)\right) - T\rho^{-2} \\ &= O\left(T\right).\end{aligned}$$

Also we have that

$$\begin{aligned}\int_{1/T}^{1-1/T} \frac{\left(\varrho^{-2T}\right)^x}{x^2} dx &= \log\left(\varrho^{-2T}\right) \mathrm{Ei}\left(\left(1-\frac{1}{T}\right)\log\left(\varrho^{-2T}\right)\right) - \frac{T\varrho^{-2(T-1)}}{T-1} \\ &\quad - \log\left(\varrho^{-2T}\right) \mathrm{Ei}\left(\frac{1}{T}\log\left(\varrho^{-2T}\right)\right) - T\varrho^{-2} \\ &= O\left(T\right)\end{aligned}$$

The other term is of the same order, implying that

$$T^{-\eta-1/2} C_{2,T} = O_p\left(T^{-1/2}\right).$$

Finally, we look at the variance of $C_{3,T} = \sum_{t=2}^{T} \frac{1}{t-1} \sum_{j=1}^{t-1} \tilde{z}_j \frac{1}{T-t+1} \sum_{j=t}^{T} u_j$:

$$\mathrm{Var}\left(C_{3,T}\right) \le \max_{s\in[0,1]} \sigma_u^2(s) \max_{s\in[0,1]} \sigma_\nu^2(s) \sum_{t=2}^{T} \frac{1}{(t-1)^2} \mathrm{E}\left[\left(\sum_{j=1}^{t-1} \tilde{z}_j\right)^2\right],$$

which can be treated as $C_{2,T}$, leading to

$$T^{-\eta-1/2} C_{3,T} = O_p\left(T^{-1/2}\right).$$

For D_T, appearing in the denominator we have

$$\begin{aligned}T^{-\eta-1} D_T &= T^{-\eta-1} \sum_{t=2}^{T} \left(\tilde{z}_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1} \tilde{z}_j\right)^2 \left(u_t^2 - \sigma_u^2(t/T)\right) \\ &\quad + T^{-\eta-1} \sum_{t=2}^{T} \left(\tilde{z}_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1} \tilde{z}_j\right)^2 \sigma_u^2(t/T) \\ &= D_{1,T} + D_{2,T}\end{aligned}$$

Examine

$$D_{2,T} = T^{-\eta-1} \sum_{t=2}^{T} \left(\tilde{z}_{t-1} - \frac{1}{t-1}\sum_{j=1}^{t-1} \tilde{z}_j\right)^2 \sigma_u^2\left(\frac{t}{T}\right)$$

$$= \frac{1}{T^{1+\eta}}\sum_{t=2}^{T}\tilde{z}_{t-1}^{2}\sigma_u^2\left(\frac{t}{T}\right) + \frac{1}{T^{1+\eta}}\sum_{t=2}^{T}\left(\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2\sigma_u^2\left(\frac{t}{T}\right) - \frac{2}{T^{1+\eta}}\sum_{t=2}^{T}\frac{\tilde{z}_{t-1}}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\sigma_u^2\left(\frac{t}{T}\right).$$

It is not difficult to show (using for example arguments analogous to the derivations for $C_{1,T}$) that, for all $2 \leq t \leq T$ and a suitable constant C, $\mathrm{Var}\left(\sum_{j=1}^{t-1}\tilde{z}_j\right) \leq CtT^{2\eta}$. Therefore,

$$\mathrm{E}\left(\left|T^{-\eta-1}\sum_{t=2}^{T}\left(\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2\sigma_u^2\left(\frac{t}{T}\right)\right|\right) \leq \frac{C}{T^{\eta+1}}\sum_{t=2}^{T}\mathrm{Var}\left(\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right) \leq \frac{C}{T^{1-\eta}}\sum_{t=2}^{T}\frac{1}{t-1}$$

and, thanks to Markov's inequality,

$$T^{-\eta-1}\sum_{t=2}^{T}\left(\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2\sigma_u^2\left(\frac{t}{T}\right) = O_p\left(T^{\eta-1}\log T\right).$$

Moreover, recall that $T^{-\eta/2}\tilde{z}_{t-1}$ is uniformly L_4-bounded, hence, with the Cauchy-Schwarz inequality, we have that

$$\mathrm{E}\left(\left|\frac{\tilde{z}_{t-1}}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right|\right) \leq \frac{CT^{\eta/2}}{t-1}\sqrt{\mathrm{E}\left(\left(\sum_{j=1}^{t-1}\tilde{z}_j\right)^2\right)} \leq \frac{CT^{3\eta/2}}{\sqrt{t}}$$

and

$$\mathrm{E}\left(\left|T^{-\eta-1}\sum_{t=2}^{T}\frac{\tilde{z}_{t-1}}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\sigma_u^2\left(\frac{t}{T}\right)\right|\right) \leq \frac{CT^{3\eta/2}}{T^{\eta+1}}\sum_{t=2}^{T}\frac{1}{\sqrt{t}} = CT^{\eta/2-1/2}.$$

Furthermore, it is not difficult to show that $D_{1,T} \xrightarrow{p} 0$ thanks to the md property of the summands. We thus have that

$$T^{-\eta-1}D_T \xrightarrow{p} \frac{1}{2a}\int_0^1 \sigma_u^2(s)\sigma_\nu^2(s)\mathrm{d}s$$

and consequently

$$\frac{C_{1,T}}{\sqrt{D_T}} + \frac{C_{2,T}}{\sqrt{D_T}} + \frac{C_{3,T}}{\sqrt{D_T}} = O_p\left(T^{-1/2}\right).$$

We are now left to consider the asymptotic behavior of $\frac{\sum_{t=2}^{T}\tilde{z}_{t-1}u_t}{\sqrt{D_T}}$. To do so we follow the proof of Proposition 1 and write

$$\frac{T^{-1/2-\eta/2}\sum_{t=2}^{T}\tilde{z}_{t-1}u_t}{\sqrt{T^{-1-\eta}\sum_{t=2}^{T}\left(\tilde{z}_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2u_t^2}} = Z_T + B_T + R_T$$

where Z_T and B_T are the same as equations (8) and (9) which have been treated in the proof of Proposition 1, and

$$R_T = \frac{3}{4}\frac{T^{-1/2-\eta/2}\sum_{t=2}^{T}\tilde{z}_{t-1}u_t}{\sqrt{\xi_T^5}}\left(T^{-1-\eta}\sum_{t=2}^{T}\left(\tilde{z}_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2u_t^2 - \frac{1}{2a}\int_0^1\sigma_u^2(s)\sigma_\nu^2(s)\mathrm{d}s\right)^2$$

$$-\frac{T^{-1/2-\eta/2}\sum_{t=2}^{T}\tilde{z}_{t-1}u_t}{2\sqrt{\left(\frac{1}{2a}\int_0^1\sigma_u^2(s)\sigma_\nu^2(s)\mathrm{d}s\right)^3}}\frac{1}{T^{1+\eta}}\sum_{t=2}^{T}\left(\left(\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2 u_t^2-\frac{2u_t^2\tilde{z}_{t-1}}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)$$

with ξ_T between $\frac{1}{2a}\int_0^1\sigma_u^2(s)\sigma_\nu^2(s)\mathrm{d}s$ and $T^{-1-\eta}\sum_{t=2}^{T}\left(\tilde{z}_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2 u_t^2$, i.e. $\xi_T=O_p(1)$. The term B_T vanishes due to the component $\left(T^{-1-\eta}\sum_{t=2}^{T}\left(\tilde{z}_{t-1}-\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2 u_t^2-\frac{1}{2a}\int_0^1\sigma_u^2(s)\sigma_\nu^2(s)\mathrm{d}s\right)$, appearing squared in the first summand of R_T which is therefore, in turn, dominated by B_T. To analyze the second summand, examine in turn

$$\mathrm{E}\left(\left|\left(\frac{1}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right)^2 u_t^2\right|\right)\leq\frac{CT^{2\eta}}{t-1}$$

thanks to independence of $\tilde{z}_j$ and u_t for $j\leq t-1$, and, like above,

$$\mathrm{E}\left(\left|\frac{2u_t^2\tilde{z}_{t-1}}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right|\right)\leq\frac{C}{t-1}\mathrm{E}\left(\left|\tilde{z}_{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right|\right)=\frac{C}{t-1}\sum_{j=1}^{t-1}\mathrm{E}\left(|\tilde{z}_{t-1}\tilde{z}_j|\right).$$

Some algebra indicates that

$$\mathrm{E}\left(|\tilde{z}_{t-1}\tilde{z}_j|\right)\leq CT^{\eta}\varrho^{t-j-1}$$

such that, summing up,

$$\mathrm{E}\left(\left|\frac{1}{T^{1+\eta}}\sum_{t=2}^{T}\frac{2u_t^2\tilde{z}_{t-1}}{t-1}\sum_{j=1}^{t-1}\tilde{z}_j\right|\right)\leq\frac{CT^{\eta}}{T^{1+\eta}}\sum_{t=2}^{T}\frac{1}{t-1}\sum_{j=1}^{t-1}\varrho^{t-j-1}=\frac{C}{T}\sum_{t=2}^{T}\frac{1}{t-1}\frac{1-\varrho^t}{1-\varrho}=O\left(T^{\eta-1}\log T\right)$$

as required for $R_T=o_p\left(T^{\eta/2-1/2}\right)$. The result follows from the proof of Proposition 1.

II A more detailed comparison of t_{vx} and t_{vx}^W

The test statistics t_{vx} and t_{vx}^W have the same leading terms, but differ with respect to some higher-order ones. Concretely, t_{vx}^W has the additional term

$$\frac{1}{2}\frac{\frac{1}{T^{1/2+\eta/2}}\sum_{t=2}^{T}z_{t-1}\left(u_t-\bar{u}\right)}{\sqrt{\zeta_T}}\left(T^{-\eta}\bar{z}^2\hat{\omega}_{u|v}^2\right)$$

with $\zeta_T=\frac{1}{T^{1+\eta}}\sum_{t=2}^{T}z_{t-1}^2\hat{u}_t^2+o_p(1)$, but, as can be seen from the proof of Proposition 3, does not exhibit

$$-\frac{1}{2}\frac{1}{\sqrt{\left(\frac{1}{2a}\int_0^1\sigma_u^2(s)\sigma_\nu^2(s)\mathrm{d}s\right)^3}}\left(\frac{1}{T^{1/2+\eta/2}}\sum_{t=2}^{T}\tilde{z}_{t-1}u_t\right)\left(\frac{-2\bar{\tilde{z}}}{T^{1+\eta}}\sum_{t=2}^{T}\tilde{z}_{t-1}u_t^2+\frac{\bar{\tilde{z}}^2}{T^{1+\eta}}\sum_{t=2}^{T}u_t^2\right)$$

and, since $\xi_T, \xi_T^W = \frac{1}{2a}\int_0^1 \sigma_u^2(s)\sigma_\nu^2(s)\mathrm{d}s + o_p(1)$, it also does not exhibit

$$\frac{3}{4}\frac{1}{\sqrt{\left(\frac{1}{2a}\int_0^1 \sigma_u^2(s)\sigma_\nu^2(s)\mathrm{d}s\right)^5}}\left(\frac{1}{T^{1/2+\eta/2}}\sum_{t=2}^{T}\tilde{z}_{t-1}u_t\right)\left(\frac{-2\bar{\tilde{z}}}{T^{1+\eta}}\sum_{t=2}^{T}\tilde{z}_{t-1}u_t^2 + \frac{\bar{\tilde{z}}^2}{T^{1+\eta}}\sum_{t=2}^{T}u_t^2\right)^2.$$

One may use arguments from the proof of Proposition 3 to show them be of order $O_p(T^{\eta-1})$ and thus, for η close to unity, they will also matter in finite samples alongside B_T and C_T (as suggested by the comparison of Figure 1 with Figure 2 for $c = 0$). An exact derivation of the expectation of the above terms is however beyond the scope of this paper.

III Further Monte Carlo results

In this section we provide further Monte Carlo results to investigate the effects of conditional heteroskedasticity on the behavior of different testing procedures. In particular, we consider a DGP based on

$$y_t = \mu + \beta x_{t-1} + u_t, \quad t = 2, \ldots, T, \tag{S.8}$$

where

$$x_t = \rho x_{t-1} + v_t, \tag{S.9}$$

with $v_t = \phi v_t + \nu_t$ for $\phi = 0.5$ and $u_t = \sigma_t \varepsilon_t$ with $\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \gamma_1 \sigma_{t-1}^2$, where $(\varepsilon_t, \nu_t) \sim iiN(0, \Sigma)$ when the diagonal elements of Σ are identity and the correlation coefficient is $\delta = -0.95$. For the GARCH(1,1) specifications we choose $\alpha_1 = 0.1$ and $\gamma_1 \in \{0.5, 0.7, 0.8, 0.85, 0.89\}$ with $\alpha_0 = 1 - \alpha_1 - \gamma_1$. Further, we set $\rho = 1 - c/T$ for various $c \in \{0, 1, 5, 10, 30, 50\}$ and $\varrho = 1 - 1/T^{0.95}$ and use standard normal critical values. Finally, we consider $T = 250$ and 500 with 10^4 replications for the Monte Carlo. Table S.1 shows the MC results which speak of a good size control when using the t-ratios proposed in this paper: t_{vx}^{rec} or t_{vx}^*. While the size figures for the two-sided versions are generally very close to the nominal value of 5% for all the three test statistics considered here, the size figures for t_{vx}^W show some deviations in the left and right tails. For $T = 500$, t_{vx}^W is undersized in the left tail for $c = 0$, 1, 5 or 10 with values ranging from 0.04% to 1.00%. The size figures for the left tail of t_{vx}^W increase to around 3% when $c = 50$. On the other hand t_{vx}^W is over-sized in the right tail whether c is small or large. All in all, the differences to Table 1 are not essential.

References

Phillips, P. C. B. and V. Solo (1992). Asymptotics for linear processes. *The Annals of Statistics 20*(2), 971–1001.

Table S.1: Size properties of different tests under short-run dynamics and a GARCH(1,1) specification for u_t in (1)

		T = 250												T = 500											
		2-sided				left-sided				right-sided				2-sided				left-sided				right-sided			
c	γ_1	t_{vx}	t_{vx}^{rec}	t_{vx}^{W}	t_{vx}^{*}	t_{vx}	t_{vx}^{rec}	t_{vx}^{W}	t_{vx}^{*}	t_{vx}	t_{vx}^{rec}	t_{vx}^{W}	t_{vx}^{*}	t_{vx}	t_{vx}^{rec}	t_{vx}^{W}	t_{vx}^{*}	t_{vx}	t_{vx}^{rec}	t_{vx}^{W}	t_{vx}^{*}	t_{vx}	t_{vx}^{rec}	t_{vx}^{W}	t_{vx}^{*}
	0.50	21.03	5.02	4.53	4.68	0.13	3.21	0.07	4.06	33.00	6.67	8.99	5.57	20.50	4.94	4.31	4.19	0.13	2.99	0.08	4.08	32.17	6.80	8.81	5.20
	0.70	21.70	4.73	4.67	4.59	0.13	3.10	0.05	4.09	33.52	6.83	9.12	5.57	20.47	4.35	4.40	4.16	0.11	3.05	0.04	3.69	32.46	6.29	8.59	5.17
0	0.80	20.29	4.82	4.77	4.56	0.14	3.02	0.06	4.30	32.45	6.72	8.84	5.50	20.69	5.45	4.51	4.50	0.13	3.08	0.09	3.92	32.59	6.87	8.76	5.58
	0.85	20.71	5.30	4.65	4.69	0.11	3.19	0.08	4.34	32.60	7.41	9.12	5.70	20.17	4.75	4.46	4.66	0.11	2.84	0.09	3.96	32.01	6.93	8.93	5.67
	0.89	20.89	5.24	5.10	4.65	0.16	2.84	0.07	4.05	32.32	7.32	9.53	5.84	19.77	5.28	4.47	4.15	0.20	3.51	0.17	4.05	30.73	6.84	8.95	5.12
	0.50	16.13	4.70	5.32	5.65	0.26	2.92	0.15	5.25	26.15	6.73	9.97	6.13	15.26	4.98	4.40	6.02	0.29	3.18	0.16	6.02	24.87	6.95	8.99	5.79
	0.70	15.35	4.70	4.80	5.36	0.23	3.16	0.15	5.55	25.81	6.58	9.77	5.94	15.77	5.26	4.72	5.43	0.26	3.14	0.14	5.38	25.06	7.20	9.33	5.80
1	0.80	15.57	4.69	4.68	5.07	0.27	3.02	0.16	5.35	24.70	6.79	9.36	5.48	16.02	4.88	4.70	5.69	0.32	2.84	0.21	5.99	25.49	7.03	9.24	5.80
	0.85	15.40	5.51	4.98	5.51	0.21	3.29	0.10	5.41	24.42	7.20	9.48	5.83	14.96	4.83	4.52	5.63	0.26	3.11	0.12	5.90	24.12	6.50	9.23	5.59
	0.89	15.78	5.01	5.27	5.82	0.28	3.26	0.14	5.50	25.00	7.19	10.54	5.84	14.60	5.29	4.84	6.26	0.24	3.24	0.20	6.10	23.39	6.87	9.40	5.88
	0.50	10.02	5.17	6.08	5.60	1.12	3.11	0.87	4.17	15.92	7.10	10.62	6.20	9.27	4.99	5.00	4.96	1.17	3.20	0.99	4.26	14.64	6.79	9.42	5.40
	0.70	9.82	5.13	5.60	5.50	1.27	3.04	1.01	4.67	15.54	7.23	10.22	5.77	8.97	5.05	4.97	5.12	1.03	3.43	0.76	4.53	14.93	6.61	9.33	5.60
5	0.80	9.44	4.89	5.22	5.04	1.08	3.22	0.80	4.11	15.59	6.69	10.14	5.54	9.34	5.07	5.57	5.06	1.13	3.14	0.82	4.26	14.86	6.91	9.93	5.67
	0.85	9.47	4.90	5.79	5.23	1.31	3.05	1.03	4.40	14.75	6.74	9.76	5.86	9.07	5.11	5.32	5.49	1.27	3.09	0.98	4.80	14.69	6.72	9.63	5.74
	0.89	9.63	5.45	5.86	5.54	1.33	3.31	0.97	4.45	15.11	7.46	10.41	6.01	8.54	5.09	4.87	5.05	1.34	3.11	1.00	4.51	14.14	6.75	9.36	5.27
	0.50	7.70	4.86	5.87	3.84	1.92	3.28	1.72	2.98	12.18	6.76	9.62	5.35	7.91	5.05	5.99	3.92	2.02	3.68	1.82	3.08	12.09	6.53	9.48	5.52
	0.70	7.66	4.92	5.66	4.21	1.96	3.28	1.66	2.97	11.84	6.49	9.48	5.17	7.32	4.65	5.44	3.59	1.80	3.26	1.51	2.55	11.65	6.13	9.40	5.13
10	0.80	7.85	4.93	5.90	4.07	1.88	3.49	1.63	2.95	11.82	6.61	9.70	5.40	7.73	4.97	5.52	4.12	2.12	3.48	1.90	3.29	11.78	6.51	9.06	5.26
	0.85	7.77	4.56	5.70	4.16	2.00	3.06	1.76	3.00	11.87	6.23	9.60	5.13	7.42	4.78	5.60	4.08	1.94	3.29	1.61	2.92	11.50	6.66	9.29	5.15
	0.89	8.03	5.06	6.23	4.47	1.86	3.23	1.68	3.09	11.99	6.69	9.74	5.65	7.08	4.97	5.38	3.88	1.70	2.99	1.39	2.70	11.71	6.87	9.61	5.20
	0.50	6.29	5.22	5.74	3.96	2.96	3.78	2.80	3.18	9.21	6.38	8.68	5.25	5.97	4.94	5.42	3.69	2.97	3.74	2.80	3.09	8.66	6.17	8.14	4.93
	0.70	6.46	5.00	5.89	4.15	3.08	3.85	2.94	3.17	8.76	6.19	8.26	5.21	6.18	4.94	5.56	3.76	2.90	4.01	2.71	3.06	8.83	6.02	8.32	5.10
30	0.80	6.30	5.18	5.77	3.95	3.14	3.92	3.05	3.50	8.76	6.23	8.34	5.09	5.53	4.59	5.04	3.58	2.96	3.89	2.81	3.15	8.05	5.54	7.30	4.32
	0.85	6.70	5.04	6.26	4.20	3.11	3.77	2.96	3.34	9.23	5.98	8.75	5.45	6.46	4.92	5.89	4.22	3.10	3.90	2.89	3.19	8.85	6.32	8.22	5.23
	0.89	6.64	4.63	6.09	3.98	3.00	3.56	2.82	3.33	9.07	6.10	8.46	5.42	6.41	5.32	5.85	4.17	3.09	3.78	2.86	3.14	8.90	6.50	8.34	5.34
	0.50	6.42	4.99	6.03	4.20	3.51	3.91	3.40	3.63	8.63	6.24	8.28	5.57	6.29	5.23	5.99	4.35	3.58	4.09	3.48	3.69	7.98	6.00	7.80	5.40
	0.70	5.91	5.25	5.61	4.16	3.28	3.90	3.10	3.43	8.06	6.24	7.76	5.20	5.93	4.92	5.63	4.06	3.30	3.95	3.22	3.37	7.73	6.06	7.54	5.02
50	0.80	6.25	5.25	5.98	4.53	3.40	3.87	3.35	3.48	8.16	6.04	7.93	5.42	6.04	4.86	5.77	4.02	3.51	4.02	3.42	3.58	7.97	5.90	7.67	5.18
	0.85	5.92	5.03	5.68	4.38	3.52	3.99	3.41	3.69	7.67	5.81	7.36	4.95	5.97	4.99	5.68	4.13	3.31	3.97	3.24	3.44	7.69	5.72	7.39	4.88
	0.89	6.03	5.11	5.77	4.26	3.38	3.73	3.29	3.52	8.07	5.99	7.88	5.32	5.62	5.08	5.36	3.83	3.50	4.17	3.35	3.61	7.64	5.83	7.41	4.87

Note: Data generated with (S.8) and (S.9) with $v_t = \phi v_t + \nu_t$ for $\phi = 0.5$ and $u_t = \sigma_t \varepsilon_t$ with $\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \gamma_1 \sigma_{t-1}^2$, where $(\varepsilon_t, \nu_t) \sim iiN(0, \Sigma)$ when the diagonal elements of Σ are identity and the correlation coefficient is $\delta = -0.95$. We set $\rho = 1 - {}^{c}/_{T}$ for various c and $\varrho = 1 - {}^{1}/_{T^{0.95}}$ and use standard normal critical values. See text for details.