Supplementary Appendices of "Large Sample Properties of Bayesian Estimation of Spatial Econometric Models"

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These supplementary appendices consist of derivatives of the log-likelihood function of the SARAR model (Appendix C), a few technical details on the MCMC sampler for the high order SARAR and SAR Tobit models, as well as additional graphs and tables. In particular, Appendix D outlines the exchange algorithm to sample spatial parameters $\psi = (\lambda', \rho')'$ in the high order SARAR model. Appendix E presents the MCMC sampler for the SAR Tobit models. Appendix F summarizes our simulation designs and results for the graphs and tables reported in Appendix G (SARAR model) and Appendix H (SAR Tobit model).[30]

C. Derivatives of the Log-likelihood Function of the SARAR Model

A log likelihood component $l_{i,n}(\theta) = -\frac{\ln 2\pi\sigma^2}{2} + \frac{1}{n}\ln|S_n(\lambda)R_n(\rho)| - \frac{\{R_{i\cdot,n}(\rho)[S_n(\lambda)Y_n - X_n\beta]\}^2}{2\sigma^2}$. Let $m_{i\cdot,kn}$ be the ith row of M_{kn}.

First order derivatives

$$\begin{aligned}
\frac{\partial l_{i,n}(\theta)}{\partial\lambda_j} &= -\frac{1}{n}\operatorname{tr}\left[S_n(\lambda)^{-1}W_{jn}\right] + \frac{1}{\sigma^2}\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}\left[R_{i\cdot,n}(\rho)W_{jn}Y_n\right],\\
\frac{\partial l_{i,n}(\theta)}{\partial\beta} &= \frac{1}{\sigma^2}X_n' R_{i\cdot,n}(\rho)'\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\},\\
\frac{\partial l_{i,n}(\theta)}{\partial\sigma^2} &= -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}^2,\\
\frac{\partial l_{i,n}(\theta)}{\partial\rho_k} &= -\frac{1}{n}\operatorname{tr}\left[R_n(\rho)^{-1}M_{kn}\right] + \frac{1}{\sigma^2}\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}\left\{m_{i\cdot,kn}\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}.
\end{aligned}$$

Second order derivatives

$$\begin{aligned}
\frac{\partial^2 l_{i,n}(\theta)}{\partial\lambda_j\partial\lambda_k} &= -\frac{1}{n}\operatorname{tr}\left[S_n(\lambda)^{-1}W_{kn}S_n(\lambda)^{-1}W_{jn}\right] - \frac{1}{\sigma^2}\left[R_{i\cdot,n}(\rho)W_{kn}Y_n\right]\left[R_{i\cdot,n}(\rho)W_{jn}Y_n\right],\\
\frac{\partial^2 l_{i,n}(\theta)}{\partial\lambda_j\partial\beta} &= -\frac{1}{\sigma^2}X_n' R_{i\cdot,n}(\rho)'\left[R_{i\cdot,n}(\rho)W_{jn}Y_n\right],\\
\frac{\partial^2 l_{i,n}(\theta)}{\partial\lambda_j\partial\sigma^2} &= -\frac{1}{\sigma^4}\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}\left[R_{i\cdot,n}(\rho)W_{jn}Y_n\right],\\
\frac{\partial^2 l_{i,n}(\theta)}{\partial\lambda_j\partial\rho_k} &= -\frac{1}{\sigma^2}(W_{jn}Y_n)'\left[R_{i\cdot,n}(\rho)'m_{i\cdot,kn} + m_{i\cdot,kn}'R_{i\cdot,n}(\rho)\right]\left[S_n(\lambda)Y_n - X_n\beta\right],\\
\frac{\partial^2 l_{i,n}(\theta)}{\partial\beta\partial\beta'} &= -\frac{1}{\sigma^2}X_n' R_{i\cdot,n}(\rho)' R_{i\cdot,n}(\rho)X_n,
\end{aligned}$$

[30] There are numerous graphs for different simulation settings and most of these graphs are quite similar. To keep the supplementary appendices at a reasonable length, we follow the suggestion of the editor to only present some representative graphs.

$$\frac{\partial^2 l_{i,n}(\theta)}{\partial\beta\partial\sigma^2} = -\frac{1}{\sigma^4} X_n' R_{i\cdot,n}(\rho)' \left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\},$$
$$\frac{\partial^2 l_{i,n}(\theta)}{\partial\beta\partial\rho_k} = -\frac{1}{\sigma^2} X_n' \left[R_{i\cdot,n}(\rho)' m_{i\cdot,kn} + m_{i\cdot,kn}' R_{i\cdot,n}(\rho)\right]\left[S_n(\lambda)Y_n - X_n\beta\right],$$
$$\frac{\partial^2 l_{i,n}(\theta)}{(\partial\sigma^2)^2} = \frac{1}{2\sigma^4} - \frac{1}{\sigma^6}\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}^2,$$
$$\frac{\partial^2 l_{i,n}(\theta)}{\partial\sigma^2\partial\rho_k} = -\frac{1}{\sigma^4}\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}\left\{m_{i\cdot,kn}\left[S_n(\lambda)Y_n - X_n\beta\right]\right\},$$
$$\frac{\partial^2 l_{i,n}(\theta)}{\partial\rho_k\partial\rho_j} = -\frac{1}{n}\operatorname{tr}\left[R_n(\rho)^{-1}M_{kn}R_n(\rho)^{-1}M_{jn}\right] - \frac{1}{\sigma^2}\left\{m_{i\cdot,jn}\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}\left\{m_{i\cdot,kn}\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}.$$

Third order derivatives

$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\lambda_j\partial\lambda_k\partial\lambda_r} = -\frac{1}{n}\operatorname{tr}\left[S_n(\lambda)^{-1}W_{rn}S_n(\lambda)^{-1}W_{kn}S_n(\lambda)^{-1}W_{jn}\right] - \frac{1}{n}\operatorname{tr}\left[S_n(\lambda)^{-1}W_{kn}S_n(\lambda)^{-1}W_{rn}S_n(\lambda)^{-1}W_{jn}\right],$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\lambda_j\partial\lambda_k\partial\beta} = 0,$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\lambda_j\partial\lambda_k\partial\sigma^2} = \frac{1}{\sigma^4}\left[R_{i\cdot,n}(\rho)W_{kn}Y_n\right]\left[R_{i\cdot,n}(\rho)W_{jn}Y_n\right],$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\lambda_j\partial\lambda_r\partial\rho_k} = \frac{1}{\sigma^2}(W_{jn}Y_n)'\left[R_{i\cdot,n}(\rho)' m_{i\cdot,kn} + m_{i\cdot,kn}' R_{i\cdot,n}(\rho)\right]W_{rn}Y_n,$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\lambda_j\partial\beta\partial\beta'} = 0,$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\lambda_j\partial\beta\partial\sigma^2} = \frac{1}{\sigma^4} X_n' R_{i\cdot,n}(\rho)'\left[R_{i\cdot,n}(\rho)W_{jn}Y_n\right],$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\lambda_j\partial\beta\partial\rho_k} = \frac{1}{\sigma^2} X_n'\left[R_{i\cdot,n}(\rho)' m_{i\cdot,kn} + m_{i\cdot,kn}' R_{i\cdot,n}(\rho)\right]W_{jn}Y_n,$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\lambda_j\partial\sigma^2\partial\sigma^2} = \frac{2}{\sigma^6}\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}\left[R_{i\cdot,n}(\rho)W_{jn}Y_n\right],$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\lambda_j\partial\sigma^2\partial\rho_k} = \frac{1}{\sigma^4}(W_{jn}Y_n)'\left[R_{i\cdot,n}(\rho)' m_{i\cdot,kn} + m_{i\cdot,kn}' R_{i\cdot,n}(\rho)\right]\left[S_n(\lambda)Y_n - X_n\beta\right],$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\lambda_j\partial\rho_k\partial\rho_r} = \frac{1}{\sigma^2}(W_{jn}Y_n)'\left(m_{i\cdot,rn}' m_{i\cdot,kn} + m_{i\cdot,kn}' m_{i\cdot,rn}\right)\left[S_n(\lambda)Y_n - X_n\beta\right],$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\beta\partial\beta'\partial\beta_j} = 0, \text{ for any } j = 1, \cdots, K$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\beta\partial\beta'\partial\sigma^2} = \frac{1}{\sigma^4} X_n' R_{i\cdot,n}(\rho)' R_{i\cdot,n}(\rho) X_n,$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\beta\partial\beta'\partial\rho_k} = \frac{1}{\sigma^2} X_n'\left[R_{i\cdot,n}(\rho)' m_{i\cdot,kn} + m_{i\cdot,kn}' R_{i\cdot,n}(\rho)\right]X_n,$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\beta\partial\sigma^2\partial\sigma^2} = \frac{2}{\sigma^6} X_n' R_{i\cdot,n}(\rho)'\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\},$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\beta\partial\sigma^2\partial\rho_k} = \frac{1}{\sigma^4} X_n'\left[R_{i\cdot,n}(\rho)' m_{i\cdot,kn} + m_{i\cdot,kn}' R_{i\cdot,n}(\rho)\right]\left[S_n(\lambda)Y_n - X_n\beta\right],$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\beta\partial\rho_j\partial\rho_k} = \frac{1}{\sigma^2} X_n'\left(m_{i\cdot,jn}' m_{i\cdot,kn} + m_{i\cdot,kn}' m_{i\cdot,jn}\right)\left[S_n(\lambda)Y_n - X_n\beta\right],$$
$$\frac{\partial^3 l_{i,n}(\theta)}{(\partial\sigma^2)^3} = -\frac{1}{\sigma^6} + \frac{3}{\sigma^8}\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}^2,$$
$$\frac{\partial^2 l_{i,n}(\theta)}{(\partial\sigma^2)^2\partial\rho_k} = \frac{2}{\sigma^6}\left\{R_{i\cdot,n}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}\left\{m_{i\cdot,kn}\left[S_n(\lambda)Y_n - X_n\beta\right]\right\},$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\sigma^2\partial\rho_j\partial\rho_k} = \frac{1}{\sigma^4}\left\{m_{i\cdot,jn}(\rho)\left[S_n(\lambda)Y_n - X_n\beta\right]\right\}\left\{m_{i\cdot,kn}\left[S_n(\lambda)Y_n - X_n\beta\right]\right\},$$
$$\frac{\partial^3 l_{i,n}(\theta)}{\partial\rho_j\partial\rho_k\partial\rho_r} = -\frac{1}{n}\operatorname{tr}\left[R_n(\rho)^{-1}M_{rn}R_n(\rho)^{-1}M_{kn}R_n(\rho)^{-1}M_{jn}\right] - \frac{1}{n}\operatorname{tr}\left[R_n(\rho)^{-1}M_{kn}R_n(\rho)^{-1}M_{rn}R_n(\rho)^{-1}M_{jn}\right].$$

D. The Exchange Algorithm to Sample ψ in the High Order SARAR Model

The MCMC algorithm to sample ψ in Subsection 4.1 might be computationally intensive when n, the sample size, is large because the Jacobian determinants $|R_n(\rho)|$ and $|S_n(\lambda)|$ need to be evaluated in each iteration of the sampler. Here we follow Han, Hsieh and Lee (2017) to briefly suggest the exchange algorithm in Murray et al (2006) to avoid the computation of $|R_n(\rho)|$ and $|S_n(\lambda)|$ in the M-H step. Recall that $\theta = (\psi', \beta', \sigma^2)'$ and $\psi = (\lambda', \rho')'$ with prior $\pi_\psi(\psi) = \pi_\lambda(\lambda) \times \pi_\rho(\rho)$. The likelihood function $L_n(\theta|Y_n)$ can be decomposed into:

$$L_n(\theta|Y_n) = f(Y_n; \psi, \beta, \sigma^2) \times D(\psi),$$

where $D(\psi) = |R_n(\rho)| \times |S_n(\lambda)|$ is the Jacobian determinant term in the likelihood function. In Subsection 4.1, we utilize an M-H step to sample ψ, with $\tilde{\psi} = (\tilde{\lambda}', \tilde{\rho}')'$ being the new candidate value generated from the AM proposal. The corresponding acceptance probability of the M-H step is

$$Pr(\psi, \tilde{\psi}) = \min\{1, \frac{L_n(\tilde{\psi}, \beta, \sigma^2|Y_n)}{L_n(\psi, \beta, \sigma^2|Y_n)}\} = \min\{1, \frac{f(Y_n; \tilde{\psi}, \beta, \sigma^2)}{f(Y_n; \psi, \beta, \sigma^2)} \times \frac{D(\tilde{\psi})}{D(\psi)}\}.$$

When n is large, evaluating $\frac{D(\tilde{\psi})}{D(\psi)}$ might be computationally demanding. So the acceptance probability can not be easily evaluated. To solve this issue, we suggest the use of the exchange algorithm in Murray et al (2006). In addition to $\tilde{\psi}$, we further simulate auxiliary sample $\tilde{Y}_n$ from the density $f(\tilde{Y}_n; \psi, \beta.\sigma)D(\psi)$, which is equal to $L_n(\psi, \beta, \sigma^2|\tilde{Y}_n)$. According to Murray et al (2006), $\tilde{Y}_n$ can be viewed as a "replacement" data, compared to the real data Y_n. The acceptance probability under the exchange algorithm is

$$\begin{aligned}\Pr(\psi, \tilde{\psi}) &= \min\left[1, \frac{L_n(\tilde{\psi}, \beta, \sigma^2|Y_n)}{L_n(\psi, \beta, \sigma^2|Y_n)} \times \frac{L_n(\psi, \beta, \sigma^2|\tilde{Y}_n)}{L_n(\tilde{\psi}, \beta, \sigma^2|\tilde{Y}_n)}\right] = \min\left[1, \frac{L_n(\psi, \beta, \sigma^2|\tilde{Y}_n)}{L_n(\psi, \beta, \sigma^2|Y_n)} \times \frac{L_n(\tilde{\psi}, \beta, \sigma^2|Y_n)}{L_n(\tilde{\psi}, \beta, \sigma^2|\tilde{Y}_n)}\right] \\ &= \min\left[1, \frac{f(\tilde{Y}_n; \psi, \beta, \sigma^2)}{f(Y_n; \psi, \beta, \sigma^2)} \times \frac{f(Y_n; \tilde{\psi}, \beta, \sigma^2)}{f(\tilde{Y}_n; \tilde{\psi}, \beta, \sigma^2)}\right]. \qquad \text{(D.1)}\end{aligned}$$

Notice that $D(\psi)$ and $D(\tilde{\psi})$ have been cancelled out in D.1. So the acceptance probability can be evaluated in a more efficient way (Murray et al, 2006).

Furthermore, simulating $\tilde{Y}_n$ from $L_n(\psi, \beta, \sigma^2|\tilde{Y}_n)$ is computationally tractable because the likelihood function $L_n(\psi, \beta, \sigma^2|Y_n)$ is a multivariate normal density function. Given λ and ρ, we adopt the contraction-mapping algorithm in Lee (2003) to simulate $\tilde{Y}_n$. Here are the simulation steps:

Step 1: Simulate the disturbance term $\tilde{U}_n$:

Recall that $U_n = \sum_{k=1}^{q} \rho_k M_{kn} U_n + V_n$ and $U_n = R_n^{-1}(\rho) V_n$. We first simulate $V_n \sim N(0, \sigma^2 I_n)$. Define the mapping:

$$Map(U_n) = M_n(\tilde{\rho}) U_n + V_n, \tag{D.2}$$

where $M_n(\tilde{\rho}) = \sum_{k=1}^{q} \tilde{\rho}_k M_{kn}$. Provided that Eq. (3.3) holds, namely, $\sup_n ||M_n(\tilde{\rho})||_\infty < 1$, D.2 is a contraction mapping such that,

$$||Map(U_{n|1}) - Map(U_{n|2})||_\infty = ||M_n(\tilde{\rho})(U_{n|1} - U_{n|2})||_\infty \leq \Delta_M ||U_{n|1} - U_{n|2}||_\infty, \tag{D.3}$$

with $\Delta_M = ||M_n(\tilde{\rho})||_\infty < 1$. Let $\tilde{U}_n$ be the fixed point of $\tilde{U}_n = Map(\tilde{U}_n)$. With any initial value $U_{n|1}$,[31] $\tilde{U}_n$ can be obtained by iterations of as $U_{n|i+1} = Map(U_{n|i}) = M_n(\tilde{\rho}) U_{n|i} + V_n$.

Step 2: Simulate $\tilde{Y}_n$ based on $\tilde{U}_n$.

Given $\tilde{U}_n$ from Step 1, we can simulate $\tilde{Y}_n$ based upon the high order SARAR model. Define the mapping:

$$Map(Y_n) = \sum_{j=1}^{p} \tilde{\lambda}_j W_{jn} Y_n + X_n \beta + \tilde{U}_n, \tag{D.4}$$

which is also a contraction mapping under Eq. (3.3), with $\tilde{Y}_n = Map(\tilde{Y}_n)$ being the fixed point.

E. Bayesian Estimation of the SAR Tobit Model

In this section we summarize the MCMC sampler of the SAR Tobit model. Let $y_{i,n}$ be the observed dependent variable of individual i, with $y_{i,n} = y_{i,n}^* I(y_{i,n}^* > 0)$. Denote $Y_n = (y_{1,n}, \cdots, y_{n,n})'$, $Y_n^* = (y_{1,n}^*, \cdots, y_{n,n}^*)'$ and $\epsilon_n = (\epsilon_1, \cdots, \epsilon_n)'$. In matrix form, the SAR Tobit model is

$$Y_n^* = \lambda W_n Y_n + X_n \beta + \epsilon_n. \tag{E.1}$$

[31] To boost the speed of the contraction mapping in the MCMC sampler, one may set initial value of $U_{n|1}$ as follows: in the first run of the Markov chain, set $U_{n|1}$ to some arbitrary values and use the contraction mapping to derive a fixed point $\tilde{U}_n$. Then use $\tilde{U}_n$ as the initial value of the mapping in the second run of the MCMC sampler. After that, always use the fixed point in the previous run as the initial values for the mapping in the current run. As long as those fixed points do not change much, the computation time of the contraction mapping step could be saved.

Without loss of generality, we decompose and order the $n \times 1$ vector Y_n into $Y_n = (Y_{1n}', Y_{2n}')'$, where Y_{1n} is a $n_1 \times 1$ subvector with $Y_{1n} = 0$, and Y_{2n} is a $(n - n_1) \times 1$ with all positive elements. The $n \times 1$ column vector Y_n^*, the $n \times k$ matrix X_n, the $n \times n$ spatial weights matrix W_n and the $n \times 1$ disturbance vector ϵ_n can confirmably be decomposed as:

$$Y_n^* = \begin{pmatrix} Y_{1n}^* \\ Y_{2n}^* \end{pmatrix}, \; X_n = \begin{pmatrix} X_{1n} \\ X_{2n} \end{pmatrix}, \; W_n = \begin{pmatrix} W_{11,n} & W_{12,n} \\ W_{21,n} & W_{22,n} \end{pmatrix}, \; \epsilon_n = \begin{pmatrix} \epsilon_{1n} \\ \epsilon_{2n} \end{pmatrix}.$$

Then the model in E.1 can be rewritten as

$$\begin{pmatrix} Y_{1n}^* \\ Y_{2n}^* \end{pmatrix} = \lambda \begin{pmatrix} W_{11,n} & W_{12,n} \\ W_{21,n} & W_{22,n} \end{pmatrix} \begin{pmatrix} Y_{1n} \\ Y_{2n} \end{pmatrix} + \begin{pmatrix} X_{1n} \\ X_{2n} \end{pmatrix} \beta + \begin{pmatrix} \epsilon_{1n} \\ \epsilon_{2n} \end{pmatrix}.$$

With $Y_{1n} = 0$ and all elements of Y_{2n} are positive, we have

$$Y_{1n}^* = \lambda W_{12,n} Y_{2n} + X_{1n}\beta + \epsilon_{1n}, \tag{E.2}$$

$$Y_{2n} = \lambda W_{22,n} Y_{2n} + X_{2n}\beta + \epsilon_{2n}, \tag{E.3}$$

where $\epsilon_{1n} \sim N_{n_1}(0, \sigma^2 I_{n_1})$, $\epsilon_{2n} \sim N_{n-n_1}(0, \sigma^2 I_{n-n_1})$, and ϵ_{1n} and ϵ_{2n} are independent of each other. Let $\theta = (\lambda, \beta', \sigma^2)'$ be the whole parameter vector of the model. Following Xu and Lee (2015), from Eq. (E.2)-(E.3), the likelihood function is

$$\begin{aligned} & L_n(\theta|Y_n) = f(Y_{1n} = 0, Y_{2n}|\theta) = f(Y_{1n} = 0|Y_{2n}, \theta) \times f(Y_{2n}|\theta) \\ =& P(Y_{1n}^* \leq 0|Y_{2n}, \theta) \times f(Y_{2n}|\theta) = \prod_{i=1}^{n_1} P(y_{i,n}^* \leq 0|Y_{2n}, \theta) \times f(Y_{2n}|\theta) \\ =& \left\{ \prod_{i=1}^{n_1} \left[1 - \Phi\left(\frac{(\lambda W_{12n} Y_{2n} + X_{1n}\beta)_i}{\sigma} \right) \right] \right\} \cdot \\ & (2\pi\sigma^2)^{-\frac{n-n_1}{2}} |S_{22}(\lambda)| \exp\left[-\frac{(S_{22}(\lambda)Y_{2n} - X_{2n}\beta)'(S_{22}(\lambda)Y_{2n} - X_{2n}\beta)}{2\sigma^2} \right], \end{aligned} \tag{E.4}$$

where $(\lambda W_{12n} Y_{2n} + X_{1n}\beta)_i$ stands for the ith row of $\lambda W_{12n} Y_{2n} + X_{1n}\beta$ and $S_{22}(\lambda) = I_{n-n_1} - \lambda W_{22n}$. By Bayes' theorem, the posterior distribution of θ is

$$p(\theta|Y_n) \propto \pi_\lambda(\lambda) \times \pi_{\beta|\sigma^2}(\beta|\sigma^2) \times \pi_{\sigma^2}(\sigma^2) \times f(Y_{1n} = 0, Y_{2n}|\theta).$$

From (E.4), $p(\theta|Y_n)$ does not take a known form. Hence, we apply the M-H step with the AM proposal in Subsection 4.1 to sample θ. Particularly, let $\tilde{\theta} = (\tilde{\lambda}, \tilde{\beta}', \tilde{\sigma}^2)'$ be a new candidate value of θ from the AM proposal. The corresponding acceptance probability is[32]

$$\Pr(\theta, \tilde{\theta}) = \min\left\{1, \frac{\pi_{\beta|\sigma^2}(\tilde{\beta}|\tilde{\sigma}^2)}{\pi_{\beta|\sigma^2}(\beta|\sigma^2)} \times \frac{\pi_{\sigma^2}(\tilde{\sigma}^2)}{\pi_{\sigma^2}(\sigma^2)} \times \frac{f(Y_{1n}=0, Y_{2n}|\tilde{\theta})}{f(Y_{1n}=0, Y_{2n}|\theta)}\right\}.$$

F. Simulation Designs and Results

F.1. Monte Carlo simulation design

We focus on the data observed on the Euclidean plane $\mathbb{R}^2$ throughout the simulation study. We first consider the high order SARAR model,

$$Y_n = \lambda_1 W_{1n} Y_n + \lambda_2 W_{2n} Y_n + l_n \beta_1 + X_{2n}\beta_2 + X_{3n}\beta_3 + X_{4n}\beta_4 + \epsilon_n,\ \epsilon_n = \rho_1 M_{1n}\epsilon_n + \rho_2 M_{2n}\epsilon_n + V_n, \tag{F.1}$$

where l_n is an $n \times 1$ vector of ones for the intercept, $X_{2n} = (x_{21}, \cdots, x_{2n})'$, $X_{3n} = (x_{31}, \cdots, x_{3n})'$ and $X_{4n} = (x_{41}, \cdots, x_{4n})'$ are $n \times 1$ column vectors of exogenous regressors. $\beta_1, \cdots, \beta_4$ are the corresponding coefficients. W_{jn}'s and M_{kn}'s are spatial weights matrices constructed from m nearest neighbors (detailed below), which may or may not be row-normalized. We study three sample sizes: $n = 200$, $n = 400$ and $n = 800$. Three data generating processes (DGPs) of parameter values are investigated, namely,

$$\textbf{DGP1: } \lambda_1 = 0.5,\ \lambda_2 = 0,\ \rho_1 = 0.4, \rho_2 = 0, \tag{F.2}$$

$$\textbf{DGP2: } \lambda_1 = 0.5,\ \lambda_2 = 0.3,\ \rho_1 = 0.4, \rho_2 = 0.2, \tag{F.3}$$

$$\textbf{DGP3: } \lambda_1 = 0.2,\ \lambda_2 = 0,\ \rho_1 = 0.06, \rho_2 = 0, \tag{F.4}$$

where the first two DGPs are designed for the model with row-normalized spatial weights matrices, whereas DGP3 refers to the model with non-row-normalized spatial weights matrices. We set $\beta_1 = 2$ and $\beta_2 = \beta_3 = \beta_4 = 1$ in all DGPs. We explore two different settings for X_{2n}. In the first setting, x_{2i}'s are i.i.d. and follow the Bernoulli(0.5) distribution. In the second setting,

[32] In the simulation study we consider a row-normalized W_n for the SAR Tobit model. So the stability condition of the M-H step is $|\lambda| < 1$ and $\sigma^2 > 0$.

$X_{2n} = (I_n - 0.2 \times W_{1n})^{-1} \cdot N_n(0, I_n)$, so x_{2i}'s are spatially correlated. In both settings, X_{3n} and X_{4n} are generated from $N_n(0, 2I_n)$. Furthermore, we augment Eq. (F.1) with an additional exogenous regressor X_{5n}, to compare the performance of MLE and QMLE with that of Bayesian and quasi-Bayesian estimators, under the scenario where there might be some multicollinearity among exogenous regressors. Specifically, the augmented model becomes

$$Y_n = \lambda_1 W_{1n} Y_n + \lambda_2 W_{2n} Y_n + l_n \beta_1 + \sum_{t=2}^{5} X_{tn} \beta_t + \epsilon_n, \quad \epsilon_n = \rho_1 M_{1n} \epsilon_n + \rho_2 M_{2n} \epsilon_n + V_n, \quad \text{(F.5)}$$

and the DGPs of parameter values of β_2 to β_5 are modified but the value of the spatial parameters λ_1, λ_2, ρ_1 and ρ_2, and the intercept term β_1 remain the same. The coefficients β_2 to β_5 are modified to be generated from a smooth normal density, as in Han and Lee (2013),

$$\beta_t = \frac{10 \exp(-(t - 2.4)^2/18)}{\sqrt{18\pi}}, \; t = 2, 3, 4, 5, \quad \text{(F.6)}$$

Particularly, we have

DGP4: $\lambda_1 = 0.5, \; \lambda_2 = 0, \; \rho_1 = 0.4, \rho_2 = 0, \; \beta_1 = 2, \; \beta_2 = 1.3180, \; \beta_3 = 1.3035,$
$\beta_4 = 1.1535, \; \beta_5 = 0.9135,$

DGP5: $\lambda_1 = 0.5, \; \lambda_2 = 0.3, \; \rho_1 = 0.4, \rho_2 = 0.2, \; \beta_1 = 2, \; \beta_2 = 1.3180, \; \beta_3 = 1.3035,$
$\beta_4 = 1.1535, \; \beta_5 = 0.9135.$

To capture multicollinearity among regressors, we generate $X_{2n} \sim N_n(0, 2I_n)$ and modify X_{3n}, X_{4n} and X_{5n} as high order spatial Durbin terms, namely,

$$X_{3n} = W_{3n} X_{2n}, \; X_{4n} = W_{3n}^2 X_{2n}, \; X_{5n} = W_{3n}^3 X_{2n}. \quad \text{(F.7)}$$

With $n = 800$, the correlation coefficient of X_{4n} and X_{5n} could be as high as 0.9533. We would like to see how the MLE and QMLE perform compared with their Bayesian counterparts under this multicollinearity scenario.

The spatial econometrics model in Eq. (F.1) and (F.5) are linear models. For a nonlinear spatial model, we study the SAR Tobit model, namely,[33]

$$y_{i,n} = \max(0,\ \lambda_1 \sum_{j=1}^{n} w_{ij,n} y_{j,n} + x'_{i,n}\beta + v_i), \tag{F.8}$$

with $x_{i,n} = (x_{i1n}, \cdots, x_{i4n})'$, $\beta = (\beta_1, \cdots, \beta_4)'$ and $w_{ij,n}$ being the ijth element of the spatial weights matrix W_n. The exogenous regressors $x_{i,n}$'s are generated in the same way as the SARAR model in Eq. (F.1). The values of λ_1 and β are set to be the same as in DGP1, except that we set the intercept $\beta_1 = -0.5$ to achieve a censoring rate about 0.42. According to Qu and Lee (2013), with a row-normalized W_n (detailed below), $|\lambda_1| < 1$ ensures a unique solution to the model given exogenous regressors and disturbances. We follow Qu and Lee (2013) to generate the dependent variable vector $Y_n = (y_{1,n}, \cdots, y_{n,n})'$ based upon the iterative contraction mapping algorithm $Y_n^{(j)} = \max(0, \lambda_1 W_n Y_n^{(j-1)} + X_n\beta + V_n)$, starting from $Y_n^{(0)} = X_n\beta + V_n$. The iteration stops when $||Y_n^{(j)} - \max(0, \lambda_1 W_n Y_n^{(j-1)} + X_n\beta + V_n)||_\infty < 10^{-6}$.[34]

For the SARAR model, the disturbances v_i's are i.i.d draws from i) $N(0, 1)$; ii) a uniform distribution over $(-\sqrt{3},\ \sqrt{3})$; iii) the standardized t_6 distribution.[35] While i) corresponds to the MLE case, ii) - iii) correspond to QMLE cases. In particular, even though ii) - iii) are different distributions, they all have zero mean and unit variance. For the SAR Tobit model, we only consider i.i.d $N(0, 1)$ v_i's.

All spatial weights matrices in Eq. (F.1), (F.5), and (F.8) are constructed from the function "makeneighborsw",[36] which generates a row-normalized spatial weights matrix based on m nearest neighbors with various m for different DGPs (described below). Taking W_{1n} and M_{1n} as examples, the procedure consists of the following three steps:

1. For each observational unit i on the Euclidean plane $\mathbb{R}^2$, generate its coordinates $xc(i)$ and $yc(i)$ from $\mathcal{X}^2(3)$, which is a chi-square random variable with three degrees of freedom, to represent its geographical location on $\mathbb{R}^2$.

2. Compute the geographical distance d_{ij} between i and j based upon coordinates $(xc(i), yc(i))$ and $(xc(j), yc(j))$; For each i, find the nearest m neighbors based upon $d(i, j)$, $j \neq i$ and denote the

[33] We thank one referee for suggesting the need of investigating the SAR Tobit model in the simulation study.

[34] There are robust checks on the normality assumption in Xu and Lee (2015). From their simulations, the estimates are not very sensitive to unimodal non-normal distribution, but can be sensitive to a bimodal mixture distribution.

[35] We have also explored the case where v_i's are i.i.d draws from the standardized t_8 distribution. The corresponding graphs turn out to be very similar to that of normal error case. Thus, we do not report them in the supplementary appendices.

[36] This function is taken from James LeSage's Matlab code for spatial econometrics, which can be found at http://www.spatial-econometrics.com/.

corresponding $W_{ij1} = 1$ ($M_{ij1} = 1$), otherwise $W_{ij1} = 0$ ($M_{ij1} = 0$).

3. Row-normalize W_{1n} (M_{1n}).

We first set $m = 1$ for W_{1n} and $m = 3$ for M_{1n} with DGP1 and DGP2, and $m = 8$ ($m = 10$) for W_{3n} with DGP4 (DGP5). We specify W_{2n} and M_{2n} as, respectively, row-normalized spatial weights matrices based upon only the 2nd and 4th nearest neighbors for DGP2.[37] These spatial weights matrices are rather sparse.[38] We also explore some denser spatial weights matrices for DGP1, where W_{1n} and M_{1n} are specified, respectively, with $m = 0.02n$ and $m = 0.04n$ nearest neighbors. Additionally, for DGP3, we specify W_{1n} with $m = 3$ and M_{1n} with $m = 12$ and do not row-normalize them.[39] Finally, we set W_n in (F.8) to be a row-normalized spatial weights matrix with $m = 5$.[40]

We implement the Bayesian MCMC sampler in Subsection 4.1, along with the exchange algorithm for high order SARAR models in Appendix D and the SAR Tobit model in Appendix E, and utilize the posterior draws of parameters to examine asymptotic properties in Theorems 1 to 3. We follow Eq. (4.3) to specify priors for parameters of the high order SARAR model.[41] In particular, we set the prior mean of β to be zero to take a neutral stance regarding its sign. The scaling factor $\frac{1}{\phi}$ is set to be 10, which is much larger than the actual variance in the DGPs. In addition to a fixed ϕ, we also implement the Bayesian ridge regression in Subsection 4.1 for (F.1) by further assuming a hierarchical gamma prior on ϕ, namely, $\phi \sim \frac{1}{\Gamma(c)}\phi^{c-1}d^c \exp(-d\phi)$, where we follow Kyung et al. (2010) to set $c = 1$ and $d = 0.1$, which leads to a relatively flat gamma prior. The prior of σ^2 is an inverse-gamma prior with shape parameter $a = 6$ and scaling parameter $b = 4$. The priors of parameters λ_1, β and σ^2 in Eq. (F.8) can be specified in a similar manner to Eq. (4.3).

[37] Specifically, we first generate $\tilde{W}_{2n}(\tilde{M}_{2n})$, which is a row-normalized spatial weights matrix based on 2 (4) nearest neighbors. We then eliminate the first (first, second and third) nearest neighbors to obtain W_{2n} (M_{2n}).

[38] We follow Assumption 3.3 to evaluate $\max_{1\le j\le 2,\ 1\le k\le 2}(||W_{jn}||_1, ||M_{kn}||_1, ||W_{jn}||_\infty, ||M_{kn}||_\infty, ||S_n^{-1}||_1, ||R_n^{-1}||_1)$ for DGP1 and DGP under different sample sizes. For DGP1, the norm is 4.5 with $n = 200$, and 4.3 with $n = 400,\ 800$. For DGP2, the norm is 14.7 with $n = 200$, 13.8 with $n = 400$ and 13.9 with $n = 800$. Though these results are not a formal justification of Assumption 3.3, they do give us more confidence that the spatial weights matrices we construct are more or less compatible with Assumption 3.3.

[39] We thank one referee for raising the issue that row-normalized spatial weights matrices have been criticized in the literature, and suggesting the need of considering spatial weights matrices that are not row-normalized. As in DGP3, the row normalization restriction of the spatial weights matrices in the SARAR model is relaxed. The stability condition in Subsection 4.1 also works for non-row-normalized spatial weights matrices.

[40] We compute all pairwise distances associated with W_n for the SAR Tobit model under different sample sizes. The minimum (maximum) pairwise distance is 0.0079 (21.66) for $n = 200$, 0.018 (24.24) for $n = 400$ and 0.0069 (22.53) for $n = 800$, with an average of 4.10 for $n = 200$, 4.16 for $n = 400$ and 4.05 for $n = 800$. Though these can not been viewed as a formal justification of Assumption 3.9, they do suggest that W_n is more or less compatible with the NED design where all units should be separated by a finite distance.

[41] As in Subsection 4.1, we impose the stability condition on λ_j's and ρ_k's through the acceptance-rejection M-H step, not through the uniform prior. When the model reduces to a SARAR model with only W_{1n} and M_{1n}, as in DGP1, one may follow LeSage and Pace (2009) to set lower bounds of the intervals to be, respectively, the most negative real eigenvalues of W_{1n} and M_{1n}.

We make use of the Bayesian and quasi-Bayesian estimates to justify Theorems 1-3. Regarding the finite sample behavior of posterior and quasi-posterior densities, we first run one Markov chain with 25000 iterations and obtain the posterior draws of λ, ρ and β. The first 20% draws are discarded as burn-in draws.[42] We thereby compare the marginal posterior density plots of remaining parameter draws, demeaned at MLEs or QMLEs, with corresponding normal density plots. For empirical distributions of Bayesian and quasi-Bayesian estimates, we first run 1000 Markov chains (repetitions) with length 25000 and 20% burn-in ratio, where V_n are regenerated in each repetition. We then obtain posterior and quasi-posterior means and medians of λ, ρ and β in each repetition. Based upon those 1000 samples, we can plot the empirical densities of posterior and quasi-posterior means and medians, and compare those plots with the corresponding normal density plots.

We treat the posterior and quasi-posterior means in each repetition as Bayesian and quasi-Bayesian point estimates. For DGP1 through DGP3, we not only compare the density plots, but we also compare the bias and root mean squared error (RMSE) of the Bayesian estimates with those of MLEs and QMLEs across 1000 repetitions. Besides, we report the computation time of the MCMC algorithm in Table 4 and the exchange algorithm in Appendix D for the high order SARAR model in a single Markov chain with 1000-iteration length, under different values of n.[43] Both algorithms are run on a 3.30GHZ server with an Intel Xeon processor and 8 GB installed memory. For DGP4 and DGP5 that involve multicollinear regressors, we explore both non-hierarchical and hierarchical normal priors for β in the MCMC sampler. We report the bias and the RMSE over 1000 repetitions for both classical and Bayesian approaches. We would like to see whether the Bayesian approach can produce a smaller RMSE than the classical ML approach.

F.2. Simulation results

Figures for marginal (quasi) posterior densities

1. Normally distributed disturbances: Figures G.3-G.5 compare plots of marginal posterior densities of λ_j's, ρ_k's and β_h's for $j = 1, 2$, $k = 1, 2$ and $h = 1, 2, 3, 4$, with plots of their corresponding normal densities under different settings, for DGP1 and DGP2. As can be seen

[42] Some trace plots are depicted in Figures G.1 and G.2 in the supplementary appendices to demonstrate the convergence of the MCMC sampler.

[43] We explore cases of small and moderate sample sizes with $n = 200$, $n = 400$ and $n = 800$, as well as cases of relatively large sample sizes with $n = 1000$, $n = 1200$ and $n = 1400$.

from those figures, plots of posterior densities and the corresponding normal densities are very close to each other in most cases. The only exception is when x_{2i}'s are spatially correlated with $n = 200$, the marginal posterior densities of λ_1, λ_2 and the intercept term β_1 are not so close to their normal densities at the peak. But as n grows to 400 and 800, those marginal posterior densities and corresponding normal densities almost overlap.

2. Non-normally distributed disturbances: Figures G.6-G.11 summarize plots of marginal quasi-posterior densities of λ_j's, ρ_k's, β_h's and corresponding normal density plots for DGP1 and DGP2, with t_6 and uniformly distributed errors. From those figures, when n increases from 200 to 400 and 800, normal densities can serve as good approximations for those quasi-posterior densities in all settings. In particular, for β_2, no matter whether x_{2i}'s are discrete or spatially correlated, the marginal quasi-posterior densities and corresponding normal densities look similar to each other.

3. Denser or non-row-normalized spatial weights matrices: Figures G.12-G.15 summarize marginal posterior and quasi-posterior densities of parameters and corresponding normal density plots for DGP1, under cases of denser or non-row-normalized spatial weights matrices. The marginal posterior and quasi-posterior densities look similar to those corresponding normal densities in most cases, particularly when $n = 800$.

4. Parameter space implied by Eq. (3.3): Figures G.16-G.17 report plots of marginal posterior and quasi-posterior densities for DGP2, under the broader stability condition implied by Eq. (3.3). Similar to the case under the stronger stability condition, those posterior densities behave like normal densities for all sample sizes.

5. SAR Tobit model: Figure H.1 compares the marginal posterior densities of λ_1 and β_h for $h = 1, 2, 3$ with corresponding normal densities, for the SAR Tobit model. Plots of posterior densities are very close to normal densities, especially for the cases of $n = 400$ and $n = 800$.

Figures for posterior means and medians

1. Normally distributed disturbances: Figures G.18-G.23 provide empirical density plots of posterior means and medians of parameters for DGP1 and DGP2, under different sample sizes. We

see that even with only $n = 200$, those empirical densities already behave similarly to their corresponding normal densities.

2. Non-normally distributed disturbances: Figure G.24-G.25 report empirical density plots of quasi-posterior means and medians of parameters for DGP1 in cases of uniform distributed errors. With $n = 800$, normal densities can provide considerably good approximation for the empirical densities of all parameters under different settings.

3. Denser or non-row-normalized spatial weights matrices: Figures G.26-G.28 display empirical density plots of posterior and quasi-posterior means and medians of parameters for DGP1, under cases of denser or non-row-normalized spatial weights matrices. One can see that normal densities can still serve as good approximations for the empirical densities of parameter estimates in both scenarios.

4. Parameter space implied by Eq. (3.3): Figure G.29 provides empirical density plots of posterior and quasi-posterior means and medians of parameters for DGP2, under the broader stability condition. The empirical densities still look similar to corresponding normal densities in most cases.

5. SAR Tobit model: Figures H.2-H.4 in Appendix H present empirical densities of Bayesian estimates for the SAR Tobit model. Those empirical densities all look similar to corresponding normal densities.

Tables

Table 4 reports the computation time of the Bayesian MCMC algorithm and the exchange algorithm in a single Markov chain of 1000 iterations, for DGP1 and DGP2. When n is small (less than 800), both algorithms can deliver well-behaved MCMC samples within about 25 seconds for all cases. In particular, with $n = 200$, the whole Bayesian estimation procedure can be done in 2 seconds. When n is relatively large (more than 1000) and the classical ML method might be computationally intensive, the exchange algorithm costs significantly less CPU time than the direct MCMC algorithm, and is able to finish estimation in 4.3 minutes for the most complicated model, namely, the high order SARAR model with two spatial lags both in the main equation and the error term.

Tables 5-8 give biases and RMSEs of MLE, QMLE, and Bayesian and Quasi-Bayesian estimates across 1000 repetitions under different sample sizes. For DGP1 and DGP2, biases and RMSEs of the classical and Bayesian approaches tend to be similar in most cases.

Tables 9-11 present estimation results of MLE, QMLE and the Bayesian estimates for DGP4 and DGP5, where some regressors are highly correlated. The results of spatial parameters λ_j's, ρ_k's, and slope coefficients β_1 through β_3 are similar in terms of biases and RMSEs for both classical and Bayesian approaches. However, the Bayesian estimates of β_4 and β_5, which are the coefficients of highly correlated regressors, tend to have smaller RMSE than their classical counterparts. This is so in particular for the case where we impose hierarchical priors on β. In this case the Bayesian and quasi-Bayesian estimates of β_5 have slightly larger biases but much smaller RMSEs than their corresponding MLEs and QMLEs. This suggests that, with highly correlated regressors, the Bayesian approach is able to produce better estimates in terms of RMSE than the classical approach.

Tables 12-13 report estimation results of MLEs, QMLEs and the Bayesian estimates under cases of denser and non-row-normalized spatial weights matrices. Tables 14-16 present the performance of QBE under the broader stability condition implied by Eq. (3.3). In all scenarios, the Bayesian estimates are very similar to classical estimates in terms of biases and RMSEs.

Table 17 summarizes biases and RMSEs of MLEs and Bayesian estimates for the SAR Tobit model. We still find biases and RMSEs of classical MLEs and Bayesian estimates for all sample sizes to be very similar.

G. Figures and Tables for the SARAR Model

G.1. Time Duration of Bayesian Estimation Algorithm

Table 4: CPU Time of Bayesian Estimation Algorithm

Direct Bayesian Estimation		$n = 200$			$n = 400$			$n = 800$		
		Normal	t_8	Uniform	Normal	t_8	Uniform	Normal	t_8	Uniform
DGP1	Setting 1	2.20	2.27	2.30	21.62	21.68	21.53	85.40	83.99	85.89
	Setting 2	2.12	2.44	2.39	21.57	21.67	21.91	85.30	85.07	83.86
DGP2	Setting 1	2.35	2.30	2.33	26.30	26.38	26.68	105.75	104.65	104.77
	Setting 2	2.39	2.30	2.34	26.34	26.30	26.38	103.83	104.17	104.06
Exchange Algorithm		$n = 200$			$n = 400$			$n = 800$		
		Normal	t_8	Uniform	Normal	t_8	Uniform	Normal	t_8	Uniform
DGP1	Setting 1	1.74	1.84	1.86	15.51	15.25	15.49	59.41	59.96	59.66
	Setting 2	1.79	1.70	1.72	15.66	15.39	15.38	59.73	59.89	59.92
DGP2	Setting 1	2.45	2.41	2.44	20.60	20.16	20.31	79.09	78.80	79.17
	Setting 2	2.34	3.09	2.40	20.03	20.33	20.07	79.23	79.13	78.87
Direct Bayesian Estimation		$n = 1000$			$n = 1200$			$n = 1400$		
		Normal	t_8	Uniform	Normal	t_8	Uniform	Normal	t_8	Uniform
DGP1	Setting 1	143.24	142.17	140.51	213.16	209.94	216.30	304.52	305.17	306.40
	Setting 2	143.94	140.44	140.06	212.49	209.91	216.31	305.86	306.62	306.53
DGP2	Setting 1	174.09	167.51	168.48	256.58	253.03	253.03	366.43	366.16	364.68
	Setting 2	170.40	168.19	168.43	257.85	257.50	255.55	366.06	371.61	371.11
Exchange Algorithm		$n = 1000$			$n = 1200$			$n = 1400$		
		Normal	t_8	Uniform	Normal	t_8	Uniform	Normal	t_8	Uniform
DGP1	Setting 1	96.31	95.68	96.12	142.11	142.79	142.40	196.02	196.96	195.48
	Setting 2	95.81	96.29	96.16	142.49	142.05	142.24	196.61	197.02	196.83
DGP2	Setting 1	127.93	127.95	127.75	191.41	188.66	188.24	259.09	258.55	258.95
	Setting 2	128.20	128.31	128.12	191.42	188.31	190.54	260.78	256.76	259.68

DGP1: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0, 0.4, 0)$; DGP2: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$.
$(\beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (2, 1, 1, 1, 1)$.
All algorithms are run on a 3.30GHZ server with an Intel Xeon processor and 8 GB installed memory.
The CPU times are in seconds.

G.2. Trace plots

Figure G.1: Trace plots of Bayesian estimates of high order SARAR model under DGP2

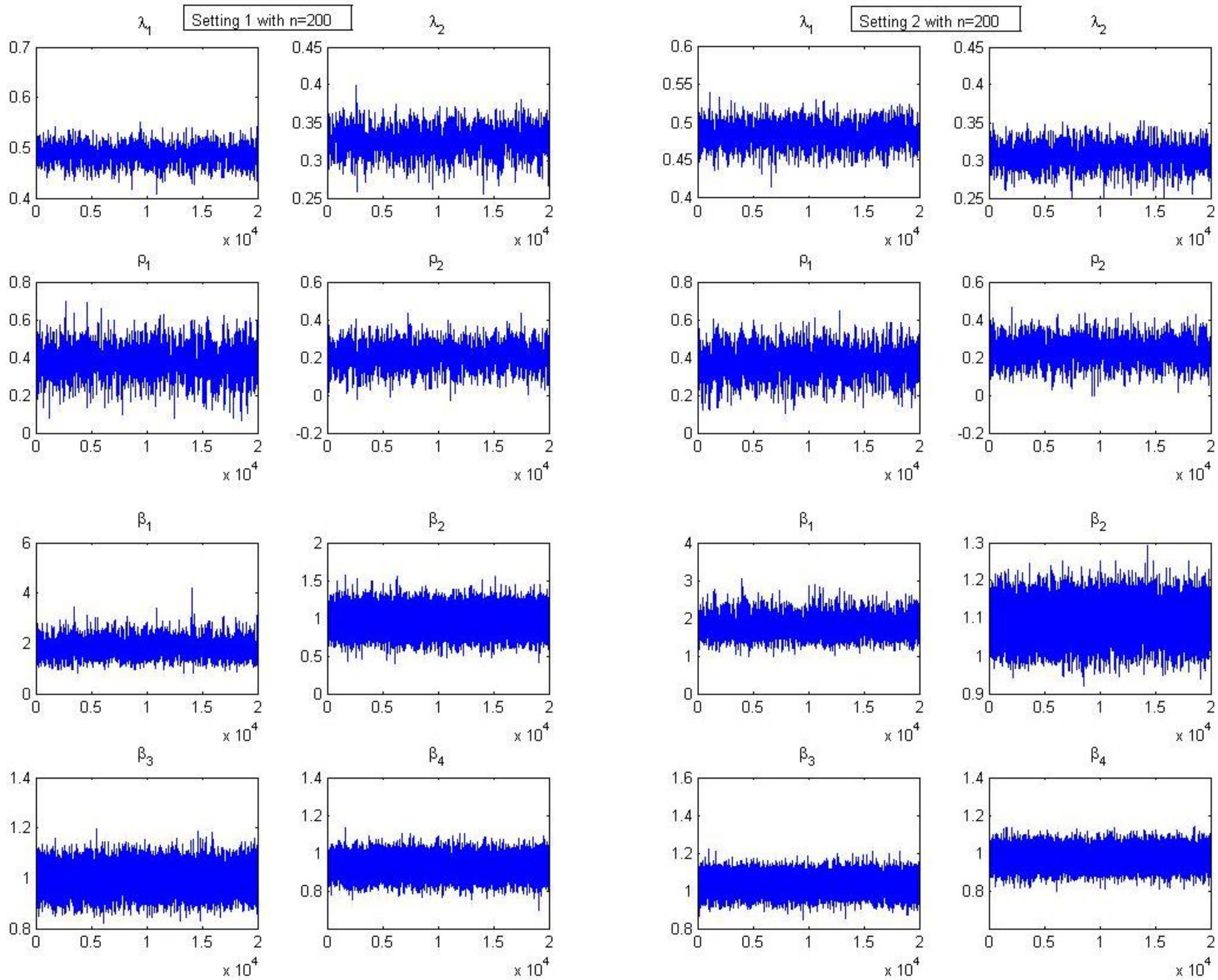
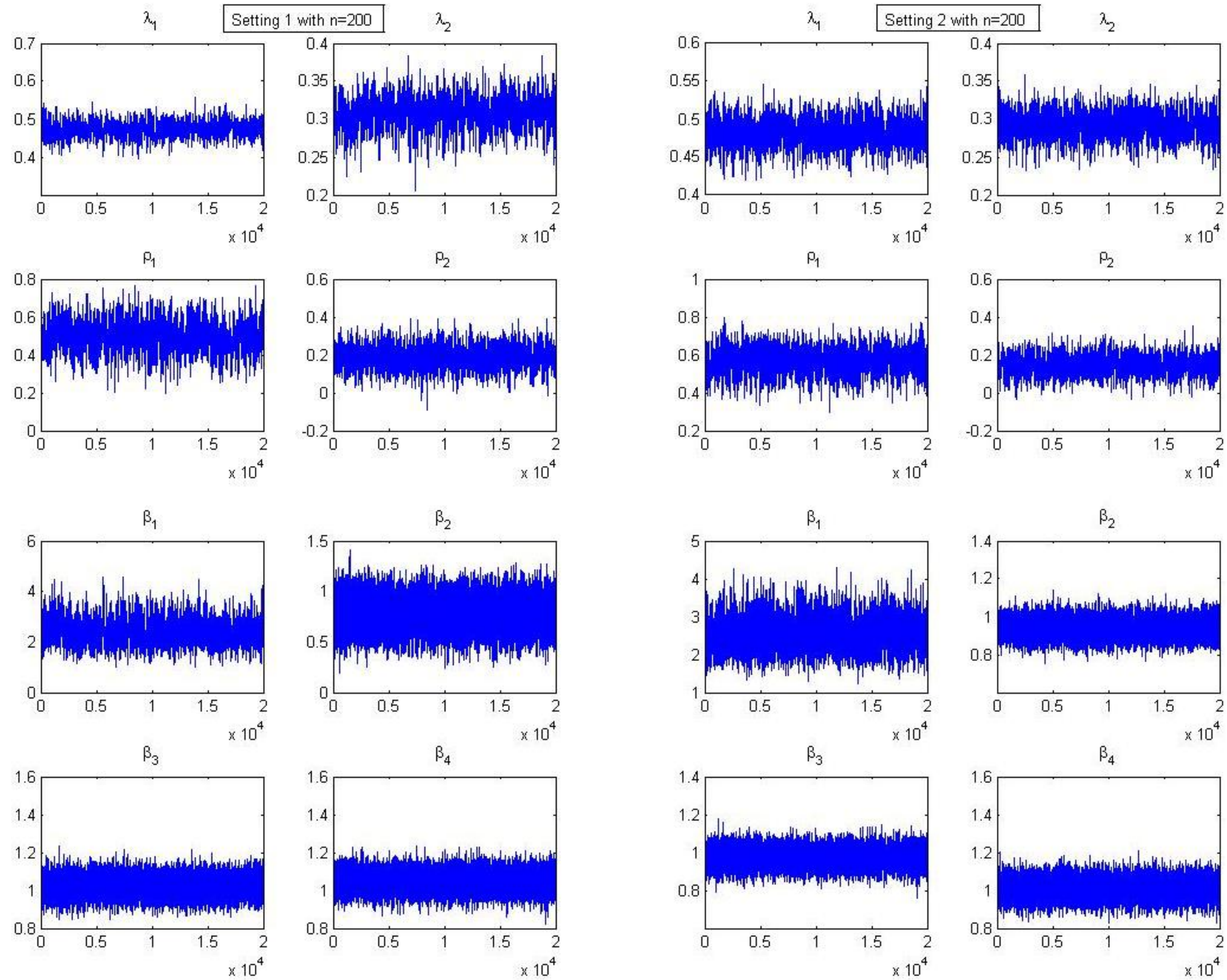


Figure G.2: Trace plots of Quasi-Bayesian estimates of high order SARAR model under DGP2: uniform errors



G.3. Marginal (quasi-)posterior densities vs normal densities

Figure G.3: Marginal posterior density vs normal density: DGP1 and DGP2 with $n = 200$

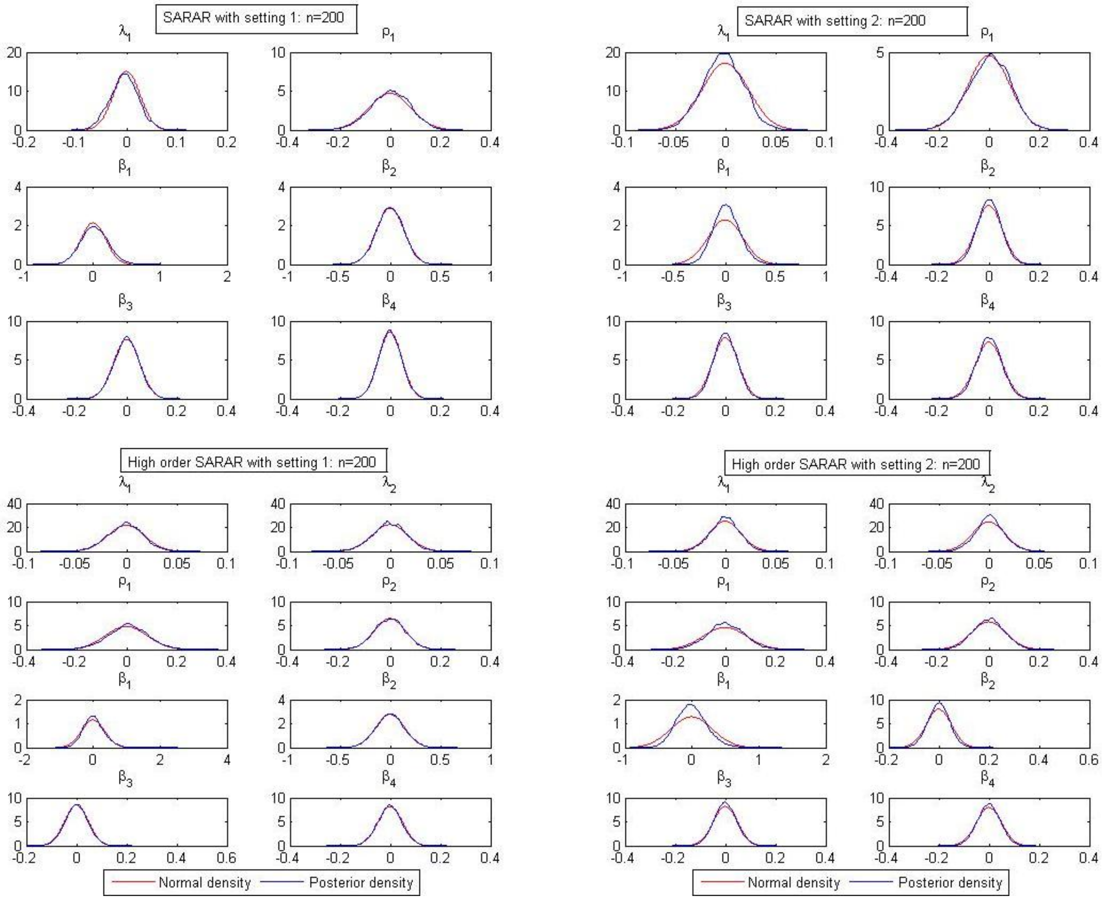


Figure G.4: Marginal posterior density vs normal density: DGP1 and DGP2 with $n = 400$

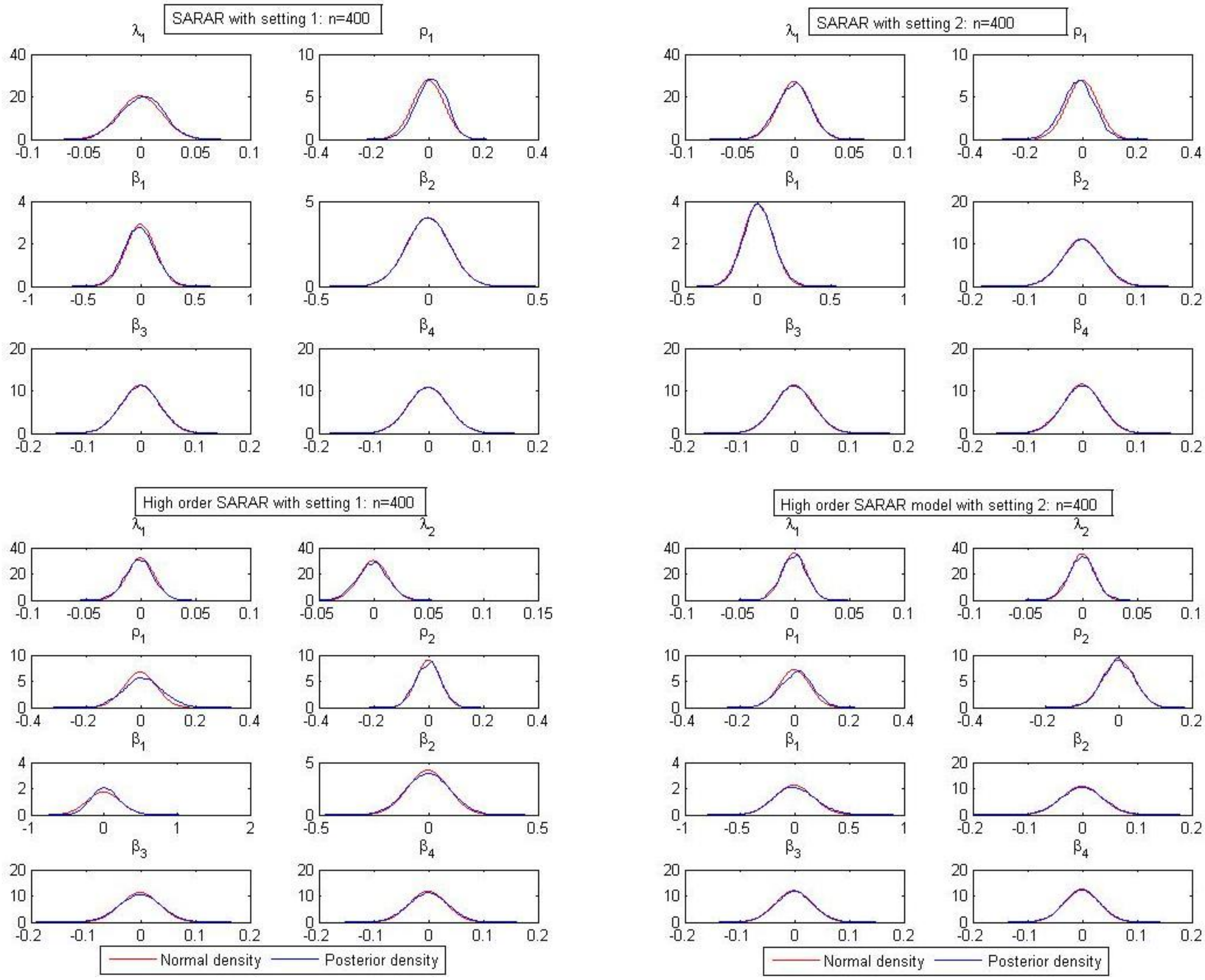


Figure G.5: Marginal posterior density vs normal density: DGP1 and DGP2 with $n = 800$

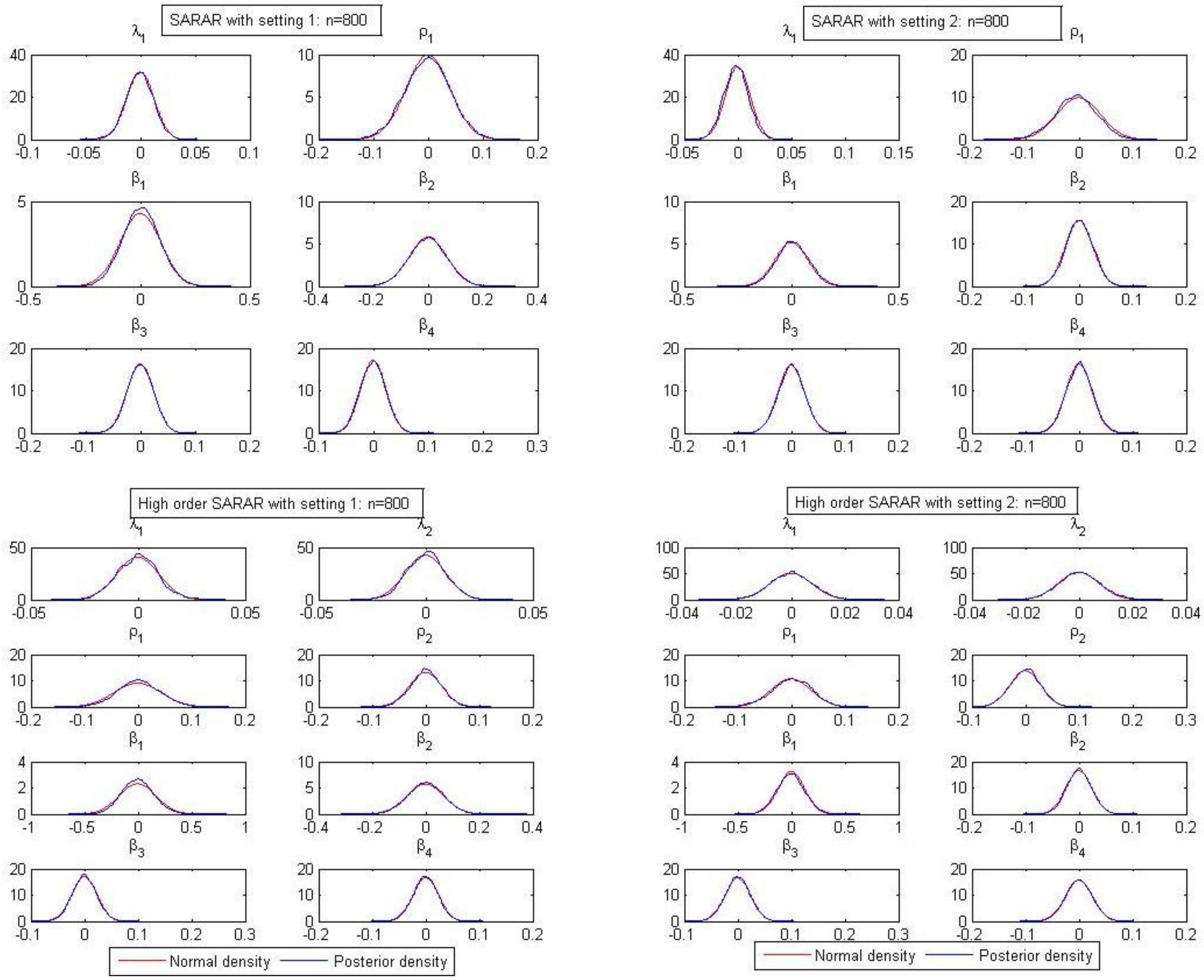


Figure G.6: Marginal quasi-posterior density vs normal density: DGP1 and DGP2 with t_6 error and $n = 200$.

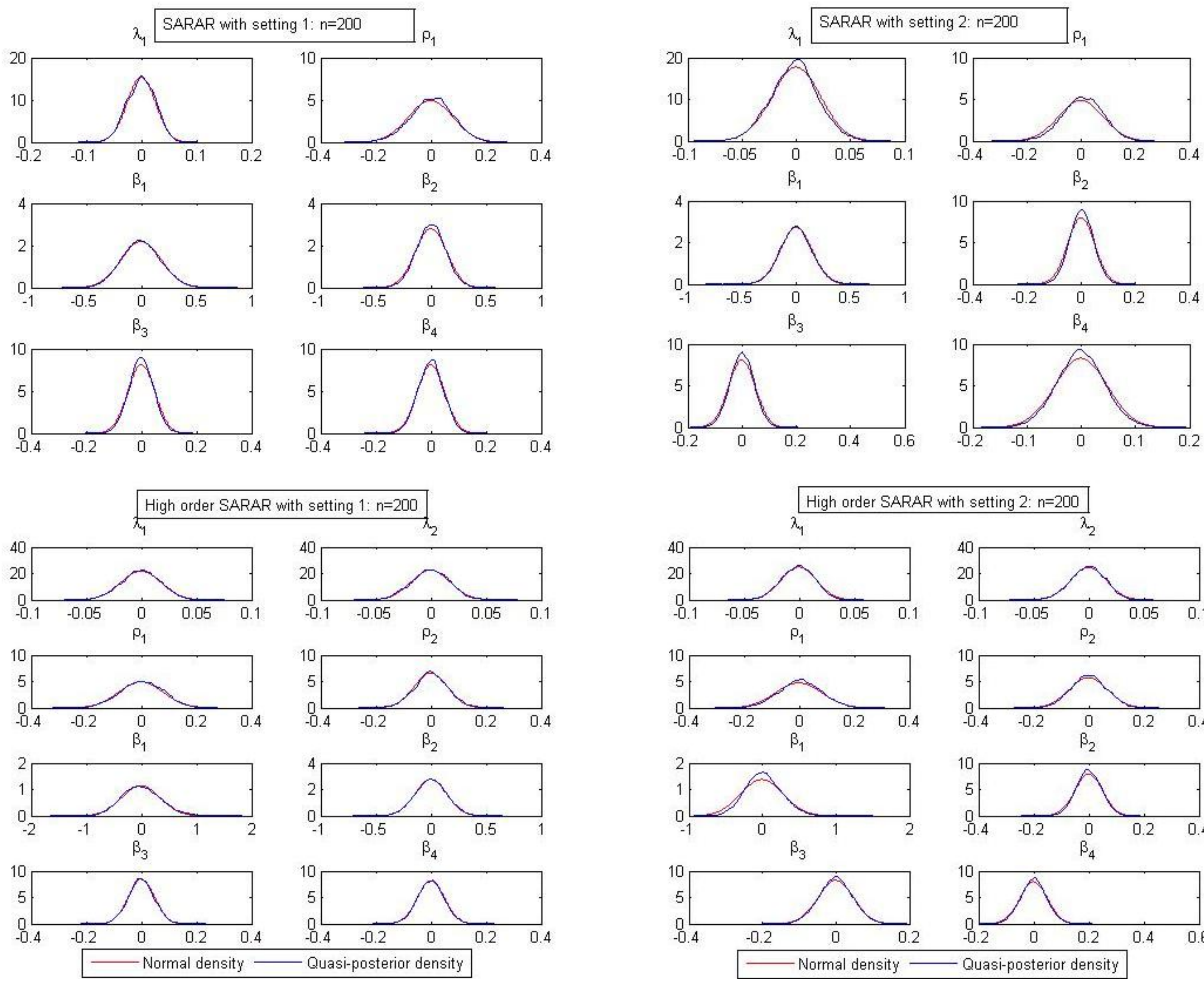


Figure G.7: Marginal quasi-posterior density vs normal density: DGP1 and DGP2 with t_6 error and $n = 400$.

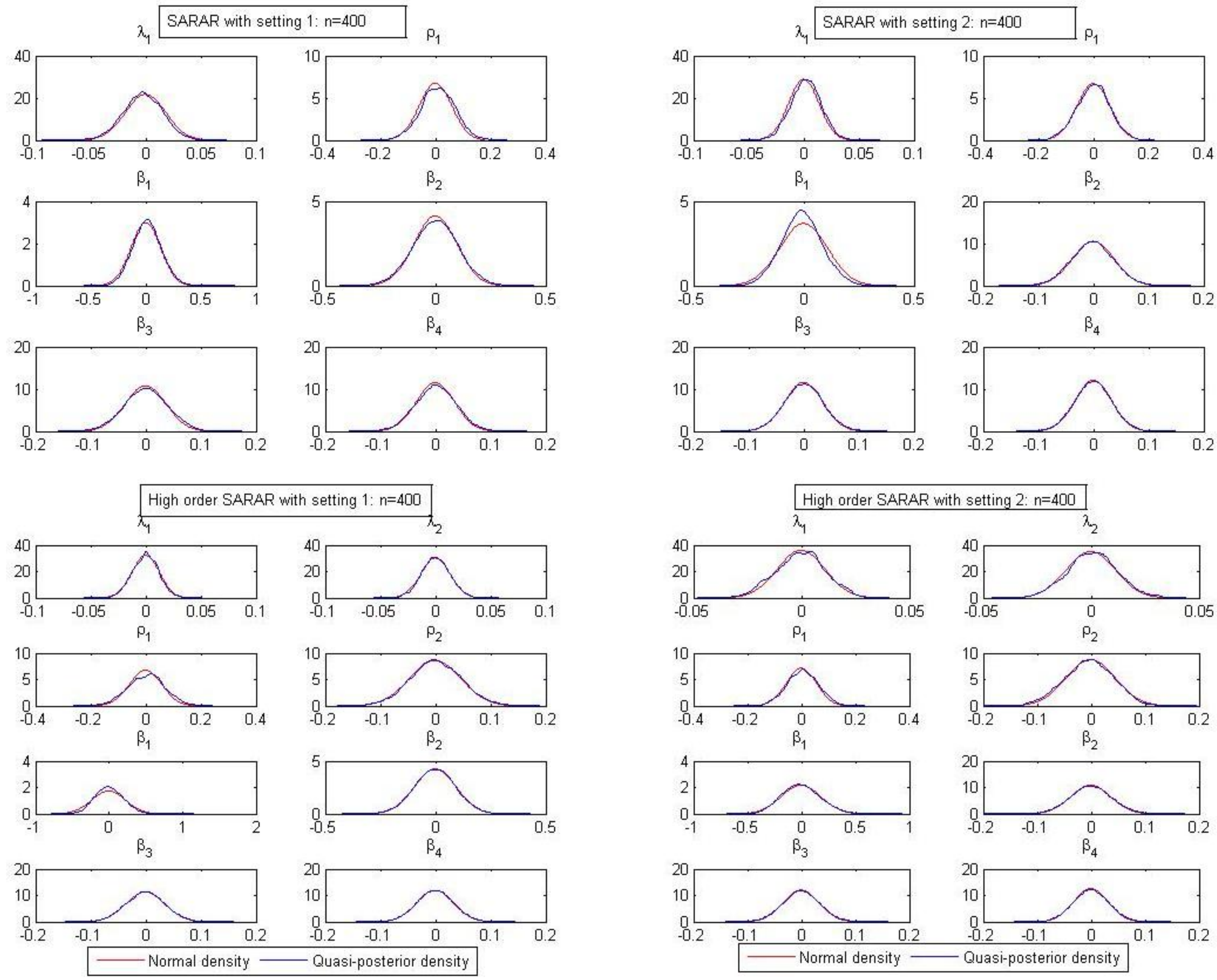


Figure G.8: Marginal quasi-posterior density vs normal density: DGP1 and DGP2 with t_6 error and $n = 800$.

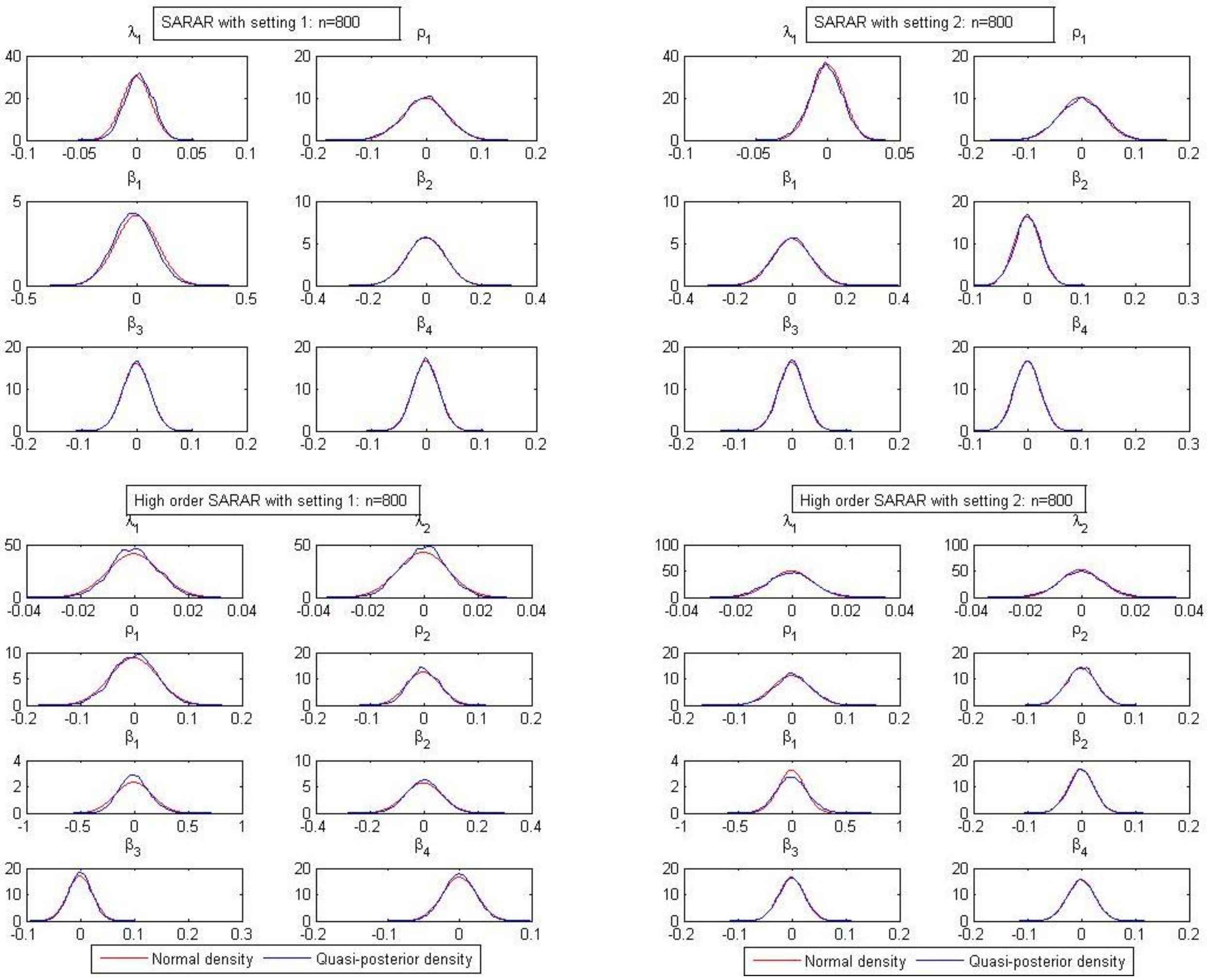


Figure G.9: Marginal quasi-posterior density vs normal density: DGP1 and DGP2 with uniform error and $n = 200$.

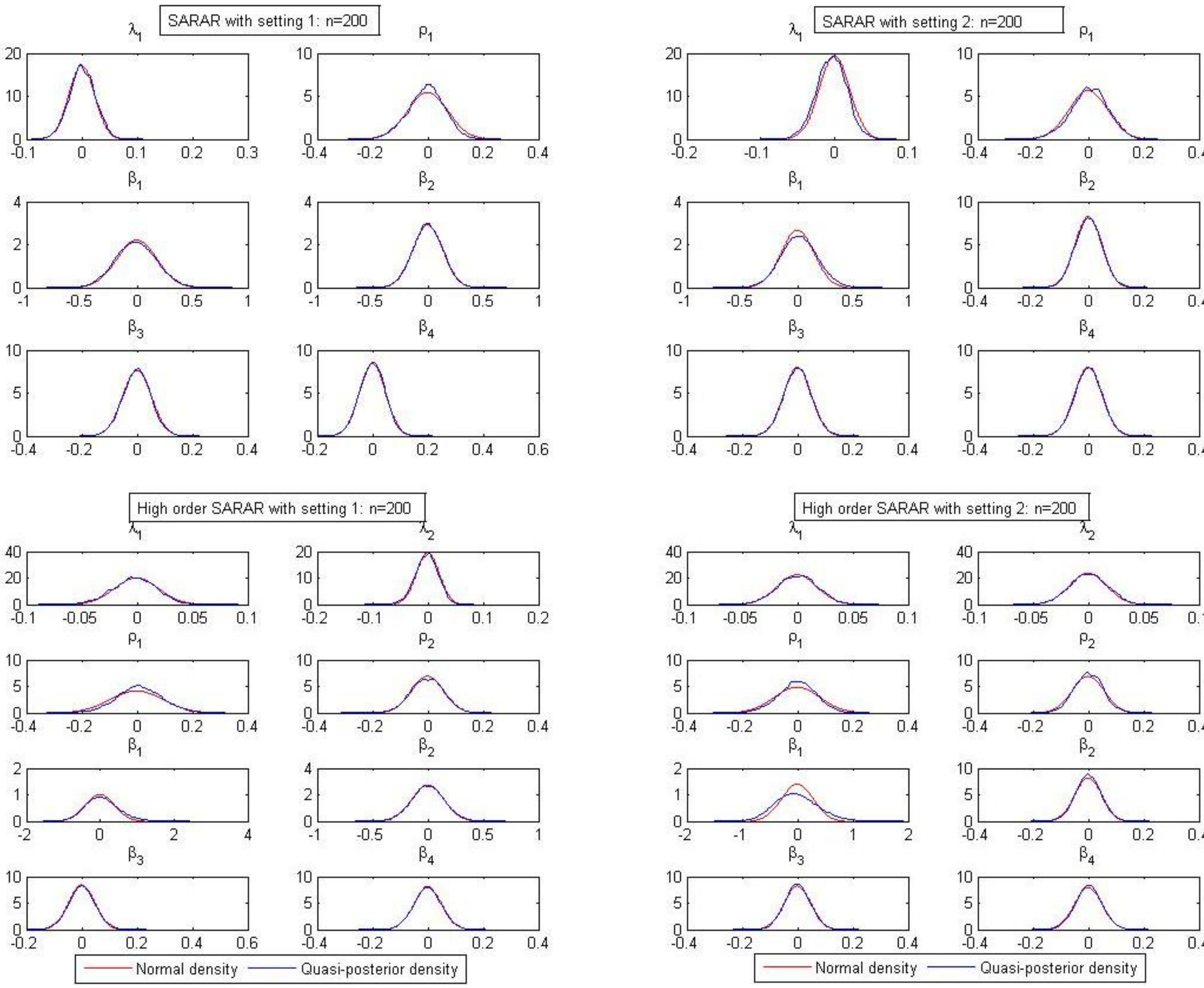


Figure G.10: Marginal quasi-posterior density vs normal density: DGP1 and DGP2 with uniform error and $n = 400$.

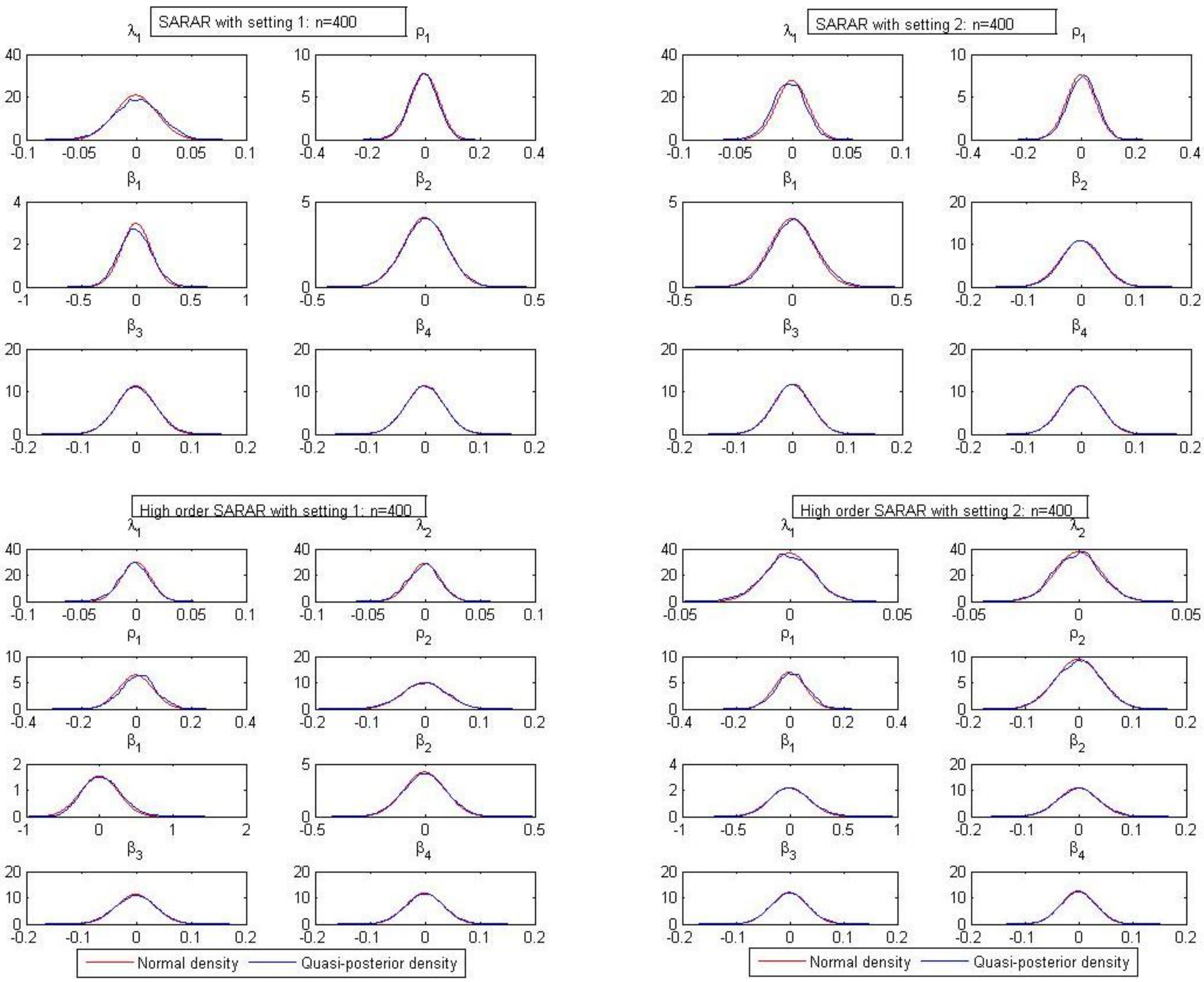


Figure G.11: Marginal quasi-posterior density vs normal density: DGP1 and DGP2 with uniform error and $n = 800$.

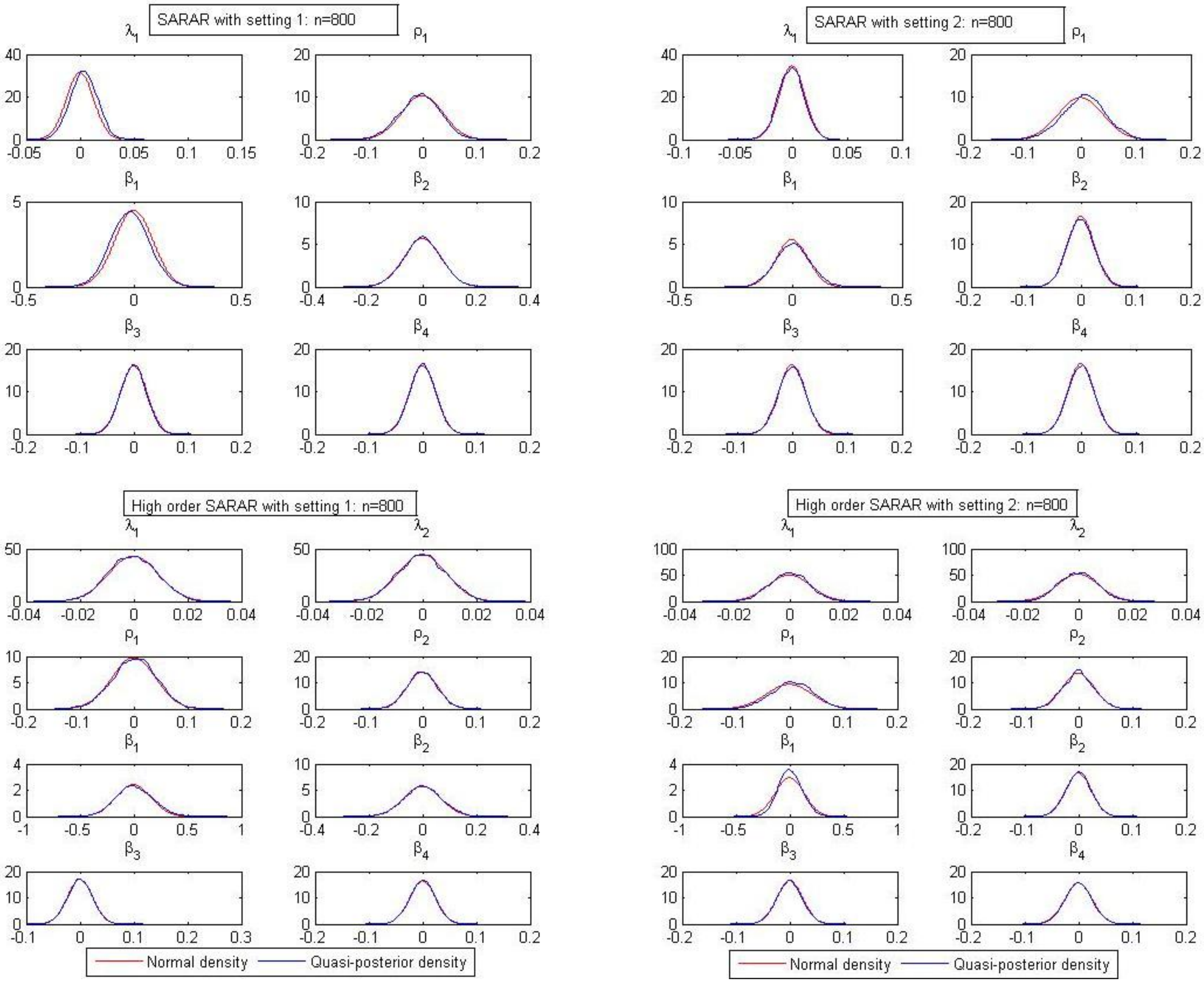


Figure G.12: Marginal posterior density vs normal density: denser spatial weights matrices

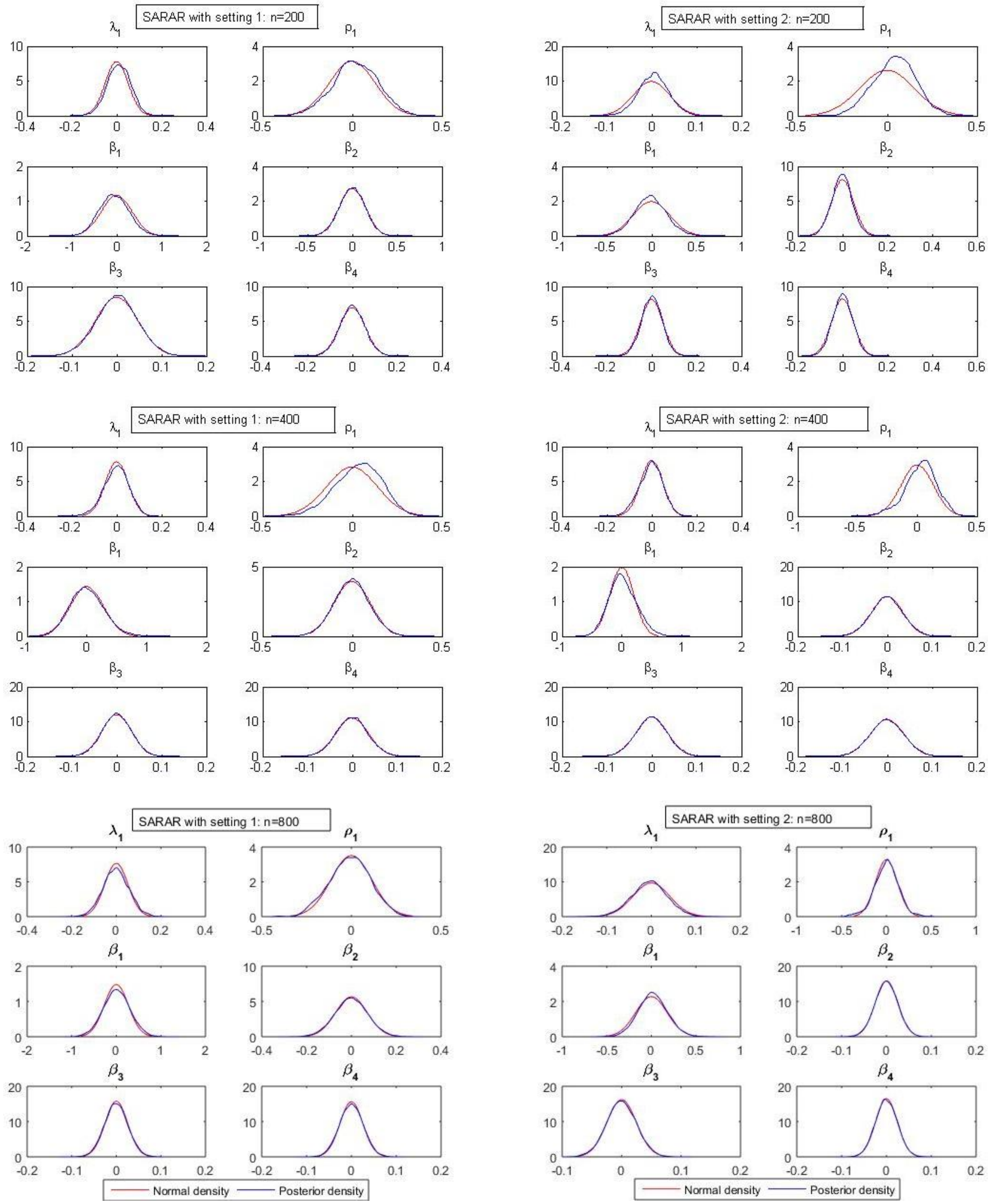


Figure G.13: Marginal quasi-posterior density vs normal density: denser spatial weights matrices with uniform error

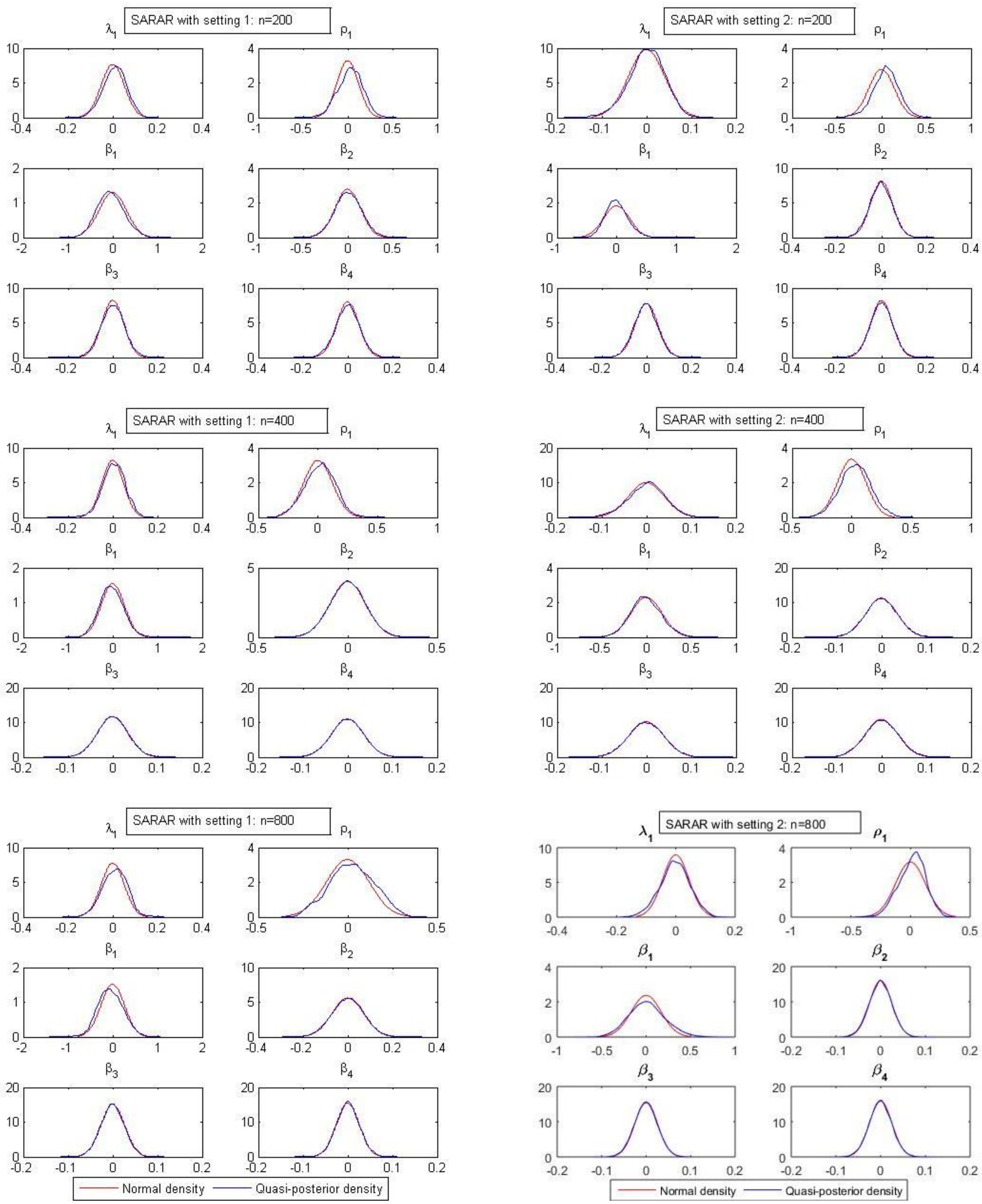


Figure G.14: Marginal posterior density vs normal density: non-row-normalized spatial weights matrices

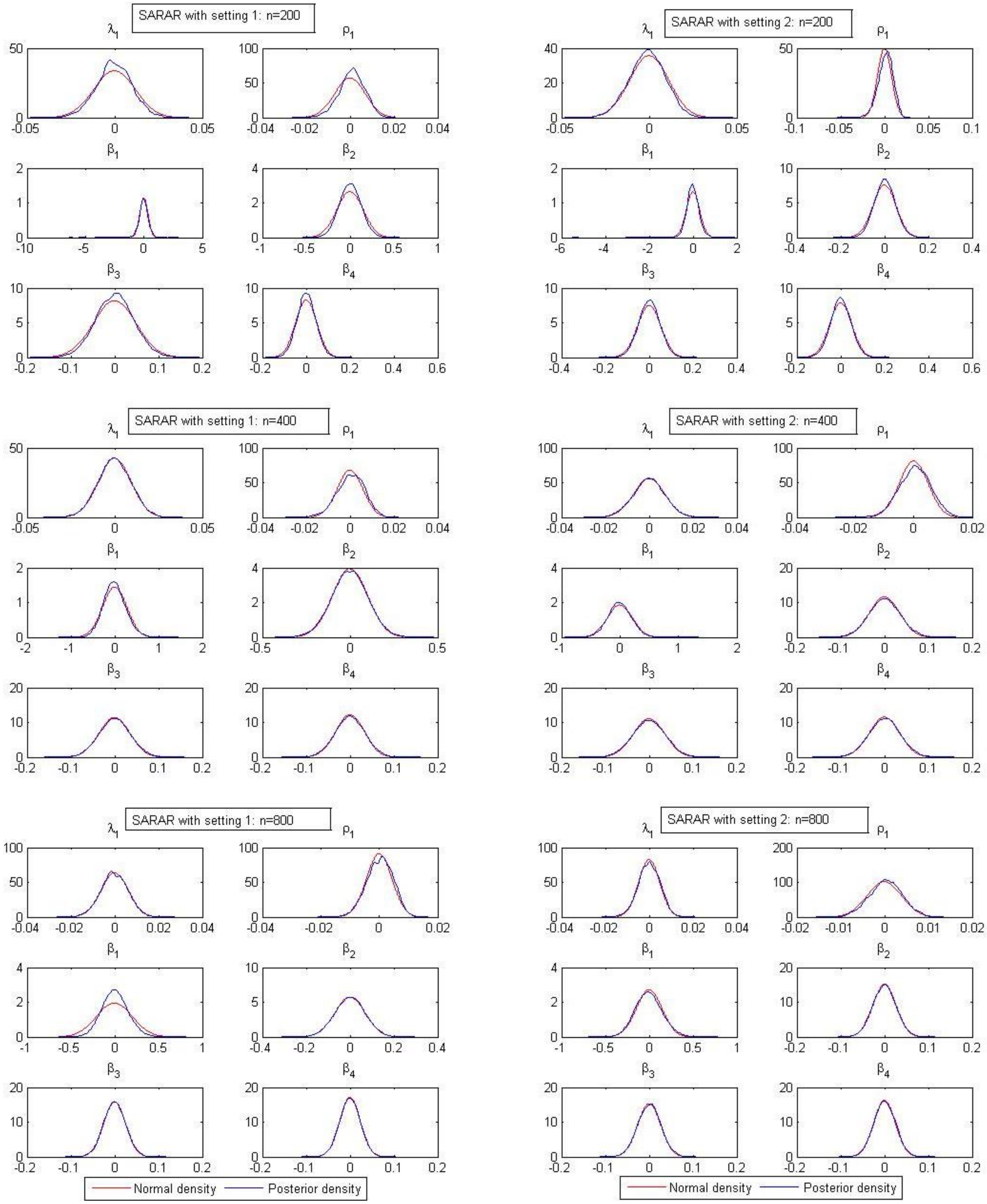


Figure G.15: Marginal quasi-posterior density vs normal density: non-row-normalized spatial weights matrices with uniform error

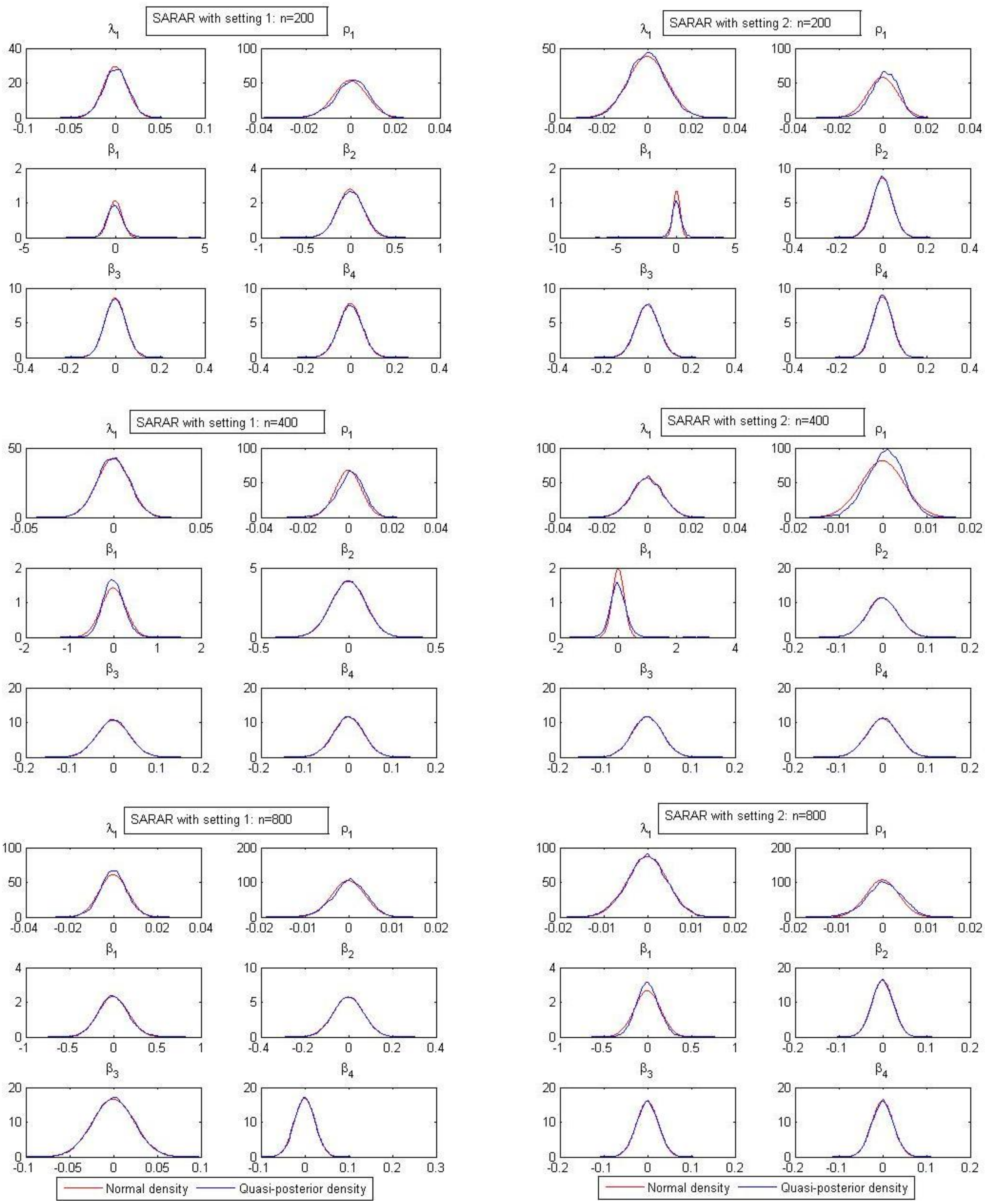


Figure G.16: Marginal posterior density vs normal density: DGP2 with broader stability condition

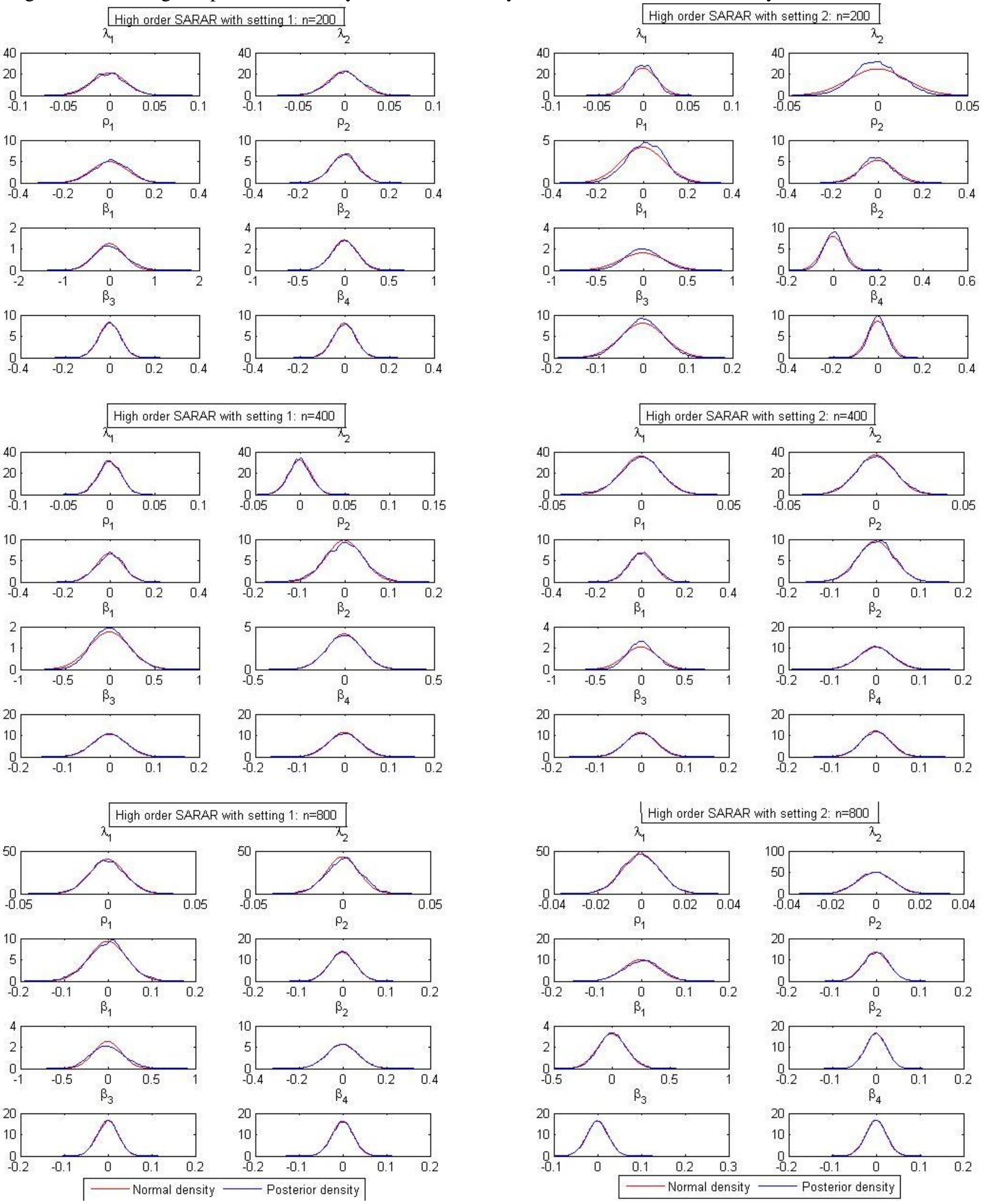
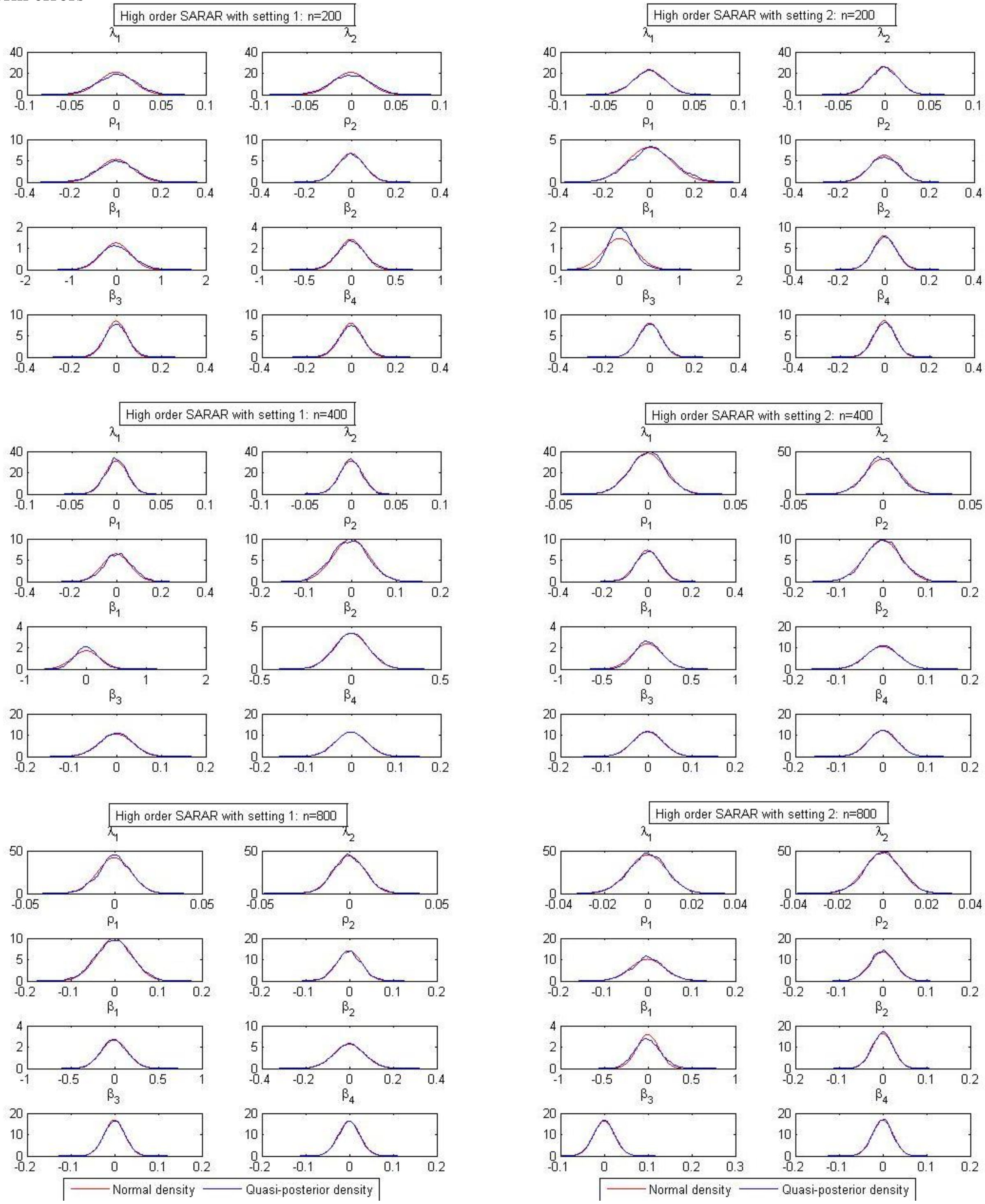


Figure G.17: Marginal quasi-posterior density vs normal density: DGP2 with broader stability condition and uniform errors



G.4. Empirical Densities of (Quasi-)Bayesian Estimates vs Normal Densities

Figure G.18: Empirical density of Bayesian estimates vs normal density: DGP1 with $n = 200$

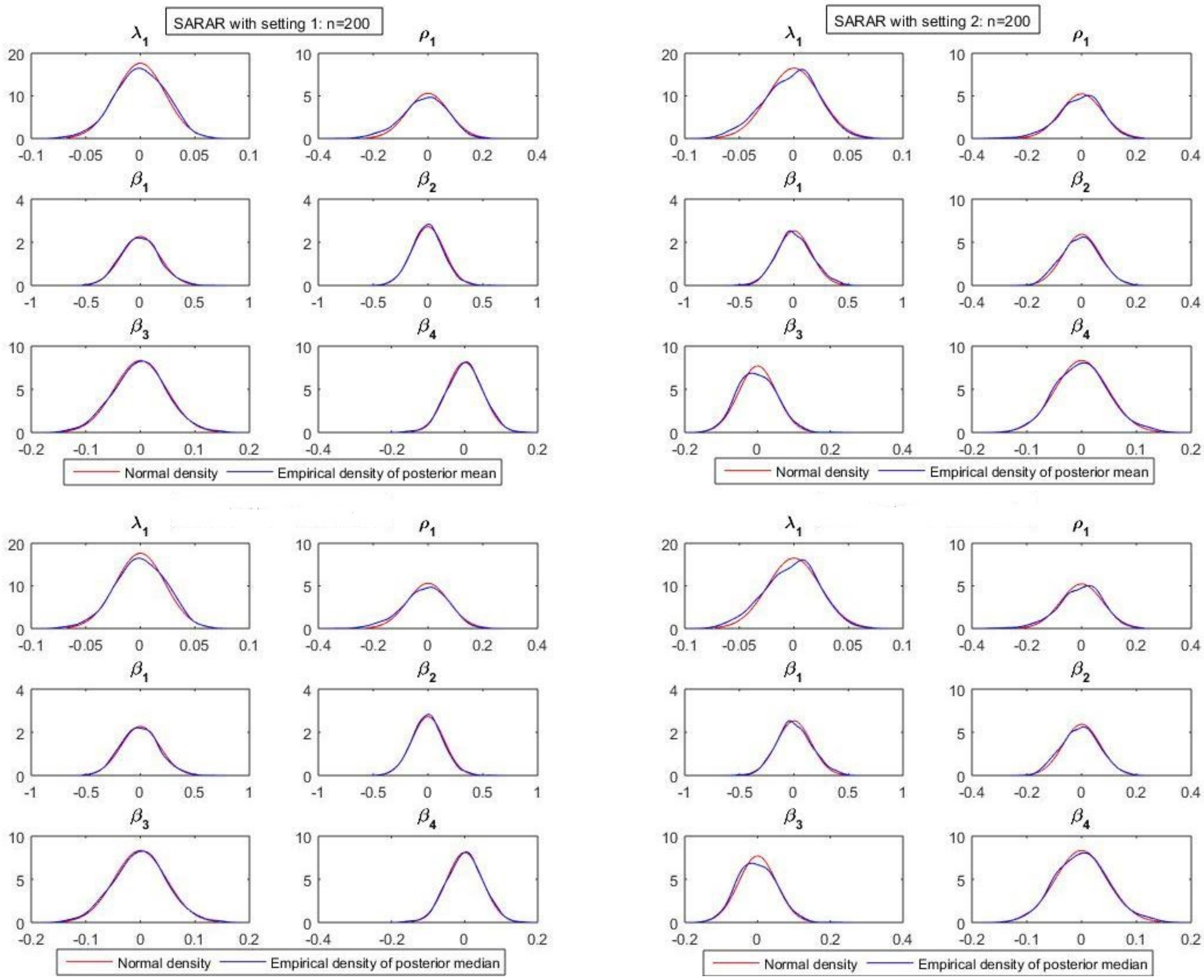


Figure G.19: Empirical density of Bayesian estimates vs normal density: DGP1 with $n = 400$

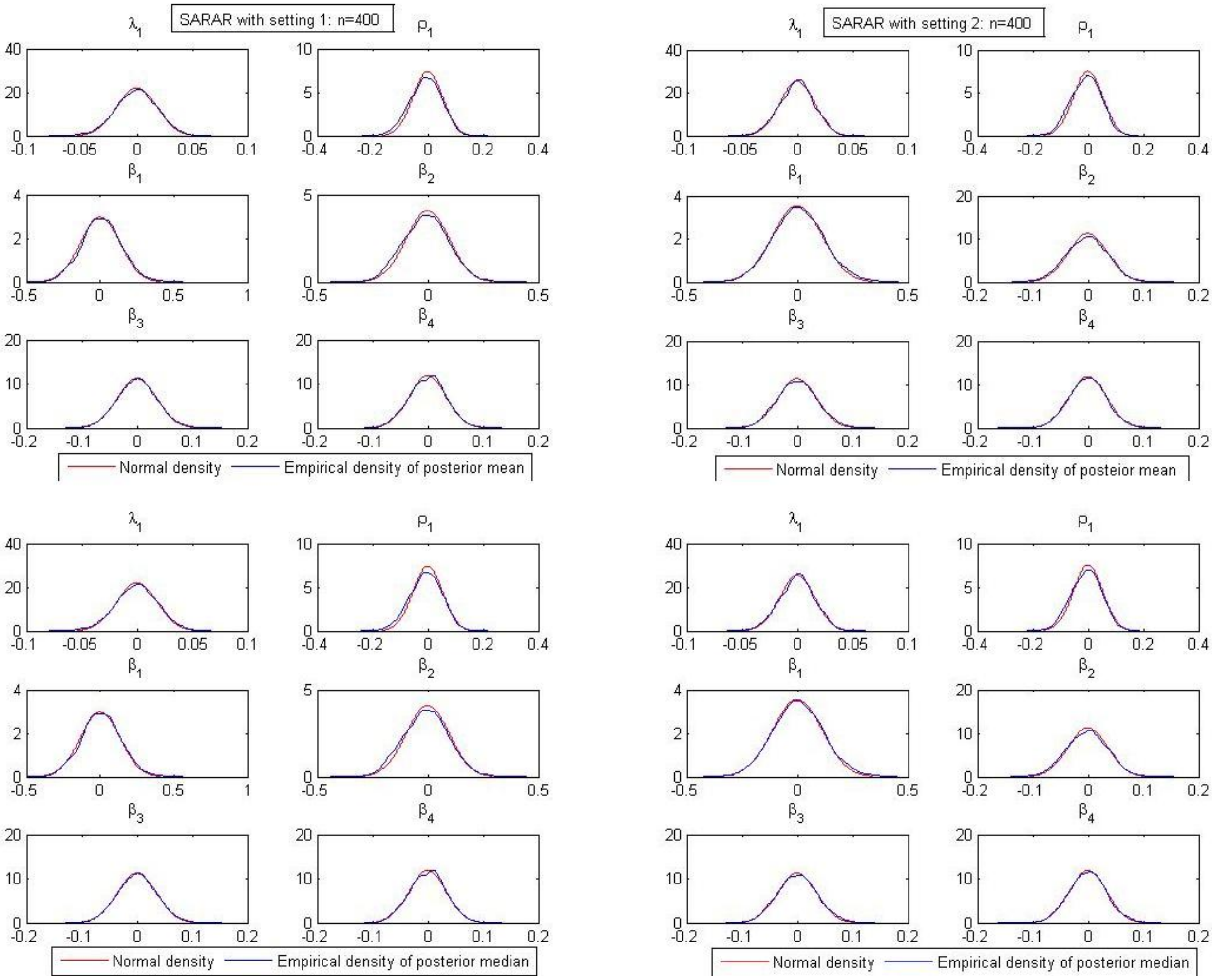


Figure G.20: Empirical density of Bayesian estimates vs normal density: DGP1 with $n = 800$

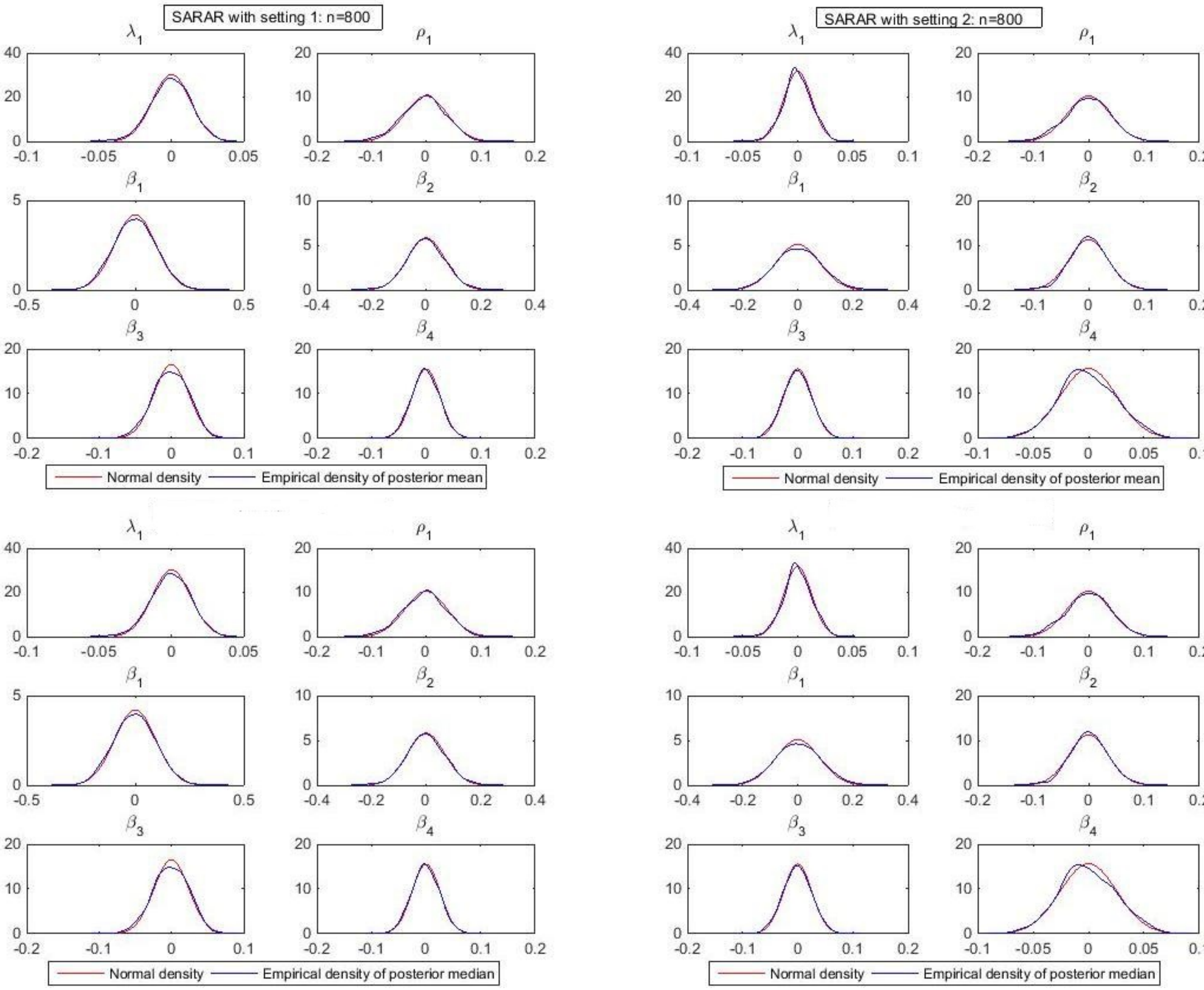


Figure G.21: Empirical density of Bayesian estimates vs normal density: DGP2 with $n = 200$

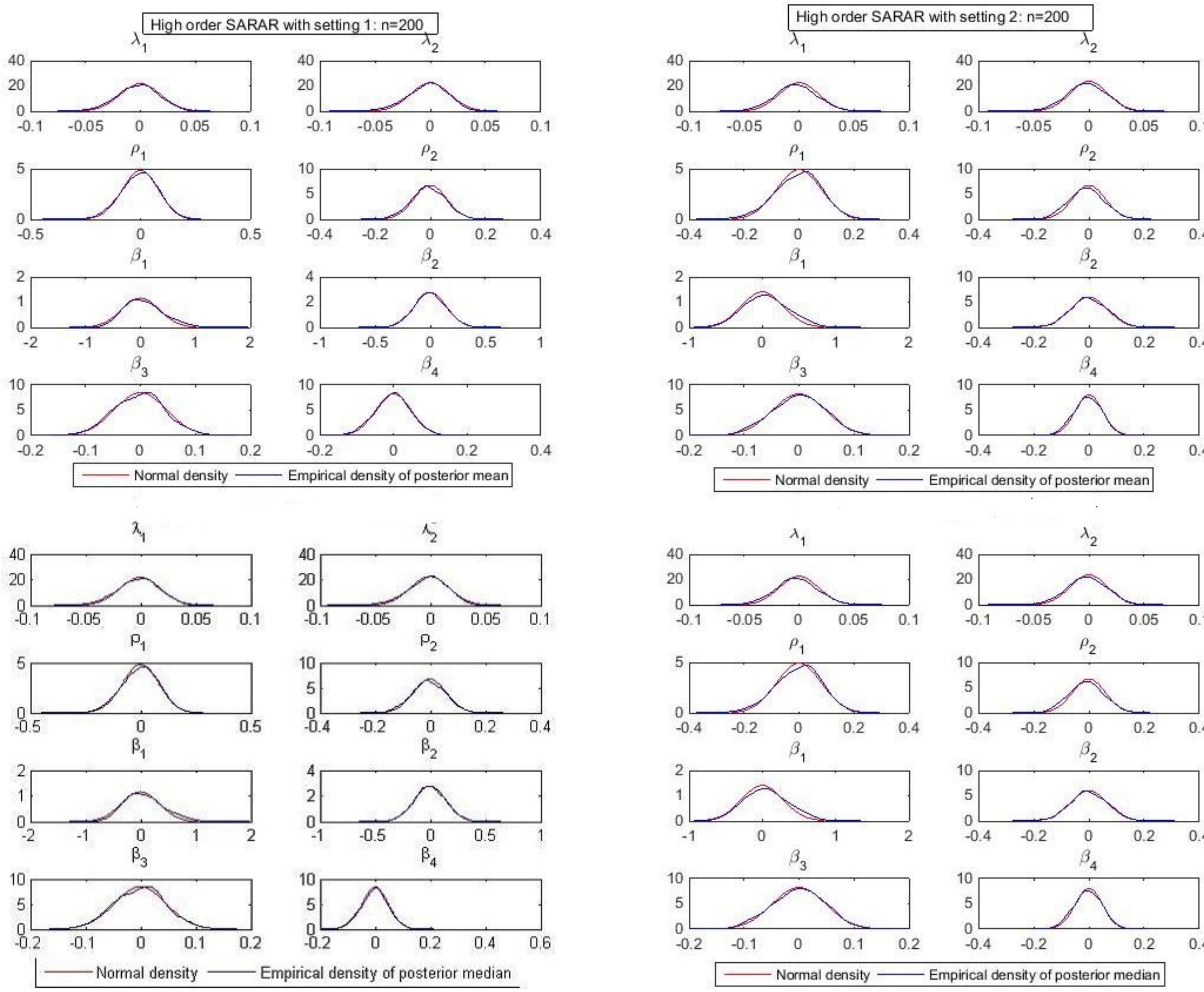


Figure G.22: Empirical density of Bayesian estimates vs normal density: DGP2 with $n = 400$

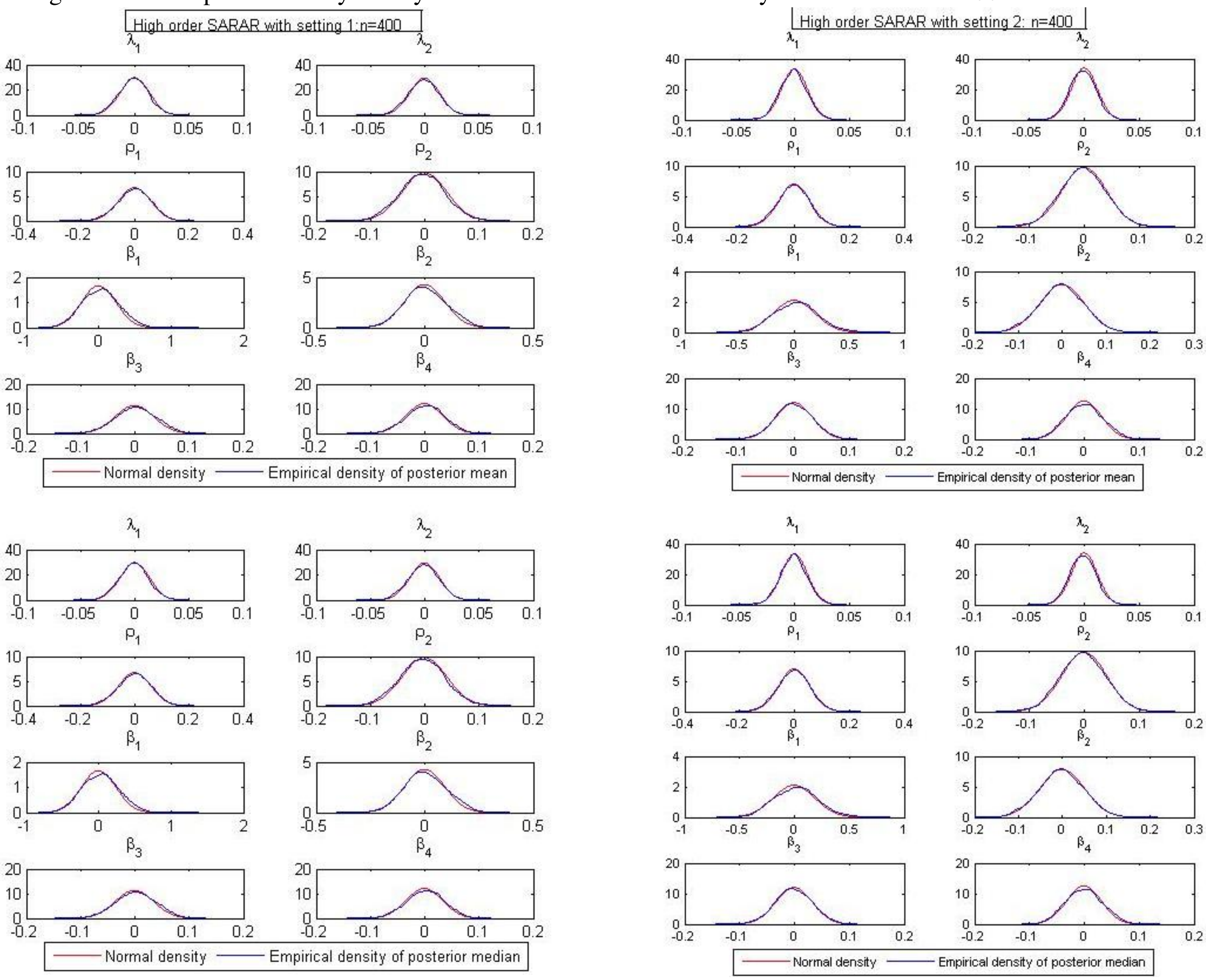


Figure G.23: Empirical density of Bayesian estimates vs normal density: DGP2 with $n = 800$

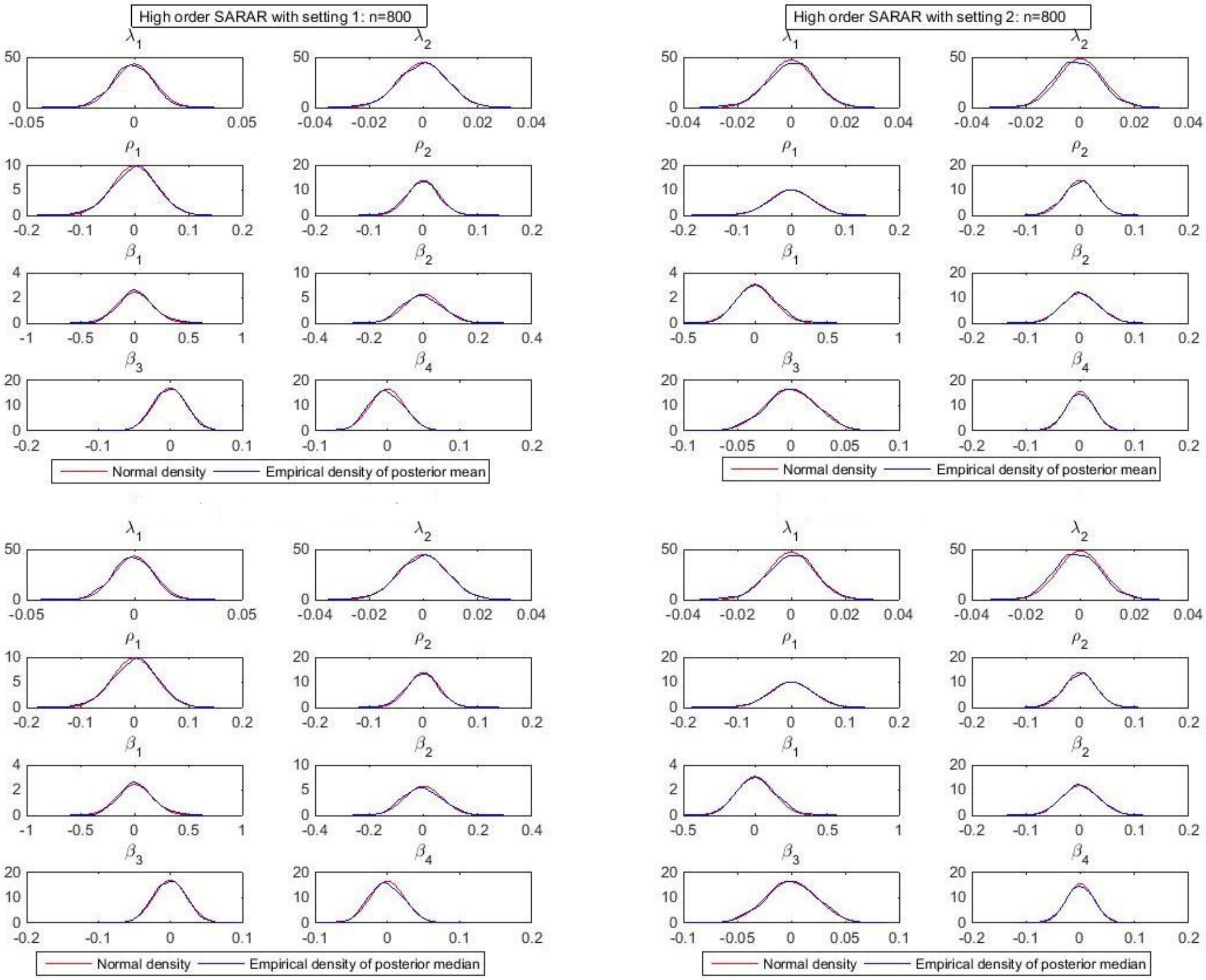


Figure G.24: Empirical density of Quasi-Bayesian estimates vs normal density: DGP1 with $n = 200$ and uniform errors

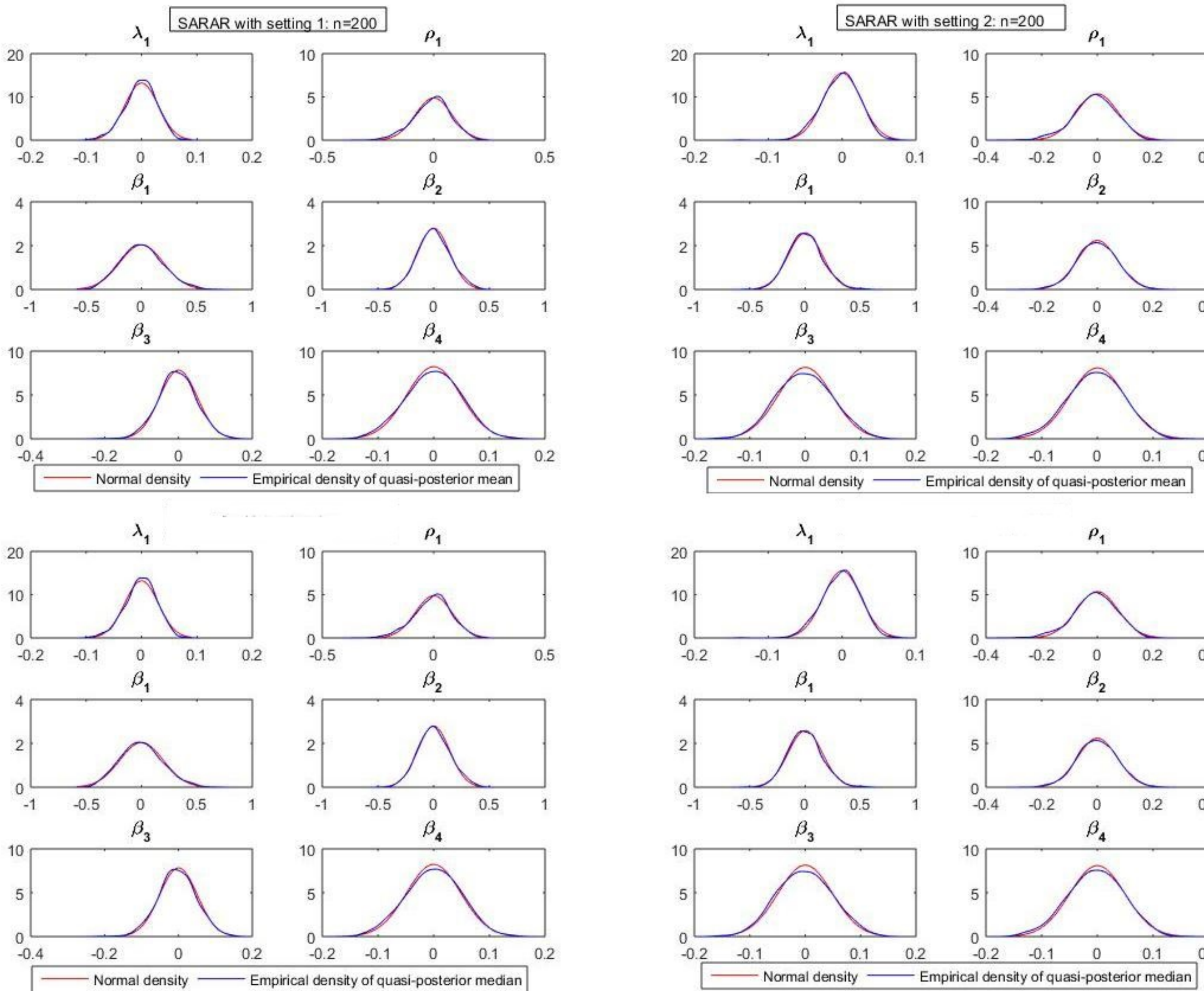


Figure G.25: Empirical density of Quasi-Bayesian estimates vs normal density: DGP1 with $n = 800$ and uniform errors

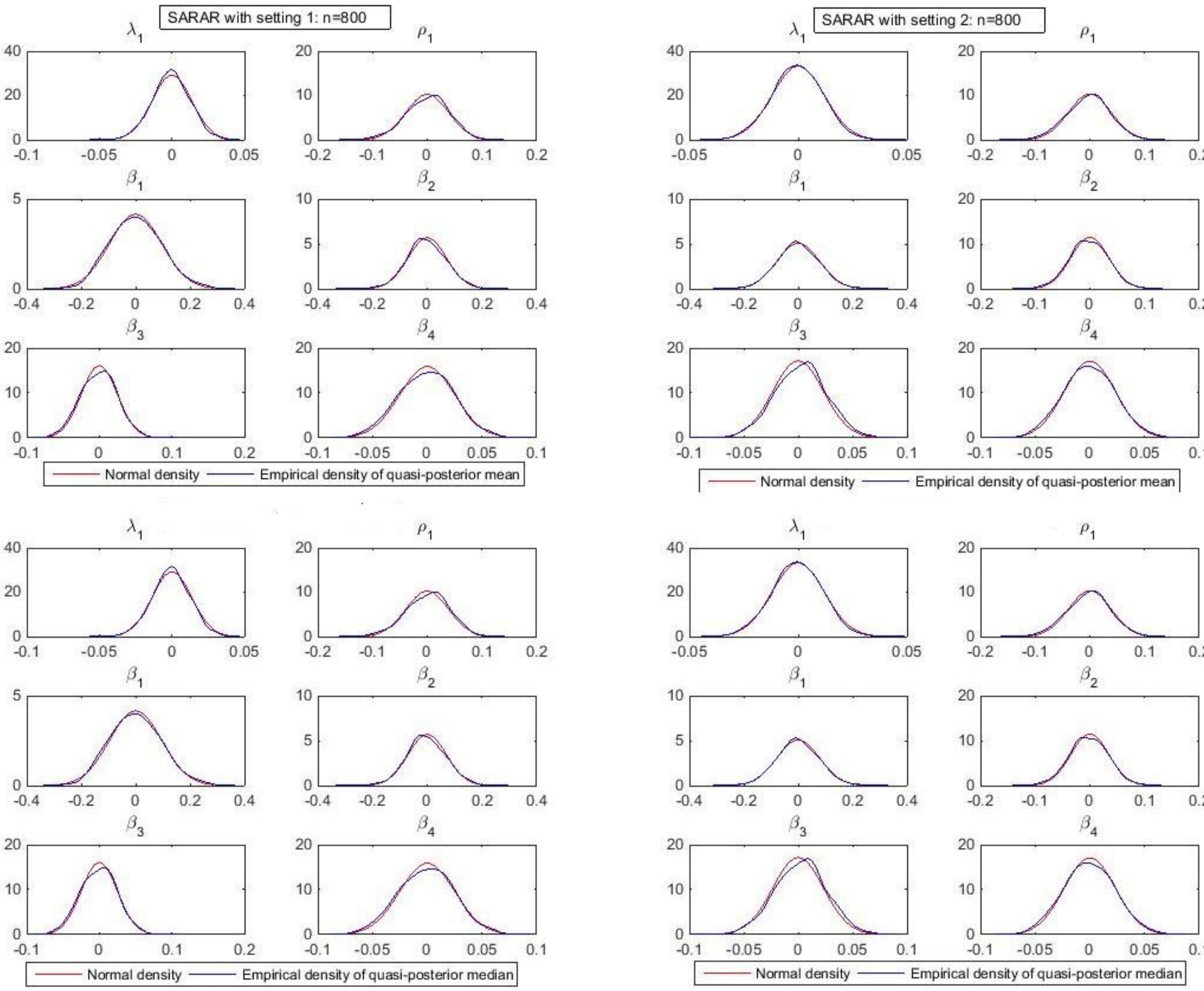


Figure G.26: Empirical density of Bayesian estimates vs normal density under denser spatial weights: DGP1 with $n = 200$

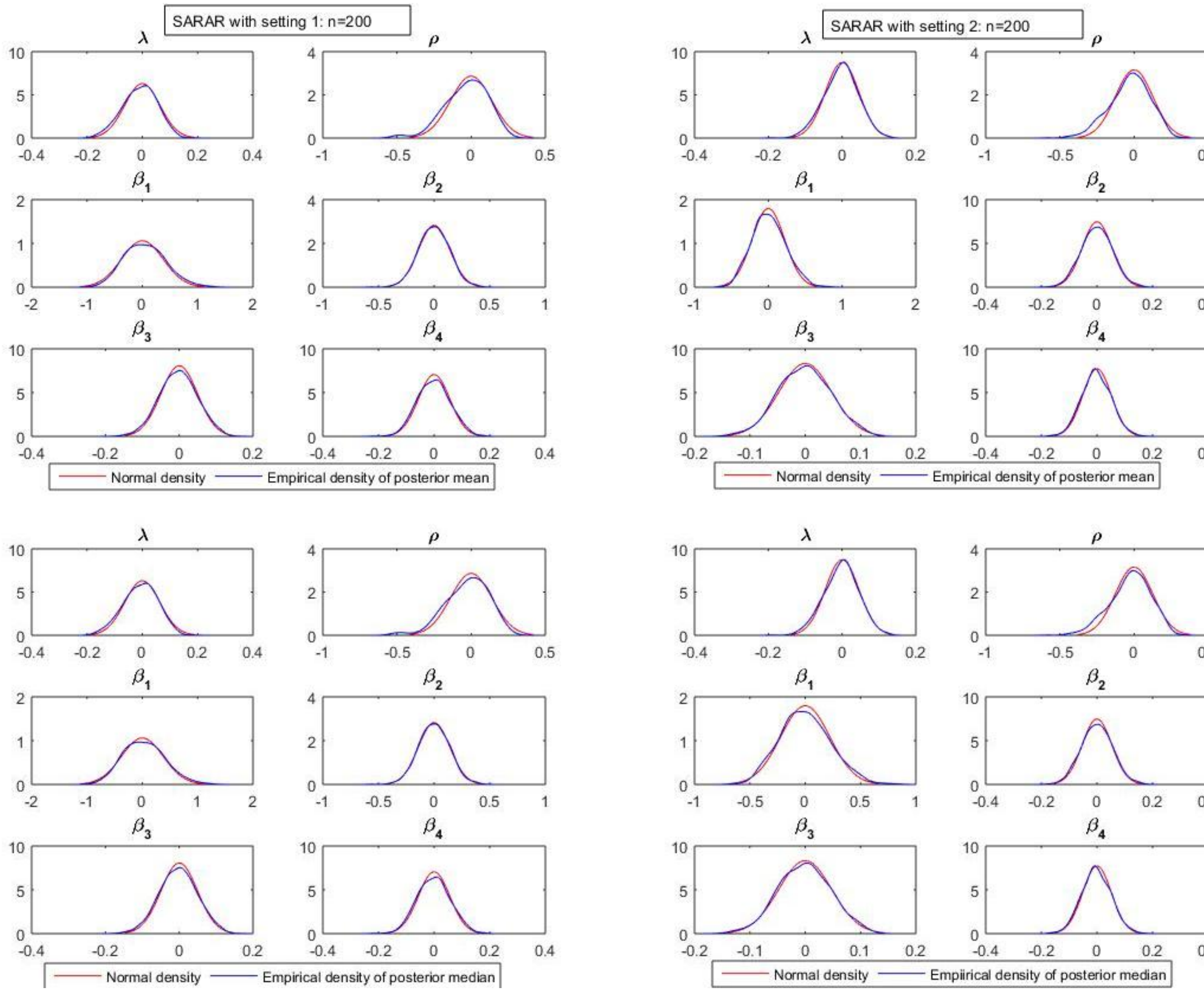


Figure G.27: Empirical density of Bayesian estimates vs normal density under non-row-normalized spatial weights matrices: DGP1 with $n = 200$

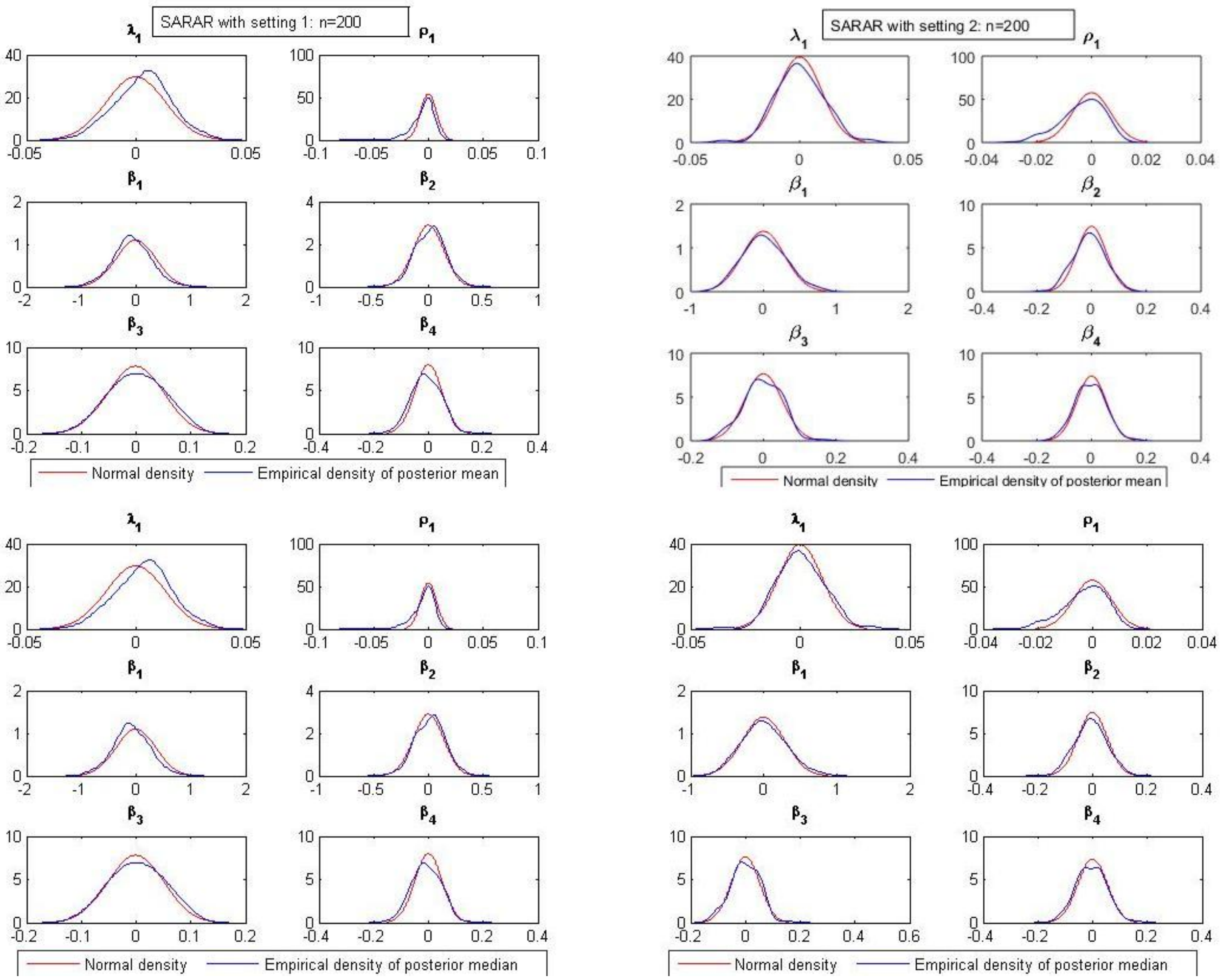


Figure G.28: Empirical density of Bayesian estimates vs normal density under non-row-normalized spatial weights matrices: DGP1 with $n = 800$

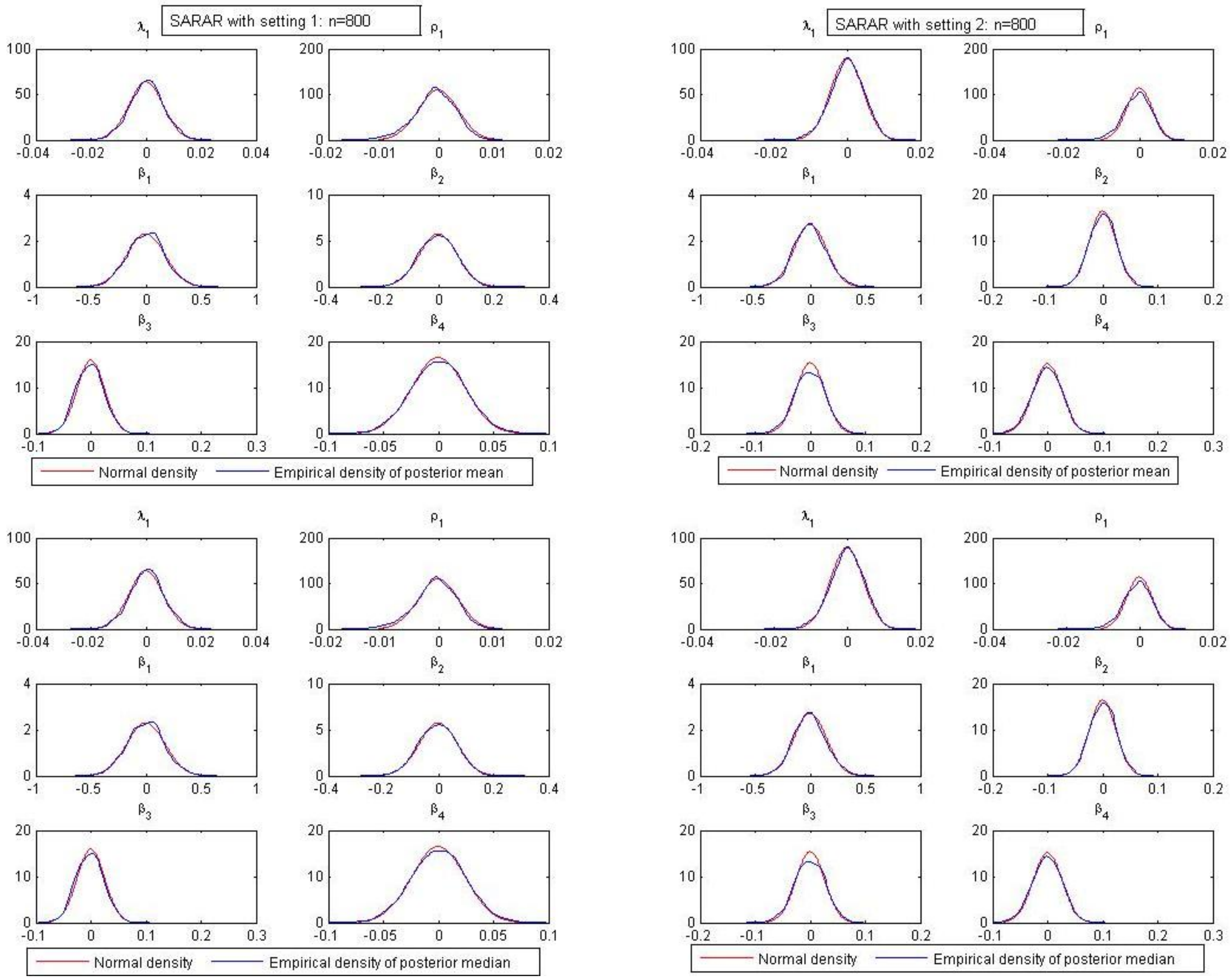
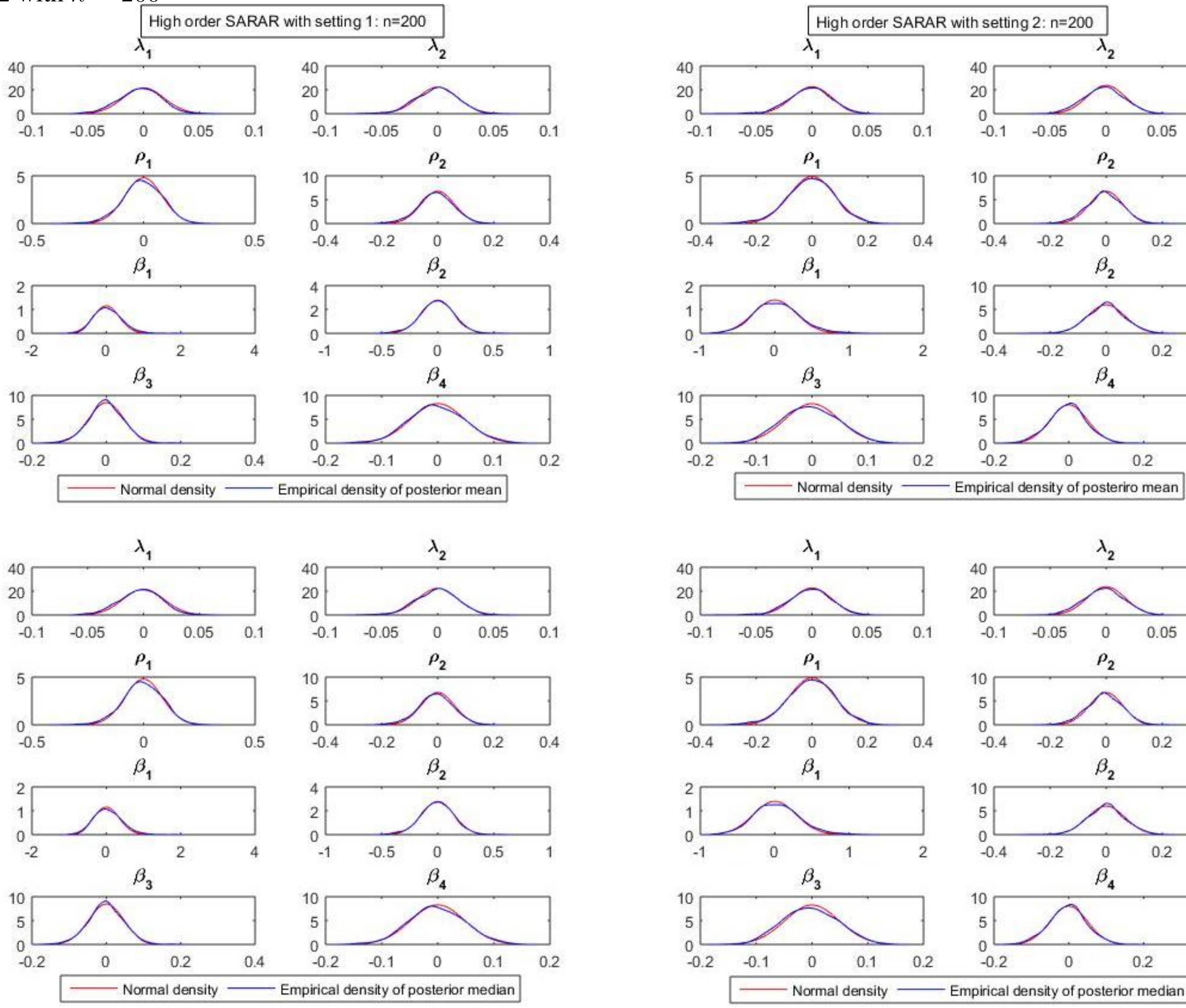


Figure G.29: Empirical density of Bayesian estimates vs normal density under broader stability condition: DGP2 with $n = 200$



MC results for SARAR model in tables

Table 5: Model estimation over 1000 repetitions: normal error

		$n = 200$				$n = 400$				$n = 800$			
		MLE		BE		MLE		BE		MLE		BE	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
DGP1	λ_1	-0.002	0.026	-0.003	0.025	0.000	0.019	-0.001	0.018	-0.000	0.012	-0.000	0.012
S1	ρ_1	-0.014	0.083	-0.010	0.080	0.01	0.059	0.008	0.057	-0.002	0.041	-0.001	0.041
	β_1	0.005	0.194	0.009	0.193	0.007	0.134	0.011	0.133	0.004	0.092	0.004	0.093
	β_2	0.003	0.146	0.002	0.146	-0.008	0.101	-0.008	0.101	-0.001	0.066	-0.001	0.066
	β_3	-0.001	0.048	-0.001	0.048	0.001	0.035	0.001	0.033	0.002	0.025	0.002	0.025
	β_4	0.000	0.053	-0.000	0.053	0.000	0.033	0.000	0.033	-0.000	0.025	-0.000	0.024
	σ^2	-0.028	0.100	-0.008	0.096	-0.013	0.074	-0.003	0.073	-0.006	0.051	-0.001	0.051
DGP2	λ_1	-0.002	0.018	-0.003	0.018	-0.001	0.013	-0.001	0.013	-0.001	0.009	-0.001	0.009
S1	λ_2	-0.000	0.018	-0.001	0.018	-0.001	0.014	-0.001	0.014	0.000	0.009	-0.0002	0.009
	ρ_1	-0.012	0.091	-0.005	0.087	-0.005	0.061	-0.001	0.059	-0.002	0.041	-0.001	0.041
	ρ_2	-0.006	0.063	-0.007	0.062	-0.004	0.041	-0.005	0.041	-0.002	0.029	-0.003	0.029
	β_1	0.020	0.363	0.048	0.365	0.018	0.249	0.030	0.249	0.011	0.167	0.017	0.166
	β_2	-0.006	0.144	-0.007	0.144	0.004	0.096	0.004	0.096	-0.004	0.069	-0.005	0.069
	β_3	-0.001	0.046	-0.002	0.046	0.001	0.037	0.001	0.037	-0.001	0.024	-0.001	0.024
	β_4	-0.001	0.048	-0.001	0.048	0.001	0.048	0.001	0.034	-0.000	0.024	-0.001	0.024
	σ^2	-0.038	0.109	-0.008	0.103	-0.016	0.076	-0.001	0.075	-0.012	0.053	-0.004	0.052
DGP1	λ_1	-0.001	0.025	-0.002	0.025	-0.000	0.016	-0.001	0.016	-0.001	0.013	0.000	0.012
S2	ρ_1	-0.014	0.085	-0.009	0.080	-0.008	0.055	-0.008	0.054	-0.000	0.041	-0.003	0.039
	β_1	-0.001	0.158	0.001	0.157	0.007	0.113	0.007	0.113	0.000	0.079	-0.003	0.079
	β_2	0.000	0.067	-0.000	0.067	-0.002	0.037	-0.002	0.037	-0.001	0.037	-0.000	0.036
	β_3	-0.002	0.050	-0.002	0.049	0.001	0.035	0.000	0.035	0.001	0.026	0.001	0.025
	β_4	0.000	0.054	-0.001	0.054	-0.001	0.034	-0.001	0.034	-0.000	0.027	0.001	0.026
	σ^2	-0.029	0.106	-0.009	0.102	-0.014	0.074	-0.004	0.072	-0.001	0.050	0.000	0.050
DGP2	λ_1	-0.001	0.018	-0.002	0.018	-0.000	0.012	-0.001	0.012	-0.000	0.009	-0.000	0.009
S2	λ_2	-0.001	0.017	-0.001	0.017	-0.001	0.012	-0.001	0.012	-0.000	0.008	-0.000	0.008
	ρ_1	-0.011	0.086	-0.004	0.082	-0.007	0.058	-0.003	0.057	-0.004	0.040	-0.002	0.040
	ρ_2	-0.006	0.063	-0.007	0.061	-0.001	0.041	-0.002	0.050	-0.001	0.029	-0.001	0.029
	β_1	0.015	0.306	0.033	0.303	0.011	0.197	0.018	0.196	0.006	0.129	0.010	0.129
	β_2	-0.001	0.066	-0.001	0.066	-0.001	0.050	-0.002	0.050	0.001	0.034	0.001	0.034
	β_3	-0.002	0.048	-0.003	0.048	-0.001	0.032	-0.002	0.032	0.002	0.025	0.001	0.025
	β_4	-0.002	0.049	-0.003	0.049	0.000	0.033	-0.000	0.033	-0.002	0.026	-0.002	0.026
	σ^2	-0.034	0.106	-0.004	0.101	-0.017	0.076	-0.002	0.074	-0.012	0.054	-0.004	0.053

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate; QBE: quasi-Bayesian estimate. S1: Setting 1; S2: Setting 2.
DGP1: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0, 0.4, 0)$; DGP2: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$.
$(\beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (2, 1, 1, 1, 1)$.

Table 6: Model estimation over 1000 repetitions: t_8 error

		$n = 200$				$n = 400$				$n = 800$			
		QMLE		QBE		QMLE		QBE		QMLE		QBE	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
DGP1	λ_1	-0.001	0.025	-0.001	0.025	-0.002	0.019	-0.002	0.019	-0.000	0.014	-0.001	0.014
S1	ρ_1	-0.012	0.084	-0.010	0.083	-0.005	0.055	-0.004	0.054	-0.001	0.039	-0.002	0.039
	β_1	0.000	0.189	0.001	0.187	0.009	0.142	0.011	0.142	-0.004	0.010	-0.004	0.099
	β_2	0.001	0.135	0.001	0.135	-0.002	0.094	-0.002	0.093	0.004	0.071	0.004	0.072
	β_3	-0.001	0.050	-0.001	0.050	-0.002	0.035	-0.002	0.035	0.000	0.025	-0.000	0.025
	β_4	-0.001	0.050	-0.001	0.050	-0.001	0.034	-0.001	0.034	0.000	0.024	0.000	0.024
	σ^2	-0.034	0.138	-0.014	0.134	-0.012	0.094	-0.001	0.093	-0.006	0.066	-0.001	0.066
DGP2	λ_1	-0.001	0.019	-0.003	0.019	-0.001	0.013	-0.002	0.013	-0.001	0.010	-0.001	0.009
S1	λ_2	-0.001	0.018	-0.002	0.018	-0.000	0.014	-0.001	0.014	-0.000	0.009	-0.000	0.009
	ρ_1	-0.012	0.091	-0.005	0.082	-0.009	0.062	-0.006	0.060	-0.004	0.043	-0.003	0.041
	ρ_2	-0.004	0.062	0.001	0.060	-0.001	0.044	-0.003	0.043	-0.001	0.030	-0.001	0.030
	β_1	0.038	0.352	0.066	0.356	0.019	0.251	0.032	0.250	0.012	0.178	0.016	0.164
	β_2	-0.007	0.147	-0.008	0.146	0.003	0.094	0.003	0.094	-0.001	0.069	0.000	0.068
	β_3	-0.000	0.048	-0.001	0.048	-0.000	0.036	-0.001	0.036	0.001	0.023	-0.001	0.023
	β_4	0.001	0.048	0.000	0.048	-0.001	0.032	-0.002	0.032	-0.000	0.024	0.001	0.024
	σ^2	-0.047	0.141	-0.019	0.135	-0.019	0.095	-0.004	0.094	-0.001	0.067	-0.007	0.068
DGP1	λ_1	-0.001	0.025	-0.002	0.025	-0.000	0.018	-0.001	0.018	-0.000	0.012	0.000	0.012
S2	ρ_1	-0.014	0.085	-0.009	0.080	-0.001	0.055	-0.001	0.054	-0.001	0.040	-0.002	0.039
	β_1	-0.001	0.158	0.001	0.157	0.007	0.113	0.007	0.113	0.001	0.076	0.000	0.076
	β_2	0.000	0.067	-0.000	0.067	-0.002	0.037	0.007	0.113	0.001	0.035	-0.000	0.034
	β_3	-0.002	0.050	-0.002	0.049	0.001	0.035	0.0003	0.035	0.000	0.025	-0.001	0.025
	β_4	0.000	0.054	-0.001	0.054	-0.001	0.034	0.000	0.035	-0.001	0.024	0.000	0.025
	σ^2	-0.029	0.106	-0.009	0.102	-0.014	0.074	-0.004	0.072	-0.008	0.067	-0.009	0.067
DGP2	λ_1	-0.001	0.018	-0.002	0.018	-0.001	0.012	-0.001	0.012	0.000	0.008	-0.000	0.008
S2	λ_2	-0.001	0.017	-0.001	0.017	-0.001	0.012	-0.001	0.012	-0.001	0.008	-0.001	0.008
	ρ_1	-0.011	0.086	-0.004	0.082	-0.009	0.060	-0.006	0.058	-0.002	0.039	-0.000	0.039
	ρ_2	-0.006	0.063	-0.007	0.061	-0.001	0.043	-0.002	0.042	-0.002	0.029	-0.002	0.029
	β_1	0.015	0.306	0.033	0.303	0.016	0.196	0.023	0.196	0.002	0.130	0.005	0.130
	β_2	-0.001	0.066	-0.001	0.066	-0.000	0.052	-0.001	0.052	0.001	0.035	0.001	0.035
	β_3	-0.002	0.048	-0.003	0.048	0.002	0.032	0.002	0.032	-0.000	0.024	-0.001	0.024
	β_4	-0.002	0.049	-0.003	0.049	0.0001	0.030	-0.000	0.030	0.000	0.027	0.000	0.027
	σ^2	-0.034	0.106	-0.004	0.101	-0.020	0.094	-0.005	0.092	-0.011	0.067	-0.004	0.066

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate; QBE: quasi-Bayesian estimate. S1: Setting 1; S2: Setting 2.
DGP1: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0, 0.4, 0)$; DGP2: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$.
$(\beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (2, 1, 1, 1, 1)$.

Table 7: Model estimation over 1000 repetitions with t_6 error

		$n = 200$				$n = 400$				$n = 800$			
		QMLE		QBE		QMLE		QBE		QMLE		QBE	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
DGP1	λ_1	0.001	0.027	-0.001	0.027	-0.001	0.020	-0.001	0.020	-0.000	0.014	-0.000	0.014
S1	ρ_1	-0.019	0.085	-0.014	0.082	-0.005	0.056	-0.005	0.056	-0.001	0.040	-0.001	0.039
	β_1	-0.005	0.207	-0.001	0.205	0.006	0.142	0.007	0.141	-0.000	0.101	0.001	0.101
	β_2	0.005	0.139	0.005	0.139	-0.003	0.010	-0.003	0.099	0.001	0.072	0.001	0.072
	β_3	-0.000	0.049	-0.001	0.049	-0.000	0.035	-0.001	0.035	0.001	0.024	0.001	0.024
	β_4	0.000	0.051	-0.001	0.051	-0.001	0.036	-0.002	0.036	-0.003	0.025	-0.003	0.025
	σ^2	-0.031	0.163	-0.011	0.160	-0.018	0.109	-0.008	0.108	-0.008	0.075	-0.002	0.074
DGP2	λ_1	-0.002	0.019	-0.003	0.019	-0.001	0.014	-0.002	0.014	-0.000	0.010	-0.001	0.010
S1	λ_2	-0.001	0.018	-0.002	0.018	-0.000	0.014	-0.001	0.014	0.000	0.009	-0.000	0.009
	ρ_1	-0.011	0.086	-0.004	0.082	-0.007	0.062	-0.004	0.061	-0.002	0.042	-0.001	0.042
	ρ_2	-0.004	0.062	-0.006	0.060	-0.001	0.044	-0.002	0.043	-0.003	0.030	-0.003	0.030
	β_1	0.035	0.344	0.062	0.347	0.016	0.246	0.029	0.246	0.003	0.167	0.009	0.167
	β_2	0.002	0.149	0.001	0.149	-0.006	0.094	-0.007	0.094	-0.001	0.069	-0.002	0.069
	β_3	-0.001	0.048	-0.002	0.048	0.001	0.037	0.000	0.036	0.000	0.023	-0.00	0.023
	β_4	-0.001	0.049	-0.002	0.049	-0.001	0.032	-0.002	0.032	0.001	0.024	0.000	0.024
	σ^2	-0.034	0.165	-0.004	0.163	-0.020	0.111	-0.005	0.109	-0.011	0.076	-0.003	0.076
DGP1	λ_1	-0.000	0.025	-0.001	0.025	-0.001	0.016	-0.001	0.016	-0.001	0.012	-0.001	0.012
S2	ρ_1	-0.013	0.086	0.008	0.083	-0.002	0.058	-0.003	0.057	-0.001	0.038	-0.002	0.037
	β_1	0.005	0.160	0.006	0.159	-0.001	0.105	-0.002	0.104	0.003	0.075	0.002	0.075
	β_2	0.000	0.067	-0.001	0.067	0.004	0.054	0.004	0.054	-0.001	0.034	-0.001	0.034
	β_3	-0.001	0.049	-0.002	0.049	-0.002	0.035	-0.003	0.035	-0.000	0.024	-0.000	0.024
	β_4	-0.001	0.053	-0.002	0.053	-0.000	0.036	-0.001	0.036	-0.001	0.024	-0.001	0.024
	σ^2	-0.029	0.156	-0.009	0.153	-0.014	0.111	-0.003	0.110	-0.008	0.077	-0.003	0.077
DGP2	λ_1	-0.001	0.019	-0.003	0.019	-0.001	0.012	-0.001	0.012	-0.000	0.008	-0.000	0.008
S2	λ_2	-0.001	0.018	-0.001	0.018	-0.000	0.012	-0.000	0.012	-0.000	0.008	-0.000	0.008
	ρ_1	-0.014	0.090	-0.008	0.086	-0.004	0.058	-0.000	0.057	-0.002	0.039	-0.001	0.039
	ρ_2	-0.006	0.062	-0.007	0.060	-0.003	0.043	-0.004	0.042	-0.001	0.030	-0.002	0.029
	β_1	0.025	0.304	0.042	0.301	0.013	0.190	0.020	0.190	0.004	0.129	0.007	0.129
	β_2	0.001	0.070	0.001	0.070	0.001	0.051	-0.000	0.051	0.001	0.034	0.000	0.034
	β_3	-0.000	0.049	-0.001	0.049	0.001	0.032	0.000	0.032	-0.001	0.023	-0.001	0.023
	β_4	-0.001	0.049	-0.001	0.049	-0.002	0.032	-0.002	0.032	-0.001	0.025	-0.001	0.025
	σ^2	-0.039	0.156	-0.010	0.152	-0.018	0.112	-0.004	0.110	-0.013	0.078	-0.006	0.077

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate; QBE: quasi-Bayesian estimate.
DGP1: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0, 0.4, 0)$; DGP2: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$.
$(\beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (2, 1, 1, 1, 1)$.

Table 8: Model estimation over 1000 repetitions: uniform error

		$n=200$				$n=400$				$n=800$			
		QMLE		QBE		QMLE		QBE		QMLE		QBE	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
DGP1	λ_1	0.001	0.024	-0.000	0.024	-0.001	0.019	-0.002	0.019	-0.000	0.013	-0.001	0.013
S1	ρ_1	-0.013	0.078	-0.010	0.076	-0.003	0.057	-0.001	0.056	-0.002	0.040	0.002	0.039
	β_1	0.004	0.168	0.004	0.166	0.000	0.129	0.003	0.129	-0.001	0.093	0.003	0.094
	β_2	-0.010	0.138	-0.010	0.138	0.005	0.095	0.005	0.095	0.003	0.070	-0.003	0.071
	β_3	-0.000	0.050	-0.001	0.050	-0.001	0.036	-0.001	0.036	-0.001	0.025	-0.001	0.025
	β_4	0.000	0.047	-0.000	0.047	0.001	0.034	0.0002	0.034	0.001	0.025	-0.001	0.025
	σ^2	-0.026	0.073	-0.006	0.068	-0.015	0.051	-0.005	0.048	-0.010	0.034	-0.007	0.034
DGP2	λ_1	-0.001	0.019	-0.003	0.019	-0.000	0.014	-0.001	0.014	0.000	0.009	0.001	0.009
S1	λ_2	-0.001	0.018	-0.002	0.018	-0.001	0.014	-0.002	0.014	-0.000	0.009	-0.001	0.009
	ρ_1	-0.012	0.086	-0.005	0.081	-0.006	0.061	-0.002	0.059	-0.004	0.040	-0.005	0.042
	ρ_2	-0.006	0.061	-0.007	0.060	-0.000	0.041	-0.001	0.041	0.002	0.029	-0.001	0.028
	β_1	0.014	0.344	0.041	0.340	0.016	0.248	0.029	0.246	0.003	0.158	0.004	0.161
	β_2	-0.000	0.146	-0.001	0.146	-0.002	0.096	-0.002	0.096	0.000	0.068	-0.004	0.069
	β_3	0.001	0.049	0.000	0.049	-0.001	0.034	-0.001	0.034	-0.000	0.023	-0.000	0.024
	β_4	0.001	0.050	-0.000	0.049	-0.002	0.035	-0.002	0.035	-0.000	0.023	-0.001	0.023
	σ^2	-0.039	0.080	-0.010	0.071	-0.019	0.053	-0.004	0.050	-0.010	0.036	-0.010	0.036
DGP1	λ_1	-0.001	0.025	-0.002	0.025	-0.000	0.018	-0.001	0.017	-0.001	0.012	-0.001	0.012
S2	ρ_1	-0.014	0.085	-0.010	0.083	-0.003	0.055	-0.002	0.054	-0.002	0.038	-0.002	0.039
	β_1	0.011	0.153	0.013	0.152	0.000	0.114	0.001	0.114	0.005	0.076	-0.000	0.078
	β_2	-0.002	0.075	-0.003	0.075	0.003	0.050	0.003	0.050	0.002	0.035	-0.002	0.035
	β_3	-0.003	0.053	-0.004	0.053	0.001	0.035	0.001	0.035	0.000	0.024	0.003	0.023
	β_4	0.002	0.049	0.001	0.049	0.001	0.037	0.001	0.037	0.001	0.026	-0.000	0.033
	σ^2	-0.028	0.072	-0.008	0.067	-0.015	0.051	-0.005	0.048	-0.007	0.033	-0.007	0.033
DGP2	λ_1	-0.001	0.018	-0.003	0.018	-0.000	0.012	-0.001	0.012	0.000	0.009	0.000	0.008
S2	λ_2	-0.001	0.017	-0.002	0.017	-0.001	0.012	-0.001	0.012	-0.000	0.008	-0.000	0.008
	ρ_1	-0.010	0.087	-0.004	0.083	-0.006	0.059	-0.002	0.058	-0.004	0.041	-0.005	0.041
	ρ_2	-0.005	0.064	-0.007	0.062	-0.003	0.042	-0.004	0.041	-0.001	0.029	-0.001	0.028
	β_1	0.025	0.286	0.042	0.285	0.010	0.196	0.017	0.196	-0.006	0.127	-0.000	0.133
	β_2	0.000	0.067	0.000	0.067	0.001	0.048	0.001	0.048	-0.001	0.034	-0.002	0.033
	β_3	-0.001	0.049	-0.002	0.049	0.001	0.033	-0.000	0.033	-0.001	0.024	-0.001	0.024
	β_4	-0.000	0.049	-0.002	0.049	0.001	0.032	0.001	0.032	0.001	0.025	0.000	0.026
	σ^2	-0.037	0.078	-0.007	0.067	-0.018	0.053	-0.003	0.050	-0.009	0.036	-0.009	0.036

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate; QBE: quasi-Bayesian estimate. S1: Setting 1; S2: Setting 2.
DGP1: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0, 0.4, 0)$; DGP2: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$.
$(\beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (2, 1, 1, 1, 1)$.

Table 9: MLE and Bayesian estimation over 1000 repetitions under multicollinearity

		MLE		BE with non-hierarchical prior		BE with hierarchical prior	
		Bias	RMSE	Bias	RMSE	Bias	RMSE
DGP4	λ_1	-0.002	0.035	-0.003	0.034	0.006	0.031
$n = 200$	ρ_1	-0.028	0.097	-0.016	0.088	-0.021	0.084
	β_2	-0.000	0.068	-0.000	0.067	-0.001	0.058
	β_3	0.023	0.417	0.019	0.348	-0.031	0.274
	β_4	0.061	1.036	0.038	0.713	-0.105	0.527
	β_5	-0.069	1.567	-0.065	1.033	0.019	0.626
DGP4	λ_1	0.000	0.021	-0.002	0.021	0.002	0.020
$n = 400$	ρ_1	-0.013	0.059	-0.006	0.056	-0.008	0.055
	β_2	-0.002	0.047	-0.002	0.046	0.001	0.038
	β_3	-0.007	0.250	-0.005	0.239	-0.014	0.195
	β_4	0.016	0.603	0.024	0.553	-0.052	0.425
	β_5	-0.006	0.850	-0.011	0.769	-0.012	0.545
DGP4	λ_1	-0.001	0.015	-0.001	0.014	0.001	0.015
$n = 800$	ρ_1	-0.004	0.041	-0.002	0.040	-0.006	0.041
	β_2	0.000	0.030	-0.001	0.031	0.001	0.028
	β_3	0.006	0.149	0.001	0.157	0.000	0.132
	β_4	-0.001	0.424	-0.000	0.396	-0.009	0.324
	β_5	0.010	0.565	0.009	0.519	-0.012	0.410
DGP5	λ_1	-0.001	0.020	-0.003	0.020	0.004	0.020
$n = 200$	λ_2	-0.001	0.019	-0.002	0.019	0.003	0.018
	ρ_1	-0.022	0.091	-0.010	0.085	-0.039	0.095
	ρ_2	-0.006	0.065	-0.004	0.062	-0.002	0.062
	β_1	0.023	0.326	0.054	0.325	-0.097	0.317
	β_2	0.001	0.056	0.000	0.056	-0.006	0.058
	β_3	0.010	0.332	0.013	0.321	-0.061	0.306
	β_4	-0.017	0.915	0.006	0.843	-0.161	0.585
	β_5	0.034	1.242	0.050	1.167	-0.023	0.639
DGP5	λ_1	-0.001	0.014	-0.001	0.014	0.003	0.013
$n = 400$	λ_2	-0.001	0.129	-0.001	0.013	0.002	0.013
	ρ_1	-0.011	0.062	-0.004	0.059	-0.017	0.063
	ρ_2	-0.002	0.043	-0.001	0.043	-0.003	0.042
	β_2	0.001	0.043	0.001	0.043	0.001	0.044
	β_3	0.006	0.292	0.008	0.280	-0.033	0.239
	β_4	0.044	0.883	0.050	0.803	-0.141	0.525
	β_5	-0.045	1.152	-0.030	1.044	-0.029	0.602
DGP5	λ_1	0.000	0.009	-0.001	0.009	0.001	0.009
$n = 800$	λ_2	-0.000	0.008	-0.000	0.008	0.000	0.008
	ρ_1	-0.005	0.040	-0.004	0.042	-0.006	0.043
	ρ_2	-0.001	0.028	-0.000	0.029	-0.001	0.029
	β_2	-0.001	0.031	-0.001	0.029	-0.000	0.029
	β_3	0.002	0.180	0.007	0.167	-0.007	0.157
	β_4	0.009	0.552	0.006	0.506	-0.062	0.409
	β_5	0.004	0.740	-0.003	0.688	-0.012	0.541

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate; QBE: quasi-Bayesian estimate.
DGP4: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0, 0.4, 0)$.
DGP5: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$.
$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \sigma^2) = (2, 1.3180, 1.3035, 1.1535, 0.9135, 1)$.

Table 10: QMLE and Quasi-Bayesian estimation over 1000 repetitions under multicollinearity: t_8 error

		QMLE		QBE with non-hierarchical error		QBE with hierarchical error	
		Bias	RMSE	Bias	RMSE	Bias	RMSE
DGP4	λ_1	0.001	0.029	-0.003	0.034	0.004	0.030
$n = 200$	ρ_1	-0.026	0.092	-0.016	0.088	-0.023	0.090
	β_2	-0.001	0.069	-0.001	0.069	-0.005	0.062
	β_3	0.012	0.336	0.019	0.348	-0.031	0.265
	β_4	0.030	0.880	0.038	0.713	-0.068	0.490
	β_5	-0.071	1.183	-0.065	0.033	-0.007	0.989
DGP4	λ_1	-0.000	0.021	-0.001	0.021	0.003	0.021
$n = 400$	ρ_1	-0.013	0.063	-0.006	0.061	-0.012	0.056
	β_2	-0.002	0.049	-0.002	0.040	-0.001	0.039
	β_3	0.002	0.208	0.003	0.203	-0.023	0.177
	β_4	0.014	0.553	0.015	0.520	-0.037	0.398
	β_5	-0.006	0.676	-0.008	0.633	-0.004	0.501
DGP4	λ_1	-0.000	0.016	-0.001	0.015	0.002	0.015
$n = 800$	ρ_1	-0.004	0.041	-0.002	0.043	-0.007	0.040
	β_2	0.000	0.031	-0.001	0.032	-0.000	0.030
	β_3	-0.006	0.156	0.004	0.155	-0.013	0.147
	β_4	-0.019	0.379	0.010	0.395	-0.036	0.339
	β_5	0.031	0.521	-0.004	0.525	0.017	0.438
DGP5	λ_1	-0.002	0.020	-0.004	0.020	0.003	0.020
$n = 200$	λ_2	-0.001	0.019	-0.002	0.019	0.003	0.017
	ρ_1	-0.017	0.091	-0.005	0.084	-0.030	0.088
	ρ_2	-0.011	0.064	-0.010	0.062	-0.005	0.061
	β_2	0.000	0.061	0.000	0.061	0.001	0.055
	β_3	0.004	0.338	0.010	0.327	-0.050	0.276
	β_4	0.009	0.929	0.035	0.845	-0.127	0.573
	β_5	0.033	1.239	0.029	1.117	-0.059	0.662
DGP5	λ_1	-0.001	0.015	-0.002	0.014	0.002	0.014
$n = 400$	λ_2	-0.000	0.014	-0.001	0.014	0.002	0.014
	ρ_1	-0.013	0.066	-0.005	0.063	-0.019	0.062
	ρ_2	-0.004	0.043	-0.003	0.042	-0.000	0.042
	β_2	-0.001	0.043	-0.001	0.043	-0.003	0.042
	β_3	-0.003	0.294	0.002	0.283	-0.039	0.238
	β_4	-0.027	0.895	-0.008	0.817	-0.151	0.526
	β_5	0.043	1.151	0.038	1.052	-0.008	0.625
DGP5	λ_1	-0.000	0.009	-0.000	0.009	0.001	0.009
$n = 800$	λ_2	0.000	0.008	-0.001	0.008	0.000	0.008
	ρ_1	-0.005	0.042	-0.003	0.042	-0.007	0.042
	ρ_2	-0.002	0.028	-0.001	0.030	0.000	0.029
	β_2	0.000	0.030	-0.001	0.030	-0.001	0.030
	β_3	-0.000	0.177	0.001	0.178	-0.014	0.159
	β_4	-0.017	0.557	0.011	0.527	-0.047	0.399
	β_5	0.010	0.753	0.009	0.713	-0.015	0.521

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate; QBE: quasi-Bayesian estimate.
DGP4: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0, 0.4, 0)$.
DGP5: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$.
$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \sigma^2) = (2, 1.3180, 1.3035, 1.1535, 0.9135, 1)$.

Table 11: QMLE and Quasi-Bayesian estimation over 1000 repetitions under multicollinearity: uniform error

		QMLE		QBE with non-hierarchical error		QBE with hierarchical error	
		Bias	RMSE	Bias	RMSE	Bias	RMSE
DGP4	λ_1	-0.001	0.030	-0.004	0.029	0.005	0.030
$n = 200$	ρ_1	-0.025	0.092	-0.009	0.085	-0.023	0.084
	β_2	-0.003	0.061	-0.003	0.060	-0.002	0.056
	β_3	0.015	0.289	0.015	0.276	-0.004	0.255
	β_4	0.063	0.895	0.053	0.766	-0.066	0.521
	β_5	-0.105	1.162	-0.088	0.972	-0.096	0.647
DGP4	λ_1	-0.001	0.020	-0.002	0.022	0.002	0.019
$n = 400$	ρ_1	-0.010	0.060	-0.007	0.060	-0.009	0.057
	β_2	-0.001	0.046	-0.002	0.045	-0.001	0.043
	β_3	0.002	0.272	0.002	0.256	-0.021	0.208
	β_4	0.047	0.749	0.043	0.665	-0.065	0.415
	β_5	-0.038	1.024	-0.035	0.887	0.035	0.522
DGP4	λ_1	0.000	0.014	-0.001	0.014	0.000	0.015
$n = 800$	ρ_1	-0.003	0.043	-0.001	0.039	-0.002	0.040
	β_2	-0.002	0.033	-0.000	0.081	0.001	0.031
	β_3	-0.001	0.149	-0.001	0.150	-0.002	0.141
	β_4	0.007	0.420	-0.004	0.393	-0.019	0.338
	β_5	-0.011	0.543	0.001	0.542	-0.003	0.453
DGP5	λ_1	-0.001	0.020	-0.003	0.020	0.004	0.020
$n = 200$	λ_2	-0.001	0.018	-0.002	0.018	0.003	0.018
	ρ_1	-0.021	0.092	-0.008	0.085	-0.034	0.093
	ρ_2	-0.005	0.064	-0.004	0.061	-0.002	0.062
	β_2	0.003	0.058	0.003	0.058	-0.008	0.058
	β_3	0.029	0.337	0.034	0.329	-0.043	0.287
	β_4	-0.007	0.907	0.020	0.838	-0.126	0.553
	β_5	-0.004	1.211	-0.004	1.147	-0.095	0.653
DGP5	λ_1	-0.000	0.014	-0.001	0.014	0.002	0.014
$n = 400$	λ_2	-0.000	0.013	-0.001	0.013	0.002	0.013
	ρ_1	-0.012	0.067	-0.005	0.064	-0.015	0.066
	ρ_2	-0.004	0.043	-0.003	0.042	-0.005	0.040
	β_2	-0.003	0.045	-0.003	0.045	-0.002	0.044
	β_3	0.005	0.293	0.009	0.282	-0.030	0.241
	β_4	0.028	0.880	0.041	0.806	-0.126	0.518
	β_5	-0.037	1.138	-0.034	1.051	-0.024	0.616
DGP5	λ_1	-0.001	0.009	-0.001	0.009	0.001	0.009
$n = 800$	λ_2	0.000	0.008	-0.001	0.008	0.001	0.008
	ρ_1	-0.004	0.041	-0.001	0.041	-0.007	0.042
	ρ_2	-0.001	0.030	-0.002	0.028	-0.001	0.028
	β_2	0.000	0.031	-0.002	0.029	0.000	0.030
	β_3	-0.001	0.183	-0.007	0.166	-0.009	0.156
	β_4	0.021	0.544	0.017	0.494	-0.040	0.408
	β_5	-0.008	0.745	0.036	0.683	-0.042	0.527

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate; QBE: quasi-Bayesian estimate.
DGP4: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0, 0.4, 0)$.
DGP5: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$.
$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \sigma^2) = (2, 1.3180, 1.3035, 1.1535, 0.9135, 1)$.

Table 12: Model estimation over 1000 repetitions: DGP1 with denser spatial weights matrices

		$n=200$				$n=400$				$n=800$			
		MLE / QMLE		BE / QBE		MLE / QMLE		BE / QBE		MLE/ QMLE		BE / QBE	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
Normal	λ_1	-0.004	0.065	-0.009	0.062	0.002	0.050	-0.004	0.050	0.000	0.060	-0.004	0.059
S1	ρ_1	-0.052	0.165	-0.032	0.148	-0.061	0.156	-0.036	0.142	-0.050	0.155	-0.033	0.148
	β_1	0.023	0.393	0.052	0.378	-0.011	0.255	0.017	0.252	0.002	0.305	0.024	0.300
	β_2	-0.000	0.138	-0.000	0.137	-0.002	0.094	-0.002	0.094	-0.005	0.071	-0.005	0.071
	β_3	-0.001	0.051	-0.002	0.051	-0.002	0.036	-0.002	0.036	0.001	0.025	0.001	0.025
	β_4	-0.000	0.059	-0.001	0.059	-0.001	0.038	-0.001	0.038	-0.001	0.025	-0.001	0.025
	σ^2	-0.026	0.104	-0.007	0.100	-0.013	0.071	-0.004	0.070	-0.008	0.052	-0.003	0.052
t_8	λ_1	-0.002	0.059	-0.007	0.058	-0.001	0.052	-0.007	0.052	-0.005	0.061	-0.010	0.060
S1	ρ_1	-0.058	0.158	-0.040	0.142	-0.055	0.161	-0.032	0.146	-0.044	0.151	-0.025	0.141
	β_1	0.012	0.344	0.036	0.333	0.011	0.272	0.037	0.268	0.026	0.316	0.052	0.310
	β_2	-0.002	0.144	-0.002	0.144	-0.002	0.099	-0.003	0.099	-0.003	0.070	-0.003	0.070
	β_3	0.001	0.054	-0.001	0.054	-0.002	0.037	-0.003	0.037	-0.000	0.026	-0.001	0.026
	β_4	-0.002	0.052	-0.003	0.052	-0.003	0.033	-0.003	0.033	0.001	0.024	0.000	0.024
	σ^2	-0.027	0.071	-0.008	0.065	-0.014	0.048	-0.005	0.046	-0.007	0.034	-0.003	0.033
Uniform	λ_1	-0.002	0.056	-0.008	0.055	-0.003	0.055	-0.008	0.054	-0.003	0.052	-0.009	0.052
S1	ρ_1	-0.054	0.162	-0.029	0.145	-0.059	0.163	-0.038	0.150	-0.059	0.160	-0.033	0.146
	β_1	0.001	0.317	0.029	0.308	0.012	0.281	0.036	0.275	0.014	0.291	0.047	0.289
	β_2	0.005	0.138	0.004	0.137	0.000	0.102	0.000	0.102	0.000	0.072	0.000	0.072
	β_3	-0.001	0.047	-0.002	0.047	-0.001	0.036	-0.002	0.036	-0.000	0.026	-0.001	0.026
	β_4	-0.003	0.051	-0.003	0.051	-0.002	0.036	-0.003	0.036	-0.000	0.026	-0.000	0.026
	σ^2	-0.036	0.133	-0.016	0.129	-0.010	0.092	-0.001	0.092	-0.008	0.065	-0.003	0.064
Normal	λ_1	0.000	0.048	-0.003	0.047	0.001	0.039	-0.003	0.039	0.000	0.044	-0.003	0.044
S2	ρ_1	-0.050	0.154	-0.030	0.139	-0.053	0.149	-0.031	0.138	-0.055	0.148	-0.035	0.138
	β_1	-0.000	0.235	0.009	0.229	-0.004	0.180	0.009	0.179	0.001	0.187	0.015	0.186
	β_2	0.000	0.055	-0.000	0.055	0.001	0.034	0.000	0.034	0.000	0.025	-0.001	0.025
	β_3	-0.000	0.047	-0.001	0.047	-0.002	0.032	-0.002	0.032	0.000	0.025	-0.001	0.025
	β_4	-0.001	0.052	-0.002	0.052	-0.001	0.039	-0.001	0.039	0.000	0.024	0.000	0.024
	σ^2	-0.031	0.103	-0.011	0.098	-0.011	0.069	-0.002	0.068	-0.010	0.051	-0.005	0.050
t_8	λ_1	-0.002	0.049	-0.007	0.048	-0.000	0.049	-0.007	0.049	0.001	0.049	-0.005	0.049
S2	ρ_1	-0.054	0.158	-0.031	0.142	-0.053	0.149	-0.026	0.135	-0.059	0.153	-0.034	0.138
	β_1	0.011	0.229	0.029	0.226	0.003	0.214	0.028	0.213	-0.001	0.199	0.021	0.196
	β_2	-0.003	0.076	-0.004	0.076	0.000	0.053	-0.000	0.053	-0.000	0.034	-0.001	0.034
	β_3	0.002	0.056	0.001	0.056	0.001	0.035	0.000	0.035	0.001	0.025	0.001	0.025
	β_4	-0.000	0.047	-0.001	0.047	-0.001	0.036	-0.002	0.036	-0.000	0.025	-0.000	0.025
	σ^2	-0.025	0.130	-0.006	0.127	-0.016	0.048	-0.006	0.045	-0.007	0.032	-0.002	0.031
Uniform	λ_1	-0.002	0.051	-0.007	0.050	-0.002	0.062	-0.009	0.061	-0.003	0.054	-0.009	0.054
S2	ρ_1	-0.053	0.156	-0.032	0.141	-0.056	0.166	-0.033	0.150	-0.049	0.153	-0.026	0.142
	β_1	0.002	0.247	0.019	0.244	0.014	0.280	0.041	0.276	0.012	0.221	0.037	0.220
	β_2	0.002	0.069	0.001	0.069	0.002	0.050	0.001	0.050	0.001	0.033	0.001	0.033
	β_3	-0.002	0.048	-0.003	0.048	-0.002	0.036	-0.003	0.036	0.000	0.026	-0.000	0.026
	β_4	-0.002	0.052	-0.003	0.052	0.001	0.035	0.001	0.035	-0.001	0.025	-0.001	0.025
	σ^2	-0.033	0.129	-0.013	0.125	-0.018	0.095	-0.008	0.093	-0.007	0.068	-0.003	0.068

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate; QBE: quasi-Bayesian estimate.
S1: Setting 1; S2: Setting 2.
DGP1: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0, 0.4, 0)$;
$(\beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (2, 1, 1, 1, 1)$.

Table 13: Model estimation over 1000 repetitions: DGP3 with non-row-normalized spatial weights

		$n = 200$				$n = 400$				$n = 800$			
		MLE / QMLE		BE / QBE		MLE / QMLE		BE / QBE		MLE / QMLE		BE / QBE	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
Normal	λ_1	0.004	0.013	0.003	0.013	0.000	0.009	-0.000	0.009	0.000	0.006	0.000	0.006
S1	ρ_1	-0.006	0.013	-0.005	0.012	-0.002	0.007	0.002	0.006	-0.001	0.004	-0.001	0.004
	β_1	-0.061	0.337	-0.074	0.335	-0.005	0.253	-0.009	0.247	0.002	0.165	0.000	0.164
	β_2	0.015	0.135	0.014	0.135	0.001	0.099	0.001	0.099	-0.003	0.068	-0.003	0.068
	β_3	0.003	0.049	0.002	0.049	0.001	0.033	0.001	0.033	-0.002	0.025	-0.002	0.025
	β_4	-0.008	0.054	-0.008	0.054	0.001	0.034	0.001	0.034	0.000	0.024	0.000	0.024
	σ^2	-0.030	0.101	-0.011	0.097	-0.012	0.071	-0.003	0.070	-0.007	0.052	-0.003	0.051
t_8	λ_1	-0.000	0.012	-0.000	0.012	-0.001	0.008	-0.001	0.008	-0.000	0.007	-0.000	0.007
S1	ρ_1	0.004	0.010	-0.004	0.010	-0.002	0.006	-0.001	0.006	-0.001	0.004	-0.001	0.004
	β_1	0.004	0.374	-0.023	0.365	0.004	0.236	-0.001	0.233	0.001	0.185	-0.000	0.184
	β_2	0.007	0.141	0.008	0.141	0.001	0.093	0.001	0.093	-0.002	0.068	-0.002	0.068
	β_3	-0.001	0.046	-0.001	0.046	-0.001	0.034	-0.001	0.034	0.000	0.025	0.000	0.025
	β_4	-0.003	0.057	-0.003	0.057	-0.000	0.035	-0.001	0.035	-0.001	0.024	-0.001	0.024
	σ^2	-0.027	0.133	-0.008	0.130	-0.013	0.092	-0.004	0.091	-0.006	0.067	-0.002	0.067
Uniform	λ_1	-0.001	0.012	-0.001	0.012	-0.001	0.01	-0.001	0.010	0.0003	0.007	0.000	0.007
S1	ρ_1	-0.005	0.010	-0.004	0.010	-0.002	0.006	-0.001	0.006	-0.001	0.004	-0.001	0.004
	β_1	0.003	0.339	-0.014	0.331	0.011	0.247	0.007	0.243	-0.005	0.179	-0.006	0.178
	β_2	-0.001	0.139	-0.001	0.138	0.000	0.100	0.000	0.100	0.001	0.068	0.001	0.068
	β_3	0.001	0.048	0.001	0.048	-0.002	0.040	-0.003	0.040	-0.001	0.024	-0.001	0.024
	β_4	0.002	0.053	-0.001	0.053	0.002	0.034	0.002	0.034	-0.001	0.025	-0.001	0.025
	σ^2	-0.025	0.070	-0.006	0.065	-0.012	0.048	-0.003	0.046	-0.006	0.034	-0.002	0.033
Normal	λ_1	0.000	0.011	0.000	0.011	-0.000	0.006	-0.000	0.006	0.000	0.004	0.000	0.004
S2	ρ_1	0.004	0.009	-0.003	0.008	-0.002	0.006	-0.001	0.005	-0.001	0.004	-0.001	0.004
	β_1	0.029	0.300	0.013	0.295	0.001	0.204	-0.004	0.203	-0.000	0.147	-0.002	0.144
	β_2	-0.007	0.058	-0.007	0.057	-0.000	0.032	0.0001	0.032	0.000	0.024	0.000	0.024
	β_3	-0.001	0.052	-0.001	0.052	0.000	0.035	-0.000	0.035	-0.001	0.027	-0.001	0.027
	β_4	-0.004	0.055	-0.005	0.055	0.003	0.032	0.002	0.032	-0.001	0.027	-0.001	0.027
	σ^2	-0.031	0.107	-0.012	0.103	-0.012	0.075	-0.003	0.074	-0.008	0.053	-0.004	0.053
t_8	λ_1	-0.000	0.011	-0.005	0.011	-0.000	0.007	-0.000	0.007	-0.000	0.006	-0.000	0.006
S2	ρ_1	-0.005	0.011	-0.004	0.010	-0.002	0.006	-0.001	0.006	-0.001	0.004	-0.0004	0.004
	β_1	-0.006	0.314	-0.024	0.311	0.004	0.223	-0.001	0.221	-0.000	0.149	-0.001	0.147
	β_2	0.000	0.074	0.000	0.073	0.002	0.048	0.003	0.048	0.001	0.035	0.002	0.035
	β_3	-0.001	0.046	-0.002	0.046	-0.000	0.037	-0.001	0.036	0.000	0.025	0.000	0.025
	β_4	0.002	0.049	0.002	0.049	0.001	0.038	0.001	0.038	0.000	0.024	0.000	0.024
	σ^2	-0.028	0.130	-0.009	0.127	-0.016	0.094	-0.007	0.093	-0.003	0.065	0.001	0.065
Uniform	λ_1	-0.000	0.012	-0.000	0.012	-0.001	0.007	-0.001	0.007	-0.000	0.006	-0.000	0.006
S2	ρ_1	0.005	0.010	-0.004	0.009	-0.002	0.006	-0.001	0.006	-0.001	0.004	-0.001	0.004
	β_1	0.000	0.307	-0.014	0.302	0.006	0.214	0.001	0.212	-0.000	0.155	-0.002	0.154
	β_2	-0.002	0.074	-0.003	0.074	0.002	0.043	0.002	0.043	-0.002	0.035	-0.001	0.035
	β_3	-0.001	0.046	-0.001	0.046	-0.000	0.037	-0.000	0.037	-0.000	0.025	-0.000	0.025
	β_4	-0.003	0.048	-0.003	0.048	-0.001	0.035	-0.001	0.035	0.000	0.024	0.000	0.024
	σ^2	-0.019	0.068	-0.001	0.065	-0.012	0.047	-0.003	0.045	-0.006	0.033	-0.002	0.033

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate; QBE: quasi-Bayesian estimate.
S1: Setting 1; S2: Setting 2.
DGP3: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.2, 0, 0.06, 0)$;
$(\beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (2, 1, 1, 1, 1)$.

Table 14: Model estimation over 1000 repetitions: normal error under broader stability condition

		$n = 200$				$n = 400$				$n = 800$			
		MLE		BE		MLE		BE		MLE		BE	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
DGP2	λ_1	-0.002	0.018	-0.003	0.018	0.0002	0.014	-0.001	0.014	-0.000	0.010	-0.001	0.010
S1	λ_2	-0.000	0.018	-0.001	0.018	-0.001	0.014	-0.001	0.014	-0.000	0.009	-0.000	0.009
	ρ_1	-0.011	0.091	-0.005	0.087	-0.011	0.062	-0.008	0.060	-0.002	0.041	-0.000	0.041
	ρ_2	-0.006	0.063	-0.007	0.062	0.001	0.043	-0.000	0.043	-0.002	0.029	-0.003	0.029
	β_1	0.021	0.363	0.049	0.365	0.001	0.245	0.014	0.242	0.001	0.160	0.006	0.160
	β_2	-0.006	0.144	-0.007	0.143	0.004	0.096	0.004	0.095	0.000	0.070	0.0002	0.070
	β_3	-0.001	0.046	-0.001	0.046	0.001	0.035	0.000	0.035	-0.001	0.024	-0.001	0.024
	β_4	-0.001	0.048	-0.001	0.048	-0.002	0.035	-0.002	0.035	-0.0001	0.023	-0.0003	0.023
	σ^2	-0.037	0.109	-0.008	0.103	-0.018	0.075	-0.003	0.073	-0.010	0.056	-0.003	0.055
DGP2	λ_1	-0.001	0.018	-0.002	0.018	-0.000	0.012	-0.001	0.012	0.000	0.008	-0.000	0.008
S2	λ_2	-0.001	0.017	-0.002	0.017	-0.001	0.012	-0.001	0.012	0.000	0.008	-0.000	0.008
	ρ_1	-0.011	0.086	-0.004	0.082	-0.007	0.058	-0.003	0.057	0.005	0.041	-0.003	0.040
	ρ_2	-0.006	0.063	-0.007	0.061	-0.001	0.041	-0.002	0.040	-0.001	0.030	-0.001	0.030
	β_1	0.015	0.306	0.033	0.303	0.012	0.197	0.019	0.196	-0.004	0.129	-0.001	0.129
	β_2	-0.001	0.066	-0.001	0.066	-0.001	0.050	-0.002	0.050	-0.000	0.034	-0.001	0.034
	β_3	-0.002	0.048	-0.003	0.048	-0.001	0.032	-0.001	0.032	-0.000	0.025	-0.001	0.025
	β_4	-0.002	0.049	-0.003	0.049	-0.000	0.033	-0.000	0.032	0.000	0.025	-0.000	0.025
	σ^2	-0.034	0.105	-0.004	0.101	-0.017	0.076	-0.003	0.074	-0.010	0.055	-0.003	0.054

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate; QBE: quasi-Bayesian estimate. S1: Setting 1; S2: Setting 2.
DGP2: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$; $(\beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (2, 1, 1, 1, 1)$.

Table 15: Model estimation over 1000 repetitions: t_8 error under broader stability condition

		$n = 200$				$n = 400$				$n = 800$			
		QMLE		QBE		QMLE		QBE		QMLE		QBE	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
DGP2	λ_1	-0.001	0.019	-0.003	0.019	-0.001	0.013	-0.002	0.013	-0.000	0.009	-0.001	0.009
S1	λ_2	-0.002	0.018	-0.003	0.018	-0.000	0.014	-0.001	0.014	-0.000	0.009	-0.000	0.009
	ρ_1	-0.013	0.087	-0.006	0.083	-0.009	0.062	-0.006	0.060	-0.004	0.040	-0.002	0.040
	ρ_2	-0.003	0.062	-0.004	0.060	-0.001	0.044	-0.002	0.043	-0.001	0.028	-0.002	0.028
	β_1	0.043	0.350	0.069	0.352	0.019	0.252	0.032	0.251	0.006	0.163	0.012	0.163
	β_2	0.001	0.144	-0.000	0.144	0.004	0.094	0.003	0.093	0.001	0.068	0.001	0.068
	β_3	-0.002	0.049	-0.003	0.049	-0.000	0.036	-0.001	0.036	0.000	0.024	-0.000	0.024
	β_4	-0.001	0.048	-0.002	0.048	-0.001	0.032	-0.001	0.032	0.000	0.023	-0.000	0.023
	σ^2	-0.041	0.136	-0.011	0.131	-0.020	0.096	-0.005	0.094	-0.013	0.067	-0.005	0.066
DGP2	λ_1	-0.002	0.019	-0.004	0.019	-0.001	0.012	-0.001	0.012	-0.001	0.009	-0.001	0.009
S2	λ_2	-0.001	0.018	-0.002	0.018	-0.001	0.012	-0.001	0.012	-0.000	0.008	-0.000	0.008
	ρ_1	-0.008	0.088	-0.002	0.084	-0.009	0.060	-0.006	0.059	-0.003	0.039	-0.001	0.039
	ρ_2	-0.007	0.063	-0.008	0.062	-0.001	0.043	-0.002	0.042	0.000	0.029	-0.000	0.028
	β_1	0.038	0.297	0.055	0.298	0.016	0.196	0.023	0.195	0.010	0.127	0.013	0.127
	β_2	0.003	0.067	0.003	0.067	0.000	0.053	-0.000	0.053	0.001	0.034	0.001	0.034
	β_3	0.002	0.048	0.001	0.048	0.002	0.032	0.002	0.032	0.001	0.024	0.001	0.024
	β_4	0.001	0.050	0.000	0.050	-0.000	0.030	-0.000	0.030	0.000	0.026	-0.000	0.026
	σ^2	-0.042	0.133	-0.012	0.128	-0.020	0.094	-0.005	0.092	-0.012	0.067	-0.004	0.066

The values reported in this table are calculated from 1000 repetitions.
QBE: quasi-Bayesian estimate. S1: Setting 1; S2: Setting 2.
DGP2: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$; $(\beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (2, 1, 1, 1, 1)$.

Table 16: Model estimation over 1000 repetitions: uniform error under broader stability condition

		$n = 200$				$n = 400$				$n = 800$			
		QMLE		QBE		QMLE		QBE		QMLE		QBE	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
DGP2	λ_1	-0.002	0.019	-0.003	0.019	-0.000	0.014	-0.001	0.014	-0.000	0.009	-0.001	0.009
S1	λ_2	-0.000	0.019	-0.001	0.018	-0.001	0.014	-0.002	0.014	-0.001	0.009	-0.001	0.009
	ρ_1	-0.013	0.083	-0.006	0.079	-0.006	0.061	-0.003	0.059	-0.002	0.042	-0.000	0.042
	ρ_2	-0.006	0.061	-0.008	0.059	-0.000	0.041	-0.001	0.041	-0.002	0.029	-0.002	0.029
	β_1	0.016	0.349	0.043	0.346	0.016	0.245	0.028	0.244	0.012	0.166	0.018	0.166
	β_2	-0.003	0.147	-0.004	0.146	-0.002	0.096	-0.003	0.096	-0.003	0.069	-0.003	0.069
	β_3	0.001	0.047	0.000	0.047	-0.001	0.034	-0.001	0.034	-0.000	0.024	-0.000	0.024
	β_4	0.000	0.050	-0.001	0.049	-0.002	0.035	-0.002	0.035	-0.000	0.024	-0.000	0.024
	σ^2	-0.038	0.081	-0.008	0.072	-0.019	0.053	-0.004	0.050	-0.008	0.036	-0.001	0.035
DGP2	λ_1	-0.001	0.018	-0.002	0.018	-0.000	0.012	-0.001	0.012	-0.000	0.009	-0.001	0.009
S2	λ_2	-0.001	0.017	-0.002	0.017	-0.001	0.012	-0.001	0.012	-0.000	0.008	-0.001	0.009
	ρ_1	-0.010	0.086	-0.004	0.083	-0.006	0.059	-0.002	0.058	-0.003	0.041	-0.002	0.040
	ρ_2	-0.006	0.062	-0.007	0.060	-0.002	0.042	-0.004	0.041	0.001	0.029	-0.001	0.029
	β_1	0.027	0.284	0.046	0.284	0.011	0.198	0.017	0.197	0.003	0.130	0.006	0.129
	β_2	-0.002	0.067	-0.002	0.067	0.001	0.048	0.001	0.048	0.000	0.033	0.000	0.033
	β_3	-0.001	0.049	-0.002	0.049	0.000	0.033	-0.000	0.033	0.001	0.024	0.000	0.024
	β_4	0.001	0.048	-0.000	0.048	0.001	0.032	0.000	0.032	-0.001	0.026	-0.001	0.026
	σ^2	-0.035	0.077	-0.006	0.069	-0.018	0.053	-0.003	0.050	-0.008	0.036	-0.001	0.034

The values reported in this table are calculated from 1000 repetitions.

QBE: quasi-Bayesian estimate. S1: Setting 1; S2: Setting 2.

DGP2: $(\lambda_1, \lambda_2, \rho_1, \rho_2) = (0.5, 0.3, 0.4, 0.2)$; $(\beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (2, 1, 1, 1, 1)$.

H. Figures and Tables for the SAR Tobit Model

Figure H.1: Marginal posterior density vs normal density: the SAR Tobit model

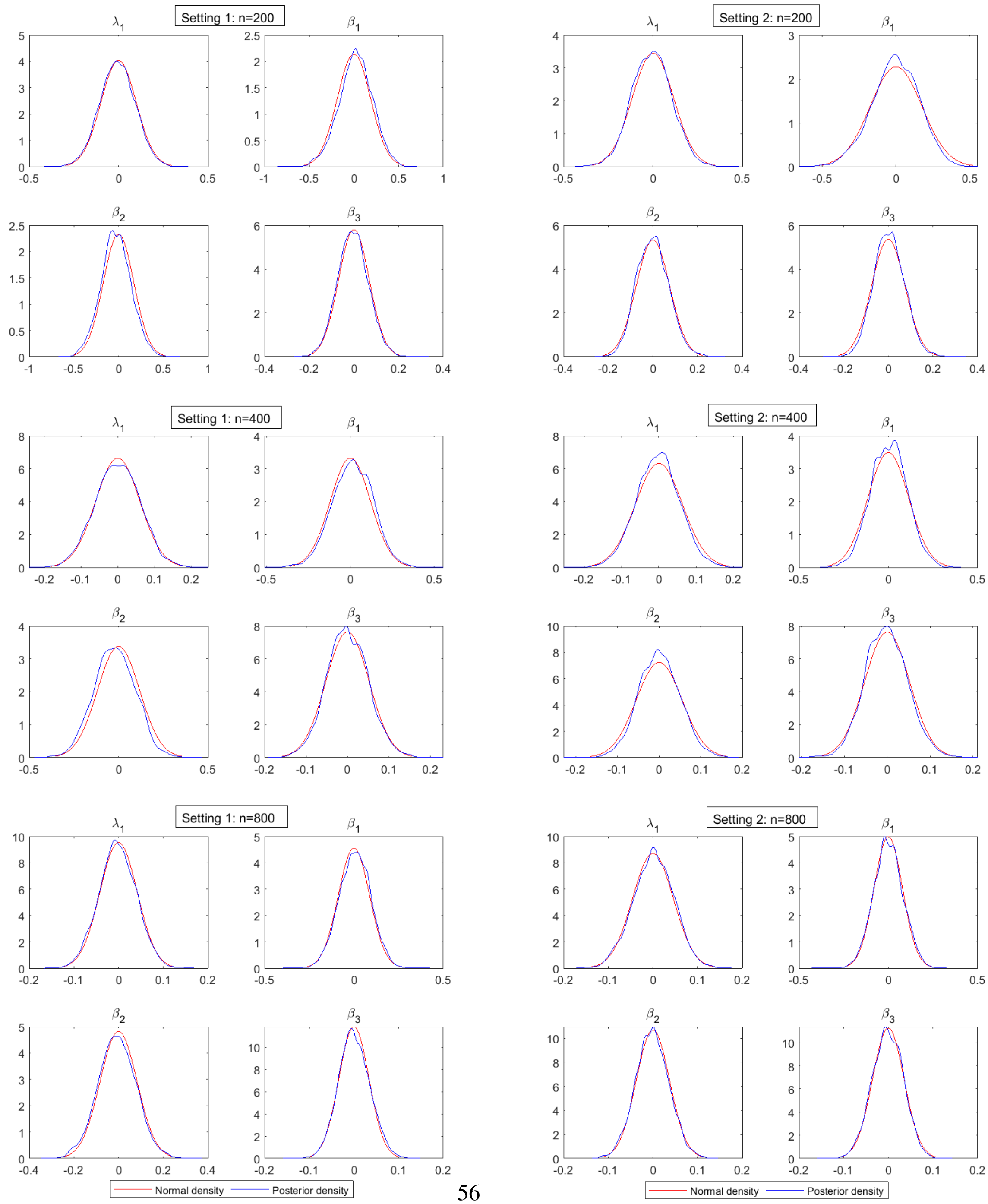


Figure H.2: Empirical density of Bayesian estimates vs normal density: SAR Tobit with $n = 200$

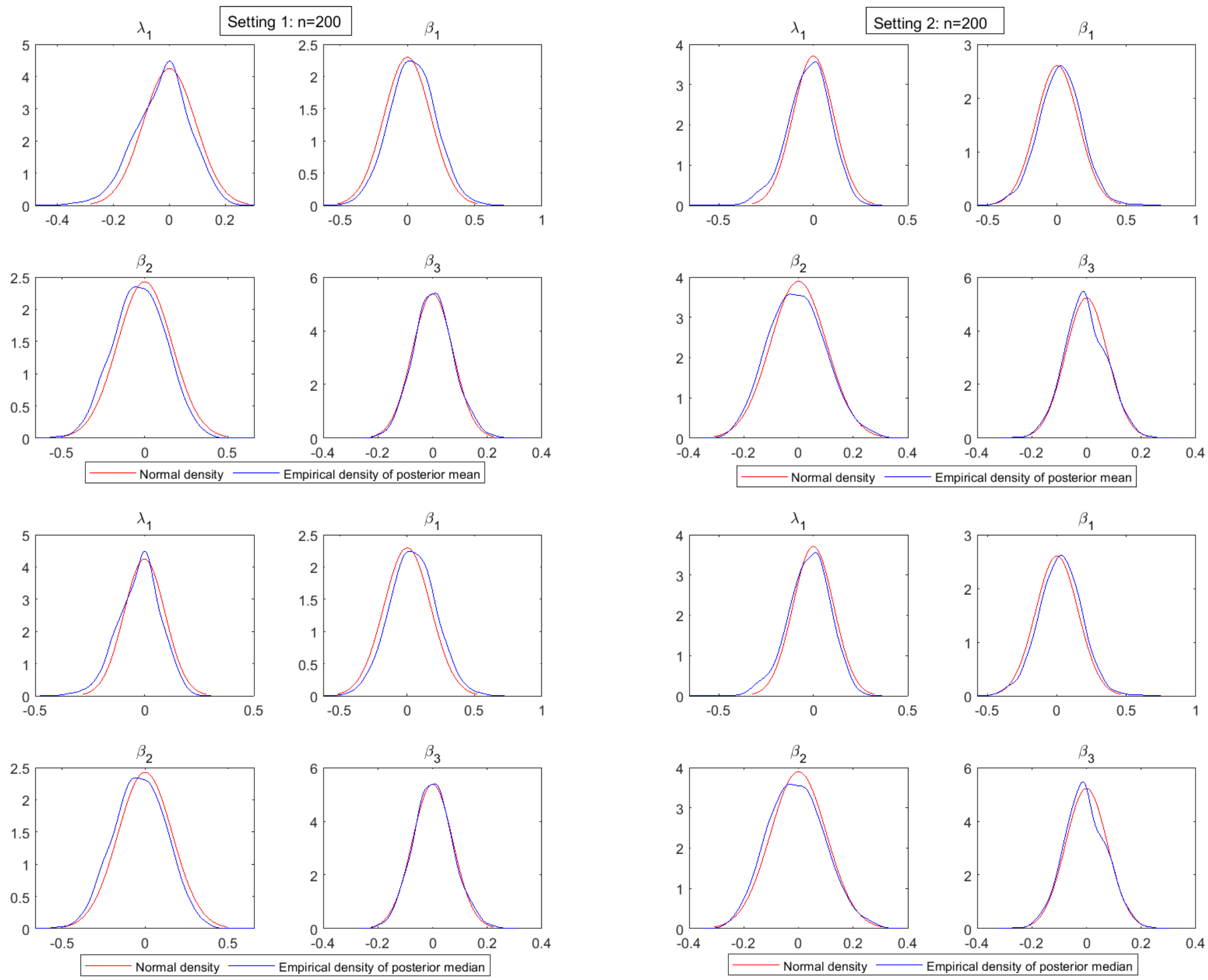


Figure H.3: Empirical density of Bayesian estimates vs normal density: SAR Tobit with $n = 400$

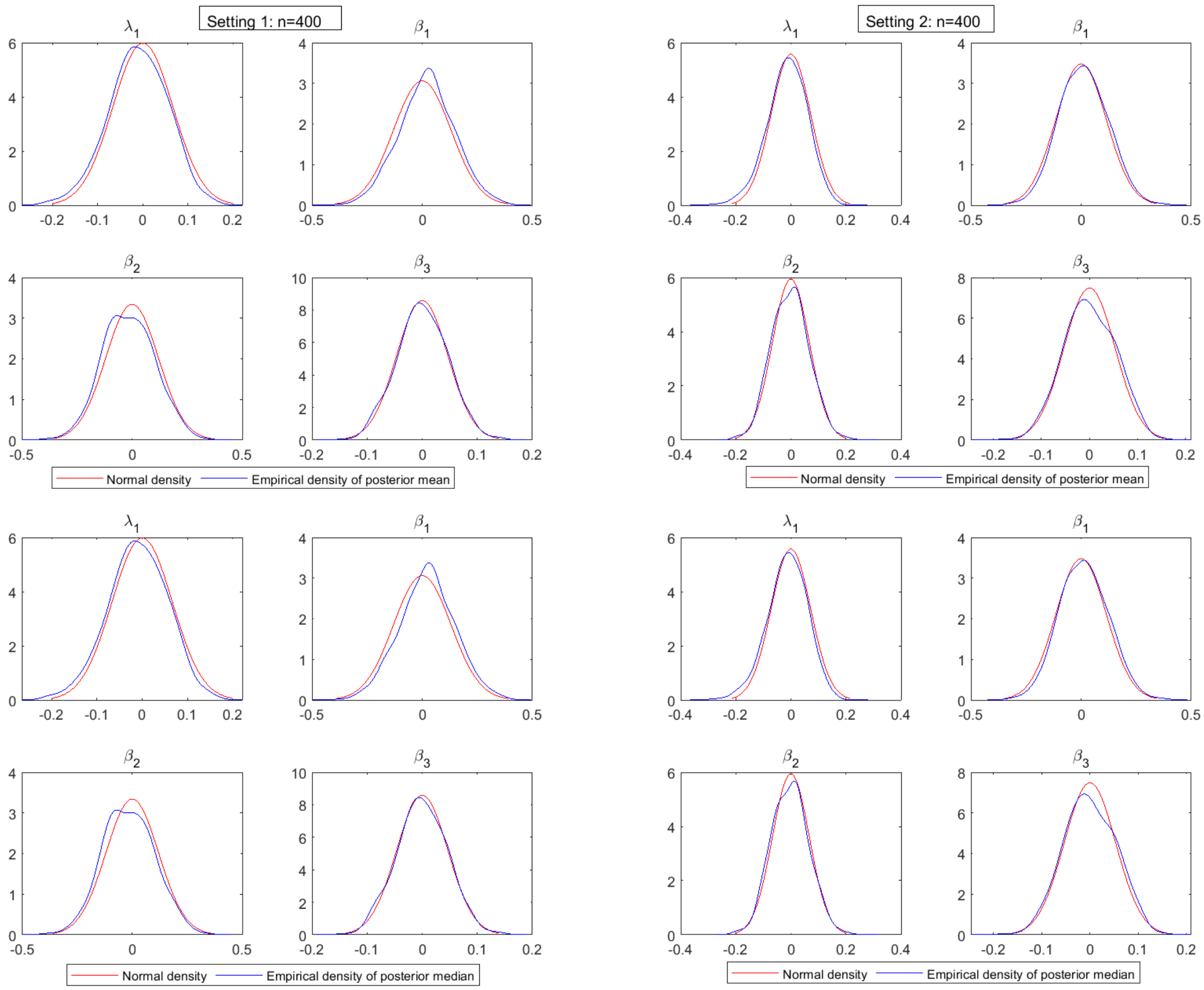


Figure H.4: Empirical density of Bayesian estimates vs normal density: SAR Tobit with $n = 800$

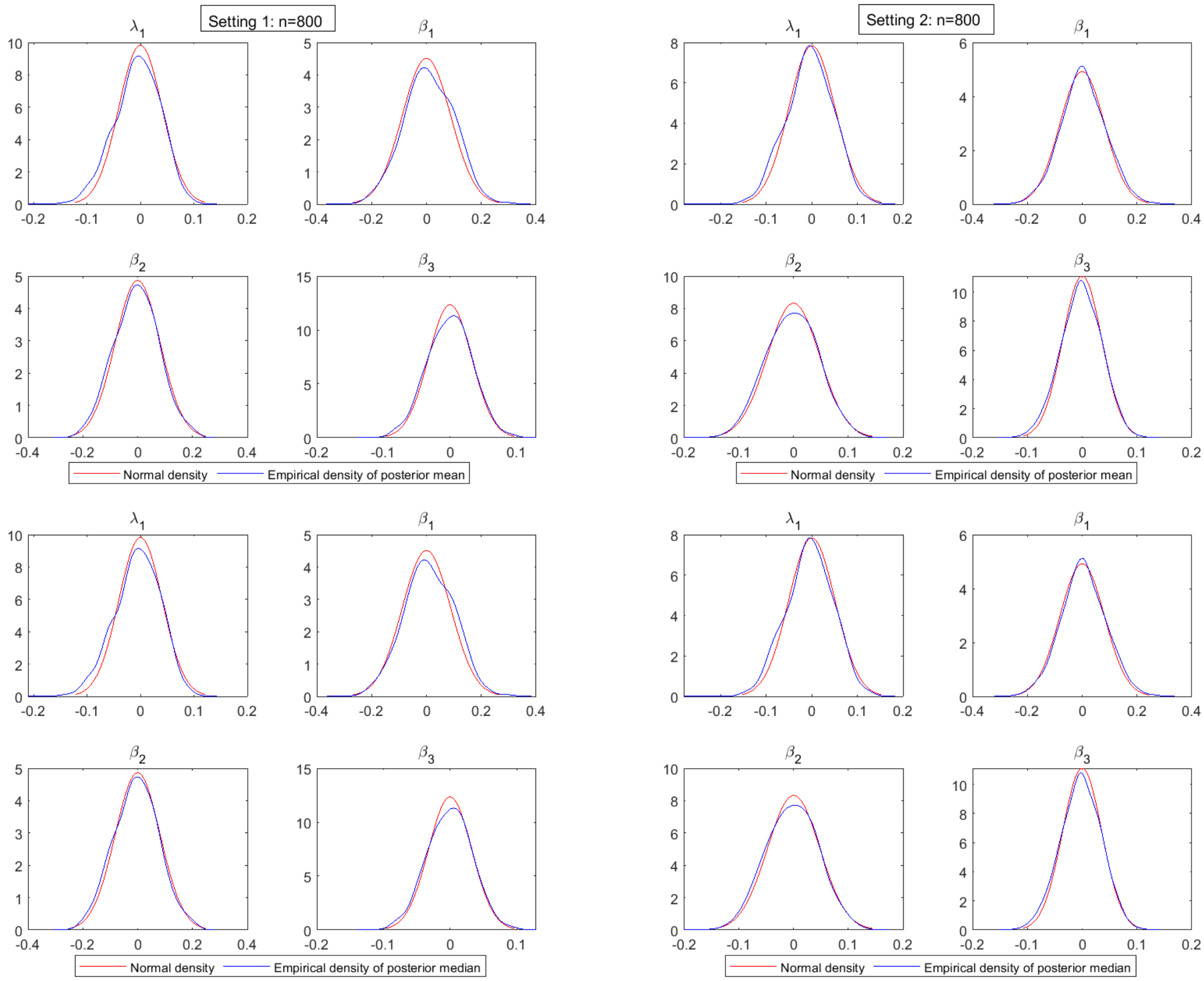


Table 17: SAR Tobit Model estimation over 1000 repetitions

		$n = 200$				$n = 400$				$n = 800$			
		MLE		BE		MLE		BE		MLE		BE	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
S1	λ_1	-0.022	0.100	-0.031	0.100	-0.009	0.067	-0.011	0.066	-0.004	0.043	-0.006	0.042
	β_1	0.030	0.193	0.055	0.190	0.007	0.128	0.016	0.121	0.006	0.091	0.011	0.089
	β_2	-0.000	0.163	-0.030	0.161	-0.002	0.123	-0.017	0.121	-0.001	0.088	-0.008	0.088
	β_3	-0.002	0.066	-0.004	0.066	0.000	0.050	-0.001	0.049	-0.002	0.033	-0.002	0.033
	β_4	0.001	0.063	-0.001	0.063	0.001	0.051	0.001	0.051	-0.0004	0.032	-0.001	0.032
	σ^2	-0.026	0.128	0.016	0.126	-0.015	0.089	0.008	0.088	-0.009	0.064	0.002	0.063
S2	λ_1	-0.017	0.111	-0.023	0.110	-0.009	0.063	-0.010	0.061	-0.005	0.045	-0.004	0.040
	β_1	0.026	0.176	0.025	0.168	0.011	0.116	0.007	0.108	0.006	0.091	0.001	0.075
	β_2	-0.002	0.072	-0.002	0.071	-0.001	0.055	-0.002	0.054	-0.001	0.037	0.001	0.037
	β_3	-0.001	0.092	-0.005	0.091	0.001	0.050	0.001	0.050	0.000	0.038	0.001	0.038
	β_4	-0.000	0.076	-0.002	0.076	0.002	0.051	0.002	0.051	-0.002	0.036	-0.002	0.035
	σ^2	-0.045	0.150	0.011	0.143	-0.021	0.102	0.006	0.100	-0.009	0.070	0.004	0.069

The values reported in this table are calculated from 1000 repetitions.
BE: Bayesian estimate.
S1: Setting 1; S2: Setting 2.
$(\lambda, \beta_1, \beta_2, \beta_3, \beta_4, \sigma^2) = (0.5, -0.5, 1, 1, 1, 1)$.
$v_i \sim N(0, 1)$.