Supplementary Material to "Inference in Dynamic, Nonparametric Models of Production: Central Limit Theorems for Malmquist Indices"

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A Empirical Applications in the Literature

As noted in Section 1, Malmquist indices have been widely applied to measure changes in productivity over time. Areas of empirical application in the literature include development economics (e.g., Thirtle et al., 2003; Kaur, 2015; Li et al., 2015; Wijesiri and Meoli, 2015), regional science and urban economics (e.g., Tengyun et al., 2009; Sun et al., 2012; Yasunaga, 2014), environmental economics (e.g, Kortelainen, 2008; Zhou et al., 2010; Macpherson et al., 2013; Sueyoshi and Goto, 2013; Lin and Fei, 2015; Molinos-Senante et al., 2016), transportation and logistics (e.g., Murillo-Melchor, 1999; Estache et al., 2004; Gitto and Mancuso, 2012; Egilmez and McAvoy, 2013; De Nicola et al., 2013; Ahn and Min, 2014), Shi and Xiao, 2015; macroeconomic growth (e.g., Färe et al., 1994 and 1997; Ray and Desli, 1997; Kumar and Russell, 2002; Krüger, 2003), natural resources (e.g., Hoff, 2006; Oliveira et al., 2009; Pyo and Kim, 2010; Kao, 2010; Korkmaz, 2011; Elhendy and Alkahtani, 2012), health economics (e.g., Burgess Jr. and Wilson, 1995; Giuffrida, 1999; Sommersguter-Reichmann, 2000; Staat, 2003; Kontodimopoulos and Niakas, 2006; Ozcan and Luke, 2011; Roh et al., 2011), energy economics (e.g., Yaisawarng and Klein, 1994; Price and Weyman-Jones, 1996; Yang and Pollitt, 2012; Sözen and Alp, 2013; Morfeldt and Silveira, 2014; Wu et al., 2014; Woo et al., 2015; Wu et al., 2015), economics of education (e.g, Rayeni et al., 2010; Ouellette and Vierstraete, 2010); Essid et al., 2014), agricultural economics (e.g., Ball et al., 2004); Bhushan, 2005; Coelli and Rao, 2005), and economics of innovation (e.g., Zheng, 2015). Malmquist indices are also used to examine specific industries, including computers and electronics (e.g, Chen and Ali, 2004; Chen and Ali, 2004; Liu and Wang, 2008; Chen et al., 2011; Lee et al., 2014), construction (e.g., Xue et al., 2008; Park et al., 2015), oil (e.g., Sueyoshi and Goto, 2015), textiles and clothing (e.g., Kapelko and Lansink, 2015), insurance and finance (e.g., Nektarios and Barros, 2010; Barros et al., 2005; Cummins and Rubio-Misas, 2006), manufacturing (e.g., Chavas and Cox, 1990; Weber and Domazlicky, 2001; Shestalova, 2003; Sowlati and Vahid, 2006), retail trade (e.g., Barros and Alves, 2004; Vaz and Camanho, 2012), chemicals (e.g., Ray and Ray, 2012; Han et al., 2014), biotechnology (e.g., Wang and Chang, 2012), accounting firms (e.g., Chang et al., 2009; Wang and Zhang, 2012), banking (e.g., Gilbert and Wilson, 1998; Wheelock and Wilson, 1999; Chen et al., 2007; Lin et al., 2007; Olgu and Weyman-Jones, 2008; Portela and Thanassoulis, 2010; Sharma and Gupta, 2010; Hadad et al., 2011), government services (e.g., Fuentes and Lillo-Bañuls, 2014), and pharmaceuticals (e.g., Yan and Yang, 2013; Song and Zhang, 2013). Many other examples can be found in the economics literature.

B Inference Based on Arithmetic Means of Logs

Much of the discussion here is analogous to the development in Section 4. Theorem 3.6 from Section 3.3 provides the basis for making inference about productivity change while working with arithmetic means of logs of estimated Malmquist indices. Recall from Section 4 that $\sigma^2_{\mathcal{M}} = \mathrm{VAR}(\log \mathcal{M}_i) = E\left((\log \mathcal{M}_i - E(\log \mathcal{M}_i))^2\right)$ where the expectations are over (X, Y) in both periods 1 and 2. Recall also that in the definition of $\mu_{\mathcal{M}}$ in (3.24) the expectation is also with respect to (X, Y) in both periods 1 and 2. Assume again that $\sigma^2_{\mathcal{M}}$ is finite.

As noted in Section 3.3 just after Lemma 3.3, it is clear from (3.34) that $\widehat{\mu}_{\mathcal{M},n}$ is a consistent estimator of $\mu_{\mathcal{M}}$, but with a bias of $C_{\mathcal{M}} n^{-\kappa}$ since $E\left(\mu_{\mathcal{M},n}\right) = \mu_{\mathcal{M}} + C_{\mathcal{M}} n^{-\kappa}$. If $\kappa > 1/2$, then the bias term as well as the remainder term $\xi_{n,\kappa}$ are dominated by the factor $\sqrt{n}$ and therefore can be ignored. Hence when $\kappa > 1/2$, a $(1-\alpha) \times 100$-percent confidence interval for $\widehat{\mu}_{\mathcal{M},n}$ is estimated by

$$\left[\widehat{\mu}_{\mathcal{M},n} \pm z_{1-\frac{\alpha}{2}} \frac{\widehat{\sigma}_{\mathcal{M},n}}{\sqrt{n}}\right], \tag{B.1}$$

where $z_{1-\frac{\alpha}{2}}$ is the corresponding quantile of the standard normal distribution function. Under the conditions of Theorem 3.6, provided $\kappa > 1/2$ (i.e., $p + q \leq 2$), the interval in (B.1) has asymptotically correct coverage. But if $\kappa = 1/2$, the bias in (3.34) is constant, and if $\kappa < 1/2$, the bias tends to infinity as $n \to \infty$.

Suppose $\kappa \leq 1/2$, and again let $n_\kappa = \min\left(\lfloor n^{2\kappa} \rfloor,\, n\right)$, where $\lfloor a \rfloor$ denotes the largest integer less than or equal to a as in Section 4. Assume that the observations in $\mathcal{X}_n$ are randomly sorted (the algorithm described by Daraio et al., 2018, Appendix D can be used to randomly sort the observations while allowing results to be replicated by other researchers using the same data and the same sorting algorithm). Let

$$\widehat{\mu}_{\mathcal{M},n_\kappa} := n_\kappa^{-1} \sum_{i=1}^{n_\kappa} \log \widehat{\mathcal{M}}_i \tag{B.2}$$

where the estimates $\widehat{\mathcal{M}}_i$ are computed using n (not n_κ) observations; i.e., the 4 estimates comprising $\widehat{\mathcal{M}}_i$ are each computed using all of the available observations in each period. The next result establishes the properties of this estimator.

Theorem B.1. *Under the conditions of Theorem 3.6, for cases where $\kappa \leq 1/2$,*

$$n^{\kappa}\left(\widehat{\mu}_{\mathcal{M},n_{\kappa}} - \mu_{\mathcal{M}} - C_{\mathcal{M}}n^{-\kappa} - \xi_{n,\kappa}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2_{\mathcal{M}}\right) \tag{B.3}$$

as $n \to \infty$, where $\xi_{n,\kappa} = O\left(n^{-\frac{3}{p+q+1}}(\log n)^{\frac{3}{p+q+1}}\right)$.

The bias term $C_{\mathcal{M}}n^{-\kappa}$ remains in (B.3), but it is now multiplied by the factor n^{κ} and hence is constant instead of exploding to infinity as before when $\kappa < 1/2$. In order to estimate the bias, a generalized jackknife estimator similar to the one described in Section 4 can be used, taking care to split the data into sub-samples appropriately for the two periods in which firms are observed.

As in Section 4, let $Z_i^t = (X_i^t, Y_i^t)$, $t \in \{1,\ 2\}$ so that the sample can be described by $\mathcal{X}_n = \{(Z_i^1, Z_i^2)\}_{i=1}^n$. Similar to Section 4, split $\mathcal{X}_n$ randomly into two sub-samples $\mathcal{X}_{m_1}^{(1)}$ and $\mathcal{X}_{m_2}^{(2)}$ of sizes $m_1 = \lfloor n/2 \rfloor$ and $m_2 = n - \lfloor n/2 \rfloor$ (respectively). Define

$$\widehat{\mu}_{\mathcal{M},m_j}^{(j)} := m_j^{-1} \sum_{(Z_i^1, Z_i^2) \in \mathcal{X}_{m_j}^{(j)}} \log \widehat{\mathcal{M}}_i(\mathcal{X}_{m_j}^{(j)}) \tag{B.4}$$

for $j \in \{1,\ 2\}$, where the notation $\widehat{\mathcal{M}}_i(\mathcal{X}_{m_j}^{(j)})$ indicates that the four estimates comprising the estimated Malmquist index $\widehat{\mathcal{M}}_i$ are each computed for observation i in the jth sub-sample using only the observations in the jth sub-sample $\mathcal{X}_{m_j}^{(j)}$. Then set

$$\widehat{\mu}^*_{\mathcal{M},n/2} = \frac{1}{2}\left(\widehat{\mu}^{(1)}_{\mathcal{M},m_1} + \widehat{\mu}^{(2)}_{\mathcal{M},m_2}\right). \tag{B.5}$$

By following arguments similar to those in Kneip et al. (2015, Section 4) it is easy to show that

$$\widetilde{B}_{n,\kappa} = (2^{\kappa} - 1)^{-1}\left(\widehat{\mu}^*_{\mathcal{M},n/2} - \widehat{\mu}_{\mathcal{M},n}\right) = C_{\mathcal{M}}n^{-\kappa} + \xi^*_{n,\kappa} + o_p\left(n^{-1/2}\right), \tag{B.6}$$

where $\xi^*_{n,\kappa}$ is of the same order as $\xi_{n,\kappa}$ appearing in (3.34), provides an estimator of the bias $C_{\mathcal{M}}n^{-\kappa}$.

As in Section 4, note that there are $\binom{n}{n/2}$ possible splits of the original n observations. To reduce the variance of the bias estimate in (B.6), the sample can be split $K << \binom{n}{n/2}$ times while randomly shuffling the observations before each split, and computing $\widetilde{B}_{n,\kappa,k}$ using (B.6) for $k = 1, \ldots, K$. Then

$$\widehat{B}_{n,\kappa} = K^{-1} \sum_{k=1}^{K} \widetilde{B}_{n,\kappa} \tag{B.7}$$

gives a generalized jackknife estimate of the bias $C_{\mathcal{M}} n^{-\kappa}$. Averaging in (B.7) reduces the variance by a factor of K^{-1} relative to the bias in (B.6).

Combining Theorem 3.6 and (B.7) leads to the following result.

Theorem B.2. *Under the conditions of Theorem 3.6, for cases where $\kappa \geq 2/5$,*

$$\sqrt{n}\left(\widehat{\mu}_{\mathcal{M},n} - \widehat{B}_{n,\kappa} - \mu_{\mathcal{M}} + \xi_{n,\kappa}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2_{\mathcal{M}}\right) \tag{B.8}$$

as $n \to \infty$.

Similar to the discussion in Section 4, the interplay between the root-n scaling factor and the remainder term $\xi_{n,\kappa}$ ensures that the result in Theorem B.8 holds for $\kappa \geq 2/5$, and hence for $(p+q) \leq 4$. However, it is important to note that Theorem B.2 does not hold in cases where $\kappa < 2/5$. In such cases, the remainder term $\xi_{n,\kappa}$, when multiplied by $\sqrt{n}$, diverges toward infinity. Alternatively, combining Theorem B.1 and (B.7) yields the following result.

Theorem B.3. *Under the conditions of Theorem 3.6, for cases where $\kappa < 1/2$,*

$$n^{\kappa}\left(\widehat{\mu}_{\mathcal{M},n_\kappa} - \widehat{B}_{n,\kappa} - \mu_{\mathcal{M}} - \xi_{n,\kappa}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2_{\mathcal{M}}\right) \tag{B.9}$$

as $n \to \infty$.

Note that in all cases (i.e., for all values of κ), $\xi_{n,\kappa} = o(n^{-\kappa})$ and hence $n^{\kappa}\xi_{n,\kappa} = o(1)$. Therefore the remainder term can be neglected.

Whenever $\kappa \geq 2/5$ and hence $(p+q) \leq 4$, Theorem B.2 can be used to construct an asymptotically correct $(1-\alpha)$ confidence interval for $\mu_{\mathcal{M}}$ given by

$$\left[\widehat{\mu}_{\mathcal{M},n} - \widehat{B}_{n,\kappa} \pm z_{1-\frac{\alpha}{2}} \frac{\widehat{\sigma}_{\mathcal{M},n}}{\sqrt{n}}\right], \tag{B.10}$$

where as in (B.1) $z_{1-\frac{\alpha}{2}}$ represents the $\left(1-\frac{\alpha}{2}\right)$ quantile of the standard normal distribution function.

Alternatively, in cases where $\kappa < 1/2$ and hence $(p + q) \geq 4$, Theorem B.3 permits construction of the asymptotically correct $(1 - \alpha)$ confidence interval

$$\left[\widehat{\mu}_{\mathcal{M},n_\kappa} - \widehat{B}_{n,\kappa} \pm z_{1-\frac{\alpha}{2}} \frac{\widehat{\sigma}_{\mathcal{M},n}}{n^\kappa}\right] \tag{B.11}$$

for $\mu_\mathcal{M}$. This interval is centered on $\widehat{\mu}_{\mathcal{M},n_\kappa} - \widehat{B}_{n,\kappa}$, and $\widehat{\mu}_{\mathcal{M},n_\kappa}$ computed from a random subset of estimates $\widehat{\mathcal{M}}_i$ (where each estimate $\widehat{\mathcal{M}}_i$ is computed using all of the sample observations in $\mathcal{X}_n$). While this may seem arbitrary, note that any confidence interval for $\mu_\mathcal{M}$ is arbitrary since any asymmetric confidence interval for $\mu_\mathcal{M}$ can be constructed simply by using different quantiles of the $\mathcal{N}(0, 1)$ distribution to establish the bounds. The main point is always to achieve a high level of coverage without making the confidence interval too wide to be informative.

In cases where $\kappa < 1/2$, the randomness of the interval in (B.11) due to centering on a mean over a subsample of size $n_\kappa < n$ can be eliminated by averaging the center of (B.11) over all possible draws (without replacement) of subsamples of size n_κ. This yields an interval

$$\left[\widehat{\mu}_{\mathcal{M},n} - \widehat{B}_{n,\kappa} \pm z_{1-\frac{\alpha}{2}} \frac{\widehat{\sigma}_{\mathcal{M},n}}{n^\kappa}\right] \tag{B.12}$$

centered on $\widehat{\mu}_{\mathcal{M},n} - \widehat{B}_{n,\kappa}$. The only difference between the intervals in (B.11) and (B.12) is the centering value. Both intervals have the same length and hence are equally informative. But the interval in (B.12) should be more accurate (i.e., should have higher coverage in finite samples) because the estimator $\widehat{\mu}_{\mathcal{M},n}$ uses more information than the estimator $\widehat{\mu}_{\mathcal{M},n_\kappa}$. Therefore, for $\kappa < 1/2$, $n_\kappa < n$ and hence the interval in (B.12) contains the true value $\mu_\mathcal{M}$ with probability greater than $(1 - \alpha)$. Due to the results given above, it is clear that the coverage of the interval in (B.12) converges to 1 as $n \to \infty$.

Note that when $(p+q) = 4$, either Theorems B.2 or B.3 can be used to provide different but asymptotically correct confidence intervals for $\mu_\mathcal{M}$. The interval in (B.10) uses the scaling factor $\sqrt{n}$ and hence neglects the term $\sqrt{n}\xi_{n,\kappa} = O\left(n^{-1/10}\right)$ in Theorem B.2. By contrast, the interval in (B.11) uses the scaling factor n^κ and hence neglects the term $n^\kappa\xi_{n,\kappa} = O\left(n^{-1/5}\right)$ in Theorem B.3. Therefore one should expect (B.11) to provide a better approximation in finite samples than (B.10) when $(p + q) = 4$.

The null hypothesis of no change in productivity versus change in productivity between periods 1 and 2 can be tested by computing the appropriate interval for $\mu_\mathcal{M}$. Under the null, $\mu_{\mathcal{M}} = 0$, while under the alternative hypothesis, $\mu_{\mathcal{M}} \neq 0$. Hence the null is rejected whenever the estimated confidence interval does not include zero.

The intervals given so far in (B.1), (B.10) and (B.11) are for $\mu_{\mathcal{M}}$ defined in (3.24). Theorem 3.7 and Remark 3.2 ensure that these intervals can be used to make inference about the geometric mean $E(M_n)$ where M_n is defined by (2.10). In particular, asymptotically valid intervals for $E(M_n)$ are obtained by taking exponentials of the bounds of the appropriate interval for $\mu_{\mathcal{M}}$.

C Additional Results, Proofs and Technical Details

C.1 Additional Results

As noted just after Theorem 3.5, the bias in (3.30) will be zero if the distributions in each period are identical, the numbers of observations n_1, n_2 available for estimation in each period are the same, and the joint density f_{12} introduced in Assumption 3.2(iii) is symmetric in its arguments. This leads to the following result.

Theorem C.1. *Assume Assumptions 2.1–2.7, 3.1 and 3.2 hold. In addition, assume that (i) $f := f^1 = f^2$ (and hence $\mathcal{D} := \mathcal{D}^1 = \mathcal{D}^2$); (ii) $n = n_1 = n_2$; and (iii) $f_{12}((x,y),\ (x^*,y^*)) = f_{12}((x^*,y^*),\ (x,y))$ for all $(x,y),\ (x^*,y^*) \in \mathcal{D}$. Then*

$$E\left(\widehat{\mu}_{\mathcal{M},n} - \mu_{\mathcal{M}}\right) = 0 \tag{C.1}$$

and as $n \to \infty$,

$$VAR\left(\widehat{\mu}_{\mathcal{M},n} - \mu_{\mathcal{M}}\right) = \frac{1}{n} VAR(\log \mathcal{M}_i) + o(n^{-1}). \tag{C.2}$$

A proof is given below in Section C.14.

Remark C.1. *Note that* (C.1) *holds for all n as seen in the proof that appears below in Section C.14, and is a consequence of a somewhat trivial fact: If (a) there are two samples with* identical *data generating processes, and (b) for both samples the same type of estimator is applied, then all resulting biases are identical (and hence cancel out when subtracting). In our context there only exists the difficulty that the roles of (X_i^1, Y_i^1) and (X_i^2, Y_i^2) are different in $\log \widehat{\gamma}_C^2(X_i^1, Y_i^1 \mid \mathcal{X}_{n_2}^2)$ and $\log \widehat{\gamma}_C^1(X_i^2, Y_i^2 \mid \mathcal{X}_{n_1}^1)$, which is resolved by the additional assumption (iii) of "symmetry" on the joint density.*

Remark C.2. *Section 3.1 of Kneip et al. (2016) overlooks the point raised in Remark C.1. Indeed the results in Section 3.1 of Kneip et al. (2016) are incomplete (but not false; the results provide bad approximations in the case of identical distributions). In Kneip et al. (2016, Section 3.1) if $n_1 = n_2$ and if both samples possess identical distributions then the biases cancel out, and in the notation of Kneip et al. (2016) $E(\widehat{\mu}_{1,n_1} - \widehat{\mu}_{2,n_2}) = 0$.*

Remark C.3. *It is possible to achieve* (C.1) *while only requiring $f^1 = f^2$ (i.e., without assuming $n_1 = n_2$ and symmetry of the joint density). This is possible by modifying the estimator and using*

$$\log \widetilde{\mathcal{M}}_i = \frac{1}{2}(\log \widehat{\gamma}^1_C(X_i^2, Y_i^2 \mid \mathcal{X}^1_{n_1,-i}) + \log \widehat{\gamma}^2_C(X_i^2, Y_i^2 \mid \mathcal{X}^2_{n_2,-i}) \\ - \log \widehat{\gamma}^1_C(X_i^1, Y_i^1 \mid \mathcal{X}^1_{n_1,-i}) - \log \widehat{\gamma}^2_C(X_i^1, Y_i^1 \mid \mathcal{X}^2_{n_2,-i})),$$

where $\mathcal{X}^s_{n_s,-i}$ is the reduced sample of size $(n-1)$ obtained by eliminating the ith observation (X_i^s, Y_i^s), $s = 0,\ 1$. In other words, for any $i = 1,\ \ldots,\ n$ the estimates $\widehat{\gamma}$ are constructed without taking into account the i-th observation. In this case everything is symmetric, and for identical distributions arguments similar to those used above lead to

$$E\left(\log \widehat{\gamma}^1_C(X_i^1, Y_i^1 \mid \mathcal{X}^1_{n_1,-i})\right) = E\left(\log \widehat{\gamma}^1_C(X_i^2, Y_i^2 \mid \mathcal{X}^1_{n_1,-i})\right) \tag{C.3}$$

and

$$E\left(\log \widehat{\gamma}^2_C(X_i^1, Y_i^1 \mid \mathcal{X}^2_{n_2,-i})\right) = E\left(\log \widehat{\gamma}^2_C(X_i^2, Y_i^2 \mid \mathcal{X}^2_{n_2,-i})\right) \tag{C.4}$$

independent of n_1 and n_2. Hence $E\left(\log \widetilde{\mathcal{M}}_i\right) = 0$.

Remark C.4. *Tests based on Theorem C.1 are tests of $f^1 = f^2$ rather than of $\mu_{\mathcal{M}} := E(\log \mathcal{M}_i) = 0$. Note that the true mean $\mu_{\mathcal{M}}$ may be zero even if $f^1 \neq f^2$. But if $f^1 \neq f^2$ then biases do not cancel out in general, and one is back to* (3.30)*. Since for large $(p+q)$ bias dominates variance, the test will (asymptotically) reject the null hypotheses even if $\mu_{\mathcal{M}} = 0$ if bias is not accounted for.*

C.2 Proof of Lemma 3.1

Consider rays $\mathcal{L}_1 = \mathcal{L}(x, y)$ and $\mathcal{L}_2 = \mathcal{L}(x, \lambda(x, y \mid \mathcal{C}(\Psi))y) \subset \mathcal{C}^\partial(\Psi)$.

Since $(\theta(x, y \mid \mathcal{C}(\Psi))x,\ y) \in \mathcal{L}_2$ and $(x,\ \lambda(x, y \mid \mathcal{C}(\Psi))y) \in \mathcal{L}_2$, $\frac{\lambda(x,y|\mathcal{C}(\Psi))\|y\|}{\|x\|} = \frac{\|y\|}{\theta(x,y|\mathcal{C}(\Psi))\|x\|}$ and hence $\lambda(x, y \mid \mathcal{C}(\Psi))^{-1} = \theta(x, y \mid \mathcal{C}(\Psi))$. In addition, $(\gamma_C(x, y \mid \mathcal{C}(\Psi))x,\ \gamma_C(x, y \mid \mathcal{C}(\Psi))^{-1}y) \in \mathcal{L}_2$. Therefore $\frac{\gamma_C(x,y|\mathcal{C}(\Psi))^{-1}\|y\|}{\gamma_C(x,y|\mathcal{C}(\Psi))\|x\|} = \frac{\|y\|}{\theta(x,y|\mathcal{C}(\Psi))\|x\|}$. Result (i) follows immediately.

To prove (ii), consider two points $(x, y) \in \mathcal{L}_1$ and $(\widetilde{x}, \widetilde{y}) \in \mathcal{L}_1$. Clearly, $(x,\ \lambda(x, y \mid \mathcal{C}(\Psi)y) \in \mathcal{L}_2$ and $(\widetilde{,}\ \lambda(\widetilde{x}, \widetilde{y} \mid \mathcal{C}(\Psi)\widetilde{y}) \in \mathcal{L}_2$. It follows that $\frac{\lambda(x,y|\mathcal{C}(\Psi)\|y\|}{\|x\|} = \frac{\lambda(\widetilde{x},\widetilde{y}|\mathcal{C}(\Psi)\|\widetilde{y}\|}{\|\widetilde{x}\|}$. Hence $\lambda(x, y \mid \mathcal{C}(\Psi) = \lambda(\widetilde{x}, \widetilde{y} \mid \mathcal{C}(\Psi)$ since $\frac{\|y\|}{\|x\|} = \frac{\|\widetilde{y}\|}{\|\widetilde{x}\|}$, establishing (ii). Results (iii) and (iv) follow from (i) and (ii). ■

C.3 Proof of Lemma 3.2

The results follow from the proof of Lemma 3.1 after replacing $\mathcal{C}(\Psi)$ with $\mathcal{C}(\widehat{\Psi}))$. ■

C.4 Some Background Material used in Proof of Theorem 3.1

The proof of Theorem 3.1 that follows relies on the structural analysis used in the proof of Theorem 3.1 in Kneip et al. (2015). Let us first recall some of the notation used there.

Consider an arbitrary point $(x, y) \in \mathcal{D}$. Let $\mathcal{V}(x)$ denote the $(p-1)$-dimensional linear space of all vectors $z \in \mathbb{R}^p$ such that $z^T x = 0$. Any input vector X_i adopts a unique decomposition of the form $X_i = \gamma_i \frac{x}{\|x\|} + Z_i$ for some $Z_i \in \mathcal{V}(x)$ and $\gamma_i = \frac{x^T X_i}{\|x\|}$, where $\|\cdot\|$ denotes the Euclidean norm. Let $\Psi^*(x)$ denote the set of all $(z, y) \in \mathcal{V}(x) \times \mathbb{R}^q$ with $(\gamma \frac{x}{\|x\|} + z, y) \in \mathcal{D}$ for some $\gamma > 0$. Note that the point of interest $(x, y) \in \Psi$ has coordinates $(0, y)$ in $\Psi^*(x)$.

The maintained assumptions imply that for any $(z, y) \in \Psi^*(x)$, there exists $\gamma > 0$ such that $(\gamma \frac{x}{\|x\|} + z, y) \in \Psi$. The efficient boundary of Ψ can therefore be described by the function $g_x(z, y) := \inf \left\{ \gamma \mid \left(\gamma \frac{x}{\|x\|} + z, y \right) \in \Psi \right\}$ defined for any $(z, y) \in \Psi^*(x)$. Furthermore, with only a small abuse of notation, one may extend the definition of g_x to all (v, y) with $\left(v - \frac{x^T v}{\|x\|^2} x, y \right) \in \Psi^*(x)$ by taking $g_x(v, y) = g_x \left(v - \frac{x^T v}{\|x\|^2} x, y \right)$.

Properties of g_x are discussed in Kneip et al. (2008). In particular, under the assumptions of the theorem, g_x is a three times continuously differentiable, strictly convex function, and there exists a constant $C_1 > 0$ such that $w^T g''_x(0, y) w \geq C_1$ for all $w \in \mathcal{V}(x) \times \mathbb{R}^q$ with $\|w\| = 1$ and all $x \in \mathbb{R}^q$ with $(x, y) \in \mathcal{D}$. Moreover, $g''_x(0, y)$ changes continuously in x. In the following we will additionally use $g''_{x;zz}(\widetilde{z}, \widetilde{y})$ to denote the $(p-1) \times (p-1)$-matrix of partial derivatives with respect to the z-coordinates at a point $(\widetilde{z}, \widetilde{y})$, while $g''_{x;yy}(\widetilde{z}, \widetilde{y})$ will denote the $q \times q$-matrix of partial derivatives with respect to the y-coordinates.

The decomposition described above establishes a new coordinate system in which each observation (X_i, Y_i) can be equivalently represented by the corresponding vector (θ_i, Z_i, Y_i), where $\theta_i := \theta(X_i, Y_i)$. Any point (x, ay) of interest has coordinates $(\theta(x, ay), 0, ay)$ in this system.

Different from Kneip et al. (2015) we will need an additional decomposition of the variable Y_i given by

$$Y_i = \alpha_i y + V_i \quad \text{for some } V_i \in \mathbb{R}^q,\ V_i^t y = 0,\ \text{and } \alpha_i = \frac{y^T Y_i}{\|y\|^2}. \tag{C.5}$$

This establishes another coordinate system with $(Z_i, V_i) \in \mathcal{V}(x, y)$, where $\mathcal{V}(x, y)$ denotes the $(p-1) \times (q-1)$-dimensional linear space of all vectors $z \in \mathbb{R}^p$ and $v \in \mathbb{R}^p$ such that $z^T x = 0$ and $v^T y = 0$. Instead of using (θ_i, Z_i, Y_i), each observation (X_i, Y_i) can be equivalently represented by the corresponding vector $(\theta_i, Z_i, \alpha_i, V_i)$, where $\theta_i := \theta(X_i, Y_i)$. Any point (x, ay) of interest has coordinates $(\theta(x, ay),\ 0,\ ay,\ 0)$ in this new system.

Let $z_x^{(1)}, \ldots, z_x^{(p-1)}$ and $v_y^{(1)}, \ldots, v_y^{(q-1)}$ be orthonormal bases of Z_i and V_i. Clearly, the $z_x^{(j)}$ and $v_y^{(j)}$ can be chosen as continuous functions of x and y, respectively. Every vector Z_i can be expressed in the form $Z_i = \mathbf{Z}_x \zeta_i$, where $\mathbf{Z}_x$ is the $p \times (p-1)$ matrix with columns $z_x^{(j)}$, $j = 1, \ldots, p-1$, and $\zeta_i \in \mathbb{R}^{p-1}$. Similarly, every vector V_i can be expressed in the form $V_i = \mathbf{V}_y v_i$, where $\mathbf{V}_y$ is the $q \times (q-1)$ matrix with columns $v_y^{(j)}$, $j = 1, \ldots, q-1$, and $v_i \in \mathbb{R}^{q-1}$.

Since $\theta_i = \theta(X_i, Y_i)$, $Z_i = X_i - \frac{x^T X_i}{\|x\|^2} x$, $\alpha_i = \frac{y^T Y_i}{\|y\|^2}$, and $V_i = Y_i - \frac{y^T Y_i}{\|y\|^2} y$ are smooth functions of (X_i, Y_i), the joint density f of (X_i, Y_i) translates into a density $\widetilde{f}_{x,y}$ on $(0, 1] \times \mathbb{R}^{p-1} \times \mathbb{R} \times \mathbb{R}^{q-1}$ of $(\theta_i, \zeta_i, \alpha_i, v_i)$. Let $\widetilde{\mathcal{D}}$ denote the support of this density. Since f is continuously differentiable, $\widetilde{f}_{x,y}(\theta, \zeta, \alpha, v)$ is also continuously differentiable on $(\theta, \zeta, \alpha, v) \in \widetilde{\mathcal{D}}$. Furthermore, compactness of $\mathcal{D}^*$, as well as $f(\theta(x, y)x, y) > 0$ for all $(x, y) \in \mathcal{D}$, imply that there exists a constant $c_{\inf} > 0$ such that

$$\widetilde{f}_{x,y}(\theta, \zeta, \alpha, v) \geq c_{\inf} \tag{C.6}$$

whenever $(\mathbf{Z}_x \zeta, \alpha y + \mathbf{V}_y v) \in \Psi^*(x)$ and $(x, y) \in \mathcal{D}$.

C.5 Proof of Theorem 3.1

Consider an arbitrary point $(x, y) \in \mathcal{D}$ and recall the notation introduced above. First note that $g_x(0, ay) = \|x\| \theta(x, ay)$ and hence

$$\theta_{\mathrm{C}}(x, y) = \frac{1}{\|x\|} \cdot \min_{a>0} \left\{ \frac{g_x(0, ay)}{a} \mid \left(\frac{g_x(0, ay)}{\|x\|} x, ay\right) \in \Psi \right\}. \tag{C.7}$$

Assumption 3.1 together with strict convexity of g_x therefore imply that $a_{min}^{x,y} \in \mathbb{R}_+$ is uniquely defined and $(\theta(x, a_{min}^{x,y} y)x, a_{min}^{x,y} y) \in \mathcal{D}$. Taking derivatives yields

$$\frac{\partial}{\partial a} \frac{g_x(0, ay)}{a}\bigg|_{a=a_{min}^{x,y}} = 0, \quad A_{x,y} := \frac{\partial^2}{\partial a^2} \frac{g_x(0, ay)}{a}\bigg|_{a=a_{min}^{x,y}} = \frac{y^T g''_{x;yy}(0, a_{min}^{x,y} y)y}{a_{min}^{x,y}} > 0. \tag{C.8}$$

Since by assumption g_x is at least three times continuously differentiable, Taylor expansions lead to

$$\left| \frac{g_x(0, ay)}{a} - \frac{g_x(0, a_{min}^{x,y} y)}{a_{min}^{x,y}} - \frac{A_{x,y}}{2}(a - a_{min}^{x,y})^2 \right| \leq D|a - a_{min}^{x,y}|^3 \tag{C.9}$$

for some $D > 0$ and all a with $(\theta(x, ay)x, ay) \in \mathcal{D}$. Since $\mathcal{D}^* = \{\theta(\widetilde{x}, \widetilde{y})\widetilde{x}, \widetilde{y})|(\widetilde{x}, \widetilde{y}) \in \mathcal{D}\}$ is compact, D can be chosen independent of $a > 0$ and $(x, y) \in \mathcal{D}$.

Let $\kappa = \frac{2}{p+q+1}$. Since by Assumption 3.1 no relevant point lies in the "observable boundary" for sufficiently large n, it follows from (A.6) and (A.9) in the proof of Theorem 3.1 in Kneip et al. (2015) that for any $a > 0$ with $|a - a_{min}^{x,y}| < \delta$ there exists some $0 < D_1, D_2 < \infty$, which can be chosen independent of (x, y), such that

$$\Pr\left(\left| \|x\| \widehat{\theta}_{\mathrm{VRS}}(x, ay \mid \mathcal{X}_n) - \|x\| \theta(x, ay) \right| \geq D_1 n^{-\kappa} (\log n)^\kappa \right) \leq D_2 n^{-2} \tag{C.10}$$

On the other hand, by (C.9) there exists a $0 < d_1 < \infty$ such that

$$\begin{aligned} \frac{g_x(0, (a_{min}^{x,y} - d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}})y)}{a_{min}^{x,y} - d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}} - \frac{g_x(0, a_{min}^{x,y} y)}{a_{min}^{x,y}} &\geq 3D_1 n^{-\kappa} (\log n)^\kappa, \\ \frac{g_x(0, (a_{min}^{x,y} + d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}})y)}{a_{min}^{x,y} + d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}} - \frac{g_x(0, a_{min}^{x,y} y)}{a_{min}^{x,y}} &\geq 3D_1 n^{-\kappa} (\log n)^\kappa. \end{aligned} \tag{C.11}$$

Since necessarily $\inf_{(x,y)\in\mathcal{D}} A_{x,y} > 0$, d_1 can be chosen independent of $(x, y) \in \mathcal{D}$. Inequalities (C.9) and (C.10) now imply that with probability converging to 1 we obtain

$$\|x\| \frac{\widehat{\theta}_{\mathrm{VRS}}(x, ay \mid \mathcal{X}_n)}{a} > \|x\| \frac{\widehat{\theta}_{\mathrm{VRS}}(x, a_{min}^{x,y} y \mid \mathcal{X}_n)}{a_{min}^{x,y}} \tag{C.12}$$

for $a = a_{min}^{x,y} - d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}$ as well as for $a = a_{min}^{x,y} + d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}$. But convexity then additionally implies that (C.12) also holds for all $a \leq a_{min}^{x,y} - d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}$ and $a \geq a_{min}^{x,y} + d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}$. More precisely, there exists a constant $0 < D_3 < \infty$, which can be chosen independent of (x, y), such that

$$1 - \Pr\left(\widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n) = \min_{a_{min}^{x,y} - d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}} \leq a \leq a_{min}^{x,y} + d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}} \frac{\widehat{\theta}_{\mathrm{VRS}}(x, ay \mid \mathcal{X}_n)}{a} \right) \leq D_3 n^{-2}. \tag{C.13}$$

Recall that $Y_i = \alpha_i y + V_i$. Representation (A.15) of the VRS-DEA estimator in the proof of Theorem 3.1 in Kneip et al. (2015) tells us that

$$\frac{\widehat{\theta}_{\mathrm{VRS}}(x, y \mid \mathcal{X}_n)}{a} = \min \Big\{ \sum_{i=1}^{n} \omega_i \frac{g_x(\theta_i Z_i, \alpha_i y + V_i)}{a\|x\|\theta_i} \mid \boldsymbol{i}_n^T \boldsymbol{\omega} = 1,\ \boldsymbol{Z}\boldsymbol{\omega} = 0,$$
$$\boldsymbol{V}\boldsymbol{\omega} = 0, \boldsymbol{\alpha}^T \boldsymbol{\omega} = a, \boldsymbol{\omega} \in \mathbb{R}_+^n \Big\}$$
$$= \theta_{\mathrm{C}}(x, y) \times \min \Big\{ \sum_{i=1}^{n} \omega_i \frac{a_{min}^{x,y} g_x(\theta_i Z_i, \alpha_i y + V_i)}{a g_x(0, a_{min}^{x,y} y)\theta_i} \mid \boldsymbol{i}_n^T \boldsymbol{\omega} = 1,\ \boldsymbol{Z}\boldsymbol{\omega} = 0,$$
$$\boldsymbol{V}\boldsymbol{\omega} = 0, \boldsymbol{\alpha}\boldsymbol{\omega} = a, \boldsymbol{\omega} \in \mathbb{R}_+^n \Big\} \tag{C.14}$$

where $\boldsymbol{i}_n = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, ω_i represents the ith element of $\boldsymbol{\omega}$, $\theta_i = \theta(X_i, Y_i)$, $Z_i = X_i - \frac{\boldsymbol{x}^T X_i}{||\boldsymbol{x}||^2}\boldsymbol{x}$ is a $(p \times 1)$ vector and $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$ is a $(p \times n)$ matrix, $\boldsymbol{V} = (V_1, \ldots, V_n)$ is a $((q-1) \times n)$ matrix, and $\boldsymbol{\alpha}^T = (\alpha_1, \ldots, \alpha_n)$.

An essential step of the proof now consists in the localization argument developed in Kneip et al. (2008) and reconsidered in Kneip et al. (2015) which states that VRS-DEA estimators are asymptotically determined by local information. In Kneip et al. (2008, 2015) the argument relies on using the coordinates (θ_i, Z_i, Y_i), but a generalization to the coordinates $(\theta_i, Z_i, \alpha_i, V_i)$ is immediate. For any $h > 0$, define the set

$$C(x, a_{min}^{x,y} y; h) = \Big\{ (\widetilde{\theta}, \widetilde{z}, \widetilde{\alpha}, \widetilde{v}) \in (0, 1] \times \mathbb{R}^{p-1} \times \mathbb{R}_+ \times \mathbb{R}^{q-1} \mid 1 - \widetilde{\theta} \leq h^2, |\widetilde{\alpha} - a_{min}^{x,y}| \leq h,$$
$$\widetilde{z} = \sum_j \zeta_j z_x^{(j)}, |\zeta_j| \leq h \; \forall\, j = 1, \ldots, p-1,\ \widetilde{v} = \sum_r v_r v_y^{(r)},\ |v_r| \leq h \; \forall\, r = 1, \ldots, q \Big\}, \tag{C.15}$$

and let $\mathcal{X}_n(x, a_{min}^{x,y} y; h) := \{(X_i, Y_i) \in \mathcal{X}_n \mid (\theta_i, Z_i, \alpha_i, V_i) \in C(x, a_{min}^{x,y} y; h)\}$.

In the following it will be necessary to distinguish between points (x, y) lying in the interior and on the observable boundary of $\mathcal{D}$. For $(x, y) \in \mathcal{D}$ let

$$\Psi^{*\partial}(x, y) = \Big\{ (\widetilde{z}, \widetilde{v}) \in \mathcal{V}(x, y) \mid (g_x(\widetilde{z}, a_{min}^{x,y} y + \widetilde{v}) \frac{x}{||x||} + \widetilde{z}, a_{min}^{x,y} y + \widetilde{v}) \in \mathcal{D}^* \text{ while for any}$$
$$\epsilon > 0 \text{ there is some } (z, v) \in \mathbb{R}^{p-1} \times \mathbb{R}^{q-1} \text{ with } \|\widetilde{z} - z\| < \epsilon \text{ and } \|\widetilde{v} - v\| < \epsilon$$
$$\text{such that } (g_x(z, a_{min}^{x,y} y + v) \frac{x}{||x||} + z, a_{min}^{x,y} y + v) \notin \mathcal{D}^* \Big\} \tag{C.16}$$

denote the boundary of possible vectors (z, v), where of course $\Psi^{*\partial}(x, y) = \emptyset$ if $\min p, q = 1$.

Then define the observable boundary as

$$\mathcal{W}(h) := \Bigg\{ (x,y) \in \mathcal{D} \mid \min \Big\{ \min_{j=1,\,\ldots,\,p-1} |\zeta_j|, \min_{r=1,\,\ldots,\,q-1} |v_r| \Big\} \le h$$
$$\text{for some } (\widetilde{z},\widetilde{v}) \in \Psi^{*\partial}(x,y) \text{ with } \widetilde{z} = \sum_j \zeta_j z_x^{(j)}, \widetilde{v} = \sum_r v_r v_y^{(r)} \Bigg\}. \tag{C.17}$$

If $p \le 1$ and $q \le 1$, then $\mathcal{W}(h) = \emptyset$; but for $p+q > 2$, compactness of $\mathcal{D}^*$ implies that for any $h > 0$ the observable boundary $\mathcal{W}(h)$ is nonempty.[1]

Recall the constant d_1 in (C.12) and choose some constant $b \ge 4(p+q)(1+d_1)$. Then set

$$\nu_n := b \left(\frac{\log n}{n \widetilde{f}_{x,y}(1, 0, a^{x,y}_{min}, 0)} \right)^{\frac{1}{p+q+1}} \tag{C.18}$$

as well as

$$\nu_n^* := b \left(\frac{\log n}{c_{\inf} n} \right)^{\frac{1}{p+q+1}}. \tag{C.19}$$

Case(i): We first consider the case where (x,y) is in the interior of $\mathcal{D}$ in the sense that $(x,y) \notin \mathcal{W}(\nu_n^*)$. In this case, by Assumption 3.1 we have $C(x, a^{x,y}_{min} y;\ \nu_n) \subset \mathcal{D}$ for all sufficiently large n.

Following the arguments in Kneip et al. (2008, 2015) one can construct $k = 2(p+q-1)$ hypercubes $B_s \subset \mathbb{R}^{p-1} \times \mathbb{R}^p$, $s = 1, \ldots, k$, of side lengths $\frac{\nu_n}{2(p-1)+2q}$ and centered at values (z_j, y_j) determined in the following way: $z_j = \sum_s \zeta_s z_x^{(s)}$, $y_j = (\alpha + a^{x,y}_{min})y + \sum_r v_r v_y^{(r)}$, where for each $j = 1,\ \ldots,\ 2(p+q-1)$ exactly one of the coordinates $(\zeta_1,\ \ldots,\ \zeta_{p-1}, \alpha, v_1, \ldots, v_{q-1})$ equals $\nu_n \cdot \frac{2(p-1)+2q-1}{2(p-1)+2q}$ or $-\nu_n \cdot \frac{2(p-1)+2q-1}{2(p-1)+2q}$, while all others are identically zero. By definition of ν_n, the probability that there exist at least k observations $(\theta_{i_1}, Z_{i_1}, Y_{i_1}), \ldots, (\theta_{i_k}, Z_{i_k}, Y_{i_k})$ with $\theta_{i_j} \ge 1 - \nu_n^2$ and $(Z_{i_j}, Y_{i_j}) \in B_j$, $j = 1, \ldots, k$, is of order $1 - n^{-2}$ as $n \to \infty$.

On the other hand, if such a set of k observations exists, then by construction for any $a \in [a^{x,y}_{min} - d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}, a^{x,y}_{min} + d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}]$ the point $(0, ay)$ is in the interior of the convex hull of (Z_{i_j}, Y_{i_j}), $j = 1,\ \ldots,\ k$. If n is sufficiently large, by the strict convexity of g_x the arguments in the proof of Theorem 1 of Kneip et al. (2008)

[1] Note that there is an error in Appendix A of Kneip et al. (2015). The concept of the boundary $\Psi^{*\partial}(x)$ used here is correct (as well as the arguments relying on $\Psi^{*\partial}(x)$). But the definition in formula (A.4) of Kneip et al. (2015) does not provide the proper boundary, and it should be replaced by an analog of C.16. The proof of Theorem 3.1 in Kneip et al. (2015) still holds after this change.

can then be used to show then for any other observation (θ_i, Z_i, Y_i) with $(\theta_i, Z_i, Y_i)) \notin C(x, a_{min}^{x,y} y; \nu_n)$ and any vector $\boldsymbol{\omega} \in \mathbb{R}_+^n$ with $\omega_i > 0$, satisfying the constraints in (C.14) for $a \in [a_{min}^{x,y} - d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}, a_{min}^{x,y} + d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}]$, there exists another vector $\boldsymbol{\omega}^* \in \mathbb{R}_+^n$ with $\omega_i^* = 0$ and $\omega_{i_j}^* \geq 0$, $j = 1, \ldots, k$, such that $\sum_{i=1}^n \omega_i \frac{g_x(\theta_i Z_i, Y_i)}{\theta_i} > \sum_{i=1}^n \omega_i^* \frac{g_x(\theta_i Z_i, Y_i)}{\theta_i}$. This implies that for arbitrary $a \in [a_{min}^{x,y} - d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}, a_{min}^{x,y} + d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}]$ the minimum in (C.14) is achieved by assigning zero weight $\omega_i = 0$ to each observation with $(\theta_i, Z_i, Y_i) \notin C(x, a_{min}^{x,y} y; \nu_n)$. This then leads to $\widehat{\theta}_{\mathrm{VRS}}(x, ay \mid \mathcal{X}_n) = \widehat{\theta}_{\mathrm{VRS}}(x, ay \mid \mathcal{X}_n(x, a_{min}^{x,y} y; \nu_n))$, where $\widehat{\theta}_{\mathrm{VRS}}(x, ay \mid \mathcal{X}_n(x, a_{min}^{x,y} y; \nu_n))$ denotes the VRS-DEA estimator only based on the subset of all observations in $\mathcal{X}_n(x, a_{min}^{x,y} y; \nu_n)$.

Therefore, there exists a $D_4 \in (0, \infty)$, which can be chosen independent of $(x, y) \in \mathcal{D}$ with $(x, y) \notin \mathcal{W}(\nu_n^*)$, such that for all sufficiently large n,

$$\Pr\left(\widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n) = \min_{a_{min}^{x,y} - d_1 (\frac{\log n}{n})^{\frac{\kappa}{2}} \leq a \leq a_{min}^{x,y} + d_1 (\frac{\log n}{n})^{\frac{\kappa}{2}}} \frac{\widehat{\theta}_{\mathrm{VRS}}(x, ay \mid \mathcal{X}_n(x, a_{min}^{x,y} y; \nu_n))}{a}\right) \geq 1 - D_4 n^{-2} \tag{C.20}$$

Now consider the sums in (C.14) with respect to the (random) number $K_n \leq \#\mathcal{X}_n(x, a_{min}^{x,y} y; \nu_n)$ of all observations with coordinates $(\theta_{i_j}, Z_{i_j}, \alpha_{i_j}, V_{i_j}) \in C(x, a_{min}^{x,y} y; \nu_n)$. Furthermore, for some $a \in [a_{min}^{x,y} - d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}, a_{min}^{x,y} + d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}]$ consider arbitrary weight vectors $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{K_n})^T \in \mathbb{R}_+^{K_n}$ such that $\sum_{j=1}^{K_n} \omega_j = 1$, $\sum_{j=1}^{K_n} \omega_j Z_{i_j} = 0$, $\sum_{j=1}^{K_n} \omega_j \alpha_{i_j} = a$, and $\sum_{j=1}^{K_n} \omega_j V_{i_j} = 0$. Let $\theta_{i_j}^* := 1 - \theta_{i_j}$, $G(ay) := g_x''(0, ay)$, and note that $\sum_{j=1}^{K_n} \omega_j (\alpha_{i_j} - a_{min}^{x,y})^2 = \sum_{j=1}^{K_n} \omega_j (\alpha_{i_j} - a)^2 + (a - a_{min}^{x,y})^2$. It then follows from Taylor expansions of g_x as well as from (C.9) that for some $0 \leq R_n, R_n^* < \infty$

$$\begin{aligned}
\sum_{j=1}^{K_n} \omega_j \frac{g_x(\theta_{i_j} Z_{i_j}, \alpha_{i_j} y + V_{i_j})}{a \theta_{i_j}} &= \frac{g_x(0, ay)}{a} + \frac{1}{a} \sum_{j=1}^{K_n} \omega_j \Bigg[\begin{pmatrix} Z_{i_j} \\ V_{i_j} \end{pmatrix}^T \frac{G(ay)}{2} \begin{pmatrix} Z_{i_j} \\ V_{i_j} \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ (a_{i_j} - a) y \end{pmatrix}^T G(ay) \begin{pmatrix} Z_{i_j} \\ V_{i_j} \end{pmatrix} + (\alpha_{i_j} - a)^2 \frac{y^T g''_{x;yy}(0, ay) y}{2} + \theta_{i_j}^* \Bigg] + R_n \nu_n^3 \\
&= \frac{g_x(0, a_{min}^{x,y} y)}{a_{min}^{x,y}} \\
&\quad + \frac{1}{a_{min}^{x,y}} \underbrace{\sum_{j=1}^{K_n} \omega_j \left[\begin{pmatrix} Z_{i_j} \\ (\alpha_{i_j} - a_{min}^{x,y}) y + V_{i_j} \end{pmatrix}^T \frac{G(a_{min}^{x,y})}{2} \begin{pmatrix} Z_{i_j} \\ (\alpha_{i_j} - a_{min}^{x,y}) y + V_{i_j} \end{pmatrix} + \theta_{i_j}^* \right]}_{=:\tau((\theta_{i_1}^*, Z_{i_1}, \alpha_{i_1}, V_{i_1}), \ldots, (\theta_{i_{K_n}}^*, Z_{i_{K_n}}, \alpha_{i_{K_n}}, V_{i_{K_n}}); \boldsymbol{\omega})} + R_n^* \nu_n^3
\end{aligned} \tag{C.21}$$

By our assumptions there exists a constant $D_5 < \infty$ such that $R_n^* < D_5$ for all possible K_n, all possible sets $\{(\theta_{i_j}, Z_{i_j}, \alpha_{i_j}, V_{i_j})\} \subset C(x, a_{min}^{x,y} y; \nu_n)$, all a and all $(x, y) \in \mathcal{D}$ with $(x, y) \notin \mathcal{W}(\nu_n^*)$.

The result in (C.21) shows that $\widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n)$ is essentially determined by minimizing $\tau(\cdot)$ over all possible $\boldsymbol{\omega}$ with $\sum_{j=1}^{K_n} \omega_i Z_{i_j} = 0$ and $\sum_{j=1}^{K_n} \omega_j V_{i_j} = 0$, independent of the corresponding value of $\sum_{j=1}^{K_n} \omega_j \alpha_{i_j} = a$ (even cases with $a \notin [a_{min}^{x,y} - d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}, a_{min}^{x,y} + d_1 n^{-\frac{\kappa}{2}} (\log n)^{\frac{\kappa}{2}}]$ need not be excluded since due to (C.9) they cannot constitute an optimal solution with probability tending to 1). Recall that $\theta_{i_j}^* := 1 - \theta_{i_j}$, and let

$$\begin{aligned} T_{K_n} &\left((\theta_{i_1}^*, Z_{i_1}, \alpha_{i_1}, V_{i_1}), \ldots, (\theta_{i_{K_n}}^*, Z_{i_{K_n}}, \alpha_{i_{K_n}}, V_{i_{K_n}})\right) \\ &= \min \Big\{ \tau((\theta_{i_1}^*, Z_{i_1}, \alpha_{i_1}, V_{i_1}), \ldots, (\theta_{i_{K_n}}^*, Z_{i_{K_n}}, \alpha_{i_{K_n}}, V_{i_{K_n}}); \boldsymbol{\omega}) \mid \\ &\qquad \boldsymbol{i}_{K_n}^T \boldsymbol{\omega} = 1, \ \sum_{j=1}^{K_n} \omega_j Z_{i_j} = \sum_{j=1}^{K_n} \omega_j V_{i_j} = 0 \Big\} \end{aligned} \tag{C.22}$$

When combining these arguments with (C.14) and (C.20) one can conclude that there are constants $0 < D_6, D_7 < \infty$ such that with probability at least $1 - D_6 n^{-2}$

$$\left| \widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(x, y) \left(1 + \frac{T_{K_n}\left((\theta_{i_1}^*, Z_{i_1}, \alpha_{i_1}, V_{i_1}), \ldots, (\theta_{i_{K_n}}^*, Z_{i_{K_n}}, \alpha_{i_{K_n}}, V_{i_{K_n}})\right)}{g_x(0, a_{min}^{x,y} y)} \right) \right| \leq D_7 \nu_n^3 \tag{C.23}$$

Here, D_6 and D_7 can be chosen independent of $(x, y) \in \mathcal{D}$ with $(x, y) \notin \mathcal{W}(\nu_n^*)$. Since necessarily, $\tau((\theta_{i_1} *, Z_{i_1}, \alpha_{i_1}, V_{i_1}), \ldots, (\theta_{i_{K_n}}^*, Z_{i_{K_n}}, \alpha_{i_{K_n}}, V_{i_{K_n}}); \boldsymbol{\omega}) \leq D_8 \nu_n^2$, (C.23) immediately implies that for some constant $D_8 < \infty$ and all $\beta > 0$

$$E\left(\left| \widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n) - \theta(x, y) \right|^{\beta} \right) \leq D_8 \max\{ n^{-\frac{2\beta}{p+q+1}} (\log n)^{\frac{2\beta}{p+q+1}}, n^{-2} \} \ \forall \ (x, y) \in \mathcal{D} \backslash \mathcal{W}(\nu_n^*). \tag{C.24}$$

More precise results are to be obtained from the distribution of T_{K_n}. When translating the results of Kneip et al. (2008, (2015)) into the alternative $(\theta, \zeta, \alpha, v)$-coordinate system it turns out that the asymptotic behavior the VRS-DEA estimator $\widehat{\theta}(x, a_{min}^{x,y} y \mid \mathcal{X}_n)$ of $\theta(x, a_{min}^{x,y} y)$ is determined by a similar random variable $T_{K_n}^{DEA}\left((\theta_{i_1}^*, Z_{i_1}, \alpha_{i_1}, V_{i_1}), \ldots, (\theta_{i_{K_n}}^*, Z_{i_{K_n}}, \alpha_{i_{K_n}}, V_{i_{K_n}})\right)$ defined by minimizing $\tau((\theta_{i_1}^*, Z_{i_1}, \alpha_{i_1}, V_{i_1}), \ldots, (\theta_{i_{K_n}}^*, Z_{i_{K_n}}, \alpha_{i_{K_n}}, V_{i_{K_n}}); \boldsymbol{\omega})$ with respect to all weight sequences with $\boldsymbol{i}_{K_n}^T \boldsymbol{\omega} = 1$, $\sum_{j=1}^{K_n} \omega_j Z_{i_j} = \sum_{j=1}^{K_n} \omega_j V_{i_j} = 0$, **and** $\sum_{j=1}^{K_n} \omega_j \alpha_{i_j} = a_{min}^{x,y}$. Therefore, the only difference between T_{K_n} and $T_{K_n}^{DEA}$ consists in the fact that (C.22) does not incorporate the additional constraint $\sum_{j=1}^{K_n} \omega_j \alpha_{i_j} = a_{min}^{x,y}$. But all arguments developed for analyzing $T_{K_n}^{DEA}$ readily generalize to T_{K_n}.

Obviously, the observations $(\theta_{i_j}^*, \zeta_{i_j}, \alpha_{i_j}, v_{i_j})$ are independent. The conditional distribution of $(\theta_{i_j}^*, \zeta_{i_j}, \alpha_{i_j}, v_{i_j})$ given $(X_{i_j}, Y_{i_j}) \in \mathcal{X}_n(x, a_{min}^{x,y} y; \nu_n)$ converges to a uniform distribution. Also note that for all (x, y) in the interior of $\mathcal{D}$ we necessarily have $(x, y) \notin \mathcal{W}(\nu_n^*)$ for all sufficiently large n. For deriving the asymptotic distribution of T_{K_n} we rely on the construction presented in Kneip et al. (2008). Let $(\widetilde{\theta}_1,\ \widetilde{\zeta}_1,\ \widetilde{\alpha}_1,\ \widetilde{v}_1),\ \ldots,\ (\widetilde{\theta}_k,\ \widetilde{\zeta}_k,\ \widetilde{\alpha}_k,\ \widetilde{v}_k)$ denote iid random variables uniformly distributed on $[0, 1] \times [-1, 1]^{p-1} \times [a_{min}^{x,y} - 1, a_{min}^{x,y} + 1] \times [-1, 1]^{q-1}$, and set $\widetilde{Z}_i = \sum_j \widetilde{\zeta}_{ij} z_x^{(j)}$, $\widetilde{V}_i = \sum_r \widetilde{v}_{ir} v_y^{(r)}$, $i = 1, \ldots, k$. Then for any integer k and $\gamma > 0$ define the following event $\mathcal{U}[\gamma, k]$: there exists a weight vector $\boldsymbol{\omega} \in \mathbb{R}_+^k$ with $\boldsymbol{i}_k^T \boldsymbol{\omega} = 1$ and $\sum_{j=1}^k \omega_j \widetilde{Z}_j = \sum_{j=1}^k \omega_j \widetilde{V}_j = 0$ such that

$$\frac{\tau((\widetilde{\theta}_1,\ \widetilde{Z}_1,\ \widetilde{\alpha}_1,\ \widetilde{V}_1),\ \ldots,\ (\widetilde{\theta}_k,\ \widetilde{Z}_k,\ \widetilde{\alpha}_k,\ \widetilde{V}_k); \boldsymbol{\omega})}{g_x(0, a_{min}^{x,y} y)} \leq \gamma. \tag{C.25}$$

Applying the same type of arguments as those used in the proof of Theorem 2 of Kneip et al. (2008) it can then be derived that for any $\gamma > 0$

$$\begin{aligned} &\lim_{n\to\infty} \Pr\left(n^\kappa \Big(\frac{\widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(x, y)}{\theta_{\mathrm{C}}(x, y)} \Big) \leq \gamma \right) \\ &= \lim_{n\to\infty} \Pr\left(n^\kappa \frac{T_{K_n}((\theta_{i_1}^*, Z_{i_1}, \alpha_{i_1}, V_{i_1}), \ldots, (\theta_{i_{K_n}}^*, Z_{i_{K_n}}, \alpha_{i_{K_n}}, V_{i_{K_n}}))}{g_x(0, a_{min}^{x,y} y)} \leq \gamma \right) = F_{x,y}(\gamma) \end{aligned} \tag{C.26}$$

where $F_{x,y}$ is a continuous distribution function with $F_{x,y}(0) = 0$ and

$$F_{x,y}(\gamma) = \lim_{k\to\infty} \Pr\left(\mathcal{U}\left[\gamma \frac{\widetilde{f}_{x,y}(1, 0, a_{min}^{x,y}, 0)^{\frac{2}{p+q+1}}}{k^{\frac{2}{p+q+1}}}, k \right] \right) \tag{C.27}$$

This proves (3.12). Analysis of expectations now relies on the techniques developed in Kneip et al. (2015).

Let $\widetilde{\nu}_n := \left(\frac{n}{\widetilde{f}_{x,y}(1,0,a_{min}^{x,y},0)} \right)^{\frac{1}{p+q+1}}$, $\widetilde{Z}_j^{(n)} = \mathbf{Z}_x \widetilde{\zeta}_j^{(n)}$, $\widetilde{V}_j^{(n)} = \mathbf{V}_y \widetilde{v}_j^{(n)}$ and let $(\widetilde{\theta}_j^{(n)}, \widetilde{\zeta}_j^{(n)}, \widetilde{\alpha}_j^{(n)}, \widetilde{v}_j^{(n)})$, $j = 1, \ldots, n$, denote iid random variables with a uniform distribution on $[0, \widetilde{\nu}_n^2] \times [-\widetilde{\nu}_n, \widetilde{\nu}_n]^{p-1} \times [a_{min}^{x,y} - \widetilde{\nu}_n, a_{min}^{x,y} + \widetilde{\nu}_n] \times [-\widetilde{\nu}_n, \widetilde{\nu}_n]^{p-1}$. Similar to T_{K_n} one can then define the r.v. $T_n\left((\widetilde{\theta}_1^{(n)}, \widetilde{Z}_1^{(n)}, \widetilde{\alpha}_1^{(n)}, \widetilde{V}_1^{(n)}),\ \ldots,\ (\widetilde{\theta}_n^{(n)}, \widetilde{Z}_n^{(n)}, \widetilde{\alpha}_n^{(n)}, \widetilde{V}_n^{(n)}) \right)$ by minimizing (C.22) with respect to the set of observations $\{(\widetilde{\theta}_j^{(n)}, \widetilde{\zeta}_j^{(n)}, \widetilde{\alpha}_j^{(n)}, \widetilde{v}_j^{(n)})\}$ instead of $\{(\theta_{i_j}^*, Z_{i_j}, \alpha_{i_j}, V_{i_j})\}$.

In a straightforward generalization of the arguments leading to relations (A.13)–(A.18) in the proof of Theorem 3.1 of Kneip et al. (2015) it can then be shown that the asymptotic distributions of $n^{\kappa}T_{K_n}\left((\theta^*_{i_1}, Z_{i_1}, \alpha_{i_1}, V_{i_1}), \ldots, (\theta^*_{i_{K_n}}, Z_{i_{K_n}}, \alpha_{i_{K_n}}, V_{i_{K_n}})\right)$ and of $T_n\left((\widetilde{\theta}_1^{(n)}, \widetilde{Z}_1^{(n)}, \widetilde{\alpha}_1^{(n)}, \widetilde{V}_1^{(n)}),\ \ldots,\ (\widetilde{\theta}_n^{(n)}, \widetilde{Z}_n^{(n)}, \widetilde{\alpha}_n^{(n)}, \widetilde{V}_n^{(n)})\right)$ coincide, and that all moments of $T_n\left((\widetilde{\theta}_1^{(n)}, \widetilde{Z}_1^{(n)}, \widetilde{\alpha}_1^{(n)}, \widetilde{V}_1^{(n)}),\ \ldots,\ (\widetilde{\theta}_n^{(n)}, \widetilde{Z}_n^{(n)}, \widetilde{\alpha}_n^{(n)}, \widetilde{V}_n^{(n)})\right)$ converge rapidly to finite, fixed values as $n \to \infty$. Additionally using (C.23), we obtain the following generalization of relations (A.16)–(A.18) in the proof of Theorem 3.1 of Kneip et al. (2015):

$$\left|E\left(\widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(x, y)\right) - \theta_{\mathrm{C}}(x, y) n^{-\frac{2}{p+q+1}} \frac{\widetilde{C}_{g''_x, \widetilde{f}_{x,y}(1,0,a^{x,y}_{min},0)}}{g_x(0, a^{x,y}_{min} y)}\right| \leq D_9 n^{-\frac{3}{p+q+1}} (\log n)^{\frac{3}{p+q+1}} \tag{C.28}$$

for all $(x, y) \in \mathcal{D}$ with $(x, y) \notin \mathcal{W}(\nu^*_n)$. and some $D_9 \in (0, \infty)$, where

$$\widetilde{C}_{g''_x, \widetilde{f}_{x,y}(1,0,a^{x,y}_{min},0)} := \lim_{n\to\infty} E\left[T_n\left((\widetilde{\theta}_1, \widetilde{Z}_1, \widetilde{\alpha}_1^{(n)}, \widetilde{V}_1^{(n)}),\ \ldots,\ (\widetilde{\theta}_n, \widetilde{Z}_n, \widetilde{\alpha}_n^{(n)}, \widetilde{V}_n^{(n)})\right)\right] \tag{C.29}$$

only depends upon g''_x and $\widetilde{f}_{x,y}(1, 0, a^{x,y}_{min}, 0)$ and changes continuously in $(x, y) \in \mathcal{D}$. Furthermore, there exists some $D_{10} \in (0, \infty)$ such that

$$E\left(\left|\widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(x, y)\right|^2\right) \leq D_{10} n^{-\frac{4}{p+q+1}} \tag{C.30}$$

for all $(x, y) \in \mathcal{D}$ with $(x, y) \notin \mathcal{W}(\nu^*_n)$.

Case (ii): For a further analysis of expectations we additionally have to consider the alternative case where $(x, y) \in \mathcal{W}(\nu^*_n)$. We again rely on arguments similar to those used in the proof of Theorem 3.1 of Kneip et al. (2015).

In this case, the problem arises that some of the sets B_j used in the above construction surpass the boundary and are no longer in $\mathcal{D}$. As a consequence, one cannot exclude that $\widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n)$ is influenced by an observation with $\theta_i \leq 1 - \nu_n^2$. But let

$$\mathcal{H}_n(x, y; \nu_n) := \{(X_i, Y_i) \in \mathcal{X}_n \mid (1, Z_i, a^{x,y}_{min}, V_i) \in C(x, a^{x,y}_{min} y; , \nu_n)\} \tag{C.31}$$

By a straightforward generalization of the arguments in the proof of Theorem 3.1 of Kneip et al. (2015) it follows that

$$\left|1 - \Pr\left(\widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n) = \widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{H}_n(x, y; \nu_n))\right)\right| \leq D_{11} n^{-2} \tag{C.32}$$

for all $(x, y) \in \mathcal{D}$, some $D_{11} \in (0, \infty)$, and all sufficiently large n.

Recall that boundary problems arise only if $p+q>2$. In such cases, for $r=1,\ \ldots,\ p-1$, define

$$v_{r;x,y} := \min_{\substack{(\zeta,v)\in\mathbb{R}^{p-1}\times\mathbb{R}^{q-1},\\ (\sum_{j=1}^{p-1}\zeta_j z_x^{(j)},v)\in\Psi^{*\partial}(x,y)}} \{\nu_n,\ |\zeta_r|\}. \tag{C.33}$$

Similarly, for $r=1,\ \ldots,\ q-1$, define $v_{p-1+r;x,y}$ by replacing $|\zeta_r|$ with $|v_r|$ in (C.33). These $v_{r;x,y}$ can be viewed as measuring a "distance" from (x,y) to the boundary, with $v_{r;x,y}\le\nu_n$.

If $\prod_{r=1}^{p+q-2} v_{r;x,y} \ge \nu_n^{p+q+1}$, i.e. (x,y) is not too near the boundary, an upper bound for $\widehat{\theta}_{\mathrm{C}}(x,y\mid\mathcal{X}_n)$ can then be obtained by relying on the observations with $1-\theta_i\le\left(\frac{\nu_n^{p+q+1}}{\prod_{r=1}^{p+q-2}v_{r;x,y}}\right)^{2/3}$ and $|\alpha_i-\alpha_{min}^{x,y}|\le\left(\frac{\nu_n^{p+q+1}}{\prod_{r=1}^{p+q-2}v_{r;x,y}}\right)^{1/3}$. Arguments similar to those used above then show that for all $(x,y)\in\mathcal{W}(\nu_n^*)$ with $\prod_{r=1}^{p+q-2} v_{r;x,y}\ge\nu_n^{p+q+1}$, we have for $\alpha\in\{1,2\}$

$$E\left(\left|\widehat{\theta}_{\mathrm{C}}(x,y\mid\mathcal{X}_n)-\theta_{\mathrm{C}}(x,y)\right|^{\alpha}\right)\le D_{12}^{\alpha}\left(\frac{\nu_n^{p+q+1}}{\prod_{r=1}^{p+q-2}v_{r;x,y}}\right)^{2\alpha/3}, \tag{C.34}$$

for some constant $D_{12}\in(0,\infty)$, and for all sufficiently large n.

Now the moments of $\widehat{\theta}_{\mathrm{C}}(X_i,Y_i\mid\mathcal{X}_n)$ can be analyzed in a way similar to Kneip et al. (2015). Let $\mathcal{X}_{n,-i}$ denote the sample of size $n-1$ obtained by eliminating the i-th observation (X_i,Y_i). When relying on $\mathcal{X}_{n,-i}$, it is clear that all constants in the above inequalities can be chosen independently of (x,y) and thus also apply for the (random) coordinate system induced by the specific choice $(x,y)=(X_i,Y_i)$. Obviously,

$$\widehat{\theta}_{\mathrm{C}}(X_i,Y_i\mid\mathcal{X}_n)=\min\left\{\widehat{\theta}_{\mathrm{C}}(X_i,Y_i\mid\mathcal{X}_{n,-i}),\ 1\right\}. \tag{C.35}$$

Since (X_i,Y_i) is independent of $\mathcal{X}_{n,-i}$, (C.24) and (C.35) imply that

$$E\left(\widehat{\theta}_{\mathrm{C}}(X_i,Y_i\mid\mathcal{X}_n)-\theta_{\mathrm{C}}(X_i,Y_i)\mid(X_i,Y_i)\notin\mathcal{W}(\nu_n^*)\right)=C_0n^{-\frac{2}{p+q+1}}+O\left(n^{-\frac{3}{p+q+1}}(\log n)^{\frac{3}{p+q+1}}\right) \tag{C.36}$$

for some $C_0\in(0,\infty)$. If $p=1$ and $q\le1$, then assertion (3.13) follows directly from (C.36), since in this case there is no boundary problem due to $\mathcal{W}(\nu_n^*)=\emptyset$.

In order to quantify the influence of boundary effects for $p+q>2$, let $\mathcal{W}_{n,1}:=\left\{(x,y)\in\mathcal{D}\mid\nu_n^{p+q-2}>\prod_{r=1}^{p+q-1}v_{r;x,y}\ge\nu_n^{p+q+1}\right\}$ contain points in $\mathcal{W}(\nu_n^*)$ but not too near the boundary, and let $\mathcal{W}_{n,2}:=\left\{(x,y)\in\mathcal{D}\mid\prod_{r=1}^{p+q-2}v_{r;x,y}<\nu_n^{p+q+1}\right\}$ contain the other points of $\mathcal{W}(\nu_n^*)$ very near the boundary where only the trivial upper bound $|\widehat{\theta}_{\mathrm{C}}(X_i,Y_i\mid\mathcal{X}_n)-\theta(X_i,Y_i)|\le1$ can be used. For points in $\mathcal{W}_{n,1}$, note that for all $r=1,\ldots,p+q-2$,

$\nu_n^4 \leq \upsilon_{r;x,y} \leq \nu_n$. Fortunately, the boundary is "smaller" than in the DEA-case, and its influence is less pronounced. Note that

$$\begin{aligned} E\left(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta(X_i, Y_i)\right) =& E\left(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(X_i, Y_i) \mid (X_i, Y_i) \notin \mathcal{W}(\nu_n^*)\right) \\ & \times \Pr((X_i, Y_i)) \notin \mathcal{W}(\nu_n^*)) \\ & + \sum_{s=1}^{2} E\left(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta(X_i, Y_i) \mid (X_i, Y_i) \in \mathcal{W}_{n,s}\right) \\ & \times \Pr((X_i, Y_i) \in \mathcal{W}_{n,s}). \end{aligned} \tag{C.37}$$

When relying on (C.34), straightforward calculations similar to those in Kneip et al. (2015) yield that with for some constants $D_{13}, D_{14} < \infty$,

$$\begin{aligned} E\left(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta(X_i, Y_i) \mid (X_i, Y_i) \in \mathcal{W}_{n,1}\right) \cdot \Pr((X_i, Y_i) \in \mathcal{W}_{n,1}) \\ \leq D_{13} \int_{\mathcal{W}_{n,1}} \left(\frac{\nu_n^{p+q+1}}{\prod_{r=1}^{p+q-2} \upsilon_{r;x,y}}\right)^{2/3} f(x,y) dx dy \\ \leq D_{14} \sum_{r=1}^{p+q-2} \int_{\mathcal{B}} \frac{\nu_n^{8/3}}{\upsilon_{r;x,y}^{2/3}} \, dx\, dy + O_p\left(n^{-\frac{4}{p+q+1}} (\log n)^{\frac{4}{p+q+1}}\right) \\ = O_p\left(n^{-\frac{4}{p+q+1}} (\log n)^{\frac{4}{p+q+1}}\right), \end{aligned} \tag{C.38}$$

where $\mathcal{B} := \{(x,y) \in \mathcal{D} \mid$
$\nu_n > \upsilon_{r,x,y} \geq \nu_n^4\}$ In addition, $\Pr((X_i, Y_i) \in \mathcal{W}_{n,2}) = O\left(n^{-\frac{4}{p+q+1}} (\log n)^{\frac{4}{p+q+1}}\right)$. Together with (C.36), this leads to (3.13).[2]

Recall (C.28) and (C.30). Assertion (3.14) follows from the fact that (C.34) implies the existence of constants $D_{15}, D_{16} < \infty$ such that

$$\begin{aligned} \mathrm{VAR}\big(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(X_i, Y_i)\big) \leq D_{15} n^{-\frac{4}{p+q+1}} \times \Pr\left((X_i, Y_i) \notin \mathcal{W}(\nu_n^*)\right) \\ + D_{16}^2 \int_{\mathcal{W}_{n,1}} \left(\left(\frac{\nu_n^{p+q+1}}{\prod_{r=1}^{p+q-2} \upsilon_{r;x,y}}\right)^{4/3}\right) f(x,y)\, dx\, dy + \Pr\left((X_i, Y_i) \in \mathcal{W}_{n,2}\right) \\ = O\left(n^{-\frac{4}{p+q+1}} + n^{-\frac{4}{p+q+1}} (\log n)^{\frac{4}{p+q+1}}\right). \end{aligned} \tag{C.39}$$

It remains to prove (3.15). Estimators $\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n)$ exhibit stronger correlations than the original VRS-DEA estimators $\widehat{\theta}_{\mathrm{VRS}}(X_i, Y_i \mid \mathcal{X}_n)$. The reason is that by (C.20), for any

[2] Note that a typographical error appears in Appendix A of Kneip et al. (2015). The quantity $E\left(\widehat{\theta}_{\mathrm{VRS}}(X_i, Y_i \mid \mathcal{X}_n) - \theta(X_i, Y_i) \mid (X_i, Y_i) \in \mathcal{W}_{n,1}\right)$ in formula (A.24) should be replaced by $E\left(\widehat{\theta}_{\mathrm{VRS}}(X_i, Y_i \mid \mathcal{X}_n) - \theta(X_i, Y_i) \mid (X_i, Y_i) \in \mathcal{W}_{n,1}\right) \cdot \Pr((X_i, Y_i) \in \mathcal{W}_{n,1})$.

$b > 0$ the estimators $\widehat{\theta}_{\mathrm{C}}(x, y \mid \mathcal{X}_n)$ and $\widehat{\theta}_{\mathrm{C}}(x, by \mid \mathcal{X}_n)$ depend on the same local observations in $\mathcal{X}_n(x, a_{min}^{x,y} y; \nu_n)$, while for sufficiently large b, DEA estimators $\widehat{\theta}_{\mathrm{DEA}}(x, y \mid \mathcal{X}_n)$ and $\widehat{\theta}_{\mathrm{DEA}}(x, by \mid \mathcal{X}_n)$ will be asymptotically uncorrelated.

However, for all $i, j \in 1, \ldots, n$, $i \neq j$, it follows from (C.32) that $\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta(X_i, Y_i)$ and $\widehat{\theta}_{\mathrm{C}}(X_j, Y_j \mid \mathcal{X}_n) - \theta(X_j, Y_j)$ are asymptotically uncorrelated if $\mathcal{H}_n(X_i, Y_i; \nu_n) \cap \mathcal{H}_n(X_j, Y_j; \nu_n) = \emptyset$. Since all observations are iid, the Cauchy-Schwarz inequality yields

$$\begin{aligned}
&\left|\mathrm{COV}\left(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(X_i, Y_i),\ \widehat{\theta}_{\mathrm{C}}(X_j, Y_j \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(X_j, Y_j)\right)\right| \\
&\quad \leq \Pr\left(\mathcal{H}_n(X_i, Y_i; \nu_n) \cap \mathcal{H}_n(X_j, Y_j; \nu_n) \neq \emptyset\right) \\
&\qquad \times \mathrm{VAR}\left(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(X_i, Y_i)\right) + O\left(n^{-2}\right). \qquad \text{(C.40)}
\end{aligned}$$

Relation (3.14) as well as

$$\Pr\left(\mathcal{H}_n(X_i, Y_i; \nu_n) \cap \mathcal{H}_n(X_j, Y_j; \nu_n) \neq \emptyset\right) = O\left(n^{-\frac{p+q-2}{p+q+1}} (\log n)^{\frac{p+q-2}{p+q+1}}\right) \qquad \text{(C.41)}$$

now lead to assertion (3.15), completing the proof of the theorem. ■

C.6 Proof of Theorem 3.2

The transformation defined by the respective function Γ is monotonic and differentiable with nonzero derivatives on $\mathbb{R}_+$. Therefore, (3.16) follows via the delta method.

By Assumption 3.1 (iii) $\Gamma(\theta_{\mathrm{C}}(X_i, Y_i))$ as well as its derivatives $\Gamma'(\theta_{\mathrm{C}}(X_i, Y_i))$ and $\Gamma''(\theta_{\mathrm{C}}(X_i, Y_i))$ are uniformly bounded for all $(X_i, Y_i) \in \mathcal{D}$. It thus follows from a Taylor expansion and (3.14) that

$$\begin{aligned}
E\left(\Gamma(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n)) - \Gamma(\theta_{\mathrm{C}}(X_i, Y_i))\right) &= E\left(\Gamma'(\theta_{\mathrm{C}}(X_i, Y_i))[\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(X_i, Y_i)]\right) \\
&\quad + O\left(n^{-\frac{4}{p+q+1}} (\log n)^{\frac{4}{p+q+1}}\right). \qquad \text{(C.42)}
\end{aligned}$$

Recall that (C.35) states that $\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) = \min\left\{\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_{n,-i}), 1\right\}$. Moreover, the arguments developed in the proof of Theorem 3.1 imply that

$$\Pr\left(\{\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) = 1\} \cap \{(X_i, Y_i) \notin \mathcal{W}(\nu_n^*)\}\right) = O\left(n^{-\frac{3}{p+q+1}} (\log n)^{\frac{3}{p+q+1}}\right), \qquad \text{(C.43)}$$

where the boundary $\mathcal{W}(\nu_n^*)$ is defined as in the proof of Theorem 3.1. Since $\Gamma'(\theta_{\mathrm{C}}(X_i, Y_i) > 0$, it follows from (C.42), (C.28), and (C.29) that similar to (C.36) we have

$$E\left(\Gamma(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n)) - \Gamma(\theta_{\mathrm{C}}(X_i, Y_i)) \mid (X_i, Y_i) \notin \mathcal{W}(\nu_n^*)\right) = \\ C_0^{\Gamma} n^{-\frac{2}{p+q+1}} + O\left(n^{-\frac{3}{p+q+1}} (\log n)^{\frac{3}{p+q+1}}\right) \tag{C.44}$$

for some $0 < C_0^{\Gamma} < \infty$. An immediate generalization of (C.37) yields

$$\begin{aligned} E&\left(\Gamma(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n)) - \Gamma(\theta_{\mathrm{C}}(X_i, Y_i))\right) = \\ &E\left(\Gamma(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n)) - \Gamma(\theta_{\mathrm{C}}(X_i, Y_i)) \mid (X_i, Y_i) \notin \mathcal{W}(\nu_n^*)\right) \cdot \Pr((X_i, Y_i) \notin \mathcal{W}(\nu_n^*)) \\ &+ \sum_{s=1}^{2} E\left(\Gamma(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n)) - \Gamma(\theta_{\mathrm{C}}(X_i, Y_i)) \mid (X_i, Y_i) \in \mathcal{W}_{n,s}\right) \cdot \Pr((X_i, Y_i) \in \mathcal{W}_{n,s}). \end{aligned} \tag{C.45}$$

With $0 < M_1 := \sup_{(x,y)\in\mathcal{D}} \Gamma'(\theta_{\mathrm{C}}(x, y)) < \infty$ a Taylor expansion leads to

$$\begin{aligned} E&\left(\Gamma(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n)) - \Gamma(\theta_{\mathrm{C}}(X_i, Y_i)) \mid (X_i, Y_i) \in \mathcal{W}_{n,1}\right) \cdot \Pr((X_i, Y_i) \in \mathcal{W}_{n,1}) \\ &\leq M_1 E\left(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(X_i, Y_i) \mid (X_i, Y_i) \in \mathcal{W}_{n,1}\right) \cdot \Pr((X_i, Y_i) \in \mathcal{W}_{n,1}), \end{aligned} \tag{C.46}$$

and Assertion (3.17) then is an immediate consequence of (C.44), (C.38), and $\Pr((X_i, Y_i) \in \mathcal{W}_{n,2}) = O\left(n^{-\frac{4}{p+q+1}} (\log n)^{\frac{4}{p+q+1}}\right)$. Similarly, (C.39) implies

$$\begin{aligned} E\left([\Gamma(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n)) - \Gamma(\theta_{\mathrm{C}}(X_i, Y_i))]^2\right) &\leq M_1^2 E\left([\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n) - \theta_{\mathrm{C}}(X_i, Y_i)]^2\right) \\ &= O\left(n^{-\frac{4}{p+q+1}} (\log n)^{\frac{4}{p+q+1}}\right), \end{aligned} \tag{C.47}$$

which proves Assertion (3.18). Analogous to (C.40) and (C.41) Assertion (3.19) finally follows from the fact that $\Gamma(\widehat{\theta}_{\mathrm{C}}(X_i, Y_i \mid \mathcal{X}_n)) - \Gamma(\theta_{\mathrm{C}}(X_i, Y_i))$ and $\Gamma(\widehat{\theta}_{\mathrm{C}}(X_j, Y_j \mid \mathcal{X}_n)) - \Gamma(\theta_{\mathrm{C}}(X_j, Y_j))$ are asymptotically uncorrelated if $\mathcal{H}_n(X_i, Y_i; \nu_n) \cap \mathcal{H}_n(X_j, Y_j; \nu_n) = \emptyset$. ■

C.7 Proof of Theorem 3.3

Note that Theorem 3.2 holds for both (x^1, y^1) and (x^2, y^2) due to Assumption 3.2. The log transformation in Theorem 3.2 is monotonic, differentiable, and invertible. Hence the result follows via the delta method. ■

C.8 Proof of Theorem 3.4

For $t = s$ Assertion (3.27) follows from (3.17). Now consider the case $t \neq s$. Following the notation introduced in (3.4) let

$$(\breve{X}_i^t, \breve{Y}_i^t) := (\tilde{g}_x(\alpha_{min}^{X_i^t,Y_i^t} \frac{Y_i^t}{\|Y_i^t\|}) \frac{X_i^t}{\|X_i^t\|}, \alpha_{min}^{X_i^t,Y_i^t} \frac{Y_i^t}{\|Y_i^t\|})$$

Since $\mathcal{D}^1_{norm} = \mathcal{D}^2_{norm}$ we have $(\breve{X}_i^t, \breve{Y}_i^t) \in \mathcal{D}^s$. Then (3.10) implies that

$$\begin{aligned} \log \widehat{\gamma}_C^s(X_i^t, Y_i^t \mid \mathcal{X}_{n_s}^s) - \log \gamma_C^s(X_i^t, Y_i^t) &= \log \widehat{\gamma}_C^s(\breve{X}_i^t, \breve{Y}_i^t \mid \mathcal{X}_{n_s}^s) - \log \gamma_C^s(\breve{X}_i^t, \breve{Y}_i^t) \\ &= \Gamma(\widehat{\theta}_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t \mid \mathcal{X}_{n_s}^s)) - \Gamma(\theta_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t)), \qquad \text{(C.48)} \end{aligned}$$

where $\Gamma(\theta) = \log \theta^{1/2}$ for all $\theta > 0$. Recall the arguments developed in the proofs of Theorems 3.1 and 3.2 and the definitions of the boundaries $\mathcal{W}(\nu_{n_s}^*) \equiv \mathcal{W}^s(\nu_{n_s}^*)$, $\mathcal{W}_{n_s,1} \equiv \mathcal{W}_{n_s,1}^s$ as well as $\mathcal{W}_{n_2,2} \equiv \mathcal{W}_{n_s,2}^s$. If $(\widehat{\theta}_{\mathrm{C}}^s(\breve{X}_i^s, \breve{Y}_i^s \mid \mathcal{X}_{n_s}^s) \neq 1$, then obviously $\widehat{\theta}_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t \mid \mathcal{X}_{n_s}^s) = \widehat{\theta}_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t \mid \mathcal{X}_{n_s,-i}^s)$, where again $\mathcal{X}_{n_s,-i}$ denote the sample of size $n-1$ obtained by eliminating the i-th observation (X_i, Y_i). Moreover, the arguments developed in the proof of Theorem 3.1 imply that $\Pr\left(\{\widehat{\theta}_{\mathrm{C}}^s(\breve{X}_i^s, \breve{Y}_i^s \mid \mathcal{X}_{n_s}^s) = 1\}\right) = O\left(n_s^{-\frac{3}{p+q+1}} (\log n_s)^{\frac{3}{p+q+1}}\right)$. Hence,

$$\begin{aligned} & E\left(\Gamma(\widehat{\theta}_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t \mid \mathcal{X}_{n_s}^s)) - \Gamma(\theta_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t))\right) \\ &= E\left(\Gamma(\widehat{\theta}_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t \mid \mathcal{X}_{n_s,-i}^s)) - \Gamma(\theta_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t)) \mid (\breve{X}_i^t, \breve{Y}_i^t) \notin \mathcal{W}^s(\nu_{n_s}^*)\right) \cdot \Pr((\breve{X}_i^t, \breve{Y}_i^t) \notin \mathcal{W}^s(\nu_{n_s}^*)) \\ &\quad + \sum_{l=1}^{2} E\left(\Gamma(\widehat{\theta}_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t \mid \mathcal{X}_{n_s,-i}^s)) - \Gamma(\theta_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t)) \mid (\breve{X}_i^t, \breve{Y}_i^t) \in \mathcal{W}_{n_s,l}^s\right) \cdot \Pr((\breve{X}_i^t, \breve{Y}_i^t) \in \mathcal{W}_{n_s,l}^s) \\ &\quad + O\left(n_s^{-\frac{3}{p+q+1}} (\log n_s)^{\frac{3}{p+q+1}}\right) \qquad \text{(C.49)} \end{aligned}$$

Note that $(\breve{X}_i^t, \breve{Y}_i^t)$ is independent of $\mathcal{X}_{n_s,-i}^s$, and that by definition of our coordinate system $(\breve{X}_i^t, \breve{Y}_i^t) \notin \mathcal{W}^s(\nu_{n_s}^*)$ if and only if $(X_i^t, Y_i^t) \notin \mathcal{W}^s(\nu_{n_s}^*)$, as well as $(\breve{X}_i^t, \breve{Y}_i^t) \in \mathcal{W}_{n_s,l}^s$ if and only if $(X_i^t, Y_i^t) \in \mathcal{W}_{n_s,l}^s$ for $l = 1, 2$. As $n_s \to \infty$, our assumptions on the densities f^1 and f^2 the probabilities of these events are of the same order of magnitude as those obtained when analyzing (X_i^s, Y_i^s). Therefore, (3.27) follows from $\Pr((\breve{X}_i^t, \breve{Y}_i^t) \in \mathcal{W}_{n_s,2}^s) = O\left(n_s^{-\frac{4}{p+q+1}} (\log n_s)^{\frac{4}{p+q+1}}\right)$ and arguments similar to (C.44) and (C.46).

In an analogous manner straightforward generalizations of the arguments in the proof of Theorem 3.1 lead to $E\left([\widehat{\theta}_{\mathrm{C}}^s(\breve{X}_i^t, \breve{Y}_i^t \mid \mathcal{X}_{n_s}^s) - \theta_{\mathrm{C}}^s(X_i^t, Y_i^t)]^2\right) = O\left(n_s^{-\frac{4}{p+q+1}} (\log n_s)^{\frac{4}{p+q+1}}\right)$, and

(3.28) is obtained by an argument similar to (C.47). Finally, (3.29) can be derived from straightforward generalizations of (C.40) and (C.41). ■

C.9 Proof of Theorem 3.6

Using (3.24), (3.26) and the definition of $\mathcal{R}_n$ in the first part of (3.33), the left-hand side of (3.32) can be written as

$$\sqrt{n}(\widehat{\mu}_{\mathcal{M},n} - \mu_{\mathcal{M}} - \mathcal{R}_n) = \frac{\sqrt{n}}{n}\sum_{i=1}^{n}(\log\widehat{\mathcal{M}}_i - \log\mathcal{M}_i - E(\log\widehat{\mathcal{M}}_i) + \mu_{\mathcal{M}}) + \frac{\sqrt{n}}{n}\sum_{i=1}^{n}(\log\mathcal{M}_i - \mu_{\mathcal{M}}). \quad (C.50)$$

Since (3.28) and (3.29) imply $\frac{\sqrt{n}}{n}\sum_{i=1}^{n}(\log\widehat{\mathcal{M}}_i - \log\mathcal{M}_i - E(\log\widehat{\mathcal{M}}_i) + \mu_{\mathcal{M}}) \xrightarrow{p} 0$, the assertion is now an immediate consequence of standard CLTs. ■

C.10 Proof of Lemma 3.3

The proof is straightforward:

$$\begin{aligned}\widehat{\sigma}^2_{\mathcal{M},n} &= n^{-1}\sum_{i=1}^{n}\left(\log\widehat{\mathcal{M}}_i - \widehat{\mu}_{\mathcal{M},n}\right)^2\\ &\xrightarrow{p} E\left[\left(\log\widehat{\mathcal{M}}_i\right)^2\right] - \mu^2_{\mathcal{M}}\\ &= \mathrm{VAR}\,(\log\mathcal{M}_i) + [E(\log\mathcal{M}_i)]^2 - \mu^2_{\mathcal{M}}\\ &= \sigma^2_{\mathcal{M}}\end{aligned}$$

since $[E(\log\mathcal{M}_i)]^2 - \mu^2_{\mathcal{M}} = 0$. ■

C.11 Proof of Theorem 3.7

The result follows from straightforward arguments based on the delta method: Indeed, a Taylor expansion yields

$$\sqrt{n}\,(\exp(\widehat{\mu}_{\mathcal{M},n}) - \exp(\mu_{\mathcal{M}} + \mathcal{R}_n)) = \exp(\mu_{\mathcal{M}} + \mathcal{R}_n)\cdot\sqrt{n}\,(\widehat{\mu}_{\mathcal{M},n} - \mu_{\mathcal{M}} - \mathcal{R}_n) + O_P(\frac{1}{\sqrt{n}}). \quad (C.51)$$

Since $R_n = O\left(n^{-\frac{2}{p+q+1}}\right)$, the desired result follows from a further Taylor expansion of $\exp(\mu_{\mathcal{M}} + \mathcal{R}_n)$ and Theorem 3.6. ■

C.12 Proof of Theorem B.1

The result follows directly from Theorem 3.6 after noting that the big-O remainder term in (3.33) is $o\left(n^{-\kappa}\right)$ and noting that $n^{\kappa} o\left(n^{-\kappa}\right) = o(1)$. Since $\widehat{\mu}_{\mathcal{M},n}$ in (3.32) has been replaced with $\widehat{\mu}_{\mathcal{M},n_\kappa}$ in (B.3), the scale factor needed to stabilize variance is n^{κ}. ■

C.13 Proof of Theorem B.2

The result follows after substituting $\widehat{B}_{n,\kappa}$ for the bias term in (3.33). For $(p+q) = 4$ we have $\kappa = 2/5$. The remainder term is $O\left(n^{-3\kappa/2}\right)$ ignoring the $(\log n)$ term which does not affect the rate. Then $\sqrt{n} O\left(n^{-3\kappa/2}\right) = O\left(n^{-1/10}\right)$. ■

C.14 Proof of Theorem C.1

We only have to show (C.1). First note that the additional assumptions (i)–(iii) imply $\mu_{\mathcal{M}} = 0$. Since for both samples $s = 1, 2$ the same algorithm is employed to determine $\log \widehat{\gamma}_C^s(x, y \mid \mathcal{X}_n^s)$, there exists a measurable function G such that $\log \widehat{\gamma}_C^s(x, y \mid \mathcal{X}_n^s) = G((x, y); (X_1^s, Y_1^s), \ldots, (X_n^s, Y_n^s))$. Since by (i) and (ii) the distributions in each period are identical, we necessarily have

$$\begin{aligned}
E\left(\log \widehat{\gamma}_C^1(X_i^1, Y_i^1 \mid \mathcal{X}_n^1)\right) &= E\left(G((X_i^1, Y_i^1); (X_1^1, Y_1^1), \ldots, (X_n^1, Y_n^1))\right) \\
&= E\left(G((X_i^2, Y_i^2); (X_1^2, Y_1^2), \ldots, (X_n^2, Y_n^2))\right) \\
&= E\left(\log \widehat{\gamma}_C^2(X_i^2, Y_i^2 \mid \mathcal{X}_n^2)\right).
\end{aligned}$$

for all $i = 1, \ldots, n$. When additionally using c) we furthermore obtain

$$\begin{aligned}
&E\left(\log \widehat{\gamma}_C^1(X_i^2, Y_i^2 \mid \mathcal{X}_n^1)\right) = E\left(G((X_i^2, Y_i^2); (X_1^1, Y_1^1), \ldots, (X_n^1, Y_n^1))\right) \\
&= \int E\left(G((x^2, y^2); (X_1^1, Y_1^1), \ldots, (x^1, y^1), \ldots, (X_n^1, Y_n^1))\right) f_{12}(x^1, y^1, x^2, y^2) dx^1 \ldots dy^2 \\
&= \int E\left(G((x^2, y^2); (X_1^1, Y_1^1), \ldots, (x^1, y^1), \ldots, (X_n^1, Y_n^1))\right) f_{12}(x^2, y^2, x^1, y^1) dx^1 \ldots dy^2 \\
&= \int E\left(G((x^1, y^1); (X_1^2, Y_1^2), \ldots, (x^2, y^2), \ldots, (X_n^2, Y_n^2))\right) f_{12}(x^1, y^1, x^2, y^2) dx^1 \ldots dy^2 \\
&= E\left(G((X_i^1, Y_i^1); (X_1^2, Y_1^2), \ldots, (X_n^2, Y_n^2))\right) = E\left(\log \widehat{\gamma}_C^2(X_i^1, Y_i^1 \mid \mathcal{X}_n^2)\right).
\end{aligned}$$

By definition of $\log \widehat{\mathcal{M}}_i$ this implies that $E\left(\widehat{\mu}_{\mathcal{M},n}\right) = E\left(\log \widehat{\mathcal{M}}_i\right) = 0$. ■

D Additional Simulation Results

The simulation results in Tables D.1–D.5 are obtained from the data-generating process described in Section 6.1, and have the same layout as Table 1 in the main paper.

Table D.1 reports rejection rates for (two-sided) tests of no change versus change in productivity using logs of estimated Malmquist indices. For $q = 1$ and $p \in \{1,\ 2,\ 3\}$, results are from tests based on Theorem B.2 and intervals computed using (B.10). Results for $p \in \{4,\ 5\}$ are based on Theorem B.3 and intervals computed from (B.11). See Section 6.2 in the main paper for discussion.

As discussed in Section 6.2 in the main paper, Tables D.2–D.3 are analogous to Tables 1 and D.1, and are identical for $q = 1$ and $p \in \{1,\ 2,\ 4, ; 5\}$. But for Tables D.2–D.3, the reported rejection rates for $p = 3$, $q = 1$ are obtained using subsamples of size n_κ based on Theorems 4.3 and B.3, respectively.

Also as discussed in Section 6.2 in the main paper, Tables D.4–D.5 report rejection rates using the re-centered interval in (4.13) when working with untransformed indices (for the results in Table D.4), and the re-centered interval in (B.12) when working with logged indices (for the results in Table D.5). Again, see Section 6.2 in the main paper for discussion.

Table D.1: Rejection Rates for Test for Productivity Change using Logs (Two-sided Test)

n	β	— $p=1,\ q=1$ —			— $p=2,\ q=1$ —			— $p=3,\ q=1$ —			— $p=4,\ q=1$ —			— $p=5,\ q=1$ —		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
25	0.000	0.116	0.058	0.012	0.124	0.063	0.014	0.130	0.068	0.016	0.123	0.064	0.014	0.126	0.069	0.016
	0.005	0.183	0.108	0.030	0.194	0.115	0.034	0.204	0.125	0.039	0.148	0.084	0.021	0.143	0.082	0.021
	0.010	0.354	0.245	0.099	0.373	0.264	0.111	0.389	0.278	0.122	0.214	0.137	0.045	0.192	0.123	0.040
	0.015	0.566	0.445	0.233	0.594	0.473	0.257	0.609	0.490	0.274	0.313	0.219	0.090	0.268	0.185	0.074
	0.020	0.756	0.650	0.419	0.782	0.681	0.453	0.795	0.698	0.474	0.429	0.325	0.158	0.362	0.266	0.124
	0.030	0.954	0.914	0.774	0.965	0.931	0.806	0.970	0.939	0.821	0.658	0.554	0.350	0.565	0.458	0.271
	0.040	0.996	0.989	0.946	0.997	0.992	0.959	0.998	0.994	0.963	0.822	0.743	0.561	0.736	0.641	0.448
	0.050	1.000	0.999	0.990	1.000	1.000	0.993	1.000	1.000	0.994	0.915	0.864	0.729	0.853	0.781	0.614
50	0.000	0.109	0.053	0.010	0.114	0.057	0.011	0.117	0.059	0.012	0.112	0.059	0.013	0.112	0.062	0.015
	0.005	0.231	0.143	0.043	0.241	0.151	0.048	0.250	0.158	0.051	0.147	0.085	0.023	0.135	0.079	0.023
	0.010	0.513	0.387	0.182	0.539	0.413	0.200	0.555	0.429	0.213	0.244	0.159	0.056	0.200	0.129	0.046
	0.015	0.789	0.684	0.442	0.814	0.717	0.480	0.827	0.734	0.502	0.379	0.274	0.120	0.298	0.208	0.088
	0.020	0.939	0.890	0.722	0.952	0.911	0.760	0.958	0.920	0.780	0.531	0.415	0.219	0.417	0.310	0.151
	0.030	0.999	0.996	0.976	0.999	0.998	0.983	1.000	0.998	0.986	0.791	0.694	0.483	0.660	0.547	0.338
	0.040	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	0.929	0.876	0.726	0.840	0.753	0.554
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.979	0.956	0.874	0.935	0.883	0.737
100	0.000	0.104	0.051	0.010	0.109	0.055	0.010	0.113	0.058	0.011	0.105	0.055	0.012	0.106	0.058	0.015
	0.005	0.331	0.222	0.080	0.351	0.239	0.090	0.362	0.251	0.095	0.159	0.094	0.028	0.138	0.082	0.025
	0.010	0.751	0.638	0.389	0.776	0.672	0.424	0.789	0.688	0.445	0.299	0.202	0.079	0.224	0.147	0.056
	0.015	0.961	0.925	0.788	0.970	0.942	0.823	0.974	0.948	0.839	0.488	0.369	0.179	0.352	0.249	0.111
	0.020	0.998	0.994	0.968	0.999	0.996	0.978	0.999	0.997	0.982	0.678	0.560	0.330	0.502	0.381	0.195
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.921	0.858	0.679	0.779	0.669	0.438
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.988	0.971	0.896	0.932	0.871	0.696
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.996	0.974	0.984	0.962	0.870

Table D.1: Rejection Rates for Test for Productivity Change using Logs (continued)

n	β	— $p=1,\ q=1$ —			— $p=2,\ q=1$ —			— $p=3,\ q=1$ —			— $p=4,\ q=1$ —			— $p=5,\ q=1$ —		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
250	0.000	0.102	0.051	0.010	0.107	0.053	0.010	0.108	0.055	0.011	0.104	0.054	0.012	0.102	0.055	0.014
	0.005	0.592	0.464	0.234	0.617	0.493	0.258	0.633	0.507	0.273	0.201	0.125	0.041	0.159	0.096	0.031
	0.010	0.977	0.953	0.851	0.983	0.963	0.878	0.985	0.968	0.892	0.437	0.320	0.144	0.306	0.208	0.085
	0.015	1.000	1.000	0.997	1.000	1.000	0.998	1.000	1.000	0.999	0.703	0.582	0.344	0.505	0.380	0.187
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.889	0.811	0.597	0.705	0.581	0.342
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.994	0.985	0.932	0.943	0.889	0.709
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.995	0.995	0.986	0.930
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.990
500	0.000	0.102	0.051	0.010	0.103	0.051	0.010	0.107	0.055	0.011	0.101	0.052	0.012	0.101	0.053	0.013
	0.005	0.842	0.753	0.522	0.862	0.779	0.558	0.871	0.793	0.580	0.256	0.167	0.060	0.185	0.114	0.039
	0.010	1.000	0.999	0.995	1.000	1.000	0.997	1.000	1.000	0.998	0.591	0.464	0.242	0.393	0.279	0.122
	0.015	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.868	0.782	0.556	0.647	0.519	0.286
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.977	0.950	0.836	0.850	0.753	0.514
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996	0.990	0.975	0.893
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.993
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1000	0.000	0.102	0.051	0.010	0.102	0.051	0.010	0.106	0.054	0.011	0.101	0.051	0.011	0.100	0.051	0.012
	0.005	0.982	0.962	0.875	0.986	0.971	0.899	0.988	0.975	0.911	0.341	0.234	0.093	0.228	0.145	0.051
	0.010	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.767	0.657	0.411	0.521	0.394	0.192
	0.015	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.969	0.936	0.807	0.806	0.699	0.454
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.996	0.975	0.954	0.907	0.740
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.986
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table D.2: Rejection Rates for Test for Productivity Change using Geometric Mean (Two-sided Test)

n	β	— $p=1,\ q=1$ —			— $p=2,\ q=1$ —			— $p=3,\ q=1$ —			— $p=4,\ q=1$ —			— $p=5,\ q=1$ —		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
25	0.000	0.118	0.061	0.014	0.126	0.066	0.016	0.125	0.064	0.013	0.125	0.066	0.014	0.127	0.070	0.017
	0.005	0.150	0.080	0.018	0.157	0.086	0.021	0.144	0.078	0.018	0.138	0.078	0.020	0.136	0.081	0.024
	0.010	0.295	0.188	0.062	0.307	0.200	0.070	0.229	0.144	0.046	0.191	0.121	0.040	0.175	0.114	0.040
	0.015	0.497	0.364	0.161	0.515	0.384	0.178	0.354	0.248	0.101	0.272	0.187	0.074	0.236	0.164	0.068
	0.020	0.692	0.562	0.314	0.712	0.586	0.341	0.494	0.376	0.185	0.369	0.271	0.125	0.312	0.226	0.105
	0.030	0.926	0.859	0.655	0.936	0.876	0.686	0.734	0.627	0.407	0.567	0.458	0.265	0.477	0.376	0.209
	0.040	0.988	0.968	0.873	0.990	0.972	0.888	0.874	0.803	0.620	0.723	0.628	0.426	0.626	0.523	0.335
	0.050	0.998	0.992	0.956	0.997	0.991	0.958	0.938	0.896	0.767	0.826	0.748	0.571	0.738	0.646	0.458
50	0.000	0.110	0.055	0.011	0.115	0.059	0.012	0.112	0.057	0.011	0.113	0.060	0.013	0.112	0.062	0.016
	0.005	0.198	0.114	0.027	0.205	0.118	0.030	0.155	0.088	0.023	0.139	0.082	0.024	0.130	0.079	0.027
	0.010	0.465	0.330	0.130	0.484	0.350	0.143	0.290	0.193	0.069	0.221	0.144	0.052	0.186	0.122	0.048
	0.015	0.750	0.625	0.355	0.772	0.653	0.386	0.470	0.350	0.161	0.335	0.239	0.103	0.266	0.186	0.082
	0.020	0.921	0.852	0.631	0.934	0.873	0.670	0.650	0.528	0.298	0.467	0.353	0.177	0.363	0.266	0.129
	0.030	0.998	0.992	0.948	0.998	0.994	0.958	0.885	0.811	0.611	0.705	0.595	0.376	0.567	0.453	0.262
	0.040	1.000	1.000	0.994	1.000	1.000	0.995	0.966	0.934	0.825	0.858	0.777	0.582	0.735	0.628	0.419
	0.050	1.000	1.000	0.999	1.000	1.000	0.998	0.988	0.974	0.920	0.933	0.882	0.737	0.846	0.760	0.567
100	0.000	0.105	0.052	0.010	0.110	0.055	0.011	0.107	0.054	0.011	0.106	0.056	0.013	0.106	0.059	0.016
	0.005	0.302	0.192	0.059	0.316	0.205	0.066	0.193	0.117	0.034	0.152	0.091	0.030	0.135	0.083	0.030
	0.010	0.723	0.596	0.326	0.747	0.627	0.358	0.413	0.296	0.125	0.274	0.185	0.074	0.210	0.140	0.058
	0.015	0.953	0.908	0.735	0.963	0.925	0.771	0.661	0.538	0.303	0.440	0.325	0.154	0.317	0.224	0.103
	0.020	0.997	0.992	0.952	0.998	0.994	0.964	0.847	0.755	0.527	0.612	0.488	0.270	0.442	0.329	0.166
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	0.982	0.959	0.864	0.864	0.774	0.557	0.688	0.564	0.340
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.994	0.970	0.963	0.922	0.784	0.856	0.762	0.541
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.991	0.990	0.975	0.904	0.941	0.883	0.710

Table D.2: Rejection Rates for Test for Productivity Change using Geometric Mean (continued)

n	β	— $p=1,\ q=1$ —			— $p=2,\ q=1$ —			— $p=3,\ q=1$ —			— $p=4,\ q=1$ —			— $p=5,\ q=1$ —		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
250	0.000	0.103	0.051	0.010	0.107	0.054	0.011	0.103	0.052	0.011	0.104	0.054	0.013	0.102	0.055	0.014
	0.005	0.572	0.438	0.204	0.594	0.463	0.225	0.288	0.190	0.068	0.194	0.122	0.043	0.156	0.097	0.035
	0.010	0.974	0.946	0.826	0.980	0.957	0.854	0.661	0.535	0.296	0.407	0.294	0.131	0.286	0.196	0.083
	0.015	1.000	1.000	0.996	1.000	1.000	0.998	0.914	0.846	0.644	0.655	0.527	0.296	0.459	0.340	0.165
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	0.989	0.972	0.890	0.845	0.747	0.508	0.640	0.510	0.285
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996	0.985	0.963	0.859	0.895	0.808	0.580
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.997	0.976	0.981	0.952	0.822
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997	0.998	0.991	0.942
500	0.000	0.102	0.051	0.010	0.103	0.052	0.010	0.102	0.052	0.010	0.101	0.052	0.012	0.100	0.054	0.014
	0.005	0.833	0.738	0.494	0.852	0.764	0.528	0.418	0.300	0.126	0.247	0.162	0.060	0.181	0.114	0.042
	0.010	1.000	0.999	0.994	1.000	1.000	0.996	0.867	0.782	0.554	0.558	0.429	0.218	0.368	0.261	0.117
	0.015	1.000	1.000	1.000	1.000	1.000	1.000	0.992	0.980	0.914	0.833	0.732	0.489	0.599	0.468	0.249
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.995	0.962	0.920	0.762	0.798	0.683	0.433
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.983	0.976	0.942	0.797
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.996	0.962
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996
1000	0.000	0.102	0.051	0.010	0.102	0.051	0.010	0.101	0.051	0.011	0.101	0.051	0.011	0.100	0.052	0.012
	0.005	0.981	0.960	0.865	0.985	0.969	0.889	0.604	0.478	0.250	0.329	0.226	0.091	0.223	0.144	0.054
	0.010	1.000	1.000	1.000	1.000	1.000	1.000	0.979	0.955	0.850	0.738	0.620	0.373	0.490	0.366	0.177
	0.015	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996	0.956	0.911	0.749	0.763	0.645	0.396
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997	0.992	0.950	0.928	0.861	0.653
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.994	0.956
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table D.3: Rejection Rates for Test for Productivity Change using Logs (Two-sided Test)

n	β	— $p=1,\ q=1$ —			— $p=2,\ q=1$ —			— $p=3,\ q=1$ —			— $p=4,\ q=1$ —			— $p=5,\ q=1$ —		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
25	0.000	0.116	0.058	0.012	0.124	0.063	0.014	0.123	0.062	0.012	0.123	0.064	0.014	0.126	0.069	0.016
	0.005	0.183	0.108	0.030	0.194	0.115	0.034	0.163	0.092	0.024	0.148	0.084	0.021	0.143	0.082	0.021
	0.010	0.354	0.245	0.099	0.373	0.264	0.111	0.269	0.177	0.063	0.214	0.137	0.045	0.192	0.123	0.040
	0.015	0.566	0.445	0.233	0.594	0.473	0.257	0.414	0.305	0.138	0.313	0.219	0.090	0.268	0.185	0.074
	0.020	0.756	0.650	0.419	0.782	0.681	0.453	0.570	0.455	0.251	0.429	0.325	0.158	0.362	0.266	0.124
	0.030	0.954	0.914	0.774	0.965	0.931	0.806	0.815	0.731	0.528	0.658	0.554	0.350	0.565	0.458	0.271
	0.040	0.996	0.989	0.946	0.997	0.992	0.959	0.935	0.892	0.761	0.822	0.743	0.561	0.736	0.641	0.448
	0.050	1.000	0.999	0.990	1.000	1.000	0.993	0.979	0.960	0.892	0.915	0.864	0.729	0.853	0.781	0.614
50	0.000	0.109	0.053	0.010	0.114	0.057	0.011	0.111	0.056	0.011	0.112	0.059	0.013	0.112	0.062	0.015
	0.005	0.231	0.143	0.043	0.241	0.151	0.048	0.171	0.100	0.027	0.147	0.085	0.023	0.135	0.079	0.023
	0.010	0.513	0.387	0.182	0.539	0.413	0.200	0.326	0.224	0.085	0.244	0.159	0.056	0.200	0.129	0.046
	0.015	0.789	0.684	0.442	0.814	0.717	0.480	0.527	0.407	0.204	0.379	0.274	0.120	0.298	0.208	0.088
	0.020	0.939	0.890	0.722	0.952	0.911	0.760	0.716	0.606	0.376	0.531	0.415	0.219	0.417	0.310	0.151
	0.030	0.999	0.996	0.976	0.999	0.998	0.983	0.933	0.882	0.730	0.791	0.694	0.483	0.660	0.547	0.338
	0.040	1.000	1.000	0.999	1.000	1.000	1.000	0.988	0.975	0.917	0.929	0.876	0.726	0.840	0.753	0.554
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.994	0.976	0.979	0.956	0.874	0.935	0.883	0.737
100	0.000	0.104	0.051	0.010	0.109	0.055	0.010	0.106	0.054	0.011	0.105	0.055	0.012	0.106	0.058	0.015
	0.005	0.331	0.222	0.080	0.351	0.239	0.090	0.209	0.128	0.039	0.159	0.094	0.028	0.138	0.082	0.025
	0.010	0.751	0.638	0.389	0.776	0.672	0.424	0.451	0.332	0.149	0.299	0.202	0.079	0.224	0.147	0.056
	0.015	0.961	0.925	0.788	0.970	0.942	0.823	0.711	0.598	0.363	0.488	0.369	0.179	0.352	0.249	0.111
	0.020	0.998	0.994	0.968	0.999	0.996	0.978	0.889	0.817	0.617	0.678	0.560	0.330	0.502	0.381	0.195
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	0.993	0.982	0.930	0.921	0.858	0.679	0.779	0.669	0.438
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.992	0.988	0.971	0.896	0.932	0.871	0.696
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.996	0.974	0.984	0.962	0.870

Table D.3: Rejection Rates for Test for Productivity Change using Logs (continued)

n	β	— $p=1,\ q=1$ —			— $p=2,\ q=1$ —			— $p=3,\ q=1$ —			— $p=4,\ q=1$ —			— $p=5,\ q=1$ —		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
250	0.000	0.102	0.051	0.010	0.107	0.053	0.010	0.103	0.052	0.010	0.104	0.054	0.012	0.102	0.055	0.014
	0.005	0.592	0.464	0.234	0.617	0.493	0.258	0.305	0.203	0.074	0.201	0.125	0.041	0.159	0.096	0.031
	0.010	0.977	0.953	0.851	0.983	0.963	0.878	0.695	0.576	0.337	0.437	0.320	0.144	0.306	0.208	0.085
	0.015	1.000	1.000	0.997	1.000	1.000	0.998	0.936	0.882	0.711	0.703	0.582	0.344	0.505	0.380	0.187
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	0.994	0.984	0.933	0.889	0.811	0.597	0.705	0.581	0.342
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.994	0.985	0.932	0.943	0.889	0.709
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.995	0.995	0.986	0.930
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.990
500	0.000	0.102	0.051	0.010	0.103	0.051	0.010	0.102	0.051	0.010	0.101	0.052	0.012	0.101	0.053	0.013
	0.005	0.842	0.753	0.522	0.862	0.779	0.558	0.435	0.316	0.136	0.256	0.167	0.060	0.185	0.114	0.039
	0.010	1.000	0.999	0.995	1.000	1.000	0.997	0.887	0.812	0.602	0.591	0.464	0.242	0.393	0.279	0.122
	0.015	1.000	1.000	1.000	1.000	1.000	1.000	0.995	0.987	0.942	0.868	0.782	0.556	0.647	0.519	0.286
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.977	0.950	0.836	0.850	0.753	0.514
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996	0.990	0.975	0.893
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.993
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1000	0.000	0.102	0.051	0.010	0.102	0.051	0.010	0.102	0.051	0.011	0.101	0.051	0.011	0.100	0.051	0.012
	0.005	0.982	0.962	0.875	0.986	0.971	0.899	0.622	0.498	0.267	0.341	0.234	0.093	0.228	0.145	0.051
	0.010	1.000	1.000	1.000	1.000	1.000	1.000	0.984	0.965	0.879	0.767	0.657	0.411	0.521	0.394	0.192
	0.015	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.969	0.936	0.807	0.806	0.699	0.454
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.996	0.975	0.954	0.907	0.740
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.986
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table D.4: Rejection Rates for Test for Productivity Change using Geometric Mean (Two-sided Test, Recentered Intervals when $\kappa < 1/2$)

n	β	— $p=1,\ q=1$ —			— $p=2,\ q=1$ —			— $p=3,\ q=1$ —			— $p=4,\ q=1$ —			— $p=5,\ q=1$ —		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
25	0.000	0.118	0.061	0.014	0.126	0.066	0.016	0.036	0.013	0.001	0.009	0.002	0.000	0.004	0.001	0.000
	0.005	0.150	0.080	0.018	0.157	0.086	0.021	0.047	0.017	0.002	0.012	0.003	0.000	0.005	0.001	0.000
	0.010	0.295	0.188	0.062	0.307	0.200	0.070	0.128	0.060	0.011	0.044	0.015	0.001	0.022	0.006	0.001
	0.015	0.497	0.364	0.161	0.515	0.384	0.178	0.276	0.157	0.041	0.124	0.053	0.008	0.069	0.025	0.003
	0.020	0.692	0.562	0.314	0.712	0.587	0.341	0.465	0.307	0.108	0.257	0.132	0.028	0.160	0.071	0.011
	0.030	0.926	0.859	0.655	0.936	0.876	0.686	0.791	0.645	0.354	0.583	0.396	0.146	0.438	0.262	0.075
	0.040	0.988	0.968	0.873	0.989	0.972	0.888	0.937	0.859	0.626	0.816	0.664	0.354	0.698	0.512	0.218
	0.050	0.998	0.992	0.955	0.997	0.991	0.958	0.977	0.942	0.803	0.917	0.826	0.566	0.845	0.710	0.404
50	0.000	0.110	0.055	0.011	0.115	0.059	0.012	0.018	0.005	0.000	0.003	0.000	0.000	0.000	0.000	0.000
	0.005	0.198	0.113	0.027	0.205	0.118	0.030	0.041	0.012	0.001	0.006	0.001	0.000	0.001	0.000	0.000
	0.010	0.465	0.330	0.130	0.484	0.350	0.143	0.177	0.077	0.009	0.048	0.012	0.000	0.013	0.002	0.000
	0.015	0.750	0.625	0.355	0.772	0.653	0.386	0.444	0.257	0.057	0.186	0.068	0.006	0.071	0.017	0.001
	0.020	0.921	0.852	0.631	0.934	0.873	0.670	0.723	0.529	0.193	0.428	0.217	0.034	0.220	0.079	0.006
	0.030	0.998	0.992	0.948	0.998	0.994	0.959	0.970	0.908	0.642	0.853	0.667	0.267	0.665	0.411	0.093
	0.040	1.000	1.000	0.994	1.000	1.000	0.995	0.996	0.984	0.899	0.969	0.905	0.622	0.901	0.750	0.347
	0.050	1.000	1.000	0.999	1.000	1.000	0.998	0.999	0.995	0.964	0.989	0.965	0.831	0.963	0.896	0.619
100	0.000	0.105	0.052	0.010	0.110	0.055	0.011	0.010	0.002	0.000	0.001	0.000	0.000	0.000	0.000	0.000
	0.005	0.302	0.192	0.059	0.316	0.205	0.066	0.061	0.018	0.001	0.005	0.000	0.000	0.000	0.000	0.000
	0.010	0.723	0.596	0.326	0.747	0.627	0.358	0.351	0.173	0.022	0.080	0.017	0.000	0.009	0.001	0.000
	0.015	0.953	0.908	0.735	0.963	0.925	0.771	0.767	0.563	0.178	0.380	0.150	0.009	0.100	0.017	0.000
	0.020	0.997	0.991	0.952	0.998	0.994	0.964	0.963	0.884	0.535	0.765	0.488	0.086	0.390	0.129	0.005
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997	0.959	0.990	0.947	0.624	0.911	0.701	0.177
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.995	0.999	0.993	0.922	0.987	0.942	0.622
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	0.998	0.977	0.996	0.983	0.861

Table D.4: Rejection Rates for Test for Productivity Change using Geometric Mean (continued)

n	β	— $p=1,\ q=1$ —			— $p=2,\ q=1$ —			— $p=3,\ q=1$ —			— $p=4,\ q=1$ —			— $p=5,\ q=1$ —		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
250	0.000	0.103	0.051	0.010	0.107	0.054	0.011	0.005	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.005	0.572	0.438	0.204	0.594	0.463	0.225	0.151	0.050	0.002	0.008	0.001	0.000	0.000	0.000	0.000
	0.010	0.974	0.946	0.826	0.980	0.957	0.854	0.789	0.574	0.154	0.279	0.071	0.001	0.027	0.001	0.000
	0.015	1.000	1.000	0.996	1.000	1.000	0.998	0.995	0.975	0.765	0.879	0.605	0.075	0.424	0.096	0.000
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.989	0.997	0.967	0.552	0.916	0.604	0.038
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.990	0.999	0.991	0.769
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	0.999	0.979
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997
500	0.000	0.102	0.051	0.010	0.103	0.052	0.010	0.003	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.005	0.833	0.738	0.494	0.852	0.764	0.528	0.347	0.152	0.011	0.020	0.001	0.000	0.000	0.000	0.000
	0.010	1.000	0.999	0.993	1.000	0.999	0.996	0.986	0.941	0.616	0.697	0.314	0.008	0.106	0.005	0.000
	0.015	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997	0.999	0.976	0.500	0.887	0.445	0.004
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.983	0.999	0.976	0.323
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.992
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1000	0.000	0.102	0.051	0.010	0.102	0.051	0.010	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.005	0.981	0.960	0.865	0.985	0.969	0.889	0.711	0.458	0.077	0.073	0.005	0.000	0.000	0.000	0.000
	0.010	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.987	0.987	0.869	0.142	0.515	0.064	0.000
	0.015	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.991	1.000	0.973	0.139
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.970
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table D.5: Rejection Rates for Test for Productivity Change using Logs (Two-sided Test, Recentered Intervals when $\kappa < 1/2$)

n	β	— $p=1,\ q=1$ —			— $p=2,\ q=1$ —			— $p=3,\ q=1$ —			— $p=4,\ q=1$ —			— $p=5,\ q=1$ —		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
25	0.000	0.116	0.058	0.012	0.124	0.063	0.014	0.032	0.011	0.001	0.007	0.002	0.000	0.003	0.001	0.000
	0.005	0.183	0.107	0.030	0.195	0.115	0.034	0.070	0.029	0.004	0.021	0.006	0.000	0.010	0.002	0.000
	0.010	0.354	0.245	0.099	0.373	0.264	0.111	0.186	0.098	0.021	0.076	0.029	0.003	0.040	0.013	0.001
	0.015	0.567	0.445	0.233	0.594	0.473	0.257	0.371	0.234	0.075	0.196	0.094	0.017	0.119	0.048	0.007
	0.020	0.756	0.649	0.419	0.782	0.681	0.453	0.584	0.424	0.182	0.373	0.217	0.057	0.255	0.128	0.025
	0.030	0.954	0.914	0.774	0.965	0.931	0.807	0.887	0.783	0.516	0.740	0.566	0.264	0.612	0.417	0.153
	0.040	0.996	0.989	0.946	0.997	0.992	0.959	0.983	0.950	0.803	0.931	0.836	0.562	0.864	0.720	0.400
	0.050	1.000	0.999	0.990	1.000	0.999	0.993	0.998	0.990	0.935	0.984	0.949	0.791	0.960	0.891	0.652
50	0.000	0.109	0.053	0.010	0.114	0.057	0.011	0.016	0.004	0.000	0.002	0.000	0.000	0.000	0.000	0.000
	0.005	0.231	0.143	0.043	0.241	0.151	0.048	0.062	0.023	0.002	0.013	0.003	0.000	0.003	0.000	0.000
	0.010	0.513	0.387	0.182	0.539	0.413	0.200	0.242	0.121	0.020	0.083	0.025	0.001	0.027	0.005	0.000
	0.015	0.789	0.684	0.442	0.814	0.717	0.480	0.541	0.355	0.107	0.278	0.124	0.015	0.128	0.040	0.002
	0.020	0.939	0.890	0.722	0.952	0.911	0.760	0.808	0.652	0.311	0.566	0.342	0.078	0.349	0.155	0.018
	0.030	0.999	0.996	0.975	0.999	0.998	0.983	0.989	0.962	0.802	0.937	0.824	0.459	0.825	0.615	0.214
	0.040	1.000	1.000	0.999	1.000	1.000	0.999	1.000	0.998	0.971	0.995	0.974	0.832	0.975	0.909	0.607
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.994	0.999	0.995	0.955	0.995	0.978	0.857
100	0.000	0.104	0.051	0.010	0.109	0.055	0.010	0.010	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.005	0.331	0.223	0.080	0.351	0.239	0.090	0.086	0.030	0.002	0.011	0.001	0.000	0.001	0.000	0.000
	0.010	0.751	0.638	0.389	0.776	0.672	0.424	0.422	0.239	0.044	0.130	0.035	0.001	0.022	0.002	0.000
	0.015	0.961	0.925	0.788	0.970	0.942	0.823	0.824	0.660	0.280	0.498	0.246	0.027	0.182	0.044	0.001
	0.020	0.998	0.994	0.968	0.999	0.996	0.978	0.979	0.932	0.680	0.858	0.646	0.189	0.559	0.254	0.019
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.988	0.998	0.985	0.825	0.972	0.873	0.400
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000	0.999	0.983	0.998	0.988	0.865
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997	1.000	0.998	0.972

Table D.5: Rejection Rates for Test for Productivity Change using Logs (continued)

n	β	— $p=1,\ q=1$ —			— $p=2,\ q=1$ —			— $p=3,\ q=1$ —			— $p=4,\ q=1$ —			— $p=5,\ q=1$ —		
		.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01	.10	.05	.01
250	0.000	0.102	0.051	0.010	0.107	0.053	0.011	0.005	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.005	0.592	0.464	0.234	0.617	0.493	0.258	0.183	0.070	0.005	0.014	0.001	0.000	0.000	0.000	0.000
	0.010	0.977	0.953	0.851	0.983	0.963	0.878	0.826	0.643	0.226	0.366	0.122	0.003	0.057	0.005	0.000
	0.015	1.000	1.000	0.997	1.000	1.000	0.998	0.997	0.984	0.846	0.926	0.729	0.167	0.577	0.201	0.003
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996	0.999	0.988	0.750	0.966	0.788	0.136
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	1.000	0.998	0.936
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.997
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
500	0.000	0.102	0.051	0.010	0.103	0.051	0.010	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.005	0.842	0.753	0.522	0.862	0.779	0.558	0.385	0.185	0.019	0.030	0.003	0.000	0.000	0.000	0.000
	0.010	1.000	0.999	0.995	1.000	1.000	0.997	0.989	0.956	0.696	0.767	0.421	0.023	0.183	0.017	0.000
	0.015	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.999	0.989	0.685	0.945	0.640	0.024
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996	1.000	0.994	0.622
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
1000	0.000	0.102	0.051	0.010	0.102	0.051	0.010	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.005	0.982	0.962	0.875	0.986	0.971	0.899	0.738	0.500	0.104	0.099	0.010	0.000	0.000	0.000	0.000
	0.010	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.992	0.992	0.915	0.254	0.644	0.141	0.000
	0.015	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	1.000	0.992	0.365
	0.020	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.996
	0.030	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.040	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	0.050	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

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