Online supplement to "Count and duration time series with equal conditional stochastic and mean orders"

Abdelhakim Aknouche and Christian Francq

Finite-sample properties of the EQMLE

Finite-sample behavior of EQMLE is examined through a simulation study. We consider three models satisfying the stochastic-equal-mean order property (cf. (1.5) in Aknouche and Francq, 2020, henceforth AF), namely the exponential conditional distribution with mean λ_t ($Y_t/\mathcal{F}_{t-1} \sim \Gamma(1, 1/\lambda_t)$), the quadratic Gamma distribution, $\Gamma(0.5, 0.5/\lambda_t)$, and the linear Gamma distribution $\Gamma(\lambda_t/2, 1/2)$. For each model, we generate $N = 1000$ replications with sample-sizes $n = 500$, $n = 1000$ and $n = 3000$. The conditional mean is generated from a linear POLI model (cf. AF, (1.1)) with $p = q = 1$ and true parameter $\theta_0 = (1, 0.6, 0.2)^\top$. EQMLE and PQMLE are computed for each model. The means of EQML and PQML estimates over the 1000 replications are reported in bold, in Table 1 for model $\Gamma(1, 1/\lambda_t)$, in Table 2 for model $\Gamma(0.5, 0.5/\lambda_t)$, and in Table 3 for model $\Gamma(0.5\lambda_t, 0.5)$. These tables also show four estimates of the mean square error $\mathrm{E}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right)^2$ (see also Ahmad and Francq, 2016). These estimates are i) the estimated standard error (ESE) given by $\mathrm{ESE}(\boldsymbol{\theta}_{0j}) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\boldsymbol{\theta}}_j^{(i)} - \boldsymbol{\theta}_{0j}\right)^2$ ($\widehat{\boldsymbol{\theta}}_j^{(i)}$being the estimate of $\boldsymbol{\theta}_{0j}$ at the ith replication, $j = 1, 2, 3$), ii) the asymptotic standard error (ASE) defined by $\mathrm{ASE}(\boldsymbol{\theta}_{0j}) = \frac{1}{N}\sum_{i=1}^{N}\sqrt{\frac{1}{n}\left(\widehat{\boldsymbol{\Sigma}}^{(i)}\right)^{-1}(j,j)}$, iii) the theoretical standard error (TSE) given by $\mathrm{TSE}(\boldsymbol{\theta}_{0j}) = \frac{1}{N}\sum_{i=1}^{N}\sqrt{\frac{1}{n}\left(\boldsymbol{\Sigma}^{(i)}\right)^{-1}(j,j)}$, where $\boldsymbol{\Sigma}$ is computed from a very large series (with sample size $n = 20000$), and finally iv) the eXponential standard error (XSE) computed similarly to ASE while replacing $\widehat{\boldsymbol{\Sigma}}^{(i)}$ by $\widehat{\boldsymbol{J}}^{(i)}$.

The same measures are considered for PQMLE but are rather based on the asymptotic results given by (4.6) in AF. In particular, XSE is replaced by the Poisson standard error (PSE) computed from (4.7) in AF with $b = 1$ (see also Ahmad and Francq, 2016).

				$\Gamma(1, 1/\lambda_t)$				
		ω_0	α_0	β_0		ω_0	α_0	β_0
n	θ_0	1	0.6	0.2		1	0.6	0.2
500	**EQMLE**	**1.1286**	**0.5918**	**0.1743**	PQMLE	**1.1450**	**0.5698**	**0.1867**
	ESE	0.3059	0.0728	0.0772	ESE	0.3034	0.0923	0.0913
	ASE	0.1945	0.0699	0.0675	ASE	0.2500	0.0932	0.0828
	TSE	0.1819	0.0706	0.0672	TSE	0.2752	0.1076	0.0996
	XSE	0.1955	0.0703	0.0663	PSE	0.1072	0.0251	0.0310
1000	EQMLE	**1.0641**	**0.5971**	**0.1860**	PQMLE	**1.0841**	**0.5827**	**0.1899**
	ESE	0.1694	0.0495	0.0494	ESE	0.1961	0.0707	0.0658
	ASE	0.1318	0.0496	0.0464	ASE	0.1837	0.0727	0.0618
	TSE	0.1286	0.0499	0.0475	TSE	0.1946	0.0761	0.0704
	XSE	0.1332	0.0498	0.0463	PSE	0.0723	0.0177	0.0210
3000	EQMLE	**1.0247**	**0.6001**	**0.1945**	PQMLE	**1.0368**	**0.5916**	**0.1960**
	ESE	0.0857	0.0305	0.0285	ESE	0.1215	0.0430	0.0425
	ASE	0.0750	0.0287	0.0266	ASE	0.1153	0.0462	0.0389
	TSE	0.0743	0.0288	0.0274	TSE	0.1124	0.0439	0.0407
	XSE	0.0749	0.0288	0.0265	PSE	0.0400	0.0101	0.0116

Table 1. Estimation results for EQMLE and PQMLE for model $\Gamma(1, 1/\lambda_t)$.

		$\Gamma(0.5, 0.5/\lambda_t)$						
		ω_0	α_0	β_0		ω_0	α_0	β_0
n	θ_0	1	0.6	0.2		1	0.6	0.2
500	EQMLE	**1.0710**	**0.5968**	**0.1836**	PQMLE	**1.2129**	**0.5960**	**0.1597**
	ESE	0.2457	0.0943	0.0789	ESE	0.3268	0.0535	0.0826
	ASE	0.1960	0.0950	0.0723	ASE	0.2236	0.0539	0.0691
	TSE	0.1954	0.0946	0.0738	TSE	0.2047	0.0553	0.0670
	XSE	0.1412	0.0681	0.0518	PSE	0.2157	0.0467	0.0648
1000	EQMLE	**1.0409**	**0.5970**	**0.1890**	PQMLE	**1.0979**	**0.5973**	**0.1827**
	ESE	0.1596	0.0704	0.0552	ESE	0.1882	0.0389	0.0517
	ASE	0.1365	0.0678	0.0514	ASE	0.1524	0.0388	0.0481
	TSE	0.1382	0.0669	0.0522	TSE	0.1447	0.0391	0.0474
	XSE	0.0980	0.0482	0.0367	PSE	0.1437	0.0328	0.0444
3000	EQMLE	**1.0182**	**0.6006**	**0.1950**	PQMLE	**1.0326**	**0.6004**	**0.1930**
	ESE	0.0852	0.0383	0.0306	ESE	0.0940	0.0231	0.0285
	ASE	0.0783	0.0394	0.0296	ASE	0.0865	0.0227	0.0276
	TSE	0.0798	0.0386	0.0301	TSE	0.0836	0.0226	0.0274
	XSE	0.0557	0.0279	0.0211	PSE	0.0799	0.0189	0.0251

Table 2. Estimation results for EQMLE and PQMLE for model $\Gamma(0.5, 0.5/\lambda_t)$.

		$\Gamma(0.5\lambda_t, 0.5)$						
		ω_0	α_0	β_0		ω_0	α_0	β_0
n	θ_0	1	0.6	0.2		1	0.6	0.2
500	EQMLE	**1.2100**	**0.6158**	**0.1395**	PQMLE	**1.1290**	**0.6018**	**0.1719**
	ESE	0.3543	0.0603	0.1042	ESE	0.2601	0.0529	0.0783
	ASE	0.2324	0.0582	0.0775	ASE	0.2049	0.0504	0.0684
	TSE	0.2093	0.0597	0.0759	TSE	0.1909	0.0516	0.0677
	XSE	0.2796	0.0840	0.1038	PSE	0.1474	0.0362	0.0490
1000	EQMLE	**0.1286**	**0.6112**	**0.1620**	PQMLE	**1.0565**	**0.6019**	**0.1868**
	ESE	0.2265	0.0426	0.0680	ESE	0.1547	0.0367	0.0500
	ASE	0.1615	0.0414	0.0548	ASE	0.1401	0.0358	0.0477
	TSE	0.1480	0.0422	0.0537	TSE	0.1350	0.0365	0.0479
	XSE	0.1891	0.0594	0.0728	PSE	0.0999	0.0255	0.0340
3000	EQMLE	**1.0433**	**0.6040**	**0.1856**	PQMLE	**1.0231**	**0.5992**	**0.1964**
	ESE	0.1044	0.0241	0.0351	ESE	0.0852	0.0208	0.0279
	ASE	0.0905	0.0240	0.0312	ASE	0.0800	0.0207	0.0274
	TSE	0.0855	0.0244	0.0310	TSE	0.0779	0.0211	0.0277
	XSE	0.1033	0.0341	0.0414	PSE	0.0567	0.0147	0.0195

Table 3. Estimation results for EQMLE and PQMLE for model $\Gamma(0.5\lambda_t, 0.5)$.

From the latter simulations some broad conclusions may be drawn. Firstly, the parameters are well estimated by the two methods regarding their small bias and their various estimated standard errors. The latter are quite close to each other implying a good reliability of the estimates. Secondly, the estimation results are consistent with asymptotic theory as their accuracies increase with the sample size. Thirdly, as expected, the EQMLE gives better results under the conditional exponential distribution but is less accurate than the PQMLE if we depart from the exponential distribution. Note finally that EQMLE largely outperforms PQMLE under the conditional exponential model but its superiority is less pronounced in the Gamma $\Gamma(0.5, 0.5/\lambda_t)$ case. However, PQMLE dominates EQMLE for the Gamma $\Gamma(0.5\lambda_t, 0.5)$ model with linear conditional variance, which is in accordance with Remark 4.3 in AF. The estimation methods were implemented in Matlab on a desktop with Intel Core i7. The optimization routines were developed using the fminunc function of Matlab.

References

[1] Ahmad, A. and Francq, C. (2016) Poisson QMLE of count time series models. *Journal of Time Series Analysis* 37, 291–314.

[2] Aknouche, A. and Francq, C. (2020). Count and duration time series with equal conditional stochastic and mean orders. To appear in *Econometric Theory*.