Online Supplement for " Testing for Structural Changes in Factor Models via a Nonparametric Regression"

Liangjun Su[a], Xia Wang[b,c]
[a] School of Economics, Singapore Management University, Singapore
[b] School of Economics, Renmin University of China
[c] Lingnan (University) College, Sun Yat-Sen University, China

This Online Supplement contains two appendices. Appendix A is a mathematical appendix that contains some technical lemmas and the proofs of the theorems and lemmas in the paper. Appendix B contains some additional simulation and application results.

A Mathematical Appendix

This Mathematical Appendix is composed of three parts. Section A.1 provides some technical lemmas that are used in the proof of the theorems in Section 3. Section A.2 provides the proofs of the theorems in Section 3. Section A.3 gives the proofs of the technical lemmas in Section A.1.

A.1 Technical Lemmas

Let V_{NT} denote the $R \times R$ diagonal matrices of the first R largest eigenvalues of $(NT)^{-1} XX'$ arranged in decreasing order along its diagonal line. Let $H = (\Lambda_0'\Lambda_0/N)(F'\tilde{F}/T)V_{NT}^{-1}$, $C_{NT} = \min\{\sqrt{T}, \sqrt{N}\}$, $\tau_{ij,s} = E\left[e_{is}e_{js}F_s'F_s\right]$, $S_{Tr} = \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\tilde{F}_t\tilde{F}_t'$ and $S_{Tr}^{(0)} = \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}F_tF_t'$. We state some technical lemmas whose proofs are relegated to Section A.3.

Lemma A.1 *Suppose Assumptions A.1–A.3 and A.5 hold. Then under* $\mathbb{H}_1(a_{NT})$ *with* $a_{NT} = T^{-1/2}N^{-1/4}h^{-1/4}$,

(i) $T^{-1}\tilde{F}'(NT)^{-1}XX'\tilde{F} = V_{NT} \xrightarrow{P} V_0$,

(ii) $(T^{-1}\tilde{F}'F)\left(N^{-1}\Lambda_0'\Lambda_0\right)(T^{-1}F'\tilde{F}) \xrightarrow{P} V_0$,

where V_{NT} *is an* $R \times R$ *diagonal matrix consisting of the* R *largest eigenvalues of* $(NT)^{-1}XX'$, *and* V_0 *is an* $R \times R$ *matrix consisting of the* R *eigenvalues of* $\Sigma_{\Lambda_0}\Sigma_F$, *both arranged in descending order.*

Lemma A.2 *Suppose Assumptions A.1–A.3 and A.5 hold. Then under* $\mathbb{H}_1(a_{NT})$ *with* $a_{NT} = T^{-1/2}N^{-1/4}h^{-1/4}$,

(i) $\frac{1}{T}\left\|\tilde{F} - FH\right\|^2 = O_P\left(C_{NT}^{-2}\right)$,

(ii) $\frac{1}{T}(\tilde{F} - FH)'FH = O_P\left(C_{NT}^{-2}\right) + o_P(a_{NT})$,

(iii) $\frac{1}{T}(\tilde{F} - FH)'\tilde{F} = O_P\left(C_{NT}^{-2}\right) + o_P(a_{NT})$,

(iv) $\frac{1}{T}(\tilde{F}'\tilde{F} - H'F'FH) = O_P\left(C_{NT}^{-2}\right) + o_P(a_{NT})$,

(v) $V_{NT} = V_0 + O_P\left(C_{NT}^{-1}\right)$,

(vi) $H = Q_0^{-1} + O_P\left(C_{NT}^{-1}\right)$,

where $Q_0 = V_0^{1/2}\Upsilon_0^{-1}\Sigma_{\Lambda_0}^{-1/2}$ *and* Υ_0 *denotes the probability limit of* Υ_{NT} *defined in the proof of (v).*

Lemma A.3 *Suppose Assumptions A.1–A.5 hold. Then under* $\mathbb{H}_1(a_{NT})$ *with* $a_{NT} = T^{-1/2}N^{-1/4}h^{-1/4}$,

(i) $\max_r \left\|\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}(\tilde{F}_t - H'F_t)(\tilde{F}_t - H'F_t)'\right\| = O_P(T^{-1}\ln T + N^{-1})$,

(ii) $\max_r \left\|\frac{1}{T}\sum_{r=1}^{T} k_{h,tr}(\tilde{F}_t - H'F_t)F_tH'\right\| = O_P(T^{-1}\ln T + N^{-1}) + o_P(a_{NT})$,

(iii) $\max_r \left\| S_{Tr}^{(0)} - \Sigma_F \right\| = O_P\left(T^{-1/2}(\ln T)^{1/2}\right)$,

(iv) $\max_r \left\| S_{Tr} - \left(Q_0^{-1}\right)' \Sigma_F Q_0^{-1} \right\| = O_P((Th)^{-1/2}(\ln T)^{1/2} + N^{-1/2})$,

(v) $\max_r \left\| H S_{Tr}^{-1} S_{Tr}^{-1} H' - \Sigma_F^{-1} Q_0' Q_0 \Sigma_F^{-1} \right\| = O_P((Th)^{-1/2}(\ln T)^{1/2} + N^{-1/2})$.

Lemma A.4 *Suppose Assumptions A.1–A.5 hold. Then under* $\mathbb{H}_1(a_{NT})$ *with* $a_{NT} = T^{-1/2}N^{-1/4}h^{-1/4}$,

(i) $\max_{i,r} \left\| \frac{1}{T}\sum_{t=1}^T k_{h,tr}\tilde{F}_t F_t' g_{it} \right\| = O_P(1)$,

(ii) $\max_{i,r} \left\| \frac{1}{T}\sum_{t=1}^T k_{h,tr} F_t' e_{it} \right\| = O_P(T^{-1/2}h^{-1/2}\ln(NT))$,

(iii) $\frac{1}{N}\sum_{i=1}^N \left\| \frac{1}{T}\sum_{s=1}^T (\tilde{F}_s - HF_s)e_{is}^{\dagger} \right\|^2 = O_P\left(C_{NT}^{-4}\right)$,

(iv) $\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T \left\| \frac{1}{T}\sum_{s=1}^T (\tilde{F}_s - H'F_s)e_{is}k_{h,st} \right\|^2 = O_P\left(N^{-3/2} + T^{-2}\right) + o_P\left(a_{NT}^2\right)$.

Lemma A.5 *Suppose Assumptions A.1 and A.3–A.5 hold. Suppose that* $\mathbb{H}_1(a_{NT})$ *holds with* $a_{NT} = T^{-1/2}N^{-1/4}h^{-1/4}$*. Then uniformly in* (i, r),

(i) $\frac{1}{T}\sum_{t=1}^T k_{h,tr}F_t F_t' g_{it} = \Sigma_F g_i\left(\frac{r}{T}\right) + o_P(1)$,

(ii) $\frac{1}{T}\sum_{t=1}^T F_t F_t' g_{it} = \Sigma_F \frac{1}{T}\sum_{t=1}^T g_i\left(\frac{t}{T}\right) + o_P(1) = o_P(1)$.

Lemma A.6 *Suppose Assumptions A.1–A.5 hold. Then under* $\mathbb{H}_1(a_{NT})$ *with* $a_{NT} = T^{-1/2}N^{-1/4}h^{-1/4}$,

(i) $\frac{1}{N}\sum_{i=1}^N \left\| \tilde{\lambda}_{i0} - H^{-1}\lambda_{i0} \right\|^2 = O_P\left(C_{NT}^{-2}\right)$,

(ii) $\frac{1}{T}\sum_{t=1}^T \left\| \tilde{F}_t \right\|^l = O_P(1)$ *for* $l = 4,\ 6$,

(iii) $\max_s \frac{1}{T}\sum_{t=1}^T k_{h,st}^2 = O\left(h^{-1}\right)$,

(iv) $\frac{1}{T^2}\sum_{t=1}^T\sum_{s=1}^T \tilde{F}_s' L_{st}\tilde{F}_s\tilde{F}_s'\tilde{F}_s = O\left(h^{-1}\right)$,

(v) $\frac{1}{T}\sum_{t=1}^T \left\| (\tilde{F}_t - H'F_t)\tilde{F}_t \right\|^2 = O_P\left(C_{NT}^{-2} + TN^{-2}\right)$,

(vi) $\frac{1}{NT}\sum_{t=1}^T \left\| \sum_{i=1}^N \lambda_{i0}' H'^{-1}(\tilde{F}_t - H'F_t)e_{it} \right\|^2 = O_P\left(C_{NT}^{-2}\right)$,

(vii) $\frac{1}{NT}\sum_{i=1}^N\sum_{s=1}^T (\tilde{F}_s - H'F_s)F_s' H e_{is}^2 = O_P(a_{NT})$.

In addition, we need the following lemma from Sun and Chiang (1997).

Lemma A.7 *Let* $\{V_t, t \geq 1\}$ *be a strong mixing process with mixing coefficient* $\alpha(\cdot)$. *Let* $G_{t_1,...,t_m}$ *denote the distribution function of* $(V_{t_1}, ..., V_{t_m})$. *For any integer* $m > 1$ *and integers* $(t_1, ..., t_m)$ *such that* $1 \leq t_1 < t_2 < \ldots < t_m$, *let* ϑ *be a Borel measurable function such that* $\max\{\int |\vartheta(v_1, \ldots, v_m)|^{1+\tilde{\eta}}\, dG_{t_1,...,t_j}(v_1, \ldots, v_j)\, dG_{t_j+1,...,t_m}(v_{j+1}, \ldots, v_m), \int |\vartheta(v_1, \ldots, v_m)|^{1+\tilde{\eta}}\, dG_{t_1,...,t_m}\} \leq M$ *for some* $\tilde{\eta} > 0$. *Then* $|\int \vartheta(v_1, \ldots, v_m)\, dG_{t_1,...,t_m}(v_1, \ldots, v_m) - \int \vartheta(v_1, \ldots, v_m)\, dG_{t_1,...,t_j}(v_1, \ldots, v_j)\, dG_{t_j+1,...,t_m}(v_{j+1}, \ldots, v_m)| \leq 4M^{1/(1+\tilde{\eta})}\alpha(t_{j+1} - t_j)^{\tilde{\eta}/(1+\tilde{\eta})}$.

A.2 Proofs of the Theorems in Section 3

Proof of Theorem 3.1. The result in Theorem 3.1 follows as a special case of Theorem 3.2 with $g_i(t/T) = 0$ for each i and t. ■

Proof of Theorem 3.2. Under $\mathbb{H}_1(a_{NT}) : \lambda_{it} = \lambda_{i0} + a_{NT}g_i(t/T)$, we can decompose $TN^{1/2}h^{1/2}\hat{M}$ as follows:

$$
\begin{aligned}
TN^{1/2}h^{1/2}\hat{M} &= N^{-1/2}h^{1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|\left(\hat{\lambda}_{it}-H^{-1}\lambda_{i0}\right)-\left(\tilde{\lambda}_{i}-H^{-1}\lambda_{i0}\right)\right\|^{2} \\
&= N^{-1/2}h^{1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\hat{\lambda}_{it}-H^{-1}\lambda_{i0}\right)'\left(\hat{\lambda}_{it}-H^{-1}\lambda_{i0}\right) \\
&+ N^{-1/2}h^{1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\tilde{\lambda}_{i}-H^{-1}\lambda_{i0}\right)'\left(\tilde{\lambda}_{i}-H^{-1}\lambda_{i0}\right) \\
&- 2N^{-1/2}h^{1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\hat{\lambda}_{it}-H^{-1}\lambda_{i0}\right)'\left(\tilde{\lambda}_{i}-H^{-1}\lambda_{i0}\right) \\
&\equiv M_1+M_2-2M_3,\ \text{say},
\end{aligned}
$$

where for notational simplicity we suppress the dependence of M_l on (N,T) for $l=1,2,3$. We complete the proof by showing that under $\mathbb{H}_1\left(a_{NT}\right)$, (i) $M_1-\mathbb{B}_{1NT}-\Pi_{1NT}\overset{d}{\to}N\left(0,\mathbb{V}_0\right)$, (ii) $M_2-\mathbb{B}_{2NT}-\Pi_{2NT}=o_P\left(1\right)$, and (iii) $M_3-\mathbb{B}_{3NT}-\Pi_{3NT}=o_P\left(1\right)$, (iv) $\hat{\mathbb{B}}_{NT}=\mathbb{B}_{NT}+o_P\left(1\right)$, and (v) $\hat{\mathbb{V}}_{NT}=\mathbb{V}_{NT}+o_P\left(1\right)$, where $\mathbb{B}_{NT}=\mathbb{B}_{1NT}+\mathbb{B}_{2NT}-2\mathbb{B}_{3NT}$, and

$$
\begin{aligned}
\mathbb{B}_{1NT} &= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^2F_s'HS_{Tt}^{-1}S_{Tt}^{-1}H'F_se_{is}^2, \\
\mathbb{B}_{2NT} &= \frac{h^{1/2}}{TN^{1/2}}\sum_{i=1}^{N}\sum_{s=1}^{T}F_s'HH'F_se_{is}^2, \\
\mathbb{B}_{3NT} &= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}F_s'HS_{Tt}^{-1}H'F_se_{is}^2, \\
\Pi_{1NT} &= \frac{1}{TN}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathrm{tr}\left(Q_0g_i\left(\frac{t}{T}\right)g_i\left(\frac{t}{T}\right)'Q_0'\right), \\
\Pi_{2NT} &= \frac{1}{N}\sum_{i=1}^{N}\mathrm{tr}\left[(Q_0^{-1})'\Sigma_F\frac{1}{T}\sum_{r=1}^{T}g_i\left(\frac{r}{T}\right)\frac{1}{T}\sum_{s=1}^{T}g_i\left(\frac{s}{T}\right)'\Sigma_FQ_0^{-1}\right], \\
\Pi_{3NT} &= \frac{1}{N}\sum_{i=1}^{N}\mathrm{tr}\left[\frac{1}{T}\sum_{r=1}^{T}g_i\left(\frac{r}{T}\right)\frac{1}{T}\sum_{t=1}^{T}g_i\left(\frac{t}{T}\right)'\Sigma_F\right],
\end{aligned}
$$

$\mathbb{V}_{NT}$ are defined in Theorem 3.2, and $\mathbb{V}_0=\lim_{(N,T)\to\infty}\mathbb{V}_{NT}$. We prove these claims in Propositions A.8-A.12 below. Noting that $\frac{1}{T}\sum_{r=1}^{T}g_i\left(\frac{r}{T}\right)=\int_0^1g_i\left(u\right)du+O\left(1/T\right)=O\left(1/T\right)$ under the normalization rule $\int_0^1g_i\left(u\right)du=0$, we have $\Pi_{lNT}=O\left(1/T\right)$ for $l=2,3$. Combining these results yields $\widehat{SM}_{NT}=\hat{\mathbb{V}}_{NT}^{-1/2}(TN^{1/2}h^{1/2}\hat{M}-\hat{\mathbb{B}}_{NT})\overset{d}{\to}N\left(\pi_0,1\right)$, where $\pi_0=\lim_{(N,T)\to\infty}\Pi_{1NT}/\mathbb{V}_{NT}^{1/2}$. ■

Proposition A.8 *Suppose that the conditions in Theorem 3.2 hold. Then* $M_1-\mathbb{B}_{1NT}-\Pi_{1NT}\overset{d}{\to}N\left(0,\mathbb{V}_0\right)$ *under* $\mathbb{H}_1\left(a_{NT}\right)$.

Proof. Using $X_{it}=F_t'\lambda_{it}+e_{it}=F_t'HH^{-1}\lambda_{i0}+e_{it}+a_{NT}F_t'g_{it}=\tilde{F}_t'H^{-1}\lambda_{i0}+e_{it}+a_{NT}F_t'g_{it}-(\tilde{F}_t-$

$H'F_t)'H^{-1}\lambda_{i0}$, we have

$$\begin{aligned}
\hat{\lambda}_{it} - H^{-1}\lambda_{i0} &= \left(\frac{1}{T}\sum_{s=1}^{T} k_{h,st}\tilde{F}_s\tilde{F}_s'\right)^{-1} \frac{1}{T}\sum_{s=1}^{T} k_{h,st}\tilde{F}_s X_{is} - H^{-1}\lambda_{i0} \\
&= S_{Tt}^{-1}H'\frac{1}{T}\sum_{s=1}^{T} k_{h,st}F_s e_{is} + a_{NT}S_{Tt}^{-1}\frac{1}{T}\sum_{s=1}^{T} k_{h,st}\tilde{F}_s F_s' g_{is} \\
&\quad - S_{Tt}^{-1}\frac{1}{T}\sum_{s=1}^{T} k_{h,st}\tilde{F}_s\left(\tilde{F}_s - H'F_s\right)' H^{-1}\lambda_{i0} + S_{Tt}^{-1}\frac{1}{T}\sum_{s=1}^{T} k_{h,st}\left(\tilde{F}_s - H'F_s\right)e_{is} \\
&\equiv D_1(i,t) + D_2(i,t) - D_3(i,t) + D_4(i,t), \quad \text{say,} \qquad \text{(A.1)}
\end{aligned}$$

where $S_{Tt} = \frac{1}{T}\sum_{s=1}^{T} k_{h,st}\tilde{F}_s\tilde{F}_s'$. By (A.1), we decompose M_1 as follows:

$$\begin{aligned}
M_1 &= h^{1/2}N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|D_1(i,t) + D_2(i,t) - D_3(i,t) + D_4(i,t)\right\|^2 \\
&= h^{1/2}N^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}[\|D_1(i,t)\|^2 + \|D_2(i,t)\|^2 + \|D_3(i,t)\|^2 + \|D_4(i,t)\|^2 \\
&\quad + 2D_1(i,t)'D_2(i,t) - 2D_1(i,t)'D_3(i,t)) + 2D_1(i,t)'D_4(i,t) \\
&\quad -2D_2(i,t)'D_3(i,t) + 2D_2(i,t)'D_4(i,t)) - 2D_3(i,t)'D_4(i,t)] \\
&\equiv M_{1,1} + M_{1,2} + M_{1,3} + M_{1,4} + 2M_{1,5} - 2M_{1,6} + 2M_{1,7} - 2M_{1,8} + 2M_{1,9} - 2M_{1,10}, \text{ say.}
\end{aligned}$$

We prove the proposition by showing that (i) $M_{1,1} - \mathbb{B}_{1NT} \xrightarrow{d} N(0, \mathbb{V}_0)$, (ii) $M_{1,2} = \Pi_{1NT} + o_P(1)$, and (iii) $M_{1,j} = o_P(1)$ for $j = 3, ..., 10$.

We first prove (i). We decompose the $M_{1,1}$ term as follows:

$$\begin{aligned}
M_{1,1} &= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|S_{Tt}^{-1}H'\frac{1}{T}\sum_{s=1}^{T} F_s e_{is} k_{h,st}\right\|^2 \\
&= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} k_{h,st}F_s' e_{is} H S_{Tt}^{-1}S_{Tt}^{-1}H'\sum_{r=1}^{T} k_{h,rt}F_r e_{ir} \\
&= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} k_{h,st}^2 F_s' H S_{Tt}^{-1}S_{Tt}^{-1}H'F_s e_{is}^2 + \frac{2h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{1\le r<s\le T} k_{h,st}k_{h,rt}F_s'\mathbb{S}F_r e_{is}e_{ir} \\
&\quad + \frac{2h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{1\le r<s\le T} k_{h,st}k_{h,rt}F_s'\left(HS_{Tt}^{-1}S_{Tt}^{-1}H' - \mathbb{S}\right)F_r e_{is}e_{ir} \equiv M_{1,1}^{(1)} + M_{1,1}^{(2)} + M_{1,1}^{(3)},
\end{aligned}$$

where $\mathbb{S} \equiv \Sigma_F^{-1} Q_0' Q_0 \Sigma_F^{-1}$. Apparently, $M_{1,1}^{(1)} = \mathbb{B}_{1NT}$. For $M_{1,1}^{(2)}$, we make the following decomposition

$$
\begin{aligned}
M_{1,1}^{(2)} &= \frac{2h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{1 \le r < s \le T} k_{h,st} k_{h,rt} F_s' \mathbb{S} F_r e_{is} e_{ir} \\
&= \frac{2}{TN^{1/2}h^{1/2}} \sum_{i=1}^{N} \sum_{1 \le r < s \le T} \bar{K}\left(\frac{s-r}{Th}\right) F_s' \mathbb{S} F_r e_{is} e_{ir} \\
&\quad + \frac{2}{TN^{1/2}h^{1/2}} \sum_{i=1}^{N} \sum_{1 \le r < s \le T} \left[\frac{h}{T} \sum_{t=1}^{T} k_{h,st} k_{h,rt} - \bar{K}\left(\frac{s-r}{Th}\right)\right] F_s' \mathbb{S} F_r e_{is} e_{ir} \equiv M_{1,1}^{(2,1)} + M_{1,1}^{(2,2)},
\end{aligned}
$$

where $\bar{K}(v) = \int_{-1}^{1} K(u) K(u-v) \, du$. Let $Z_{NT,s} = T^{-1} N^{-1/2} h^{-1/2} \sum_{r=1}^{s-1} \bar{K}\left(\frac{s-r}{Th}\right) F_s' \mathbb{S} F_r e_s' e_r$, then $M_{1,1}^{(2,1)} = 2\sum_{s=2}^{T} Z_{NT,s}$ and $E(Z_{NT,s} | \mathcal{F}_{NT,s-1}) = T^{-1} N^{-1/2} h^{-1/2} \sum_{r=1}^{s-1} \bar{K}\left(\frac{s-r}{Th}\right) F_s' \mathbb{S} F_r E(e_s' | \mathcal{F}_{NT,s-1}) e_r = 0$. By the martingale central limit theorem (e.g., Pollard, 1984, p.171), it suffices to prove $\mathbb{V}_{NT}^{-1/2} M_{1,1}^{(2,1)} \xrightarrow{d} N(0,1)$ by showing that

$$
\mathcal{Z} \equiv \sum_{s=2}^{T} E\left(Z_{NT,s}^4 | \mathcal{F}_{NT,s-1}\right) = o_P(1) \text{ and } \sum_{s=2}^{T} Z_{NT,s}^2 - \mathbb{V}_{NT} = o_P(1). \tag{A.2}
$$

First, we verify the first part of (A.2). Observing that $\mathcal{Z} \ge 0$, it suffices to show $\mathcal{Z} = o_P(1)$ by showing that $E(\mathcal{Z}) = o(1)$ by Markov inequality. Let $\bar{k}_{sr} = \bar{K}\left(\frac{s-r}{Th}\right)$ and $\phi_{sr} = F_s' \mathbb{S} F_r e_s' e_r$. We have

$$
\begin{aligned}
E(\mathcal{Z}) &= \sum_{s=2}^{T} E\left\{\left[\frac{2}{TN^{1/2}h^{1/2}} \sum_{r=1}^{s-1} \bar{k}_{sr} \phi_{sr}\right]^4\right\} \\
&= \frac{16}{T^4 N^2 h^2} \sum_{s=2}^{T} E\left[\sum_{r=1}^{s-1} \bar{k}_{sr}^4 \phi_{sr}^4 + 2 \sum_{1 \le r_1 < r_2 \le s-1} \bar{k}_{sr_1}^2 \bar{k}_{sr_2}^2 \phi_{sr_1}^2 \phi_{sr_2}^2\right. \\
&\quad \left. + 4 \sum_{t=1}^{s-1} \sum_{1 \le r_1 < r_2 \le s-1} \bar{k}_{st}^2 \bar{k}_{sr_1} \bar{k}_{sr_2} \phi_{st}^2 \phi_{sr_1} \phi_{sr_2} + 4 \sum_{1 \le r_1 < r_2 \le s-1, 1 \le t_1 < t_2 \le s-1} \bar{k}_{r_1 s} \bar{k}_{r_2 s} \bar{k}_{t_1 s} \bar{k}_{t_2 s} \phi_{sr_1} \phi_{sr_2} \phi_{st_1} \phi_{st_2}\right] \\
&\equiv \mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3 + \mathcal{Z}_4, \text{ say.}
\end{aligned}
$$

Noting that $\max_{r<s} \left\| N^{-1/2} \phi_{sr} \right\|_4^4 \le C$ under Assumption A.3(v), we can readily show that under Assumption A.4

$$
\begin{aligned}
\mathcal{Z}_1 &\le \max_{r<s} \left\| N^{-1/2} \phi_{sr} \right\|_4^4 \frac{16}{T^4 h^2} \sum_{s=2}^{T} \sum_{r=1}^{s-1} \bar{k}_{sr}^4 = O\left(T^{-2} h^{-1}\right) \\
\mathcal{Z}_2 &\le \max_{r<s} \left\| N^{-1/2} \phi_{sr} \right\|_4^4 \frac{32}{T^4 h^2} \sum_{s=2}^{T} \sum_{1 \le r_1 < r_2 \le s-1} \bar{k}_{sr_1}^2 \bar{k}_{sr_2}^2 = O\left(T^{-1}\right), \\
\mathcal{Z}_3 &\le \max_{r<s} \left\| N^{-1/2} \phi_{sr} \right\|_4^4 \frac{64}{T^4 h^2} \sum_{s=2}^{T} \sum_{t=1}^{s-1} \sum_{1 \le r_1 < r_2 \le s-1} \bar{k}_{st}^2 \bar{k}_{sr_1} \bar{k}_{sr_2} = O(h).
\end{aligned}
$$

For $\mathcal{Z}_4$, we can apply Assumptions A.3(iii) and (v) and A.5 along with the Davydov inequality to show that

$$
\mathcal{Z}_4 = \frac{64}{T^4 N^2 h^2} \sum_{s=2}^{T} \sum_{1 \le r_1 < r_2 \le s-1, 1 \le t_1 < t_2 \le s-1} \bar{k}_{r_1 s} \bar{k}_{r_2 s} \bar{k}_{t_1 s} \bar{k}_{t_2 s} E\left(\phi_{sr_1} \phi_{sr_2} \phi_{st_1} \phi_{st_2}\right) = O(h).
$$

Thus $E(\mathcal{Z}) = o(1)$ and $\mathcal{Z} = o_P(1)$.

To verify the second part of (A.2), it suffices to show (I) $\sum_{s=2}^{T} E\left(Z_{NT,s}^2\right) = \mathbb{V}_{NT} + o(1)$, and (II) $\text{Var}(\sum_{s=2}^{T} Z_{NT,s}^2) = o_P(1)$ by Chebyshev inequality. These two claims can be easily proved if we also assume independence of $\{e_{i\cdot} = (e_{i1}, ..., e_{iT})'\}$ across i conditional on the factor. Here we prove them without imposing such a cross-sectional independence condition. We first prove (I). Observe that

$$\begin{aligned}\text{Var}(M_{1,1}^{(2,1)}) &= \sum_{s=2}^{T} E\left(Z_{NT,s}^2\right) = 4T^{-2}N^{-1}h^{-1}\sum_{s=2}^{T}\sum_{r=1}^{s-1}\bar{k}_{sr}^2 E\left(F_s'\mathbb{S}F_r e_s' e_r\right)^2 \\ &\quad + 4T^{-2}N^{-1}h^{-1}\sum_{s=2}^{T}\sum_{1\le r_1\ne r_2\le s-1}\bar{k}_{sr_1}\bar{k}_{sr_2}E\left(F_s'\mathbb{S}F_{r_1}e_s'e_{r_1}F_{r_2}'\mathbb{S}F_s e_{r_2}'e_s\right) \\ &\equiv \mathbb{V}_{NT} + b_{NT}.\end{aligned}$$

To study b_{NT}, let $\mathbb{S} = \Sigma_F^{-1}Q_0'Q_0\Sigma_F^{-1} = \{s_{mn}\}$. Then $\phi_{sr} = F_s'\mathbb{S}F_r e_s'e_r = \sum_{m=1}^{R}\sum_{n=1}^{R} s_{mn}F_{sm}F_{rn}e_s'e_r$, and

$$\begin{aligned}b_{NT} &= 4T^{-2}N^{-1}h^{-1}\sum_{s=2}^{T}\sum_{1\le r_1\ne r_2\le s-1}\bar{k}_{sr_1}\bar{k}_{sr_2}E\left(F_s'\mathbb{S}F_{r_1}e_s'e_{r_1}F_{r_2}'\mathbb{S}F_s e_{r_2}'e_s\right) \\ &= 4T^{-2}N^{-1}h^{-1}\sum_{1\le m_1,m_2\le R}\sum_{1\le n_1,n_2\le R}\sum_{s=2}^{T}\sum_{1\le r_1\ne r_2\le s-1}\bar{k}_{sr_1}\bar{k}_{sr_2}s_{m_1n_1}s_{m_2n_2}E\left(F_{sm_1}F_{r_1n_1}e_s'e_{r_1}F_{sm_2}F_{r_2n_2}e_s'e_{r_2}\right) \\ &= 4\sum_{1\le m_1,m_2\le R}\sum_{1\le n_1,n_2\le R}s_{m_1n_1}s_{m_2n_2}b_{NT}(m_1,m_2,n_1,n_2)\end{aligned}$$

where $b_{NT}(m_1,m_2,n_1,n_2) = T^{-2}N^{-1}h^{-1}\sum_{s=2}^{T}\sum_{1\le r_1\ne r_2\le s-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\bar{k}_{sr_1}\bar{k}_{sr_2}E(F_{sm_1}F_{r_1n_1}e_{is}e_{jr_1}F_{sm_2}$ $\times F_{r_2n_2}e_{is}e_{jr_2})$. Since R is fixed and s_{mn}'s are finite, $b_{NT} = o(1)$ provided $b_{NT}(m_1,m_2,n_1,n_2) = o(1)$ for each quadruple (m_1,m_2,n_1,n_2). We consider three cases (1) $|s-r_2| > T_0$, (2) $|s-r_2| \le T_0$ and $|r_2-r_1| > T_0$, and (3) $|s-r_2| \le T_0$ and $|r_2-r_1| \le T_0$. We use $b_{NT}^{(l)}(m_1,m_2,n_1,n_2)$ to denote $b_{NT}(m_1,m_2,n_1,n_2)$ when the time indices are restricted to case (l) for $l = 1,2,3$. In case (1), we apply Lemma A.7 and the fact that $E(F_{r_1}F_{r_2}'e_{ir_1}e_{ir_2}) = 0$ for $r_1 \ne r_2$ under Assumption A.3(iii) to obtain

$$|b_{NT}^{(l)}(m_1,m_2,n_1,n_2)| \le CT^{-2}N^{-1}h^{-1}\sum_{r_1<r_2<s}\sum_{i=1}^{N}\sum_{j=1}^{N}\bar{k}_{sr_1}\bar{k}_{sr_2}\alpha(T_0)^{\delta/(1+\delta)} = O\left(NTh\alpha(T_0)^{\delta/(1+\delta)}\right) = o(1)$$

In case (2), we apply Lemma A.7 and the fact that $E(F_{r_1}e_{ir_1}) = 0$ to obtain

$$|b_{NT}^{(l)}(m_1,m_2,n_1,n_2)| \le CT^{-2}N^{-1}h^{-1}\sum_{r_1<r_2<s}\sum_{i=1}^{N}\sum_{j=1}^{N}\bar{k}_{sr_1}\bar{k}_{sr_2}\alpha(T_0)^{\delta/(1+\delta)} = O\left(NTh\alpha(T_0)^{\delta/(1+\delta)}\right) = o(1)$$

In case (3), we have

$$\begin{aligned}\left|b_{NT}^{(3)}(m_1,m_2,n_1,n_2)\right| &= T^{-2}N^{-1}h^{-1}\sum_{r_1<r_2<s,\text{ case (3)}}\bar{k}_{sr_1}\bar{k}_{sr_2}\left|E\left(F_sF_se_{r_1}'e_se_{r_2}'e_sF_{r_1}F_{r_2}\right)\right| \\ &\le \max_{m,n}\max_{r<s}\left\|N^{-1/2}F_rF_se_r'e_s\right\|_2^2 T^{-2}h^{-1}\sum_{r_1<r_2<s,\text{ case (3)}}\bar{k}_{sr_1}\bar{k}_{sr_2} = O\left(T^{-1}T_0^2h\right) = o(1),\end{aligned}$$

where we use the fact that the total number of terms in the summation over three time indices for $b_{NT}^{(3)}$ are of order $O\left(TT_0^2\right)$. In sum, we have shown that $b_{NT} = o(1)$ and $\sum_{s=2}^{T} E\left(Z_{NT,s}^2\right) = \mathbb{V}_{NT} + o(1)$.

Now, we want to prove (II) by showing that $E(\sum_{s=2}^{T} Z_{NT,s}^2)^2 = \mathbb{V}_{NT}^2 + o(1)$. Noting that

$$\begin{aligned} E\left(\sum_{s=2}^{T} Z_{NT,s}^2\right)^2 &= \frac{1}{T^4N^2h^2} E\left(\sum_{s=2}^{T}\left[\sum_{r=1}^{s-1}\bar{k}_{sr}\phi_{sr}\right]^2\right)^2 \\ &= \frac{1}{T^4N^2h^2} E\left(\sum_{s=2}^{T}\sum_{r=1}^{s-1}\bar{k}_{sr}^2\phi_{sr}^2\right)^2 + \frac{1}{T^4N^2h^2} E\left(\sum_{s=2}^{T}\sum_{1\le r_1\neq r_2\le s-1}\bar{k}_{sr_1}\bar{k}_{sr_2}\phi_{sr_1}\phi_{sr_2}\right)^2 \\ &\quad + \frac{2}{T^4N^2h^2} E\left[\left(\sum_{s=2}^{T}\sum_{r=1}^{s-1}\bar{k}_{sr}^2\phi_{sr}^2\right)\sum_{s=2}^{T}\sum_{1\le r_1\neq r_2\le s-1}\bar{k}_{sr_1}\bar{k}_{sr_2}\phi_{sr_1}\phi_{sr_2}\right] \\ &\equiv b_{1NT} + b_{2NT} + b_{3NT}, \text{ say,} \end{aligned}$$

it suffices to show that (a) $b_{1NT} = \mathbb{V}_{NT}^2 + o_P(1)$ and (b) $b_{2NT} = o_P(1)$, because then $b_{3NT} \le 2\{b_{1NT}b_{2NT}\}^{1/2} = o_P(1)$ by Cauchy-Schwarz (CS) inequality. Note that $b_{1NT} = \frac{1}{T^4N^2h^2}\sum_{1\le r_1<s_1\le T, 1\le r_1<s_1\le T}\bar{k}_{s_1r_1}^2\bar{k}_{s_2r_2}^2 \times E(\phi_{s_1r_1}^2\phi_{s_2r_2}^2)$ and $\mathbb{V}_{NT}^2 = \frac{1}{T^4N^2h^2}\sum_{1\le r_1<s_1\le T, 1\le r_2<s_2\le T}\bar{k}_{s_1r_1}^2\bar{k}_{s_2r_2}^2 E(\phi_{s_1r_1}^2) E(\phi_{s_2r_2}^2)$. Let $\mathcal{S}_3 = \{r_1, s_1, r_2, s_2\}$. We consider two cases: (1) for each $t \in \mathcal{S}_3$, $|t-l| > T_0$ for all $l \in \mathcal{S}_3$ with $l \neq t$, and (2) all the other remaining cases. Let $\mathcal{S}_{3,1}$ and $\mathcal{S}_{3,2}$ denote the subsets of $\mathcal{S}_3$ corresponding to these two cases, respectively. For $l = 1, 2$, let $b_{1NT}(l)$ and $\mathbb{V}_{NT}^2(l)$ denote b_{1NT} and $\mathbb{V}_{NT}^2$ when the time indices are restricted to lie in $\mathcal{S}_{3,l}$, respectively. Note that $b_{1NT} = b_{1NT}(1) + b_{1NT}(2)$ and $\mathbb{V}_{NT}^2 = \mathbb{V}_{NT}^2(1) + \mathbb{V}_{NT}^2(2)$. In case (2), we have by Assumptions A.3(iii), (v) and A.4

$$\begin{aligned} b_{1NT}(2) &\le \max_{s<r}\left\|N^{-1}\phi_{sr}^2\right\|_2^2 \frac{1}{T^4h^2}\sum_{\substack{1\le r_1<s_1\le T, 1\le r_2<s_2\le T, \\ \text{case (2)}}}\bar{k}_{s_1r_1}^2\bar{k}_{s_2r_2}^2 = O\left(T_0T^{-1}\right) = o(1), \\ \mathbb{V}_{NT}^2(2) &\le \max_{s<r}\left[E\left(N^{-1}\phi_{sr}^2\right)\right]^2 \frac{1}{T^4h^2}\sum_{\substack{1\le r_1<s_1\le T, 1\le r_2<s_2\le T \\ \text{case (2)}}}\bar{k}_{s_1r_1}^2\bar{k}_{s_2r_2}^2 = O\left(T_0T^{-1}\right) = o(1), \end{aligned}$$

where we use the fact that there are at most T^3T_0 terms in the above displayed summations. In case (1), we consider six subcases: (1a) $r_1 < s_1 < r_2 < s_2$, (1b) $r_2 < s_2 < r_1 < s_1$, (1c) $r_1 < r_2 < s_1 < s_2$, (1d) $r_2 < r_1 < s_1 < s_2$, (1e) $r_1 < r_2 < s_2 < s_1$, and (1f) $r_2 < r_1 < s_2 < s_1$. We use $b_{1NT}(1, v)$ and $\mathbb{V}_{NT}^2(1, v)$ to denote $b_{1NT}(1)$ and $\mathbb{V}_{NT}^2(1)$, respectively, when the summation over the time indices are restricted to satisfy the conditions in subcase $(1v)$ for $v = a, b, c, d, e, f$. First, we study subcase (1a). By Lemma A.7, Assumptions A.3(iii), (v) and A.4

$$\begin{aligned} b_{1NT}(1,a) &= \frac{1}{T^4N^2h^2}\sum_{r_1<s_1<r_2<s_2,\ \mathcal{S}_{3,1}}\bar{k}_{s_1r_1}^2\bar{k}_{s_2r_2}^2 E\left(\phi_{s_1r_1}^2\phi_{s_2r_2}^2\right) \\ &= \frac{1}{T^4N^2h^2}\sum_{r_1<s_1<r_2<s_2,\ \mathcal{S}_{3,1}}\sum_{i_1,j_1,i_2,j_2}\bar{k}_{s_1r_1}^2\bar{k}_{s_2r_2}^2 E(f_{s_1r_1}^2 e_{i_1s_1}e_{i_1r_1}e_{j_1s_1}e_{j_1r_1}f_{s_2r_2}^2 e_{i_2s_2}e_{i_2r_2}e_{j_2s_2}e_{j_2r_2}) \\ &\le \frac{1}{T^4N^2h^2}\sum_{r_1<s_1<r_2<s_2,\ \mathcal{S}_{3,1}}\sum_{i_1,j_1,i_2,j_2}\bar{k}_{s_1r_1}^2\bar{k}_{s_2r_2}^2\{E\left(f_{s_1r_1}^2 e_{i_1s_1}e_{i_1r_1}e_{j_1s_1}e_{j_1r_1}\right) \\ &\quad \times E\left(f_{s_2r_2}^2 e_{i_2s_2}e_{i_2r_2}e_{j_2s_2}e_{j_2r_2}\right) + C\alpha(T_0)^{\delta/(1+\delta)}\} \\ &= \frac{1}{T^4N^2h^2}\sum_{r_1<s_1<r_2<s_2,\ \mathcal{S}_{3,1}}\sum_{i_1,j_1,i_2,j_2}\bar{k}_{s_1r_1}^2\bar{k}_{s_2r_2}^2 E\left(\phi_{s_1r_1}^2\right)E\left(\phi_{s_2r_2}^2\right) + O(N^2\alpha(T_0)^{\delta/(1+\delta)}) \\ &= \mathbb{V}_{NT}^2(1,a) + o(1), \end{aligned}$$

where $f_{sr} = F_s'\mathbb{S}F_r$, $\sum_{i_1,j_1,i_2,j_2}$ denotes $\sum_{i_1=1}^{N}\sum_{j_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j_2=1}^{N}$, and $\sum_{r_1<s_1<r_2<s_2,\ \mathcal{S}_{3,1}}$ indicates the summation is done over the four time indices satisfying the condition in case (1) (corresponding to $\mathcal{S}_{3,1}$).

By the same token, $b_{1NT}(1,b) = \mathbb{V}^2_{NT}(1,b) + o(1)$. Now, consider subcase (1c). For notational simplicity, we assume that $R = 1$ so that each term in $F_s'\mathbb{S}F_t$ is a scalar. [Otherwise, we need to utilize $F_s'\mathbb{S}F_t = \sum_{m=1}^R \sum_{n=1}^R s_{mn} F_{s,m} F_{s,n}$ as in the analysis of Part (I)]. By applying Lemma A.7 three times, we have

$$\begin{aligned}
b_{1NT}(1,c) &= \frac{1}{T^4N^2h^2} \sum_{r_1<r_2<s_1<s_2,\mathcal{S}_{3,1}} \bar{k}^2_{s_1r_1}\bar{k}^2_{s_2r_2} E\left(\phi^2_{s_1r_1}\phi^2_{s_2r_2}\right) \\
&= \frac{\mathbb{S}^4}{T^4N^2h^2} \sum_{r_1<r_2<s_1<s_2,\mathcal{S}_{3,1}} \sum_{i_1,j_1,i_2,j_2} \bar{k}^2_{s_1r_1}\bar{k}^2_{s_2r_2} E\left(F^2_{s_1}F^2_{s_2}F^2_{r_1}F^2_{r_2} e_{i_1s_1}e_{i_1r_1}e_{j_1s_1}e_{j_1r_1}e_{i_2s_2}e_{i_2r_2}e_{j_2s_2}e_{j_2r_2}\right) \\
&\leq \frac{\mathbb{S}^4}{T^4N^2h^2} \sum_{r_1<r_2<s_1<s_2,\mathcal{S}_{3,1}} \sum_{i_1,j_1,i_2,j_2} \bar{k}^2_{s_1r_1}\bar{k}^2_{s_2r_2} \{E\left(F^2_{r_1}F^2_{r_2} e_{i_1r_1}e_{j_1r_1}e_{i_2r_2}e_{j_2r_2}\right) \\
&\quad \times E\left(F^2_{s_1}F^2_{s_2} e_{i_1s_1}e_{j_1s_1}e_{i_2s_2}e_{j_2s_2}\right) + C\alpha(T_0)^{\delta/(1+\delta)}\} \\
&\leq \frac{\mathbb{S}^4}{T^4N^2h^2} \sum_{r_1<r_2<s_1<s_2,\mathcal{S}_{3,1}} \sum_{i_1,j_1,i_2,j_2} \bar{k}^2_{s_1r_1}\bar{k}^2_{s_2r_2} \{E\left(F^2_{r_1} e_{i_1r_1}e_{j_1r_1}\right) E\left(F^2_{r_2} e_{i_2r_2}e_{j_2r_2}\right) \\
&\quad \times E\left(F^2_{s_1} e_{i_1s_1}e_{j_1s_1}\right) E\left(F^2_{s_2} e_{i_2s_2}e_{j_2s_2}\right) + 2C\alpha(T_0)^{\delta/(1+\delta)}\} \\
&= \frac{\mathbb{S}^4}{T^4N^2h^2} \sum_{r_1<r_2<s_1<s_2,\mathcal{S}_{3,1}} \sum_{i_1,j_1,i_2,j_2} \bar{k}^2_{s_1r_1}\bar{k}^2_{s_2r_2} E\left(F^2_{r_1} e_{i_1r_1}e_{j_1r_1}\right) E\left(F^2_{r_2} e_{i_2r_2}e_{j_2r_2}\right) \\
&\quad \times E\left(F^2_{s_1} e_{i_1s_1}e_{j_1s_1}\right) E\left(F^2_{s_2} e_{i_2s_2}e_{j_2s_2}\right) + o(1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{V}^2_{NT}(1,c) &= \frac{1}{T^4N^2h^2} \sum_{r_1<r_2<s_1<s_2,\mathcal{S}_{3,1}} \bar{k}^2_{s_1r_1}\bar{k}^2_{s_2r_2} E\left(\phi^2_{s_1r_1}\right) E\left(\phi^2_{s_2r_2}\right) \\
&= \frac{\mathbb{S}^4}{T^4N^2h^2} \sum_{r_1<r_2<s_1<s_2,\mathcal{S}_{3,1}} \sum_{i_1,j_1,i_2,j_2} \bar{k}^2_{s_1r_1}\bar{k}^2_{s_2r_2} E\left(F^2_{r_1}F^2_{r_2} e_{i_1r_1}e_{j_1r_1}e_{i_2r_2}e_{j_2r_2}\right) \\
&\quad \times E\left(F^2_{s_1}F^2_{s_2} e_{i_1s_1}e_{j_1s_1}e_{i_2s_2}e_{j_2s_2}\right) \\
&\leq \frac{\mathbb{S}^4}{T^4N^2h^2} \sum_{r_1<r_2<s_1<s_2,\mathcal{S}_{3,1}} \sum_{i_1,j_1,i_2,j_2} \bar{k}^2_{s_1r_1}\bar{k}^2_{s_2r_2} \{E\left(F^2_{r_1} e_{i_1r_1}e_{j_1r_1}\right) E\left(F^2_{r_2} e_{i_2r_2}e_{j_2r_2}\right) \\
&\quad \times E\left(F^2_{s_1} e_{i_1s_1}e_{j_1s_1}\right) E\left(F^2_{s_2} e_{i_2s_2}e_{j_2s_2}\right) + C\alpha(T_0)^{\delta/(1+\delta)}\} \\
&= \frac{\mathbb{S}^4}{T^4N^2h^2} \sum_{r_1<r_2<s_1<s_2,\mathcal{S}_{3,1}} \sum_{i_1,j_1,i_2,j_2} \bar{k}^2_{s_1r_1}\bar{k}^2_{s_2r_2} E\left(F^2_{r_1} e_{i_1r_1}e_{j_1r_1}\right) E\left(F^2_{r_2} e_{i_2r_2}e_{j_2r_2}\right) \\
&\quad \times E\left(F^2_{s_1} e_{i_1s_1}e_{j_1s_1}\right) E\left(F^2_{s_2} e_{i_2s_2}e_{j_2s_2}\right) + o(1).
\end{aligned}$$

It follows that $b_{1NT}(1,c) = \mathbb{V}^2_{NT}(1,c) + o(1)$. Analogously, we can show that $b_{1NT}(1,v) = \mathbb{V}^2_{NT}(1,v) + o(1)$ for $v = d,e,f$. Consequently, we have $b_{1NT}(1) = \mathbb{V}^2_{NT}(1) + o(1)$ and $b_{1NT} = \mathbb{V}^2_{NT} + o(1)$. Using arguments as used in the analysis of b_{1NT} and Lemma A.7, we can also show that

$$\begin{aligned}
b_{2NT} &= \frac{1}{T^4N^2h^2} \sum_{s_1=2}^T \sum_{s_2=2}^T \sum_{1\leq r_1\neq r_2\leq s_1-1} \sum_{1\leq r_3\neq r_4\leq s_2-1} \bar{k}_{s_1r_1}\bar{k}_{s_1r_2}\bar{k}_{s_2r_3}\bar{k}_{s_2r_4} E\left(\phi_{s_1r_1}\phi_{s_1r_2}\phi_{s_2r_3}\phi_{s_2r_4}\right) \\
&= O\left(T^{-1}h^{-2} + N^2Th^2\alpha(T_0)^{\delta/(1+\delta)} + T^{-2}T_0^4 + T^{-2}T_0^3h^{-1} + T^{-2}T_0^2h^{-2}\right) = o(1).
\end{aligned}$$

It follows that $E(\sum_{s=2}^T Z^2_{NT,s})^2 = \mathbb{V}^2_{NT} + o(1)$ and $\mathrm{Var}(\sum_{s=2}^T Z^2_{NT,s}) = o(1)$. Then the second part of (A.2) follows by Chebyshev inequality. In addition, by straightforward moment calculations, we can show that

$M_{1,1}^{(2,2)} = o_P(1)$. It follows that $M_{1,1}^{(2)} \xrightarrow{d} N(0, \mathbb{V}_0)$. For $M_{1,1}^{(3)}$, by the matrix version of Cauchy-Schwarz inequality, Jensen inequality, and Lemma A.3(v), we have

$$\begin{aligned}
\left|M_{1,1}^{(3)}\right| &= \frac{h^{1/2}}{T^2 N^{1/2}} \left|\sum_{t=1}^{T} \operatorname{tr}\left[\left(HS_{Tt}^{-1}S_{Tt}^{-1}H' - \mathbb{S}\right)\sum_{i=1}^{N}\sum_{1\le s<r\le T} k_{h,st}k_{h,rt}F_rF_s'e_{is}e_{ir}\right]\right| \\
&\le \max_t \left\|HS_{Tt}^{-1}S_{Tt}^{-1}H' - \mathbb{S}\right\| \frac{h^{1/2}}{T^2N^{1/2}} \sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\sum_{1\le s<r\le T} k_{h,st}k_{h,rt}F_rF_s'e_{is}e_{ir}\right\| \\
&\le \max_t \left\|HS_{Tt}^{-1}S_{Tt}^{-1}H' - \mathbb{S}\right\| \left\{\frac{h}{T^3N}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\sum_{1\le s<r\le T} k_{h,st}k_{h,rt}F_rF_s'e_{is}e_{ir}\right\|^2\right\}^{1/2} \\
&= O_P\left((Th)^{-1/2}(\ln T)^{1/2} + N^{-1/2}\right) O_P(1) = o_P(1),
\end{aligned}$$

where we also use the fact that $E\left(\frac{h}{T^3N}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\sum_{1\le s<r\le T} k_{h,st}k_{h,rt}F_rF_s'e_{is}e_{ir}\right\|^2\right) = O(1)$ by using Lemma A.7 and arguments as used in the above study of b_{1NT}. Consequently, we have shown that $M_{1,1} - \mathbb{B}_{1NT} \xrightarrow{d} N(0, \mathbb{V}_0)$.

Next, we prove (ii). Using $a_{NT} = T^{-1/2}N^{-1/4}h^{-1/4}$, $\tilde{F}_s = H'F_s + (\tilde{F}_s - H'F_s)$, and Lemmas A.3(i), (v) and A.5(i), we have

$$\begin{aligned}
M_{1,2} &= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\|D_2(i,t)\|^2 = \frac{a_{NT}^2 h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|S_{Tt}^{-1}\frac{1}{T}\sum_{s=1}^{T}k_{h,st}\tilde{F}_sF_s'g_{is}\right\|^2 \\
&= \frac{1}{TN}\sum_{i=1}^{N}\sum_{t=1}^{T}\operatorname{tr}\left(S_{Tt}^{-1}\frac{1}{T}\sum_{s=1}^{T}k_{h,st}\tilde{F}_sF_s'g_{is}\frac{1}{T}\sum_{r=1}^{T}k_{h,rt}g_{ir}'F_r\tilde{F}_r'S_{Tt}^{-1}\right) \\
&= \frac{1}{TN}\sum_{i=1}^{N}\sum_{t=1}^{T}\operatorname{tr}\left(HS_{Tt}^{-1}S_{Tt}^{-1}H'\frac{1}{T}\sum_{s=1}^{T}k_{h,st}F_sF_s'g_{is}\frac{1}{T}\sum_{r=1}^{T}k_{h,rt}g_{ir}'F_rF_r'\right) + O_P\left(C_{NT}^{-2}\right) \\
&= \frac{1}{TN}\sum_{i=1}^{N}\sum_{t=1}^{T}\operatorname{tr}\left(\Sigma_F^{-1}Q_0'Q_0\Sigma_F^{-1}\Sigma_F g_i\left(\frac{t}{T}\right)g_i\left(\frac{t}{T}\right)'\Sigma_F\right) + o_P(1) \\
&= \frac{1}{TN}\sum_{i=1}^{N}\sum_{t=1}^{T}\operatorname{tr}\left(Q_0g_i\left(\frac{t}{T}\right)g_i\left(\frac{t}{T}\right)'Q_0'\right) + o_P(1) = \Pi_{1NT} + o_P(1).
\end{aligned}$$

Now, we prove (iii). For $M_{1,3}$, we apply Lemmas A.3(i)-(ii) and (iv) and triangle inequality to obtain

$$\begin{aligned}
M_{1,3} &= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\|D_3(i,t)\|^2 = \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|S_{Tt}^{-1}\frac{1}{T}\sum_{s=1}^{T}k_{h,st}\tilde{F}_s\left(\tilde{F}_s - H'F_s\right)'H^{-1}\lambda_{i0}\right\|^2 \\
&\le TN^{1/2}h^{1/2}\left\|H^{-1}\right\|\max_t\left\|S_{Tt}^{-1}\right\|^2\left\{\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{i0}\|^2\right\}\max_t\left\|\frac{1}{T}\sum_{s=1}^{T}k_{h,st}\tilde{F}_s\left(\tilde{F}_s - H'F_s\right)'\right\|^2 \\
&= TN^{1/2}h^{1/2}O_P(1)\left(O_P\left(T^{-1}\ln T + N^{-1}\right)^2 + o_P\left(a_{NT}^2\right)\right) = o_P(1).
\end{aligned}$$

By Lemma A.4(iv) and triangle inequality, we obtain

$$\begin{aligned}
M_{1,4} &= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|D_4(i,t)\right\|^2=\frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|S_{Tt}^{-1}\frac{1}{T}\sum_{s=1}^{T}\left(\tilde{F}_s-H'F_s\right)e_{is}k_{h,st}\right\|^2\\
&\leq TN^{1/2}h^{1/2}\max_t\left\|S_{Tt}^{-1}\right\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}\left(\tilde{F}_s-H'F_s\right)e_{is}k_{h,st}\right\|^2\\
&= TN^{1/2}h^{1/2}O_P(1)\left(O_P\left(N^{-3/2}+T^{-2}\right)+o_P\left(a_{NT}^2\right)\right)=o_P(1).
\end{aligned}$$

For $M_{1,5}$, we have

$$\begin{aligned}
|M_{1,5}| &= \frac{a_{NT}h^{1/2}}{T^2N^{1/2}}\left|\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}F_s'e_{is}HS_{Tt}^{-1}S_{Tt}^{-1}\sum_{r=1}^{T}k_{h,rt}\tilde{F}_rF_r'g_{ir}\right|\\
&\leq \max_{i,t}\left\|\frac{1}{T}\sum_{r=1}^{T}k_{h,rt}\tilde{F}_rF_r'g_{ir}\right\|\frac{h^{1/4}}{T^{3/2}N^{3/4}}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\sum_{s=1}^{T}F_s'e_{is}k_{h,st}HS_{Tt}^{-1}S_{Tt}^{-1}\right\|.
\end{aligned}$$

By Lemma A.4(i), $\max_{i,t}\left\|\frac{1}{T}\sum_{r=1}^{T}k_{h,rt}\tilde{F}_rF_r'g_{ir}\right\|=O_P(1)$. In addition,

$$\begin{aligned}
&\frac{h^{1/4}}{T^{3/2}N^{3/4}}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}F_s'e_{is}HS_{Tt}^{-1}S_{Tt}^{-1}H'H'^{-1}\right\|\\
\leq\ &\frac{h^{1/4}}{T^{3/2}N^{3/4}}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}F_s'e_{is}\Sigma_F^{-1}Q_0'Q_0\Sigma_F^{-1}\right\|\left\|H'^{-1}\right\|\\
&+\frac{h^{1/4}}{T^{3/2}N^{3/4}}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}F_s'e_{is}\left(HS_{Tt}^{-1}S_{Tt}^{-1}H'-\Sigma_F^{-1}Q_0'Q_0\Sigma_F^{-1}\right)\right\|\left\|H'^{-1}\right\|\\
\equiv\ &\left\{II_1+II_2\right\}\left\|H'^{-1}\right\|,\ \text{say.}
\end{aligned}$$

Noting that under Assumptions A.3(ii), (v) and A.4

$$\begin{aligned}
\frac{h}{T^2N}\sum_{t=1}^{T}E\left\|\sum_{i=1}^{N}\sum_{s=1}^{T}F_s'e_{is}k_{h,st}\right\|^2 &= \frac{h}{T^2N}\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{s=1}^{T}\sum_{j=1}^{N}k_{h,st}^2E\left(F_s'F_se_{is}e_{js}\right)\\
&\leq \max_s\left\|\frac{h}{T}\sum_{t=1}^{T}k_{h,st}^2\right\|\left\|\Sigma_F\right\|^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}|\tau_{ij,s}|=O(1),
\end{aligned}$$

we have $\frac{h^{1/2}}{TN^{1/2}}\sum_{i=1}^{N}\left\|\sum_{t=1}^{T}\sum_{s=1}^{T}F_s'e_{is}k_{h,st}\right\|\leq\left\{\frac{h}{T^2N}\sum_{i=1}^{N}\left\|\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}F_s'e_{is}\right\|^2\right\}^{1/2}=O_P(1)$ and $II_1=O_P(a_{NT})$. For II_2, we have by Lemmas A.4(ii) and A.3(v)

$$\begin{aligned}
II_2 &\leq \frac{h^{1/4}}{T^{3/2}N^{3/4}}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}F_s'e_{is}\left(HS_{Tt}^{-1}S_{Tt}^{-1}H'-\Sigma_F^{-1}Q_0'Q_0\Sigma_F^{-1}\right)\right\|\\
&\leq h^{1/4}T^{1/2}N^{1/4}\max_{i,t}\left\|\frac{1}{T}\sum_{s=1}^{T}k_{h,st}F_s'e_{is}\right\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|HS_{Tt}^{-1}S_{Tt}^{-1}-\Sigma_F^{-1}Q_0'Q_0\Sigma_F^{-1}\right\|\\
&= h^{1/4}T^{1/2}N^{1/4}O_P\left(T^{-1/2}h^{-1/2}\ln(NT)\right)O_P\left(T^{-1/2}h^{-1/2}(\ln T)^{1/2}+N^{-1/2}\right)\\
&= O_P\left(T^{-1/2}h^{-3/4}N^{1/4}\ln(NT)(\ln T)^{1/2}+N^{-1/4}h^{-1/4}\ln(NT)\right)=o_P(1).
\end{aligned}$$

It follows that $M_{1,5} = o_P(1)$.

For $M_{1,6}$, we have by Lemmas A.3(i)-(ii) and (iv),

$$\begin{aligned}
|M_{1,6}| &= \frac{h^{1/2}}{T^2N^{1/2}}\left|\sum_{i=1}^N\sum_{t=1}^T\sum_{s=1}^T F_s' e_{is}k_{h,st}HS_{Tt}^{-1}S_{Tt}^{-1}\sum_{r=1}^T k_{h,rt}\tilde{F}_r\left(\tilde{F}_r - H'F_r\right)' H^{-1}\lambda_{i0}\right| \\
&= \frac{h^{1/2}}{T^2N^{1/2}}\left|\sum_{t=1}^T \operatorname{tr}\left[\left(\sum_{i=1}^N\sum_{s=1}^T \lambda_{i0}F_s' e_{is}k_{h,st}\right)HS_{Tt}^{-1}S_{Tt}^{-1}\sum_{r=1}^T k_{h,rt}\tilde{F}_r\left(\tilde{F}_r - H'F_r\right)' H^{-1}\right]\right| \\
&\leq \max_t\left\|S_{Tt}^{-1}\right\|\left\|H^{-1}\right\|\|H\|\max_t\left\|\frac{1}{T}\sum_{r=1}^T k_{h,rt}\tilde{F}_r\left(\tilde{F}_r - H'F_r\right)'\right\| \\
&\quad\times\left\{\frac{h}{TN}\sum_{t=1}^T\left\|\sum_{i=1}^N\sum_{s=1}^T \lambda_{i0}F_s' e_{is}k_{h,st}\right\|^2\right\}^{1/2} \\
&= \left[O_P\left(T^{-1}\ln T + N^{-1}\right) + o_P(a_{NT})\right]O(T^{1/2}) = o_P(1),
\end{aligned}$$

where we also use the fact that under Assumptions A.3(ii), (v) and A.4

$$\begin{aligned}
\frac{h}{TN}\sum_{t=1}^T E\left\|\sum_{i=1}^N\sum_{s=1}^T \lambda_{i0}F_s' e_{is}k_{h,st}\right\|^2 &= \frac{h}{TN}\sum_{t=1}^T\sum_{i=1}^N\sum_{j=1}^N\sum_{s=1}^T k_{h,st}^2 E\left(F_s'F_s e_{is}e_{js}\right)\lambda_{j0}'\lambda_{i0} \\
&\leq CT\left(\max_s\frac{h}{T}\sum_{t=1}^T k_{h,st}^2\right)\frac{1}{NT}\sum_{i=1}^N\sum_{j=1}^N\sum_{s=1}^T|\tau_{ij,s}| \\
&= TO(1)O(1) = O(T).
\end{aligned}$$

For $M_{1,7}$, we apply (A.8) in the supplementary appendix to make the following decomposition

$$\begin{aligned}
M_{1,7} &= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^N\sum_{t=1}^T F_s' e_{is}k_{h,st}HS_{Tt}^{-1}\sum_{r=1}^T k_{h,rt}e_{ir}(\tilde{F}_r - H'F_r) \\
&= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^N\sum_{t=1}^T F_s' e_{is}k_{h,st}HS_{Tt}^{-1}\sum_{r=1}^T k_{h,rt}e_{ir}\left[A_1(r) + A_2(r) + A_3(r) + A_4(r)\right] \\
&\equiv M_{1,7}^{(1)} + M_{1,7}^{(2)} + M_{1,7}^{(3)} + M_{1,7}^{(4)},\ \text{say.}
\end{aligned}$$

By straightforward calculations, we can show that $M_{1,7}^{(l)} = o_P(1)$ for $l = 1, 2, 3, 4$. It follows that $M_{1,7} = o_P(1)$.

Finally, $M_{1,8} \leq \{M_{1,2}M_{1,3}\}^{1/2} = o_P(1)$, $M_{1,9} \leq \{M_{1,2}M_{1,4}\}^{1/2} = o_P(1)$, and $M_{1,10} \leq \{M_{1,3}M_{1,4}\}^{1/2} = o_P(1)$ by CS inequality and the fact that $M_{1,2} = O_P(1)$ and $M_{1,j} = o_P(1)$ for $j = 3, 4$. Consequently, $M_1 - \mathbb{B}_{1NT} - \Pi_{1NT} \xrightarrow{d} N(0, \mathbb{V}_0)$. ■

Proposition A.9 *Suppose that the conditions in Theorem 3.2 hold. Then* $M_2 - \mathbb{B}_{2NT} - \Pi_{2NT} = o_P(1)$ *under* $\mathbb{H}_1(a_{NT})$.

Proof. Using $X_{it} = F_t'\lambda_{it} + e_{it} = F_t'\lambda_{i0} + (e_{it} + a_{NT}F_t'g_{it}) = F_t'\lambda_{i0} + e_{it}^{\dagger}$ with $e_{it}^{\dagger} = e_{it} + a_{NT}F_t'g_{it}$ and by Bai (2003, p.165), we have

$$\begin{aligned}
\tilde{\lambda}_{i0} - H^{-1}\lambda_{i0} &= H'\frac{1}{T}\sum_{s=1}^T F_s e_{is}^{\dagger} + \frac{1}{T}\sum_{s=1}^T(\tilde{F}_s - HF_s)e_{is}^{\dagger} - \frac{1}{T}\tilde{F}'(\tilde{F}H^{-1} - F)\lambda_{i0} \\
&\equiv D_5(i) + D_6(i) - D_7(i),\ \text{say.} \qquad \text{(A.3)}
\end{aligned}$$

By (A.3), we make the following decomposition for M_2 :

$$\begin{aligned}
M_2 &= N^{-1/2}h^{1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left\|\tilde{\lambda}_{i0}-H^{-1}\lambda_{i0}\right\|^2 = TN^{-1/2}h^{1/2}\sum_{i=1}^{N}\left\|D_5(i)+D_6(i)-D_7(i)\right\|^2 \\
&= TN^{-1/2}h^{1/2}\sum_{i=1}^{N}\left[\|D_5(i)\|^2+\|D_6(i)\|^2+\|D_7(i)\|^2+2D_5(i)'D_6(i)-2D_5(i)'D_7(i)-2D_6(i)'D_7(i)\right] \\
&\equiv M_{2,1}+M_{2,2}+M_{2,3}+2M_{2,4}-2M_{2,5}-2M_{2,6},\ \text{say.}
\end{aligned}$$

We prove the proposition by showing that (i) $M_{2,1}-\mathbb{B}_{2NT}-\Pi_{2NT}=o_P(1)$ and (ii) $M_{2,j}=o_P(1)$ for $j=2,3,...,6.$

To prove (i), we use $e_{is}^{\dagger}=e_{is}+a_{NT}F_s'g_{is}$ and further make the following decomposition:

$$\begin{aligned}
M_{2,1} &= \frac{h^{1/2}}{TN^{1/2}}\sum_{i=1}^{N}\sum_{s=1}^{T}e_{is}^{\dagger}F_s'HH'\sum_{r=1}^{T}F_re_{ir}^{\dagger} \\
&= \frac{h^{1/2}}{TN^{1/2}}\sum_{i=1}^{N}\sum_{s=1}^{T}\sum_{r=1}^{T}F_s'HH'F_re_{is}e_{ir}+\frac{a_{NT}^2h^{1/2}}{TN^{1/2}}\sum_{i=1}^{N}\sum_{s=1}^{T}\sum_{r=1}^{T}F_s'F_sg_{is}'HH'F_rF_r'g_{ir} \\
&\quad+\frac{2a_{NT}h^{1/2}}{TN^{1/2}}\sum_{i=1}^{N}\sum_{s=1}^{T}\sum_{r=1}^{T}F_s'HH'F_rF_r'g_{ir}e_{is}\equiv M_{2,1}^{(1)}+M_{2,1}^{(2)}+2M_{2,1}^{(3)},\quad\text{say.}
\end{aligned}$$

For $M_{2,1}^{(1)}$ we make the following decomposition:

$$\begin{aligned}
M_{2,1}^{(1)} &= \frac{h^{1/2}}{TN^{1/2}}\sum_{i=1}^{N}\sum_{s=1}^{T}F_s'HH'F_se_{is}^2+2T^{-1}N^{-1/2}h^{1/2}\sum_{i=1}^{N}\sum_{1\le s<r\le T}F_s'Q_0^{-1}Q_0^{-1\prime}F_re_{is}e_{ir} \\
&\quad+2T^{-1}N^{-1/2}h^{1/2}\sum_{i=1}^{N}\sum_{1\le s<r\le T}F_s'(HH'-Q_0^{-1}Q_0^{-1\prime})F_re_{is}e_{ir} \\
&\equiv M_{2,1}^{(1,1)}+2M_{2,1}^{(1,2)}+2M_{2,1}^{(1,3)},\quad\text{say.}
\end{aligned}$$

Apparently, $M_{21}^{(1,1)}=\mathbb{B}_{2NT}$. Using the fact that $H-Q_0^{-1}=O_P(C_{NT}^{-1})$ under $\mathbb{H}_1\left(a_{NT}\right)$, we can show that $M_{2,1}^{(1,2)}=o_P(1)$ and $M_{2,1}^{(1,3)}=o_P(1)$ by arguments as used in the analyses of $M_{1,1}^{(2)}$ and $M_{1,1}^{(3)}$, respectively. By Lemmas A.2(vi) and A.5(ii), we have

$$\begin{aligned}
M_{2,1}^{(2)} &= \frac{a_{NT}^2h^{1/2}}{TN^{1/2}}\sum_{i=1}^{N}\sum_{s=1}^{T}F_s'F_sg_{is}'HH'\sum_{r=1}^{T}F_rF_r'g_{ir} \\
&= \operatorname{tr}\left[HH'\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{T}\sum_{r=1}^{T}F_rF_r'g_{ir}\right)\left(\frac{1}{T}\sum_{s=1}^{T}F_s'F_sg_{is}'\right)\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\operatorname{tr}\left[Q_0^{-1}(Q_0^{-1})'\Sigma_F\frac{1}{T}\sum_{r=1}^{T}g_{ir}\frac{1}{T}\sum_{s=1}^{T}g_{is}'\Sigma_F\right]+o_P\left(1\right)=\Pi_{2NT}+o_P\left(1\right).
\end{aligned}$$

For $M_{2,1}^{(3)}$, we have

$$\begin{aligned}\left|M_{2,1}^{(3)}\right| &= \frac{a_{NT}h^{1/2}}{TN^{1/2}}\left|\operatorname{tr}\left(HH'\sum_{r=1}^{T}F_rF_r'\sum_{i=1}^{N}\sum_{s=1}^{T}g_{ir}e_{is}F_s'\right)\right| \\ &\leq \|H\|^2\frac{h^{1/4}}{T^{3/2}N^{3/4}}\left\|\sum_{r=1}^{T}E\left(F_rF_r'\right)\sum_{i=1}^{N}\sum_{s=1}^{T}g_{ir}e_{is}F_s'\right\| \\ &\quad+\|H\|^2\frac{h^{1/4}}{T^{3/2}N^{3/4}}\left\|\sum_{r=1}^{T}\left[F_rF_r'-E\left(F_rF_r'\right)\right]\sum_{i=1}^{N}\sum_{s=1}^{T}g_{ir}e_{is}F_s'\right\| \equiv M_{2,1}^{(3,1)}+M_{2,1}^{(3,2)}.\end{aligned}$$

Noting that under Assumptions A.1(i), A.3(ii), (v) and A.5(i),

$$\begin{aligned}E\left(\frac{1}{T^{3/2}N^{3/4}}\left\|\sum_{r=1}^{T}\Sigma_F\sum_{i=1}^{N}\sum_{s=1}^{T}g_{ir}e_{is}F_s'\right\|\right)^2 &= \frac{1}{T^3N^{3/2}}\sum_{i=1}^{N}\sum_{r=1}^{T}\sum_{s=1}^{T}\sum_{j=1}^{N}\sum_{r_1=1}^{T}\operatorname{tr}(\Sigma_Fg_{ir}\tau_{ij,s}g_{jr_1}'\Sigma_F) \\ &\leq C\frac{1}{TN^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}|\tau_{ij,s}| = O(N^{-1/2}),\end{aligned}$$

we have $M_{2,1}^{(3,1)} = \|H\|^2\frac{h^{1/4}}{T^{3/2}N^{3/4}}\left\|\sum_{r=1}^{T}\Sigma_F\sum_{i=1}^{N}\sum_{s=1}^{T}g_{ir}e_{is}F_s'\right\| = h^{1/4}O_P(N^{-1/4}) = o_P(1)$. Similarly, noting that $E\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\sum_{s=1}^{T}e_{is}F_s'\right\|\right)^2 = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}\sum_{r=1}^{T}E\left(F_s'F_re_{jr}e_{is}\right) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}\tau_{ij,s}$ $= O(T/N)$, we have

$$\begin{aligned}M_{2,1}^{(3,2)} &= \|H\|^2\frac{h^{1/4}}{T^{3/2}N^{3/4}}\left\|\sum_{i=1}^{N}\sum_{r=1}^{T}\left[F_rF_r'-E\left(F_rF_r'\right)\right]g_{ir}\sum_{s=1}^{T}e_{is}F_s'\right\| \\ &\leq \left\{\max_i\left\|\frac{1}{T}\sum_{r=1}^{T}\left[F_rF_r'-E\left(F_rF_r'\right)\right]g_{ir}\right\|\right\}\|H\|^2\frac{h^{1/4}}{T^{1/2}N^{3/4}}\sum_{i=1}^{N}\left\|\sum_{s=1}^{T}e_{is}F_s'\right\| \\ &= O_P(T^{-1/2}\ln N)O_P(h^{1/4}N^{-1/4}) = o_P(1).\end{aligned}$$

Thus $M_{2,1} = \mathbb{B}_{2NT}+\Pi_{2NT}+o_P(1)$.

Now we prove (ii). By Lemma A.4(iii) and Lemma A.2 (iii)-(vi),

$$\begin{aligned}M_{2,2} &= TN^{-1/2}h^{1/2}\sum_{i=1}^{N}\|D_6(i)\|^2 = TN^{1/2}h^{1/2}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s-HF_s)e_{is}^{\dagger}\right\|^2 \\ &= TN^{1/2}h^{1/2}O_P(C_{NT}^{-4}) = o_P(1) \text{ and} \\ M_{2,3} &= TN^{-1/2}h^{1/2}\sum_{i=1}^{N}\|D_7(i)\|^2 \leq TN^{1/2}h^{1/2}\left\|\frac{1}{T}\tilde{F}'(\tilde{F}H^{-1}-F)\right\|^2\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{i0}\|^2 \\ &= TN^{1/2}h^{1/2}O_P(C_{NT}^{-4}) = o_P(1).\end{aligned}$$

By CS inequality $M_{2,6} \leq \{M_{2,2}M_{2,3}\}^{1/2} = o_P(1)$. For $M_{2,4}$, we apply Lemma A.4(iii) to obtain

$$\begin{aligned}|M_{2,4}| &= TN^{-1/2}h^{1/2}\left|\sum_{i=1}^{N}D_5(i)'D_6(i)\right| = TN^{-1/2}h^{1/2}\left|\sum_{i=1}^{N}\frac{1}{T}\sum_{s=1}^{T}e_{is}^{\dagger}F_s'H\frac{1}{T}\sum_{t=1}^{T}(\tilde{F}_t-HF_t)e_{it}^{\dagger}\right| \\ &\leq TN^{1/2}h^{1/2}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}e_{is}^{\dagger}F_s'\right\|^2\right\}^{1/2}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{t=1}^{T}(\tilde{F}_t-HF_t)e_{it}^{\dagger}\right\|^2\right\}^{1/2} \\ &= TN^{1/2}h^{1/2}O_P(T^{-1/2})O_P(C_{NT}^{-2}) = o_P(1),\end{aligned}$$

where we use the fact that $\frac{1}{N}\sum_{i=1}^{N}E||\frac{1}{T}\sum_{s=1}^{T}e_{is}^{\dagger}F_s'||^2 \leq \frac{2}{N}\sum_{i=1}^{N}E||\frac{1}{T}\sum_{s=1}^{T}e_{is}F_s'||^2+\frac{a_{NT}^2}{N}\sum_{i=1}^{N}E||\frac{1}{T}\sum_{s=1}^{T} g_{is}'F_sF_s'||^2 = O(T^{-1}+a_{NT}^2)=O(T^{-1})$. Now,

$$\begin{aligned}
M_{2,5} &= TN^{-1/2}h^{1/2}\left|\sum_{i=1}^{N}D_5(i)'D_7(i)\right| = TN^{-1/2}h^{1/2}\left|\sum_{i=1}^{N}\frac{1}{T}\sum_{s=1}^{T}e_{is}F_s'H\frac{1}{T}\tilde{F}'(\tilde{F}H^{-1}-F)\lambda_{i0}\right| \\
&= T^{-1}N^{-1/2}h^{1/2}\left|\text{tr}\left(H\tilde{F}'(\tilde{F}H^{-1}-F)\sum_{i=1}^{N}\sum_{s=1}^{T}\lambda_{i0}F_s'e_{is}\right)\right| \\
&\leq T^{1/2}h^{1/2}\left\|H\right\|\left\{T^{-1}\left\|\tilde{F}'(\tilde{F}H^{-1}-F)\right\|\right\}\left\|N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{s=1}^{T}\lambda_{i0}F_s'e_{is}\right\| \\
&= T^{1/2}h^{1/2}O_P(C_{NT}^{-2}) = o_P(1).
\end{aligned}$$

Thus we have shown that $M_{2,j}=o_P(1)$ for $j=2,3,...,6$ and the second part of the lemma follows. ■

Proposition A.10 *Suppose that the conditions in Theorem 3.2 hold. Then* $M_3-\mathbb{B}_{3NT}-\Pi_{3NT}=o_P(1)$ *under* $\mathbb{H}_1(a_{NT})$.

Proof. By (A.1) and (A.3), we can write M_3 as follows:

$$\begin{aligned}
M_3 &= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\left(\hat{\lambda}_{it}-H^{-1}\lambda_{i0}\right)'\left(\tilde{\lambda}_i-H^{-1}\lambda_{i0}\right) \\
&= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[D_1(i,t)+D_2(i,t)-D_3(i,t)+D_4(i,t)\right]'\left[D_5(i)+D_6(i)-D_7(i)\right] \\
&= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}[D_1(i,t)'D_5(i)+D_1(i,t)'D_6(i)-D_1(i,t)'D_7(i)+D_2(i,t)'D_5(i) \\
&+D_2(i,t)'D_6(i)-D_2(i,t)'D_7(i)-D_3(i,t)'D_5(i)-D_3(i,t)'D_6(i)+D_3(i,t)'D_7(i) \\
&+D_4(i,t)'D_5(i)+D_4(i,t)'D_6(i)-D_4(i,t)'D_7(i)] \\
&\equiv \sum_{i=1}^{12}M_{3,i}, \quad \text{say.}
\end{aligned}$$

We prove the proposition by showing that (i) $M_{3,1}=\mathbb{B}_{3NT}+o_P(1)$, (ii) $M_{3,4}=\Pi_{3NT}+o_P(1)$ and (iii) $M_{3,j}=o_P(1)$ for $j=2,3,5,6,...,12$.

First, we show (i). We decompose $M_{3,1}$ as follows:

$$\begin{aligned}
M_{3,1} &= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}D_1(i,t)'D_5(i) = \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}k_{h,st}F_s'HS_{Tt}^{-1}H'F_re_{ir}^{\dagger}e_{is} \\
&= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}k_{h,st}F_s'HS_{Tt}^{-1}H'F_re_{ir}e_{is}+\frac{a_{NT}h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}k_{h,st}F_s'HS_{Tt}^{-1}H'F_rF_r'g_{ir}e_{is} \\
&= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}F_s'HS_{Tt}^{-1}H'F_se_{is}^2 + \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{1\leq s\neq r\leq T}k_{h,st}F_s'HS_{Tt}^{-1}H'F_re_{ir}e_{is} \\
&+\frac{a_{NT}h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}k_{h,st}F_s'HS_{Tt}^{-1}H'F_rF_r'g_{ir}e_{is} \\
&\equiv M_{3,1}^{(1)}+M_{3,1}^{(2)}+M_{3,1}^{(3)}, \text{ say.}
\end{aligned}$$

Apparently $M_{3,1}^{(1)} = \mathbb{B}_{3NT}$. Following the analysis of $M_{1,1}$, we can readily show that $M_{3,1}^{(2)} = O_P(h^{1/2})$ and $M_{3,1}^{(2)} = O_P(T^{-1/2}N^{1/4}h^{-1/4}) = o_P(1)$. It follows that $M_{3,1} = \mathbb{B}_{3NT} + o_P(1)$.

Next, we show (ii). Using $e_{ir}^{\dagger} = e_{ir} + a_{NT}F_r' g_{ir}$ we decompose $M_{3,4}$ as follows:

$$\begin{aligned} M_{3,4} &= \frac{h^{1/2}}{N^{1/2}} \sum_{i=1}^{N}\sum_{t=1}^{T} D_2(i,t)' D_5(i) = \frac{a_{NT}h^{1/2}}{T^2N^{1/2}} \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} k_{h,st} g_{is}' F_s \tilde{F}_s' S_{Tt}^{-1} H' \sum_{r=1}^{T} F_r e_{ir}^{\dagger} \\ &= \frac{a_{NT}h^{1/2}}{T^2N^{1/2}} \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} k_{h,st} g_{is}' F_s \tilde{F}_s' S_{Tt}^{-1} H' \sum_{r=1}^{T} F_r e_{ir} \\ &\quad + \frac{a_{NT}^2 h^{1/2}}{T^2N^{1/2}} \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} k_{h,st} g_{is}' F_s \tilde{F}_s' S_{Tt}^{-1} H' \sum_{r=1}^{T} F_r F_r' g_{ir} \equiv M_{3,4}^{(1)} + M_{3,4}^{(2)}, \text{ say.} \end{aligned}$$

For $M_{3,4}^{(1)}$, by Lemmas A.2(iv) and A.3(iv) we have

$$\begin{aligned} \left| M_{3,4}^{(1)} \right| &\leq \frac{a_{NT}h^{1/2}}{T^2N^{1/2}} \left| \sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{s=1}^{T} k_{h,st} g_{is}' F_s F_s' H S_{Tt}^{-1} H' \sum_{r=1}^{T} F_r e_{ir} \right| + o_P(1) \\ &= \frac{a_{NT}h^{1/2}}{T^2N^{1/2}} \left| \sum_{t=1}^{T} \operatorname{tr}\left(H S_{Tt}^{-1} H' \sum_{i=1}^{N}\sum_{s=1}^{T}\sum_{r=1}^{T} F_r e_{ir} k_{h,st} g_{is}' F_s F_s' \right) \right| + o_P(1) \\ &= \frac{h^{1/4}}{T^{5/2}N^{3/4}} \|H\|^2 \max_t \left\| S_{Tt}^{-1} \right\| \sum_{t=1}^{T} \left\| \sum_{i=1}^{N}\sum_{s=1}^{T} k_{h,st} g_{is}' F_s F_s' \sum_{r=1}^{T} F_r e_{ir} \right\| + o_P(1) \\ &= \frac{h^{1/4}}{T^{5/2}N^{3/4}} O_P(N^{1/2}T^{5/2}h^{-1/2}) + o_P(1) = o_P(1). \end{aligned}$$

Noting that $H S_{Tt}^{-1} H' = Q_0^{-1} \left((Q_0^{-1})' \Sigma_F Q_0^{-1} \right)^{-1} (Q_0^{-1})' + o_P(1) = \Sigma_F^{-1} + o_P(1)$ uniformly in t by Lemmas A.2(vi) and A.3(iv), we have by Lemmas A.2(vi) and A.5(i)-(ii)

$$\begin{aligned} M_{3,4}^{(2)} &= \frac{1}{T^3 N} \sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T} k_{h,st} g_{is}' F_s \tilde{F}_s' S_{Tt}^{-1} H' \sum_{r=1}^{T} F_r F_r' g_{ir} \\ &= \frac{1}{N} \sum_{i=1}^{N} \frac{1}{T^2} \sum_{t=1}^{T}\sum_{s=1}^{T} k_{h,st} g_{is}' F_s F_s' H S_{Tt}^{-1} H' \frac{1}{T} \sum_{r=1}^{T} F_r F_r' g_{ir} + o_P(1) \\ &= \frac{1}{NT} \sum_{i=1}^{N}\sum_{t=1}^{T} \operatorname{tr}\left[H S_{Tt}^{-1} H' \left(\frac{1}{T} \sum_{r=1}^{T} F_r F_r' g_{ir} \right) \left(\frac{1}{T} \sum_{s=1}^{T} k_{h,st} g_{is}' F_s F_s' \right) \right] \\ &= \frac{1}{N} \sum_{i=1}^{N} \operatorname{tr}\left[\Sigma_F^{-1} \Sigma_F \frac{1}{T} \sum_{r=1}^{T} g_i\left(\frac{r}{T}\right) \frac{1}{T} \sum_{t=1}^{T} g_i\left(\frac{t}{T}\right)' \Sigma_F \right] + o_P(1) = \Pi_{3NT} + o_P(1). \end{aligned}$$

It follows that $M_{3,4} = \Pi_{3NT} + o_P(1)$.

Now, we prove (iii). We first consider $M_{3,2}$ and $M_{3,3}$. Note that by Lemmas A.2(vi) and A.3(iv),

$$\begin{aligned}
&\frac{h^{1/2}}{T^2}\sum_{i=1}^{N}\left\|\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}F_s'HS_{Tt}^{-1}e_{is}\right\|^2 \\
\leq\ & \frac{2h^{1/2}}{T^2}\sum_{i=1}^{N}\left\|\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}F_s'\Sigma_F^{-1}Q_0'e_{is}\right\|^2+\frac{2h^{1/2}}{T^2}\sum_{i=1}^{N}\left\|\sum_{t=1}^{T}\left(\sum_{s=1}^{T}k_{h,st}F_s'e_{is}\left(HS_{Tt}^{-1}-\Sigma_F^{-1}Q_0'\right)\right)\right\|^2 \\
\leq\ & \frac{2h^{1/2}}{T^2}\sum_{i=1}^{N}\left\|\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}F_s'\Sigma_F^{-1}Q_0'e_{is}\right\|^2 \\
&+2h^{1/2}N\max_{i,t}\left\|\frac{1}{T}\sum_{s=1}^{T}k_{h,st}F_s'e_{is}\right\|^2\left\|\sum_{t=1}^{T}\left\|HS_{Tt}^{-1}-\Sigma_F^{-1}Q_0'\right\|\right\|^2 \\
=\ & O_P(NTh^{1/2})+h^{1/2}NO_P(T^{-1}h^{-1}\ln(NT))O_P(T^2((Th)^{-1}\ln T+N^{-1}))=O_P(NTh^{1/2}).
\end{aligned}$$

By analogous analysis as used in the study of $M_{1,1}^{(2)}$ and $M_{1,1}^{(3)}$ and Lemma A.4(ii), we have

$$\begin{aligned}
M_{3,2} &= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}D_1(i,t)'D_6(i)=\frac{h^{1/2}}{TN^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}F_s'HS_{Tt}^{-1}e_{is}\frac{1}{T}\sum_{r=1}^{T}(\tilde{F}_r-HF_r)e_{ir}^{\dagger} \\
&\leq \left\{\frac{h}{T^2}\sum_{i=1}^{N}\left\|\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}F_s'HS_{Tt}^{-1}e_{is}\right\|^2\right\}^{1/2}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{r=1}^{T}(\tilde{F}_r-HF_r)e_{ir}^{\dagger}\right\|^2\right\}^{1/2} \\
&= O_P(N^{1/2}T^{1/2}h^{1/2})O_P(C_{NT}^{-2})=o_P(1).
\end{aligned}$$

Similarly, noting that

$$\begin{aligned}
&\frac{h^{1/2}}{TN^{1/2}}\left\|\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}\lambda_{i0}F_s'HS_{Tt}^{-1}e_{is}\right\| \\
\leq\ & \frac{h^{1/2}}{TN^{1/2}}\left\|\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}\lambda_{i0}F_s'\Sigma_F^{-1}Q_0'e_{is}\right\|+\frac{h^{1/2}}{TN^{1/2}}\left\|\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}\lambda_{i0}F_s'\left(HS_{Tt}^{-1}-\Sigma_F^{-1}Q_0'\right)e_{is}\right\| \\
=\ & O_P(T^{1/2}h^{1/2}),
\end{aligned}$$

we have by Lemma A.2(iii)

$$\begin{aligned}
M_{3,3} &= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}D_1(i,t)'D_7(i)=\frac{h^{1/2}}{TN^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}F_s'HS_{Tt}^{-1}e_{is}\frac{1}{T}\tilde{F}'\left[\tilde{F}H^{-1}-F\right]\lambda_{i0} \\
&= \frac{h^{1/2}}{TN^{1/2}}\mathrm{tr}\left(\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}\lambda_{i0}F_s'HS_{Tt}^{-1}e_{is}\frac{1}{T}\tilde{F}'\left[\tilde{F}H^{-1}-F\right]\right) \\
&\leq \left\|\frac{1}{T}\tilde{F}'\left[\tilde{F}H^{-1}-F\right]\right\|\frac{h^{1/2}}{TN^{1/2}}\left\|\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{s=1}^{T}k_{h,st}\lambda_{i0}F_s'HS_{Tt}^{-1}e_{is}\right\| \\
&= O_P\left(C_{NT}^{-2}+a_{NT}\right)O_P(T^{1/2}h^{1/2})=o_P(1).
\end{aligned}$$

For $M_{3,5}$, $M_{3,6}$, $M_{3,8}$, $M_{3,9}$, $M_{3,11}$, and $M_{3,12}$, we apply CS inequality and the fact that $M_{1,2}=O_P(1)$, $M_{1,l}=o_P(1)$ for $l=3,4$, and $M_{2,j}=o_P(1)$ for $j=2,3$ to obtain

$$\begin{aligned}
|M_{3,5}| &\leq \{M_{1,2}M_{2,2}\}^{1/2}=o_P(1),\ |M_{3,6}|\leq\{M_{1,2}M_{2,3}\}^{1/2}=o_P(1),\ |M_{3,8}|\leq\{M_{1,3}M_{2,2}\}^{1/2}=o_P(1), \\
|M_{3,9}| &\leq \{M_{1,3}M_{2,3}\}^{1/2}=o_P(1),\ |M_{3,11}|\leq\{M_{1,4}M_{2,2}\}^{1/2}=o_P(1),\ |M_{3,12}|\leq\{M_{1,4}M_{2,3}\}^{1/2}=o_P(1).
\end{aligned}$$

For $M_{3,7}$, we have

$$
\begin{aligned}
|M_{3,7}| &= \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}D_3(i,t)'D_5(i) = \frac{a_{NT}h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}\lambda_{i0}'H'^{-1}\left(\tilde{F}_s - H'F_s\right)\tilde{F}_s'S_{Tt}^{-1}H'\sum_{r=1}^{T}F_re_{ir}^{\dagger} \\
&= \frac{h^{1/4}}{T^{5/2}N^{3/4}}\left|\sum_{t=1}^{T}\sum_{s=1}^{T}\operatorname{tr}\left(k_{h,st}H'^{-1}\left(\tilde{F}_s - H'F_s\right)\tilde{F}_s'S_{Tt}^{-1}H'\sum_{i=1}^{N}\sum_{r=1}^{T}F_re_{ir}^{\dagger}\lambda_{i0}'\right)\right| \\
&= \frac{h^{1/4}}{T^{5/2}N^{3/4}}\left|\sum_{t=1}^{T}\sum_{s=1}^{T}\operatorname{tr}\left(k_{h,st}H'^{-1}\left(\tilde{F}_s - H'F_s\right)\tilde{F}_s'S_{Tt}^{-1}H'\sum_{i=1}^{N}\sum_{r=1}^{T}F_re_{ir}^{\dagger}\lambda_{i0}'\right)\right| \\
&\leq N^{-1/4}h^{1/4}\left\|H^{-1}\right\|\left\{\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}k_{h,st}\left(\tilde{F}_s - H'F_s\right)\tilde{F}_s'\right\|^2\right\}^{1/2}\left\{\frac{1}{T^2N}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\sum_{r=1}^{T}F_re_{ir}^{\dagger}\lambda_{i0}'\right\|^2\right\}^{1/2} \\
&= N^{-1/4}h^{1/4}O_P\left(C_{NT}^{-2}\right)O_P\left(1+N^{1/2}T^{1/2}a_{NT}\right) = o_P(1),
\end{aligned}
$$

Similarly, we can show that $M_{3,10} = \frac{h^{1/2}}{N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}D_4(i,t)'D_5(i) = \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}e_{is}(\tilde{F}_s - H'F_s)'\ S_{Tt}^{-1}H'\frac{1}{T}\sum_{r=1}^{T}F_re_{ir}^{\dagger} = o_P(1)$. Consequently, we have $M_3 = \mathbb{B}_{3NT} + \Pi_{3NT} + o_P(1)$. ■

Proposition A.11 *Suppose that the conditions in Theorem 3.2 hold. Then* $\hat{\mathbb{B}}_{NT} - \mathbb{B}_{NT} = o_P(1)$ *under* $\mathbb{H}_1(a_{NT})$.

Proof. Let $L_{st} = \left(k_{h,st}S_{Tt}^{-1} - \mathbb{I}_R\right)\left(k_{h,st}S_{Tt}^{-1} - \mathbb{I}_R\right)$. Using $\tilde{e}_{is}^2 - e_{is}^2 = \left(\tilde{e}_{is} - e_{is}\right)^2 + 2\left(\tilde{e}_{is} - e_{is}\right)e_{is}$, we have

$$
\begin{aligned}
\hat{\mathbb{B}}_{NT} - \mathbb{B}_{NT} &= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\operatorname{tr}\left[L_{st}(\tilde{F}_s\tilde{F}_s'\tilde{e}_{is}^2 - H'F_sF_s'He_{is}^2)\right] \\
&= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\{\operatorname{tr}(L_{st}\tilde{F}_s\tilde{F}_s')\left(\tilde{e}_{is} - e_{is}\right)^2 + 2\operatorname{tr}(L_{st}\tilde{F}_s\tilde{F}_s')\left(\tilde{e}_{is} - e_{is}\right)e_{is} \\
&\quad +\operatorname{tr}[L_{st}(\tilde{F}_s\tilde{F}_s' - H'F_sF_s'H)]e_{is}^2\} \\
&\equiv B_1 + 2B_2 + B_3,\ \text{say.}
\end{aligned}
$$

It suffices to show that (i1) $B_1 = o_P(1)$, (i2) $B_2 = o_P(1)$, and (i3) $B_3 = o_P(1)$.

We first show (i1). We make the following decomposition:

$$
\begin{aligned}
e_{is} - \tilde{e}_{is} &= \tilde{\lambda}_{i0}'\tilde{F}_s - \lambda_{is}'F_s = \tilde{\lambda}_{i0}'\tilde{F}_s - \lambda_{i0}'H'^{-1}H'F_s - a_{NT}F_t'g_{is} \\
&= (\tilde{\lambda}_{i0} - H^{-1}\lambda_{i0})'\tilde{F}_s + \lambda_{i0}'H'^{-1}(\tilde{F}_s - H'F_s) - a_{NT}F_s'g_{is} \equiv d_{1is} + d_{2is} - d_{3is},\ \text{say.} \qquad \text{(A.4)}
\end{aligned}
$$

By CS inequality, $B_1 \leq \frac{3h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\operatorname{tr}\left(L_{st}\tilde{F}_s\tilde{F}_s'\right)\left(d_{1is}^2 + d_{2is}^2 + d_{3is}^2\right) \equiv 3B_{1,1} + 3B_{1,2} + 3B_{1,3}$, say. By Lemmas A.6(i) and (iv),

$$
\begin{aligned}
B_{1,1} &= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{F}_s'L_{st}\tilde{F}_s\tilde{F}_s'(\tilde{\lambda}_i - H^{-1}\lambda_{i0})(\tilde{\lambda}_i - H^{-1}\lambda_{i0})'\tilde{F}_s \\
&\leq N^{1/2}h^{1/2}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\tilde{\lambda}_i - H^{-1}\lambda_{i0}\right\|^2\right\}\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{F}_s'L_{st}\tilde{F}_s\tilde{F}_s'\tilde{F}_s \\
&= N^{1/2}h^{1/2}O_P(C_{NT}^{-2})O_P(h^{-1}) = o_P(1).
\end{aligned}
$$

Noting that $L_{st} \le k_{h,st}^2 S_{Tt}^{-1} S_{Tt}^{-1} + \mathbb{I}_R$, by Lemmas A.3(iv) and A.6(iii) and (v)

$$\begin{aligned}
B_{1,2} &= \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}\left[L_{st} \tilde{F}_s (\tilde{F}_s - H' F_s)' H^{-1} \lambda_{i0} \lambda_{i0}' H'^{-1} (\tilde{F}_s - H' F_s) \tilde{F}_s' \right] \\
&\le \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}\left[\left(k_{h,st}^2 S_{Tt}^{-1} S_{Tt}^{-1} + \mathbb{I}_R \right) \tilde{F}_s (\tilde{F}_s - H' F_s)' H^{-1} \lambda_{i0} \lambda_{i0}' H'^{-1} (\tilde{F}_s - H' F_s) \tilde{F}_s' \right] \\
&\le N^{1/2} h^{1/2} \left\| H^{-1} \right\|^2 c_{1NT} \left\{ \frac{1}{N} \sum_{i=1}^{N} \| \lambda_{i0} \|^2 \right\} \frac{1}{T} \sum_{s=1}^{T} \left\| \tilde{F}_s (\tilde{F}_s - H' F_s)' \right\|^2 \\
&= N^{1/2} h^{1/2} O_P\left(h^{-1}\right) O(1) O_P(C_{NT}^{-2} + T^{-1} N^{-2}) = o_P(1),
\end{aligned}$$

and

$$\begin{aligned}
B_{1,3} &= \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \tilde{F}_s' L_{st} \tilde{F}_s d_{3is}^2 = \frac{a_{NT}^2 h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \tilde{F}_s' L_{st} \tilde{F}_s \left(F_s' g_{is} \right)^2 \\
&\le \frac{1}{T^3 N} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}\left[\left(k_{h,st}^2 S_{Tt}^{-1} S_{Tt}^{-1} + \mathbb{I}_R \right) \tilde{F}_s \tilde{F}_s' \left(F_s' g_{is} \right)^2 \right] \\
&\le c_{1NT} \frac{1}{T^2 N} \sum_{i=1}^{N} \sum_{s=1}^{T} \left\| \tilde{F}_s \right\|^2 \| F_s \|^2 \| g_{is} \|^2 \le \frac{\bar{c}_g^2 c_{1NT}}{T} \left\{ \frac{1}{T} \sum_{s=1}^{T} \left\| \tilde{F}_s \right\|^4 \frac{1}{T} \sum_{s=1}^{T} \| F_s \|^4 \right\}^{1/2} \\
&= O_P\left(T^{-1} h^{-1} \right) O_P(1) = o_P(1),
\end{aligned}$$

where $c_{1NT} \equiv \max_t \left\| S_{Tt}^{-1} \right\|^2 \max_s \frac{1}{T} \sum_{t=1}^{T} k_{h,st}^2 + 1 = O_P\left(h^{-1}\right)$.

Next, we show (i2). Using (A.4), we decompose B_2 as follows

$$\begin{aligned}
B_2 &= \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') \left(\tilde{e}_{is} - e_{is} \right) e_{is} \\
&= \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') \left(-d_{1is} - d_{2is} + d_{3is} \right) e_{is} \equiv -B_{2,1} - B_{2,2} + B_{2,3}, \text{ say.}
\end{aligned}$$

By (A.3), we further decompose $B_{2,1}$:

$$\begin{aligned}
B_{2,1} &= \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') \tilde{F}_s' \left(\tilde{\lambda}_i - H^{-1} \lambda_{i0} \right) e_{is} \\
&= \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') \tilde{F}_s' \left[D_5(i) + D_6(i) - D_7(i) \right] e_{is} \equiv B_{2,1}^{(1)} + B_{2,1}^{(2)} - B_{2,1}^{(3)}, \text{ say.}
\end{aligned}$$

For $B_{2,1}^{(1)}$, we have

$$\begin{aligned}
B_{2,1}^{(1)} &= \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') \tilde{F}_s' H' \left(\frac{1}{T} \sum_{i=1}^{N} \sum_{r=1}^{T} F_r e_{ir}^{\dagger} e_{is} \right) \\
&\le \| H \| \left\{ \frac{h}{T^3} \sum_{s=1}^{T} \left\| \sum_{t=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') \tilde{F}_s' \right\|^2 \right\}^{1/2} \left\{ \frac{1}{NT} \sum_{s=1}^{T} \left\| \frac{1}{T} \sum_{i=1}^{N} \sum_{r=1}^{T} F_r e_{ir}^{\dagger} e_{is} \right\|^2 \right\}^{1/2}.
\end{aligned}$$

Using $L_{st} \leq k_{h,st}^2 S_{Tt}^{-1} S_{Tt}^{-1} + \mathbb{I}_R$ and Lemma A.6(ii),

$$\begin{aligned}
\frac{h}{T^3}\sum_{s=1}^{T}\left\|\sum_{t=1}^{T}\mathrm{tr}(L_{st}\tilde{F}_s\tilde{F}_s')\tilde{F}_s'\right\|^2 &\leq \frac{h}{T^3}\sum_{s=1}^{T}\left\|\sum_{t=1}^{T}\mathrm{tr}[\left(k_{h,st}^2 S_{Tt}^{-1}S_{Tt}^{-1}+\mathbb{I}_R\right)\tilde{F}_s\tilde{F}_s']\tilde{F}_s'\right\|^2 \leq c_{1NT}^2\frac{h}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s\right\|^6 \\
&= O_P((Th)^{-1}).
\end{aligned}$$

In addition, $\frac{1}{NT}\sum_{s=1}^{T} E\left\|\frac{1}{T}\sum_{i=1}^{N}\sum_{r=1}^{T} F_r e_{ir}^{\dagger} e_{is}\right\|^2 = O\left(T^{-1}+NT^{-2}+a_{NT}^2(1+N/T)\right) = o\left(1\right).$ It follows that $B_{2,1}^{(1)} = o_P\left(1\right)$. For $B_{2,1}^{(2)}$ and $B_{2,1}^{(3)}$, we have by Lemmas A.2(iii) and A.6(ii), and the proof of Lemma A.6(i),

$$\begin{aligned}
B_{2,1}^{(2)} &= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathrm{tr}(L_{st}\tilde{F}_s\tilde{F}_s')\tilde{F}_s' D_6(i)e_{is} \\
&= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathrm{tr}(L_{st}\tilde{F}_s\tilde{F}_s')\tilde{F}_s'\sum_{i=1}^{N}\frac{1}{T}\sum_{r=1}^{T}(\tilde{F}_r - HF_r)e_{ir}^{\dagger}e_{is} \\
&\leq \frac{h^{1/2}}{T^2N^{1/2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathrm{tr}[\left(k_{h,st}^2 S_{Tt}^{-1}S_{Tt}^{-1}+\mathbb{I}_R\right)\tilde{F}_s\tilde{F}_s']\tilde{F}_s'\sum_{i=1}^{N}\frac{1}{T}\sum_{r=1}^{T}(\tilde{F}_r - HF_r)e_{ir}^{\dagger}e_{is} \\
&= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathrm{tr}[\left(k_{h,st}^2 S_{Tt}^{-1}S_{Tt}^{-1}+\mathbb{I}_R\right)\tilde{F}_s\tilde{F}_s']\tilde{F}_s'\left\|\sum_{i=1}^{N}\frac{1}{T}\sum_{r=1}^{T}(\tilde{F}_r - HF_r)e_{ir}^{\dagger}e_{is}\right\| \\
&\leq c_{1NT}N^{1/2}h^{1/2}\left\{\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s\right\|^3\right\}\left\{\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{r=1}^{T}(\tilde{F}_r - HF_r)e_{ir}^{\dagger}\right\|^2\right\}^{1/2}\max_s\left\{\frac{1}{N}\sum_{i=1}^{N}e_{is}^2\right\}^{1/2} \\
&= O_P(N^{1/2}h^{-1/2})O_P\left(1\right)O_P\left(C_{NT}^{-2}\right)O_P\left(1\right) = o_P\left(1\right),
\end{aligned}$$

and

$$\begin{aligned}
B_{2,1}^{(3)} &= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathrm{tr}(L_{st}\tilde{F}_s\tilde{F}_s')\tilde{F}_s' D_7(i)e_{is} \\
&= \frac{h^{1/2}}{T^2N^{1/2}}\sum_{t=1}^{T}\sum_{s=1}^{T}\mathrm{tr}(L_{st}\tilde{F}_s\tilde{F}_s')\tilde{F}_s'\frac{1}{T}\tilde{F}'(\tilde{F}H^{-1}-F)\sum_{i=1}^{N}\lambda_{i0}e_{is} \\
&\leq N^{1/2}h^{-1/2}\frac{1}{T}\left\|\tilde{F}'(\tilde{F}H^{-1}-F)\right\|\max_s\left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_{i0}e_{is}\right\|\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left\|\mathrm{tr}(L_{st}\tilde{F}_s\tilde{F}_s')\tilde{F}_s'\right\| \\
&\leq c_{1NT}N^{1/2}h^{1/2}\frac{1}{T}\left\|\tilde{F}'(\tilde{F}H^{-1}-F)\right\|\max_s\left\|\frac{1}{N}\sum_{i=1}^{N}\lambda_{i0}e_{is}\right\|\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s\right\|^3 \\
&= O_P(N^{1/2}h^{-1/2})\left(O_P(C_{NT}^{-2})+o_P\left(a_{NT}\right)\right)O_P(N^{-1/2}\ln T)O_P\left(1\right) = o_P\left(1\right).
\end{aligned}$$

Thus $B_{2,1} = o_P(1)$. By Lemma A.6(vi),

$$\begin{aligned}
|B_{2,2}| &= \frac{h^{1/2}}{T^2 N^{1/2}} \left| \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') \lambda_{i0}' {H'}^{-1} (\tilde{F}_s - H' F_s) e_{is} \right| \\
&\leq \frac{h^{1/2}}{T N^{1/2}} \left| \sum_{s=1}^{T} \left[\frac{1}{T} \sum_{t=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') \right] \left[\sum_{i=1}^{N} \lambda_{i0}' {H'}^{-1} (\tilde{F}_s - H' F_s) e_{is} \right] \right| \\
&\leq h^{1/2} \left\{ \frac{1}{T} \sum_{s=1}^{T} \left\| \frac{1}{T} \sum_{t=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') \right\|^2 \right\}^{1/2} \left\{ \frac{1}{NT} \sum_{s=1}^{T} \left\| \sum_{i=1}^{N} \lambda_{i0}' {H'}^{-1} (\tilde{F}_s - H' F_s) e_{is} \right\|^2 \right\}^{1/2} \\
&\leq c_{1NT} h^{1/2} \left\{ \frac{1}{T} \sum_{s=1}^{T} \left\| \tilde{F}_s \right\|^2 \right\}^{1/2} \left\{ \frac{1}{NT} \sum_{s=1}^{T} \left\| \sum_{i=1}^{N} \lambda_{i0}' {H'}^{-1} (\tilde{F}_s - H' F_s) e_{is} \right\|^2 \right\}^{1/2} \\
&= O_P(h^{-1/2}) O_P(1) O_P(C_{NT}^{-1}) = o_P(1).
\end{aligned}$$

In addition,

$$\begin{aligned}
B_{2,3} &= \frac{a_{NT} h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') F_s' g_{is} e_{is} = \frac{N^{1/4} h^{1/4}}{T^{5/2}} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}(L_{st} \tilde{F}_s \tilde{F}_s') F_s' \left(\frac{1}{N} \sum_{i=1}^{N} g_{is} e_{is} \right) \\
&\leq c_{1NT} \frac{N^{1/4} h^{1/4}}{T^{1/2}} \max_s \left| \frac{1}{N} \sum_{i=1}^{N} g_{is} e_{is} \right| \left\{ \frac{1}{T} \sum_{s=1}^{T} \left\| \tilde{F}_s \right\|^4 \frac{1}{T} \sum_{s=1}^{T} \| F_s \|^2 \right\}^{1/2} \\
&= N^{1/4} h^{-3/4} T^{-1/2} O_P(N^{-1/2} \ln T) O_P(1) = o_P(1).
\end{aligned}$$

Thus $B_2 = o_P(1)$.

Now, we show (i3). For B_3, we use the definition of L_{st} and make the following decomposition:

$$\begin{aligned}
B_3 &= \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}\left[L_{st} (\tilde{F}_s \tilde{F}_s' - H' F_s F_s' H) \right] e_{is}^2 \\
&= \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} k_{h,st}^2 \operatorname{tr}\left[S_{Tt}^{-1} S_{Tt}^{-1} (\tilde{F}_s \tilde{F}_s' - H' F_s F_s' H) \right] e_{is}^2 \\
&\quad - \frac{2h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} k_{h,st} \operatorname{tr}\left[S_{Tt}^{-1} (\tilde{F}_s \tilde{F}_s' - H' F_s F_s' H) \right] e_{is}^2 + \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} \operatorname{tr}(\tilde{F}_s \tilde{F}_s' - H' F_s F_s' H) e_{is}^2 \\
&\equiv B_{3,1} + B_{3,2} + B_{3,3}, \text{ say.}
\end{aligned}$$

Using $\tilde{F}_s \tilde{F}_s' - H' F_s F_s' H = (\tilde{F}_s - H' F_s)(\tilde{F}_s - H' F_s)' + (\tilde{F}_s - H' F_s) F_s' H + H' F_s (\tilde{F}_s - H' F_s)'$, we can decompose $B_{3,1}$ as follows

$$\begin{aligned}
|B_{3,1}| &\leq \frac{h^{1/2}}{T^2 N^{1/2}} \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} k_{h,st}^2 \operatorname{tr}\left[S_{Tt}^{-1} S_{Tt}^{-1} (\tilde{F}_s - H' F_s)(\tilde{F}_s - H' F_s)' \right] e_{is}^2 \\
&\quad + \frac{2h^{1/2}}{T^2 N^{1/2}} \left| \sum_{i=1}^{N} \sum_{t=1}^{T} \sum_{s=1}^{T} k_{h,st}^2 \operatorname{tr}\left[S_{Tt}^{-1} S_{Tt}^{-1} (\tilde{F}_s - H' F_s) F_s' H \right] e_{is}^2 \right| \equiv B_{3,1}^{(1)} + 2B_{3,1}^{(2)}.
\end{aligned}$$

By Lemma A.2(i) and the fact that $\max_i \frac{1}{N}\sum_{i=1}^N e_{is}^2 = O_P(1)$, we have

$$\begin{aligned} B_{3,1}^{(1)} &\leq c_{1NT} N^{1/2} h^{1/2} \left\{ \max_i \frac{1}{N} \sum_{i=1}^N e_{is}^2 \right\} \left\{ \frac{1}{T} \sum_{s=1}^T \left\| \tilde{F}_s - H' F_s \right\|^2 \right\} \\ &= O_P(N^{1/2} h^{-1/2}) O_P(1) O_P(C_{NT}^{-2}) = o_P(1). \end{aligned}$$

In addition, by Lemmas A.3(iv), A.6 (iii) and (vii), we can readily show that

$$\begin{aligned} B_{3,1}^{(2)} &= \frac{h^{1/2}}{T^2 N^{1/2}} \frac{1}{T} \sum_{t=1}^T k_{h,st}^2 \mathrm{tr} \left(S_{Tt}^{-1} S_{Tt}^{-1} \sum_{i=1}^N \sum_{s=1}^T (\tilde{F}_s - H' F_s) F_s' H e_{is}^2 \right) \\ &\leq N^{1/2} h^{1/2} \left\{ \max_t \left\| S_{Tt}^{-1} \right\|^2 \right\} \left\{ \max_s \frac{1}{T} \sum_{t=1}^T k_{h,st}^2 \right\} \left\| \frac{1}{NT} \sum_{i=1}^N \sum_{s=1}^T (\tilde{F}_s - H' F_s) F_s' H e_{is}^2 \right\| \\ &\leq N^{1/2} h^{1/2} O_P(1) O\left(h^{-1}\right) O_P(1) O_P(a_{NT}) = O_P(T^{-1/2} N^{-1/4} h^{-3/4}) = o_P(1). \end{aligned}$$

Thus $B_{3,1} = o_P(1)$. Similarly, we have $B_{3,l} = o_P(1)$ for $l = 2, 3$. Then $B_3 = o_P(1)$. This completes the proof of Proposition A.11. ■

Proposition A.12 *Suppose that the conditions in Theorem 3.2 hold. Then* $\hat{\mathbb{V}}_{NT} = \mathbb{V}_{NT} + o_P(1)$ *under* $\mathbb{H}_1(a_{NT})$.

Proof. Let $\bar{k}_{sr} = \bar{K}\left(\frac{s-r}{Th}\right)$. Observe that $\hat{\mathbb{V}}_{NT} - \mathbb{V}_{NT} = \mathbb{V}_{1NT} + \mathbb{V}_{2NT}$, where

$$\begin{aligned} \mathbb{V}_{1NT} &= 2T^{-2} N^{-1} h^{-1} \sum_{1 \leq s \neq r \leq T} \bar{k}_{sr}^2 \left[\tilde{F}_r' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_s \tilde{F}_s' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_r (\tilde{e}_r' \tilde{e}_s)^2 - F_r' \mathbb{S} F_s F_s' \mathbb{S} F_r (e_r' e_s)^2 \right], \\ \mathbb{V}_{2NT} &= 2T^{-2} N^{-1} h^{-1} \sum_{1 \leq s \neq r \leq T} \bar{k}_{sr}^2 \left[F_r' \mathbb{S} F_s F_s' \mathbb{S} F_r (e_r' e_s)^2 - E(F_r' \mathbb{S} F_s F_s' \mathbb{S} F_r (e_r' e_s)^2) \right]. \end{aligned}$$

Using $(\tilde{e}_r' \tilde{e}_s)^2 - (e_r' e_s)^2 = (\tilde{e}_r' \tilde{e}_s - e_r' e_s)^2 + 2(\tilde{e}_r' \tilde{e}_s - e_r' e_s) e_r' e_s$ we can decompose $\mathbb{V}_{1NT}$ as follows:

$$\begin{aligned} \mathbb{V}_{1NT} &= 2T^{-2} N^{-1} h^{-1} \sum_{1 \leq s \neq r \leq T} \bar{k}_{sr}^2 \tilde{F}_r' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_s \tilde{F}_s' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_r (\tilde{e}_r' \tilde{e}_s - e_r' e_s)^2 \\ &\quad + 4T^{-2} N^{-1} h^{-1} \sum_{1 \leq s \neq r \leq T} \bar{k}_{sr}^2 \tilde{F}_r' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_s \tilde{F}_s' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_r (\tilde{e}_r' \tilde{e}_s - e_r' e_s) e_r' e_s \\ &\quad + 2T^{-2} N^{-1} h^{-1} \sum_{1 \leq s \neq r \leq T} \bar{k}_{sr}^2 (\tilde{F}_r' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_s \tilde{F}_s' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_r - F_r' \mathbb{S} F_s F_s' \mathbb{S} F_r)(e_r' e_s)^2 \\ &\equiv 2\mathbb{V}_{1NT,1} + 4\mathbb{V}_{1NT,2} + 2\mathbb{V}_{1NT,3}, \text{ say.} \end{aligned}$$

Using (A.4) and following the analysis in proving (i), we can readily show that $\mathbb{V}_{1NT,l} = o_P(1)$ for $l = 1, 2$. For $\mathbb{V}_{1NT,3}$, using $\tilde{a}'\tilde{a} - a'a = (\tilde{a} - a)'(\tilde{a} - a) + (\tilde{a} - a)' a + a'(\tilde{a} - a)$, we decompose it as follows:

$$\begin{aligned} \mathbb{V}_{1NT,3} &= T^{-2} N^{-1} h^{-1} \sum_{1 \leq s \neq r \leq T} \bar{k}_{sr}^2 (\tilde{F}_r' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_s - F_r' \mathbb{S} F_s)(\tilde{F}_r' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_s - F_r' \mathbb{S} F_s)' (e_r' e_s)^2 \\ &\quad + 2T^{-2} N^{-1} h^{-1} \sum_{1 \leq s \neq r \leq T} \bar{k}_{sr}^2 (\tilde{F}_r' \tilde{\Sigma}_F^{-1} \tilde{\Sigma}_F^{-1} \tilde{F}_s - F_r' \mathbb{S} F_s) F_r' \mathbb{S} F_s (e_r' e_s)^2 \\ &\equiv \mathbb{V}_{1NT,3}^{(1)} + 2\mathbb{V}_{1NT,3}^{(2)}, \text{ say.} \end{aligned}$$

Noting that $\tilde{F}_r'\tilde{\Sigma}_F^{-1}\tilde{\Sigma}_F^{-1}\tilde{F}_s - F_r'\mathbb{S}F_s = \tilde{F}_r'\tilde{\Sigma}_F^{-1}\tilde{\Sigma}_F^{-1}(\tilde{F}_s - H'F_s) + \tilde{F}_r'H^{-1}(H\tilde{\Sigma}_F^{-1}\tilde{\Sigma}_F^{-1}H' - \mathbb{S})F_s + (\tilde{F}_r'H^{-1} - F_r)\mathbb{S}F_s$, we have

$$\begin{aligned}\mathbb{V}_{1NT,3}^{(1)} \leq\ & 3T^{-2}N^{-1}h^{-1}\sum_{1\leq s\neq r\leq T}\bar{k}_{sr}^2\left\|\tilde{F}_r'\tilde{\Sigma}_F^{-1}\tilde{\Sigma}_F^{-1}(\tilde{F}_s - H'F_s)\right\|^2(e_r'e_s)^2 \\ & +3T^{-2}N^{-1}h^{-1}\sum_{1\leq s\neq r\leq T}\bar{k}_{sr}^2\left\|\tilde{F}_r'H^{-1}(H\tilde{\Sigma}_F^{-1}\tilde{\Sigma}_F^{-1}H' - \mathbb{S})F_s\right\|^2(e_r'e_s)^2 \\ & +3T^{-2}N^{-1}h^{-1}\sum_{1\leq s\neq r\leq T}\bar{k}_{sr}^2\left\|(\tilde{F}_r'H^{-1} - F_r)\mathbb{S}F_s\right\|^2(e_r'e_s)^2.\end{aligned}$$

Using Lemma A.2, we can readily show that each term in the last expression is $o_P(1)$. Then we have $\mathbb{V}_{1NT,3}^{(1)} = o_P(1)$. Similarly, we can show $\mathbb{V}_{1NT,3}^{(2)} = o_P(1)$. So $\mathbb{V}_{1NT,3} = o_P(1)$ and $\mathbb{V}_{1NT} = o_P(1)$.

In addition, noting that $E(\mathbb{V}_{2NT}) = 0$ and $\text{Var}(\mathbb{V}_{2NT}) = o(1)$, we have $\mathbb{V}_{2NT} = o_P(1)$. Thus, $\hat{\mathbb{V}}_{NT} = \mathbb{V}_{NT} + o_P(1)$. ■

Proof of Theorem 3.3. Let P^* denote the probability measure induced by the modified parametric bootstrap conditional on the original sample $\mathcal{W}_{NT}$. Let E^* and Var^* denote the expectation and variance under P^*. Let $O_{P^*}(\cdot)$ and $o_{P^*}(\cdot)$ denote the probability order under P^*, e.g., $b_{NT} = o_{P^*}(1)$ if for any $\epsilon > 0$, $P^*(\|b_{NT}\| > \epsilon) = o_P(1)$. The proof is similar to but much simpler than that of Theorem 3.2 for three reasons: (1) the null hypothesis is satisfied in the bootstrap world, (2) e_t^*'s are independent over t conditional on $\mathcal{W}_{NT}$, and (3) both $\tilde{\lambda}_{i0}$ and $\tilde{F}_t$ are fixed given $\mathcal{W}_{NT}$. Even though $\tilde{\lambda}_{i0}$ and $\tilde{F}_t$ are not uniformly bounded over i or t, we can use arguments as used in the proof of Lemma A.6(i) to demonstrate that $\frac{1}{T}\sum_{t=1}^T ||\tilde{F}_t||^8 = O_P(1) + O_P(T^3C_{NT}^{-8}) = O_P(1)$ and that $\frac{1}{N}\sum_{i=1}^N ||\tilde{\lambda}_{i0}||^8 = O_P(1)$. These are sufficient for the analysis of $\hat{J}_{NT}^*$.

Let $\tilde{\lambda}_{i0}^*$, $\tilde{F}_t^*$, and $\tilde{\lambda}_{it}^*$ denote the bootstrap analogue of $\tilde{\lambda}_{i0}$, $\tilde{F}_t$, and $\tilde{\lambda}_{it}$, respectively. Let $\hat{M}^*$, J_{NT}^*, $\mathbb{B}_{NT}^*$, $\mathbb{V}_{NT}^*$, $\hat{J}_{NT}^*$, $\hat{\mathbb{B}}_{NT}^*$, and $\hat{\mathbb{V}}_{NT}^*$ denote the bootstrap analogue of $\hat{M}$, J_{NT}, $\mathbb{B}_{NT}$, $\mathbb{V}_{NT}$, $\hat{J}_{NT}$, $\hat{\mathbb{B}}_{NT}$, and $\hat{\mathbb{V}}_{NT}$, respectively. Then $J_{NT}^* \equiv (TN^{1/2}h^{1/2}\hat{M}^* - \mathbb{B}_{NT}^*)/\sqrt{\mathbb{V}_{NT}^*}$ and $\hat{J}_{NT}^* \equiv (N^{-1/2}\hat{M}^* - \hat{\mathbb{B}}_{NT}^*)/\sqrt{\hat{\mathbb{V}}_{NT}^*}$. Following the proof of Theorem 3.2, we can show that $TN^{1/2}h^{1/2}\hat{M}^* - \mathbb{B}_{NT}^* = \sum_{s=2}^T Z_{NT,s}^* + o_{P^*}(1)$, where $Z_{NT,s}^* = 2T^{-1}N^{-1/2}h^{-1/2}\sum_{r=1}^{s-1}\bar{k}_{sr}\tilde{F}_s'\mathbb{S}^*\tilde{F}_r e_s^{*\prime}e_r^*$, $e_s^* = (e_{N1}^*, ..., e_{Ns}^*)'$, and $\mathbb{S}^* = HS_{Tt}^{-1}S_{Tt}^{-1}H'$. Then we can prove the theorem by showing that: (i) $\sum_{s=2}^T Z_{NT,s}^*/\sqrt{\mathbb{V}_{NT}^*} \xrightarrow{D^*} N(0,1)$, (ii) $\hat{\mathbb{B}}_{NT}^* = \mathbb{B}_{NT}^* + o_{P^*}(1)$, and (iii) $\hat{\mathbb{V}}_{NT}^* = \mathbb{V}_{NT}^* + o_{P^*}(1)$.

We only outline the proof of (i) as those of other parts are analogous to the corresponding parts in the proof of Theorem 3.2. Noting that $\{Z_{NT,t}^*, \mathcal{F}_{NT,t}^*\}$ is an m.d.s., we can continue to apply the martingale CLT by showing that

$$\mathcal{Z}^* \equiv \sum_{t=2}^T E_{\mathcal{F}_{NT,t-1}^*}^*\left|Z_{NT,t}^*\right|^4 = o_{P^*}(1), \text{ and } \sum_{t=2}^T Z_{NT,t}^{*2} - \mathbb{V}_{NT}^* = o_{P^*}(1). \tag{A.5}$$

As in the proof of Proposition A.8,

$$\begin{aligned}E^*(\mathcal{Z}^*) = & \frac{16}{T^4N^2h^2}\sum_{s=2}^T E^*\left[\sum_{r=1}^{s-1}\bar{k}_{sr}^4\phi_{sr}^{*4} + 2\sum_{1\leq r_1<r_2\leq s-1}\bar{k}_{sr_1}^2\bar{k}_{sr_2}^2\phi_{sr_1}^{*2}\phi_{sr_2}^{*2}\right. \\ & \left. +4\sum_{t=1}^{s-1}\sum_{1\leq r_1<r_2\leq s-1}\bar{k}_{st}^2\bar{k}_{sr_1}\bar{k}_{sr_2}\phi_{st}^{*2}\phi_{sr_1}^*\phi_{sr_2}^* + 4\sum_{1\leq r_1<r_2\leq s-1, 1\leq t_1<t_2\leq s-1}\bar{k}_{r_1s}\bar{k}_{r_2s}\bar{k}_{t_1s}\bar{k}_{t_2s}\phi_{sr_1}^*\phi_{sr_2}^*\phi_{st_1}^*\phi_{st_2}^*\right] \\ \equiv\ & \mathcal{Z}_1^* + \mathcal{Z}_2^* + \mathcal{Z}_3^* + \mathcal{Z}_4^*,\end{aligned}$$

where $\phi^*_{sr} = \tilde{F}'_s \mathbb{S}^* \tilde{F}_r e^{*\prime}_s e^*_r$. Using the *i.i.d.* property of ς_{it} and the conditions in Theorem 3.3, we can readily verify that $\mathcal{Z}^*_l = o_P(1)$ for $l = 1, 2, 3, 4$. For example, noting that $E[\varsigma_{i_1 s}\varsigma_{i_2 s}\varsigma_{i_3 s}\varsigma_{i_4 s}] = 3$ if $i_1 = i_2 = i_3 = i_4$, $= 1$ if $i_1 = i_2 \neq i_3 = i_4$, $i_1 = i_3 \neq i_2 = i_4$, or $i_1 = i_4 \neq i_2 = i_4$, and zero otherwise, we have for any $s \neq r$,

$$
\begin{aligned}
E^*[(e^{*\prime}_s e^*_r)^4] &= E^*[(\varsigma'_s \tilde{\Sigma} \varsigma_r)^4] = \sum_{i_1,\ldots,i_4,j_1,\ldots,j_4} \tilde{\sigma}_{i_1 j_1}\tilde{\sigma}_{i_2 j_2}\tilde{\sigma}_{i_3 j_3}\tilde{\sigma}_{i_4 j_4} E[\varsigma_{i_1 s}\varsigma_{i_2 s}\varsigma_{i_3 s}\varsigma_{i_4 s}] E[\varsigma_{j_1 r}\varsigma_{j_2 r}\varsigma_{j_3 r}\varsigma_{j_4 r}] \\
&= 9\sum_{i,j}\tilde{\sigma}^4_{ij} + 9\sum_i \sum_{j_1 \neq j_2} \tilde{\sigma}^2_{ij_1}\tilde{\sigma}^2_{ij_2} + 9\sum_{i_1\neq i_2}\sum_j \tilde{\sigma}^2_{i_1 j}\tilde{\sigma}^2_{i_2 j} \\
&\quad + \sum_{i_1\neq i_2}\sum_{j_1\neq j_2} [\tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_2j_2}\tilde{\sigma}_{i_2j_2} + \tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_1j_2}\tilde{\sigma}_{i_2j_1}\tilde{\sigma}_{i_2j_2} + \tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_1j_2}\tilde{\sigma}_{i_2j_2}\tilde{\sigma}_{i_2j_1} \\
&\quad + \tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_2j_1}\tilde{\sigma}_{i_1j_2}\tilde{\sigma}_{i_2j_2} + \tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_2j_2}\tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_2j_2} + \tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_2j_2}\tilde{\sigma}_{i_1j_2}\tilde{\sigma}_{i_2j_1} \\
&\quad + \tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_2j_1}\tilde{\sigma}_{i_2j_2}\tilde{\sigma}_{i_1j_2} + \tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_2j_2}\tilde{\sigma}_{i_2j_1}\tilde{\sigma}_{i_1j_2} + \tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_2j_2}\tilde{\sigma}_{i_2j_2}\tilde{\sigma}_{i_1j_1}] \\
&= 9\sum_{i,j}\tilde{\sigma}^4_{ij} + 18\sum_i\sum_{j_1\neq j_2}\tilde{\sigma}^2_{ij_1}\tilde{\sigma}^2_{ij_2} + 3\sum_{i_1\neq i_2}\sum_{j_1\neq j_2}[\tilde{\sigma}^2_{i_1j_1}\tilde{\sigma}^2_{i_2j_2} + 2\tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{i_1j_2}\tilde{\sigma}_{i_2j_1}\tilde{\sigma}_{i_2j_2}] \\
&= O_P\left(\xi^3_{NT}N + N\xi^2_{NT} + N^2\xi^2_{NT}\right) = O_P\left(N^2\xi^2_{NT}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathcal{Z}^*_1 &= \frac{16}{T^4N^2h^2}\sum_{s=2}^{T}\sum_{r=1}^{s-1}\bar{k}^4_{sr}\left(\tilde{F}'_s\mathbb{S}^*\tilde{F}_r\right)^4 E^*\left(e^{*\prime}_s e^*_r\right)^4 \\
&= \frac{16}{T^4N^2h^2}\sum_{s=2}^{T}\sum_{r=1}^{s-1}\bar{k}^4_{sr}\left(\tilde{F}'_s\mathbb{S}^*\tilde{F}_r\right)^4 O_P\left(N^2\xi^2_{NT}\right) = O_P\left(T^{-2}h^{-2}\xi^2_{NT}\right),
\end{aligned}
$$

where we use the fact that $\frac{1}{T^2h}\sum_{s=2}^T\sum_{r=1}^{s-1}\bar{k}^4_{sr}||\tilde{F}_s||^8 = O_P(1)$ under Assumption A.3 and the extra conditions in the theorem. Similarly, noting that for any $r_1 < r_2 < s$,

$$
\begin{aligned}
E^*\left[\left(e^{*\prime}_s e^*_{r_1}\right)^2\left(e^{*\prime}_s e^*_{r_2}\right)^2\right] &= E^*\left[(\varsigma'_s\tilde{\Sigma}\varsigma_{r_1}\varsigma'_{r_1}\tilde{\Sigma}\varsigma_s)(\varsigma'_s\tilde{\Sigma}\varsigma_{r_2}\varsigma'_{r_1}\tilde{\Sigma}\varsigma_s)\right] = E^*[\varsigma'_s\tilde{\Sigma}\tilde{\Sigma}\varsigma_s\varsigma'_s\tilde{\Sigma}\tilde{\Sigma}\varsigma_s] \\
&= \sum_{i_1,\ldots,i_4,j_1,j_2}\tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{j_1i_2}\tilde{\sigma}_{i_3j_2}\tilde{\sigma}_{j_2i_4}E[\varsigma_{i_1s}\varsigma_{i_2s}\varsigma_{i_3s}\varsigma_{i_4s}] \\
&= 3\sum_{i,j_1,j_2}\tilde{\sigma}^2_{ij_1}\tilde{\sigma}^2_{ij_2} + \sum_{i_1,i_2,j_1,j_2}[\tilde{\sigma}^2_{i_1j_1}\tilde{\sigma}^2_{i_2j_2} + 2\tilde{\sigma}_{i_1j_1}\tilde{\sigma}_{j_1i_2}\tilde{\sigma}_{i_1j_2}\tilde{\sigma}_{j_2i_2}] \\
&= O_P\left(N\xi^3_{NT}\right) + O_P\left(N^2\xi^2_{NT}\right) = O_P\left(N^2\xi^2_{NT}\right),
\end{aligned}
$$

where we use the fact that $\tilde{\sigma}_{ij} = \tilde{\sigma}_{ji}$ and $\xi_{NT} = o\left(T^{1/2}\right) = o(N)$, we have

$$
\begin{aligned}
\mathcal{Z}^*_4 &= \frac{64}{T^4N^2h^2}\sum_{s=2}^{T}\sum_{1\leq r_1<r_2\leq s-1, 1\leq t_1<t_2\leq s-1}\bar{k}_{r_1s}\bar{k}_{r_2s}\bar{k}_{t_1s}\bar{k}_{t_2s}\tilde{F}'_s\mathbb{S}^*\tilde{F}_{r_1}\tilde{F}'_s\mathbb{S}^*\tilde{F}_{r_2}\tilde{F}'_s\mathbb{S}^*\tilde{F}_{t_1}\tilde{F}'_s\mathbb{S}^*\tilde{F}_{t_2} \\
&\quad \times E^*\left[\left(e^{*\prime}_s e^*_{r_1}\right)\left(e^{*\prime}_s e^*_{r_2}\right)\left(e^{*\prime}_s e^*_{t_1}\right)\left(e^{*\prime}_s e^*_{t_2}\right)\right] \\
&= \frac{64}{T^4N^2h^2}\sum_{s=2}^{T}\sum_{1\leq r_1<r_2\leq s-1}\bar{k}^2_{r_1s}\bar{k}^2_{r_2s}\left(\tilde{F}'_s\mathbb{S}^*\tilde{F}_{r_1}\right)^2\left(\tilde{F}'_s\mathbb{S}^*\tilde{F}_{r_2}\right)^2 O_P\left(N^2\xi^2_{NT}\right) = O_P\left(T^{-2}h^{-2}\xi^2_{NT}\right),
\end{aligned}
$$

Then $\mathcal{Z}^* = o_{P^*}(1)$ by the conditional Markov inequality. Now $\sum_{t=2}^T E^*(Z^{*2}_{NT,t}) = 4T^{-2}N^{-1}h^{-1}E^*[\sum_{r=1}^{s-1}\bar{k}_{sr} \times\tilde{F}'_s\mathbb{S}^*\tilde{F}_r e^{*\prime}_s e^*_r]^2 = \mathbb{V}^*_{NT}$. Straightforward moment calculations yield that $E^*(\sum_{t=2}^T Z^{*2}_{NT,t})^2 = \mathbb{V}^{*2}_{NT} + o_P(1)$. Thus $\mathrm{Var}^*(\sum_{t=2}^T Z^{*2}_{NT,t}) = o_P(1)$ and $\sum_{t=2}^T Z^{*2}_{NT,t} - \mathbb{V}^*_{NT} = o_P(1)$. This completes the proof of (i). ■

A.3 Proofs of the Technical Lemmas

Recall that $\max_i$, $\max_t$, and $\max_{s,t}$ denote $\max_{1\leq i\leq N}$, $\max_{1\leq t\leq T}$, and $\max_{1\leq s,t\leq T}$, respectively. Let $\|A\|_q = \{E\|A\|^q\}^{1/q}$ for $q \geq 1$.

Proof of Lemma A.1. (i) From the principal component analysis, we have the identity $(NT)^{-1} XX'\tilde{F} = \tilde{F}V_{NT}$. Pre-multiplying both sides by $T^{-1}\tilde{F}'$ and using the normalization $T^{-1}\tilde{F}'\tilde{F} = \mathbb{I}_R$ yields $T^{-1}\tilde{F}'(NT)^{-1} XX'\tilde{F} = V_{NT}$. By Bai (2003, Lemma A.3) and following the proof of (ii) below, V_{NT} has probability limit V_0 that is a diagonal matrix consisting of the R eigenvalues of $\Sigma_{\Lambda_0}\Sigma_F$ under Assumptions A.1-A.3 and A.5.

(ii) Noting that $X = F\Lambda_0' + e^\dagger$, where $e^\dagger = e + a_{NT}g^\dagger$, $g^\dagger = (g_1^\dagger, ..., g_T^\dagger)'$, $g_t^\dagger = (F_t'g_{1t}, ..., F_t'g_{Nt})'$ and $g_{it} = g_i(t/T)$, (i) implies that

$$(T^{-1}\tilde{F}'F)\left(N^{-1}\Lambda_0'\Lambda_0\right)(T^{-1}F'\tilde{F}) + d_{NT} = V_{NT} \xrightarrow{P} V_0, \tag{A.6}$$

where $d_{NT} = N^{-1}T^{-2}\tilde{F}'e^\dagger e^{\dagger\prime}\tilde{F} + (T^{-1}\tilde{F}'F)(N^{-1}T^{-1}\Lambda_0'e^{\dagger\prime}\tilde{F}) + (N^{-1}T^{-1}\tilde{F}'e^\dagger\Lambda_0)(T^{-1}F'\tilde{F})$. Noting that

$$\begin{aligned} N^{-1}T^{-2}||\tilde{F}'e^{\dagger\prime}e^\dagger\tilde{F}|| &\leq 2N^{-1}T^{-1}\{T^{-1}||\tilde{F}||^2\}\left(R\|e\|_{\text{sp}}^2 + a_{NT}^2\left\|g^\dagger\right\|^2\right) \\ &= O_P(T^{-1} + N^{-1} + a_{NT}^2), \\ N^{-1}T^{-1}||\Lambda_0'e^{\dagger\prime}\tilde{F}|| &\leq N^{-1}T^{-1/2}\{T^{-1/2}||\tilde{F}||\}\left(\|e\Lambda_0\| + a_{NT}\|\bar{g}\Lambda_0\|\right) \\ &= N^{-1}T^{-1/2}O_P(1)\left(N^{1/2}T^{1/2} + a_{NT}NT^{1/2}\right) = O_P(N^{-1/2} + a_{NT}), \end{aligned}$$

and $T^{-1}||F'\tilde{F}|| = O_P(1)$ under Assumptions A.1-A.3 and A.5, we have

$$\|d_{NT}\| = O_P(N^{-1} + T^{-1} + a_{NT}^2 + N^{-1/2} + a_{NT}) = o_P(1). \tag{A.7}$$

It follows that $(\tilde{F}'F/T)(\Lambda_0'\Lambda_0/N)(F'\tilde{F}/T) \xrightarrow{P} V_0$. ■

Proof of Lemma A.2. (i) Let $e_t^\dagger = (e_{1t}^\dagger, ..., e_{Nt}^\dagger)'$ and $\Lambda_0 = (\lambda_{10}, ..., \lambda_{N0})'$. Noting that $(NT)^{-1}XX'\tilde{F} = \tilde{F}V_{NT}$ and $X_{it} = \lambda_{it}'F_t + e_{it} = \lambda_{i0}'F_t + e_{it}^\dagger$ with $e_{it}^\dagger = e_{it} + a_{NT}F_t'g_{it}$, we can decompose $\tilde{F}_t - H'F_t$ as follows:

$$\begin{aligned} \tilde{F}_t - H'F_t &= V_{NT}^{-1}\frac{1}{NT}\sum_{s=1}^T \tilde{F}_s X_s'X_t - H'F_t \\ &= V_{NT}^{-1}\frac{1}{NT}\sum_{s=1}^T \tilde{F}_s\left[\Lambda_0F_s + e_s^\dagger\right]'\left[\Lambda_0F_t + e_t^\dagger\right] - H'F_t \\ &= V_{NT}^{-1}\left\{\frac{1}{T}\sum_{s=1}^T \tilde{F}_sE(e_s^{\dagger\prime}e_t^\dagger/N) + \frac{1}{T}\sum_{s=1}^T \tilde{F}_s\left[e_s^{\dagger\prime}e_t^\dagger/N - E(e_s^{\dagger\prime}e_t^\dagger/N)\right]\right. \\ &\quad \left. + \frac{1}{T}\sum_{s=1}^T \tilde{F}_sF_s'\Lambda_0'e_t^\dagger/N + \frac{1}{T}\sum_{s=1}^T \tilde{F}_sF_t'\Lambda_0'e_s^\dagger/N\right\} \\ &\equiv A_1(t) + A_2(t) + A_3(t) + A_4(t), \quad \text{say.} \end{aligned} \tag{A.8}$$

By (A.8) and the inequality $(a+b+c+d)^2 \leq 4(a^2+b^2+c^2+d^2)$, we have

$$\frac{1}{T}\sum_{t=1}^T\left\|\tilde{F}_t - H'F_t\right\|^2 \leq \frac{4}{T}\left\|V_{NT}^{-1}\right\|^2\sum_{t=1}^T\left[\|V_{NT}A_1(t)\|^2 + \|V_{NT}A_2(t)\|^2 + \|V_{NT}A_3(t)\|^2 + \|V_{NT}A_4(t)\|^2\right].$$

By Lemma A.1(i), it suffices to bound $\frac{1}{T}\sum_{t=1}^T\|V_{NT}A_l(t)\|^2$ for $l = 1, 2, 3, 4$. Let $g_t^\dagger \equiv (F_t'g_{1t}, ..., F_t'g_{Nt})'$. Using $e_s^{\dagger\prime}e_t^\dagger = (e_s + a_{NT}g_s^\dagger)'(e_t + a_{NT}g_t^\dagger) = e_s'e_t + a_{NT}e_s'g_t^\dagger + a_{NT}g_s^{\dagger\prime}e_t + a_{NT}^2g_s^{\dagger\prime}g_t^\dagger$ and Cauchy-Schwarz (CS hereafter) inequality, we have

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right]^2 \leq \frac{4}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[E\left(e_t^{\prime}e_s/N\right)\right]^2+\frac{8a_{NT}^2}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[E(e_s^{\prime}g_t^{\dagger}/N)\right]^2+\frac{4a_{NT}^4}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[E(g_s^{\dagger\prime}g_t^{\dagger}/N)\right]^2.$$

As in Bai (2003), the first term is bounded above by $4\max_{s,t}\gamma_N(s,t)\max_t\sum_{t=1}^{T}|\gamma_N(s,t)| = O(1)$ by Assumption A.3(iv). By Davydov inequality and Assumptions A.1(ii), A.3(i) and (iii) and A.5(i)

$$\begin{aligned}
\frac{a_{NT}^2}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[E(e_s^{\prime}g_t^{\dagger}/N)\right]^2 &= \frac{a_{NT}^2}{TN^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i=1}^{N}\sum_{j=1}^{N}E\left(e_{is}F_t^{\prime}g_{it}\right)E\left(e_{js}F_t^{\prime}g_{jt}\right)\\
&\leq C\max_{i,s}E\left(e_{is}F_t^{\prime}g_{it}\right)\|e_{is}\|_{2+\delta}\|F_t\|_{2+\delta}\frac{8a_{NT}^2}{TN^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_j\left(|t-s|\right)^{\delta/(2+\delta)}\\
&\leq C\max_{i,s}E\left(e_{is}F_t^{\prime}g_{it}\right)\|e_{is}\|_{2+\delta}\|F_t\|_{2+\delta}\frac{8a_{NT}^2}{N}\sum_{j=1}^{N}\sum_{s=1}^{\infty}\alpha_j\left(s\right)^{\delta/(2+\delta)} = O(a_{NT}^2).
\end{aligned}$$

In addition, we can show that $\frac{a_{NT}^4}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[E(g_s^{\dagger\prime}g_t^{\dagger}/N)\right]^2 = O(a_{NT}^4T) = o(1)$ under Assumptions A.1(ii) and A.5(i). It follows that $\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right]^2 \leq \frac{4}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[E\left(e_t^{\prime}e_s/N\right)\right]^2+o(1) = O(1)$ under Assumption A.3(i)-(ii). Then by the submultiplicative property of the Frobenius norm, CS inequality, and the fact that $T^{-1}\tilde{F}^{\prime}\tilde{F} = \mathbb{I}_R$, we have

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\|V_{NT}A_1(t)\|^2 &= \frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_sE(e_s^{\dagger\prime}e_t^{\dagger}/N)\right\|^2 \leq \frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_sE(e_s^{\dagger\prime}e_t^{\dagger}/N)\right\|\right\}^2\\
&\leq \frac{1}{T}\sum_{r=1}^{T}\left\|\tilde{F}_r\right\|^2\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[E\left(e_s^{\dagger\prime}e_t^{\dagger}/N\right)\right]^2 = O_P(1)O(T^{-1}) = O_P(T^{-1}).
\end{aligned}$$

Now, we consider the second term. Recall that $\xi_{st} = e_s^{\prime}e_t/N - E\left(e_s^{\prime}e_t/N\right)$. Let $\xi_{st}^{\dagger} = e_s^{\dagger\prime}e_t^{\dagger}/N - E(e_s^{\dagger\prime}e_t^{\dagger}/N)$. Then

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\|V_{NT}A_2(t)\|^2 &= \frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\xi_{st}^{\dagger}\right\|^2 = \frac{1}{T^3}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}\tilde{F}_s^{\prime}\tilde{F}_r\xi_{st}^{\dagger}\xi_{rt}^{\dagger}\\
&\leq \frac{1}{T}\left[\frac{1}{T^2}\sum_{s=1}^{T}\sum_{r=1}^{T}(\tilde{F}_s^{\prime}\tilde{F}_r)^2\right]^{1/2}\left[\frac{1}{T^2}\sum_{s=1}^{T}\sum_{l=1}^{T}\left(\sum_{t=1}^{T}\xi_{st}^{\dagger}\xi_{lt}^{\dagger}\right)^2\right]^{1/2}\\
&= \frac{1}{T}\left[\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s\right\|^2\right]\left[\frac{1}{T^2}\sum_{s=1}^{T}\sum_{l=1}^{T}\left(\sum_{t=1}^{T}\xi_{st}^{\dagger}\xi_{rt}^{\dagger}\right)^2\right]^{1/2}.
\end{aligned}$$

In addition, using $\xi_{st}^{\dagger} = \xi_{st}+a_{NT}N^{-1}[e_s^{\prime}g_t^{\dagger}-E(e_s^{\prime}g_t^{\dagger})]+a_{NT}N^{-1}[g_s^{\dagger\prime}e_t-E\left(g_s^{\dagger\prime}e_t\right)]+a_{NT}^2N^{-1}[g_s^{\dagger\prime}g_t^{\dagger}-E(g_s^{\dagger\prime}g_t^{\dagger})]$, we can readily show that $\frac{1}{T^2}\sum_{s=1}^{T}\sum_{l=1}^{T}E(\sum_{t=1}^{T}\xi_{st}^{\dagger}\xi_{rt}^{\dagger})^2 = O(T^2N^{-2})$ under Assumptions A.3 and A.5. It follows that $\frac{1}{T}\sum_{t=1}^{T}\|V_{NT}A_2(t)\|^2 = \frac{1}{T}O_P\left(T/N\right) = O_P(N^{-1})$. For the third term, noting that $E(e_{it}g_{jt}^{\dagger}) = 0$, we have by Assumptions A.1(i)-(ii), A.2(i), A.3(vi) and A.5(i)

$$\begin{aligned}(NT)^{-1}\sum_{t=1}^{T}E\left\|\Lambda_0'e_t^{\dagger}\right\|^2 &= (NT)^{-1}\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{i0}'\lambda_{j0}E(e_{it}^{\dagger}e_{jt}^{\dagger})\\ &= (NT)^{-1}\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{i0}'\lambda_{j0}E\left(e_{it}e_{jt}\right)+a_{NT}^2(NT)^{-1}\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{i0}'\lambda_{j0}E(g_{it}^{\dagger}g_{jt}^{\dagger})\\ &\leq C(NT)^{-1}\sum_{t=1}^{T}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|E\left(e_{it}e_{jt}\right)\right|+O(T^{-1}N^{1/2}h^{-1/2})=O\left(1\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\left\|V_{NT}A_3(t)\right\|^2 &= \frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_sF_s'\Lambda_0'e_t^{\dagger}/N\right\|^2\leq\frac{1}{N}\frac{1}{T}\sum_{t=1}^{T}\left\|\frac{\Lambda_0'e_t^{\dagger}}{\sqrt{N}}\right\|^2\left[\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s\right\|^2\right]\left[\frac{1}{T}\sum_{s=1}^{T}\left\|F_s\right\|^2\right]\\ &= O_P(N^{-1}).\end{aligned}$$

For the fourth term, we have

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\left\|V_{NT}A_4(t)\right\|^2 &= \frac{1}{T}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_sF_t'\Lambda_0'e_s^{\dagger}/N\right\|^2\leq\frac{1}{N^2T^3}\sum_{t=1}^{T}\left\{\sum_{s=1}^{T}\left\|\tilde{F}_sF_t'\right\|\left\|\Lambda_0'e_s^{\dagger}\right\|\right\}^2\\ &\leq\frac{1}{N}\left\{\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left\|\tilde{F}_sF_t'\right\|^2\right\}\frac{1}{NT}\sum_{s=1}^{T}\left\|\Lambda_0'e_s^{\dagger}\right\|^2=N^{-1}O_P\left(1\right)O_P\left(1\right)=O_P\left(N^{-1}\right),\end{aligned}$$

as we have shown that $\frac{1}{NT}\sum_{s=1}^{T}E\left\|\Lambda_0'e_s^{\dagger}\right\|^2=O(1+N^{1/2}T^{-1}h^{-1/2})=O\left(1\right)$. Combining these results, we have $\frac{1}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t-H'F_t\right\|^2=O_P(C_{NT}^{-2})$.

(ii) By (A.8), we have $\frac{1}{T}(\tilde{F}-FH)'FH=\frac{1}{T}\sum_{t=1}^{T}[A_1(t)+A_2(t)+A_3(t)+A_4(t)]F_t'H$. We first decompose $V_{NT}\frac{1}{T}\sum_{t=1}^{T}\left[A_1(t)+A_2\left(t\right)\right]F_t'H$ as follows

$$\begin{aligned}V_{NT}\frac{1}{T}\sum_{t=1}^{T}\left[A_1(t)+A_2\left(t\right)\right]F_t'H &= \frac{1}{NT^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{F}_se_s^{\dagger\prime}e_t^{\dagger}F_t'H\\ &= \frac{1}{NT^2}\sum_{t=1}^{T}\sum_{s=1}^{T}(\tilde{F}_s-H'F_s)e_s^{\dagger\prime}e_t^{\dagger}F_t'H+\frac{1}{NT^2}H'\sum_{t=1}^{T}\sum_{s=1}^{T}F_se_s^{\dagger\prime}e_t^{\dagger}F_t'H\\ &\equiv \bar{A}_1+\bar{A}_2,\ \text{say.}\end{aligned}$$

For $\bar{A}_1$, we apply CS inequality and the result in part (i) to obtain

$$\begin{aligned}\left\|\bar{A}_1\right\| &\leq \left\|H\right\|\left\{\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s-H'F_s\right\|^2\right\}^{1/2}\left\{\frac{1}{N^2T^3}\sum_{s=1}^{T}\left\|\sum_{t=1}^{T}e_s^{\dagger\prime}e_t^{\dagger}F_t'\right\|^2\right\}^{1/2}\\ &= O_P\left(C_{NT}^{-1}\right)O_P(N^{-1/2})=O_P(C_{NT}^{-1}N^{-1/2}),\end{aligned}$$

provided that $\frac{1}{N^2T^3}\sum_{s=1}^{T}\left\|\sum_{t=1}^{T}e_s^{\dagger\prime}e_t^{\dagger}F_t^{\prime}\right\|^2=O_P\left(N^{-1}\right)$. To see why the last claim is true, note that

$$\begin{aligned}\frac{1}{N^2T^3}\sum_{s=1}^{T}E\left\|\sum_{t=1}^{T}e_s^{\dagger\prime}e_t^{\dagger}F_t^{\prime}\right\|^2 &\leq \frac{4}{N^2T^3}\sum_{s=1}^{T}E\left\|\sum_{t=1}^{T}e_s^{\prime}e_tF_t^{\prime}\right\|^2+\frac{4a_{NT}}{N^2T^3}\sum_{s=1}^{T}E\left\|\sum_{t=1}^{T}e_s^{\prime}g_t^{\dagger}F_t^{\prime}\right\|^2\\ &\quad+\frac{4a_{NT}}{N^2T^3}\sum_{s=1}^{T}E\left\|\sum_{t=1}^{T}g_s^{\dagger\prime}e_tF_t^{\prime}\right\|^2+\frac{4a_{NT}^2}{N^2T^3}\sum_{s=1}^{T}E\left\|\sum_{t=1}^{T}g_s^{\dagger\prime}g_t^{\dagger}F_t^{\prime}\right\|^2\\ &\equiv 4A_{1,1}+4A_{1,2}+4A_{1,3}+4A_{1,4}.\end{aligned}$$

For $A_{1,1}$, we have by Assumptions A.1(ii) and A.3(iv),

$$\begin{aligned}A_{1,1} &\leq \frac{2}{N^2T^3}\sum_{s=1}^{T}E\left\|\sum_{t=1}^{T}E(e_s^{\prime}e_t)F_t^{\prime}\right\|^2+\frac{2}{T^2}\sum_{s=1}^{T}E\|\left\|\sum_{t=1}^{T}\xi_{st}F_t^{\prime}\right\|^2\\ &=\frac{2}{N^2T^3}\sum_{s=1}^{T}\sum_{t=1}^{T}\sum_{l=1}^{T}E(e_s^{\prime}e_t)E(e_s^{\prime}e_l)E(F_t^{\prime}F_l)+\frac{2}{T^3}\sum_{s=1}^{T}\sum_{t=1}^{T}\sum_{l=1}^{T}E(\xi_{st}\xi_{sl}F_t^{\prime}F_l)\\ &\leq 2T^{-2}\left(\max_s\sum_{t=1}^{T}\gamma_N(s,t)\right)^2\max_t\|F_t\|_2^2+2N^{-1}\max_{s,t}\|N^{1/2}\xi_{st}\|^2\max_t\|F_t\|_4^2\\ &=O(T^{-2}+N^{-1}).\end{aligned}$$

For $A_{1,2}$, we have under Assumptions A.2-A.3 and A.5(i),

$$\begin{aligned}A_{1,2} &= \frac{a_{NT}}{N^2T^3}\sum_{s=1}^{T}E\left\|e_s^{\prime}\sum_{t=1}^{T}g_t^{\dagger}F_t^{\prime}\right\|^2=\frac{a_{NT}}{N^2T^3}E\left[\sum_{s=1}^{T}e_s^{\prime}e_s\left\|\sum_{t=1}^{T}g_t^{\dagger}F_t^{\prime}\right\|^2\right]\\ &\leq a_{NT}N^{-1}E\left\|\frac{1}{NT}\sum_{s=1}^{T}e_s^{\prime}e_s\right\|_2\left\{\frac{1}{T^4}E\left[\left\|\sum_{t=1}^{T}g_t^{\dagger}F_t^{\prime}\right\|^4\right]\right\}^{1/2}=O\left(a_{NT}N^{-1}\right).\end{aligned}$$

Similarly, we can show that $A_{1,3}=O\left(a_{NT}(N^{-1}+T^{-1})\right)$ and $A_{1,4}=O\left(a_{NT}^2\right)$. As a result, $\frac{1}{N^2T^3}\sum_{s=1}^{T}E\left\|\sum_{t=1}^{T}e_s^{\dagger\prime}e_t^{\dagger}F_t^{\prime}\right\|^2=O\left(N^{-1}\right)$ and $\bar{A}_1=O_P\left(C_{NT}^{-1}N^{-1/2}\right)$. Now, let $\breve{A}_2=\frac{1}{NT^2}\sum_{t=1}^{T}\sum_{s=1}^{T}F_se_s^{\dagger\prime}e_t^{\dagger}F_t^{\prime}$. Let $a_{2,mn}$ denote the (m,n)th element of $\breve{A}_2$ for $m,n=1,...,R$. By CS inequality, it is easy to see that $|a_{2,mn}|\leq\{a_{2,mm}a_{2,nn}\}^{1/2}$. This, in conjunction with the Markov inequality, implies that it suffices to show that $\breve{A}_2=O_P\left(T^{-1}\right)$ by showing that $E\left|a_{2,mm}\right|=O\left(T^{-1}\right)$. In fact, by CS inequality, Assumptions A.1(ii), A.3(v) and A.5(i), we can readily show that

$$\begin{aligned}E\left|a_{2,mm}\right| &= E\left(a_{2,mm}\right)=\frac{1}{NT^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\iota_m^{\prime}E\left(F_se_s^{\dagger\prime}e_t^{\dagger}F_t^{\prime}\right)\iota_m\\ &\leq \frac{2}{NT^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\iota_m^{\prime}E\left(F_se_s^{\prime}e_tF_t^{\prime}\right)\iota_m+\frac{2a_{NT}^2}{NT^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\iota_m^{\prime}E\left(F_sg_s^{\dagger\prime}g_t^{\dagger}F_t^{\prime}\right)\iota_m\\ &\leq \frac{2}{T}\max_s\left(\sum_{t=1}^{T}\gamma_{N,FF}(s,t)\right)+\frac{2a_{NT}^2}{NT^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\iota_m^{\prime}E\left(F_sg_s^{\dagger\prime}g_t^{\dagger}F_t^{\prime}\right)\iota_m\\ &= O\left(T^{-1}\right)+O\left(a_{NT}^2\right)=O\left(T^{-1}\right),\end{aligned}$$

where ι_m is the mth column of R-dimensional identity matrix $\mathbb{I}_R$. It follows that $\bar{A}_2 = O_P\left(T^{-1}\right)$ and $\frac{1}{T}\sum_{t=1}^T V_{NT}\left[A_1(t)+A_2(t)\right] F_t'H = O_P\left(C_{NT}^{-2}\right).$

Now, we consider $V_{NT}\frac{1}{T}\sum_{t=1}^T A_3(t)F_t'H$. Note that

$$\begin{aligned}\left\|V_{NT}\frac{1}{T}\sum_{t=1}^T A_3(t)F_t'H\right\| &= \frac{1}{NT^2}\left\|\sum_{s=1}^T \tilde{F}_s F_s' \sum_{t=1}^T \Lambda_0' e_t^\dagger F_t' H\right\| \le \|H\| \left\|\frac{1}{T}\sum_{s=1}^T \tilde{F}_s F_s'\right\| \left\|\frac{1}{NT}\sum_{t=1}^T \Lambda_0' e_t^\dagger F_t'\right\| \\ &= o\left(a_{NT}\right)\end{aligned}$$

provided $\left\|\frac{1}{NT}\sum_{t=1}^T \Lambda_0' e_t^\dagger F_t'\right\| = o\left(a_{NT}\right)$. To see this, we write

$$\begin{aligned}E\left\|\frac{1}{NT}\sum_{t=1}^T \Lambda_0' e_t^\dagger F_t'\right\|^2 &= E\left\|\frac{1}{NT}\sum_{i=1}^N\sum_{t=1}^T \lambda_{i0}\left(e_{it}+a_{NT}F_t'g_{it}\right)F_t'\right\|^2 \\ &\le \frac{2}{N^2T^2}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^T\sum_{s=1}^T E\left(e_{it}e_{js}F_t'F_s\right)\lambda_{j0}'\lambda_{i0} \\ &\quad + \frac{2a_{NT}^2}{N^2T^2}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^T\sum_{s=1}^T g_{js}'E\left(F_sF_s'F_tF_t'\right)g_{it}\lambda_{j0}'\lambda_{i0} \\ &\equiv 2A_{3,1}+2A_{3,2}, \text{ say.}\end{aligned}$$

It is easy to show that $A_{3,1} = O\left(N^{-1}T^{-1}\right)$ under Assumptions A.2(i) and A.3(v). For $A_{3,2}$, using $E\left(F_sF_s'F_tF_t'\right) = E\left(F_sF_s'\right)E\left(F_tF_t'\right) + \mathrm{Cov}(F_sF_s', F_tF_t')$, we have

$$\begin{aligned}A_{3,2} &= \frac{a_{NT}^2}{N^2T^2}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^T\sum_{s=1}^T g_{js}'E\left(F_sF_s'F_tF_t'\right)g_{it}\lambda_{j0}'\lambda_{i0} \\ &= \frac{a_{NT}^2}{N^2T^2}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^T\sum_{s=1}^T g_{js}'\Sigma_F\Sigma_F g_{it}\lambda_{j0}'\lambda_{i0} + \frac{a_{NT}^2}{N^2T^2}\sum_{i=1}^N\sum_{j=1}^N\sum_{t=1}^T\sum_{s=1}^T g_{js}'\mathrm{Cov}\left(F_sF_s', F_tF_t'\right)g_{it}\lambda_{j0}'\lambda_{i0} \\ &\equiv A_{3,2}^{(1)} + A_{3,2}^{(2)}.\end{aligned}$$

By local normalization $\int_0^1 g_i(u)du = 0$, $\lambda_{i0}'\frac{1}{T}\sum_{t=1}^T g_{it} = \lambda_{i0}'\frac{1}{T}\sum_{t=1}^T g_{NT,i}\left(t/T\right) = \lambda_{i0}'\int_0^1 g_{NT,i}\left(\tau\right)d\tau + O\left(\frac{1}{T}\right) = o\left(1\right)$ uniformly in i. Thus $A_{3,2}^{(1)} = o(a_{NT}^2)$. By Davydov inequality and Assumptions A.1(ii), A.2(i), A.3(iii), and A.5(i), we can readily show that $A_{3,2}^{(2)} = O\left(a_{NT}^2T^{-1}\right)$. It follows that $A_{3,2} = o\left(a_{NT}^2\right)$ and $V_{NT}\frac{1}{T}\sum_{t=1}^T A_3(t)F_t'H = o_P\left(a_{NT}\right).$

Now, we consider $V_{NT}\frac{1}{T}\sum_{t=1}^T A_4(t)F_t'H$:

$$\begin{aligned}V_{NT}\frac{1}{T}\sum_{t=1}^T A_4(t)F_t'H &= \frac{1}{NT^2}\sum_{t=1}^T\sum_{s=1}^T(\tilde{F}_s - H'F_s)F_t'\Lambda_0 e_s^\dagger F_t'H + \frac{1}{NT^2}H'\sum_{t=1}^T\sum_{s=1}^T F_sF_t'\Lambda_0 e_s^\dagger F_t'H \\ &\equiv A_{4,1}+A_{4,2}, \text{ say.}\end{aligned}$$

For $A_{4,1}$, we apply CS inequality and the result in part (i) to obtain

$$\begin{aligned}\left\|A_{4,1}\right\| &\le \|H\|\frac{1}{NT^2}\sum_{t=1}^T\sum_{s=1}^T\left\|\tilde{F}_s - H'F_s\right\|\left\|F_t\right\|^2\left\|\Lambda_0 e_s^\dagger\right\| \\ &\le \|H\|\left\{\frac{1}{T}\sum_{t=1}^T\left\|F_t\right\|^2\right\}\left\{\frac{1}{T}\sum_{s=1}^T\left\|\tilde{F}_s - H'F_s\right\|^2\right\}^{1/2}\left\{\frac{1}{N^2T}\sum_{s=1}^T\left\|\Lambda_0' e_s^\dagger\right\|^2\right\}^{1/2} \\ &= O_P\left(1\right)O_P\left(C_{NT}^{-1}\right)O_P(N^{-1/2}+a_{NT}) = O_P\left(C_{NT}^{-2}\right)\end{aligned}$$

as $\frac{1}{N^2T}\sum_{s=1}^{T} E\left\|\Lambda_0' e_s^{\dagger}\right\|^2 = O\left(N^{-1}+a_{NT}^2\right)$ under Assumptions A.2, A.3 and A.5. Let $\bar{A}_{4,2}=\frac{1}{NT^2}\sum_{t=1}^{T}\sum_{s=1}^{T} F_sF_t'\Lambda_0 e_s^{\dagger}F_t'$. Let $a_{4,2,mn}$ denote the (m,n)th element of $\bar{A}_{4,2}$. Then

$$\begin{aligned} |a_{4,2,mn}| &= \left|\frac{1}{NT^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\iota_m' F_sF_t'\Lambda_0 e_s^{\dagger}F_t'\iota_n\right| = \left|\frac{1}{NT^2}\sum_{t=1}^{T}\left(F_t'\iota_n\right)F_t'\sum_{s=1}^{T}\Lambda_0' e_s^{\dagger}\iota_m' F_s\right| \\ &\leq \frac{1}{T}\sum_{t=1}^{T}\|F_t\|^2\frac{1}{NT}\left\|\sum_{s=1}^{T}\Lambda_0' e_s^{\dagger}\iota_m' F_s\right\| = o_P\left(a_{NT}\right), \end{aligned}$$

because

$$\begin{aligned} E\left\|\frac{1}{NT}\sum_{s=1}^{T}\Lambda_0' e_s^{\dagger}\iota_m' F_s\right\|^2 &\leq \frac{2}{N^2T^2}E\left\|\sum_{i=1}^{N}\sum_{t=1}^{T}\lambda_{ir}e_{it}\iota_m' F_t\right\|^2 + \frac{2a_{NT}^2}{N^2T^2}E\left\|\sum_{i=1}^{N}\sum_{t=1}^{T}\lambda_{ir}F_t' g_{it}\iota_m' F_t\right\|^2 \\ &= O(N^{-1}T^{-1}) + o(a_{NT}^2) = o(a_{NT}^2) \end{aligned}$$

by using arguments as used in the analysis of $A_{3,2}$. It follows that $\bar{A}_{4,2}=o_P\left(a_{NT}\right)$ and $V_{NT}\frac{1}{T}\sum_{t=1}^{T}A_4(t)F_t'H$ $= o_P\left(a_{NT}\right)$. Combining the above results yields the claim in part (ii) of the lemma.

(iii) This follows from the results in (i) and (ii) and the triangle inequality.

(iv) Observing that $\frac{1}{T}(\tilde{F}'\tilde{F}-HF'FH)=\frac{1}{T}(\tilde{F}-FH)'(\tilde{F}-FH)+\frac{1}{T}(\tilde{F}-FH)'FH+\frac{1}{T}\left(FH\right)'\left(\tilde{F}-FH\right)$, the results follows from (i) and (ii).

(v) By (A.6) in the proof of Lemma A.1(ii),

$$(T^{-1}\tilde{F}'F)\left(N^{-1}\Lambda_0'\Lambda_0\right)(T^{-1}F'\tilde{F}) + d_{NT} = V_{NT}.$$

Premultiplying both sides by $\left(N^{-1}\Lambda_0'\Lambda_0\right)^{1/2}(T^{-1}F'\tilde{F})$ and using the fact that $T^{-1}\tilde{F}'\tilde{F}=\mathbb{I}_R$, we have

$$\left(N^{-1}\Lambda_0'\Lambda_0\right)^{1/2}(T^{-1}F'F)\left(N^{-1}\Lambda_0'\Lambda_0\right)(T^{-1}F'\tilde{F}) + \bar{d}_{NT} = \left(N^{-1}\Lambda_0'\Lambda_0\right)^{1/2}(T^{-1}F'\tilde{F})V_{NT}$$

where $\bar{d}_{NT}=(T^{-1}\tilde{F}'F)\left(N^{-1}\Lambda_0'\Lambda_0\right)d_{NT}=O_P(N^{-1/2})$. Let $D_{NT}=\left(N^{-1}\Lambda_0'\Lambda_0\right)^{1/2}(T^{-1}F'F)\left(N^{-1}\Lambda_0'\Lambda_0\right)^{1/2}$, $R_{NT}=\left(N^{-1}\Lambda_0'\Lambda_0\right)^{1/2}(T^{-1}F'\tilde{F})$, and $D_0=\Sigma_{\Lambda_0}^{1/2}\Sigma_F\Sigma_{\Lambda_0}^{1/2}$. Then as in Bai (2003, p.161),

$$\left[D_{NT}+\bar{d}_{NT}R_{NT}\right]\Upsilon_{NT}=\Upsilon_{NT}V_{NT}$$

where $\Upsilon_{NT}=R_{NT}V_{NT}^{*-1/2}$ with V_{NT}^{*} being a diagonal matrix that contains the diagonal elements of $R_{NT}'R_{NT}$. That is, V_{NT} contains the eigenvalues of $D_{NT}+\bar{d}_{NT}R_{NT}$ with the corresponding normalized eigenvectors contained in Υ_{NT}. It is trivial to show that

$$\left\|D_{NT}+\bar{d}_{NT}R_{NT}-D_0\right\| = O_P\left(C_{NT}^{-1}\right). \tag{A.9}$$

By the perturbation theory for eigenvalue problem,

$$\left|\mu_j\left(D_{NT}+\bar{d}_{NT}R_{NT}\right)-\mu_j\left(D_0\right)\right| \leq \left\|D_{NT}+\bar{d}_{NT}R_{NT}-D_0\right\| = O_P\left(C_{NT}^{-1}\right),$$

where $\mu_j\left(A\right)$ denotes the jth largest eigenvalue of a symmetric matrix A. That is, $V_{NT}-V_0=O_P\left(C_{NT}^{-1}\right)$.

(vi) Let Υ_0 denote the probability limit of Υ_{NT}. By (A.9) and the eigenvector perturbation theory that requires distinctness of eigenvalues (see, e.g., Steward and Sun (1990), $\|\Upsilon_{NT}-\Upsilon_0\|=O_P\left(C_{NT}^{-1}\right)$. [Let $\left(\phi_j,\mu_j\right)$ and $(\tilde{\phi}_j,\tilde{\mu}_j)$ be the eigenvector-eigenvalue pairs of a symmetric matrix A and its symmetric perturbation version $\tilde{A}=A+\Delta A$, respectively, where the eigenvectors are properly normalized. Then (i) $\tilde{\mu}_j=\mu_j+\phi_j'\Delta A\phi_j+o(\|\Delta A\|^2)$, and (ii) $\tilde{\phi}_j=\phi_j+\sum_{j\neq i}\left[\phi_j'\Delta A\phi_i/(\phi_j-\phi_i)\right]\phi_j+o(\|\Delta A\|^2)$ if $\mu_j\neq\mu_i$ for all $j\neq i$.] This, in conjunction with the definition of R_{NT}, implies that

$$T^{-1}F'\tilde{F}=(N^{-1}\Lambda_0'\Lambda_0)^{-1/2}\Upsilon_{NT}V_{NT}^{*1/2}=\Sigma_{\Lambda_0}^{-1/2}\Upsilon_0V_0^{1/2}+O_P\left(C_{NT}^{-1}\right).$$

It follows that $H = (N^{-1}\Lambda_0'\Lambda_0)(T^{-1}F^{'}\tilde{F})V_{NT}^{-1} = \Sigma_{\Lambda_0}\Sigma_{\Lambda_0}^{-1/2}\Upsilon_0 V_0^{1/2}V_0^{-1} + O_P\left(C_{NT}^{-1}\right) = \Sigma_{\Lambda_0}^{1/2}\Upsilon_0 V_0^{-1/2}$ $+O_P\left(C_{NT}^{-1}\right) = Q_0^{-1} + O_P\left(C_{NT}^{-1}\right)$, where $Q_0 = V_0^{1/2}\Upsilon_0^{-1}\Sigma_{\Lambda_0}^{-1/2}$. ■

Proof of Lemma A.3. (i) The proof parallels that of Lemma A.2(i) and we only sketch it. By (A.8)

$$\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\left\|\tilde{F}_t - H^{'}F_t\right\|^2 \leq 4\left\|V_{NT}^{-1}\right\|^2\sum_{l=1}^{4}\frac{1}{T}\sum_{t=1}^{T}k_{h,tr}\left\|V_{NT}A_l(t)\right\|^2 \equiv 4\left\|V_{NT}^{-1}\right\|^2\sum_{l=1}^{4}A_{lNT}(r), \text{ say.}$$

We prove (i) by finding the bound for $A_{lNT}(r)$, $l = 1,2,3,4$, uniformly in r. Using the fact that $\max_s \sum_{s=1}^{T}[E(e_s^{\dagger\prime} e_t^{\dagger}/N)]^2 = O(1)$ and that

$$\frac{1}{T}\sum_{t=1}^{T}k_{h,tr} = \frac{1}{Th}\sum_{t=1}^{T}K_r\left(\frac{t-r}{Th}\right) = 1 + O\left(\frac{1}{Th}\right) \text{ uniformly in } r \text{ under Assumption A.4,}$$

we have

$$\begin{aligned}
\max_r A_{1NT}(r) &= \max_r \frac{1}{T}\sum_{t=1}^{T}k_{h,tr}\left\|V_{NT}A_1(t)\right\|^2 \leq \max_r \frac{1}{T}\sum_{t=1}^{T}k_{h,tr}\left\{\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right\|\right\}^2 \\
&\leq \frac{1}{T}\left\{\frac{1}{T}\sum_{l=1}^{T}\left\|\tilde{F}_l\right\|^2\right\}\max_t\sum_{s=1}^{T}\left[E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right]^2\max_r\left\{\frac{1}{T}\sum_{t=1}^{T}k_{h,tr}\right\} \\
&= T^{-1}O(1)O(1)O(1) = O_P(T^{-1}).
\end{aligned}$$

For $A_{2NT}(r)$, using notations defined in the proof of Lemma A.2(i) and by CS inequality, we have

$$\begin{aligned}
A_{2NT}(r) &= \frac{1}{T}\sum_{t=1}^{T}k_{h,tr}\left\|V_{NT}A_2(t)\right\|^2 = \frac{1}{T}\sum_{t=1}^{T}k_{h,tr}\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\xi_{st}^{\dagger}\right\|^2 \\
&\leq \frac{2}{T}\sum_{t=1}^{T}k_{h,tr}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s - H'F_s)\xi_{st}^{\dagger}\right\|^2 + \frac{2}{T}\sum_{t=1}^{T}k_{h,tr}\left\|\frac{1}{T}\sum_{s=1}^{T}H'F_s\xi_{st}^{\dagger}\right\|^2 \\
&\equiv 2A_{2NT,1}(r) + 2A_{2NT,2}(r), \text{ say.}
\end{aligned}$$

For $A_{2NT,1}(r)$ we apply Lemma A.2(i) to obtain the rough bound

$$\begin{aligned}
\max_r A_{2NT,1}(r) &\leq \frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s - H'F_s\right\|^2\max_r\frac{1}{T}\sum_{t=1}^{T}k_{h,tr}\max_t\frac{1}{T}\sum_{s=1}^{T}\left\|\xi_{st}^{\dagger}\right\|^2 \\
&= O_P\left(C_{NT}^{-2}\right)O(1)O_P(1) = O_P\left(C_{NT}^{-2}\right)
\end{aligned}$$

as we can readily show that $\max_t \frac{1}{T}\sum_{s=1}^{T}||\xi_{st}^{\dagger}||^2 \leq \max_t \frac{1}{T}\sum_{s=1}^{T}E||\xi_{st}^{\dagger}||^2 + \max_t|\frac{1}{T}\sum_{s=1}^{T}(||\xi_{st}^{\dagger}||^2 - E||\xi_{st}^{\dagger}||^2)| = O(1) + o_P(1)$ by a simple application of Bernstein inequality for strong mixing processes. Let $\bar{A}_{2NT,2}(r) = \frac{1}{T^3}\sum_{t=1}^{T}k_{h,tr}\sum_{s=1}^{T}\sum_{l=1}^{T}F_sF_l'\xi_{st}^{\dagger}\xi_{lt}^{\dagger}$. Observing that $A_{2NT,2}(r) = \text{tr}(HH'\bar{A}_{2NT,2}(r))$, we can bound $A_{2NT,2}(r)$ by bounding each element of $\bar{A}_{2NT,2}(r)$. Let $a_{mn}(r)$ denote the (m,n)th element of $\bar{A}_{2NT,2}(r)$. Noting that $a_{mn}(r) \leq \{a_{mm}(r)a_{nn}(r)\}^{1/2}$, it suffices to bound $a_{mm}(r)$ for $m = 1,...,R$. Observe that

$$\begin{aligned}
a_{mm}(r) &= \frac{1}{T^3}\sum_{t=1}^{T}k_{h,tr}\sum_{s=1}^{T}\sum_{l=1}^{T}\iota_m'F_sF_l'\iota_m\xi_{st}^{\dagger}\xi_{lt}^{\dagger} = \frac{1}{T}\sum_{t=1}^{T}k_{h,tr}\left\{\frac{1}{T}\sum_{s=1}^{T}\iota_m'F_s\xi_{st}^{\dagger}\right\}^2 \\
&\leq \max_t\left|\frac{1}{T}\sum_{s=1}^{T}\iota_m'F_s\xi_{st}^{\dagger}\right|^2\max_r\frac{1}{T}\sum_{t=1}^{T}k_{h,tr} = O_P\left(T^{-1}\ln T + a_{NT}^2\right)O(1).
\end{aligned}$$

It follows that $\max_r A_{2NT,2}(r) = O_P\left(T^{-1}\ln T\right)$ and $\max_r A_{2NT}(r) = O_P\left(T^{-1}\ln T + N^{-1}\right)$. To study $A_{3NT}(r)$, we first study $\bar{A}_{3NT}(r) \equiv \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\left\|\frac{\Lambda_0' e_t^\dagger}{\sqrt{N}}\right\|^2$.

$$\begin{aligned}\bar{A}_{3NT}(r) &= \frac{1}{NT}\sum_{t=1}^{T} k_{h,tr}\left\|\Lambda_0' e_t^\dagger\right\|^2 \leq \frac{2}{NT}\sum_{t=1}^{T} k_{h,tr}\left\|\Lambda_0' e_t\right\|^2 + \frac{2a_{NT}^2}{NT}\sum_{t=1}^{T} k_{h,tr}\left\|\Lambda_0' g_t^\dagger\right\|^2 \\ &\equiv 2\bar{A}_{3NT,1}(r) + \bar{A}_{3NT,2}(r),\ \text{say.}\end{aligned}$$

For $\bar{A}_{3NT,1}(r)$, we have under Assumptions A.2-A.4

$$\begin{aligned}\max_r \bar{A}_{3NT,1}(r) &\leq \max_t\left[N^{-1}E\left\|\Lambda_0' e_t\right\|^2 + N^{-1}\left(\left\|\Lambda_0' e_t\right\|^2 - E\left\|\Lambda_0' e_t\right\|^2\right)\right]\max_r \frac{1}{T}\sum_{t=1}^{T} k_{h,tr} \\ &= O\left(1 + o_P(1)\right)O(1) = O_P(1).\end{aligned}$$

Similarly, $\max_r \bar{A}_{3NT,2}(r) = O_P\left(a_{NT}^2 N\right)$. Then $\max_r \bar{A}_{3NT}(r) = O_P(1)$ and

$$\begin{aligned}\max_r A_{3NT}(r) &= \max_r \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s F_s' \Lambda_0' e_t^\dagger / N\right\|^2 \\ &\leq \frac{1}{N}\max_r \bar{A}_{3NT}(r)\left[\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s\right\|^2\right]\left[\frac{1}{T}\sum_{s=1}^{T}\left\|F_s\right\|^2\right] = O_P(N^{-1}).\end{aligned}$$

For the fourth term, we have

$$\begin{aligned}\max_r A_{4NT}(r) &= \max_r \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s F_t' \Lambda_0' e_s^\dagger / N\right\|^2 \leq \max_r \frac{1}{N^2 T^3}\sum_{t=1}^{T} k_{h,tr}\left\{\sum_{s=1}^{T}\left\|\tilde{F}_s F_t'\right\|\left\|\Lambda_0' e_s^\dagger\right\|\right\}^2 \\ &\leq \frac{1}{N}\max_r\left\{\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\left\|F_t\right\|^2\right\}\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s\right\|^2 \frac{1}{NT}\sum_{s=1}^{T}\left\|\Lambda_0' e_s^\dagger\right\|^2 \\ &= N^{-1}O_P(1)\,O_P(1)\,O_P(1) = O_P(N^{-1}),\end{aligned}$$

as we can show that $\max_r \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\left\|F_t\right\|^2 \leq \max_r \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}E\left\|F_t\right\|^2 + \max_r |\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}[\left\|F_t\right\|^2 - E\left\|F_t\right\|^2]\} = O_P(1)$. Combining these results, we have $\max_r ||\frac{1}{T}\sum_{r=1}^{T} k_{h,tr}(\tilde{F}_r - H'F_r)\left(\tilde{F}_r - H'F_r\right)'|| = O_P(T^{-1}\ln T + N^{-1})$.

(ii) The proof of (ii) is analogous to that of (i) with some modifications similar to those used in the proof of Lemma A.2(ii).

(iii) Write $S_{Tr}^{(0)} = \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}E\left(F_t F_t'\right) + \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\left[F_t F_t' - E\left(F_t F_t'\right)\right] \equiv S_{T,1}^{(r,0)} + S_{T,2}^{(r,0)}$, say. Using $E\left(F_t F_t'\right) = \Sigma_F$ and the Riemann sum approximation of integral, we have

$$\max_r\left\|S_{T,1}^{(r,0)} - \Sigma_F\right\| = \max_r\left|\frac{1}{Th}\sum_{t=1}^{T} K_r\left(\frac{t-r}{Th}\right) - 1\right|\left\|\Sigma_F\right\| = O\left(\frac{1}{Th}\right).$$

By Bernstein inequality for strong mixing processes, we can readily show that $\max_r ||S_{T,2}^{(r,0)}|| = O_P\left(T^{-1/2}(\ln T)^{1/2}\right)$. It follows that $\max_r ||S_{Tr}^{(0)} - \Sigma_F|| = O_P\left(T^{-1/2}(\ln T)^{1/2}\right)$.

(iv) Using $\tilde{F}_s = H'F_s + (\tilde{F}_s - H'F_s)$, we make the following decomposition:

$$
\begin{aligned}
S_{Tr} &= \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\tilde{F}_t\tilde{F}_t' \\
&= H'S_{Tr}^{(0)}H + \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}H'F_t(\tilde{F}_t - H'F_t)' + \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}(\tilde{F}_t - H'F_t)F_t'H \\
&\quad + \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}(\tilde{F}_t - H'F_t)(\tilde{F}_t - H'F_t)' \\
&\equiv S_{Tr,1} + S_{Tr,2} + S_{Tr,3} + S_{Tr,4}, \text{ say.}
\end{aligned}
$$

By Lemmas A.2(vi) and A.3(iii), $\max_r ||S_{Tr,1} - \left(Q_0^{-1}\right)' \Sigma_F Q_0^{-1}|| = O_P((Th)^{-1/2} (\ln T)^{1/2} + N^{-1/2})$. By Lemma A.3(i)-(ii), $\max_r ||S_{Tr,2}|| = \max_r ||S_{Tr,3}|| = O_P(T^{-1}\ln T + N^{-1} + a_{NT})$ and $\max_r ||S_{Tr,4}|| = O_P(T^{-1}\ln T + N^{-1})$. Combining these results yields the desired result.

(v) This follows from Lemmas A.2(vi) and A.3 (iv) above. ■

Proof Lemma A.4. (i) First, using $\tilde{F}_tF_t' = H'F_tF_t' + (\tilde{F}_t - H'F_t)F_t' = H'\Sigma_F + H'\left(F_tF_t' - \Sigma_F\right) + (\tilde{F}_t - H'F_t)F_t'$, we have:

$$
\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\tilde{F}_tF_t'g_{it} &= H'\Sigma_F\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}g_{it} + H'\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\left(F_tF_t' - \Sigma_F\right)g_{it} + \frac{1}{T}\sum_{t=1}^{T} k_{h,tr}(\tilde{F}_t - H'F_t)F_t'g_{it} \\
&\equiv L_1(i,r) + L_2(i,r) + L_3(i,r), \text{ say.}
\end{aligned}
$$

By the uniform approximation property for Riemann integral and Bernstein inequality for mixing processes, we have that under Assumptions A.1 and A.3-A.5

$$
\begin{aligned}
\max_{i,r}\|L_1(i,r)\| &\leq \|H'\Sigma_F\|\,\bar{c}_g\max_r\frac{1}{T}\sum_{t=1}^{T} k_{h,tr} = \|H'\Sigma_F\|\,\bar{c}_g\left\{1 + O\left(\frac{1}{Th}\right)\right\} = O(1), \\
\max_{i,r}\|L_2(i,r)\| &\leq \|H'\|\max_r\left|\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}\left(F_tF_t' - \Sigma_F\right)g_{it}\right| = o_p(1).
\end{aligned}
$$

In addition, by arguments as used in the proof of Lemma A.3(i), we can readily show $\max_{i,r}\|L_3(i,r)\| = o_P(1)$. Alternatively, we can apply CS inequality and Lemma A.2(i)

$$
\begin{aligned}
\max_{i,r}\|L_3(i,r)\| &\leq \max_{i,r}\left\{\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}^2\|F_tg_{it}\|^2\right\}^{1/2}\left\{\frac{1}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t - H'F_t\right\|^2\right\}^{1/2} \\
&= O_P(h^{-1/2})O_P\left(C_{NT}^{-1}\right) = o_P(1).
\end{aligned}
$$

(ii) It is standard to show that $\max_{i,r}\left\|\frac{1}{T}\sum_{t=1}^{T} k_{h,tr}F_t'e_{it}\right\| = O_P(T^{-1/2}h^{-1/2}\ln(NT))$ by using Bernstein inequality for strong mixing processes.

(iii) Using $e_{is}^{\dagger} = e_{is} + a_{NT}F_s'g_{is}$ and CS inequality,

$$
\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s - HF_s)e_{is}^{\dagger}\right\|^2 \leq \frac{2}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s - HF_s)e_{is}\right\|^2 + \frac{2a_{NT}^2}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s - HF_s)F_s'g_{is}\right\|^2.
$$

It is standard to show that the first term is $O_P\left(C_{NT}^{-4}\right)$ and $\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s - HF_s)F_s'g_{is}\right\|^2 = O_P\left(C_{NT}^{-4}\right) + o_P\left(a_{NT}^2\right)$. It follows that $\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s - HF_s)e_{is}^{\dagger}\right\|^2 = O_P\left(C_{NT}^{-4}\right)$.

(iv) By (A.8) and CS inequality

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|\frac{1}{T}\sum_{t=1}^{T}\left(\tilde{F}_t-H'F_t\right)e_{it}k_{h,tr}\right\|^2 \le 4\sum_{l=1}^{4}\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|\frac{1}{T}\sum_{t=1}^{T}A_l\left(t\right)e_{it}k_{h,tr}\right\|^2 \equiv 4\sum_{l=1}^{4}I_l, \text{ say.}$$

Using $\tilde{F}_s=(\tilde{F}_s-H'F_s)+H'F_s$ and CS inequality,

$$\begin{aligned}
I_1 &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_sE(e_s^{\dagger\prime}e_t^{\dagger}/N)\right\}e_{it}k_{h,tr}\right\|^2\\
&\le 2\left\|V_{NT}^{-1}\right\|^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s-H'F_s)E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right\}e_{it}k_{h,tr}\right\|^2\\
&\quad +2\left\|V_{NT}^{-1}H'\right\|^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}F_sE(e_s^{\dagger\prime}e_t^{\dagger}/N)\right\}e_{it}k_{h,tr}\right\|^2 \equiv 2I_{1,1}+2I_{1,2}, \text{ say.}
\end{aligned}$$

For $I_{1,1}$, we have

$$\begin{aligned}
I_{1,1} &= \left\|V_{NT}^{-1}\right\|^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s-H'F_s)\left(\frac{1}{T}\sum_{t=1}^{T}E(e_s^{\dagger\prime}e_t^{\dagger}/N)e_{it}k_{h,tr}\right)\right\|^2\\
&\le \left\|V_{NT}^{-1}\right\|^2\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s-H'F_s\right\|^2\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\frac{1}{T}\sum_{s=1}^{T}\left\{\frac{1}{T}\sum_{t=1}^{T}E(e_s^{\dagger\prime}e_t^{\dagger}/N)e_{it}k_{h,tr}\right\}^2\\
&= O_P\left(C_{NT}^{-2}\right)O_P\left(N^{-1}\right)=O_P\left(N^{-1}C_{NT}^{-2}\right),
\end{aligned}$$

as one can readily show that $\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\frac{1}{T}\sum_{s=1}^{T}E[\frac{1}{T}\sum_{t=1}^{T}E(e_s^{\dagger\prime}e_t^{\dagger}/N)e_{it}k_{h,tr}]^2=O\left(N^{-1}+a_{NT}^2\right).$ For $I_{1,2}$, by straightforward moment calculations, we can show that

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}E\left\|\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}F_sE(e_s^{\dagger\prime}e_t^{\dagger}/N)\right\}e_{it}k_{h,tr}\right\|^2\\
&= \frac{1}{N^3T^5}\sum_{i=1}^{N}\sum_{r=1}^{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t_1=1}^{T}\sum_{s_1=1}^{T}E(e_s^{\dagger\prime}e_t^{\dagger})E(e_{s_1}^{\dagger\prime}e_{t_1}^{\dagger})E\left(F_s'F_{s_1}e_{it}e_{it_1}\right)k_{h,tr}k_{h,t_1r}\\
&= \frac{1}{N^3T^5}\sum_{i=1}^{N}\sum_{r=1}^{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{s_1=1}^{T}E(e_s^{\dagger\prime}e_t^{\dagger})E(e_{s_1}^{\dagger\prime}e_t^{\dagger})E\left(F_s'F_{s_1}e_{it}^2\right)k_{h,tr}^2=O\left(T^{-1}N^{-2}h^{-1}+a_{NT}^2T^{-1}h^{-1}\right).
\end{aligned}$$

So $I_{1,2}=O\left(T^{-1}N^{-2}h^{-1}+a_{NT}^2T^{-1}h^{-1}\right)$ and $I_1=O_P\left(N^{-1}C_{NT}^{-2}+T^{-1}N^{-2}h^{-1}+a_{NT}^2T^{-1}h^{-1}\right)$. For I_2, we have

$$\begin{aligned}
I_2 &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left[e_s^{\dagger\prime}e_t^{\dagger}/N-E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right]\right\}e_{it}k_{h,tr}\right\|^2 \\
&\leq \frac{4}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left[e_s^{\prime}e_t/N-E(e_s^{\prime}e_t/N)\right]\right\}e_{it}k_{h,tr}\right\|^2 \\
&\quad +\frac{4a_{NT}^2}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left[g_s^{\dagger\prime}g_t^{\dagger}/N-E(g_s^{\dagger\prime}g_t^{\dagger}/N)\right]\right\}e_{it}k_{h,tr}\right\|^2 \\
&\quad +\frac{4a_{NT}^2}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left[e_s^{\prime}g_t^{\dagger}/N-E(e_s^{\prime}g_t^{\dagger}/N)\right]\right\}e_{it}k_{h,tr}\right\|^2 \\
&\quad +\frac{4a_{NT}^4}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left[g_s^{\dagger\prime}g_t^{\dagger}/N-E(g_s^{\dagger\prime}g_t^{\dagger}/N)\right]\right\}e_{it}k_{h,tr}\right\|^2 \\
&\equiv 4I_{2,1}+4I_{2,2}+4I_{2,3}+4I_{2,4},\ \text{say.}
\end{aligned}$$

It is trivial to show that $I_{2,j}=o_P\left(a_{NT}^2\right)$ for $j=2,3$ and $I_{2,4}=O_P\left(a_{NT}^4\right)$. To bound $I_{2,1}$, notice that

$$\begin{aligned}
I_{2,1} &\leq \frac{2}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s-H^{\prime}F_s)\xi_{st}\right\}e_{it}k_{h,tr}\right\|^2 \\
&\quad +\frac{2}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}H^{\prime}\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}F_s\xi_{st}\right\}e_{it}k_{h,tr}\right\|^2\equiv 2I_{2,1}^{(1)}+2I_{2,1}^{(2)}.
\end{aligned}$$

One can readily show that $I_{2,1}^{(1)}=O_P\left(C_{NT}^{-2}N^{-1}\right)$. Noting that under Assumptions A.1(ii), A.3(iv) and A.4,

$$\begin{aligned}
&\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}E\left\|\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}F_s\xi_{st}\right\}e_{it}k_{h,tr}\right\|^2 \\
&= \frac{1}{N^3T^5}\sum_{r=1}^{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t_1=1}^{T}\sum_{s_1=1}^{T}E\left(\xi_{st}\xi_{s_1t_1}e_t^{\prime}e_{t_1}F_s^{\prime}F_{s_1}\right)k_{h,tr}k_{h,t_1r} \\
&= \frac{1}{T^5}\sum_{r=1}^{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t_1=1}^{T}\sum_{s_1=1}^{T}\left[E\left(\xi_{st}\xi_{s_1t_1}\xi_{tt_1}F_s^{\prime}F_{s_1}\right)+E\left(\xi_{st}\xi_{s_1t_1}F_s^{\prime}F_{s_1}\right)\gamma_N\left(t,t_1\right)\right]k_{h,tr}k_{h,t_1r} \\
&\leq N^{-3/2}\max_{s,t}\left\|N^{1/2}\xi_{st}\right\|_4^3\frac{1}{T^5}\sum_{r=1}^{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t_1=1}^{T}\sum_{s_1=1}^{T}\left\|F_s^{\prime}F_{s_1}\right\|_4k_{h,tr}k_{h,t_1r} \\
&\quad +N^{-1}\max_{s,t}\left\|N^{1/2}\xi_{st}\right\|_4^2\frac{1}{T^5}\sum_{r=1}^{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{t_1=1}^{T}\sum_{s_1=1}^{T}\gamma_N\left(t,t_1\right)\left\|F_s^{\prime}F_{s_1}\right\|_2k_{h,tr}k_{h,t_1r} \\
&= O(N^{-3/2}+N^{-1}T^{-1}),
\end{aligned}$$

we have

$$I_{2,1}^{(2)}\leq\left\|V_{NT}^{-1}H^{\prime}\right\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}F_s\left[e_s^{\prime}e_t/N-E(e_s^{\prime}e_t/N)\right]\right\}e_{it}k_{h,tr}\right\|^2=O_P(N^{-3/2}+N^{-1}T^{-1}).$$

It follows that $I_{2,1}=O_P(C_{NT}^{-2}N^{-1})$ and $I_2=O_P(C_{NT}^{-2}N^{-1})+o_P(a_{NT}^2)$. In addition, we can readily show that

$$\begin{aligned}
I_3 &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_sF_s'(\Lambda_0'e_t^{\dagger}/N)\right\}e_{it}k_{h,tr}\right\|^2\\
&\le 2\left\|V_{NT}^{-1}\right\|\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_sF_s'\right\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|\frac{1}{NT}\sum_{t=1}^{T}\Lambda_0'e_te_{it}k_{h,tr}\right\|^2\\
&\quad +2a_{NT}^2\left\|V_{NT}^{-1}\right\|\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_sF_s'\right\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|\frac{1}{NT}\sum_{t=1}^{T}\Lambda_0'g_t^{\dagger}e_{it}k_{h,tr}\right\|^2\\
&= O_P(N^{-3/2}+a_{NT}^2T^{-1}h^{-1}),
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \frac{1}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_sF_t'\Lambda_0'e_s^{\dagger}/N\right\}e_{it}k_{h,tr}\right\|^2\\
&\le \frac{2}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\left(\tilde{F}_s-H'F_s\right)e_s^{\dagger\prime}\Lambda_0F_t/N\right\}e_{it}k_{h,tr}\right\|^2\\
&\quad +\frac{2}{NT}\sum_{i=1}^{N}\sum_{r=1}^{T}\left\|V_{NT}^{-1}H'\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}F_se_s^{\dagger\prime}\Lambda_0F_t/N\right\}e_{it}k_{h,tr}\right\|^2\\
&= O_P(C_{NT}^{-4})+O_P(N^{-3/2}+a_{NT}^2N^{-1/2})
\end{aligned}$$

It follows that $\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}||\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s-H'F_s)e_{is}k_{h,st}||^2=O_P(N^{-3/2}+T^{-1})+o_P(a_{NT}^2)$. ■

Proof Lemma A.5. (i) Using $F_tF_t'=E\left(F_tF_t'\right)+F_tF_t'-E(F_tF_t')$, we have

$$\frac{1}{T}\sum_{t=1}^{T}k_{h,tr}F_tF_t'g_{it}=\Sigma_F\frac{1}{T}\sum_{t=1}^{T}k_{h,tr}g_{it}+\frac{1}{T}\sum_{t=1}^{T}k_{h,tr}\left(F_tF_t'-\Sigma_F\right)g_{it}.$$

The second term is $o_P(1)$ uniformly in (i,r) under Assumptions A.1(ii), A.3(iii), A.4 and A.5(i). For the first term, we consider three cases: (i1) $r\in(Th,T(1-h)]$, (i2) $r\in(1,Th]$, and (i3) $r\in(T(1-h),T]$. In case (i1), by the fact that the kernel function K has compact support on $[-1,1]$, the uniform approximation of Riemann integral, and the dominated convergence theorem, we have

$$\begin{aligned}
\Sigma_F\frac{1}{T}\sum_{t=1}^{T}k_{h,tr}g_{it} &= \Sigma_F\frac{1}{Th}\sum_{t=1}^{T}K_r\left(\frac{t-r}{Th}\right)g_i\left(\frac{t-r+r}{T}\right)\\
&= \Sigma_F\frac{1}{Th}\sum_{t=r-Th}^{r+Th}K\left(\frac{t-r}{Th}\right)g_i\left(\frac{t-r+r}{T}\right)\\
&= \Sigma_F\int_0^1K\left(u\right)g_i\left(uh+\frac{r}{T}\right)du+O\left(\frac{1}{Th}\right)\\
&= \Sigma_Fg_i\left(\frac{r}{T}\right)+o\left(1\right).
\end{aligned}$$

In case (i2),

$$\begin{aligned}
\Sigma_F \frac{1}{T}\sum_{t=1}^{T} k_{h,tr} g_{it} &= \Sigma_F \frac{1}{Th}\sum_{t=1}^{T} K_r\left(\frac{t-r}{Th}\right) g_i\left(\frac{t-r+r}{T}\right) \\
&= \Sigma_F \frac{1}{Th\int_{-r/(Th)}^{1} K(u)\,du}\sum_{t=1}^{r+Th} K\left(\frac{t-r}{Th}\right) g_i\left(\frac{t-r+r}{T}\right) \\
&= \Sigma_F \frac{1}{\int_{-r/(Th)}^{1} K(u)\,du}\int_{-r/(Th)}^{1} K(u)\, g_i\left(uh+\frac{r}{T}\right) du + O\left(\frac{1}{Th}\right) \\
&= \Sigma_F g_i\left(\frac{r}{T}\right) + o(1).
\end{aligned}$$

A similar result holds in case (i3). It follows that $\frac{1}{T}\sum_{t=1}^{T} k_{h,tr} F_t F_t' g_{it} = \Sigma_F g_i\left(\frac{r}{T}\right) + o_P(1)$ uniformly in (i,r).

(ii) As in the above analysis, $\frac{1}{T}\sum_{t=1}^{T} F_t F_t' g_{it} = \Sigma_F \frac{1}{T}\sum_{t=1}^{T} g_{it} + o_P(1) = o_P(1)$. ■

Proof Lemma A.6. (i) By (A.3) and CS inequality,

$$\frac{1}{N}\sum_{i=1}^{N}\left\|\tilde{\lambda}_{i0} - H^{-1}\lambda_{i0}\right\|^2 \leq \frac{3}{N}\sum_{i=1}^{N}\left\{\|D_5(i)\|^2 + \|D_6(i)\|^2 + \|D_7(i)\|^2\right\}.$$

By straightforward moment calculations and Chebyshev inequality,

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\|D_5(i)\|^2 &= \frac{1}{NT^2}\sum_{i=1}^{N}\operatorname{tr}\left(HH'\sum_{s=1}^{T}\sum_{r=1}^{T} F_s e_{is}^{\dagger} e_{ir}^{\dagger\prime} F_r'\right) \\
&\leq \|H\|^2 \frac{1}{NT^2}\left\|\sum_{i=1}^{N}\sum_{s=1}^{T}\sum_{r=1}^{T} F_s e_{is}^{\dagger} e_{ir}^{\dagger\prime} F_r'\right\| = O_P\left(C_{NT}^{-2} + a_{NT}^2\right).
\end{aligned}$$

Using $e_{is}^{\dagger} = e_{is} + a_{NT} F_s' g_{is}$, we can readily show that

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\|D_6(i)\|^2 &= \frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s - HF_s)e_{is}^{\dagger}\right\|^2 \\
&\leq \frac{2}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s - HF_s)e_{is}\right\|^2 + \frac{2a_{NT}^2}{N}\sum_{i=1}^{N}\left\|\frac{1}{T}\sum_{s=1}^{T}(\tilde{F}_s - HF_s)F_s' g_{is}\right\|^2. \\
&= O_P\left(C_{NT}^{-2} + a_{NT}^2\right) = O_P\left(C_{NT}^{-2}\right).
\end{aligned}$$

By Lemma A.2(i), $\frac{1}{N}\sum_{i=1}^{N}\|D_7(i)\|^2 \leq \frac{1}{T}\left\|\tilde{F}'(\tilde{F}H^{-1} - F)\right\| \frac{1}{N}\sum_{i=1}^{N}\|\lambda_{i0}\|^2 = O_P\left(C_{NT}^{-2}\right)$. It follows that $\frac{1}{N}\sum_{i=1}^{N}\left\|\tilde{\lambda}_{i0} - H^{-1}\lambda_{i0}\right\|^2 = O_P\left(C_{NT}^{-2}\right)$.

(ii) By the CS inequality, $\frac{1}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t\right\|^4 \leq \frac{8}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t - H'F_t\right\|^4 + \frac{8}{T}\sum_{t=1}^{T}\|H'F_s\|^4$. Apparently, the second term is bounded from above by $8\|H\|^4 \frac{1}{T}\sum_{s=1}^{T}\|F_s\|^4 = O_P(1)$ under Assumption A.1(ii). For the first term, we apply Lemma A.2(i) to obtain a rough bound

$$\begin{aligned}
\frac{8}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t - H'F_t\right\|^4 &\leq \max_s\left\|\tilde{F}_s - H'F_s\right\|^2 \frac{1}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t - H'F_t\right\|^2 \\
&\leq T\left\{\frac{1}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t - H'F_t\right\|^2\right\}^2 = O_P\left(TC_{NT}^{-4}\right).
\end{aligned}$$

It follows that $\frac{1}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t\right\|^4 = O_P\left(TC_{NT}^{-4}\right)+O_P(1)=O_P(1)$. Similarly,

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t\right\|^6 &\leq \frac{32}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t-H'F_t\right\|^6+\frac{32}{T}\sum_{t=1}^{T}\left\|H'F_s\right\|^6 \\
&\leq \max_t\left\|\tilde{F}_t-H'F_t\right\|^2\frac{32}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t-H'F_t\right\|^4+O_P(1) \\
&= T\left\{\frac{1}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t-H'F_t\right\|^2\right\}\frac{32}{T}\sum_{t=1}^{T}\left\|\tilde{F}_t-H'F_t\right\|^4+O_P(1) \\
&= TO_P\left(C_{NT}^{-2}\right)TO_P\left(C_{NT}^{-4}\right)+O_P(1)=O_P(1).
\end{aligned}$$

(iii) Noting that $\int_{-t/(Th)}^{1}K(u)\,du\geq\int_{0}^{1}K(u)\,du=\frac{1}{2}$ for any $t\in(0,\lfloor Th\rfloor)$ and $\int_{-1}^{1-t/(Th)}K(u)\,du\geq\int_{-1}^{0}K(u)\,du=\frac{1}{2}$ for any $t\in(\lfloor T(1-h)\rfloor,T]$, we have

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}k_{h,st}^2 &= \frac{1}{Th^2}\sum_{t=1}^{T}K_t\left(\frac{s-t}{Th}\right)^2 \\
&= \frac{1}{Th^2}\left\{\sum_{t=\lfloor Th\rfloor}^{\lfloor T(1-h)\rfloor}+\sum_{t=1}^{\lfloor Th\rfloor-1}+\sum_{t=\lfloor T(1-h)\rfloor+1}^{T}\right\}K_t\left(\frac{s-t}{Th}\right)^2 \\
&= \frac{1}{Th^2}\sum_{t=\lfloor Th\rfloor}^{\lfloor T(1-h)\rfloor}K\left(\frac{t-s}{Th}\right)^2+\frac{1}{Th^2}\sum_{t=1}^{\lfloor Th\rfloor-1}\frac{1}{\left(\int_{-t/(Th)}^{1}K(u)\,du\right)^2}K\left(\frac{s-t}{Th}\right)^2 \\
&\quad+\frac{1}{Th^2}\sum_{t=\lfloor T(1-h)\rfloor+1}^{T}\frac{1}{\left(\int_{-1}^{1-t/(Th)}K(u)\,du\right)^2}K_t\left(\frac{s-t}{Th}\right)^2 \\
&\leq \frac{1}{Th^2}\sum_{t=\lfloor Th\rfloor}^{\lfloor T(1-h)\rfloor}K\left(\frac{t-s}{Th}\right)^2+\frac{4}{Th^2}\sum_{t=1}^{\lfloor Th\rfloor-1}K\left(\frac{t-s}{Th}\right)^2+\frac{4}{Th^2}\sum_{t=\lfloor T(1-h)\rfloor+1}^{T}K\left(\frac{t-s}{Th}\right)^2.
\end{aligned}$$

By the uniform approximation property of Riemann integral, $\frac{1}{Th}\sum_{t=\lfloor Th\rfloor}^{\lfloor T(1-h)\rfloor}K\left(\frac{t-s}{Th}\right)^2=\int_{-1}^{1}K(u)^2\,du+O\left(\frac{1}{Th}\right)$ uniformly in s under Assumption A.4. So the first term is $O\left(\frac{1}{Th}\right)$ uniformly in s. Similar results hold for the other two terms. Thus $\max_s\frac{1}{T}\sum_{t=1}^{T}k_{h,st}^2=O\left(h^{-1}\right)$.

(iv) Observe that $L_{st}=\left(k_{h,st}S_{Tt}^{-1}-\mathbb{I}_R\right)\left(k_{h,st}S_{Tt}^{-1}-\mathbb{I}_R\right)\leq k_{h,st}^2S_{Tt}^{-1}S_{Tt}^{-1}+\mathbb{I}_R$, we have

$$\begin{aligned}
\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{F}_s'L_{st}\tilde{F}_s\tilde{F}_s'\tilde{F}_s &\leq \frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^2\tilde{F}_s'S_{Tt}^{-1}S_{Tt}^{-1}\tilde{F}_s\tilde{F}_s'\tilde{F}_s+\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s'\tilde{F}_s\tilde{F}_s'\tilde{F}_s \\
&\leq \max_t\left\|S_{Tt}^{-1}\right\|^2\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}k_{h,st}^2\left\|\tilde{F}_s\right\|^4+\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s\right\|^4 \\
&\leq \left[\max_t\left\|S_{Tt}^{-1}\right\|^2\max_s\left(\frac{1}{T}\sum_{t=1}^{T}k_{h,st}^2\right)+1\right]\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s\right\|^4 \\
&= \left[O_P(1)\,O\left(h^{-1}\right)+1\right]O_P(1)=O\left(h^{-1}\right).
\end{aligned}$$

(v) First $\frac{1}{T}\sum_{t=1}^{T}||(\tilde{F}_t-H'F_t)(\tilde{F}_t-H'F_t)'||^2\leq T\{\frac{1}{T}\sum_{t=1}^{T}||\tilde{F}_t-H'F_t||^2\}^2=O\left(TC_{NT}^{-4}\right)$. By (A.8), $\frac{1}{T}\sum_{t=1}^{T}||(\tilde{F}_t-H'F_t)F_t'H||^2\leq\left\|V_{NT}^{-1}\right\|\|H\|\{\frac{3}{T}\sum_{t=1}^{T}\|V_{NT}\left(A_{1t}+A_{2t}\right)F_t'\|^2+\frac{3}{T}\sum_{t=1}^{T}\|V_{NT}A_{3t}F_t'\|^2+\frac{3}{T}\sum_{t=1}^{T}$

$\|V_{NT}A_{4t}F_t'\|^2\}$. We bound each term in the last pair of curly brackets. The first term satisfies

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\|V_{NT}(A_{1t}+A_{2t})F_t'\|^2 &= \frac{1}{T^3N^2}\sum_{t=1}^{T}\left\|\sum_{s=1}^{T}\tilde{F}_s e_s^{\dagger\prime}e_t^{\dagger}F_t'\right\|^2\\ &\leq \frac{2}{T^3N^2}\sum_{t=1}^{T}\left\|\sum_{s=1}^{T}(\tilde{F}_s-H'F_s)e_s^{\dagger\prime}e_t^{\dagger}F_t\right\|^2+\frac{2}{T^3N^2}\sum_{t=1}^{T}\left\|H'\sum_{s=1}^{T}F_s e_s^{\dagger\prime}e_t^{\dagger}F_t\right\|^2\\ &\equiv 2A_1+2A_2,\ \text{say.}\end{aligned}$$

For A_1, we only consider its rough bound. Noting that $\frac{1}{T^2N^2}\sum_{t=1}^{T}\sum_{s=1}^{T}E\|e_s^{\dagger\prime}e_t^{\dagger}F_t'\|^2=O(1)$, we have by Lemma A.2(i),

$$A_1\leq\left\{\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s-H'F_s\right\|^2\right\}\left\{\frac{1}{T^2N^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left\|e_s^{\dagger\prime}e_t^{\dagger}F_t\right\|^2\right\}=O_P\left(C_{NT}^{-2}\right)O_P(1)=O_P\left(C_{NT}^{-2}\right).$$

For A_2, we observe that $A_2\leq 2\|H\|^2\bar{A}_2$ where $\bar{A}_2=\frac{1}{T^3N^2}\sum_{t=1}^{T}\left\|\sum_{s=1}^{T}F_s e_s^{\dagger\prime}e_t^{\dagger}F_t\right\|^2$. By CS inequality,

$$\begin{aligned}E\left(\bar{A}_2\right) &\leq \frac{4}{T^3N^2}\sum_{t=1}^{T}E\left\|\sum_{s=1}^{T}F_s e_s'e_tF_t'\right\|^2+\frac{4\gamma_{NT}^2}{T^3N^2}\sum_{t=1}^{T}E\left\|\sum_{s=1}^{T}F_s g_s^{\dagger\prime}e_tF_t'\right\|^2\\ &\quad+\frac{4\gamma_{NT}^2}{T^3N^2}\sum_{t=1}^{T}E\left\|\sum_{s=1}^{T}F_s e_s'g_t^{\dagger}F_t'\right\|^2+\frac{4\gamma_{NT}^4}{T^3N^2}\sum_{t=1}^{T}E\left\|\sum_{s=1}^{T}F_s g_s^{\dagger\prime}g_t^{\dagger}F_t'\right\|^2\\ &\equiv 4A_{2,1}+4A_{2,2}+4A_{2,3}+4A_{2,4}.\end{aligned}$$

Noting that under Assumptions A.1(ii) and A.3(iv)

$$\begin{aligned}\frac{1}{TN^2}\sum_{t=1}^{T}E\left\|\frac{1}{T}\sum_{s=1}^{T}F_sE(e_s'e_t)F_t'\right\|^2 &= \frac{1}{T^3N^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}E(e_s'e_t)E(e_r'e_t)E(F_t'F_tF_r'F_s)\\ &\leq \frac{1}{T}\max_t\|F_t\|_4^4\left\{\max_t\sum_{s=1}^{T}\gamma_N(s,t)\right\}^2=O\left(T^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{TN^2}\sum_{t=1}^{T}E\left\|\frac{1}{T}\sum_{s=1}^{T}F_s\left[e_s'e_t-E(e_s'e_t)\right]F_t'\right\|^2 &= \frac{1}{T^3}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}E(\xi_{st}\xi_{rt}F_t'F_tF_r'F_s)\\ &\leq N^{-1}\left\|N^{1/2}\xi_{st}\right\|_4^2\frac{1}{T^3}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}\|F_t'F_t\|_4\|F_r'F_s\|_4=O\left(N^{-1}\right)\end{aligned}$$

we have

$$\begin{aligned}A_{2,1} &\leq \frac{2}{TN^2}\sum_{t=1}^{T}E\left\|\frac{1}{T}\sum_{s=1}^{T}F_sE(e_s'e_t)F_t'\right\|^2+\frac{2}{TN^2}\sum_{t=1}^{T}E\left\|\frac{1}{T}\sum_{s=1}^{T}F_s\left[e_s'e_t-E(e_s'e_t)\right]F_t'\right\|^2\\ &= O\left(T^{-1}\right)+O\left(N^{-1}\right)=O\left(C_{NT}^{-2}\right).\end{aligned}$$

For $A_{1,2}$, $A_{1,3}$, and $A_{1,4}$, one can readily obtain their rough bounds given by $O(a_{NT}^2)$, $O(a_{NT}^2)$, and $O(a_{NT}^4)$, respectively. It follows that $A_2=O_P(C_{NT}^{-2})$ and $\frac{1}{T}\sum_{t=1}^{T}\|V_{NT}(A_{1t}+A_{2t})F_t'\|^2=O_P(C_{NT}^{-2})$. In addition, noting that $\frac{1}{TN^2}\sum_{t=1}^T E||\Lambda_0' e_t^\dagger F_t'||^2 = O(N^{-1}+a_{NT}^2) = O(N^{-1})$, we have

$$\frac{1}{T}\sum_{t=1}^T \|V_{NT}A_{3t}F_t'\|^2 = \frac{1}{T^3N^2}\sum_{t=1}^T \left\|\sum_{s=1}^T \tilde F_s F_s' \Lambda_0' e_t^\dagger F_t'\right\|^2 \le \left\{\frac{1}{T}\sum_{s=1}^T \left\|\tilde F_s F_s'\right\|^2\right\}\frac{1}{TN^2}\sum_{t=1}^T \left\|\Lambda_0' e_t^\dagger F_t'\right\|^2 = O_P(N^{-1}).$$

Similarly,

$$\frac{1}{T}\sum_{t=1}^T \|V_{NT}A_{4t}F_t'\|^2 = \frac{1}{T^3N^2}\sum_{t=1}^T \left\|\sum_{s=1}^T \tilde F_s e_s^{\dagger\prime}\Lambda_0 F_t F_t'\right\|^2 \le \frac{1}{T^2N^2}\left\|\sum_{s=1}^T \tilde F_s e_s^{\dagger\prime}\Lambda_0\right\|^2 \frac{1}{T}\sum_{t=1}^T \|F_t\|^2 = O_P(N^{-1}),$$

because

$$\begin{aligned}
\frac{1}{T^2N^2}\left\|\sum_{s=1}^T \tilde F_s e_s^{\dagger\prime}\Lambda_0\right\|^2 &\le \frac{2}{T^2N^2}\left\|\sum_{s=1}^T \left(\tilde F_s - H'F_s\right) e_s^{\dagger\prime}\Lambda_0\right\|^2 + \frac{2}{T^2N^2}\left\|H'\sum_{s=1}^T F_s e_s^{\dagger\prime}\Lambda_0\right\|^2 \\
&\le \frac{2}{N}\frac{1}{T}\sum_{s=1}^T \left\|\tilde F_s - H'F_s\right\|^2 \frac{1}{TN}\sum_{s=1}^T \left\|e_s^{\dagger\prime}\Lambda_0\right\|^2 + \|H\|^2 \frac{2}{T^2N^2}\left\|\sum_{s=1}^T F_s e_s^{\dagger\prime}\Lambda_0\right\| \\
&= N^{-1}O_P(C_{NT}^{-2}) + O_P(N^{-1}) = O_P(N^{-1}).
\end{aligned}$$

It follows that $\frac{1}{T}\sum_{t=1}^T ||(\tilde F_t - H'F_t)\tilde F_t'||^2 \le \frac{2}{T}\sum_{t=1}^T ||(\tilde F_t - H'F_t)(\tilde F_t - H'F_t)'||^2 + \frac{1}{T}\sum_{t=1}^T ||(\tilde F_t - H'F_t)F_t H||^2$ $= O_P(TC_{NT}^{-4}) + O_P(C_{NT}^{-2}) = O_P(C_{NT}^{-2} + TN^{-2})$.

(vi) By (A.8), $\frac{1}{NT}\sum_{t=1}^T \left\|\sum_{i=1}^N \lambda_{i0}' H'^{-1}(\tilde F_t - H'F_t)e_{it}\right\|^2 \le \sum_{j=1}^4 \frac{1}{NT}\sum_{t=1}^T \left\|\sum_{i=1}^N \lambda_{i0}' H'^{-1} A_j(t) e_{it}\right\|^2 \equiv$ $4\sum_{j=1}^4 II_j$, say. For II_1, we have

$$\begin{aligned}
II_1 &= \frac{1}{NT}\sum_{t=1}^T \left\|\sum_{i=1}^N \lambda_{i0}' H'^{-1} V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^T \tilde F_s E(e_s^{\dagger\prime}e_t^\dagger/N)e_{it}\right\|^2 \\
&= \frac{1}{NT}\sum_{t=1}^T \left\|\mathrm{tr}\left(H'^{-1}V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^T \tilde F_s E(e_s^{\dagger\prime}e_t^\dagger/N)\sum_{i=1}^N e_{it}\lambda_{i0}'\right)\right\|^2 \\
&\le \left\|H'^{-1}V_{NT}^{-1}\right\|^2 \frac{1}{NT}\sum_{t=1}^T \left\|\frac{1}{T}\sum_{s=1}^T \tilde F_s E(e_s^{\dagger\prime}e_t^\dagger/N)\sum_{i=1}^N e_{it}\lambda_{i0}'\right\|^2.
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{1}{NT}\sum_{t=1}^T \left\|\frac{1}{T}\sum_{s=1}^T \tilde F_s E(e_s^{\dagger\prime}e_t^\dagger/N)\sum_{i=1}^N e_{it}\lambda_{i0}'\right\|^2 \\
\le\ & \frac{4}{NT}\sum_{t=1}^T \left\|\frac{1}{T}\sum_{s=1}^T \tilde F_s E(e_s'e_t/N)\sum_{i=1}^N e_{it}\lambda_{i0}'\right\|^2 + \frac{4a_{NT}}{NT}\sum_{t=1}^T \left\|\frac{1}{T}\sum_{s=1}^T \tilde F_s E(e_s'\breve g_t/N)\sum_{i=1}^N e_{it}\lambda_{i0}'\right\|^2 \\
&+\frac{4a_{NT}}{NT}\sum_{t=1}^T \left\|\frac{1}{T}\sum_{s=1}^T \tilde F_s E(\breve g_s'e_t/N)\sum_{i=1}^N e_{it}\lambda_{i0}'\right\|^2 + \frac{a_{NT}^2}{NT}\sum_{t=1}^T \left\|\frac{1}{T}\sum_{s=1}^T \tilde F_s E(\breve g_s'\breve g_t/N)\sum_{i=1}^N e_{it}\lambda_{i0}'\right\|^2 \\
\equiv\ & 4II_{1,1} + 4II_{1,2} + 4II_{1,3} + 4II_{1,4},\ \text{say.}
\end{aligned}$$

One can readily show that $II_{1,1} = O_P(T^{-1})$, $II_{1,2} = O_P(a_{NT}N^{-1}\ln(NT))$, $II_{1,3} = O_P(a_{NT}N^{-1}\ln(NT))$, and $II_{1,4} = O_P(a_{NT}^2)$. It follows that $II_1 = O_P(C_{NT}^{-1})$. Similarly,

$$\begin{aligned}
II_2 &= \frac{1}{NT}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\lambda_{i0}' H'^{-1} V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left[e_s^{\dagger\prime}e_t^{\dagger}/N - E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right]e_{it}\right\|^2 \\
&= \frac{1}{NT}\sum_{t=1}^{T}\left\|\mathrm{tr}\left(H'^{-1}V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left[e_s^{\dagger\prime}e_t^{\dagger}/N - E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right]\sum_{i=1}^{N}e_{it}\lambda_{i0}'\right)\right\|^2 \\
&\leq \left\|H'^{-1}V_{NT}^{-1}\right\|\frac{1}{NT}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left[e_s^{\dagger\prime}e_t^{\dagger}/N - E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right]\sum_{i=1}^{N}e_{it}\lambda_{i0}'\right\|^2 \\
&\leq \left\|H'^{-1}V_{NT}^{-1}\right\|\frac{1}{NT}\sum_{t=1}^{T}\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left[e_s^{\dagger\prime}e_t^{\dagger}/N - E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right]\right\|^2\left\|\sum_{i=1}^{N}e_{it}\lambda_{i0}'\right\|^2 \\
&\leq \left\|H'^{-1}V_{NT}^{-1}\right\|\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{F}_s\right\|^2\frac{1}{NT}\sum_{t=1}^{T}\left\{\frac{1}{T}\sum_{s=1}^{T}\left[e_s^{\dagger\prime}e_t^{\dagger}/N - E(e_s^{\dagger\prime}e_t^{\dagger}/N)\right]^2\left\|\sum_{i=1}^{N}e_{it}\lambda_{i0}'\right\|^2\right\} \\
&= O_P(1)\,O_P(1)\,O_P\left(N^{-1}\right) = O_P\left(N^{-1}\right),
\end{aligned}$$

$$\begin{aligned}
II_3 &= \frac{1}{NT}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\lambda_{i0}' H'^{-1} V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s F_s'\left(\Lambda_0' e_t^{\dagger}/N\right)e_{it}\right\|^2 \\
&= \frac{1}{NT}\sum_{t=1}^{T}\left\|\mathrm{tr}\left(H'^{-1}V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s F_s'\left(\Lambda_0' e_t^{\dagger}/N\right)\sum_{i=1}^{N}e_{it}\lambda_{i0}'\right)\right\|^2 \\
&\leq \left\|H'^{-1}V_{NT}^{-1}\right\|\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s F_s'\right\|^2\frac{1}{NT}\sum_{t=1}^{T}\left\|\left(\Lambda_0' e_t^{\dagger}/N\right)\sum_{i=1}^{N}e_{it}\lambda_{i0}'\right\|^2 \\
&= O_P(1)\,O_P(1)\,O_P\left(N^{-1}\right) = O_P\left(N^{-1}\right),
\end{aligned}$$

and

$$\begin{aligned}
II_4 &= \frac{1}{NT}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}\lambda_{i0}' H'^{-1} V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s F_t'\left(\Lambda_0' e_s^{\dagger}/N\right)e_{it}\right\|^2 \\
&= \frac{1}{NT}\sum_{t=1}^{T}\left\|\mathrm{tr}\left(H'^{-1}V_{NT}^{-1}\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left(e_s^{\dagger\prime}\Lambda_0/N\right)\sum_{i=1}^{N}F_t e_{it}\lambda_{i0}'\right)\right\|^2 \\
&\leq \left\|H'^{-1}V_{NT}^{-1}\right\|\left\|\frac{1}{T}\sum_{s=1}^{T}\tilde{F}_s\left(e_s^{\dagger\prime}\Lambda_0/N\right)\right\|^2\frac{1}{NT}\sum_{t=1}^{T}\left\|\sum_{i=1}^{N}F_t e_{it}\lambda_{i0}'\right\|^2 \\
&= O_P(1)\,O_P(N^{-1})O_P(1) = O_P(N^{-1}).
\end{aligned}$$

Consequently, $\frac{1}{NT}\sum_{t=1}^{T}||\sum_{i=1}^{N}\lambda_{i0}' H'^{-1}(\tilde{F}_t - H'F_t)e_{it}||^2 = O_P\left(C_{NT}^{-2}\right)$.

(vii) The proof is analogous to that of Lemma A.2(ii) and thus omitted. ■

B Some Additional Simulation and Application Results

In this appendix, we report some additional simulation and applications results.

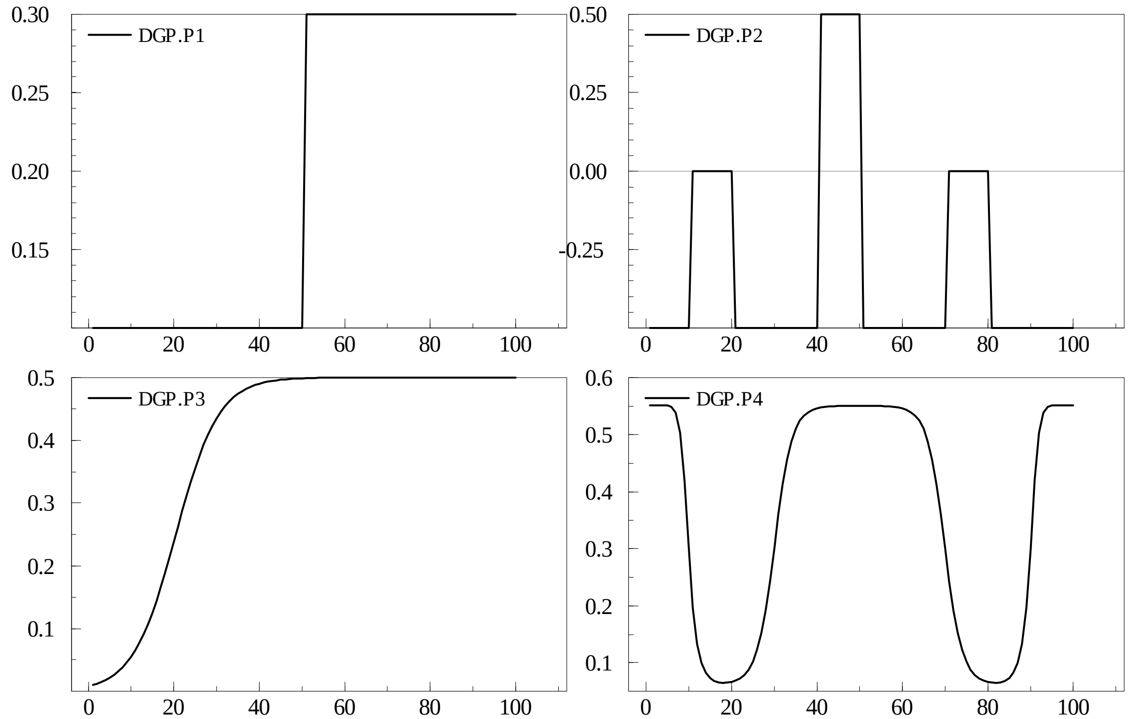

Figure 1: The factor loadings' paths for DGP.P1-P4 when $T = 100$

B.1 Some additional simulation results

First, following the suggestion of an anonymous referee, we plot λ_{it} for DGP.P1-P4 as a function of t for a representative cross-sectional unit. As we mentioned in the paper, DGP.P1-P2 have a single and multiple structural breaks, respectively, while DGP.P3-P4 describe two kinds of smooth structural changes. Among them, the factor loadings given by DGP.P3 are monotonic functions of t/T (or t), while the factor loadings given by DGP.P4 are smooth transition functions of t/T with multiple regime shifts. Figure 1 plots the paths of factor loadings under DGP.P1-P4 as functions of t when $T = 100$.

Second, to examine the sensitivity of our nonparametric test to the choice of the bandwidth parameter h, we set

$$h = c \cdot \frac{2.35}{\sqrt{12}} T^{-1/5}$$

for $c = 0.5$, 1 and 1.5. Tables A.1 and A.2 report the empirical rejection rates of our test at the 5% and 10% significance levels when the number of common factors is fixed as the true value and determined by BN's information criterion, respectively. As shown in Table A.1, the size of our test is robust to the choice of bandwidth. However, the power of our test reported in Table A.2 is a bit sensitive to the choice of bandwidth. For DGPs P1, P3, P5, and P7, the larger the bandwidth, the higher the power. In contrast, the power of the test for DGPs P2, P4, P6 and P8 tends to decrease as the bandwidth increases. Moreover, the power increases quickly as either N or T increases.

Third, we consider the tests when the number of factors are estimated by using Su and Wang's (2017) local-PCA-based information criterion. As mentioned in the paper, Su and Wang's (2017) information criterion can consistently estimate the true number of breaks under both the null and alternative hypotheses.

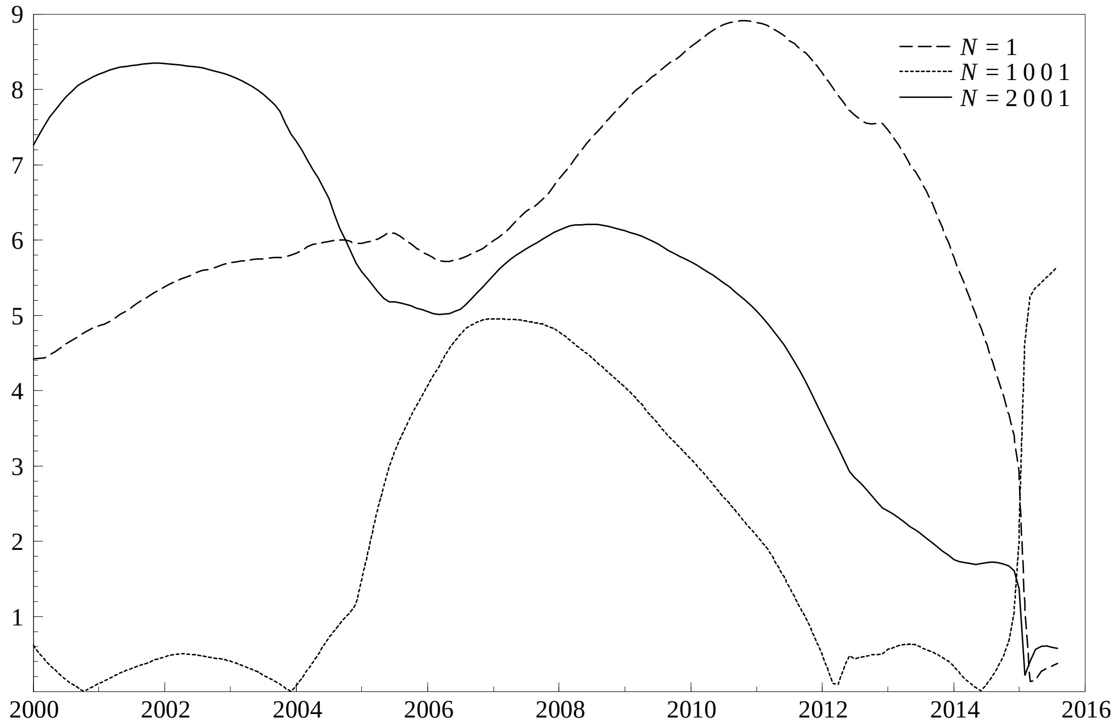

Figure 2: Some representative factor loadings estimated by local PCA

Tables A.3 and A.4 report the empirical rejection rates of several tests considered in the paper. As expected, the results in these two tables are quite similar to those in Tables 1 and 2.

B.2 Some additional application results

Following the suggestion of an anonymous referee, we use Su and Wang's (2017) local PCA to estimate the time-varying factor loadings in the empirical study. Since there are $N = 2684$ stocks and the factor loadings for these stocks are quite different from each other, it is impossible to plot them one by one. For this reason, we only plot the estimates of some representative factor loadings in Figure 2. From the figure we can see that the estimated factor loadings show significant structural changes that very likely appear to be smooth structural changes.

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Table A.1 The size of our test with different bandwidth sequences under DGP.S1-S4

DGP	N	T	R is fixed to the true value						R is determined from the data					
			$c = 0.5$		$c = 1$		$c = 1.5$		$c = 0.5$		$c = 1$		$c = 1.5$	
			5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
S1	100	100	6.6	12.0	5.0	10.8	4.4	10.8	6.6	12.0	5.0	10.8	4.4	10.8
	100	200	4.8	10.2	5.8	12.4	5.8	13.2	4.8	10.2	5.8	12.4	5.8	13.2
	200	100	4.6	10.4	4.8	8.8	4.4	10.6	4.6	10.4	4.8	8.8	4.4	10.6
	200	200	5.0	10.0	5.4	10.8	5.4	10.4	5.0	10.0	5.4	10.8	5.4	10.4
S2	100	100	6.2	11.6	5.2	9.6	4.8	10.4	6.2	11.6	5.2	9.6	4.8	10.4
	100	200	6.4	9.6	4.6	9.8	5.0	10.0	6.4	9.6	4.6	9.8	5.0	10.0
	200	100	6.8	13.2	5.4	10.6	6.6	10.6	6.8	13.2	5.4	10.6	6.6	10.6
	200	200	6.0	11.4	6.6	11.2	6.4	12.0	6.0	11.4	6.6	11.2	6.4	12.0
S3	100	100	4.4	10.0	5.6	10.8	4.6	10.6	4.4	10.0	5.6	10.8	4.6	10.6
	100	200	5.0	9.8	4.8	9.8	5.6	10.8	5.0	9.8	4.8	9.8	5.6	10.8
	200	100	4.8	11.0	6.8	12.2	7.2	13.0	4.8	11.0	6.8	12.2	7.2	13.0
	200	200	5.6	11.6	7.4	13.4	7.8	13.4	5.6	11.6	7.4	13.4	7.8	13.4
S4	100	100	6.8	11.0	5.2	12.0	5.6	10.0	6.8	11.0	5.2	12.0	5.6	10.0
	100	200	6.4	12.4	5.2	10.4	4.8	11.2	6.4	12.4	5.2	10.4	4.8	11.2
	200	100	6.2	13.4	6.0	12.0	5.8	11.6	6.2	13.4	6.0	12.0	5.8	11.6
	200	200	5.2	9.2	5.0	9.6	4.6	10.8	5.2	9.2	5.0	9.6	4.6	10.8

Note: (i) The results are obtained by setting $h = c(2.35/\sqrt{12})T^{-1/5}$ for $c = 0.5, 1$, and 1.5; (ii) R is the number of common factors.

Table A.2 The power of our test with different bandwidth sequences under DGP.P1-P8

DGP	N	T	R is fixed to the true value						R is determined from the data					
			$c = 0.5$		$c = 1$		$c = 1.5$		$c = 0.5$		$c = 1$		$c = 1.5$	
			5%	10%	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
P1	100	100	47.8	60.4	72.2	81.4	81.4	87.8	46.0	60.8	72.8	80.8	81.2	87.6
	100	200	91.6	97.4	98.4	99.6	99.4	100	91.2	96.6	98.8	99.6	99.4	99.8
	200	100	73.0	84.4	94.0	97.2	98.0	98.8	72.8	83.0	94.2	97.2	98.0	98.8
	200	200	98.8	99.6	100	100	100	100	98.6	99.8	100	100	100	100
P2	100	100	52.2	62.0	29.4	41.4	10.0	18.0	51.2	61.6	29.8	41.6	9.6	17.0
	100	200	94.6	97.6	82.2	86.8	37.2	48.4	94.0	98.4	85.2	88.6	28.8	42.6
	200	100	66.4	77.6	41.0	51.8	12.2	22.0	65.4	77.6	37.8	50.0	12.8	19.8
	200	200	99.8	99.8	93.0	95.8	55.8	67.0	99.6	99.6	92.4	94.6	53.8	65.4
P3	100	100	30.6	41.8	37.2	47.8	49.2	60.0	29.0	44.0	36.0	41.0	51.0	60.0
	100	200	54.0	68.2	64.8	73.8	78.4	86.8	55.2	67.4	65.2	74.8	77.8	87.2
	200	100	28.6	38.6	42.4	53.8	71.2	79.4	29.4	40.0	42.4	52.0	71.6	78.4
	200	200	60.0	67.8	76.0	82.2	95.8	97.2	60.8	67.8	76.0	81.6	95.8	97.2
P4	100	100	59.4	71.8	25.0	38.0	11.0	19.4	60.0	72.4	25.4	35.8	11.2	19.0
	100	200	99.8	100	74.2	83.6	34.6	46.0	99.8	100	73.4	83.6	34.2	46.2
	200	100	82.6	88.6	40.6	52.8	16.2	24.6	81.0	88.6	40.0	52.6	15.6	24.4
	200	200	100	100	92.0	94.4	51.4	62.0	100	100	91.0	94.6	51.6	62.8
P5	100	100	42.8	55.0	67.8	79.8	79.6	86.2	42.0	54.0	69.0	80.2	78.4	87.0
	100	200	90.0	95.0	97.4	99.2	100	100	89.8	94.4	97.6	99.4	100	100
	200	100	69.0	78.6	90.0	94.2	95.6	97.4	69.4	79.4	90.4	94.8	94.8	97.0
	200	200	99.6	99.6	100	100	100	100	99.2	99.8	100	100	100	100
P6	100	100	48.8	59.8	29.6	38.6	11.6	19.8	48.2	60.6	27.8	39.8	11.0	19.4
	100	200	95.6	97.4	81.2	86.0	36.6	48.8	95.2	97.6	80.8	85.2	36.0	49.2
	200	100	67.8	78.6	38.4	52.6	15.2	22.2	69.2	79.8	40.4	52.6	15.8	23.0
	200	200	99.6	99.8	92.4	95.8	53.6	65.0	99.8	99.8	92.4	95.4	54.0	63.2
P7	100	100	29.4	38.2	34.0	45.8	50.8	62.6	30.4	38.0	33.2	46.0	49.0	62.8
	100	200	57.4	63.6	62.4	72.2	77.2	85.8	58.2	64.2	63.6	72.2	77.8	85.6
	200	100	32.4	42.2	44.0	53.0	69.2	77.2	31.8	41.8	44.8	52.2	70.2	77.6
	200	200	62.4	73.2	78.8	85.0	96.0	98.2	62.2	72.4	78.2	84.4	95.6	98.2
P8	100	100	64.4	76.0	38.2	50.6	11.8	19.2	65.0	76.8	38.6	51.2	12.0	20.6
	100	200	99.0	99.6	91.2	94.8	48.6	60.6	98.2	99.6	90.6	95.4	48.4	60.4
	200	100	78.6	85.4	49.4	60.8	15.4	23.0	78.2	86.2	49.4	60.8	14.4	23.8
	200	200	100	100	97.8	99.0	66.4	76.2	100	100	97.8	99.2	66.0	76.0

Note: See the note in Table A.1.

Table A.3 Size of tests under DGP.S1-S4 when the number of factors is determined by Su and Wang's (2017) IC

DGP	N	T	SM_B		$SW17$		HI_{LM}		CDG_{LM}		BE_{LM}	
			5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
S1	100	100	5.0	10.8	6.6	13.4	0.6	3.8	3.4	8.2	2.8	6.5
	100	200	5.8	12.4	7.4	13.0	2.4	6.8	4.8	7.4	3.4	7.5
	200	100	4.8	8.8	5.2	10.2	1.6	4.4	1.4	7.0	2.7	6.3
	200	200	5.4	10.8	5.8	12.0	1.6	6.8	3.6	8.8	3.4	7.5
S2	100	100	5.2	9.6	7.4	12.0	0.4	2.4	2.0	8.2	2.8	6.4
	100	200	4.6	9.8	5.0	11.4	1.0	5.8	2.0	6.6	3.7	7.8
	200	100	5.4	10.6	6.4	14.0	0.4	1.8	1.0	4.6	2.8	6.4
	200	200	6.6	11.2	7.0	14.0	0.6	5.4	2.6	6.8	3.6	7.7
S3	100	100	5.6	10.8	7.2	11.2	0.4	2.2	2.2	8.8	11.9	20.3
	100	200	4.8	9.8	6.0	11.4	1.6	5.2	2.0	6.0	15.3	24.7
	200	100	6.8	12.2	7.8	11.6	0.4	1.8	1.2	5.0	11.9	20.2
	200	200	7.4	13.4	8.2	13.0	0.8	5.2	2.4	7.0	15.3	24.8
S4	100	100	5.2	12.0	6.2	12.2	0.4	4.6	2.8	8.0	2.8	6.4
	100	200	5.2	10.4	4.2	10.4	2.0	6.8	4.6	8.6	3.4	7.5
	200	100	6.0	12.0	6.8	12.0	1.6	3.2	2.6	6.6	2.8	6.3
	200	200	5.0	9.6	5.6	10.2	2.2	7.0	4.0	8.4	3.4	7.4

Note: (i) SM_B denotes the results of our $\widehat{SM}_{NT}$ test using bootstrap critical values; (ii) $SW17$ denotes the results of Su and Wang's (2017) bootstrap-based test; (iii) HI_{LM} denotes Han and Inoue's (2014) sup-LM test; (iv) CDG_{LM} denotes Chen et al.'s (2014) sup-LM test; (v) BE_{LM} denotes Breitung and Eickmeier's (2011) N variable-specific sup-LM test. The main entries report the average percentage of rejection.

Table A.4 Power of tests under DGP.P1-P8 when the number of factors is determined by Su and Wang's (2017) IC

DGP	N	T	SM_B		$SW17$		HI_{LM}		CDG_{LM}		BE_{LM}	
			5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
P1	100	100	72.2	81.2	67.8	79.4	0.8	4.4	2.4	7.2	5.9	11.1
	100	200	98.4	99.6	98.4	99.4	4.2	10.6	2.0	6.8	11.2	17.8
	200	100	94.0	97.2	92.2	96.4	0.8	4.0	2.4	6.6	5.7	10.7
	200	200	100	100	100	100	5.0	12.2	2.2	6.6	11.1	17.5
P2	100	100	29.6	41.4	26.2	40.4	0.6	2.0	2.2	8.6	3.8	8.3
	100	200	82.6	87.0	77.2	84.2	1.6	6.4	2.2	6.4	6.7	12.7
	200	100	40.8	51.8	27.6	40.6	0.8	2.8	1.8	8.6	3.7	8.1
	200	200	93.0	95.8	85.2	91.6	1.6	5.8	1.8	7.6	6.5	12.4
P3	100	100	37.0	46.8	46.2	56.2	35.6	66.4	6.8	16.8	4.9	10.3
	100	200	65.0	74.2	76.8	86.4	97.4	99.8	10.2	18.4	9.8	17.2
	200	100	42.4	53.4	45.2	60.2	37.4	71.4	6.6	15.4	5.2	10.7
	200	200	76.0	82.2	84.2	92.0	99.2	100	10.2	20.0	9.8	17.7
P4	100	100	25.2	38.0	25.8	36.4	0.4	1.6	1.0	4.0	3.5	7.9
	100	200	74.0	83.6	72.2	81.4	0.6	4.0	3.0	5.6	5.4	10.6
	200	100	40.6	52.8	34.2	45.2	0.4	1.4	1.0	5.8	3.5	7.8
	200	200	92.0	94.4	86.8	92.8	0.2	3.8	3.2	6.4	5.5	10.7
P5	100	100	68.0	79.8	63.0	75.8	1.4	5.8	3.2	8.8	4.9	10.1
	100	200	97.4	99.2	96.8	99.0	6.0	12.8	4.4	8.4	9.8	16.6
	200	100	90.0	94.2	88.0	92.0	2.0	6.6	1.2	6.6	4.9	9.9
	200	200	100	100	99.6	99.8	3.8	11.4	4.8	10.6	9.4	15.8
P6	100	100	29.4	38.8	27.2	36.0	0.8	5.0	3.6	9.2	3.7	8.1
	100	200	81.0	85.8	75.8	82.6	3.2	10.4	5.6	10.8	6.2	12.1
	200	100	38.6	52.6	27.6	38.2	1.4	4.6	1.6	7.6	3.6	7.9
	200	200	92.4	95.8	85.2	90.8	3.0	9.8	4.6	11.0	6.2	11.9
P7	100	100	33.8	45.8	36.6	54.6	32.4	65.0	7.4	14.6	5.0	10.5
	100	200	62.6	72.2	74.4	86.2	98.2	99.6	12.0	18.0	9.5	16.9
	200	100	44.2	53.0	43.8	60.0	36.6	68.8	7.0	15.2	5.0	10.5
	200	200	78.6	85.0	86.4	92.6	99.0	99.8	10.8	19.6	9.7	17.5
P8	100	100	38.2	50.8	35.4	47.4	0.4	2.4	2.0	8.6	4.2	10.0
	100	200	91.0	94.8	88.4	92.2	1.4	6.0	2.2	6.6	10.5	18.0
	200	100	49.4	60.8	32.8	43.8	0.8	2.8	1.8	8.8	4.7	9.5
	200	200	97.8	99.0	93.2	95.4	1.6	5.8	2.0	7.6	9.3	15.9

Note: See the note in Table A.3.