# Supplementary material on "Exact Local Whittle estimation in long memory time series with multiple poles"

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This supplementary material contains complementary Monte Carlo results, detailed proofs of Theorems 1 and 2 and technical lemmas needed to complete the proofs.

## 1 More finite sample results

Tables I and II show the results for $\Phi(L) = 1 + 1.06L - 0.6L^2$ for the case of a single pole at $\pi/4$. In that case the spectral peak at $\pi/4$ caused by the AR(2) induces a bias in the estimation of the memory parameter if a large bandwidth is used. This biasing effect seems to be stronger for the ELW than for the LW in those cases where the properties of the LW are theoretically supported ($d = 0.4, 0.8$), which can be observed in both tables with $m = 32$ and $m = 63$. In those cases the LW estimator performs better than the ELW. However, the bias is quite controlled with $m = 8$, especially for the ELW, which shows a significantly lower bias than the LW estimator even for $d = 0.4$ and $d = 0.8$. In any case, the ELW estimator with a low $m$ is the best option in terms of MSE, which can also be observed in the estimated

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kernel densities in Figure 1. The better performance of the misspecified ELW with large $m$ is due to compensation between the positive bias induced by the AR(2) term and the negative one caused by the wrong specification of the local behaviour of the spectral density function around $\pi/4$. As $m$ increases the effect of the AR(2) term dominates and the bias becomes positive.

TABLE I. Finite sample results, $\Phi(L) = 1 + 1.06L - 0.6L^2$, $n = 512$

| | $d = -3$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $m = 8$ | | | $m = 32$ | | | $m = 63$ | | |
| | bias | s.d. | MSE | bias | s.d. | MSE | bias | s.d. | MSE |
| LW | 2.9931 | 0.0364 | 8.9598 | 2.6780 | 0.2324 | 7.2259 | 2.4233 | 0.2287 | 5.9247 |
| ELW | 0.0194 | 0.1968 | 0.0391 | 0.2088 | 0.0785 | 0.0497 | 0.4523 | 0.0580 | 0.2080 |
| ELW-sine | -0.3012 | 0.1912 | 0.1273 | -0.0480 | 0.0807 | 0.0088 | 0.1416 | 0.0552 | 0.0231 |
| TLW (HC) | 1.7711 | 0.3475 | 3.2574 | 1.0687 | 0.1851 | 1.1764 | 1.0452 | 0.1408 | 1.1122 |
| TLW (V) | 2.2564 | 0.3213 | 5.1946 | 1.3629 | 0.1894 | 1.8933 | 1.2810 | 0.1555 | 1.6652 |
| | $d = -1.5$ | | | | | | | | |
| | $m = 8$ | | | $m = 32$ | | | $m = 63$ | | |
| | bias | s.d. | MSE | bias | s.d. | MSE | bias | s.d. | MSE |
| LW | 0.8831 | 0.3106 | 0.8763 | 0.5867 | 0.1628 | 0.3707 | 0.6295 | 0.1079 | 0.4080 |
| ELW | 0.0228 | 0.2022 | 0.0414 | 0.2066 | 0.0780 | 0.0488 | 0.4519 | 0.0565 | 0.2074 |
| ELW-sine | -0.3007 | 0.1966 | 0.1290 | -0.0492 | 0.0825 | 0.0092 | 0.1418 | 0.0545 | 0.0231 |
| TLW (HC) | 0.1433 | 0.2479 | 0.0820 | 0.2675 | 0.0966 | 0.0809 | 0.4607 | 0.0665 | 0.2166 |
| TLW (V) | 0.1659 | 0.2468 | 0.0885 | 0.2542 | 0.0987 | 0.0744 | 0.4489 | 0.0719 | 0.2067 |
| | $d = 0.4$ | | | | | | | | |
| | $m = 8$ | | | $m = 32$ | | | $m = 63$ | | |
| | bias | s.d. | MSE | bias | s.d. | MSE | bias | s.d. | MSE |
| LW | 0.0144 | 0.2102 | 0.0444 | 0.1807 | 0.0794 | 0.0390 | 0.3167 | 0.0546 | 0.1033 |
| ELW | 0.0074 | 0.2003 | 0.0402 | 0.2047 | 0.0789 | 0.0481 | 0.4539 | 0.0568 | 0.2093 |
| ELW-sine | -0.3046 | 0.1850 | 0.1270 | -0.0538 | 0.0837 | 0.0099 | 0.1407 | 0.0550 | 0.0228 |
| TLW (HC) | 0.0961 | 0.2253 | 0.0600 | 0.2085 | 0.0935 | 0.0522 | 0.3352 | 0.0648 | 0.1165 |
| TLW (V) | 0.0816 | 0.2406 | 0.0646 | 0.1979 | 0.1002 | 0.0492 | 0.3269 | 0.0710 | 0.1119 |

*Note:* Results for $\Phi(L)(1 - 2\cos\frac{\pi}{4}L + L^2)^{d_0}X_t = u_t I(t \geq 1)$, $u_t \sim N(0,1)$. LW, ELW, ELW-sine, TLW (HC), TLW (V) denote the original Local Whittle estimator (Arteche and Robinson, 2000), the Exact Local Whittle, the misspecified ELW without the sine term and tapered versions of the Local Whittle estimator with the "efficient" taper in Hurvich and Chen (2000) and the triangular Barlett taper ($p = 2$ in Velasco, 1999) respectively.

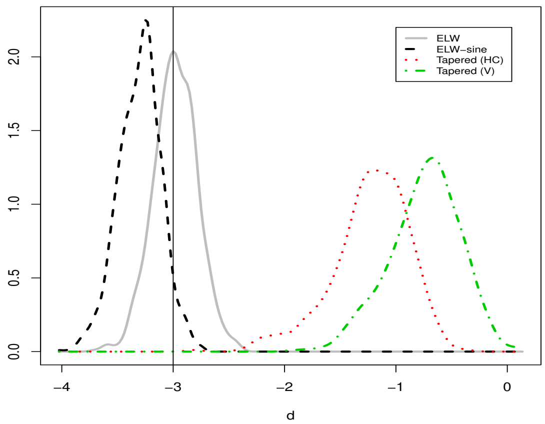
TABLE II. Finite sample results, $\Phi(L) = 1 + 1.06L - 0.6L^2$, $n = 512$

| | $d = 0.8$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $m = 8$ | | | $m = 32$ | | | $m = 63$ | | |
| | bias | s.d. | MSE | bias | s.d. | MSE | bias | s.d. | MSE |
| LW | 0.0374 | 0.2056 | 0.0437 | 0.1739 | 0.0759 | 0.0360 | 0.2394 | 0.0652 | 0.0616 |
| ELW | 0.0081 | 0.2045 | 0.0419 | 0.2125 | 0.0772 | 0.0511 | 0.4550 | 0.0590 | 0.2105 |
| ELW-sine | -0.3098 | 0.1981 | 0.1352 | -0.0474 | 0.0791 | 0.0085 | 0.1441 | 0.0556 | 0.0239 |
| TLW (HC) | 0.2046 | 0.2415 | 0.1002 | 0.2607 | 0.0963 | 0.0773 | 0.3518 | 0.0702 | 0.1287 |
| TLW (V) | 0.1885 | 0.2435 | 0.0948 | 0.2356 | 0.0954 | 0.0646 | 0.3270 | 0.0722 | 0.1121 |
| | $d = 1.5$ | | | | | | | | |
| | $m = 8$ | | | $m = 32$ | | | $m = 63$ | | |
| | bias | s.d. | MSE | bias | s.d. | MSE | bias | s.d. | MSE |
| LW | -0.3926 | 0.1345 | 0.1722 | -0.4321 | 0.0803 | 0.1932 | -0.4529 | 0.0861 | 0.2126 |
| ELW | 0.0055 | 0.2051 | 0.0421 | 0.2074 | 0.0794 | 0.0493 | 0.4551 | 0.0581 | 0.2105 |
| ELW-sine | -0.3109 | 0.1902 | 0.1329 | -0.0531 | 0.0816 | 0.0095 | 0.1422 | 0.0548 | 0.0232 |
| TLW (HC) | 0.3821 | 0.2186 | 0.1938 | 0.3832 | 0.1053 | 0.1580 | 0.4115 | 0.0711 | 0.1744 |
| TLW (V) | 0.3831 | 0.2255 | 0.1977 | 0.3194 | 0.1023 | 0.1125 | 0.3318 | 0.0656 | 0.1144 |
| | $d = 3$ | | | | | | | | |
| | $m = 8$ | | | $m = 32$ | | | $m = 63$ | | |
| | bias | s.d. | MSE | bias | s.d. | MSE | bias | s.d. | MSE |
| LW | -1.9780 | 0.0621 | 3.9162 | -1.9913 | 0.0302 | 3.9660 | -1.9966 | 0.0670 | 3.9908 |
| ELW | 0.0108 | 0.1947 | 0.0380 | 0.2038 | 0.0806 | 0.0480 | 0.4549 | 0.0586 | 0.2104 |
| ELW-sine | -0.3041 | 0.1950 | 0.1305 | -0.0548 | 0.0832 | 0.0099 | 0.1404 | 0.0567 | 0.0229 |
| TLW (HC) | -0.9268 | 0.1189 | 0.8731 | -0.9192 | 0.0671 | 0.8494 | -0.9430 | 0.0799 | 0.8955 |
| TLW (V) | -0.8874 | 0.0911 | 0.7958 | -0.9661 | 0.0434 | 0.9351 | -1.0046 | 0.0833 | 1.0162 |

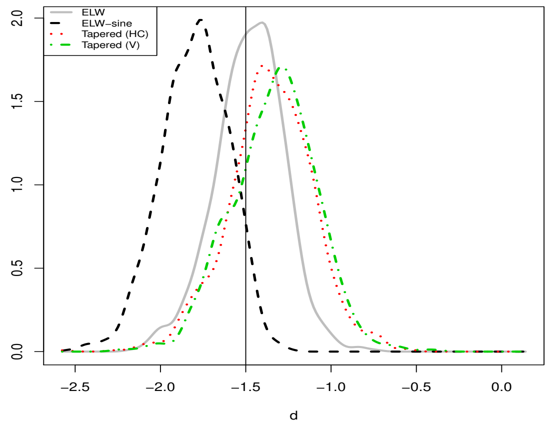
*Note:* Results for $\Phi(L)(1 - 2\cos\frac{\pi}{4}L + L^2)^{d_0}X_t = u_t I(t \geq 1)$, $u_t \sim N(0,1)$. LW, ELW, ELW-sine, TLW (HC), TLW (V) denote the original Local Whittle estimator (Arteche and Robinson, 2000), the Exact Local Whittle, the misspecified ELW without the sine term and tapered versions of the Local Whittle estimator with the "efficient" taper in Hurvich and Chen (2000) and the triangular Barlett taper ($p = 2$ in Velasco, 1999) respectively.

Figure 1. Monte Carlo probability density functions, $m = 8$, $u_t \sim AR(2)$

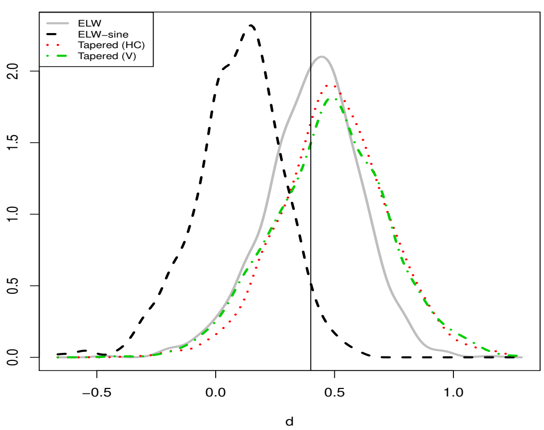
(a) $d = -3.0$



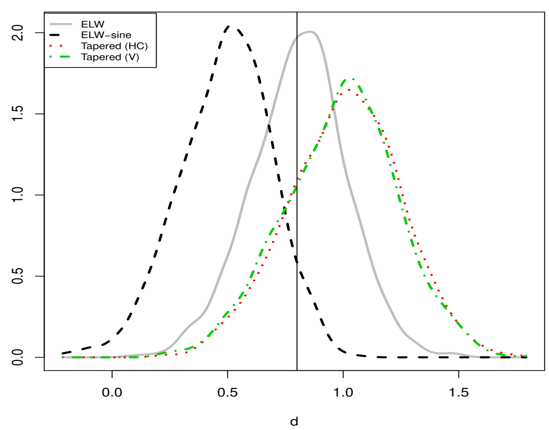

(b) $d = -1.5$



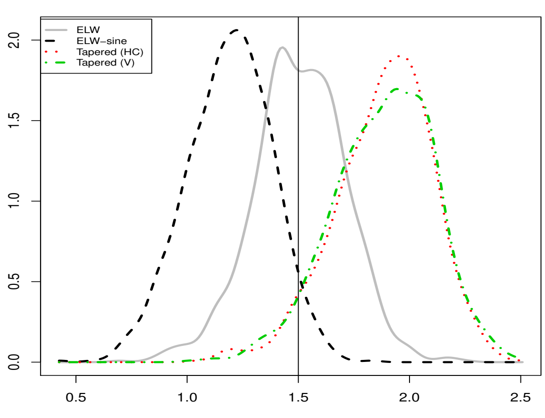

(c) $d = 0.4$



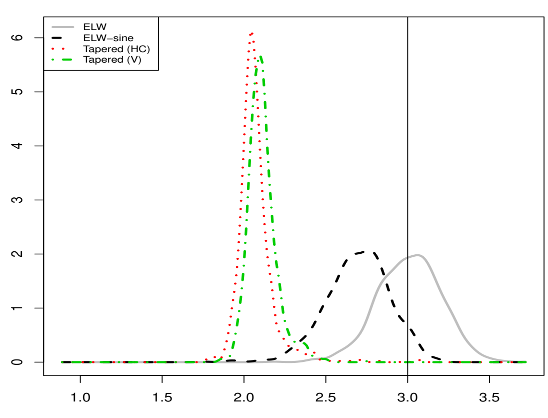

(d) $d = 0.8$




(e) $d = 1.5$




(f) $d = 3.0$



## 2 Proofs of the theorems

**Proof of Theorem 1**: Define $d = (d_1, ..., d_H)'$, $d_0 = (d_{10}, ..., d_{H0})'$ and $S(d) = R_H(d) - R_H(d_0) = \sum_{h=1}^{H} S_h(d)$, $S_h(d) = U_h(d_h) - T_h(d)$ where

$$\begin{aligned} U_h(d_h) &= 2(d_h - d_{h0}) - \log[2(d_h - d_{h0}) + 1] \\ T_h(d) &= \log \frac{\hat{C}_h(d_0)}{G_{h0}} - \log \frac{\hat{C}_h(d)}{G_h(d_h)} - \log \left( \frac{1}{2\delta_h m_h} \sum_j \left| \frac{j}{m} \right|^{2(d_h - d_{h0})} \{2(d_h - d_{h0}) + 1\} \right) \\ &+ 2(d_h - d_{h0}) \left( \frac{1}{2\delta_h m_h} \sum_j \log |j| - \log m_h + 1 \right) \end{aligned}$$

for

$$G_h(d_h) = G_{h0} \frac{1}{2\delta_h m_h} \sum_j |\lambda_j|^{2(d_h - d_{h0})} \text{ , and}$$

$$\hat{C}_h(d) = \frac{1}{2\delta_h m_h} \sum_j \frac{I_{\Delta_H^d x}(w_h + \lambda_j)}{|2g_h|^{2(d_h - d_{h0})}} \prod_{\substack{l=1 \\ l \neq h}}^{H} A_{l,h}^{-2\delta_l (d_l - d_{l0})}.$$

Define $\Theta = \prod_{h=1}^{H} [\Delta_{h1}, \Delta_{h2}]$ and $\bar{N}_\delta = \{d : ||d - d_0|| \geq \delta\}$ for some small $\delta > 0$ where $|| \cdot ||$ denotes the supremum norm. Consider also the following sets for $h = 1, 2, ..., H$, $\theta_h = d_h - d_{h0}$, where the dependence on $d$ is omitted to simplify notation:

$$\begin{aligned} \Theta_0^h &= \{d : -\frac{1}{2} + \Delta \leq \theta_h \leq \frac{1}{2}\}; \\ \Theta_1^h &= \{d : -\frac{1}{2} \leq \theta_h \leq -\frac{1}{2} + \Delta\}; \\ \Theta_2^h &= \{d : \frac{1}{2} \leq \theta_h \leq \frac{3}{2}\}; \\ \Theta_3^h &= \{d : \frac{3}{2} \leq \theta_h \leq \frac{5}{2}\}; \\ \Theta_4^h &= \{d : \frac{5}{2} \leq \theta_h \leq \frac{7}{2}\}; \\ \Theta_5^h &= \{d : \frac{7}{2} \leq \theta_h \leq \frac{9}{2}\}; \\ \Theta_6^h &= \{d : -\frac{3}{2} \leq \theta_h \leq -\frac{1}{2}\}; \\ \Theta_7^h &= \{d : -\frac{5}{2} \leq \theta_h \leq -\frac{3}{2}\}; \\ \Theta_8^h &= \{d : -\frac{7}{2} \leq \theta_h \leq -\frac{5}{2}\}; \\ \Theta_9^h &= \{d : -\frac{9}{2} \leq \theta_h \leq -\frac{7}{2}\} \end{aligned}$$

for $1/4 > \Delta > 0$ and denote $\Theta_{ij}^h = \Theta_i^h \cup \Theta_j^h$ for $i, j = 0, ..., 9$, $\Theta_1(h) = \Theta_0^h \times \prod_{l \neq h}^{H} [\Delta_{l1}, \Delta_{l2}]$ and $\Theta_2(h) = \Theta \backslash \Theta_1(h)$. Consistency holds if $P(||\hat{d} - d_0|| > \delta) \to 0$ as $n \to \infty$. Since $P(||\hat{d} -$

$d_0|| > \delta) \leq \sum_{h=1}^{H} P(\inf_{\bar{N}_\delta \cap \Theta} S_h(d) \leq 0)$ and $P(\inf_{\bar{N}_\delta \cap \Theta} S_h(d) \leq 0) \leq P(\inf_{\bar{N}_\delta \cap \Theta_1(h)} S_h(d) \leq 0) + P(\inf_{\Theta_2(h)} S_h(d) \leq 0)$ consistency holds if, for every $h = 1, ..., H$,

$$P(\inf_{\bar{N}_\delta \cap \Theta_1(h)} S_h(d) \leq 0) \to 0 \qquad \text{and} \qquad P(\inf_{\Theta_2(h)} S_h(d) \leq 0) \to 0$$

Since $\inf_{\overline{N_\delta} \cap \Theta_1(h)} U_h(d_h) \geq \delta^2/2$ (see formula (2.2) in Arteche, 2000 ), the first probability holds if $\sup_{\Theta_1(h)} |T_h(d)| \xrightarrow{p} 0$, which is satisfied if

$$\sup_{\Theta_1(h)} \left| \frac{2\theta_h + 1}{2\delta_h m_h} \sum_j \left| \frac{j}{m_h} \right|^{2\theta_h} - 1 \right| = o(1) \tag{1}$$

$$\sup_{\Theta_1(h)} \left| \frac{1}{2\delta_h m_h} \sum_j \log |j| - \log m_h + 1 \right| = o(1) \tag{2}$$

$$\sup_{\Theta_1(h)} \left| \frac{\hat{C}_h(d) - G_h(d_h)}{G_h(d_h)} \right| = o_p(1) \tag{3}$$

By Lemmas 1 and 2 in Robinson (1995), (1) and (2) are satisfied under Assumption A.4. Consider now $Y_t(\theta) = \Delta^H(L, d)X_t = \Delta^H(L, \theta)u_t I(t \geq 1)$ where $\theta = d - d_0 = (\theta_1, ..., \theta_H)'$, and denote $I_{yhj}^d = I_{\Delta^H(L,d)x}(w_h + \lambda_j)$ (note the dependence of $I_{yhj}^d$ on $d$). To prove (3),

$$\begin{aligned}
\frac{\hat{C}_h(d) - G_h(d_h)}{G_h(d_h)} &= \frac{\frac{2\theta_h+1}{2\delta_h m_h} \sum_j \left| \frac{j}{m_h} \right|^{2\theta_h} \left[ |\lambda_j|^{-2\theta_h} |2g_h|^{-2\theta_h} \prod_{\substack{l=1 \\ l \neq h}}^{H} A_{l,h}^{-2\delta_l \theta_l} I_{yhj}^d - G_{h0} \right]}{\frac{2\theta_h+1}{2\delta_h m_h} G_{h0} \sum_j \left| \frac{j}{m_h} \right|^{2\theta_h}} \\
&= \frac{A_h(d)}{B_h(d_h)}.
\end{aligned}$$

Noting that $B_h(d_h) + |B_h(d_h) - G_{h0}| \geq G_{h0}$ then

$$\inf_{\Theta_1(h)} B_h(d_h) \geq G_{h0} - \sup_{\Theta_1(h)} |B_h(d_h) - G_{h0}| \geq G_{h0}/2$$

using (1) for large enough $m_h$. It only remains to be shown that $\sup_{\Theta_1(h)} |A_h(d)| = o_p(1)$. To that end we split $\Theta_1(h)$ in a finite number of subsets based on the partition $\Theta_i^h$, $i = 0, ..., 9$, and prove that the desired bound holds in each of them. Consider the following cases:

**Case 1.1:** $\Theta_{11}(h) = \Theta_0^h \times \prod_{k \neq h} \Theta_{01}^k$

$A_h(d)$ involves positive and negative frequencies in those cases $w_h \neq 0, \pi$. Denote by $A_{h1}(d)$ as $A_h(d)$ but with the sums running from $j = 1$ to $j = m_h$ (the analysis is similar for $j = -1, .., -m_h$). Now, by summation by parts $\sup_{\Theta_{11}(h)} |A_{h1}(d)|$ is bounded by

$$\frac{1}{m_h} \sup_{\Theta_{11}(h)} \left| \sum_{r=1}^{m_h - 1} \left( \left| \frac{r}{m_h} \right|^{2\theta_h} - \left| \frac{r+1}{m} \right|^{2\theta_h} \right) \sum_{j=1}^{r} \left\{ \lambda_j^{-2\theta_h} |2g_h|^{-2\theta_h} \prod_{\substack{l=1 \\ l \neq h}}^{H} A_{l,h}^{-2\delta_l \theta_l} I_{yhj}^d - G_{h0} \right\} \right| \tag{4}$$

$$+\frac{1}{m_h}\sup_{\Theta_{11}(h)}\left|\sum_{j=1}^{m_h}\left\{\lambda_j^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-2\delta_l\theta_l}I_{yhj}^d - G_{h0}\right\}\right|. \tag{5}$$

Now, since

$$\left|\frac{r}{m_h}\right|^{2\theta_h}-\left|\frac{r+1}{m_h}\right|^{2\theta_h}=\left|\frac{r}{m_h}\right|^{2\theta_h}\left(1-\left|\frac{r+1}{r}\right|^{2\theta_h}\right)\leq\left|\frac{r}{m_h}\right|^{2\theta_h}\frac{2|\theta_h|}{r}$$

then (4) is bounded by

$$\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sup_{\Theta_{11}(h)}\left|\sum_{j=1}^{r}\left\{\lambda_j^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-2\delta_l\theta_l}I_{yhj}^d - G_{h0}\right\}\right|. \tag{6}$$

By Lemma 4

$$W_{\Delta^H(L,d)X_t}(w_h+\lambda_j)=W_{yhj}^d=D_n(e^{i(w_h+\lambda_j)},\theta)W_{uhj}-\frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}}\tilde{U}_{nj}^h(\theta) \tag{7}$$

where $\tilde{U}_{nj}^h(\theta)=\tilde{U}_n(\theta,w_h+\lambda_j)=\sum_{p=0}^{n-1}\sum_{k=p+1}^{n}d_k(\theta)\exp\{i(k-p)(w_h+\lambda_j\}u_{n-p}$. Then $\lambda_j^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-2\delta_l\theta_l}I_{yhj}^d-G_{h0}$ is equal to

$$\lambda_j^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-2\delta_l\theta_l}\left[I_{yhj}^d-|D_n(e^{i(w_h+\lambda_j)},\theta)|^2I_{uhj}\right] \tag{8}$$

$$+\quad\left[\lambda_j^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-2\delta_l\theta_l}|D_n(e^{i(w_h+\lambda_j)},\theta)|^2-\frac{G_{h0}}{f_{uhj}}\right]I_{uhj} \tag{9}$$

$$+\quad\left[I_{uhj}-|B(e^{i(w_h+\lambda_j)})|^2I_{\epsilon hj}\right]\frac{G_{h0}}{f_{uhj}}+G_{h0}(2\pi I_{\epsilon hj}-1) \tag{10}$$

By Lemma 3 and Assumptions A1 and A3,

$$\begin{aligned}&\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sup_{\Theta_{11}(h)}\left|\sum_{j=1}^{r}\left\{\left[\lambda_j^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-2\delta_l\theta_l}|D_n(e^{i(w_h+\lambda_j)},\theta)|^2-\frac{G_{h0}}{f_{uhj}}\right]I_{uhj}\right\}\right|\\ &=\quad O_p\left(\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\left[\eta+\lambda_j+j^{-1/2}\right]\right)\\ &=\quad O_p\left(\eta+\left[\frac{m_h}{n}\right]+\frac{\log m_h}{m_h^{2\Delta}}\right).\end{aligned} \tag{11}$$

Using now formula (2.10) in Arteche (2000)

$$E\left|I_{uhj}-|B(e^{i(w_h+\lambda_j)})|^2I_{\epsilon hj}\right|=O\left(\frac{\log^{1/2}j}{j^{1/2}}\right) \tag{12}$$

and then

$$\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sup_{\Theta_{11}(h)}\left|I_{uhj}-|B(e^{i(w_h+\lambda_j)})|^2 I_{\epsilon hj}\right|\left|\frac{G_{h0}}{f_{uhj}}\right|$$

$$= O_p\left(\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\left[\frac{\log^{1/2} j}{j^{1/2}}\right]\right)$$

$$= O_p\left(\frac{\log^{1/2} n}{m_h^{2\Delta}}\right) = o_p(1).$$

Now $\sum_{r=1}^{m_h-1}(r/m_h)^{2\Delta}r^{-2}|\sum_{j=1}^{r}(2\pi I_{\epsilon hj}-1)| = o_p(1)$ as shown in pages 286-287 in Arteche (2000). Finally, regarding the terms involving (8), using $||a|^2-|b|^2| \le |a+b||a-b|$, Cauchy-Schwarz inequality, Lemmas 3, 4 and 5

$$E\sup_{\Theta_{11}(h)}\left|\lambda_j^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-2\delta_l\theta_l}I_{yhj}^d-\lambda_j^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-2\delta_l\theta_l}|D_n(e^{i(w_h+\lambda_j)},\theta)|^2 I_{uhj}\right|$$

$$\le \left(E\sup_{\Theta_{11}(h)}\left||2g_h|^{-\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-\delta_l\theta_l}\left[2\lambda_j^{-\theta_h}D_n(e^{i(w_h+\lambda_j)},\theta)W_{uhj}-\frac{\lambda_j^{-\theta_h}}{\sqrt{2\pi n}}e^{in(w_h+\lambda_j)}\tilde{U}_{nj}^h\right]\right|^2\right)^{1/2}$$

$$\times\left(E\sup_{\Theta_{11}(h)}\left|\frac{\lambda_j^{-\theta_h}}{\sqrt{2\pi n}}|2g_h|^{-\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-\delta_l\theta_l}e^{in(w_h+\lambda_j)}\tilde{U}_{nj}^h\right|^2\right)^{1/2} = O\left(\frac{\log^2 n}{j^{1/2}}\right) \qquad (13)$$

and thus,

$$\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sup_{\Theta_{11}(h)}\left|\sum_{j=1}^{r}\left\{\lambda_j^{-2\theta}|2g_h|^{-2\theta_h}\prod_{\substack{l=1\\l\neq h}}^{H}A_{l,h}^{-2\delta_l\theta_l}\left[I_{yhj}^d-|D_n(e^{i(w_h+\lambda_j)},\theta)|^2 I_{uhj}\right]\right\}\right|$$

$$= O_p\left(\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\left[\frac{\log^2 n}{j^{1/2}}\right]\right) = O_p\left(\frac{\log^2 n}{m_h^{2\Delta}}\right) \qquad (14)$$

because $\Delta < 1/4$. Thus (6), and consequently also (4), are $o_p(1)$. Proceeding similarly (5) is shown to be $o_p(1)$ and the details are omitted. Then $\sup_{\Theta_{11}(h)}|A_{h1}(d)| = o_p(1)$ and thus $\sup_{\Theta_{11}(h)}|A_h(d)| = o_p(1)$.

**Case 1.2:** $\Theta_{12}(h) = \Theta_0^h \times \Theta_2^s \times \prod_{\substack{k\neq s\\k\neq h}}\Theta_{01}^k$, $s \neq h$.

In this case Lemma 5 cannot be used directly as in Case 1.1. The desired bound can however be obtained by considering $Z_t = \Delta_s^{-1}Y_t(\theta) = \Delta_s^{\theta_s-1}\prod_{k\neq s}\Delta_k^{\theta_k}u_t I(t\ge 1)$ for $\Delta_s =$

$(1-2\cos w_s L+L^2)^{\delta_s}$, $s=1,...,H$. Then $\Delta_s=(1-2\cos w_s L+L^2)$ if $w_s\in(0,\pi)$, $\Delta_s=(1-L)$ if $w_s=0$ and $\Delta_s=(1+L)$ if $w_s=\pi$. Denote $\mathbb{1}_s$ a vector of dimension $H$ with zeros except a one in the $s$-th element and $I_s=I(\delta_s=1)$ the indicator function of $\delta_s=1$. Using Lemma 4

$$W_{zhj}=D_n(e^{i(w_h+\lambda_j)},\theta-\mathbb{1}_s)W_{uhj}-\frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}}\tilde{U}^h_{nj}(\theta-\mathbb{1}_s)$$

and

$$\begin{aligned}W_{yhj} &= (1-2\cos w_s e^{i(w_h+\lambda_j)}+e^{2i(w_h+\lambda_j)})^{\delta_s}W_{zhj}\\ &- \frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}}\left(I_s e^{i2(w_h+\lambda_j)}-2\delta_s\cos w_s e^{i(w_h+\lambda_j)}\right)Z_n-\frac{e^{i(n-1)(w_h+\lambda_j)}I_s}{\sqrt{2\pi n}}Z_{n-1}\end{aligned}$$

such that

$$|\lambda_j|^{-\theta_h}|2g_h|^{-\theta_h}\prod_{l\neq h}A_{l,h}^{-\delta_l\theta_l}W_{yhj}=F^h_{nj}(\theta)W_{uhj}-\bar{U}^h_{nj}(\theta)-\bar{Z}^h_{nj}(\theta), \tag{15}$$

where

$$\begin{aligned}F^h_{nj}(\theta) &= |\lambda_j|^{-\theta_h}|2g_h|^{-\theta_h}\prod_{l\neq h}A_{l,h}^{-\delta_l\theta_l}(1-2\cos w_s e^{i(w_h+\lambda_j)}+e^{2i(w_h+\lambda_j)})D_n(e^{i(w_h+\lambda_j)},\theta-\mathbb{1}_s),\\ \bar{U}^h_{nj}(\theta) &= |\lambda_j|^{-\theta_h}|2g_h|^{-\theta_h}\prod_{l\neq h}A_{l,h}^{-\delta_l\theta_l}(1-2\cos w_s e^{i(w_h+\lambda_j)}+e^{2i(w_h+\lambda_j)})\frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}}\tilde{U}^h_{nj}(\theta-\mathbb{1}_s),\\ \bar{Z}^h_{nj}(\theta) &= |\lambda_j|^{-\theta_h}|2g_h|^{-\theta_h}\prod_{l\neq h}A_{l,h}^{-\delta_l\theta_l}\frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}}\left[\left(I_s e^{i2(w_h+\lambda_j)}-2\delta_s\cos w_s e^{i(w_h+\lambda_j)}\right)Z_n\right.\\ &+ \left. e^{-i(w_h+\lambda_j)}I_s Z_{n-1}\right].\end{aligned}$$

Then

$$\begin{aligned}&|\lambda_j|^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{l\neq h}A_{l,h}^{-2\theta_l}I^d_{yhj}-G_{h0}\\ =\ &|\lambda_j|^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{l\neq h}A_{l,h}^{-2\delta_l\theta_l}I^d_{yhj}-\left|F^h_{nj}(\theta)\right|^2 I_{uhj} \quad (16)\\ +\ &\left|F^h_{nj}(\theta)\right|^2 I_{uhj}-G_{h0}. \quad (17)\end{aligned}$$

Note that by Lemma 3,

$$\left|F^h_{nj}(\theta)\right|^2=1+O(|\lambda_j|)+O(|j|^{-1/2}) \tag{18}$$

in $\Theta_{12}(h)$ uniformly in $j=\pm1,..,\pm m_h$ and

$$\sup_{\Theta_{12}(h)}\left|\frac{2\theta_h+1}{\delta_h m_h}\sum_j\left|\frac{j}{m_h}\right|^{2\theta_h}\left\{\left|F^h_{nj}(\theta)\right|^2 I_{uhj}-G_{h0}\right\}\right|=O_p\left(\eta+\frac{m_h}{n}+\frac{\log m_h}{m_h^{2\Delta}}\right)$$

using summation by parts, Lemma 3, the result in (18) and proceeding as in the bounds for formulae (9) and (10).

Now, using (15), (16) is equal to:

$$\begin{aligned} & \left|\bar{U}_{nj}^h(\theta)\right|^2 + \left|\bar{Z}_{nj}^h(\theta)\right|^2 - 2Re\left\{F_{nj}^{h*}(\theta)W_{uhj}^*\bar{U}_{nj}^h(\theta)\right\} \\ - & \ 2Re\left\{F_{nj}^{h*}(\theta)W_{uhj}^*\bar{Z}_{nj}^h(\theta)\right\} + 2Re\left\{\bar{Z}_{nj}^{h*}(\theta)\bar{U}_{nj}^h(\theta)\right\}, \end{aligned}$$

where $*$ denotes complex conjugation. To get the desired bound for (16) we use summation by parts as in (4) and (5), focusing again on positive $\lambda_j$. By Lemma 5, $E\sup_{\Theta_{12}(h)}\left|\bar{U}_{nj}^h(\theta)\right|^2 = O(|j|^{-1}\log^2 n)$ and thus

$$\sup_{\Theta_{12}(h)}\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\left|\bar{U}_{nj}^h(\theta)\right|^2 = O_p\left(\frac{\log^2 m_h \log^2 n}{m_h^{2\Delta}}\right) = o_p(1).$$

Now, by covariance stationarity of $Z_n$, $E|Z_n|^2$ and $E|Z_{n-1}|^2$ are $O(1)$ and thus

$$\begin{aligned} \sup_{\Theta_{12}(h)}\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\left|\bar{Z}_{nj}^h(\theta)\right|^2 &= O_p\left(\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\frac{\lambda_j^{-1}}{n}\right) \\ &= O_p\left(\frac{\log^2 m_h}{m_h^{2\Delta}}\right), \end{aligned}$$

which is $o_p(1)$ under assumption A.4. Next,

$$\begin{aligned} & E\sup_{\Theta_{12}(h)}\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\left|F_{nj}^{h*}(\theta)W_{uhj}^*\bar{U}_{nj}^h(\theta)\right| \\ \leq & \sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\left(\sup_{\Theta_{12}(h)}|F_{nj}^{h*}(\theta)|^2 EI_{uhj}\right)^{1/2}\left(\sup_{\Theta_{12}(h)}E|\bar{U}_{nj}^h(\theta)|^2\right)^{1/2} \\ = & \ O\left(\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\frac{\log n}{\sqrt{j}}\right) = O\left(\frac{\log m_h \log n}{m_h^{2\Delta}}\right) = o(1) \end{aligned}$$

by Lemma 5 and (18). Similarly

$$\begin{aligned} & E\sup_{\Theta_{12}(h)}\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\left|F_{nj}^{h*}(\theta)W_{uhj}^*\bar{Z}_{nj}^h(\theta)\right| \\ \leq & \sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\sum_{j=1}^{r}\left(\sup_{\Theta_{12}(h)}|F_{nj}^{h*}(\theta)|^2 EI_{uhj}\right)^{1/2}\left(\sup_{\Theta_{12}(h)}E|\bar{Z}_{nj}^h(\theta)|^2\right)^{1/2} \\ = & \ O\left(\sum_{r=1}^{m_h-1}\left(\frac{r}{m_h}\right)^{2\Delta}\frac{1}{r^2}\frac{1}{\sqrt{n}}\sum_{j=1}^{r}\left(\frac{j}{n}\right)^{-1/2}\right) = O\left(\frac{\log m_h}{m_h^{2\Delta}}\right) = o(1). \end{aligned}$$

Finally

$$
\begin{aligned}
& E \sup_{\Theta_{12}(h)} \sum_{r=1}^{m_h-1} \left(\frac{r}{m_h}\right)^{2\Delta} \frac{1}{r^2} \sum_{j=1}^{r} \left| \bar{U}_{nj}^h(\theta) \bar{Z}_{nj}^{h*}(\theta) \right| \\
\leq & \sum_{r=1}^{m_h-1} \left(\frac{r}{m_h}\right)^{2\Delta} \frac{1}{r^2} \sum_{j=1}^{r} \left( E \sup_{\Theta_{12}(h)} |\bar{U}_{nj}^h(\theta)|^2 \right)^{1/2} \left( E \sup_{\Theta_{12}(h)} |\bar{Z}_{nj}^h(\theta)|^2 \right)^{1/2} \\
= & \; O\left( \sum_{r=1}^{m_h-1} \left(\frac{r}{m_h}\right)^{2\Delta} \frac{1}{r^2} \frac{1}{\sqrt{n}} \sum_{j=1}^{r} \left(\frac{j}{n}\right)^{-1/2} \frac{\log n}{\sqrt{j}} \right) = O\left( \frac{\log m_h \log n}{m_h^{2\Delta}} \right) = o(1).
\end{aligned}
$$

It can be similarly proved that

$$
\frac{1}{m_h} \sup_{\Theta_{12}(h)} \sum_j \left\{ |\lambda_j|^{-2\theta_h} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} I_{yhj}^d - \left| F_{nj}^h(\theta) \right|^2 I_{uhj} \right\} = o_p(1)
$$

which completes the proof that $\sup_{\Theta_{12}(h)} |A_h(d)| = o_p(1)$. The rest of values in $\Theta_1(h)$ can be considered similarly using successive applications of $\Delta_k^{-1}$ and $\Delta_k$, $k \neq h$, and the product of them, to complete the proof that $\sup_{\Theta_1(h)} |A_h(d)| = o_p(1)$.

It remains to be shown that $P(\inf_{\Theta_2(h)} S_h(d) \leq 0) \to 0$ as $n \to \infty$. Consider $S_h(d) = \log(\hat{D}_h(d)) - \log(\hat{D}_h(d_0))$ for

$$
\hat{D}_h(d) = \frac{1}{2\delta_h m_h} \sum_j \frac{I_{yhj}^d}{|2g_h|^{2\theta_h}} \prod_{\substack{l=1 \\ l \neq h}}^{H} A_{l,h}^{-2\delta_l \theta_l} |\lambda_j|^{-2\theta_h} \left| \frac{j}{q} \right|^{2\theta_h}
$$

where $q = \exp\left(\sum_j \log|j| / 2\delta_h m_h\right) \sim m_h/e$. Now, as in Shimotsu and Phillips (2005), page 1904, $\log(\hat{D}_h(d_0)) - \log G_{h0} = o_p(1)$. Therefore $P(\inf_{\Theta_2(h)} S_h(d) \leq 0) \to 0$ if for some $\delta > 0$ $P(\inf_{\Theta_2(h)} \log \hat{D}_h(d) - \log G_{h0} \leq \log(1+\delta)) = P(\inf_{\Theta_2(h)} \hat{D}_h(d) - G_{h0} \leq \delta G_{h0}) \to 0$ as $n \to \infty$. For $w_h \in (0, \pi)$ write $\hat{D}_h(d) = (\hat{D}_{h1}(d) + \hat{D}_{h2}(d))/2$ where

$$
\begin{aligned}
\hat{D}_{h1}(d) &= \frac{1}{2\delta_h m_h} \sum_{j=1}^{m_h} \frac{I_{yhj}^d}{|2g_h|^{2\theta_h}} \prod_{\substack{l=1 \\ l \neq h}}^{H} A_{l,h}^{-2\delta_l \theta_l} \lambda_j^{-2\theta_h} \left(\frac{j}{q}\right)^{2\theta_h}, \\
\hat{D}_{h2}(d) &= \frac{1}{2\delta_h m_h} \sum_{j=1}^{m_h} \frac{I_{yh,-j}^d}{|2g_h|^{2\theta_h}} \prod_{\substack{l=1 \\ l \neq h}}^{H} A_{l,h}^{-2\delta_l \theta_l} \lambda_j^{-2\theta_h} \left(\frac{j}{q}\right)^{2\theta_h},
\end{aligned}
$$

$\hat{D}_h(d) = \hat{D}_{h1}(d)$ if $w_h = 0$ and $\hat{D}_h(d) = \hat{D}_{h2}(d)$ if $w_h = \pi$. Let us focus on $\hat{D}_{h1}(d)$, the analysis with $\hat{D}_{h2}(d)$ is similar. Now, with $\sum'$ denoting the sum over $j = \lfloor \kappa m_h \rfloor, ..., m_h$ for any fixed $\kappa \in (0,1)$, $\hat{D}_{h1}(d) - G_{h0}$ is greater or equal to

$$
\frac{1}{2\delta_h m_h} \sum{}' \left(\frac{j}{q}\right)^{2\theta_h} \left( \frac{I_{yhj}^d}{|2g_h|^{2\theta_h}} \prod_{\substack{l=1 \\ l \neq h}}^{H} A_{l,h}^{-2\delta_l \theta_l} \lambda_j^{-2\theta_h} - G_{h0} \right) + G_{h0} \left( \frac{1}{2\delta_h m_h} \sum{}' \left(\frac{j}{q}\right)^{2\theta_h} - 1 \right)
$$

Since $|\theta_h| \geq 1/2 - \Delta$, Lemma 5.5 in Shimotsu and Phillips (2005) applies and thus $\inf_{\Theta_2(h)} (2\delta_h m_h)^{-1} \sum' j^{2\theta_h} q^{-2\theta_h} - 1 \geq 4\delta G_{h0}$ for $k$ sufficiently small and large enough $m_h$. Then the desired result holds if

$$P\left(\inf_{\Theta_2(h)} \frac{1}{2\delta_h m_h} \sum{}' \left(\frac{j}{q}\right)^{2\theta_h} \left(I^d_{yhj} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} \lambda_j^{-2\theta_h} - G_{h0}\right) \leq -3\delta G_{h0}\right) \to 0. \tag{19}$$

Let us split again the proof of (19) in different sets.

**Case 2.1:** $\Theta_{21}(h) = \Theta_1^h \times \prod_{k \neq h} \Theta_{01}^k$. In this case

$$\begin{aligned}
& \frac{1}{2\delta_h m_h} \sum{}' \left(\frac{j}{q}\right)^{2\theta_h} \left(I^d_{yhj} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} \lambda_j^{-2\theta_h} - G_{h0}\right) \\
= \;& \frac{1}{2\delta_h m_h} \sum{}' \left(\frac{j}{q}\right)^{2\theta_h} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} \lambda_j^{-2\theta_h} \left(I^d_{yhj} - |D_n(e^{i(w_h+\lambda_j)}, \theta|^2 I_{uhj}\right) \\
+ \;& \frac{1}{2\delta_h m_h} \sum{}' \left(\frac{j}{q}\right)^{2\theta_h} \left(|2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} \lambda_j^{-2\theta_h} |D_n(e^{i(w_h+\lambda_j)}, \theta|^2 I_{uhj} - G_{h0}\right) \\
= \;& \Lambda_{1n}(\theta) + \Lambda_{2n}(\theta).
\end{aligned}$$

Lemma 5 is still applicable and all the bounds used in Case 1.1 are still valid to show that $\sup_{\Theta_{21}(h)} |\Lambda_{1n}(\theta)| = o_p(1)$ and $\sup_{\Theta_{21}(h)} |\Lambda_{2n}(\theta)| = o_p(1)$.

**Case 2.2:** $\Theta_{22}(h) = \Theta_1^h \times \Theta_2^s \times \prod_{k \neq h} \Theta_{01}^k$.

This case is treated in a similar way to Case 1.2 by applying $\Delta_s^{-1}$ to get that

$$\sup_{\Theta_{22}(h)} \left| \frac{1}{2\delta_h m_h} \sum{}' \left(\frac{j}{q}\right)^{2\theta_h} \left(I^d_{yhj} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} \lambda_j^{-2\theta_h} - G_{h0}\right) \right| = o_p(1).$$

The bounds in the rest of sets in $\Theta_2^h \times [\Delta_1, \Delta_2]^{H-1}$ are similarly obtained by successive and combined applications of $\Delta_k^{-1}$ and $\Delta_k$, $k \neq h$.

The analysis with the rest of values for $\theta_h$ requires also the application of different combinations of $\Delta_h^{-1}$ and $\Delta_h$. Consider, as an example, the following case:

**Case 3.3:** $\Theta_{33}(h) = \Theta_2^h \times \Theta_2^s \times \prod_{\substack{k \neq h \\ k \neq s}} \Theta_{01}^k$.

Consider $Z_t = \Delta_h^{-1} \Delta_s^{-1} Y_t(\theta) = \Delta_h^{\theta_h - 1} \Delta_s^{\theta_s - 1} \prod_{\substack{k \neq h \\ k \neq s}} \Delta_k^{\theta_k} u_t I(t \geq 1)$. Denote $\mathbb{1}_{hs} = \mathbb{1}_h + \mathbb{1}_s$. By Lemma 2

$$W_{zhj} = D_n(e^{i(w_h+\lambda_j)}, \theta - \mathbb{1}_{hs}) W_{uhj} - \frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}} \tilde{U}^h_{nj}(\theta - \mathbb{1}_{hs})$$

and

$$W_{yhj} = \Delta_{shj}\Delta_{khj}W_{zhj} - \frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}}\tilde{Z}_{nhj}$$

where

$$\begin{aligned}
\Delta_{abc} &= (1 - 2\cos w_a e^{i(w_b+\lambda_c)} + e^{2i(w_b+\lambda_c)})^{\delta_a} \\
\tilde{Z}_{nhj} &= \sum_{p=0}^{3}\sum_{k=p+1}^{4} p_k e^{i(k-p)(w_h+\lambda_j)} Z_{n-p}
\end{aligned}$$

for $p_k$ the coefficients satisfying $\Delta_h\Delta_s = \sum_{k=0}^{4} p_k L^k$, that is $p_0 = 1$, $p_1 = -2\delta_s \cos w_s - 2\delta_h \cos w_h$, $p_2 = I_h + I_s + 4\delta_h\delta_s \cos w_s \cos w_h$, $p_3 = -2\delta_s I_h \cos w_s - 2\delta_h I_s \cos w_h$ and $p_4 = I_h I_s$. Then

$$|2g_h|^{-\theta_h}\prod_{l\neq h} A_{l,h}^{-\theta_l}\lambda_j^{-\theta_h}W_{yhj} = F_{nj}^{hs}(\theta) - \bar{U}_{nj}^{hs}(\theta) - \bar{Z}_{nj}^{hs}(\theta)$$

where

$$\begin{aligned}
F_{nj}^{hs}(\theta) &= |2g_h|^{-\theta_h}\prod_{l\neq h} A_{l,h}^{-\delta_l\theta_l}\lambda_j^{-\theta_h}\Delta_{shj}\Delta_{khj}D_n(e^{i(w_h+\lambda_j)}, \theta - \mathbb{1}_{hs}) \\
\bar{U}_{nj}^{hs}(\theta) &= |2g_h|^{-\theta_h}\prod_{l\neq h} A_{l,h}^{-\delta_l\theta_l}\lambda_j^{-\theta_h}\Delta_{shj}\Delta_{khj}\frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}}\tilde{U}_{nj}^{h}(\theta - \mathbb{1}_{hs}) \\
\bar{Z}_{nj}^{hs}(\theta) &= |2g_h|^{-\theta_h}\prod_{l\neq h} A_{l,h}^{-\delta_l\theta_l}\lambda_j^{-\theta_h}\frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}}\tilde{Z}_{nhj}.
\end{aligned}$$

Using Lemmas 2 and 3:

$$|F_{nj}^{hs}(\theta)|^2 = 1 + O(|\lambda_j|) + O(j^{-1/2}) \tag{20}$$

in $\Theta_{33}(h)$. Also $E|\tilde{Z}_{nhj}|^2 = O(1)$ and by Lemma 5 $E\sup_{\Theta_{33}(h)}\left|\bar{U}_{nj}^{hs}(\theta)\right|^2 = O(|j|^{-1}\log^2 n)$. It is then straightforward (see Case 1.1) to show that

$$\sup_{\Theta_{33}(h)}\left|\frac{1}{2\delta_h m_h}\sum{}'\left(\frac{j}{q}\right)^{2\theta_h}\left(|F_{nj}^{hs}(\theta)|^2 I_{uhj} - G_{h0}\right)\right| = o_p(1).$$

Now

$$\frac{1}{2\delta_h m_h}\sum{}' \left(\frac{j}{q}\right)^{2\theta_h} \left(|2g_h|^{-2\theta_h}\prod_{l\neq h} A_{l,h}^{-2\delta_l\theta_l}\lambda_j^{-2\theta_h} I_{yuj}^d - |F_{nj}^{hs}(\theta)|^2 I_{uhj}\right)$$

$$= \frac{1}{2\delta_h m_h}\sum{}' \left(\frac{j}{q}\right)^{2\theta_h} |\bar{U}_{nj}^{hs}(\theta)|^2 \tag{21}$$

$$+ \frac{1}{2\delta_h m_h}\sum{}' \left(\frac{j}{q}\right)^{2\theta_h} |\bar{Z}_{nj}^{hs}(\theta)|^2 \tag{22}$$

$$- 2Re\left\{\frac{1}{2\delta_h m_h}\sum{}' \left(\frac{j}{q}\right)^{2\theta_h} F_{nj}^{hs*}(\theta)W_{uhj}^*\bar{U}_{nj}^{hs}(\theta)\right\} \tag{23}$$

$$- 2Re\left\{\frac{1}{2\delta_h m_h}\sum{}' \left(\frac{j}{q}\right)^{2\theta_h} F_{nj}^{hs*}(\theta)W_{uhj}^*\bar{Z}_{nj}^{hs}(\theta)\right\} \tag{24}$$

$$+ 2Re\left\{\frac{1}{2\delta_h m_h}\sum{}' \left(\frac{j}{q}\right)^{2\theta_h} \bar{U}_{nj}^{hs^*}(\theta)\bar{Z}_{nj}^{hs}(\theta)\right\}. \tag{25}$$

Since $E\sup_{\Theta_{33}(h)}\left|\bar{U}_{nj}^{hs}(\theta)\right|^2 = O(|j|^{-1}\log^2 n)$, then $E\sup_{\Theta_{33}(h)}|(21)| = o(1)$ using Lemma 5.4 in Shimotsu and Phillips (2005). Also $E\sup_{\Theta_{33}(h)}|(23)| = o(1)$ because of (20) and $E|\tilde{Z}_{nhj}|^2 = O(1)$.

Finally, proceeding as in Shimotsu and Phillips (2005), pages 1907-1908, $P\left(\inf_{\Theta_{33}(h)}[(22) +(24)+(25)] \leq -\zeta\right) \to 0$ as $n\to\infty$ for any $\zeta > 0$, which implies that

$$P\left(\inf_{\Theta_{33}(h)}\frac{1}{2\delta_h m_h}\sum{}' \left(\frac{j}{q}\right)^{2\theta_h} \left(|2g_h|^{-2\theta_h}\prod_{l\neq h} A_{l,h}^{-2\delta_l\theta_l}\lambda_j^{-2\theta_h} I_{yuj}^d - |F_{nj}^{hs}(\theta)|^2 I_{uhj}\right) \leq -2G_{h0}\right) \to 0$$

such that (19) is satisfied in $\Theta_{33}(h)$. For the remaining subsets in $\Theta_2(h)$ the proof that (19) holds is similarly based on sequential applications of $\Delta_k$ and $\Delta_k^{-1}$, $k = 1,2,...,H$, noting Lemma 6 in a similar fashion as in Shimotsu and Phillips (2005), pages 1909-1912.

■

**Proof of Theorem 2**: With probability approaching 1, as $n\to\infty$,

$$\Lambda_m(\hat{d}-d_0) = -\left[\frac{\mathrm{d}^2}{\mathrm{d}d\,\mathrm{d}d'}R_H(\bar{d})\right]^{-1}\Lambda_m\frac{\mathrm{d}}{\mathrm{d}d}R_H(d_0) \tag{26}$$

for $||\bar{d}-d_0|| \leq ||\hat{d}-d_0||$, where $\frac{\mathrm{d}}{\mathrm{d}d}R_H(d_0)$ is a $H\times 1$ vector with $s$-th element

$$\frac{\partial}{\partial d_s}R_H(d_0) = \sum_{h=1}^{H}\frac{F_h^s(d_0)}{F_h(d_0)} - 2\log|2g_s| - 2\sum_{\substack{h=1\\h\neq s}}^{H}\delta_h\log A_{s,h} - \frac{1}{\delta_s m_s}\sum\log|\lambda_j|$$

and $\frac{\mathrm{d}^2}{\mathrm{d}d\,\mathrm{d}d'}R_H(\bar{d})$ is a $H\times H$ matrix with $(s,r)$-th element

$$\frac{\partial^2}{\partial d_s\partial d_r}R_H(\bar{d}) = \sum_{h=1}^{H}\frac{F_h^{sr}(\bar{d})F_h(\bar{d}) - F_h^s(\bar{d})F_h^r(\bar{d})}{F_h(\bar{d})^2} = \sum_{h=1}^{H}\frac{H_h^{sr}(\bar{d})H_h(\bar{d}) - H_h^s(\bar{d})H_h^r(\bar{d})}{H_h(\bar{d})^2}$$

for

$$\begin{aligned}
F_h(d) &= \frac{1}{2\delta_h m_h}\sum_j I^d_{yhj} \\
F^s_h(d) &= \frac{1}{2\delta_h m_h}\sum_j 2Re(W_{\log\Delta_s yhj}W^*_{yhj}) \\
F^{sr}_h(d) &= \frac{1}{2\delta_h m_h}\sum_j Z^{sr}_{yhj}(d) \\
&\quad \text{for } Z^{sr}_{yhj}(d) = 2Re(W_{\log\Delta_s\log\Delta_r yhj}W^*_{yhj}) + 2Re(W_{\log\Delta_s yhj}W^*_{\log\Delta_r yhj}) \\
H_h(d) &= \frac{1}{2\delta_h m_h}\sum_j |j|^{2\theta_h}|\lambda_j|^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{l\neq h} A^{-2\delta_l\theta_l}_{l,h} I^d_{yhj} \\
H^s_h(d) &= \frac{1}{2\delta_h m_h}\sum_j |j|^{2\theta_h}|\lambda_j|^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{l\neq h} A^{-2\delta_l\theta_l}_{l,h} 2Re(W_{\log\Delta_s yhj}W^*_{yhj}) \\
H^{sr}_h(d) &= \frac{1}{2\delta_h m_h}\sum_j |j|^{2\theta_h}|\lambda_j|^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{l\neq h} A^{-2\delta_l\theta_l}_{l,h} Z^{sr}_{yhj}(d),
\end{aligned}$$

where $\Delta_s = \Delta_s(L,1) = (1-2\cos w_s L + L^2)^{\delta_s}$. $\hat{d}$ has the stated limiting distribution if for any $H\times 1$ vector $\eta = (\eta_1,...,\eta_H)'$

$$\eta'\Lambda_m \frac{\mathrm{d}}{\mathrm{dd}} R_H(d_0) = \sum_{s=1}^{H}\eta_s\sqrt{\delta_s m_s}\frac{\partial}{\partial d_s}R_H(d_0) \xrightarrow{d} N\left(0, 4\eta'\eta\right) \tag{27}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{dd\,dd}'}R_H(\bar{d}) \xrightarrow{p} 4\mathbb{I}_H \tag{28}$$

as $n\to\infty$.

Using Lemma 4 in Arteche (2000) and Lemma 5 we get, as in Shimotsu and Phillips (2005, page 1913), that $P(\bar{d}\notin M)\to 0$ for $M = \{d : \log^4 n||d-d_0|| < \varepsilon)$ for $\varepsilon > 0$ under the more restrictive assumption B.1. The analysis can thus be restricted to values of $d\in M$.

*Hessian approximation*

First $\sup_M \left|H_h(d) - (2\delta_h m_h)^{-1}\sum_j I_{uhj}\right|$ is bounded by

$$\sup_M \left|\frac{1}{2\delta_h m_h}\sum_j |j|^{2\theta_h}\left(|\lambda_j|^{-2\theta_h}|2g_h|^{-2\theta_h}\prod_{l\neq h}A^{-2\delta_l\theta_l}_{l,h} I^d_{yhj} - I_{uhj}\right)\right| \tag{29}$$

$$+\ \sup_M \left|\frac{1}{2\delta_h m_h}\sum_j \left(|j|^{2\theta_h} - 1\right) I_{uhj}\right| \tag{30}$$

Since $\sup_M \left||j|^{2\theta_h} - 1\right| = O(\log^{-3} n)$ then (30) is $o_p(\log^{-2} n)$. Regarding (29), it is bounded by

$$\sup_M \left| \frac{1}{2\delta_h m_h} \sum_j |j|^{2\theta_h} |\lambda_j|^{-2\theta_h} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} \left( I^d_{yhj} - |D_n(e^{i(w_h+\lambda_j)}, \theta)|^2 I_{uhj} \right) \right| \tag{31}$$

$$+ \sup_M \left| \frac{1}{2\delta_h m_h} \sum_j |j|^{2\theta_h} \left( |\lambda_j|^{-2\theta_h} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} |D_n(e^{i(w_h+\lambda_j)}, \theta)|^2 - 1 \right) I_{uhj} \right|. \tag{32}$$

Using (13) and $\sup_M \left| |j|^{2\theta_h} \right| = O(1)$, then

$$E|(31)| = O\left( \frac{1}{2\delta_h m_h} \sum_j \sup_M |j|^{2\theta_h} \frac{\log^2 n}{\sqrt{j}} \right) = O\left( \frac{\log^2 n}{\sqrt{m_h}} \right) = o(\log^{-2} n),$$

and using also Lemma 3, $E|(32)|$ is bounded by

$$O\left( \sup_M \frac{1}{2\delta_h m_h} \sum_j \left[ |\lambda_j| + |j|^{-\theta_h - 1} + n^{-\underline{\theta}_h - 1} \right] \right) = o(\log^2_n)$$

by assumption B4. Thus

$$\sup_M \left| H_h(d) - \frac{1}{2\delta_h m_h} \sum_j I_{uhj} \right| = o_p(\log^{-2} n). \tag{33}$$

We prove now that

$$\sup_M \left| H^s_h(d) + \frac{2\delta_s}{2\delta_h m_h} \sum_j Re\{\bar{J}_{nj}(w_h, w_s)\} I_{uhj} \right| = o_p(\log^{-1} n) \tag{34}$$

By Lemmas 4, 9 and equation (7), $W_{\log \Delta_s yhj} W^*_{yhj}$ is equal to

$$- \quad \delta_s \bar{J}_{nj}(w_h, w_s) |D_n(e^{i(w_h+\lambda_j)}, \theta)|^2 I_{uhj} \tag{35}$$

$$+ \quad \delta_s D_n(e^{i(w_h+\lambda_j)}, \theta) \bar{J}_{nj}(w_h, w_s) \frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}} \tilde{U}^h_{nj}(\theta) W_{uhj} \tag{36}$$

$$+ \quad \frac{1}{\sqrt{n}} V_{nj} \frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}} \tilde{U}^h_{nj}(\theta) \tag{37}$$

$$- \quad \frac{1}{\sqrt{n}} V_{nj}(\theta) D_n(e^{i(w_h+\lambda_j)}, \theta) W_{uhj}. \tag{38}$$

Using now Lemma 3, $E|I_{uhj}| = O(1)$ and $\bar{J}_{nj}(w_h, w_s) = O(\log n)$, we have that $|\lambda_j|^{-2\theta_h} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} Re\{(35)\}$ is equal to

$$-\delta_s Re\{\bar{J}_{nj}(w_h, w_s)\} I_{uhj} + O_p(|\lambda_j| \log n + |j|^{-\theta_h - 1} \log n + n^{-\underline{\theta}_h - 1} \log n)$$

Using also Lemma 5, $E \sup_M \left| |\lambda_j|^{-2\theta_h} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} Re\{(36)\} \right|^2 = O(|j|^{-1} \log^4 n)$. Now, by Lemmas 5 and 9, $E \sup_M \left| |\lambda_j|^{-2\theta_h} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} Re\{(37)\} \right|^2 = O(|j|^{-2} \log^6 n)$. Finally, $E \sup_M \left| |\lambda_j|^{-2\theta_h} |2g_h|^{-2\theta_h} \prod_{l \neq h} A_{l,h}^{-2\delta_l \theta_l} Re\{(38)\} \right|^2 = O(|j|^{-1} \log^4 n)$.

Then (34) is proved because $\sup_M |j|^{2\theta} = 1 + O(\log^{-3} n)$ using assumption B4.

We also need to show the approximation

$$\sup_M \left| H_h^{sr}(d) - \frac{4\delta_s\delta_r}{2\delta_h m_h} \sum_j Re\{\bar{J}_{nj}(w_h, w_s)\} Re\{\bar{J}_{nj}(w_h, w_r)\} I_{uhj} \right| = O_p(\log^{-1} n) \qquad (39)$$

which is similarly proven using Lemmas 3, 5, 9, $\bar{J}_{nj}(w_h, w_s) = O(\log n)$ and the fact that

$$Re\{\bar{J}_{nj}(w_h, w_s)\bar{J}_{nj}(w_h, w_r)\} + Re\{\bar{J}_{nj}(w_h, w_s)\bar{J}^*_{nj}(w_h, w_r)\}$$

$$= 2Re\{\bar{J}_{nj}(w_h, w_s)\} Re\{\bar{J}_{nj}(w_h, w_r)\}.$$

Using now Lemma 1 in Arteche (2000) and Assumption B.1

$$\begin{aligned} E|I_{uhj} - G_{h0} 2\pi I_{\epsilon hj}| &\leq E|I_{uhj} - |B(e^{i(w_h+\lambda_j)}|^2 I_{\epsilon hj}| + 2\pi |f_{uhj} - f_u(w_h)| E|I_{\epsilon hj}| \\ &= O\left(\frac{\log^{1/2}|j|}{|j|^{1/2}} + |\lambda_j|^{\beta_h}\right) \end{aligned} \qquad (40)$$

by (12). Thus, by (33) and because $\bar{d} \in M$, $H_h(\bar{d})$ is equal to

$$G_{h0} \frac{2\pi}{2\delta_h m_h} \sum_j I_{\epsilon hj} + O_p\left(\frac{1}{2m_h} \sum_j \left(\frac{\log^{1/2}|j|}{|j|^{1/2}} + |\lambda_j|^{\beta_h}\right)\right) + o_p(\log^{-2} n)$$

$$= G_{h0} + o_p(\log^{-2} n) \qquad (41)$$

because $EI_{\epsilon hj} = 1/2\pi$ and $cov(I_{\epsilon hj}, I_{\epsilon hk}) = O(1)$ for $j = k$ and $O(n^{-1})$ for $j \neq k$ such that $E|(2\pi)^{-1} \sum_j I_{\epsilon hj}/(2\delta_h m_h) - 1|^2 = O(m_h^{-1} + n^{-1})$.

Similarly, by (34) $H_h^s(\bar{d})$ is equal to

$$-\frac{2\delta_s}{2\delta_h m_h} \sum_j Re\{\bar{J}_{nj}(w_h, w_s)\} 2\pi G_{h0} I_{\epsilon hj} + O_p\left(\frac{\log^{3/2} n}{\sqrt{m_h}} + \frac{m_h^{\beta_h} \log n}{n^{\beta_h}}\right) + o_p(\log^{-1} n)$$

$$= -\frac{2\delta_s G_{h0}}{2\delta_h m_h} \sum_j Re\{\bar{J}_{nj}(w_h, w_s)\} + o_p(\log^{-1} n). \qquad (42)$$

Also

$$H_h^{sr}(\bar{d}) = \frac{4G_{h0}\delta_s\delta_r}{2\delta_h m_h} \sum_j Re\{\bar{J}_{nj}(w_h, w_s)\} Re\{\bar{J}_{nj}(w_h, w_r)\} + o_p(1),$$

which leads to

$$\begin{aligned}
\frac{\partial^2}{\partial d_s \partial d_r} R_H(\bar{d}) &= 4\delta_s\delta_r \sum_{h=1}^{H} \left\{ \frac{1}{2\delta_h m_h} \sum_j Re\{\bar{J}_{nj}(w_h, w_s)\} Re\{\bar{J}_{nj}(w_h, w_r)\} \right. \\
&- \left. \frac{1}{2\delta_h m_h} \sum_j Re\{\bar{J}_{nj}(w_h, w_s)\} \frac{1}{2\delta_h m_h} \sum_j Re\{\bar{J}_{nj}(w_h, w_r)\} \right\} + o_p(1) \\
&= \begin{cases} 4 + o_p(1) & \text{if } w_r = w_s \\ o_p(1) & \text{in any other case} \end{cases} \qquad (43)
\end{aligned}$$

where the second equality comes from Lemma 8 because

$$\begin{aligned}
Re\{\bar{J}_{nj}(w_h, w_s)\} &= -\log A_{s,h} + O(|\lambda_j|) + (n^{-1}) \text{ if } s \neq h \ , \\
&= -\log|2\sin w_s| - \log|\lambda_j| + O(|\lambda_j|) + O(|j|^{-1}) \text{ if } s = h \neq 0, \pi \ , \\
&= -2\log|\lambda_j| + O(|\lambda_j|^2) + O(|j|^{-1}) \text{ if } s = h = 0, \pi \ . \qquad (44)
\end{aligned}$$

which leads to (43) after trivial operations.

*Score convergence*

By Lemma 9

$$\begin{aligned}
\sqrt{2\delta_s m_s} F_h^s(d_0) &= \frac{\sqrt{2\delta_s m_s}}{2\delta_h m_h} \sum_j 2Re\{W_{\log \Delta_s uhj} W_{uhj}^*\} \\
&= -\frac{2\delta_s \sqrt{2\delta_s m_s}}{2\delta_h m_h} \sum_j Re\{\bar{J}_{nj}(w_h, w_s)\} I_{uhj} \qquad (45) \\
&+ \frac{2\delta_s \sqrt{2\delta_s m_s}}{2\delta_h m_h} \sum_j Re\left\{ \frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}} B(1) \tilde{J}_{nj}(w_h, w_s, L) \epsilon_n W_{uhj}^* \right\} \qquad (46) \\
&- \frac{2\sqrt{2\delta_s m_s}}{2\delta_h m_h} \sum_j Re\left\{ r_{nj} W_{uhj}^* \right\} \qquad (47)
\end{aligned}$$

By Lemma 9 (47) $= o_p(m_s^{1/2} m_h^{-1/2}) = o_p(1)$ under assumption B.4. Now (46) is equal to

$$\frac{2\delta_s \sqrt{2\delta_s m_s}}{2\delta_h m_h} \sum_j Re\left\{ \frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}} B(1) \tilde{J}_{nj}(w_h, w_s, L) \epsilon_n \left[ W_{uhj}^* - B^*(e^{i(w_h+\lambda_j)}) W_{\epsilon hj}^* \right] \right\} \qquad (48)$$

$$+ \frac{2\delta_s \sqrt{2\delta_s m_s}}{2\delta_h m_h} \sum_j Re\left\{ \frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}} B(1) \tilde{J}_{nj}(w_h, w_s, L) \epsilon_n B^*(e^{i(w_h+\lambda_j)}) W_{\epsilon hj}^* \right\}. \qquad (49)$$

Using Lemma 1 in Arteche (2000), $E|W_{uhj}^* - B^*(e^{i(w_h+\lambda_j)}) W_{\epsilon hj}^*|^2 = O(|j|^{-1}\log|j|)$ and as in the proof in equation (61) below, $E|\tilde{J}_{nj}(w_h, w_s, L)\epsilon_n|^2 = O(|\lambda_j|^{-1}\log^4 n)$ such that (48) =

$O_p(m_h^{-1}\sqrt{m_s}\log^3 n) = o_p(1)$ under assumption B4. Now, for $\bar{\gamma}(w_h, w_r)$ defined in Lemma 8, (49) is equal to

$$\frac{2\delta_s\sqrt{2\delta_s m_s}}{2\delta_h m_h}\sum_j Re\left\{\frac{e^{in(w_h+\lambda_j)}}{\sqrt{2\pi n}}B(1)\sum_{p=0}^{n-1}\bar{\gamma}(w_h, w_r)e^{ipw_s}\epsilon_{n-p}B^*(e^{i(w_h+\lambda_j)})W^*_{\epsilon hj}\right\}$$
$$= O_P\left(\sqrt{\frac{m_s}{m_h}}\left[\log n\sqrt{\frac{m_h}{n}} + \frac{\log^2 n}{\sqrt{m_h}}\right]\right) = O_P\left(\frac{\log n\sqrt{m_s}}{\sqrt{n}} + \frac{\log^2 n\sqrt{m_s}}{m_h}\right) = o_p(1)$$

using Lemma 8 as in the proof of the bound for formula (63) in Shimotsu and Phillips (2005). Thus

$$\sqrt{2\delta_s m_s}F_h^s(d_0) = -\frac{2\delta_s\sqrt{2\delta_s m_s}}{2\delta_h m_h}\sum_j Re\{\bar{J}_{nj}(w_h, w_s)\}I_{uhj} + o_p(1).$$

Let us consider first the case $s \neq h$. Then using (44)

$$\begin{aligned}\sqrt{2\delta_s m_s}F_h^s(d_0) &= \frac{2\delta_s\sqrt{2\delta_s m_s}}{2\delta_h m_h}\sum_j \log A_{s,h}I_{uhj} + O_p\left(\frac{m_s^{1/2}m_h}{n}\right) + o_p(1)\\ &= \frac{2\delta_s\sqrt{2\delta_s m_s}}{2\delta_h m_h}\log A_{s,h}\sum_j I_{uhj} + o_p(1).\end{aligned}$$

If $s = h$, again using (44)

$$\begin{aligned}\sqrt{2\delta_s m_s}F_s^s(d_0) &= 2\log|2g_s|\frac{\delta_s}{\sqrt{2\delta_s m_s}}\sum_j I_{usj} + \frac{2}{\sqrt{2\delta_s m_s}}\sum_j \log|\lambda_j|I_{usj}\\ &+ O_p\left(\frac{m_s^{3/2}}{n}\right) + O_p\left(\frac{\log m_s}{m_s}\right) + o_p(1)\\ &= 2\log|2g_s|\frac{1}{\sqrt{2\delta_s m_s}}\sum_j I_{usj} + \frac{2}{\sqrt{2\delta_s m_s}}\sum_j \log|\lambda_j|I_{usj} + o_p(1),\end{aligned}$$

where it should be noted that $\log|2g_s| = 0$ for $w_s = 0, \pi$. Note now that $F_h(d_0)$ is equal to

$$\frac{1}{2\delta_h m_h}\sum_j I_{uhj} = G_{h0} + \frac{1}{2\delta_h m_h}\sum(I_{uhj} - G_{h0}2\pi I_{\epsilon hj}) + \frac{G_{h0}}{2\delta_h m_h}\sum_j(2\pi I_{\epsilon hj} - 1)$$

$$= G_{h0} + O_p\left(\frac{\log m_h}{\sqrt{m_h}} + \left(\frac{m_h}{n}\right)^{\beta_h}\right) = G_{h0} + o_p(1) \tag{50}$$

using Lemma 4 (with $l = 0$) in Arteche (2000) and (40). Then,

$$
\begin{aligned}
\sqrt{2\delta_s m_s}\frac{\partial}{\partial d_s}R_H(d_0) &= \sqrt{2\delta_s m_s}\sum_{h=1}^{H}\frac{F_h^s(d_0)}{F_h(d_0)} - 2\sqrt{2\delta_s m_s}\log|2g_s| \\
&- 2\sqrt{2\delta_s m_s}\sum_{\substack{h=1\\h\neq s}}^{H}\delta_h \log A_{s,h} - \frac{2}{\sqrt{2\delta_s m_s}}\sum \log|\lambda_j| \\
&= \frac{\frac{2}{\sqrt{2\delta_s m_s}}\sum_j \log|\lambda_j| I_{usj}}{F_s(d_0)} - \frac{2}{\sqrt{2\delta_s m_s}}\sum_j \log|\lambda_j| + o_p(1) \\
&= \frac{2}{G_{s0}\sqrt{2\delta_s m_s}}\sum_j v_j I_{usj} + o_p(1) \text{ for } v_j = \log|j| - \frac{1}{2\delta_s m_s}\sum_j \log|j| \\
&\xrightarrow{d} N(0,4),
\end{aligned}
$$

using the fact that, for $w_s \in (0,\pi)$ $\delta_s = 1$ such that

$$
\frac{1}{G_{s0}}\frac{2}{\sqrt{2\delta_s m_s}}\sum_j v_j I_{usj} = \sqrt{2}\left[\frac{1}{G_{s0}\sqrt{m_s}}\sum_{j=1}^{m_s} v_j I_{usj} + \frac{1}{G_{s0}\sqrt{m_s}}\sum_{j=1}^{m_s} v_j I_{us-j}\right] \xrightarrow{d} N(0,4),
$$

because each one of the two components between brackets converge to a $N(0,1)$ and both are asymptotically independent due to the asymptotic uncorrelation of periodogram ordinates $I_{usj}$ and $I_{us-j}$ (see formula (8.3) in Arteche and Robinson, 2000). By Assumption B.5 and the asymptotic uncorrelation of different periodogram ordinates of $u_t$, $\sqrt{\delta_s m_s}\frac{\partial}{\partial d_s}R_H(d_0)$ and $\sqrt{\delta_r m_r}\frac{\partial}{\partial d_r}R_H(d_0)$ are asymptotically independent for $r \neq s$ and thus the result in (27) follows.

■

# 3 Technical lemmas

Some of the lemmas are based on results in Phillips (1999), Phillips and Shimotsu (2004) and Shimotsu and Phillips (2005), but extended to the kind of processes defined in (1) in the paper. This relation is stressed when appropriate in the heading of the lemmas to facilitate comparison and consultation.

**Lemma 1** *Consider $w_h \in (0,\pi)$. As $k \to \infty$, uniformly in $d$,*

$$
c_k(w,d) = M(d,w)\cos[(k-d)w + \pi d/2]k^{-d-1}(1+O(k^{-1}))
$$

*where*

$$
M(d,w) = \frac{\Gamma(1/2-d)}{\sqrt{\pi}\Gamma(-2d)}e^{d+1/2}2^{-d}\sin^d w
$$

**Proof**: By formula (4.7.1) in Szego (1975)

$$c_k(w,d) = \frac{\Gamma(1/2-d)}{\Gamma(-2d)} \frac{\Gamma(k-2d)}{\Gamma(k-d+1/2)} P_k^{(-1/2-d,-1/2-d)}(\cos w)$$

and, by Theorem 8.21.8 in Szego (1975) the Jacobi polynomial $P_k^{(-1/2-d,-1/2-d)}(\cos w)$ satisfies

$$P_k^{(-1/2-d,-1/2-d)}(\cos w) = \frac{1}{\sqrt{k}} \frac{1}{\sqrt{\pi}} \sin^d \frac{w}{2} \cos^d \frac{w}{2} \cos[(k-d)w + \pi d/2] + O(k^{-3/2}).$$

The result then follows by applying Stirling's approximation of the Gamma function.

■

**Lemma 2** *Let $d_h > -1$. Then, as $n \to \infty$,*

$$D_n^h(e^{i(w_h+\lambda_j)}, d_h) = \sum_{k=0}^{n} c_k(w_h, d_h) e^{ik(w_h+\lambda_j)} = \Delta_h(e^{i(w_h+\lambda_j)}, d_h) + O(n^{-d_h}|j|^{-1})$$

*and for $w_g \neq w_h$,*

$$D_n^h(e^{i(w_g+\lambda_j)}, d_h) = \sum_{k=0}^{n} c_k(w_h, d_h) e^{ik(w_g+\lambda_j)} = \Delta_h(e^{i(w_g+\lambda_j)}, d_h) + O(n^{-d_h-1}).$$

**Proof**: For $w_h = 0$ see Lemma A.2 in Phillips and Shimotsu (2004). The results for $w_h = \pi$ are similarly obtained. For $w_h \in (0, \pi)$ note that

$$\begin{aligned} D_n^h(e^{i(w_g+\lambda_j)}, d_h) &= \sum_{k=0}^{\infty} c_k(w_h, d_h) e^{ik(w_g+\lambda_j)} - \sum_{k=n+1}^{\infty} c_k(w_h, d_h) e^{ik(w_g+\lambda_j)} \\ &= (1 - 2\cos w_h e^{i(w_g+\lambda_j)} + e^{2i(w_g+\lambda_j)})^{d_h} - \sum_{k=n+1}^{\infty} c_k(w_h, d_h) e^{ik(w_g+\lambda_j)}. \end{aligned} \quad (51)$$

By Lemma 1 the second element in (51) is

$$\begin{aligned} & M(d_h, w_h) \sum_{k=n+1}^{\infty} \cos[(k-d_h)w_h + \pi d_h/2] e^{ik(w_g+\lambda_j)} k^{-1-d_h} + O(\sum_{k=n+1}^{\infty} k^{-d_h-2}) \\ = \; & M(d_h, w_h) 2^{-1} \sum_{k=n+1}^{\infty} (e^{i[(k-d_h)w_h+\pi d_h/2]} + e^{-i[(k-d_h)w_h+\pi d_h/2]}) e^{ik(w_g+\lambda_j)} k^{-1-d_h} \\ + \; & O(n^{-d_h-1}) \\ = \; & O(n^{-d_h}|j|^{-1}) + O(n^{-d_h-1}) \text{ if } w_g = w_h, \\ = \; & O(n^{-d_h-1}) \text{ if } w_g \neq w_h, \end{aligned}$$

because, by Theorem 2.2 in Zygmund (2002),

$$\begin{aligned}
\left|\sum_{k=n+1}^{\infty} e^{ik(w_h+w_g+\lambda_j)} k^{-1-d_h}\right| &\leq (n+1)^{-1-d_h} \max_N \left|\sum_{k=n+1}^{n+N} e^{ik(w_h+w_g+\lambda_j)}\right| \\
&= O(n^{-d_h-1}) , \quad \text{similarly} \\
\left|\sum_{k=n+1}^{\infty} e^{ik(w_g-w_h+\lambda_j)} k^{-1-d_h}\right| &= O(n^{-d_h-1}) , \quad \text{for } w_g \neq w_h\text{, and if } w_g = w_h \\
\left|\sum_{k=n+1}^{\infty} e^{ik\lambda_j} k^{-1-d_h}\right| &\leq (n+1)^{-1-d_h} \max_N \left|\sum_{k=n+1}^{n+N} e^{ik\lambda_j}\right| \\
&= O(n^{-d_h-1}|\lambda_j|^{-1}) = O(n^{-d_h}|j|^{-1}).
\end{aligned}$$

■

**Lemma 3** *Consider $w_l = w_h$ for some $l \in \{1, 2, ..., H\}$. Then:*

*a) Uniformly in $d = (d_1, ..., d_H)' \in [-C, C]^H$ for some positive constant $C < \infty$, as $\lambda \to 0$,*

$$\prod_{l=1}^{H} \Delta_l(e^{i(w_h+\lambda_j)}, d_l) = |\lambda|^{d_h} |2g_h|^{d_h} e^{i\delta_h d_h (w_h-\pi)} \prod_{\substack{l=1 \\ l\neq h}}^{H} A_{l,h}^{\delta_l d_l} e^{i\delta_l d_l (w_l-\pi)} (1 + O(\lambda)).$$

*where $g_h = g(\sin w_h)$ for $g()$ a function in $[0, 1]$ defined as $g(x) = x$ if $x \in (0, 1]$ and $g(0) = 0.5$, $\delta_l = 0.5$ if $w_l = 0, \pi$ and 1 otherwise and $A_{l,h} = |4 \sin(0.5[w_h + w_l]) \sin(0.5[w_h - w_l])|$.*

*b) Uniformly in $d = (d_1, ..., d_H)' \in [-1+\varepsilon, C]^H$ for some $\varepsilon > 0$, and in $j = \pm 1, \pm 2, ..., \pm m_h$ if $w_h \neq 0, \pi$, $j = 1, 2, ..., m_h$ if $w_h = 0$, $j = -1, -2, ..., -m_h$ if $w_h = \pi$, $m_h/n \to 0$, $h = 1, 2, ..., H$, as $n \to \infty$,*

$$D_n(e^{i(w_h+\lambda_j)}, d) = |\lambda_j|^{d_h} |2g_h|^{d_h} e^{id_h\delta_h(w_h-\pi)} \prod_{\substack{l=1 \\ l\neq h}}^{H} A_{l,h}^{\delta_l d_l} e^{i\delta_l d_l (w_l-\pi)} (1+O(|\lambda_j|)+O(|j|^{-d_h-1})+O(n^{-\underline{d}_h-1})),$$

*where $\underline{d}_h = \min\{d_l\}_{\substack{l=1 \\ l\neq h}}^{H}$, and*

$$|D_n(e^{i(w_h+\lambda_j)}, d)|^2 = |\lambda_j|^{2d_h} |2g_h|^{2d_h} \prod_{\substack{l=1 \\ l\neq h}}^{H} A_{l,h}^{2\delta_l d_l} (1 + O(|\lambda_j|) + O(|j|^{-d_h-1}) + O(n^{-\underline{d}_h-1})).$$

**Proof**: To prove a), note that using the polar form of complex numbers results in that $\Delta_l(e^{i(w_h+\lambda_j)}, d_l) = (1 - 2\cos w_l e^{i(w_h+\lambda)} + e^{i2(w_h+\lambda)})^{\delta_l d_l}$ is

$$\begin{aligned}
&(1 - e^{i(w_l+w_h+\lambda)})^{\delta_l d_l} (1 - e^{i(w_h-w_l+\lambda)})^{\delta_l d_l} \\
= &\left|2\sin\left(\frac{w_l + w_h + \lambda}{2}\right)\right|^{\delta_l d_l} \left|2\sin\left(\frac{w_h - w_l + \lambda}{2}\right)\right|^{\delta_l d_l} \exp\{i\delta_l d_l (w_h + \lambda - \pi)\},
\end{aligned}$$

which equals $A_{l,h}^{\delta_l d_l} \exp\{i\delta_l d_l(w_h - \pi)\} + O(|\lambda|)$ if $w_h \neq w_l$. The result in a) follows because $2\sin\lambda/2 = \lambda + O(\lambda^2)$, $2\sin(2w_h + \lambda)/2 = 2\sin w_h + O(\lambda)$ and $\exp\{i\delta_l d_l(w_h + \lambda - \pi)\} = \exp\{i\delta_l d_l(w_h - \pi)\} + O(\lambda)$.

To prove b) note that $D_n(e^{i(w_h+\lambda_j)}, d) = \prod_{l=1}^{H} D_n^l(e^{i(w_h+\lambda_j)}, d_l)$. The result then follows from Lemma 2.

■

**Lemma 4** *For $X_t$ defined in equation (1) in the text of the paper,*

*a)*

$$W_u(\lambda) = D_n(e^{i\lambda}, d)W_x(\lambda) - \frac{e^{in\lambda}}{\sqrt{2\pi n}}\tilde{X}_n(\lambda)$$

*where $\tilde{X}_n(\lambda) = \sum_{p=0}^{n-1} \tilde{c}_p(\lambda)e^{-i\lambda p}X_{n-p}$, $\tilde{c}_p(\lambda) = \sum_{k=p+1}^{n} d_k(d)e^{ik\lambda}$.*

*b) As a particular case, if $H = 1$ and $d_1 = 1$,*

$$\begin{aligned} W_u(\lambda) &= (1 - 2\cos w_1 e^{i\lambda} + e^{i2\lambda})^{\delta_1} W_x(\lambda) - \frac{e^{in\lambda}}{\sqrt{2\pi n}}(I_1 e^{i2\lambda} - 2\delta_1 \cos w_1 e^{i\lambda})X_n \\ &- \frac{e^{i(n-1)\lambda} I_1}{\sqrt{2\pi n}} X_{n-1} \end{aligned}$$

*where $I_1 = I(\delta_1 = 1)$ is the indicator function.*

**Proof**: The result in a) comes directly from Lemma 2.1 in Phillips (1999). The result in b) comes from a) taking into account that $d_1(1) = -2\delta_1 \cos w_1$, $d_2(1) = I_1$ and $d_k(1) = 0$ for $k > 2$.

■

**Lemma 5** *(Extension of Lemma 5.3 in Shimotsu and Phillips, 2005)*
*Let $\tilde{U}_{nj}^h(d) = \tilde{U}_n(d; w_h + \lambda_j) = \sum_{p=0}^{n-1} \sum_{k=p+1}^{n} d_k(d) \exp\{i(k-p)(w_h + \lambda_j)\}u_{n-p}$. Under the assumptions in Theorem 1 and uniformly in $j = \pm 1, ..., \pm m_h$, as $n \to \infty$*

$$E \sup_{d \in [-1/2, 1/2]^H} \left| \frac{|\lambda_j|^{-d_h}}{\sqrt{2\pi n}} \tilde{U}_{nj}^h(d) \right|^2 = O\left( \frac{\log^2 n}{|j|} \right).$$

**Proof**: By formula (4) in Giraitis and Leipus (1995),

$$d_k(d) = \sum_{\substack{0 \le k_1 ... \le k_H \le k \\ k_1 + k_2 + ... + k_H = k}} \cdots \sum \prod_{s=1}^{H} c_{k_s}(w_s, d_s).$$

For example, for $H=2$, $d_k(d)=\sum_{k_1=0}^{k} c_{k_1}(w_1,d_1)c_{k-k_1}(w_2,d_2)$. Note now that by formulae (4.7.23) and (8.21.14) in Szego (1975) and after some straightforward manipulation we have $c_k(w,d)=\sum_{i=1}^{5} c_{ik}(w,d)$ for

$$\begin{aligned} c_{1k}(w,d) &= \frac{\Gamma(k-d)}{\Gamma(k+1)}A_1(w,d)e^{ikw} \\ c_{2k}(w,d) &= \frac{\Gamma(k-d)}{\Gamma(k+1)}A_2(w,d)e^{-ikw} \\ c_{3k}(w,d) &= \frac{\Gamma(k-d)}{\Gamma(k+1)}\frac{1}{k-d-1}A_3(w,d)e^{ikw} \\ c_{4k}(w,d) &= \frac{\Gamma(k-d)}{\Gamma(k+1)}\frac{1}{k-d-1}A_4(w,d)e^{-ikw} \\ c_{5k}(w,d) &= O(k^{-d-3}) \end{aligned}$$

where $A_i(w,d)$ are finite valued functions of $w$ and $d$. Using this result and formula (14) in Giraitis and Leipus (1995), $d_k(d)=\sum_{i=1}^{5} d_{ik}(d)$ for

$$\begin{aligned} d_{1k}(d) &= \sum_{s=1}^{H}\frac{\Gamma(k-d_s)}{\Gamma(k+1)}A_{s1}(w_s,d_s)e^{ikw_s} \\ d_{2k}(d) &= \sum_{s=1}^{H}\frac{\Gamma(k-d_s)}{\Gamma(k+1)}A_{s2}(w_s,d_s)e^{-ikw_s} \\ d_{3k}(d) &= \sum_{s=1}^{H}\frac{\Gamma(k-d_s)}{\Gamma(k+1)}\frac{1}{k-d-1}A_{s3}(w_s,d_s)e^{ikw_s} \\ d_{4k}(d) &= \sum_{s=1}^{H}\frac{\Gamma(k-d_s)}{\Gamma(k+1)}\frac{1}{k-d_s-1}A_{s4}(w_s,d_s)e^{-ikw_s} \end{aligned}$$

where $A_{si}(w_s,d_s)$, $i=1,2,3,4$, are finite valued functions of $w_s$ and $d_s$ and $d_{5k}(d)=O(|k|^{-3-\underline{d}})$ for $\underline{d}=\min\{d_1,...,d_H\}$. Then $\tilde{U}_{nj}^{h}=\sum_{i=1}^{5}\tilde{U}_{nj}^{(hi)}$ for

$$\tilde{U}_{nj}^{(hi)}=\sum_{p=0}^{n-1}\sum_{k=p+1}^{n} d_{ik}(d)e^{i(k-p)(w_h+\lambda_j)}u_{n-p}.$$

The result in the lemma is shown by proving the desired bound for every $\tilde{U}_{nj}^{(hi)}$. First denote $a_{s1,p}=\sum_{k=p+1}^{n}\Gamma(k+1)^{-1}\Gamma(k-d_s)\exp\{i(k-p)(w_h+w_s+\lambda_j)\}$. Then, by summation by parts

$$\begin{aligned} \tilde{U}_{nj}^{(h1)} &= \sum_{s=1}^{H}A_{s1}(w_s,d_s)\sum_{p=0}^{n-1}a_{s1,p}u_{n-p}e^{ipws} \\ &= \sum_{s=1}^{H}A_{s1}(w_s,d_s)\sum_{p=0}^{n-2}(a_{s1,p}-a_{s1,p+1})\sum_{r=0}^{p}u_{n-r}e^{irws} \\ &+ \sum_{s=1}^{H}A_{s1}(w_s,d_s)a_{s1,n-1}\sum_{r=0}^{n-1}u_{n-r}e^{irw_s} \end{aligned}$$

Now

$$\begin{aligned}
a_{s1,p} - a_{s1,p+1} &= \sum_{k=p+1}^{n} \frac{\Gamma(k-d_s)}{\Gamma(k+1)} e^{i(k-p)(w_h+w_s+\lambda_j)} - \sum_{k=p+2}^{n} \frac{\Gamma(k-d_s)}{\Gamma(k+1)} e^{i(k-p-1)(ws+w_h+\lambda_j)} \\
&= \sum_{k=p+1}^{n-1} \left[ \frac{\Gamma(k-d_s)}{\Gamma(k+1)} - \frac{\Gamma(k+1-d_s)}{\Gamma(k+2)} \right] e^{i(k-p)(w_h+w_s+\lambda_j)} \\
&+ \frac{\Gamma(n-d_s)}{\Gamma(n+1)} e^{i(n-p)(w_h+w_s+\lambda_j)} \\
&= (1+d_s) b_{1sp} + \frac{\Gamma(n-d_s)}{\Gamma(n+1)} e^{i(n-p)(w_h+w_s+\lambda_j)}
\end{aligned}$$

for $b_{1sp} = \sum_{k=p+1}^{n-1} \frac{\Gamma(k-d_s)}{\Gamma(k+2)} e^{i(k-p)(w_h+w_s+\lambda_j)}$ and then $\tilde{U}_{nj}^{(h1)}$ is equal to

$$\sum_{s=1}^{H} (1+d_s) A_{s1}(w_s, d_s) \sum_{p=0}^{n-2} b_{1sp} \sum_{r=0}^{p} u_{n-r} e^{irw_s} \tag{52}$$

$$+ \sum_{s=1}^{H} A_{s1}(w_s, d_s) \frac{\Gamma(n-d_s)}{\Gamma(n+1)} \sum_{p=0}^{n-1} e^{i(n-p)(w_h+w_s+\lambda_j)} \sum_{r=0}^{p} u_{n-r} e^{irw_s} \tag{53}$$

By Stirling's approximation $\Gamma(k-d_s)/\Gamma(k+1) = e^{d_s+2} k^{-d_s-2} [1 + O(k^{-1})]$ and thus

$$\begin{aligned}
|b_{1sp}| &= | \left| e^{d_s+2} \sum_{k=p+1}^{n-1} k^{-d_s-2} e^{i(k-p)(w_h+w_s+\lambda_j)} \right| + O\left( \sum_{k=p+1}^{n-1} k^{-d_s-3} \right) \\
&\leq e^{d_s+2} (p+1)^{-d_s-2} e^{-ip(w_h+w_s+\lambda_j)} \max_N \left| \sum_{k=p+1}^{p+N} e^{ik(w_h+w_s+\lambda_j)} \right| + O\left( (p+1)^{-d_s-2} \right) \\
&= O\left( (p+1)^{-d_s-2} \right)
\end{aligned}$$

by Theorem 2.2 of Zygmund (2002) because $0 < w_s + w_h + \lambda_j < 2\pi$ for $n$ large enough. Then each of the $H$ summands in (52) has mean zero and variance bounded by Minkowski's inequality by

$$\begin{aligned}
& (1+d_s)^2 A_{s1}^2(w_s, d_s) \left\{ \sum_{p=0}^{n-2} |b_{1sp}| \left[ E \left| \sum_{r=0}^{p} u_{n-r} e^{irw_s} \right|^2 \right]^{1/2} \right\}^2 \\
= \; & O\left( \left[ \sum_{p=0}^{n-2} (p+1)^{-d_s-3/2} \right]^2 \right) = O(\log^2 n)
\end{aligned}$$

for $d_s \geq -1/2$ if $E \left| \sum_{r=0}^{p} u_{n-r} e^{irw_s} \right|^2 = O(p+1)$. This bound holds because
$\sum_{q=-\infty}^{\infty} e^{-iqw_s} E u_t u_{t-q} = 2\pi f_u(w_s) = 2\pi G_{s0} < \infty$ by assumption A1 and covariance stationarity in assumption A5 such that

$$E \left| \sum_{r=0}^{p} u_{n-r} e^{irw_s} \right|^2 = (p+1) \sum_{q=-p}^{p} \left( 1 - \frac{|q|}{(p+1)} \right) e^{-iqw_s} E u_t u_{t-q} = O(p+1).$$

Now, (53) is

$$\sum_{s=1}^{H} A_{s1}(w_s,d_s)\frac{\Gamma(n-d_s)}{\Gamma(n+1)}\sum_{k=1}^{n} u_k e^{i(n-k)w_s}\sum_{q=1}^{k} e^{iq(w_s+w_h+\lambda_j)}$$
$$=\sum_{s=1}^{H} A_{s1}(w_s,d_s)\frac{\Gamma(n-d_s)}{\Gamma(n+1)}e^{inw_s}\sum_{k=1}^{n} u_k e^{-ikw_s} e^{i(w_h+w_s+\lambda_j)}\frac{1-e^{ik(w_h+w_s+\lambda_j)}}{1-e^{i(w_h+w_s+\lambda_j)}}$$
$$=\sum_{s=1}^{H} A_{s1}(w_s,d_s)\frac{\Gamma(n-d_s)}{\Gamma(n+1)}\frac{e^{inw_s}e^{i(w_h+w_s+\lambda_j)}}{1-e^{i(w_h+w_s+\lambda_j)}}\left(\sum_{k=1}^{n} u_k e^{-ikw_s}-\sum_{k=1}^{n} u_k e^{ik(w_h+\lambda_j)}\right) \qquad (54)$$

and thus $E|(53)|^2 = O(n^{-2\underline{d}-2}n) = O(1)$ for $\underline{d} \geq -1/2$. This proves the bound in the lemma for $\tilde{U}_{nj}^{(h1)}$ noting that $d_h \leq 1/2$.

To get the bound for $\tilde{U}_{nj}^{(h2)}$, the analysis for the summands in $\tilde{U}_{nj}^{(h2)}$ with $w_s \neq w_h$ is similar to get the same bounds as for $\tilde{U}_{nj}^{(h1)}$. The summand for $w_s = w_h$ is equal to

$$(1+d_s)A_{2s}(w_s,d_s)\sum_{p=0}^{n-2} b_{2sp}\sum_{r=0}^{p} u_{n-r}e^{irw_s} \qquad (55)$$
$$+\quad A_{2s}(w_s,d_s)\frac{\Gamma(n-d_s)}{\Gamma(n+1)}\sum_{p=0}^{n-1} e^{i(n-p)\lambda_j}\sum_{r=0}^{p} u_{n-r}e^{irw_s} \qquad (56)$$

for $b_{2sp} = \sum_{k=p+1}^{n-1}\frac{\Gamma(k-d_s)}{\Gamma(k+2)}e^{i(k-p)\lambda_j}$. Now $|b_{2sp}| = O\left((\min\left\{(p+1)^{-d_s-1},(p+1)^{-d_s-2}|\lambda_j|^{-1}\right\}\right)$ because obviously $|b_{2sp}| = O\left([p+1]^{-1-d_s}\right)$ but also

$$\begin{aligned}
|b_{2sp}| &= \left|e^{d_s+2}\sum_{k=p+1}^{n-1} k^{-d_s-2}e^{i(k-p)\lambda_j}\right| + O\left(\sum_{k=p+1}^{n-1} k^{-d_s-3}\right)\\
&\leq e^{d_s+2}(p+1)^{-d_s-2}e^{-ip\lambda_j}\max_N\left|\sum_{k=p+1}^{p+N} e^{ik\lambda_j}\right| + O\left((p+1)^{-d_s-2}\right)\\
&= O\left((p+1)^{-d_s-2}|\lambda_j|^{-1}\right).
\end{aligned}$$

Then

$$\frac{|\lambda_j|^{-d_s}}{\sqrt{n}}|b_{2sp}| = O\left(\min\left\{(p+1)^{-d_s-1}\frac{|\lambda_j|^{1/2-d_s}}{\sqrt{|j|}},(p+1)^{-d_s-2}\frac{|\lambda_j|^{-1/2-d_s}}{\sqrt{|j|}}\right\}\right) = O\left(\frac{(p+1)^{-3/2}}{\sqrt{|j|}}\right)$$

because if $p+1 \leq |\lambda_j|^{-1}$ the first bound in the min applies and if $p+1 > |\lambda_j|^{-1}$ the second one, leading to the desired result because $-1/2 \leq d_s \leq 1/2$. Then

$$E\sup_{d_s\in[1/2,1/2]}\left|\frac{|\lambda_j|^{-d_s}}{\sqrt{n}}(55)\right|^2$$

$$\leq \sup_{d_s}(1+d_s)^2A_{s2}^2(w_s,d_s)\left\{\sum_{p=0}^{n-2}\frac{|\lambda_j|^{-d_s}}{\sqrt{n}}|b_{2sp}|\left[E\left(\sum_{r=0}^{p}u_{n-r}e^{irw_s}\right)^2\right]^{1/2}\right\}^2$$

$$= O\left(\left[\sum_{p=0}^{n-2}\frac{(p+1)^{-3/2}}{\sqrt{|j|}}(p+1)^{1/2}\right]^2\right)=O\left(\frac{\log^2 n}{|j|}\right).$$

Now, proceeding as in the proof for the bound for (53) we obtain that $E|(56)|^2=O(n^{-2d_s-1}|\lambda_j|^{-1})$, which together with the result for (55) leads to $E\sup_{d\in[-1/2,1/2]^H}\left|\frac{|\lambda_j|^{-d_h}}{\sqrt{2\pi n}}\tilde{U}_{nj}^{h2}(d)\right|^2=O\left(|j|^{-1}\log^2 n\right)$.

The bounds for $\tilde{U}_{nj}^{(h3)}$ and $\tilde{U}_{nj}^{(h4)}$ are similarly obtained noting that

$$\frac{\Gamma(k-d_s)}{\Gamma(k+1)}\frac{1}{k-d_s-1}=\frac{\Gamma(k-d_s-1)}{\Gamma(k+1)}.$$

Finally, write $\tilde{U}_{nj}^{(h5)}=\sum_{p=0}^{n-1}a_{5s,p}u_{n-p}$ for $a_{5h,p}=\sum_{k=p+1}^{n}d_{5k}(d)e^{i(k-p)(w_h+\lambda_j)}$. Clearly $a_{5h,p}=O([p+1]^{-\underline{d}-2})$ and thus using again Minkowski's inequality

$$\begin{aligned}E\sup_{d\in[-1/2,1/2]^H}\left|\frac{|\lambda_j|^{-d_h}}{\sqrt{n}}\tilde{U}_{nj}^{(h5)}\right|^2 &= O\left(\sup\frac{|\lambda_j|^{-2d_h}}{n}E\left|\sum_{p=0}^{n-1}a_{5h,p}u_{n-p}\right|^2\right)\\ &= O\left(\sup\frac{|\lambda_j|^{1-2d_h}}{|j|}\left\{\sum_{p=0}^{n-1}(p+1)^{-\underline{d}-2}\left[Eu_{n-p}^2\right]^{1/2}\right\}^2\right)\\ &= O\left(\frac{1}{|j|}\right).\end{aligned}$$

■

**Lemma 6** *(Extension of Lemma 5.10 in Shimotsu and Phillips, 2005)*

*Let $Q_k$, $k=0,1,2,3$, be any real numbers, $\kappa\in(0,1/8)$ and $1/m+m/n\to 0$ as $n\to\infty$, and denote $\Delta_{wj}=(1-2\cos we^{i(w+\lambda_j)}+e^{i2(w+\lambda_j)})^{\delta}$ for $\delta=0.5$ if $w=0,\pi$ and $\delta=1$ if $w\in(0,\pi)$ and $D_w=1-e^{i2w}$ if $w\in(0,\pi)$, $D_w=1$ otherwise. Then there exists a finite constant $\eta>0$ not depending on $Q_k$, such that for $n$ sufficiently large*

*a)*
$$\frac{1}{m}\sum{}'\left|\Delta_{wj}^3Q_3+\Delta_{wj}^2Q_2+\Delta_{wj}Q_1+Q_0\right|^2 \geq \eta\left[\left(\frac{m}{n}\right)^6D_w^6Q_3^2+\left(\frac{m}{n}\right)^4D_w^4Q_2^2+\left(\frac{m}{n}\right)^2D_w^2Q_1^2+Q_0^2\right]$$

*b)*
$$\frac{1}{m}\sum{}'\left|\Delta_{wj}^{-1}Q_3+\Delta_{wj}^{-2}Q_2+\Delta_{wj}^{-3}Q_1+\Delta_{wj}^{-4}Q_0\right|^2 \geq \eta\left[\left(\frac{m}{n}\right)^{-2}D_w^{-2}Q_3^2+\left(\frac{m}{n}\right)^{-4}D_w^{-4}Q_2^2+\left(\frac{m}{n}\right)^{-6}D_w^{-6}Q_1^2+\left(\frac{m}{n}\right)^{-8}D_w^{-8}Q_0^2\right]$$

**Proof:** For $w = 0$ see Lemma 5.10 in Shimotsu and Phillips (2005). Similarly for $w = \pi$. For $w \in (0, \pi)$ define

$$A_j = \Delta_{wj}^3 Q_3 + \Delta_{wj}^2 Q_2 + \Delta_{wj} Q_1 + Q_0.$$

Since $1 - e^{i\lambda} = -i\lambda + O(\lambda^2)$ then

$$\Delta_{wj} = (1 - e^{i(2w+\lambda_j)})(1 - e^{i\lambda_j}) = -i\lambda_j D_w + O(\lambda_j^2)$$

such that

$$\begin{aligned} A_j &= D_w^3 (i\lambda_j^3) Q_3 - D_w^2 \lambda_j^2 Q_2 + D_w(-i\lambda_j) Q_1 + Q_0 \\ &+ Q_3 O(\lambda_j^4) + Q_2 O(\lambda_j^3) + Q_1 O(\lambda_j^2) \end{aligned}$$

and

$$\begin{aligned} |A_j|^2 &= |D_w^2 \lambda_j^2 Q_2 - Q_0|^2 + |D_w^3 \lambda_j^3 Q_3 - D_w \lambda_j Q_1|^2 \\ &+ Q_3^2 O(\lambda_j^7) + Q_2^2 O(\lambda_j^5) + Q_1 O(\lambda_j^3). \end{aligned}$$

The rest of the proof follows the same steps as in the proof of Lemma 5.10 in Shimotsu and Phillips (2005) and is thus omitted. For b) note that the term in the summation is $|\Delta_{wj}^{-4} A_j|^2$.

∎

**Lemma 7** *(Extension of Lemma 5.7 in Shimotsu and Phillips, 2005)*
*Define* $J_n(L, w) = \sum_{k=1}^n \frac{e^{ikw}}{k} L^k$ *and* $D_n(L, d) = \sum_{k=0}^n d_k(d) L^k$. *Then*

*a)* $J_n(L, w) = J_n(e^{i\lambda}, w) + \tilde{J}_n(e^{-i\lambda}L, w)(e^{-i\lambda}L - 1)$,

*b)* $J_n(L, w) D_n(L, d) = J_n(e^{i\lambda}, w) D_n(e^{i\lambda}, d) + \tilde{J}_n(e^{-i\lambda}L, w) D_n(e^{i\lambda}, d)(e^{-i\lambda}L - 1)$ $+ J_n(L, w) \tilde{D}_{n\lambda}(e^{-i\lambda}L, d)(e^{-i\lambda}L - 1)$,

*c)* $J_n(L, w_s) J_n(L, w_z) D_n(L, d) = J_n(e^{i\lambda}, w_s) J_n(e^{i\lambda}, w_z) D_n(e^{i\lambda}, d) + [J_n(e^{i\lambda}, w_z) \tilde{J}_n(e^{-i\lambda}L, w_s) + \tilde{J}_n(e^{-i\lambda}L, w_z) J_n(L, w_s)] D_n(e^{i\lambda}, d)(e^{-i\lambda}L - 1) + J_n(L, w_s) J_n(L, w_z) \tilde{D}_{n\lambda}(e^{-i\lambda}L, d)(e^{-i\lambda}L - 1)$,

*where*

$$\tilde{J}_n(e^{-i\lambda}L, w) = \sum_{p=0}^{n-1} \tilde{\gamma}_p(\lambda, w) e^{-ip\lambda} L^p \quad , \quad \tilde{\gamma}_p(\lambda, w) = \sum_{k=p+1}^{n} \frac{e^{ik(w+\lambda)}}{k}$$

$$\tilde{D}_{n\lambda}(z, d) = \sum_{p=0}^{n-1} \sum_{k=p+1}^{n} d_k(d) e^{ik\lambda} z^p.$$

**Proof:** Direct application of Lemma 2.1 in Phillips (1999).

■

**Lemma 8** *(Extension of Lemma 5.8 in Shimotsu and Phillips, 2005)*
*Let* $\bar{J}_{n,j}(w_r, w_s) = J_n(e^{i(w_r+\lambda_j)}, w_s) + J_n(e^{i(w_r+\lambda_j)}, -w_s)$ *and* $\bar{\gamma}_{p,j}(w_r, w_s) = \tilde{\gamma}_p(w_r + \lambda_j, w_s) + \tilde{\gamma}_p(w_r + \lambda_j, -w_s)$*. Then, uniformly in* $p = 0, ..., n-1$ *and* $j = \pm 1, ..., \pm m$ *with* $m = o(n)$*, as* $n \to \infty$*,*

*a)* $\bar{J}_{n,j}(w_r, w_s) = -\log\left|2\sin\left(\frac{w_r+w_s+\lambda_j}{2}\right)\right| - \log\left|2\sin\left(\frac{w_r-w_s+\lambda_j}{2}\right)\right| + \frac{i}{2}[\pi - w_r - w_s - \lambda_j + sign(w_r - w_s + \lambda_j)(\pi - |w_r - w_s + \lambda_j|)] + A_j$*, where* $A_j = O(n^{-1})$ *if* $w_r \neq w_s$ *and* $A_j = O(|j|^{-1})$ *if* $w_r = w_s$*.*

*b)*

$$\bar{\gamma}_{p,j}(w_r, w_s) = \begin{cases} O((p+1)^{-1}) & \text{if} \quad w_r \neq w_s \\ O(\min\{(p+1)^{-1}|\lambda_j|^{-1}, \log n\}) & \text{if} \quad w_r = w_s \end{cases}$$

.

**Proof:** For a) note that

$$J_n(e^{i(w_r+\lambda_j)}, w_s) = \sum_{k=1}^{\infty} \frac{e^{ik(w_r+w_s+\lambda_j)}}{k} - \sum_{k=n+1}^{\infty} \frac{e^{ik(w_r+w_s+\lambda_j)}}{k} \tag{57}$$

Now $0 < w_r + w_s + \lambda_j < 2\pi$ for $n$ large enough and then by formula (2.8) in Zygmund (2002), the first term on the right hand side of (57) is equal to $-2\log|2\sin(2^{-1}(w_r + w_s + \lambda_j)| + i(\pi - w_r - w_s - \lambda_j)/2$. For the second term, using summation by parts, $\left|\sum_{k=n+1}^{\infty} \frac{e^{ik(w_r+w_s+\lambda_j)}}{k}\right|$ is equal to

$$\left|\lim_{N\to\infty}\left[\sum_{r=n+1}^{n+N-1}\left(\frac{1}{r} - \frac{1}{r+1}\right)\sum_{k=n+1}^{r} e^{ik(w_r+w_s+\lambda_j)} + \frac{1}{n+N}\sum_{k=n+1}^{n+N} e^{ik(w_r+w_s+\lambda_j)}\right]\right|$$

$$= O(\lim_{N\to\infty}[\sum_{r=n+1}^{n+N-1} r^{-2} + (n+N)^{-1}]) = O(n^{-1})$$

if $w_r \neq 0, \pi$ and/or $w_s \neq 0, \pi$,

$$= O(\lim_{N\to\infty}[\sum_{r=n+1}^{n+N-1} r^{-2}|\lambda_j|^{-1} + (n+N)^{-1}|\lambda_j|^{-1}]) = O(|j|^{-1})$$

if $w_r = w_s = 0$ or $w_r = w_s = \pi$.

Proceeding similarly

$$J_n(e^{i(w_r+\lambda_j)}, -w_s) = \sum_{k=1}^{\infty} \frac{e^{ik(w_r-w_s+\lambda_j)}}{k} - \sum_{k=n+1}^{\infty} \frac{e^{ik(w_r-w_s+\lambda_j)}}{k}. \quad (58)$$

Using again formula (2.8) in Zygmund (2002), the first term on the right hand side of (58) is

$$\sum_{k=1}^{\infty} \frac{\cos(k(w_r - w_s + \lambda_j))}{k} + i \sum_{k=1}^{\infty} \frac{\sin(k(w_r - w_s + \lambda_j))}{k}$$
$$= -\log \left| 2 \sin \frac{(w_r - w_s + \lambda_j)}{2} \right| + sign(w_r - w_s + \lambda_j) \frac{i}{2} (\pi - |w_r - w_s + \lambda_j|)$$

and the second term is $O(n^{-1})$ if $w_r \neq w_s$ and $O(|j|^{-1})$ if $w_r = w_s$, using again summation by parts and the fact that $\sum_{k=n+1}^{n+N} e^{ik\lambda_j} = O(|\lambda_j|^{-1})$. Now b) is proved by noting that $\tilde{\gamma}_p(w_r + \lambda_j, w_s) = O(\min\{(p+1)^{-1}|\lambda_j|^{-1}, \log n\})$ if $w_r = w_s =, 0, \pi$, $O((p+1)^{-1})$ in any other case and $\tilde{\gamma}_p(w_r + \lambda_j, -w_s) = O((p+1)^{-1})$ if $w_r \neq w_s$ and $= O(\min\{(p+1)^{-1}|\lambda_j|^{-1}, \log n\})$ if $w_r = w_s$.

■

**Lemma 9** *(Extension of Lemma 5.9 in Shimotsu and Phillips, 2005)*
*Let* $Y_t = \Delta^H(L, \theta)u_t I(t \geq 1) = \prod_{h=1}^{H} \Delta_h(L, d_h)u_t I(t \geq 1)$ *for* $\Delta_h(L, d_h) = (1 - 2\cos w_h L + L^2)^{\delta_h d_h}$ *with* $\delta_l = 0.5$ *if* $w_h = 0, \pi$ *and* $\delta_l = 1$ *if* $w_h \in (0, \pi)$*. Denote* $W_{grj} = W_g(w_r + \lambda_j)$ *the Discrete Fourier Transform of a general series* $g_t$*,* $t = 1, 2, ..., n$*, at frequency* $w_r + \lambda_j$*,* $r = 1, 2, ..., H$*. Then, under the assumptions in Theorem 2, for* $s, z = 1, 2, ..., H$*:*

*a)* $-W_{\log \Delta_s yrj} = \delta_s \bar{J}_{n,j}(w_r, w_s) D_n(e^{i(w_r+\lambda_j)}, \theta) W_{urj} + n^{-1/2} V_{nj}(\theta)$,

*b)* $-W_{\log \Delta_s urj} = \delta_s \bar{J}_{n,j}(w_r, w_s) W_{urj} - e^{in(w_r+\lambda_j)} \delta_s \frac{B(1)}{\sqrt{2\pi n}} \tilde{J}_{nj}(w_r, w_s, L)\epsilon_n + r_{nj}$ *for* $\tilde{J}_{nj}(w_r, w_s, L) = \sum_{p=0}^{n-1} \sum_{k=p+1}^{n} k^{-1}(e^{iw_s k} + e^{-iw_s k}) e^{i(k-p)(w_r+\lambda_j)} L^p$,

*c)* $W_{\log \Delta_s \log \Delta_z yrj} = \delta_s \delta_z \bar{J}_{n,j}(w_r, w_s) \bar{J}_{n,j}(w_r, w_z) D_n(e^{i(w_r+\lambda_j)}, \theta) W_{urj} + n^{-1/2} \Psi_{nj}(\theta)$,

*where, uniformly in* $j = \pm 1, \pm 2, ..., \pm m_r$*, as* $n \to \infty$*,*

$$E \sup_{\theta} ||\lambda_j|^{1/2-\theta_r} V_{nj}(\theta)|^2 = O(\log^4 n)$$
$$E||j|^{1/2} r_{nj}|^2 = o(1) + O(|j|^{-1})$$
$$E \sup_{\theta} ||\lambda_j|^{1/2-\theta_r} \Psi_{nj}(\theta)|^2 = O(\log^6 n).$$

**Proof:** a) Since $\Delta_s = \Delta_s(L,1) = (1-2\cos w_s L + L^2)^{\delta_s} = (1-e^{-iw_s}L)^{\delta_s}(1-e^{iw_s}L)^{\delta_s}$ and noting that $|e^{iw_s}|=1$, and $Y_t = 0$ for $t \le 0$, then

$$\log \Delta_s Y_t = -\delta_s \bar{J}_n(L,w_s)Y_t \ , \quad \text{where } \bar{J}_n(L,w_s) = J_n(L,w_s) + J_n(L,-w_s)$$

and thus $\log \Delta_s Y_t = -\delta_s \bar{J}_n(L,w_s)Y_t = -\delta_s \bar{J}_n(L,w_s)D_n(L,\theta)\bar{u}_t$ where $\bar{u}_t = u_t I(t \ge 1)$. Then

$$-W_{\log \Delta_s yrj} = \frac{\delta_s}{\sqrt{2\pi n}} \sum_{t=1}^{n} \bar{J}_n(L,w_s)D_n(L,\theta)\bar{u}_t e^{it(w_r+\lambda_j)}.$$

By Lemma 2.1 in Phillips (1999) $\bar{J}_n(L,w_s)D_n(L,\theta)$ is equal to

$$\begin{aligned} & \bar{J}_{n,j}(w_r,w_s)D_n(e^{i(w_r+\lambda_j)},\theta)\bar{u}_t + \tilde{J}_{nj}(w_r,w_s,L)D_n(e^{i(w_r+\lambda_j)},\theta)(Le^{-i(w_r+\lambda_j)}-1)\bar{u}_t \\ + \quad & \bar{J}_n(L,w_s)\tilde{D}_n(e^{i(w_r+\lambda_j)}L)(Le^{-i(w_r+\lambda_j)}-1)\bar{u}_t \end{aligned}$$

where $\bar{J}_{n,j}(w_r,w_s) = \bar{J}_n(e^{i(w_r+\lambda_j}, w_s)$ and

$$\tilde{D}_n(e^{i(w_r+\lambda_j)}L) = \sum_{p=0}^{n-1}\sum_{k=p+1}^{n} d_k(\theta)e^{i(k-p)(w_r+\lambda_j)}L^p.$$

Then, since $\sum_{t=1}^{n} e^{it(w+\lambda_j)}(Le^{-i(w+\lambda_j)}-1)\bar{u}_t = -e^{in(w+\lambda_j)}\bar{u}_n$,

$$\begin{aligned} -W_{\log \Delta_s yrj} &= \delta_s \bar{J}_{n,j}(w_r,w_s)D_n(e^{i(w_r+\lambda_j)},\theta)W_{urj} \\ &- \frac{\delta_s}{\sqrt{2\pi n}}\tilde{J}_{nj}(w_r,w_s,L)D_n(e^{i(w_r+\lambda_j)},\theta)e^{in(w_r+\lambda_j)}\bar{u}_n && (59) \\ &- \frac{\delta_s}{\sqrt{2\pi n}}\bar{J}_n(L,w_s)\tilde{D}_n(e^{i(w_r+\lambda_j)}L)e^{in(w_r+\lambda_j)}\bar{u}_n && (60) \end{aligned}$$

Using Lemma 3, the desired bound for (59) is obtained if

$$E\sup_{\theta}\left||\lambda_j|^{1/2}\tilde{J}_{nj}(w_r,w_s,L)\bar{u}_n\right|^2 = O(\log^4 n). \quad (61)$$

Now

$$\begin{aligned} \tilde{J}_{nj}(w_r,w_s,L)\bar{u}_n &= \sum_{p=0}^{n-1}\sum_{k=p+1}^{n}\frac{1}{k}e^{i(k-p)(w_r+w_s+\lambda_j)}e^{ipw_s}u_{n-p} && (62) \\ &+ \sum_{p=0}^{n-1}\sum_{k=p+1}^{n}\frac{1}{k}e^{i(k-p)(w_r-w_s+\lambda_j)}e^{-ipw_s}u_{n-p} && (63) \end{aligned}$$

Let us focus first on (62). Denoting $a_p = \sum_{k=p+1}^{n} k^{-1}e^{i(k-p)(w_r+w_s+\lambda_j)}$ and by summation by parts, (62) is equal to

$$\begin{aligned} & \sum_{p=0}^{n-2}(a_p - a_{p+1})\sum_{q=0}^{p} e^{iqw_s}u_{n-q} + a_{n-1}\sum_{q=0}^{n-1} e^{iqw_s}u_{n-q} \\ = \; & \sum_{p=0}^{n-2} b_p \sum_{q=0}^{p} e^{iqw_s}u_{n-q} + \frac{1}{n}\sum_{p=0}^{n-1} e^{i(n-p)(w_r+w_s+\lambda_j)}\sum_{q=0}^{p} e^{iqw_s}u_{n-q} && (64) \end{aligned}$$

where

$$
\begin{aligned}
a_p - a_{p+1} &= b_p + \frac{1}{n} e^{i(n-p)(w_r+w_s+\lambda_j)} \\
b_p &= \sum_{k=p+1}^{n-1} \frac{1}{k(k+1)} e^{i(k-p)(w_r+w_s+\lambda_j)} \\
a_{n-1} &= \frac{1}{n} e^{i(w_r+w_s+\lambda_j)}.
\end{aligned}
$$

Now, the second part of (64) is equal to

$$
\begin{aligned}
& \frac{1}{n} e^{inw_s} \sum_{k=1}^{n} u_k e^{-ikw_s} \sum_{q=1}^{k} e^{iq(w_r+w_s+\lambda_j)} \\
= & \frac{1}{n} \frac{e^{inw_s} e^{i(w_r+w_s+\lambda_j)}}{1 - e^{i(w_r+w_s+\lambda_j)}} \left( \sum_{k=1}^{n} u_k e^{-ikw_s} - \sum_{k=1}^{n} u_k e^{-ik(w_r+\lambda_j)} \right) \qquad (65)
\end{aligned}
$$

such that $E|(65)|^2 = O(n^{-1})$ for $w_r \neq w_s$ or $w_r = w_s \neq 0, \pi$ and $O(n^{-1}|\lambda_j|^{-2}) = O(|j|^{-1}|\lambda_j|^{-1})$ if $w_r = w_s = 0, \pi$.

Regarding the first part of (64), and noting that $b_p = O((p+1)^{-2})$ for $w_r \neq w_s$ or $w_r = w_s \neq 0, \pi$ and $b_p = O(\min\{(p+1)^{-1}, (p+1)^{-2}|\lambda_j|^{-1}\})$ if $w_r = w_s = 0, \pi$, then $E|\sum_{p=0}^{n-2} b_p \sum_{q=0}^{p} e^{iqw_s} u_{n-q}|^2$ by Minkowski's inequality

$$
\begin{aligned}
&= O\left( \left\{ \sum_{p=0}^{n-2} |b_p| \left[ E \left( \sum_{q=0}^{p} e^{iqw_s} u_{n-q} \right)^2 \right]^{1/2} \right\}^2 \right) \\
&= O\left( \left\{ \sum_{p=0}^{n-2} (p+1)^{-3/2} \right\}^2 \right) = O(1) \text{ if } w_r \neq w_s \text{ or } w_r = w_s \neq 0, \pi \\
&= O\left( \left\{ \sum_{p=0}^{n/j} (p+1)^{-1/2} + \sum_{p=n/j+1}^{n} (p+1)^{-3/2} |\lambda_j|^{-1} \right\}^2 \right) \\
&= O(|\lambda_j|^{-1}) \text{ if } w_r = w_s = 0, \pi.
\end{aligned}
$$

The bounds for (63) are similarly obtained to get that (61) is satisfied. We move now to obtain the bound for (60). We need to show that

$$
E \sup_{\theta} \left| |\lambda_j|^{1/2-\theta_r} \bar{J}_n(L, w_s) \tilde{D}_n(e^{i(w_r+\lambda_j)}L) \bar{u}_n \right|^2 = O(\log^4 n). \qquad (66)
$$

If $\theta$ is a vector of zeros then $\tilde{D}_n(e^{i(w_r+\lambda_j)}L) = 0$ and the result is obvious. If some element of $\theta$ is different form zero the left hand side of (66) is bounded by

$$
\left\{ \sum_{k=1}^{n} \left| \frac{e^{ikw_s} + e^{-ikw_s}}{k} \right| \left( E \sup_{\theta} \left| |\lambda_j|^{1/2-\theta_r} \tilde{D}_n(e^{i(w_r+\lambda_j)}L) u_{n-k} \right|^2 \right)^{1/2} \right\}^2
$$

and thus (66) is satisfied if

$$E\sup_{\theta}\left||\lambda_j|^{1/2-\theta_r}\tilde{D}_n(e^{i(w_r+\lambda_j)}L)u_{n-k}\right|^2 = O(\log^2 n) \tag{67}$$

uniformly in $k = 1, ..., n-1$. But $|\lambda_j|^{1/2-\theta_r}\tilde{D}_n(e^{i(w_r+\lambda_j)}L)u_{n-k}$ is equal to

$$\begin{aligned} & |\lambda_j|^{1/2-\theta_r}L^k\sum_{p=0}^{n-1}\sum_{k=p+1}^{n}d_k(\theta)e^{i(k-p)(w_r+\lambda_j)}u_{n-p} \\ = \;& |\lambda_j|^{1/2-\theta_r}L^k\tilde{U}_{nj}^r \end{aligned}$$

and (67) follows from Lemma 5.

b) Using Lemma 7 and the fact that $\sum_{t=1}^{n}e^{it(w_r+\lambda_j)}(Le^{-i(w_r+\lambda_j)}-1)\bar{u}_t = -e^{in(w_r+\lambda_j)}\bar{u}_n$ we have that

$$\begin{aligned} -W_{\log\Delta_s urj} &= \frac{\delta_s}{\sqrt{2\pi n}}\sum_{t=1}^{n}\bar{J}_n(L,w_s)\bar{u}_t e^{it(w_r+\lambda_j)} \\ &= \delta_s\bar{J}_{nj}(w_r,w_s)W_{urj} - \frac{B(1)}{\sqrt{2\pi n}}\delta_s\tilde{J}_{nj}(w_r,w_s,L)e^{in(w_r+\lambda_j)}\epsilon_n \\ &- \frac{\delta_s}{\sqrt{2\pi n}}\tilde{J}_{nj}(w_r,w_s,L)e^{in(w_r+\lambda_j)}[\bar{u}_n - B(1)\epsilon_n] \end{aligned} \tag{68}$$

and thus b) is proved if

$$E\left|\left|\frac{j}{n}\right|^{1/2}\tilde{J}_{nj}(w_r,w_s,L)[\bar{u}_n - B(1)\epsilon_n]\right|^2 = o(1) + O(|j|^{-1}) \tag{69}$$

uniformly in $j = \pm1, ..., \pm m$. Now $|j/n|^{1/2}\tilde{J}_{nj}(w_r,w_s,L)[\bar{u}_n - B(1)\epsilon_n]$ is equal to

$$\left|\frac{j}{n}\right|^{1/2}\sum_{p=0}^{n-1}\sum_{k=p+1}^{n}\frac{e^{i(k-p)(w_r+w_s+\lambda_j)}}{k}Z_{n-p}e^{ipw_s} \tag{70}$$

$$+ \left|\frac{j}{n}\right|^{1/2}\sum_{p=0}^{n-1}\sum_{k=p+1}^{n}\frac{e^{i(k-p)(w_r-w_s+\lambda_j)}}{k}Z_{n-p}e^{-ipw_s} \tag{71}$$

for $Z_{n-p} = \bar{u}_{n-p} - B(1)\epsilon_{n-p}$. Denoting again $a_p = \sum_{k=p+1}^{n}e^{i(k-p)(w_r+w_s+\lambda_j)}/k$ and using summation by parts in the same way as from (62) to (64), (70) is equal to

$$\left|\frac{j}{n}\right|^{1/2}\sum_{p=0}^{n-2}b_p\sum_{q=0}^{p}Z_{n-q}e^{iqw_s} + \left|\frac{j}{n}\right|^{1/2}\frac{1}{n}\sum_{p=0}^{n-1}e^{i(n-p)(w_r+w_s+\lambda_j)}\sum_{q=0}^{p}e^{iqw_s}Z_{n-q}$$

$$= \left|\frac{j}{n}\right|^{1/2}\sum_{p=0}^{n-2}b_p\sum_{q=0}^{p}Z_{n-q}e^{iqw_s} \tag{72}$$

$$+ \left|\frac{j}{n}\right|^{1/2}\frac{1}{n}\frac{e^{inw_s}e^{i(w_r+w_s+\lambda_j)}}{1-e^{i(w_r+w_s+\lambda_j)}}\sqrt{2\pi n}[W_u(w_s) - B(1)W_\epsilon(w_s)] \tag{73}$$

$$+ \left|\frac{j}{n}\right|^{1/2}\frac{1}{n}\frac{e^{inw_s}e^{i(w_r+w_s+\lambda_j)}}{1-e^{i(w_r+w_s+\lambda_j)}}\sqrt{2\pi n}[W_{urj} - B(1)W_{\epsilon rj}]. \tag{74}$$

Now $E|(63)|^2$ and $E|(64)|^2$ are both $O(|j|n^{-2}) = O(|j|^{-1})$ if $w_r \neq w_s$ or $w_r = w_s \neq 0, \pi$ and $O(|j||\lambda_j|^{-2}n^{-2}) = O(|j|^{-1})$ if $w_r = w_s = 0, \pi$.

Now using formula (87) in Shimotsu and Phillips (2005) and Minkowski's innequality

$$\begin{aligned}
E|(72)|^2 &= O\left(\frac{|j|}{n}\left\{\sum_{p=0}^{n-2}|b_p|\left[E(\sum_{q=0}^{p}[Z_{n-q}e^{iqw_s})^2\right]^{1/2}\right\}^2\right) \\
&= O\left(\frac{|j|}{n}\left\{\sum_{p=0}^{n-2}(p+1)^{-3/2}\right\}^2\right) = o(1) \text{ if } w_r \neq w_s \text{ or } w_r = w_s \neq 0, \pi \\
&= o\left(\frac{|j|}{n}|\lambda_j|^{-1}\right) = o(1) \text{ if } w_r = w_s = 0, \pi.
\end{aligned}$$

Proceeding similarly we get the same bounds for (71) and thus (69) holds.

c) Note that

$$W_{\log\Delta_s \log\Delta_z yrj} = \frac{\delta_s\delta_z}{\sqrt{2\pi n}}\sum_{t=1}^{n}\bar{J}_n(L, w_s)\bar{J}_n(L, w_z)D_n(L, \theta)\bar{u}_t e^{it(w_r+\lambda_j)}$$

and using Lemma 7

$$\begin{aligned}
W_{\log\Delta_s \log\Delta_z yrj} &= \delta_s\delta_z\bar{J}_{n,j}(w_r, w_s)\bar{J}_{n,j}(w_r, w_z)D_n(e^{i(w_r+\lambda_j)}, \theta)W_{urj} \\
&- \frac{e^{in(w_r+\lambda_j)}}{\sqrt{2\pi n}}\delta_s\delta_z\bar{J}_{n,j}(w_r, w_z)\tilde{J}_{nj}(w_r, w_s, L)D_n(e^{i(w_r+\lambda_j)}, \theta)\bar{u}_n \quad (75) \\
&- \frac{e^{in(w_r+\lambda_j)}}{\sqrt{2\pi n}}\delta_s\delta_z\bar{J}_n(L, w_s)\tilde{J}_{nj}(w_r, w_z, L)D_n(e^{i(w_r+\lambda_j)}, \theta)\bar{u}_n \quad (76) \\
&- \frac{e^{in(w_r+\lambda_j)}}{\sqrt{2\pi n}}\delta_s\delta_z\bar{J}_n(L, w_s)\bar{J}_n(L, w_z)\tilde{D}_n(e^{i(w_r+\lambda_j)}L)\bar{u}_n. \quad (77)
\end{aligned}$$

Now

$$\bar{J}_n(L, w_s)\bar{J}_n(L, w_z) = \sum_{k=1}^{n}\sum_{l=1}^{n}h_{ks}h_{lz}L^{k+l} \quad \text{for} \quad h_{ab} = \frac{e^{iaw_b} + e^{-iaw_b}}{K}.$$

Now $E\sup_\theta \left||\lambda_j|^{1/2-\theta_r}\tilde{D}_n(e^{i(w_r+\lambda_j)}L)\bar{u}_{n-k}\right|^2 = O(\log^2 n)$ uniformly in $k = 0, 1, ..., n-1$, and then, ignoring constants $E\sup_\theta \left||\lambda_j|^{1/2-\theta_r}\sqrt{n}(77)\right|^2$ is

$$\begin{aligned}
&E\sup_\theta\left|\sum_{k=1}^{n}\sum_{l=1}^{n}h_{ks}h_{lz}|\lambda_j|^{1/2-\theta_r}\tilde{D}_n(e^{i(w_r+\lambda_j)}L)\bar{u}_{n-k-l}\right|^2 \\
&= O\left(\left\{\sum_{k=1}^{n}\sum_{l=1}^{n}|h_{ks}h_{lz}|\left[E\sup_\theta\left||\lambda_j|^{1/2-\theta_r}\tilde{D}_n(e^{i(w_r+\lambda_j)}L)\bar{u}_{n-k-l}\right|^2\right]^{1/2}\right\}^2\right) \\
&= O\left(\left\{\sum_{k=1}^{n}\sum_{l=1}^{n}|h_{ks}h_{lz}|\log n\right\}^2\right) = O(\log^6 n)
\end{aligned}$$

The bound for (75) is obtained as that for (59) using also the fact that $\bar{J}_{nj}(w_r, w_s) = O(\log n)$. Similarly, using (61), we get that $E \sup_\theta \left||\lambda_j|^{1/2-\theta_r}\sqrt{n}(76)\right|^2 = O(\log^6 n)$.

■

**Lemma 10** *Let $z_t(d) = \Delta^H(L,d)\cos(\bar{w}t)\, I(t \geq 1)$ for $\Delta^H(L,d) = \prod_{h=1}^H \Delta_h(L,d_h)$, $\Delta_h(L,d_h) = (1 - 2\cos w_h L + L^2)^{\delta_h d_h}$ with $\delta_l = 0.5$ if $w_h = 0, \pi$ and $\delta_l = 1$ if $w_h \in (0,\pi)$. Then,*

*a)*

$$\begin{aligned} W_z(\lambda) &= \frac{1}{\sqrt{2\pi n}} \frac{e^{i(\bar{w}+\lambda)}}{1 - e^{i(\bar{w}+\lambda)}} \left[D_{n-1}(e^{i\lambda}, d) - e^{i(\bar{w}+\lambda)n} D_{n-1}(e^{-i\bar{w}}, d)\right] \\ &+ \frac{1}{\sqrt{2\pi n}} \frac{e^{-i(\bar{w}-\lambda)}}{1 - e^{-i(\bar{w}-\lambda)}} \left[D_{n-1}(e^{i\lambda}, d) - e^{-i(\bar{w}-\lambda)n} D_{n-1}(e^{i\bar{w}}, d)\right] \end{aligned}$$

*b) For $\underline{d} = \min\{d_1, ..., d_H\}$*

$$\begin{aligned} W_z(w_h + \lambda_j) &= O\left(\frac{1}{\sqrt{n}} \lambda_j^{-1} \left[\lambda_j^{d_h} + n^{-\underline{d}} \log n\right]\right) \quad \text{if } w_h = \bar{w} \\ &= O\left(\frac{1}{\sqrt{n}} \left[\lambda_j^{d_h} + n^{-\underline{d}} \log n\right]\right) \quad \text{if } w_h \neq \bar{w} \end{aligned}$$

**Proof:** Note that $z_t(d) = \sum_{k=0}^{t-1} d_k(d) \cos(\bar{w}[t-k]) = 0.5 \sum_{k=0}^{t-1} d_k(d) \left(e^{i\bar{w}(t-k)} + e^{-i\bar{w}(t-k)}\right)$. The DFT of $z_t(d)$ is then

$$\begin{aligned} W_z(\lambda) &= \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \sum_{k=0}^{t-1} d_k(d) \cos(\bar{w}[t-k])\, e^{it\lambda} \\ &= \frac{1}{\sqrt{2\pi n}} \sum_{k=0}^{n-1} d_k(d) \sum_{t=1}^{n-k} \cos(\bar{w}t)\, e^{i(t+k)\lambda} \\ &= \frac{e^{i(\bar{w}+\lambda)}}{2\sqrt{2\pi n}} \sum_{k=0}^{n-1} d_k(d) e^{ik\lambda} \sum_{t=0}^{n-k-1} e^{it(\bar{w}+\lambda)} + \frac{e^{-i(\bar{w}-\lambda)}}{2\sqrt{2\pi n}} \sum_{k=0}^{n-1} d_k(d) e^{ik\lambda} \sum_{t=0}^{n-k-1} e^{-it(\bar{w}-\lambda)} \\ &= \frac{e^{i(\bar{w}+\lambda)}}{2\sqrt{2\pi n}} \sum_{k=0}^{n-1} d_k(d) e^{ik\lambda} \frac{1 - e^{i(n-k)(\bar{w}+\lambda)}}{1 - e^{i(\bar{w}+\lambda)}} + \frac{e^{-i(\bar{w}-\lambda)}}{2\sqrt{2\pi n}} \sum_{k=0}^{n-1} d_k(d) e^{ik\lambda} \frac{1 - e^{-i(n-k)(\bar{w}-\lambda)}}{1 - e^{-i(\bar{w}-\lambda)}} \\ &= \frac{e^{i(\bar{w}+\lambda)}}{1 - e^{i(\bar{w}+\lambda)}} \frac{1}{2\sqrt{2\pi n}} \left[D_{n-1}(e^{i\lambda}, d) - e^{in(\bar{w}+\lambda)} D_{n-1}(e^{-i\bar{w}}, d)\right] \\ &+ \frac{e^{-i(\bar{w}-\lambda)}}{1 - e^{-i(\bar{w}-\lambda)}} \frac{1}{2\sqrt{2\pi n}} \left[D_{n-1}(e^{i\lambda}, d) - e^{-in(\bar{w}-\lambda)} D_{n-1}(e^{i\bar{w}}, d)\right] \end{aligned}$$

Part b) is proven noting that $(1 - e^{-i\lambda_j})^{-1} = O(\lambda_j^{-1})$, $D_{n-1}(e^{i(w_h+\lambda_j)}, d) = O(\lambda_j^{d_h})$ by Lemmas 2 and 3, and that $D_{n-1}(e^{i\bar{w}}, d) = O(n^{-\underline{d}} \log n)$. This last bound can be shown because $d_k(d) = O(k^{-\underline{d}-1})$ (see Theorem 1 in Giraitis and Leipus, 1995) and thus the bound is straightforward for $\underline{d} \leq 0$. If $\underline{d} > 0$ then

$$D_{n-1}(e^{i\bar{w}}, d) = \Delta^H(e^{i\bar{w}}, d) + O\left(\sum_{k=n}^{\infty} k^{-\underline{d}-1}\right) = O(n^{-\underline{d}})$$

because $(1 - 2\cos\bar{w}e^{i\bar{w}} + e^{i2\bar{w}})^d = (1 - e^{i\bar{w}}e^{i\bar{w}})^d(1 - e^{i\bar{w}}e^{-i\bar{w}})^d = 0$ for $d > 0$.

■

**Lemma 11** *Consider $Y_t$ in (10) in the main paper and denote $\bar{d}_0 = \max\{d_{h0} + I(w_h = \bar{w})\}_{h=1}^H$ where $I(w_h = \bar{w}) = 1$ if $w_h = \bar{w}$ and zero otherwise. If $1/2 < \bar{d}_0 < 3/2$ then $\hat{\mu}$ converges in mean square to $\mu$ as $n \to \infty$ such that $Var(\hat{\mu}) = O(n^{2\bar{d}_0 - 3})$.*

**Proof:** Note that

$$\hat{\mu} - \mu = \frac{1}{n}\sum_{k=0}^{n-1} 2\cos(\bar{w}k)L^k X_n$$

and since $2\cos(\bar{w}k) = e^{i\bar{w}k} + e^{-i\bar{w}k}$ and $\sum_{k=0}^{\infty} e^{\pm i\bar{w}k}L^k = (1 - e^{\pm i\bar{w}}L)^{-1}$ then

$$\begin{aligned}
\hat{\mu} - \mu &= \frac{1}{n}\left[(1 - e^{i\bar{w}}L)^{-1} + (1 - e^{-i\bar{w}}L)^{-1}\right] X_n \\
&= \frac{1}{n}(1 - e^{i\bar{w}}L)^{-1}(1 - e^{-i\bar{w}}L)^{-1}\left[(1 - e^{-i\bar{w}}L) + (1 - e^{i\bar{w}}L)\right] X_n \\
&= \frac{1}{n}(1 - 2\cos(\bar{w})L + L^2)^{-1}\left[2(1 - \cos(\bar{w}))\right]\Delta^H(L, -d_0)u_n \\
&= \frac{1}{n}\sum_{k=0}^{n-1}\theta_k(d_0)u_{n-k}
\end{aligned}$$

where $\theta_k(d_0) = O(k^{\bar{d}_0 - 1})$ using Lemma 1. The mean is clearly zero and, if $\bar{d}_0 > 1/2$ then

$$Var(\hat{\mu}) = O\left(\frac{1}{n}\sum_{k=0}^{n-1} k^{2\bar{d}_0 - 2}\right) = O(n^{2\bar{d}_0 - 3}).$$

■

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