Online Supplementary Material for 'Asymptotic Theory for Kernel Estimators under Moderate Deviations from a Unit Root' by J. A. Duffy

Throughout the following, Assumptions DGP and SM are always maintained, even when not explicitly referenced. Section S.1 provides the proofs of Lemmas A.1–A.3, and Section S.2 provides the proofs of Lemmas C.1 and C.3–C.5.

S.1 Proofs of auxiliary lemmas from Appendix A

Proof of Lemma A.1. Since $d_n^2 = \operatorname{var}(x_n)$ is bounded away from zero in all cases, it suffices to prove that $d_n \lesssim n^{1/2}$ when $\{\rho_n\} \in \mathcal{P}$ is mildly integrated or local to unity. To that end, recall from (C.2) that

$$x_n = \sum_{k=1}^{n-1} a_k \varepsilon_{t-k} + \sum_{k=n}^{\infty} a_{t,k}\varepsilon_{t-k}$$

where $a_{t,k} = \sum_{l=0}^{k\wedge(t-1)} \rho^l \phi_{k-l}$. Hence

$$\begin{aligned}\operatorname{var}(x_n) = \sum_{k=1}^{n-1} a_{t,k}^2 + \sum_{k=n}^{\infty} a_{t,k}^2 &\leq \sum_{k=1}^{n-1}\left(\sum_{l=0}^{k}\rho_n^l\phi_{k-l}\right)^2 + \sum_{k=n}^{\infty}\left(\sum_{l=0}^{n-1}\rho_n^l\phi_{k-l}\right)^2 \\ &\leq n\left(\sum_{i=0}^{\infty}|\phi_i|\right)^2 + \sum_{k=n}^{\infty}\left(\sum_{l=0}^{n-1}|\phi_{k-l}|\right)^2\end{aligned}$$

For the second r.h.s. term, we have

$$\begin{aligned}\sum_{k=n}^{\infty}\left(\sum_{l=0}^{n-1}|\phi_{k-l}|\right)^2 \lesssim \sum_{k=n}^{\infty}\sum_{l=0}^{n-1}|\phi_{k-l}| &= \left(\sum_{k=n}^{2n} + \sum_{k=2n+1}^{\infty}\right)\sum_{l=0}^{n-1}|\phi_{k-l}| \\ &\leq \sum_{k=0}^{n}\sum_{l=k}^{\infty}|\phi_l| + n\sum_{k=n}^{\infty}|\phi_k| = o(n).\end{aligned}$$ □

Proof of Lemma A.2. As noted in the text, the stated convergence follows immediately from Theorem 3.2: see also Remark 3.1. Regarding the strict positivity of $\tau(x)$: when $\{\rho_n\}$ is local to unity, this follows from Ray's (1963) theorem; when $\{\rho_n\}$ is mildly integrated this is immediate from φ being the standard normal density; and when $\{\rho_n\}$ is stationary, this follows from the density f_ε of ε_t having been assumed strictly positive (see DGP2). □

Proof of Lemma A.3. We first show that $\hat{m}_n(x) = m(x) + o_p(1)$. To that end, decompose

$$\hat{m}_n(x) - m(x) = \frac{A_{n,1} + A_{n,2}}{\frac{1}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x)}$$

where:

$$|A_{n,1}| \coloneqq \frac{1}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x)|m(x_t) - m(x)|$$
$$\leq \frac{Ch_n}{e_n}\sum_{t=1}^{n}\frac{1}{h_n}K\left(\frac{x_t - x}{h_n}\right)\left|\frac{x_t - x}{h_n}\right|$$
$$\lesssim_p h_n \tag{S.1}$$

by Lemma A.2 and the Lipschitz continuity of m; and

$$A_{n,2} \coloneqq \frac{1}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x)u_{t+1} = o_p(1)$$

where the claimed negligibility follows since $A_{n,2}$ is a martingale with variance

$$\mathbb{E}A_{n,2}^2 = \frac{1}{e_n^2 h_n^2}\sum_{t=1}^{n}\mathbb{E}K^2\left(\frac{x_t - x}{h_n}\right)u_{t+1}^2$$
$$= \frac{1}{e_n h_n}\cdot\frac{\sigma^2}{e_n}\sum_{t=1}^{n}\mathbb{E}\frac{1}{h_n}\sum_{t=1}^{n}K^2\left(\frac{x_t - x}{h_n}\right) \lesssim_p \frac{1}{e_n h_n} = o(1)$$

by Lemma A.2 and $n^{1/2}h_n \to \infty$ (see SM2). Since by Lemma A.2

$$\frac{1}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x) \rightsquigarrow \tau(x)$$

which is a.s. positive, we have $\hat{m}_n(x) = m(x) + o_p(1)$ as claimed.

The remainder of the proof follows similar lines to the proof of Theorem 3.2 in Wang and Phillips (2009). Recalling

$$\hat{\sigma}_u^2(x) = \frac{\frac{1}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x)[y_{t+1} - \hat{m}_n(x)]^2}{\frac{1}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x)}$$

we decompose the numerator as

$$\frac{1}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x)[y_{t+1} - \hat{m}_n(x)]^2$$
$$= \frac{1}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x)u_{t+1}^2 + \frac{2}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x)[m(x_t) - \hat{m}_n(x)]u_{t+1}$$
$$+ \frac{1}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x)[m(x_t) - \hat{m}_n(x)]^2$$
$$\eqqcolon B_{n,1} + 2B_{n,2} + B_{n,3}.$$

Letting $\zeta_t \coloneqq u_t^2 - \sigma_u^2$, we claim that

$$\begin{aligned} B_{n,1} &= \frac{\sigma_u^2}{e_n} \sum_{t=1}^{n} K_{h_n}(x_t - x) + \frac{1}{e_n} \sum_{t=1}^{n} K_{h_n}(x_t - x)\zeta_{t+1} \\ &\rightsquigarrow \sigma_u^2 \tau(x). \end{aligned} \tag{S.2}$$

The convergence of the first r.h.s. term in (S.2) follows from Lemma A.2. Regarding the second r.h.s. term, we note that since $\zeta_{t+1} \coloneqq u_{t+1}^2 - \sigma_u^2$ is a martingale difference under DGP4, this term is a martingale with conditional variance

$$\frac{1}{e_n h_n} \cdot \frac{1}{e_n} \sum_{t=1}^{n} \frac{1}{h_n} K^2\left(\frac{x_t - x}{h_n}\right) \mathbb{E}[\zeta_{t+1}^2 \mid \mathcal{G}_t] \lesssim_p \frac{1}{e_n h_n} = o(1)$$

by Lemma A.2 and $\sup_t \mathbb{E}[\zeta_{t+1}^2 \mid \mathcal{G}_t] < \infty$ a.s. (under DGP4). It follows by Corollary 3.1 of Hall and Heyde (1980) that, indeed,

$$\frac{1}{e_n} \sum_{t=1}^{n} K_{h_n}(x_t - x)\zeta_{t+1} \xrightarrow{p} 0.$$

Next, we have

$$\begin{aligned} B_{n,3} &\leq C \frac{1}{e_n} \sum_{t=1}^{n} K_{h_n}(x_t - x)\big\{[m(x_t) - m(x)]^2 + [\hat{m}_n(x) - m(x)]^2\big\} \\ &= O_p(h_n^2) + o_p(1) \\ &= o_p(1) \end{aligned}$$

by an analogous argument as was used to prove (S.1), and $\hat{m}_n(x) = m(x) + o_p(1)$. Finally

$$B_{n,2} \leq (B_{n,1})^{1/2}(B_{n,3})^{1/2},$$

by the Cauchy-Schwarz inequality; whence by Lemma A.2 and the preceding,

$$\hat{\sigma}_u^2(x) = \frac{B_{n,1} + B_{n,2} + B_{n,3}}{\frac{1}{e_n}\sum_{t=1}^{n} K_{h_n}(x_t - x)} \rightsquigarrow \frac{\sigma_u^2 \tau(x)}{\tau(x)} = \sigma_u^2. \qquad \square$$

S.2 Proofs of auxiliary lemmas from Appendix C

Proof of Lemma C.1. Letting $c_n \coloneqq n(\rho_n - 1) \to -\infty$, we note that for every $M < \infty$, we may take n sufficiently large such that $c_n < -M$, whence

$$\rho_n^{n\epsilon} = \left(1 + \frac{c_n}{n}\right)^{n\epsilon} \leq \left(1 - \frac{M}{n}\right)^{n\epsilon} \to \mathrm{e}^{-M\epsilon} \to 0$$

as $n \to \infty$ and then $M \to \infty$. Thus (i) holds.

Now taking $s = 1$ in (C.2), we have

$$x_t = \sum_{k=0}^{t-1} a_k \varepsilon_{t-k} + \sum_{k=t}^{\infty} a_{t,k} \varepsilon_{t-k} = x_{1,t} + x'_{0,t}$$

where $x_{1,t}$ and $x'_{0,t}$ are independent, with variances $\varsigma_{1,t}^2 \coloneqq \operatorname{var}(x_{1,t})$ and $\varsigma_{2,t}^2 \coloneqq \operatorname{var}(x'_{0,t})$ respectively. Let $\{t_n\} \subseteq [n\epsilon, n]$ be as in the statement of part (iii) of the lemma. We shall prove below that

$$(1-\rho_n^2)\operatorname{var}(x_{t_n}) = (1-\rho_n^2)(\varsigma_{1,t_n}^2 + \varsigma_{2,t_n}^2) = (1-\rho_n^2)\varsigma_{1,t_n}^2 + o(1) \to \phi^2,$$

from which both parts (ii) and (iii) of the lemma immediately follow.

Some tedious algebra (verified immediately below this proof) yields

$$\varsigma_{1,t_n}^2 = \sum_{k=0}^{t_n-1} \left(\sum_{l=0}^{k} \rho_n^{k-l} \phi_l \right)^2 = \sum_{i=0}^{t_n-1} \phi_i^2 \sum_{k=0}^{t_n-i-1} \rho_n^{2k} + 2 \sum_{i=0}^{t_n-1} \sum_{j=i+1}^{t_n-1} \phi_i \phi_j \sum_{k=0}^{t_n-j-1} \rho_n^{2k+(j-i)} \tag{S.3}$$

whence, since $\rho_n \in (0,1)$,

$$(1-\rho_n^2)\varsigma_{1,t_n}^2 = \sum_{i=0}^{t_n-1} \phi_i^2 (1-\rho_n^{2(t_n-i)}) + 2 \sum_{i=0}^{t_n-1} \sum_{j=i+1}^{t_n-1} \phi_i \phi_j (1-\rho_n^{2(t_n-j)+(j-i)})$$

Since $\rho_n^{2(t_n-i)} \le \rho_n^{2(\lfloor n\epsilon \rfloor - i)} \to 0$ as $n \to \infty$ for each *fixed* $i \in \mathbb{N}$ by part (i), and $\sum_{i=0}^{\infty} |\phi_i| < \infty$, it follows that

$$(1-\rho_n^2)\varsigma_{1,t_n}^2 \to \sum_{i=0}^{\infty} \phi_i^2 + 2 \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \phi_i \phi_j = \phi^2.$$

Regarding ς_{2,t_n}^2, we note that since $|\rho_n| \le 1$ and $C_\phi \coloneqq \sum_{i=0}^{\infty} |\phi_i| < \infty$

$$\varsigma_{2,t_n}^2 = \sum_{k=t_n}^{\infty} \left(\sum_{l=0}^{t_n-1} \rho^l \phi_{k-l} \right)^2 \le C_\phi \sum_{k=t_n}^{\infty} \sum_{l=0}^{t_n-1} \rho_n^l |\phi_{k-l}| \le C_\phi \sum_{l=0}^{t_n-1} \rho_n^l \tilde{\phi}_{t_n-l},$$

where $\tilde{\phi}_j \coloneqq \sum_{i=j}^{\infty} |\phi_i|$. Further,

$$\sum_{l=0}^{t_n-1} \rho_n^l \tilde{\phi}_{t_n-l} = \left(\sum_{l=0}^{\lfloor t_n/2 \rfloor - 1} + \sum_{l=\lfloor t_n/2 \rfloor}^{t_n-1} \right) \rho_n^l \tilde{\phi}_{t_n-l}$$
$$\le \left(\tilde{\phi}_{\lfloor t_n/2 \rfloor - 1} + C_\phi \rho_n^{\lfloor t_n/2 \rfloor} \right) \sum_{l=0}^{\lfloor t_n/2 \rfloor - 1} \rho_n^l = o[(1-\rho_n^2)^{-1}],$$

since $\tilde{\phi}_{\lfloor t_n/2\rfloor} \to 0$ and $\rho_n^{\lfloor t_n/2\rfloor} \to 0$ by part (i), and

$$\sum_{l=0}^{\lfloor t_n/2\rfloor} \rho_n^l \leq (1-\rho_n)^{-1} \asymp (1-\rho_n^2)^{-1},$$

whence $\varsigma_{2,t_n}^2 = o[(1-\rho_n^2)^{-1}]$. □

Verification of (S.3). Dropping the n subscript from t_n and ρ_n for simplicity, and setting $m := t-1$, we have

$$\begin{aligned}\sum_{k=0}^{m}\left(\sum_{l=0}^{k}\rho^{k-l}\phi_l\right)^2 &= \sum_{k=0}^{m}\sum_{i=0}^{k}\sum_{j=0}^{k}\rho^{2k-i-j}\phi_i\phi_j \\ &= \sum_{i=0}^{m}\sum_{j=0}^{m}\phi_i\phi_j\sum_{k=i\vee j}^{m}\rho^{2k-i-j} \\ &= \sum_{i=0}^{m}\phi_i^2\sum_{k=i}^{m}\rho^{2(k-i)} + 2\sum_{i=0}^{m}\sum_{j=i+1}^{m}\phi_i\phi_j\sum_{k=j}^{m}\rho^{2(k-j)+(j-i)} \\ &= \sum_{i=0}^{m}\phi_i^2\sum_{k=0}^{m-i}\rho^{2k} + 2\sum_{i=0}^{m}\sum_{j=i+1}^{m}\phi_i\phi_j\sum_{k=0}^{m-j}\rho^{2k+(j-i)}.\end{aligned}$$ □

Proof of Lemma C.3. When $\{\rho_n\}$ is mildly integrated, $\rho_n \in (0,1)$ and the upper bound in (C.9) follows trivially from $|a_k(\rho_n)| \leq \sum_{i=0}^{\infty}|\phi_i|$. Further, for any $0 \leq k \leq 2k_n$,

$$\rho_n^{2k_n} \leq \rho_n^k \leq \rho_n^{-k} \leq \rho_n^{-2k_n}.$$

Noting that $\rho^{(1-\rho)^{-1}} \to \mathrm{e}^{-1}$ as $\rho \to 1$, and $2k_n \sim (1-\rho_n)^{-1}$, it follows that $(\rho_n^{2k_n}, \rho_n^{-2k_n}) \to (\mathrm{e}^{-1}, \mathrm{e})$ as $n \to \infty$. Thus there exists an $n_0 \in \mathbb{N}$ and $C_1, C_2 \in (0,\infty)$ such that $\rho_n^k, \rho_n^{-k} \in [C_1, C_2]$ for all $n \geq n_0$ and $0 \leq k \leq 2k_n$.

Now $a_k(\rho_n) = \rho_n^k \sum_{l=0}^k \rho_n^{-l}\phi_l$, and for any $m \leq k \leq 2k_n$,

$$\sum_{l=0}^{k}\rho_n^{-l}\phi_l = \sum_{l=0}^{m}\phi_l - \sum_{l=0}^{m}(1-\rho_n^{-l})\phi_l + \sum_{l=m+1}^{k}\rho_n^{-l}\phi_l.$$

Therefore, since $|\rho_n^k| \leq 1$,

$$\left|a_k(\rho_n) - \rho_n^k\sum_{l=0}^{m}\phi_l\right| \leq \sum_{l=0}^{m}|1-\rho_n^{-l}||\phi_l| + \sum_{l=m+1}^{k}|\phi_l|$$

Let m_0 be chosen such that both

$$\rho_n^k\left|\sum_{l=0}^{m_0}\phi_l\right| \geq C_1\left|\sum_{l=0}^{m_0}\phi_l\right| \geq \frac{C_1}{2}|\phi| =: 3\underline{a}$$

for all $n \geq n_0$, and $\sum_{l=m_0+1}^{\infty}|\phi_l| \leq \underline{a}$. Since $\rho_n^{-l} \to 1$ for each l, there exists an $n_1 \geq n_0$ such that

$$|a_k(\rho_n)| \geq \rho_n^k \left|\sum_{l=0}^{m_0} \phi_l\right| - \sum_{l=0}^{m_0} |1-\rho_n^{-l}||\phi_l| - \sum_{l=m_0+1}^{k} |\phi_l| \geq \underline{a}$$

for all $n \geq n_1$. Taking $k_0 \coloneqq 2m_0$ and re-designating n_1 as n_0 gives the claimed lower bound in (C.9).

Finally, since $a_0 = \phi_0$ is nonzero by DGP3, replacing $\underline{a}$ by $\underline{a} \wedge |\phi_0|$ yields a lower bound that also applies to $|a_0|$. □

Proof of Lemma C.4. Since $\psi_\varepsilon \in L^1$, ε_0 has a bounded continuous density. Thus by the Riemann-Lebesgue lemma (Feller, 1971, Lem. XV.3.3) $\limsup_{|\lambda|\to\infty}|\psi_\varepsilon(\lambda)| = 0$. Further, $\psi_\varepsilon \in L^1$ cannot be periodic, and so $|\psi_\varepsilon(\lambda)| < 1$ for all $\lambda \neq 0$ (Feller, 1971, Lem. XV.1.4); since ψ_ε is necessarily continuous (Feller, 1971, Lem. XV.1.1), it follows that $\sup_{|\lambda|\geq 1}|\psi_\epsilon(\lambda)| \geq \mathrm{e}^{-\gamma_0}$ for some $\gamma_0 \in (0,\infty)$. By the moments theorem for characteristic functions (Feller, 1971, Lem. XV.4.2), we have $\psi_\varepsilon(\lambda) = 1 - \frac{1}{2}\lambda^2(1+o(1))$ as $\lambda \to 0$. Thus there exists a $\gamma_1 \in (0,\infty)$ such that $|\psi_\epsilon(\lambda)| \leq \mathrm{e}^{-\gamma_1\lambda^2}$. Taking $\gamma \coloneqq \gamma_0 \wedge \gamma_1$ thus gives

$$|\psi_\epsilon(\lambda)| \leq \begin{cases} \mathrm{e}^{-\gamma\lambda^2} & \text{if } |\lambda| \in [0,1], \\ \mathrm{e}^{-\gamma} & \text{if } |\lambda| \geq 1. \end{cases} \tag{S.4}$$

Let $\psi_\vartheta(\lambda) \coloneqq \mathbb{E}\exp(\mathrm{i}\lambda\sum_{k=1}^{\infty}\vartheta_k\varepsilon_k) = \prod_{k=1}^{\infty}\psi_\epsilon(\vartheta_k\lambda)$; we want to control the integral of (the modulus of) this function over $[A,\infty)$. Without loss of generality, assume the coefficients $\{\vartheta_k\}$ are ordered such that $|\vartheta_i| \geq |\vartheta_{i+1}|$. Since

$$\sum_{k=1}^{\infty} \frac{3\sigma_\vartheta^2}{\pi} \cdot k^{-2} = \frac{\sigma_\vartheta^2}{2} = \frac{1}{2}\sum_{k=1}^{\infty}\vartheta_k^2,$$

the set

$$\mathcal{K} \coloneqq \left\{k \in \mathbb{N} \mid \vartheta_k^2 \geq \frac{3\sigma_\vartheta^2}{\pi} \cdot k^{-2}\right\}$$

must be nonempty; let k^* denote the smallest element of $\mathcal{K}$.

We will bound the integral of $|\psi_\vartheta|$ separately over each of the two r.h.s. sets in

$$[A,\infty) = [A, A \vee \vartheta_{k^*}^{-1}] \cup [A \vee \vartheta_{k^*}^{-1}, \infty).$$

We first have

$$\begin{aligned} \int_{\{|\lambda|\in[A,A\vee\vartheta_{k^*}^{-1}]\}} |\psi_\vartheta(\lambda)|\,\mathrm{d}\lambda &\leq \int_{\{|\lambda|\in[A,A\vee\vartheta_{k^*}^{-1}]\}} \prod_{k\in\mathcal{K}} |\psi_\epsilon(\vartheta_k\lambda)|\,\mathrm{d}\lambda \\ &\leq_{(2)} \int_{\{|\lambda|\geq A\}} \exp\left(-\gamma\lambda^2\sum_{k\in\mathcal{K}}\vartheta_k^2\right)\mathrm{d}\lambda \end{aligned}$$

$$\leq_{(3)} \int_{\{|\lambda| \geq A\}} \exp(-\gamma\lambda^2\sigma_\vartheta^2/2)\,\mathrm{d}\lambda$$

where $\leq_{(2)}$ follows from (S.4) and

$$|\lambda| \in [A, A \vee \vartheta_{k^*}^{-1}] \implies |\vartheta_{k^*}\lambda| \leq 1 \implies |\vartheta_k\lambda| \leq 1, \quad \forall k \geq k^*;$$

while $\leq_{(3)}$ follows from

$$\sum_{k\in\mathcal{K}} \vartheta_k^2 = \sigma_\vartheta^2 - \sum_{k\notin\mathcal{K}} \vartheta^2 \geq \sigma_\vartheta^2 - \frac{3\sigma_\vartheta^2}{\pi} \cdot \sum_{k\notin\mathcal{K}} k^{-2} \geq \frac{\sigma_\vartheta^2}{2}.$$

Next, we have

$$\begin{aligned}
\int_{\{|\lambda|\in[A\vee\vartheta_{k^*}^{-1},\infty)\}} |\psi_\vartheta(\lambda)|\,\mathrm{d}\lambda &\leq \int_{\{|\lambda|\in[A\vee\vartheta_{k^*}^{-1},\infty)\}} \prod_{k=1}^{k^*} \psi_\epsilon(\vartheta_k\lambda)\,\mathrm{d}\lambda \\
&\leq_{(2)} \mathrm{e}^{-\gamma(k^*-1)} \int_{\{|\lambda|\in[A\vee\vartheta_{k^*}^{-1},\infty)\}} |\psi_\epsilon(\vartheta_{k^*}\lambda)|\,\mathrm{d}\lambda \\
&\leq \mathrm{e}^{-\gamma(k^*-1)} \int_{\{|\lambda|\geq A\}} |\psi_\epsilon(\vartheta_{k^*}\lambda)|\,\mathrm{d}\lambda \\
&= \mathrm{e}^{-\gamma(k^*-1)}\vartheta_{k^*}^{-1} \int_{\{|\lambda|\geq\vartheta_{k^*}A\}} |\psi_\epsilon(\lambda)|\,\mathrm{d}\lambda \\
&\leq_{(5)} c_0^{-1}\sigma_\vartheta^{-1}\mathrm{e}^{-\gamma(k^*-1)}k^* \int_{\{|\lambda|\geq c_0\sigma_\vartheta A/k^*\}} |\psi_\epsilon(\lambda)|\,\mathrm{d}\lambda,
\end{aligned}$$

for $c_0 := (3/\pi)^{1/2}$, where $\leq_{(2)}$ holds trivially if $k^* = 1$, and otherwise follows from

$$|\lambda| \in [A \vee \vartheta_{k^*}^{-1}, \infty) \implies |\vartheta_{k^*}\lambda| \geq 1 \implies |\vartheta_k\lambda| \geq 1, \quad \forall k \leq k^*;$$

while $\leq_{(5)}$ follows from $\vartheta_{k^*}^2 \geq (3\sigma_\vartheta^2/\pi) \cdot (k^*)^{-2}$.

Finally, define

$$\begin{aligned}
&G(A; \sigma^2, \psi_\epsilon) \\
&\qquad := \int_{\{|\lambda|\geq A\}} \exp(-\gamma\lambda^2\sigma^2/2)\,\mathrm{d}\lambda + c_0^{-1}\sigma^{-1} \sup_{k\geq 1} \mathrm{e}^{-\gamma(k-1)}k \int_{\{|\lambda|\geq c_0\sigma A/k\}} |\psi_\epsilon(\lambda)|\,\mathrm{d}\lambda,
\end{aligned}$$

which clearly satisfies the first inequality in (C.10), and is decreasing in σ^2; the second inequality in (C.10) follows by evaluating $G(0; \sigma^2, \psi_\epsilon)$, and noting $\sup_{k\geq 1} \mathrm{e}^{-\gamma(k-1)}k < \infty$. It thus remains to show that $G(A; \sigma^2, \psi_\epsilon) \to 0$ as $A \to \infty$. To that end, let $\epsilon > 0$ and note that there exists a k' such that

$$\mathrm{e}^{-\gamma(k'-1)}k' \int_{\mathbb{R}} |\psi_\epsilon(\lambda)|\,\mathrm{d}\lambda < \epsilon.$$

Since
$$e^{-\gamma(k-1)}k\int_{\{|\lambda|\geq c_0\sigma A/k\}}|\psi_\epsilon(\lambda)|\,d\lambda\to 0$$
as $A\to\infty$, for each fixed $k\in\{1,\dots,k'\}$, the claim follows. □

Proof of Lemma C.5. Making the change of variables $u=\rho^x$, we have
$$\int_1^a\frac{1}{(1-\rho^x)^{1/2}}\,dx=\frac{1}{-\log\rho}\int_{\rho^a}^{\rho}\frac{1}{(1-u)^{1/2}u}\,du=\frac{1}{-\log\rho}\Big[-2\tanh^{-1}\{(1-u)^{1/2}\}\Big]_{\rho^a}^{\rho}.$$
for $\rho\in(0,1)$, where $\tanh^{-1}(x):=\frac{1}{2}\log\{(1+x)/(1-x)\}$ is inverse hyperbolic tangent function. Now set $\rho=\rho_n$, for $\{\rho_n\}$ mildly integrated, and $a=n\eta$: and note that $\rho_n\to 1$, whereas $\rho_n^{\eta n}\to 0$ by Lemma C.1. Then
$$\begin{aligned}\frac{1}{n}\int_1^{\eta n}\frac{1}{(1-\rho_n^x)^{1/2}}\,dx&=\frac{1}{n}\cdot\frac{1}{-\log\rho_n}\Big\{2\tanh^{-1}[(1-\rho_n^{\eta n})^{1/2}]+o(1)\Big\}\\&\sim\frac{1}{n}\cdot\frac{\log[1-(1-\rho_n^{\eta n})^{1/2}]}{\log\rho_n}.\end{aligned}$$
Next, note that by two applications of L'Hôpital's rule
$$\begin{aligned}\lim_{x\to 0}\frac{\log[1-(1-x)^{1/2}]}{\log x}&=\lim_{x\to 0}\frac{\frac{1}{2}(1-x)^{-1/2}/[1-(1-x)^{1/2}]}{1/x}\\&=\frac{1}{2}\lim_{x\to 0}\frac{x}{1-(1-x)^{1/2}}=\frac{1}{2}\lim_{x\to 0}\frac{1}{\frac{1}{2}(1-x)^{-1/2}}=1,\end{aligned}$$
whence
$$\frac{1}{n}\cdot\frac{\log[1-(1-\rho_n^{\eta n})^{1/2}]}{\log\rho_n}\sim\frac{1}{n}\cdot\frac{\log(\rho_n^{\eta n})}{\log\rho_n}=\eta$$
and the result follows. □

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