Online supplementary appendix to "Truncated sum of squares estimation of fractional time series models with deterministic trends" *Econometric Theory*

Javier Hualde*
Universidad Pública de Navarra

Morten Ørregaard Nielsen†
Queen's University and CREATES

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S.1 Introduction

In this supplement to Hualde and Nielsen (2019) we provide proofs of all technical results. This includes proofs of the main theorems and also some auxiliary and technical lemmas and their respective proofs. Note that the proofs of the auxiliary lemmas rely on the technical lemmas, but not vice versa. Equation references (S.n) for $n \geq 1$ refer to equations in this supplement and other equation references are to the main paper, Hualde and Nielsen (2019).

S.2 Proofs of theorems

S.2.1 Proof of Theorem 1(i): the $\gamma_0 + 1/2 > \delta_0$ case

S.2.1.1 Overall design of the proof

Throughout, ϵ will denote a generic arbitrarily small positive constant, and K a generic arbitrarily large positive constant. Fix $\varepsilon > 0$ and let $M_\varepsilon = \{\boldsymbol{\vartheta} \in \Xi : \|\boldsymbol{\tau} - \boldsymbol{\tau}_0\| < \varepsilon\}$, $\overline{M}_\varepsilon = \{\boldsymbol{\vartheta} \in \Xi : \|\boldsymbol{\tau} - \boldsymbol{\tau}_0\| \geq \varepsilon\}$, $N_\varepsilon = \{\boldsymbol{\vartheta} \in \Xi : |\gamma - \gamma_0| < \varepsilon\}$ and $\overline{N}_\varepsilon = \{\boldsymbol{\vartheta} \in \Xi : |\gamma - \gamma_0| \geq \varepsilon\}$. Then $\Pr(||\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0|| \geq \varepsilon) \to 0$ as $T \to \infty$, is implied by

$$\Pr(\widehat{\boldsymbol{\vartheta}} \in \overline{M}_\varepsilon) \to 0 \text{ as } T \to \infty, \tag{S.1}$$

$$\Pr(\widehat{\boldsymbol{\vartheta}} \in \overline{N}_\varepsilon \cap M_\varepsilon) \to 0 \text{ as } T \to \infty. \tag{S.2}$$

Strictly, ε should be $\varepsilon/\sqrt{2}$ in (S.1) and (S.2), but since ε is arbitrary this is irrelevant and we continue without the $\sqrt{2}$ factor.

*Email: javier.hualde@unavarra.es
†Email: mon@econ.queensu.ca

We decompose the objective function as $R_T(\boldsymbol{\vartheta}) = \frac{1}{T}\sum_{t=1}^{T}(d_t(\boldsymbol{\vartheta}) + s_t(\boldsymbol{\vartheta}))^2$ with

$$d_t(\boldsymbol{\vartheta}) = \mu_0\left(c_t(\gamma_0,\delta,\boldsymbol{\varphi}) - h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\sum_{j=1}^{T}c_j(\gamma_0,\delta,\boldsymbol{\varphi})\,h_{j,T}(\gamma,\delta,\boldsymbol{\varphi})\right),$$

$$s_t(\boldsymbol{\vartheta}) = \varepsilon_t(\boldsymbol{\tau}) - h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\sum_{j=1}^{T}\varepsilon_j(\boldsymbol{\tau})\,h_{j,T}(\gamma,\delta,\boldsymbol{\varphi}),$$

where, as in Hualde and Robinson (2011),

$$\varepsilon_t(\boldsymbol{\tau}) = \sum_{j=0}^{t-1}a_j(\delta_0-\delta,\boldsymbol{\varphi})\,u_{t-j},$$

and where we also defined the coefficient

$$h_{t,T}(d_1,d_2,\boldsymbol{\varphi}) = \frac{c_t(d_1,d_2,\boldsymbol{\varphi})}{(\sum_{j=1}^{T}c_j^2(d_1,d_2,\boldsymbol{\varphi}))^{1/2}}, \tag{S.3}$$

which clearly satisfies $\sum_{t=1}^{T}h_{t,T}^2(d_1,d_2,\boldsymbol{\varphi}) = 1$.

The strategy of proof relies on recognizing the competition between the stochastic term $s_t(\boldsymbol{\vartheta})$ and deterministic term $d_t(\boldsymbol{\vartheta})$ in $R_T(\boldsymbol{\vartheta})$, taking into account that when considering (S.1), just $\boldsymbol{\tau}$ is for sure "far" from $\boldsymbol{\tau}_0$, whereas when dealing with (S.2), just γ is "far" from γ_0. As will be seen, an important feature of the problem is that when $\gamma = \gamma_0$ we have $d_t(\boldsymbol{\vartheta}) = 0$, which complicates the treatment of (S.1). In any case, as in Hualde and Robinson (2011), we need to carefully consider the cases where $R_T(\boldsymbol{\vartheta})$ shows distinct behaviors, noting that either the deterministic or the stochastic term might dominate, and below we partition the parameter space accordingly.

S.2.1.2 Proof of (S.1)

To prove (S.1) we use

$$\Pr(\widehat{\boldsymbol{\vartheta}} \in \overline{M}_\varepsilon) = \Pr\left(\inf_{\boldsymbol{\vartheta}\in\overline{M}_\varepsilon}R_T(\boldsymbol{\vartheta}) \le \inf_{\boldsymbol{\vartheta}\in M_\varepsilon}R_T(\boldsymbol{\vartheta})\right) \le \Pr\left(\inf_{\boldsymbol{\vartheta}\in\overline{M}_\varepsilon}S_T(\boldsymbol{\vartheta}) \le 0\right), \tag{S.4}$$

where $S_T(\boldsymbol{\vartheta}) = R_T(\boldsymbol{\vartheta}) - R_T(\boldsymbol{\vartheta}_0)$. Fix an arbitrarily small $\eta > 0$ such that $\eta < (\gamma_0 - \delta_0 + 1/2)/2$ and suppose that $\triangledown_1 < \delta_0 - 1/2 - \eta$ and $\triangledown_2 > \gamma_0 - \eta$. Our proof will cover trivially the situation where any of these conditions does not hold, in which case some of the steps below are superfluous. Let $\mathcal{I}_1 = \{\delta : \triangledown_1 \le \delta \le \delta_0 - 1/2 - \eta\}$, $\mathcal{I}_2 = \{\delta : \delta_0 - 1/2 - \eta \le \delta \le \delta_0 - 1/2\}$, $\mathcal{I}_3 = \{\delta : \delta_0 - 1/2 \le \delta \le \delta_0 - 1/2 + \eta\}$, $\mathcal{I}_4 = \{\delta : \delta_0 - 1/2 + \eta \le \delta \le \gamma_0 - \eta\}$, and $\mathcal{I}_5 = \{\delta : \gamma_0 - \eta \le \delta \le \triangledown_2\}$, noting that the upper bound for η guarantees that $\mathcal{I}_4$ is non-empty. Correspondingly define $\mathcal{T}_i = \mathcal{I}_i \times \Psi$ and, fixing $\xi > 0$ and $\varrho > 0$, such that $\varrho < \eta/2$, also define $\mathcal{H}_i = \{\boldsymbol{\vartheta} \in \overline{M}_\varepsilon : \boldsymbol{\tau} \in \mathcal{T}_i, |\gamma - \gamma_0| \le \xi T^{-\varkappa_i}\}$, $\overline{\mathcal{H}}_i = \{\boldsymbol{\vartheta} \in \overline{M}_\varepsilon : \boldsymbol{\tau} \in \mathcal{T}_i, \xi T^{-\varkappa_i} \le |\gamma - \gamma_0| \le \varrho\}$ and $\overline{\overline{\mathcal{H}}}_i = \{\boldsymbol{\vartheta} \in \overline{M}_\varepsilon : \boldsymbol{\tau} \in \mathcal{T}_i, |\gamma - \gamma_0| \ge \varrho\}$, $i = 1, \ldots, 5$, where $\varkappa_i > 0$ will be defined subsequently, noting that $\overline{\mathcal{H}}_i$ is non-empty for any ξ, ϱ, for T large enough . Then, by (S.4), (S.1) is justified by showing

$$\Pr\left(\inf_{\overline{\overline{\mathcal{H}}}_i} S_T(\boldsymbol{\vartheta}) \le 0\right) \to 0 \text{ as } T \to \infty \text{ for } i = 1, \ldots, 5, \tag{S.5}$$

$$\Pr\left(\inf_{\overline{\mathcal{H}}_i} S_T(\boldsymbol{\vartheta}) \le 0\right) \to 0 \text{ as } T \to \infty \text{ for } i = 1, \ldots, 5, \tag{S.6}$$

$$\Pr\left(\inf_{\mathcal{H}_i} S_T(\boldsymbol{\vartheta}) \le 0\right) \to 0 \text{ as } T \to \infty \text{ for } i = 1, \ldots, 5. \tag{S.7}$$

We note that $\mathcal{H}_i$, $\overline{\mathcal{H}}_i$, and $\overline{\overline{\mathcal{H}}}_i$ are designed exactly such that in $\mathcal{H}_i$ the stochastic term dominates $S_T(\boldsymbol{\vartheta})$, while in $\overline{\mathcal{H}}_i \cup \overline{\overline{\mathcal{H}}}_i$ it is the deterministic term that dominates. As will be seen, the analysis on $\overline{\overline{\mathcal{H}}}_i$ is much simpler because γ is "far" from γ_0, whereas a much more delicate treatment is necessary for $\overline{\mathcal{H}}_i$. This motivates a separate analysis of (S.5), (S.6) and (S.7), at least for $i = 1, \ldots, 4$.

Proof of (S.5), (S.6), and (S.7) for $i = 5$ In this case, we give just one proof that covers the whole set $\mathcal{H}_5 \cup \overline{\mathcal{H}}_5 \cup \overline{\overline{\mathcal{H}}}_5$, where $\delta_0 - \delta \le \delta_0 - \gamma_0 + \eta < 1/2$, so $\Delta_+^{\delta-\delta_0} u_t$ is asymptotically stationary. Let

$$S_T(\boldsymbol{\vartheta}) = U(\boldsymbol{\tau}) - r_T(\boldsymbol{\vartheta}), \tag{S.8}$$

where $U(\boldsymbol{\tau}) = E((\phi(L;\boldsymbol{\varphi})\Delta^{\delta-\delta_0} u_t)^2) - \sigma_0^2$ and

$$\begin{aligned} r_T(\boldsymbol{\vartheta}) = {} & \frac{1}{T}\sum_{t=1}^{T}\left((\phi(L;\boldsymbol{\varphi}_0)\{u_t \mathbb{I}((t \ge 1)\})^2 - \sigma_0^2\right) \\ & - \frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_t^2(\boldsymbol{\tau}) - E\left((\phi(L;\boldsymbol{\varphi})\Delta^{\delta-\delta_0} u_t)^2\right)\right) \\ & - \frac{1}{T}\left(\sum_{t=1}^{T} \phi(L;\boldsymbol{\varphi}_0)\{u_t \mathbb{I}(t \ge 1)\} h_{t,T}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0)\right)^2 \\ & + \frac{1}{T}\left(\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) h_{t,T}(\gamma, \delta, \boldsymbol{\varphi})\right)^2 \\ & - \frac{2}{T}\sum_{t=1}^{T} d_t(\boldsymbol{\vartheta}) s_t(\boldsymbol{\vartheta}) - \frac{1}{T}\sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta}), \end{aligned}$$

noting that $\varepsilon_t(\boldsymbol{\tau}_0) = \phi(L;\boldsymbol{\varphi}_0)\{u_t\mathbb{I}(t \geq 1)\}$. It follows that (S.5), (S.6), and (S.7) for $i = 5$ hold if we show that

$$\inf_{\|\boldsymbol{\tau}-\boldsymbol{\tau}_0\|\geq\varepsilon, \boldsymbol{\tau}\in\mathcal{T}_5} U(\boldsymbol{\tau}) > \epsilon, \tag{S.9}$$

$$\frac{1}{T}\sum_{t=1}^{T}\left((\phi(L;\boldsymbol{\varphi}_0)\{u_t\mathbb{I}(t\geq 1)\})^2 - \sigma_0^2\right) = o_p(1), \tag{S.10}$$

$$\sup_{\|\boldsymbol{\tau}-\boldsymbol{\tau}_0\|\geq\varepsilon, \boldsymbol{\tau}\in\mathcal{T}_5} \frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_t^2(\boldsymbol{\tau}) - E\left((\phi(L;\boldsymbol{\varphi})\Delta^{\delta-\delta_0}u_t)^2\right)\right) = o_p(1), \tag{S.11}$$

$$\sup_{\mathcal{H}_5\cup\overline{\mathcal{H}}_5\cup\overline{\overline{\mathcal{H}}}_5} \frac{1}{T}\left|\sum_{t=1}^{T} d_t(\boldsymbol{\vartheta})\, s_t(\boldsymbol{\vartheta})\right| = o_p(1), \tag{S.12}$$

$$\sup_{\mathcal{H}_5\cup\overline{\mathcal{H}}_5\cup\overline{\overline{\mathcal{H}}}_5} \frac{1}{T}\left(\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau})\, h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\right)^2 = o_p(1). \tag{S.13}$$

First, (S.9), (S.10), and (S.11) follow by identical arguments to those in the proofs of (2.8) and (2.9) in Hualde and Robinson (2011). Next, by (S.227) of Lemma S.19 with $\gamma_0 - \delta \leq \eta$ and $\delta_0 - \delta \leq \delta_0 - \gamma_0 + \eta$, the left-hand side of (S.12) is $O_p(T^{\max\{\theta,\delta_0-\gamma_0+\eta\}+2\theta-1/2+\eta})$, and by (S.222) of Lemma S.18, the left-hand side of (S.13) is $O_p(T^{2\max\{\theta,\delta_0-\gamma_0+\eta\}-1})$. Both are $o_p(1)$ for θ and η sufficiently small, to conclude the proof of (S.5), (S.6), and (S.7) for $i = 5$.

Proof of (S.5) for $i = 1,\ldots,4$ First we show (S.5) which, in view of Lemma S.1 and that $d_t(\boldsymbol{\vartheta}_0) = 0$, holds if, for $i = 1,\ldots,4$,

$$\Pr\left(\inf_{\overline{\overline{\mathcal{H}}}_i} \frac{1}{T}\sum_{t=1}^{T}(d_t(\boldsymbol{\vartheta}) + s_t(\boldsymbol{\vartheta}))^2 \leq \sigma_0^2 + \epsilon\right) \to 0 \text{ as } T \to \infty.$$

For $\delta \in \cup_{i=1}^4 \mathcal{I}_i$ it holds that $\gamma_0 - \delta \geq \eta$, so the probability above is bounded by

$$\begin{aligned}
&\Pr\left(\inf_{\overline{\overline{\mathcal{H}}}_i}\frac{T^{2(\gamma_0-\delta)+1}}{T}\inf_{\overline{\overline{\mathcal{H}}}_i}\frac{1}{T^{2(\gamma_0-\delta)+1}}\sum_{t=1}^{T}(d_t(\boldsymbol{\vartheta}) + s_t(\boldsymbol{\vartheta}))^2 \leq \sigma_0^2 + \epsilon\right)\\
&= \Pr\left(\inf_{\overline{\overline{\mathcal{H}}}_i}\frac{1}{T^{2(\gamma_0-\delta)+1}}\sum_{t=1}^{T}(d_t(\boldsymbol{\vartheta}) + s_t(\boldsymbol{\vartheta}))^2 \leq \frac{\sigma_0^2+\epsilon}{T^{2\eta}}\right)\\
&\leq \Pr\left(\inf_{\overline{\overline{\mathcal{H}}}_i}\frac{1}{T^{2(\gamma_0-\delta)+1}}\sum_{t=1}^{T}d_t^2(\boldsymbol{\vartheta}) - \sup_{\overline{\overline{\mathcal{H}}}_i}\frac{2}{T^{2(\gamma_0-\delta)+1}}\left|\sum_{t=1}^{T}d_t(\boldsymbol{\vartheta})\,s_t(\boldsymbol{\vartheta})\right| \leq \frac{\sigma_0^2+\epsilon}{T^{2\eta}}\right).
\end{aligned}$$

Thus, (S.5) for $i = 1,\ldots,4$ follows for θ small enough by (S.229) of Lemma S.19, noting also that when $\delta \in \cup_{i=1}^4\mathcal{I}_i$, $\delta_0 - \delta \geq \delta_0 - \gamma_0 + \eta$, and by Lemma S.2.

Proof of (S.6) and (S.7) for $i = 4$ Fix ζ such that $0 < \zeta < \eta$ and let $\varkappa_4 = \gamma_0 - \delta - \zeta$, noting that $\varkappa_4 \geq \eta - \zeta > 0$ when $\delta \in \mathcal{I}_4$. Then, because $d_t(\boldsymbol{\vartheta}_0) = 0$, (S.6) holds if

$$\Pr\left(\inf_{\overline{\mathcal{H}}_4}\frac{T^{2\varkappa_4}}{T^{2(\gamma_0-\delta)+1}}\left(\sum_{t=1}^{T}d_t^2(\boldsymbol{\vartheta}) - 2\left|\sum_{t=1}^{T}d_t(\boldsymbol{\vartheta})\,s_t(\boldsymbol{\vartheta})\right| - \sum_{t=1}^{T}s_t^2(\boldsymbol{\vartheta}_0)\right) \leq 0\right) \to 0 \tag{S.14}$$

as $T \to \infty$, noting the change in the normalization from (S.6) to (S.14), which is justified because the right-hand side of the inequality inside the probability in (S.6) is 0, so multiplying the left- and right-hand sides of the inequality by the same positive number does not alter the probability. Because $\sum_{t=1}^{T} d_t(\boldsymbol{\vartheta}) c_t(\gamma, \delta, \boldsymbol{\varphi}) = 0$, it holds that

$$\sum_{t=1}^{T} d_t(\boldsymbol{\vartheta}) s_t(\boldsymbol{\vartheta}) = \sum_{t=1}^{T} d_t(\boldsymbol{\vartheta}) \varepsilon_t(\boldsymbol{\tau}). \tag{S.15}$$

By the Cauchy-Schwarz inequality and (S.15), the probability in (S.14) is bounded by

$$\Pr\left(\inf_{\overline{\mathcal{H}}_4} \frac{T^{2\varkappa_4}}{T^{2(\gamma_0-\delta)+1}} \left(\sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta})(1-2v_T(\boldsymbol{\vartheta})) - \sum_{t=1}^{T} s_t^2(\boldsymbol{\vartheta}_0)\right) \le 0\right), \tag{S.16}$$

where $v_T(\boldsymbol{\vartheta}) = (\sum_{t=1}^{T} \varepsilon_t^2(\boldsymbol{\tau}) / \sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta}))^{1/2}$. Then (S.14) holds if

$$\sup_{\overline{\mathcal{H}}_4} \frac{T^{2\varkappa_4}}{T^{2(\gamma_0-\delta)+1}} \sum_{t=1}^{T} s_t^2(\boldsymbol{\vartheta}_0) = o_p(1), \tag{S.17}$$

$$\Pr\left(\inf_{\overline{\mathcal{H}}_4} \frac{T^{2\varkappa_4}}{T^{2(\gamma_0-\delta)+1}} \sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta})(1-2v_T(\boldsymbol{\vartheta})) \le \epsilon\right) \to 0 \text{ as } T \to \infty. \tag{S.18}$$

First, given that $T^{2\varkappa_4-2(\gamma_0-\delta)-1} = T^{-1-2\zeta}$, (S.17) follows immediately by Lemma S.1. Next, fixing c such that $0 < c < 1/2$, the probability in (S.18) equals

$$\begin{aligned} &\Pr\left(\inf_{\overline{\mathcal{H}}_4} \frac{T^{2\varkappa_4}}{T^{2(\gamma_0-\delta)+1}} \sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta})(1-2v_T(\boldsymbol{\vartheta})) \le \epsilon, \sup_{\overline{\mathcal{H}}_4} v_T(\boldsymbol{\vartheta}) \le c\right) \\ &+ \Pr\left(\inf_{\overline{\mathcal{H}}_4} \frac{T^{2\varkappa_4}}{T^{2(\gamma_0-\delta)+1}} \sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta})(1-2v_T(\boldsymbol{\vartheta})) \le \epsilon, \sup_{\overline{\mathcal{H}}_4} v_T(\boldsymbol{\vartheta}) > c\right) \\ &\le \Pr\left(\inf_{\overline{\mathcal{H}}_4} \frac{T^{2\varkappa_4}}{T^{2(\gamma_0-\delta)+1}} \sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta})(1-2c) \le \epsilon\right) + \Pr\left(\sup_{\overline{\mathcal{H}}_4} v_T(\boldsymbol{\vartheta}) > c\right), \end{aligned} \tag{S.19}$$

so (S.18) holds on showing

$$\lim_{T\to\infty} \inf_{\overline{\mathcal{H}}_4} \frac{T^{2\varkappa_4}}{T^{2(\gamma_0-\delta)+1}} \sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta}) > \epsilon, \tag{S.20}$$

$$\sup_{\overline{\mathcal{H}}_4} v_T(\boldsymbol{\vartheta}) = o_p(1). \tag{S.21}$$

By the Cauchy-Schwarz inequality, $\sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta}) \ge T^{-1}\overline{d}_T^2(\boldsymbol{\vartheta})$, where $\overline{d}_T(\boldsymbol{\vartheta}) = \sum_{t=1}^{T} d_t(\boldsymbol{\vartheta})$, so that (S.20) holds by (S.109) of Lemma S.3. To show (S.21), note that

$$\sup_{\overline{\mathcal{H}}_4} v_T(\boldsymbol{\vartheta}) \le \left(\frac{\sup_{\overline{\mathcal{H}}_4} T^{-1-2\zeta} \sum_{t=1}^{T} \varepsilon_t^2(\boldsymbol{\tau})}{\inf_{\overline{\mathcal{H}}_4} T^{2\varkappa_4-2(\gamma_0-\delta)-1} \sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta})}\right)^{1/2} \tag{S.22}$$

using $\varkappa_4 = \gamma_0 - \delta - \zeta$, where $\sup_{\overline{\mathcal{H}}_4} T^{-1-2\zeta} \sum_{t=1}^{T} \varepsilon_t^2(\boldsymbol{\tau}) = o_p(1)$ by Lemma S.17 because $\delta_0 - \delta \leq 1/2 - \eta$. Then (S.22) is $o_p(1)$ by (S.20), which concludes the proof of (S.6) for $i = 4$.

Next we show (S.7) for $i = 4$. A potential problem here is that $\gamma = \gamma_0$ is admissible, so we cannot directly exploit the lower bound for the normalized $\sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta})$ as in (S.20) because $d_t(\boldsymbol{\vartheta}) = 0$ when $\gamma = \gamma_0$. However, we can instead take advantage of $|\gamma - \gamma_0| \leq \xi T^{-\varkappa_4}$ in $\mathcal{H}_4$ and apply the mean value theorem. First note that $\delta \in \mathcal{I}_4$ implies that $\delta_0 - \delta \leq 1/2 - \eta$ and $\gamma_0 - \delta \geq \eta$, so that $\Delta_+^{\delta-\delta_0} u_t$ is asymptotically stationary as in the proof for $i = 5$. Then, given (S.8), the result follows by (S.9), (S.10), (S.11) (whose proofs apply also for $\delta \in \mathcal{I}_4$), and showing also that

$$\sup_{\mathcal{H}_4} \frac{1}{T} \left(\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) h_{t,T}(\gamma, \delta, \boldsymbol{\varphi}) \right)^2 = o_p(1), \tag{S.23}$$

$$\sup_{\mathcal{H}_4} \frac{1}{T} \left| \sum_{t=1}^{T} d_t(\boldsymbol{\vartheta}) s_t(\boldsymbol{\vartheta}) \right| = o_p(1). \tag{S.24}$$

From (S.222) of Lemma S.18, the left-hand side of (S.23) is $O_p(T^{-2\eta}) = o_p(1)$ by choosing $\theta < 1/2 - \eta$. Next, because $|\gamma - \gamma_0| < \xi T^{-\varkappa_4}$ in $\mathcal{H}_4$, by (S.226) and (S.228) of Lemma S.19 the left-hand side of (S.24) is $O_p\left(T^{\zeta-\eta+2\theta}\right) = o_p(1)$ for θ small enough because $\zeta < \eta$.

Proof of (S.6) and (S.7) for $i = 3$ Fix $\varkappa_3 = \gamma_0 - \delta$, so noting that $\delta \in \mathcal{I}_3$, $\varkappa_3 \geq \gamma_0 - \delta_0 + 1/2 - \eta > 0$. Then, by the Cauchy-Schwarz inequality,

$$\Pr\left(\inf_{\overline{\mathcal{H}}_3} S_T(\boldsymbol{\vartheta}) \leq 0 \right) \leq \Pr\left(\inf_{\overline{\mathcal{H}}_3} \frac{1}{T^2} \left(\overline{d}_T(\boldsymbol{\vartheta}) + \overline{s}_T(\boldsymbol{\vartheta}) \right)^2 - \frac{1}{T} \sum_{t=1}^{T} s_t^2(\boldsymbol{\vartheta}_0) \leq 0 \right), \tag{S.25}$$

where $\overline{d}_t(\boldsymbol{\vartheta}) = \sum_{j=1}^{t} d_j(\boldsymbol{\vartheta})$ and $\overline{s}_t(\boldsymbol{\vartheta}) = \sum_{j=1}^{t} s_j(\boldsymbol{\vartheta})$, so

$$\overline{s}_T(\boldsymbol{\vartheta}) = \varepsilon_T(\delta_0 - \delta + 1, \boldsymbol{\varphi}) - \sum_{t=1}^{T} h_{t,T}(\gamma, \delta, \boldsymbol{\varphi}) \sum_{j=1}^{T} \varepsilon_j(\boldsymbol{\tau}) h_{j,T}(\gamma, \delta, \boldsymbol{\varphi}), \tag{S.26}$$

where, denoting $\varepsilon_t(\delta_0 - \delta, \boldsymbol{\varphi}) = \varepsilon_t(\boldsymbol{\tau})$,

$$\varepsilon_t(\delta_0 - \delta + 1, \boldsymbol{\varphi}) = \sum_{j=1}^{t} \varepsilon_j(\boldsymbol{\tau}) = \sum_{j=1}^{t} \sum_{k=0}^{j-1} a_k(\delta_0 - \delta, \boldsymbol{\varphi}) u_{j-k} = \sum_{j=0}^{t-1} a_j(\delta_0 - \delta + 1, \boldsymbol{\varphi}) u_{t-j}, \tag{S.27}$$

because

$$\pi_{j+1}(d) - \pi_j(d) = \pi_{j+1}(d-1). \tag{S.28}$$

The right-hand side of (S.25) is thus bounded by

$$\Pr\left(\inf_{\overline{\mathcal{H}}_3} \frac{1}{T^2} \overline{d}_T^2(\boldsymbol{\vartheta}) \left(1 - 2\left|\overline{v}_T(\boldsymbol{\vartheta})\right|\right) - \frac{1}{T} \sum_{t=1}^{T} s_t^2(\boldsymbol{\vartheta}_0) \leq 0 \right), \tag{S.29}$$

where $\overline{v}_T(\boldsymbol{\vartheta}) = \overline{s}_T(\boldsymbol{\vartheta}) / \overline{d}_T(\boldsymbol{\vartheta})$. Applying Lemma S.1, (S.6) for $i = 3$ would then hold if

$$\Pr\left(\inf_{\overline{\mathcal{H}}_3} \frac{1}{T^2} \overline{d}_T^2(\boldsymbol{\vartheta}) \left(1 - 2\left|\overline{v}_T(\boldsymbol{\vartheta})\right|\right) \leq K \right) \to 0 \text{ as } T \to \infty, \tag{S.30}$$

for an arbitrarily large K. As in (S.19), fixing c such that $0 < c < 1/2$, the probability in (S.30) is bounded by

$$\Pr\left(\inf_{\overline{\mathcal{H}}_3}\frac{1}{T^2}\overline{d}_T^2\left(\boldsymbol{\vartheta}\right)\left(1-2c\right)\leq K\right)+\Pr\left(\sup_{\overline{\mathcal{H}}_3}\left|\overline{v}_T\left(\boldsymbol{\vartheta}\right)\right|>c\right), \tag{S.31}$$

so, as in (S.22), (S.30) holds if

$$\sup_{\overline{\mathcal{H}}_3}\frac{1}{T}\left|\overline{s}_T\left(\boldsymbol{\vartheta}\right)\right| = O_p\left(1\right), \tag{S.32}$$

$$\lim_{T\to\infty}\inf_{\overline{\mathcal{H}}_3}\frac{1}{T^2}\overline{d}_T^2\left(\boldsymbol{\vartheta}\right)>K. \tag{S.33}$$

For $\delta\in\mathcal{I}_3$ it holds that $\delta_0-\delta\leq 1/2$, so in view of (S.26) the proof of (S.32) is immediate using (S.213) in Lemma S.16 together with Lemmas S.17 and S.18 with $\theta<1/2$. Finally the proof of (S.33) follows by Lemma S.3, to conclude the proof of (S.6) for $i=3$.

Next we show (S.7) for $i=3$, which holds if

$$\Pr\left(\inf_{\mathcal{H}_3}\frac{1}{T}\left(\sum_{t=1}^T s_t^2\left(\boldsymbol{\vartheta}\right)-2\left|\sum_{t=1}^T d_t\left(\boldsymbol{\vartheta}\right)s_t\left(\boldsymbol{\vartheta}\right)\right|-\sum_{t=1}^T s_t^2\left(\boldsymbol{\vartheta}_0\right)\right)\leq 0\right)\to 0 \text{ as } T\to\infty,$$

where

$$\sum_{t=1}^T s_t^2\left(\boldsymbol{\vartheta}\right)=\sum_{t=1}^T\varepsilon_t^2(\boldsymbol{\tau})-\left(\sum_{t=1}^T\varepsilon_t(\boldsymbol{\tau})h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right)^2. \tag{S.34}$$

In the proof of their (2.7) for $i=3$, Hualde and Robinson (2011) showed that

$$\Pr\left(\inf_{\|\boldsymbol{\tau}-\boldsymbol{\tau}_0\|\geq\varepsilon,\boldsymbol{\tau}\in\mathcal{T}_3}\frac{1}{T}\sum_{t=1}^T\varepsilon_t^2(\boldsymbol{\tau})>K\right)\to 1 \text{ as } T\to\infty \tag{S.35}$$

for any arbitrarily large fixed constant K (for small enough η). Thus, noting (S.34), (S.7) for $i=3$ holds by (S.35) and Lemma S.1 on showing

$$\sup_{\mathcal{H}_3}\frac{1}{T}\left(\sum_{t=1}^T\varepsilon_t(\boldsymbol{\tau})h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right)^2=O_p\left(1\right), \tag{S.36}$$

$$\sup_{\mathcal{H}_3}\frac{1}{T}\left|\sum_{t=1}^T d_t\left(\boldsymbol{\vartheta}\right)s_t\left(\boldsymbol{\vartheta}\right)\right|=O_p\left(1\right). \tag{S.37}$$

In (S.37), the bound will depend on ξ, which had to be set very large in the proof of (S.33) (see the proof of Lemma S.3). However, this can be dominated by the constant K in (S.35), which can be chosen arbitrarily large by setting η small enough. First, by (S.222) of Lemma S.18, the left-hand side of (S.36) holds by choosing $\theta<1/2$ because $\delta_0-\delta\leq 1/2$ when $\delta\in\mathcal{I}_3$. Next, noting that $\sup_{\mathcal{H}_3}|\gamma-\gamma_0|\leq\xi T^{-\varkappa_3}$ and that $\delta\in\mathcal{I}_3$ implies $\gamma_0-\delta\geq\gamma_0-\delta_0+1/2-\eta>0$ and $\delta_0-\delta\leq 1/2$, it follows by (S.226) and (S.230) of Lemma S.19 (noting that for T large enough $\gamma-\delta>0$) that the left-hand side of (S.37) is $O_p\left(1\right)$ by choosing $\theta<1/2$.

Proof of (S.6) and (S.7) for $i = 2$ Fix $\varkappa_2 = \gamma_0 - \delta_0 + 1/2 > 0$. Changing the normalization ($T^{2(\delta_0-\delta)}$ instead of T), by the Cauchy-Schwarz inequality as in (S.25), and proceeding as in (S.29), the left-hand side of (S.6) is bounded by

$$\Pr\left(\inf_{\overline{\mathcal{H}}_2} \frac{1}{T^{2(\delta_0-\delta)+1}} \overline{d}_T^2(\boldsymbol{\vartheta})\left(1 - 2\left|\overline{v}_T(\boldsymbol{\vartheta})\right|\right) - \sup_{\overline{\mathcal{H}}_2} \frac{1}{T^{2(\delta_0-\delta)}} \sum_{t=1}^{T} s_t^2(\boldsymbol{\vartheta}_0) \leq 0\right).$$

Then, given Lemma S.1,

$$\sup_{\overline{\mathcal{H}}_2} \frac{1}{T^{2(\delta_0-\delta)}} \sum_{t=1}^{T} s_t^2(\boldsymbol{\vartheta}_0) = \sup_{\overline{\mathcal{H}}_2} \frac{T}{T^{2(\delta_0-\delta)}} \frac{1}{T} \sum_{t=1}^{T} s_t^2(\boldsymbol{\vartheta}_0) = O_p(1), \tag{S.38}$$

because when $\delta \in \mathcal{I}_2$, $\delta_0 - \delta \geq 1/2$. Thus (S.6) for $i = 2$ would hold if

$$\Pr\left(\inf_{\overline{\mathcal{H}}_2} \frac{1}{T^{2(\delta_0-\delta)+1}} \overline{d}_T^2(\boldsymbol{\vartheta})\left(1 - 2\left|\overline{v}_T(\boldsymbol{\vartheta})\right|\right) \leq K\right) \to 0 \text{ as } T \to \infty \tag{S.39}$$

for an arbitrarily large K, which, as in (S.31), follows if

$$\sup_{\overline{\mathcal{H}}_2} \frac{1}{T^{\delta_0-\delta+1/2}} \left|\overline{s}_T(\boldsymbol{\vartheta})\right| = O_p(1), \tag{S.40}$$

$$\lim_{T\to\infty} \inf_{\overline{\mathcal{H}}_2} \frac{1}{T^{2(\delta_0-\delta)+1}} \overline{d}_T^2(\boldsymbol{\vartheta}) > K. \tag{S.41}$$

The proof of (S.40) is almost identical to that of (S.32), again applying Lemmas S.16, S.17, and S.18. Finally (S.41) follows by Lemma S.3, to conclude the proof of (S.6) for $i = 2$.

Next we show (S.7) for $i = 2$, which holds if

$$\Pr\left(\inf_{\mathcal{H}_2} \frac{1}{T^{2(\delta_0-\delta)}} \left(\sum_{t=1}^{T} s_t^2(\boldsymbol{\vartheta}) - 2\left|\sum_{t=1}^{T} d_t(\boldsymbol{\vartheta})\, s_t(\boldsymbol{\vartheta})\right| - \sum_{t=1}^{T} s_t^2(\boldsymbol{\vartheta}_0)\right) \leq 0\right) \to 0 \tag{S.42}$$

as $T \to \infty$. In the proof of their (2.7) for $i = 2$, Hualde and Robinson (2011) showed that

$$\Pr\left(\inf_{\|\boldsymbol{\tau}-\boldsymbol{\tau}_0\|\geq\varepsilon, \boldsymbol{\tau}\in\mathcal{T}_2} \frac{1}{T^{2(\delta_0-\delta)}} \sum_{t=1}^{T} \varepsilon_t^2(\boldsymbol{\tau}) > K\right) \to 1 \tag{S.43}$$

as $T \to \infty$ for any arbitrarily large fixed constant K (for small enough η). Thus, in view of (S.34), (S.38), and (S.43), it follows that (S.42) holds if

$$\sup_{\mathcal{H}_2} \frac{1}{T^{2(\delta_0-\delta)}} \left(\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) h_{t,T}(\gamma, \delta, \boldsymbol{\varphi})\right)^2 = O_p(1), \tag{S.44}$$

$$\sup_{\mathcal{H}_2} \frac{1}{T^{2(\delta_0-\delta)}} \left|\sum_{t=1}^{T} d_t(\boldsymbol{\vartheta})\, s_t(\boldsymbol{\vartheta})\right| = O_p(1). \tag{S.45}$$

Again, (S.44) holds by (S.223) of Lemma S.18 with $\theta < 1/2$ noting that $\delta \in \mathcal{I}_2$ implies $\delta_0 - \delta \geq 1/2$. Since $\sup_{\mathcal{H}_2} |\gamma - \gamma_0| \leq \xi T^{-\varkappa_2}$ and $\delta \in \mathcal{I}_2$ implies $\gamma_0 - \delta \geq \gamma_0 - \delta_0 + 1/2 > 0$ and $\delta_0 - \delta \geq 1/2$, (S.45) follows from (S.226) and (S.231) of Lemma S.19 setting $\theta < 1/2$, noting that for T large enough $\gamma - \delta > 0$.

Proof of (S.6) and (S.7) for $i=1$ Fix $\varkappa_1 = \gamma_0 - \delta_0 + 1/2 > 0$. As in the treatment of (S.14), (S.6) for $i=1$ holds if

$$\sup_{\mathcal{I}_1} \frac{T^{2\varkappa_1}}{T^{2(\gamma_0-\delta)+1}} \sum_{t=1}^{T} s_t^2\left(\boldsymbol{\vartheta}_0\right) = o_p\left(1\right), \tag{S.46}$$

$$\Pr\left(\inf_{\overline{\mathcal{H}}_1} \frac{T^{2\varkappa_1}}{T^{2(\gamma_0-\delta)+1}} \sum_{t=1}^{T} d_t^2\left(\boldsymbol{\vartheta}\right)\left(1 - 2v_T\left(\boldsymbol{\vartheta}\right)\right) \leq \epsilon \right) \to 0 \text{ as } T \to \infty, \tag{S.47}$$

for an arbitrarily small ϵ. First, (S.46) holds by Lemma S.1, noting that $2\varkappa_1 - 2\left(\gamma_0 - \delta\right) - 1 = 2\left(\delta - \delta_0\right)$ and $\sup_{\mathcal{I}_1} 2\left(\delta - \delta_0\right) = -1 - 2\eta < -1$. Next, as in the proof of (S.39), see also (S.18) and (S.22), (S.47) follows if

$$\sup_{\overline{\mathcal{H}}_1} \frac{1}{T^{2(\delta_0-\delta)}} \sum_{t=1}^{T} \varepsilon_t^2(\boldsymbol{\tau}) = O_p\left(1\right), \tag{S.48}$$

$$\lim_{T\to\infty} \inf_{\overline{\mathcal{H}}_1} \frac{1}{T^{2(\delta_0-\delta)}} \sum_{t=1}^{T} d_t^2\left(\boldsymbol{\vartheta}\right) > K. \tag{S.49}$$

First, (S.48) follows immediately from (S.219) of Lemma S.17, noting that $\delta_0 - \delta \geq 1/2 + \eta$. Next, by the Cauchy-Schwarz inequality, (S.49) follows by Lemma S.3.

Finally we show (S.7) for $i=1$, which holds if

$$\Pr\left(\inf_{\mathcal{H}_1} \frac{1}{T^{2(\delta_0-\delta)}} \left(\sum_{t=1}^{T} \left(d_t\left(\boldsymbol{\vartheta}\right) + s_t\left(\boldsymbol{\vartheta}\right)\right)^2 - \sum_{t=1}^{T} s_t^2\left(\boldsymbol{\vartheta}_0\right) \right) \leq 0 \right) \to 0 \tag{S.50}$$

as $T\to\infty$. By (S.46) and Lemma S.12 with $Z_t = d_t\left(\boldsymbol{\vartheta}\right) + s_t\left(\boldsymbol{\vartheta}\right)$, (S.50) follows if there exists an $\epsilon > 0$ such that

$$\Pr\left(\inf_{\mathcal{H}_1} \frac{1}{T} \sum_{t=1}^{T} \left(\frac{1}{T^{\delta_0-\delta+1/2}} \overline{d}_t\left(\boldsymbol{\vartheta}\right) + \frac{1}{T^{\delta_0-\delta+1/2}} \overline{s}_t\left(\boldsymbol{\vartheta}\right) \right)^2 > \epsilon \right) \to 1 \tag{S.51}$$

as $T\to\infty$. Note that in $\mathcal{H}_1$, $\gamma_0 - \delta \geq \eta$, so there exists $\alpha > 0$ such that for T sufficiently large it holds that $\gamma - \delta \geq \alpha$. Define the set $\mathcal{G}_1 = \{\boldsymbol{\vartheta} : \boldsymbol{\tau} \in \mathcal{T}_1, \gamma - \delta \geq \alpha, \gamma \in [\Box_1, \Box_2]\}$. Let $[\cdot]$ denote the integer part of the argument and consider $S_T\left(r,\boldsymbol{\vartheta}\right) = T^{\delta-\delta_0-1/2}\overline{s}_{[Tr]}\left(\boldsymbol{\vartheta}\right)$ a process indexed by $(r,\boldsymbol{\vartheta})$ that is càdlàg in r and continuous in $\boldsymbol{\vartheta}$. We next show that

$$\begin{aligned} S_T\left(r,\boldsymbol{\vartheta}\right) \Rightarrow S\left(r,\boldsymbol{\vartheta}\right) =& \phi\left(1;\boldsymbol{\varphi}\right)\omega\left(1;\boldsymbol{\varphi}_0\right) W\left(r; 1+\delta_0-\delta\right) \\ & - \frac{\phi\left(1;\boldsymbol{\varphi}\right)\omega\left(1;\boldsymbol{\varphi}_0\right)\left(2\left(\gamma-\delta\right)+1\right)}{\gamma-\delta+1} r^{\gamma-\delta+1} \\ & \times \left(W\left(1;1+\delta_0-\delta\right) - \int_0^1 u^{\gamma-\delta-1} W\left(u;1+\delta_0-\delta\right) du \right), \end{aligned} \tag{S.52}$$

where $\Rightarrow$ means weak convergence in the product space of functions that are càdlàg in $r\in[0,1]$ and continuous in $\boldsymbol{\vartheta}\in\mathcal{G}_1$ endowed with the Skorokhod topology in r and the uniform topology in $\boldsymbol{\vartheta}$, and where $W\left(r;d\right) = \Gamma\left(d\right)^{-1}\int_0^r \left(1-s\right)^{d-1} dB\left(s\right)$ and $B\left(s\right)$ denote fractional (Type II) and regular scalar Brownian motions, respectively, both with variance

σ_0^2. Because $\overline{d}_t(\boldsymbol{\vartheta})$ is deterministic and $\overline{s}_t(\boldsymbol{\vartheta})$ is stochastic, and in view of the square in (S.51), (S.51) and hence (S.50) follows from (S.52) (also note Assumption A1(iv) and (7)). We note that a different approach for the case $i = 1$ was taken by Hualde and Robinson (2011) in their eqn. (2.36) based on the Cauchy-Schwarz inequality, but that approach does not appear sufficient; see Johansen and Nielsen (2018) for details on this point and for an argument very similar to that for our (S.50)–(S.52). In our (S.126) below, we give the result of Hualde and Robinson (2011) with an alternative proof based on our Lemma S.12.

We thus need to prove (S.52). Note (S.27) and

$$c_t(d_1, d_2, \boldsymbol{\varphi}) - c_{t-1}(d_1, d_2, \boldsymbol{\varphi}) = c_t(d_1, d_2 + 1, \boldsymbol{\varphi}). \tag{S.53}$$

By summation by parts and (S.53) we find

$$\begin{aligned}\sum_{j=1}^{T} c_j(\gamma, \delta, \boldsymbol{\varphi})\, \varepsilon_j(\boldsymbol{\tau}) =& c_T(\gamma, \delta, \boldsymbol{\varphi})\, \varepsilon_T(\delta_0 - \delta + 1, \boldsymbol{\varphi}) \\ & - \sum_{t=1}^{T-1} c_{t+1}(\gamma, \delta + 1, \boldsymbol{\varphi})\, \varepsilon_t(\delta_0 - \delta + 1, \boldsymbol{\varphi}).\end{aligned} \tag{S.54}$$

Also, by summation by parts on (10), noting (S.28),

$$a_j(d, \boldsymbol{\varphi}) = \phi(1; \boldsymbol{\varphi})\, \pi_j(d) - \pi_j(d) \sum_{k=j+1}^{\infty} \phi_k(\boldsymbol{\varphi}) - \sum_{k=0}^{j-1} \pi_{k+1}(d-1) \sum_{l=0}^{k} \phi_{j-l}(\boldsymbol{\varphi}), \tag{S.55}$$

where for $j = 0$ the last term on the right-hand side of (S.55) is 0. Thus, noting (S.27),

$$\begin{aligned}\varepsilon_t(\delta_0 - \delta + 1, \boldsymbol{\varphi}) &= \sum_{j=0}^{t-1} a_j(\delta_0 - \delta + 1, \boldsymbol{\varphi})\, u_{t-j} \\ &= \phi(1; \boldsymbol{\varphi})\, \omega(1; \boldsymbol{\varphi}_0) \sum_{j=0}^{t-1} \pi_j(\delta_0 - \delta + 1)\, \varepsilon_{t-j} + m_{1t}(\boldsymbol{\tau}),\end{aligned} \tag{S.56}$$

where

$$\begin{aligned}m_{1t}(\boldsymbol{\tau}) =& \phi(1; \boldsymbol{\varphi}) \sum_{j=0}^{t-1} \pi_j(\delta_0 - \delta + 1)\, (u_{t-j} - \omega(1; \boldsymbol{\varphi}_0)\, \varepsilon_{t-j}) \\ & - \sum_{j=0}^{t-1} \pi_j(\delta_0 - \delta + 1) \sum_{k=j+1}^{\infty} \phi_k(\boldsymbol{\varphi})\, u_{t-j} - \sum_{j=1}^{t-1} \sum_{k=0}^{j-1} \pi_{k+1}(\delta_0 - \delta) \sum_{l=0}^{k} \phi_{j-l}(\boldsymbol{\varphi})\, u_{t-j}.\end{aligned} \tag{S.57}$$

Substituting (S.27), (S.54), (S.56) and (S.200) into $s_j(\boldsymbol{\vartheta})$, we get

$$\begin{aligned}\frac{1}{T^{\delta_0-\delta+1/2}}\sum_{j=1}^{t} s_j(\boldsymbol{\vartheta}) =& \frac{1}{T^{\delta_0-\delta+1/2}}\varepsilon_t(\delta_0-\delta+1,\boldsymbol{\varphi})\\ &- \frac{c_T(\gamma,\delta,\boldsymbol{\varphi})\,\varepsilon_T(\delta_0-\delta+1,\boldsymbol{\varphi})\sum_{j=1}^{t}c_j(\gamma,\delta,\boldsymbol{\varphi})}{T^{\delta_0-\delta+1/2}\sum_{j=1}^{T}c_j^2(\gamma,\delta,\boldsymbol{\varphi})}\\ &+ \frac{\sum_{j=1}^{t}c_j(\gamma,\delta,\boldsymbol{\varphi})\sum_{k=1}^{T-1}c_{k+1}(\gamma,\delta+1,\boldsymbol{\varphi})\,\varepsilon_k(\delta_0-\delta+1,\boldsymbol{\varphi})}{T^{\delta_0-\delta+1/2}\sum_{j=1}^{T}c_j^2(\gamma,\delta,\boldsymbol{\varphi})}\\ =& \phi(1;\boldsymbol{\varphi})\,\omega(1;\boldsymbol{\varphi}_0)\,\widetilde{V}_t(\gamma,\delta)+m_{2t}(\boldsymbol{\vartheta}), \end{aligned} \tag{S.58}$$

where

$$\begin{aligned}\widetilde{V}_t(\gamma,\delta) =& \frac{1}{T^{\delta_0-\delta+1/2}}\sum_{j=0}^{t-1}\pi_j(\delta_0-\delta+1)\,\varepsilon_{t-j}\\ &- \frac{1}{T^{\delta_0-\delta+1/2}}\frac{b_T(\gamma,\delta)\sum_{k=0}^{T-1}\pi_k(\delta_0-\delta+1)\,\varepsilon_{T-k}}{\sum_{k=1}^{T}b_k^2(\gamma,\delta)}\sum_{j=1}^{t}b_j(\gamma,\delta)\\ &+ \frac{1}{T^{\delta_0-\delta+1/2}}\frac{\sum_{k=1}^{T-1}b_{k+1}(\gamma,\delta+1)\sum_{l=0}^{k-1}\pi_l(\delta_0-\delta+1)\,\varepsilon_{k-l}}{\sum_{k=1}^{T}b_k^2(\gamma,\delta)}\sum_{j=1}^{t}b_j(\gamma,\delta), \end{aligned}$$

and $m_{2t}(\boldsymbol{\vartheta})$ collects remainder terms arising from (S.56) and (S.200). By relatively straightforward arguments, it can be shown that

$$\sup_{\mathcal{G}_1}\frac{1}{T}\sum_{t=1}^{T}m_{2t}^2(\boldsymbol{\vartheta}) = o_p(1), \tag{S.59}$$

$$\sup_{\mathcal{G}_1}\frac{1}{T}\sum_{t=1}^{T}\left|m_{2t}(\boldsymbol{\vartheta})\,\widetilde{V}_t(\gamma,\delta)\right| = o_p(1), \tag{S.60}$$

The proofs of (S.59) and (S.60) involve several results. First, in order to deal with the first term on the right-hand side of (S.57), note that $u_t=\omega(1;\boldsymbol{\varphi}_0)\,\varepsilon_t+\widetilde{\varepsilon}_{t-1}-\widetilde{\varepsilon}_t$, where $\widetilde{\varepsilon}_t=\sum_{j=0}^{\infty}\widetilde{\omega}_j(\boldsymbol{\varphi}_0)\,\varepsilon_{t-j}$, $\widetilde{\omega}_j(\boldsymbol{\varphi}_0)=\sum_{k=j+1}^{\infty}\omega_k(\boldsymbol{\varphi}_0)$. By Assumptions A1 and A2, $\widetilde{\varepsilon}_t$ is well defined in the mean-square sense and $|\widetilde{\omega}_j(\boldsymbol{\varphi}_0)|=O(j^{-\varsigma})$. We also apply Lemmas S.9, S.11, S.13, S.14, S.15, S.16, and S.17.

Because $S_T(r,\boldsymbol{\vartheta})$ is defined on a product space, we can prove weak convergence in $r\in[0,1]$ and $\boldsymbol{\vartheta}\in\mathcal{G}_1$ separately. Thus, suppose first that $\boldsymbol{\vartheta}\in\mathcal{G}_1$ is fixed. Letting $t=[Tr]$, weak convergence of $\widetilde{V}_{[Tr]}(\gamma,\delta)$ as $T\to\infty$ then follows by first applying Theorem 2 of Hosoya (2005), noting that our Assumption A2 implies conditions A(i), A(ii) and A(iii) in Hosoya (2005). We then apply Lemmas S.10, S.11, and S.13 and the continuous mapping theorem, as in Robinson and Marinucci (2000), noting that for fixed $\boldsymbol{\vartheta}\in\mathcal{G}_1$ and in particular $\gamma-\delta\geq\alpha>0$, it holds that $\int_0^1 u^{\gamma-\delta-1}W(u;1+\delta_0-\delta)\,du$ is a well-defined random variable with zero mean and finite variance (e.g., for $\delta=\delta_0$ this variance is $2\sigma_0^2((\gamma-\delta+1)(2(\gamma-\delta)+1))^{-1}$). To prove weak convergence in $\boldsymbol{\vartheta}\in\mathcal{G}_1$, we note that finite-dimensional convergence follows by weak convergence in $r \in [0,1]$. Tightness of the process $\widetilde{V}_{[Tr]}(\gamma,\delta)$ on the compact set $\mathcal{G}_1$ follows from Lemmas A.2 and C.3 of Johansen and Nielsen (2010) noting, in particular, that $\widetilde{V}_{[Tr]}(\gamma,\delta)$ is continuously differentiable for $\gamma-\delta \geq \alpha > 0$. Because $\boldsymbol{\varphi}$ only enters through the multiplicative function $\phi(1;\boldsymbol{\varphi})$, which is bounded and bounded away from zero for $\boldsymbol{\vartheta} \in \mathcal{G}_1$, tightness in $\boldsymbol{\varphi}$ follows straightforwardly. This proves (S.52) to conclude the proof of (S.7) for $i=1$ and therefore that of (S.1).

S.2.1.3 Proof of (S.2)

Here, let $R_T(\boldsymbol{\tau},\gamma) = R_T(\boldsymbol{\vartheta}), d_t(\boldsymbol{\tau},\gamma) = d_t(\boldsymbol{\vartheta})$, and $s_t(\boldsymbol{\tau},\gamma) = s_t(\boldsymbol{\vartheta}) = s_{1t}(\boldsymbol{\tau}) - s_{2t}(\boldsymbol{\vartheta})$ with $s_{1t}(\boldsymbol{\tau}) = \varepsilon_t(\boldsymbol{\tau})$ and $s_{2t}(\boldsymbol{\tau},\gamma) = s_{2t}(\boldsymbol{\vartheta}) = h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\sum_{j=1}^{T} s_{1j}(\boldsymbol{\tau}) h_{j,T}(\gamma,\delta,\boldsymbol{\varphi})$, so that, noting $\sum_{t=1}^{T} h_{t,T}^2(\gamma,\delta,\boldsymbol{\varphi}) = 1$,

$$\sum_{t=1}^{T} s_{2t}^2(\boldsymbol{\vartheta}) = \sum_{t=1}^{T} s_{1t}(\boldsymbol{\tau}) s_{2t}(\boldsymbol{\vartheta}) = \left(\sum_{j=1}^{T} s_{1j}(\boldsymbol{\tau}) h_{j,T}(\gamma,\delta,\boldsymbol{\varphi})\right)^2 .$$

Noting also (S.15),

$$\begin{aligned} R_T(\boldsymbol{\vartheta}) =& \frac{1}{T}\sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta}) + \frac{1}{T}\sum_{t=1}^{T} s_{1t}^2(\boldsymbol{\tau}) \\ & - \frac{1}{T}\left(\sum_{j=1}^{T} s_{1j}(\boldsymbol{\tau}) h_{j,T}(\gamma,\delta,\boldsymbol{\varphi})\right)^2 + \frac{2}{T}\sum_{t=1}^{T} d_t(\boldsymbol{\vartheta}) s_{1t}(\boldsymbol{\vartheta}) . \end{aligned} \tag{S.61}$$

Clearly, if $\widehat{\boldsymbol{\vartheta}} \in \overline{N}_\varepsilon \cap M_\varepsilon$, then $\inf_{\overline{N}_\varepsilon \cap M_\varepsilon} R_T(\widehat{\boldsymbol{\tau}},\gamma) \leq R_T(\widehat{\boldsymbol{\tau}},\gamma_0)$, so that

$$\Pr(\widehat{\boldsymbol{\vartheta}} \in \overline{N}_\varepsilon \cap M_\varepsilon) \leq \Pr\left(\widehat{\boldsymbol{\vartheta}} \in \overline{N}_\varepsilon \cap M_\varepsilon, \inf_{\overline{N}_\varepsilon \cap M_\varepsilon} R_T(\widehat{\boldsymbol{\tau}},\gamma) - R_T(\widehat{\boldsymbol{\tau}},\gamma_0) \leq 0\right). \tag{S.62}$$

Recalling that $d_t(\boldsymbol{\tau},\gamma_0) = 0$, $R_T(\widehat{\boldsymbol{\tau}},\gamma_0) = T^{-1}\sum_{t=1}^{T} s_{1t}(\widehat{\boldsymbol{\tau}})$ and this cancels with the corresponding term in $R_T(\widehat{\boldsymbol{\tau}},\gamma)$, see (S.61). Thus, (S.2) holds if

$$\lim_{T\to\infty} \inf_{\boldsymbol{\vartheta}\in\overline{N}_\varepsilon \cap M_\varepsilon} \frac{1}{T^{2(\gamma_0-\delta)+1}} \sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta}) > \epsilon, \tag{S.63}$$

$$\sup_{\boldsymbol{\vartheta}\in\overline{N}_\varepsilon \cap M_\varepsilon} \frac{1}{T^{2(\gamma_0-\delta)+1}} \left|\sum_{t=1}^{T} d_t(\boldsymbol{\vartheta}) s_{1t}(\boldsymbol{\tau})\right| = o_p(1), \tag{S.64}$$

$$\sup_{\boldsymbol{\vartheta}\in\overline{N}_\varepsilon \cap M_\varepsilon} \frac{1}{T^{2(\gamma_0-\delta)+1}} \left(\sum_{j=1}^{T} s_{1j}(\boldsymbol{\tau}) h_{j,T}(\gamma,\delta,\boldsymbol{\varphi})\right)^2 = o_p(1), \tag{S.65}$$

noting the change in the normalization compared with (S.62) ($T^{2(\gamma_0-\delta_0)+1}$ instead of T), which is justified because the right-hand side of the inequality inside the probability in (S.62) is 0, so multiplying the left- and right-hand sides of the inequality by a positive number does not alter the probability.

First, (S.63) follows from Lemma S.2, noting that in $\overline{N}_\varepsilon \cap M_\varepsilon$, $\gamma_0-\delta \geq \gamma_0-\delta_0-\varepsilon > -1/2$ setting ε small enough. Next, letting both ε and θ be sufficiently small and noting that in $\overline{N}_\varepsilon \cap M_\varepsilon$, $\delta_0 - \delta \geq -\varepsilon$, by (S.229) of Lemma S.19 the left-hand side of (S.64) is $O_p(T^{-1/2+\delta_0-\gamma_0+3\theta+\varepsilon}) = o_p(1)$. Finally, by (S.223) of Lemma S.18 the left-hand side of (S.65) is $O_p(T^{-2(\gamma_0-\delta_0+1/2-\theta-\varepsilon)}) = o_p(1)$, to conclude the proof of (S.2) and therefore that of consistency of $\widehat{\boldsymbol{\vartheta}}$.

S.2.2 Proof of Theorem 1(ii): the $\gamma_0 + 1/2 < \delta_0$ case

Clearly

$$\Pr\left(\|\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau}_0\| \geq \varepsilon\right) = \Pr\left(\inf_{\boldsymbol{\vartheta}\in\overline{M}_\varepsilon} R_T\left(\boldsymbol{\vartheta}\right) \leq \inf_{\boldsymbol{\vartheta}\in M_\varepsilon} R_T\left(\boldsymbol{\vartheta}\right)\right),$$

so, as in the proof for $\gamma_0 + 1/2 > \delta_0$, the result follows by showing that the right-hand side of (S.4) is $o\left(1\right)$, which, in view of Lemma S.1, holds if

$$\Pr\left(\inf_{\boldsymbol{\vartheta}\in\overline{M}_\varepsilon} R_T\left(\boldsymbol{\vartheta}\right) \leq \sigma_0^2 + \epsilon\right) \to 0 \text{ as } T \to \infty.$$

This result is given in Lemma S.4, whose proof uses very similar techniques to those employed in the proof of (S.1). This completes the proof of Theorem 1.

S.2.3 Proof of Theorem 2(i): the $\gamma_0 + 1/2 > \delta_0$ case

Let $\boldsymbol{M}_T = \operatorname{diag}(I_{p+1}, T^{\delta_0-\gamma_0})$, c.f. (15). We first show that

$$T^{1/2}\boldsymbol{M}_T^{-1}(\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0) \to_d N\left(0, \sigma_0^2\boldsymbol{V}^{-1}\right). \tag{S.66}$$

By the mean value theorem,

$$\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0 = -\left(\frac{\partial^2 R_T(\overline{\boldsymbol{\vartheta}})}{\partial\boldsymbol{\vartheta}\partial\boldsymbol{\vartheta}'}\right)^{-1}\frac{\partial R_T\left(\boldsymbol{\vartheta}_0\right)}{\partial\boldsymbol{\vartheta}}, \tag{S.67}$$

where $\overline{\boldsymbol{\vartheta}}$ represents an intermediate point which is allowed to vary across the different rows of $\partial^2 R_T(\cdot)/\partial\boldsymbol{\vartheta}\partial\boldsymbol{\vartheta}'$.

We first analyze the score in (S.67). It can be easily seen that $\partial d_t\left(\boldsymbol{\vartheta}_0\right)/\partial\boldsymbol{\tau} = 0$ and $\partial s_{1t}\left(\boldsymbol{\tau}\right)/\partial\gamma = 0$, so, recalling that $d_t\left(\boldsymbol{\vartheta}_0\right) = 0$ and the decomposition (S.61),

$$\frac{\partial R_T\left(\boldsymbol{\vartheta}_0\right)}{\partial\boldsymbol{\vartheta}} = \frac{2}{T}\sum_{t=1}^{T} s_t\left(\boldsymbol{\vartheta}_0\right)\left(\begin{pmatrix} \frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial\boldsymbol{\tau}} \\ \frac{\partial d_t(\boldsymbol{\vartheta}_0)}{\partial\gamma} \end{pmatrix} - \frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\boldsymbol{\vartheta}}\right).$$

Then, by Lemma S.5(a) it holds that

$$\frac{T^{1/2}}{2}\boldsymbol{M}_T\frac{\partial R_T\left(\boldsymbol{\vartheta}_0\right)}{\partial\boldsymbol{\vartheta}} = \begin{pmatrix} T^{-1/2}I_{p+1} & 0 \\ 0 & T^{-1/2-(\gamma_0-\delta_0)} \end{pmatrix}\sum_{t=1}^{T}\varepsilon_t\begin{pmatrix} \frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial\boldsymbol{\tau}} \\ \frac{\partial d_t(\boldsymbol{\vartheta}_0)}{\partial\gamma} \end{pmatrix} + o_p\left(1\right). \tag{S.68}$$

Next, as in (2.54) of Hualde and Robinson (2011),

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T}\varepsilon_t\frac{\partial s_{1t}\left(\boldsymbol{\tau}_0\right)}{\partial\boldsymbol{\tau}} = \frac{1}{T^{1/2}}\sum_{t=2}^{T}\varepsilon_t\sum_{j=1}^{\infty}\boldsymbol{m}_j\left(\boldsymbol{\varphi}_0\right)\varepsilon_{t-j} + o_p\left(1\right),$$

where $\boldsymbol{m}_j(\boldsymbol{\varphi}_0) = (-j^{-1}, \boldsymbol{b}_j'(\boldsymbol{\varphi}_0))'$. Also,

$$\frac{\partial d_t(\boldsymbol{\vartheta}_0)}{\partial \gamma} = -\mu_0 c_t^{(1)}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) + \mu_0 c_t(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) \frac{\sum_{j=1}^T c_j(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) c_j^{(1)}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0)}{\sum_{j=1}^T c_j^2(\gamma_0, \delta_0, \boldsymbol{\varphi}_0)},$$

where $c_t^{(1)}(\cdot,\cdot,\cdot)$ is the derivative of $c_t(\cdot,\cdot,\cdot)$ with respect to the first argument, so that

$$\frac{1}{T^{\gamma_0-\delta_0+1/2}} \sum_{t=1}^T \varepsilon_t \frac{\partial d_t(\boldsymbol{\vartheta}_0)}{\partial \gamma} = \frac{\mu_0}{T^{\gamma_0-\delta_0+1/2}} \sum_{t=1}^T \varepsilon_t c_t(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) \frac{\sum_{j=1}^T c_j(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) c_j^{(1)}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0)}{\sum_{j=1}^T c_j^2(\gamma_0, \delta_0, \boldsymbol{\varphi}_0)} - \frac{\mu_0}{T^{\gamma_0-\delta_0+1/2}} \sum_{t=1}^T \varepsilon_t c_t^{(1)}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0). \tag{S.69}$$

By (11), $c_t^{(1)}(d_1, d_2, \boldsymbol{\varphi}) = \sum_{j=0}^{t-1} \phi_j(\boldsymbol{\varphi}) b_{t-j}^{(1)}(d_1, d_2)$, where $b_j^{(1)}(\cdot,\cdot)$ is the derivative of $b_j(\cdot,\cdot)$ with respect to the first argument. Then, noting that $\gamma_0 + 1/2 > \delta_0$, by a similar analysis to that in the proof of Lemma S.15, the right-hand side of (S.69) equals

$$\frac{\mu_0 \phi(1; \boldsymbol{\varphi}_0)}{T^{\gamma_0-\delta_0+1/2}} \frac{\sum_{t=1}^T \varepsilon_t b_t(\gamma_0, \delta_0) \sum_{j=1}^T b_j(\gamma_0, \delta_0) b_j^{(1)}(\gamma_0, \delta_0)}{\sum_{j=1}^T b_j^2(\gamma_0, \delta_0)} - \frac{\mu_0 \phi(1; \boldsymbol{\varphi}_0)}{T^{\gamma_0-\delta_0+1/2}} \sum_{t=1}^T \varepsilon_t b_t^{(1)}(\gamma_0, \delta_0) + o_p(1). \tag{S.70}$$

Substituting (S.185) (evaluated at (γ_0, δ_0)) into (S.70), the first two terms of (S.70) become

$$\frac{\mu_0 \phi(1; \boldsymbol{\varphi}_0)}{T^{\gamma_0-\delta_0+1/2}} \frac{\sum_{t=1}^T \varepsilon_t b_t(\gamma_0, \delta_0) \sum_{j=1}^T \log(\frac{j}{T}) b_j^2(\gamma_0, \delta_0)}{\sum_{j=1}^T b_j^2(\gamma_0, \delta_0)} - \frac{\mu_0 \phi(1; \boldsymbol{\varphi}_0)}{T^{\gamma_0-\delta_0+1/2}} \sum_{t=1}^T \log(\frac{t}{T}) b_t(\gamma_0, \delta_0) \varepsilon_t + o_p(1). \tag{S.71}$$

Using (S.184) and (S.185), (S.71) equals $\mu_0 \phi(1; \boldsymbol{\varphi}_0) \Gamma(\gamma_0 + 1) T^{-1/2+\delta_0-\gamma_0} \sum_{t=1}^T \varepsilon_t g_{t,T}(\gamma_0 - \delta_0 + 1) + o_p(1)$ with

$$g_{t,T}(d) = \frac{\pi_t(d) \sum_{j=1}^T \log(\frac{j}{T}) \pi_j^2(d) - \log(\frac{t}{T}) \pi_t(d) \sum_{j=1}^T \pi_j^2(d)}{\sum_{j=1}^T \pi_j^2(d)}.$$

Collecting these terms shows that

$$\frac{T^{1/2}}{2} \boldsymbol{M}_T \frac{\partial R_T(\boldsymbol{\vartheta}_0)}{\partial \boldsymbol{\vartheta}} = \sum_{t=1}^T \varepsilon_t \boldsymbol{\eta}_{t,T} + o_p(1), \tag{S.72}$$

where

$$\boldsymbol{\eta}_{t,T} = \begin{pmatrix} \frac{1}{T^{1/2}} \sum_{j=1}^{\infty} \boldsymbol{m}_j(\boldsymbol{\varphi}_0) \varepsilon_{t-j} \\ \frac{1}{T^{\gamma_0-\delta_0+1/2}} \mu_0 \phi(1;\boldsymbol{\varphi}_0) \Gamma(\gamma_0+1) g_{t,T}(\gamma_0-\delta_0+1) \end{pmatrix}.$$

Defining $\mathcal{F}_{t,T} = \mathcal{F}_t$ for any $1 \le t \le T$, Assumption A2 implies that $\{\varepsilon_t \boldsymbol{\eta}_{t,T}, \mathcal{F}_{t,T}, 1 \le t \le T, T \ge 1\}$ is a martingale difference array. For any $(p+2)$-dimensional vector $\boldsymbol{\xi}$, define $\xi_{t,T} = \varepsilon_t \boldsymbol{\xi}' \boldsymbol{\eta}_{t,T} / \sigma_0 (\boldsymbol{\xi}' \boldsymbol{V} \boldsymbol{\xi})^{1/2}$ and $B_T^2 = \sum_{t=2}^T E(\xi_{t,T}^2 | \mathcal{F}_{t-1,T})$. Then, by Corollary 3.1 of Hall and Heyde (1980), if

$$B_T^2 \to_p 1, \tag{S.73}$$

and, for all $\epsilon > 0$,

$$\sum_{t=2}^T E\left(\xi_{t,T}^2 \mathbb{I}(|\xi_{t,T}| > \epsilon) | \mathcal{F}_{t-1,T}\right) \to_p 0, \tag{S.74}$$

it holds that $\sum_{t=2}^T \xi_{t,T} \to_d N(0,1)$, and hence

$$\sum_{t=2}^T \varepsilon_t \boldsymbol{\eta}_{t,T} \to_d N\left(0, \sigma_0^2 \boldsymbol{V}\right) \tag{S.75}$$

by direct application of the Cramér-Wold device. First we note that

$$E\left(\xi_{t,T}^2 | \mathcal{F}_{t-1,T}\right) = \frac{\boldsymbol{\xi}' \boldsymbol{\eta}_{t,T} \boldsymbol{\eta}_{t,T}' \boldsymbol{\xi}}{\boldsymbol{\xi}' \boldsymbol{V} \boldsymbol{\xi}},$$

so that (S.73) holds if $\sum_{t=2}^T \boldsymbol{\eta}_{t,T} \boldsymbol{\eta}_{t,T}' \to_p \boldsymbol{V}$. However, this follows straightforwardly by the same arguments as in the proof of (2.55) of Hualde and Robinson (2011) and Lemma S.10 because $\sum_{l=1}^t \sum_{j=1}^\infty \boldsymbol{m}_j(\boldsymbol{\varphi}_0) \varepsilon_{l-j} = O_p(t^{1/2})$, which implies, by summation by parts, that

$$\frac{1}{T^{\gamma_0-\delta_0+1}} \sum_{t=2}^T g_{t,T}(\gamma_0-\delta_0+1) \sum_{j=1}^\infty \boldsymbol{m}_j(\boldsymbol{\varphi}_0) \varepsilon_{t-j} = o_p(1). \tag{S.76}$$

Now (S.74) holds if, e.g., $\sum_{t=2}^T E(\xi_{t,T}^4 | \mathcal{F}_{t-1,T}) \to_p 0$, which, given that the fourth moment of ε_t is finite, holds if $\sum_{t=2}^T (\boldsymbol{\xi}' \boldsymbol{\eta}_{t,T} \boldsymbol{\eta}_{t,T}' \boldsymbol{\xi})^2 \to_p 0$, and this can be easily justified by previous arguments. This completes the proof of (S.75).

Next, noting (S.67), (S.72), and (S.75), the proof of (S.66) is completed by showing

$$\boldsymbol{M}_T \left(\frac{\partial^2 R_T(\overline{\boldsymbol{\vartheta}})}{\partial \boldsymbol{\vartheta} \partial \boldsymbol{\vartheta}'} - \frac{\partial^2 R_T(\boldsymbol{\vartheta}_0)}{\partial \boldsymbol{\vartheta} \partial \boldsymbol{\vartheta}'} \right) \boldsymbol{M}_T = o_p(1) \tag{S.77}$$

and

$$\frac{1}{2} \boldsymbol{M}_T \frac{\partial^2 R_T(\boldsymbol{\vartheta}_0)}{\partial \boldsymbol{\vartheta} \partial \boldsymbol{\vartheta}'} \boldsymbol{M}_T \to_p \boldsymbol{V}. \tag{S.78}$$

By Lemma S.6 it holds that, for some fixed $\varkappa > 0$, $T^{\varkappa}(\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0) \to_p 0$, and in light of this the proof of (S.77) is relatively straightforward. It consists of deriving all terms in $\partial^2 R_T(\overline{\boldsymbol{\vartheta}}) / \partial \boldsymbol{\vartheta} \partial \boldsymbol{\vartheta}'$ and checking that the differences with respect the corresponding ones in $\partial^2 R_T(\boldsymbol{\vartheta}_0) / \partial \boldsymbol{\vartheta} \partial \boldsymbol{\vartheta}'$ satisfy (S.77). This requires the use of the mean value theorem and Assumption A4(ii), where typically the derivatives involve additional $\log T$ factors which are compensated by the factor $T^{-\varkappa}$ that arises because $\overline{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0 = O_p(T^{-\varkappa})$.

Now we show (S.78). Recalling $d_t(\boldsymbol{\vartheta}_0) = 0$ and noting $\partial^2 d_t(\boldsymbol{\vartheta}_0)/\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}' = 0$, Lemma S.5(b) implies that

$$\frac{1}{2}\boldsymbol{M}_T\frac{\partial^2 R_T(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\vartheta}\partial\boldsymbol{\vartheta}'}\boldsymbol{M}_T = \boldsymbol{M}_T\frac{1}{T}\sum_{t=1}^{T}\begin{pmatrix} \frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial\boldsymbol{\tau}}\frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial\boldsymbol{\tau}'} & 0 \\ 0 & \left(\frac{\partial d_t(\boldsymbol{\vartheta}_0)}{\partial\gamma}\right)^2 \end{pmatrix}\boldsymbol{M}_T + o_p(1), \tag{S.79}$$

so (S.78) holds by showing that, as $T \to \infty$,

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial\boldsymbol{\tau}}\frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial\boldsymbol{\tau}'} \to_p \sigma_0^2\boldsymbol{A}, \tag{S.80}$$

$$\frac{1}{T^{2(\gamma_0-\delta_0)+1}}\sum_{t=1}^{T}\left(\frac{\partial d_t(\boldsymbol{\vartheta}_0)}{\partial\gamma}\right)^2 \to \frac{\mu_0^2\phi^2(1;\boldsymbol{\varphi}_0)\Gamma^2(\gamma_0+1)}{\Gamma^2(\gamma_0-\delta_0+1)(2(\gamma_0-\delta_0)+1)^3}. \tag{S.81}$$

Here, (S.80) follows from (2.53) of Hualde and Robinson (2011) and (S.81) follows by arguments used in the proof of (S.73).

Next, given (S.66), the remaining part of (16) is justified as follows. Noting

$$\phi(L;\boldsymbol{\varphi})\Delta_+^{\delta}x_t = \mu_0 c_t(\gamma_0,\delta,\boldsymbol{\varphi}) + \varepsilon_t(\boldsymbol{\tau}), \tag{S.82}$$

it follows from (12) that

$$\widehat{\mu} = \widehat{\mu}(\widehat{\boldsymbol{\vartheta}}) = \mu_0\sum_{t=1}^{T}c_t(\gamma_0,\widehat{\delta},\widehat{\boldsymbol{\varphi}})k_{t,T}(\widehat{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}) + \sum_{t=1}^{T}\varepsilon_t(\widehat{\boldsymbol{\tau}})k_{t,T}(\widehat{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}),$$

where $k_{t,T}(\gamma,\delta,\boldsymbol{\varphi}) = c_t(\gamma,\delta,\boldsymbol{\varphi})/\sum_{t=1}^{T}c_t^2(\gamma,\delta,\boldsymbol{\varphi})$. By straightforward application of the mean value theorem,

$$k_{t,T}(\widehat{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}) = k_{t,T}(\gamma_0,\widehat{\delta},\widehat{\boldsymbol{\varphi}}) + k_{t,T}^{(1)}(\overline{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}})(\widehat{\gamma}-\gamma_0),$$

where $k_{t,T}^{(1)}(\cdot,\cdot,\cdot)$ is the derivative of $k_{t,T}(\cdot,\cdot,\cdot)$ with respect to the first argument and $|\overline{\gamma}-\gamma_0| \le |\widehat{\gamma}-\gamma_0|$. Thus,

$$\widehat{\mu} = \mu_0 + \mu_0(\widehat{\gamma}-\gamma_0)\sum_{t=1}^{T}c_t(\gamma_0,\widehat{\delta},\widehat{\boldsymbol{\varphi}})k_{t,T}^{(1)}(\overline{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}) + \sum_{t=1}^{T}\varepsilon_t(\widehat{\boldsymbol{\tau}})k_{t,T}(\widehat{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}),$$

which implies that

$$\begin{aligned}\frac{T^{\gamma_0-\delta_0+1/2}}{\log T}(\widehat{\mu}-\mu_0) =& \mu_0 T^{\gamma_0-\delta_0+1/2}(\widehat{\gamma}-\gamma_0)\frac{1}{\log T}\sum_{t=1}^{T}c_t(\gamma_0,\widehat{\delta},\widehat{\boldsymbol{\varphi}})k_{t,T}^{(1)}(\overline{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}) \\ &+ \frac{T^{\gamma_0-\delta_0+1/2}}{\log T}\sum_{t=1}^{T}\varepsilon_t(\widehat{\boldsymbol{\tau}})k_{t,T}(\widehat{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}).\end{aligned} \tag{S.83}$$

Then, the remaining part of (16) holds on showing

$$\frac{1}{\log T}\sum_{t=1}^{T}c_t(\gamma_0,\widehat{\delta},\widehat{\boldsymbol{\varphi}})k_{t,T}^{(1)}(\overline{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}) \to_p -1, \tag{S.84}$$

$$\frac{T^{\gamma_0-\delta_0+1/2}}{\log T}\sum_{t=1}^{T}\varepsilon_t(\widehat{\boldsymbol{\tau}})k_{t,T}(\widehat{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}) = o_p(1), \tag{S.85}$$

noting that joint asymptotic distribution of $\hat{\mu}$ and $\hat{\gamma}$ follows easily from (S.83)–(S.85) by the Cramér-Wold device.

Clearly, (S.84) follows if

$$\sum_{t=1}^{T} c_t(\gamma_0, \widehat{\delta}, \widehat{\boldsymbol{\varphi}}) k_{t,T}^{(1)}(\overline{\gamma}, \widehat{\delta}, \widehat{\boldsymbol{\varphi}}) - \sum_{t=1}^{T} c_t(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) k_{t,T}^{(1)}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) = o_p\left(\log T\right), \tag{S.86}$$

$$\frac{1}{\log T} \sum_{t=1}^{T} c_t(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) k_{t,T}^{(1)}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) \to -1. \tag{S.87}$$

First, (S.86) can be easily justified by applying Lemmas S.7 and S.14, noting that

$$k_{t,T}^{(1)}(\gamma, \delta, \boldsymbol{\varphi}) = \frac{c_t^{(1)}\left(\gamma, \delta, \boldsymbol{\varphi}\right)}{\sum_{j=1}^{T} c_j^2\left(\gamma, \delta, \boldsymbol{\varphi}\right)} - \frac{2 c_t\left(\gamma, \delta, \boldsymbol{\varphi}\right) \sum_{j=1}^{T} c_j^{(1)}\left(\gamma, \delta, \boldsymbol{\varphi}\right) c_j\left(\gamma, \delta, \boldsymbol{\varphi}\right)}{\left(\sum_{j=1}^{T} c_j^2\left(\gamma, \delta, \boldsymbol{\varphi}\right)\right)^2}.$$

Next, the left-hand side of (S.87) is

$$-\frac{1}{\log T} \frac{\sum_{t=1}^{T} c_t^{(1)}\left(\gamma_0, \delta_0, \boldsymbol{\varphi}_0\right) c_t\left(\gamma_0, \delta_0, \boldsymbol{\varphi}_0\right)}{\sum_{t=1}^{T} c_t^2\left(\gamma_0, \delta_0, \boldsymbol{\varphi}_0\right)} = -\frac{1}{\log T} \frac{\sum_{t=1}^{T} b_t^2\left(\gamma_0, \delta_0\right) \log t}{\sum_{t=1}^{T} b_t^2\left(\gamma_0, \delta_0\right)} + o\left(1\right),$$

by (S.185), noting that the remainder is of smaller order. Thus (S.87) follows immediately using (S.184), Lemma S.11, and noting that by simple application of summation by parts, the mean value theorem and Lemma S.10 for $d > -1/2$, it can be easily shown that

$$\frac{1}{T^{2d+1} \log T} \sum_{t=1}^{T} \log\left(t\right) t^{2d} \to \frac{1}{2d+1}.$$

Next, the left-hand side of (S.85) is

$$\frac{T^{\gamma_0 - \delta_0 + 1/2}}{\log T} \sum_{t=1}^{T} \phi\left(L; \boldsymbol{\varphi}_0\right) \left\{u_t \mathbb{I}(t \geq 1)\right\} k_{t,T}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) \tag{S.88}$$

$$+ \frac{T^{\gamma_0 - \delta_0 + 1/2}}{\log T} \sum_{t=1}^{T} (\varepsilon_t(\widehat{\boldsymbol{\tau}}) - \phi\left(L; \boldsymbol{\varphi}_0\right) \left\{u_t \mathbb{I}(t \geq 1)\right\}) k_{t,T}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) \tag{S.89}$$

$$+ \frac{T^{\gamma_0 - \delta_0 + 1/2}}{\log T} \sum_{t=1}^{T} \phi\left(L; \boldsymbol{\varphi}_0\right) \left\{u_t \mathbb{I}(t \geq 1)\right\} \left(k_{t,T}(\widehat{\gamma}, \widehat{\delta}, \widehat{\boldsymbol{\varphi}}) - k_{t,T}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0)\right) \tag{S.90}$$

$$+ \frac{T^{\gamma_0 - \delta_0 + 1/2}}{\log T} \sum_{t=1}^{T} (\varepsilon_t(\widehat{\boldsymbol{\tau}}) - \phi\left(L; \boldsymbol{\varphi}_0\right) \left\{u_t \mathbb{I}(t \geq 1)\right\}) (k_{t,T}(\widehat{\gamma}, \widehat{\delta}, \widehat{\boldsymbol{\varphi}}) - k_{t,T}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0)). \tag{S.91}$$

Note that

$$\phi(L; \boldsymbol{\varphi}_0) \left\{u_t \mathbb{I}(t \geq 1)\right\} = \varepsilon_t - \sum_{j=t}^{\infty} \phi_j\left(\boldsymbol{\varphi}_0\right) u_{t-j}, \tag{S.92}$$

where, by Assumptions A1 and A2, it can be easily shown that

$$\sum_{j=t}^{\infty} \phi_j\left(\boldsymbol{\varphi}_0\right) u_{t-j} = O_p(t^{-1/2-\varsigma}). \tag{S.93}$$

Using summation by parts, (S.92), (S.93) and noting that, as in Lemmas S.14 and S.15,

$$k_{t,T}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0) = O_p(t^{\gamma_0-\delta_0} T^{-1-2(\gamma_0-\delta_0)}), \tag{S.94}$$

it follows that (S.88) is $O_p(\log^{-1} T)$. Next, by (S.94) and Lemma S.7 with $\varkappa = 1/2$ (because $\widehat{\boldsymbol{\tau}}$ is $T^{1/2}$-consistent), (S.89) is $O_p(T^{1/2-\varkappa} \log^{-1} T) = O_p(\log^{-1} T)$. Next, by summation by parts, (S.92), (S.93), the mean value theorem and Lemmas S.7 and S.16, it can be easily shown that (S.90) is $O_p(T^{\theta-1/2-(\gamma_0-\delta_0)}) = o_p(1)$, setting $\theta < \gamma_0 - \delta_0 + 1/2$. Finally, combining the arguments for (S.89) and (S.90), it is straightforward to show that (S.91) is $o_p(1)$, to conclude the proof of (S.85).

S.2.4 Proof of Theorem 2(ii): the $\gamma_0 + 1/2 < \delta_0$ case

First, noting (S.61), the loss function $R_T(\boldsymbol{\vartheta})$ can be decomposed in the sum of two terms, $R_T(\boldsymbol{\vartheta}) = Q_T(\boldsymbol{\tau}) + S_T(\boldsymbol{\vartheta})$, where $Q_T(\boldsymbol{\tau}) = T^{-1} \sum_{t=1}^{T} s_{1t}^2(\boldsymbol{\tau})$ and

$$S_T(\boldsymbol{\vartheta}) = \frac{1}{T} \sum_{t=1}^{T} \left(d_t(\boldsymbol{\vartheta}) - s_{2t}(\boldsymbol{\vartheta})\right)^2 + \frac{2}{T} \sum_{t=1}^{T} s_{1t}(\boldsymbol{\tau}) \left(d_t(\boldsymbol{\vartheta}) - s_{2t}(\boldsymbol{\vartheta})\right). \tag{S.95}$$

Thus, $Q_T(\boldsymbol{\tau})$ is the loss function in Hualde and Robinson (2011). Now

$$\frac{\partial R_T(\widehat{\boldsymbol{\vartheta}})}{\partial \boldsymbol{\tau}} = 0 = \frac{\partial Q_T(\widehat{\boldsymbol{\tau}})}{\partial \boldsymbol{\tau}} + \frac{\partial S_T(\widehat{\boldsymbol{\vartheta}})}{\partial \boldsymbol{\tau}}, \tag{S.96}$$

and by the mean value theorem

$$\frac{\partial Q_T(\widehat{\boldsymbol{\tau}})}{\partial \boldsymbol{\tau}} = \frac{\partial Q_T(\boldsymbol{\tau}_0)}{\partial \boldsymbol{\tau}} + \frac{\partial^2 Q_T(\overline{\boldsymbol{\tau}})}{\partial \boldsymbol{\tau} \partial \boldsymbol{\tau}'} \left(\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau}_0\right), \tag{S.97}$$

where $\overline{\boldsymbol{\tau}}$ represents an intermediate point between $\widehat{\boldsymbol{\tau}}$ and $\boldsymbol{\tau}_0$ which is allowed to vary in the different rows of $\partial^2 Q_T(\cdot)/\partial \boldsymbol{\tau} \partial \boldsymbol{\tau}'$. Inserting (S.97) in (S.96) we then find

$$T^{1/2}\left(\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau}_0\right) = -\left(\frac{\partial^2 Q_T(\overline{\boldsymbol{\tau}})}{\partial \boldsymbol{\tau} \partial \boldsymbol{\tau}'}\right)^{-1} T^{1/2} \frac{\partial Q_T(\boldsymbol{\tau}_0)}{\partial \boldsymbol{\tau}} - \left(\frac{\partial^2 Q_T(\overline{\boldsymbol{\tau}})}{\partial \boldsymbol{\tau} \partial \boldsymbol{\tau}'}\right)^{-1} T^{1/2} \frac{\partial S_T(\widehat{\boldsymbol{\vartheta}})}{\partial \boldsymbol{\tau}}. \tag{S.98}$$

Now, by Hualde and Robinson (2011) (see the proof of their Theorem 2.2), the first term on the right-hand side of (S.98) has a $N\left(0, \boldsymbol{A}^{-1}\right)$ limiting distribution, and $\partial^2 Q_T(\overline{\boldsymbol{\tau}})/\partial \boldsymbol{\tau} \partial \boldsymbol{\tau}'$ converges in probability to a nonsingular matrix. Thus, in view of (S.98), Theorem 2(ii) follows because $T^{1/2} \partial S_T(\widehat{\boldsymbol{\vartheta}})/\partial \boldsymbol{\tau} = o_p(1)$ by Lemma S.8.

S.2.5 Proof of Corollary 1

First, by very similar steps to those in the proof of Theorem 1 it is possible to show that $\widehat{\boldsymbol{\tau}}_\gamma \to_p \boldsymbol{\tau}_0$ as $T \to \infty$, when $\gamma_0 + 1/2 > \delta_0$ or $\gamma_0 + 1/2 < \delta_0$. Given this result, the proof for case (ii) is basically identical to that of Theorem 2(ii), so we focus on case (i). By the mean value theorem,

$$\widehat{\boldsymbol{\tau}}_\gamma - \boldsymbol{\tau}_0 = -\left(\frac{\partial^2 R_T(\overline{\boldsymbol{\tau}}, \gamma_0)}{\partial \boldsymbol{\tau} \partial \boldsymbol{\tau}'}\right)^{-1} \frac{\partial R_T(\boldsymbol{\tau}_0, \gamma_0)}{\partial \boldsymbol{\tau}},$$

where $\overline{\boldsymbol{\tau}}$ represents an intermediate point which is allowed to vary across the different rows of $\partial^2 R_T(\cdot)/\partial \boldsymbol{\tau} \partial \boldsymbol{\tau}'$. Then, following the proof of Theorem 2(i), it is immediate to show that $T^{1/2}(\widehat{\boldsymbol{\tau}}_\gamma - \boldsymbol{\tau}_0) \to_d N\left(\mathbf{0}_{p+1}, \boldsymbol{A}^{-1}\right)$.

Next, noting (12), we have the mean value theorem expansion

$$
\begin{aligned}
T^{\gamma_0-\delta_0+1/2}(\hat{\mu}_\gamma-\mu_0) &= T^{\gamma_0-\delta_0+1/2}\sum_{t=1}^{T}\varepsilon_t(\widehat{\boldsymbol{\tau}}_\gamma)k_{t,T}(\gamma_0,\widehat{\delta}_\gamma,\widehat{\boldsymbol{\varphi}}_\gamma) \\
&= T^{\gamma_0-\delta_0+1/2}\sum_{t=1}^{T}\varepsilon_t(\boldsymbol{\tau}_0)k_{t,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0) \\
&\quad + T^{\gamma_0-\delta_0+1/2}(\widehat{\boldsymbol{\tau}}_\gamma-\boldsymbol{\tau}_0)'\sum_{t=1}^{T}\frac{\partial}{\partial\boldsymbol{\tau}}\,\varepsilon_t(\boldsymbol{\tau})k_{t,T}(\gamma_0,\delta,\boldsymbol{\varphi})|_{\boldsymbol{\tau}=\overline{\boldsymbol{\tau}}}\,, \qquad \text{(S.99)}
\end{aligned}
$$

where $\overline{\boldsymbol{\tau}}$ represents again an intermediate value. Here, noting that $(\widehat{\boldsymbol{\tau}}_\gamma-\boldsymbol{\tau}_0)$ is $O_p(T^{-1/2})$, it can easily be shown that the second term on the right-hand side of (S.99) is $o_p(1)$ by the same arguments as for (S.85). Also, by the same central limit theorem as applied in the proof of Theorem 2(i), the first term on the right-hand side of (S.99) is asymptotically normal with mean zero and variance given by

$$
\begin{aligned}
\lim_{T\to\infty} T^{2(\gamma_0-\delta_0+1/2)}\sigma_0^2\sum_{t=1}^{T}k_{t,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0)^2 &= \lim_{T\to\infty}\frac{\sigma_0^2}{T^{-2(\gamma_0-\delta_0+1/2)}\sum_{t=1}^{T}c_t(\gamma_0,\delta_0,\boldsymbol{\varphi}_0)^2} \\
&= \frac{\sigma_0^2\Gamma^2\left(\gamma_0-\delta_0+1\right)2\left(\gamma_0-\delta_0+1/2\right)}{\phi^2\left(1;\boldsymbol{\varphi}_0\right)\Gamma^2\left(\gamma_0+1\right)}
\end{aligned}
$$

using, in turn, Lemmas S.15, S.13, S.11, and S.10.

Finally, the joint convergence in (18) can be immediately established by direct application of the Cramér-Wold device, noting that

$$
T^{1/2}\boldsymbol{P}_{\gamma,T}^{-1}\begin{pmatrix}\widehat{\boldsymbol{\tau}}_\gamma-\boldsymbol{\tau}_0\\ \widehat{\mu}_\gamma-\mu_0\end{pmatrix} = \begin{pmatrix}\frac{\partial^2 R_T(\boldsymbol{\tau}_0,\gamma_0)}{\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}'} & 0\\ 0 & 1\end{pmatrix}^{-1}\begin{pmatrix}T^{-\frac{1}{2}}\sum_{t=1}^{T}\varepsilon_t\sum_{j=1}^{\infty}\boldsymbol{m}_j\left(\boldsymbol{\varphi}_0\right)\varepsilon_{t-j}\\ T^{\gamma_0-\delta_0+1/2}\sum_{t=1}^{T}\varepsilon_t k_{t,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0)\end{pmatrix}+o_p\left(1\right),
$$

and

$$
T^{\gamma_0-\delta_0+1/2}\sum_{t=1}^{T}k_{t,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0)\sum_{j=1}^{\infty}\boldsymbol{m}_j\left(\boldsymbol{\varphi}_0\right)\varepsilon_{t-j} = o_p\left(1\right),
$$

as in the proof of (S.140).

S.3 Auxiliary lemmas

Lemma S.1 *Under Assumptions A1–A3,* $T^{-1}\sum_{t=1}^{T}s_t^2\left(\boldsymbol{\vartheta}_0\right)\to_p\sigma_0^2$.

Proof. Clearly

$$
\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}s_t^2\left(\boldsymbol{\vartheta}_0\right) =& \frac{1}{T}\sum_{t=1}^{T}\left(\phi(L;\boldsymbol{\varphi}_0)\left\{u_t\mathbb{I}(t\geq 1)\right\}\right)^2 \\
&-\frac{1}{T}\left(\sum_{t=1}^{T}\phi(L;\boldsymbol{\varphi}_0)\left\{u_t\mathbb{I}(t\geq 1)\right\}h_{t,T}\left(\gamma_0,\delta_0,\boldsymbol{\varphi}_0\right)\right)^2. \qquad \text{(S.100)}
\end{aligned}
$$

In view of (S.92), (S.93), by Assumptions A1 and A2 and simple application of Lemma S.16, the second term on the right-hand side of (S.100) is $O_p\left(T^{2\theta-1}\right)=o_p\left(1\right)$ by choosing $\theta<1/2$. Then the required result holds by (S.10). ■

Lemma S.2 *Under Assumptions A1 and A3, for any $g>0$,*

$$\lim_{T\to\infty}\inf_{\gamma_0-\delta\geq -1/2+g,|\gamma-\gamma_0|\geq g,\boldsymbol{\varphi}\in\Psi}\frac{1}{T^{2(\gamma_0-\delta)+1}}\sum_{t=1}^{T}d_t^2\left(\boldsymbol{\vartheta}\right)>\epsilon.$$

Proof. Letting $\alpha>0$ be arbitrarily small (in particular $\alpha<(\varsigma-1/2)/3$, which implies $\alpha<1/2$ and also $\alpha<g$) and defining $\Phi_1=\{\boldsymbol{\vartheta}\in\Xi:\gamma-\delta\leq -1/2-\alpha\}$, $\Phi_2=\{\boldsymbol{\vartheta}\in\Xi:-1/2-\alpha\leq\gamma-\delta\leq -1/2+\alpha\}$, and $\Phi_3=\{\boldsymbol{\vartheta}\in\Xi:\gamma-\delta\geq -1/2+\alpha\}$, the result holds on showing

$$\lim_{T\to\infty}\inf_{\{\gamma_0-\delta\geq -1/2+g,|\gamma-\gamma_0|\geq g\}\cap\Phi_j}\frac{1}{T^{2(\gamma_0-\delta)+1}}\sum_{t=1}^{T}d_t^2\left(\boldsymbol{\vartheta}\right)>\epsilon, \tag{S.101}$$

for $j=1,2,3$. We first deal with $j=1,2$. Clearly

$$\begin{aligned}\frac{1}{T^{2(\gamma_0-\delta)+1}}\sum_{t=1}^{T}d_t^2\left(\boldsymbol{\vartheta}\right)=&\frac{\mu_0^2}{T^{2(\gamma_0-\delta)+1}}\sum_{t=1}^{T}c_t^2\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)\\&-\frac{\mu_0^2}{T^{2(\gamma_0-\delta)+1}}\left(\sum_{t=1}^{T}c_t\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right)^2,\end{aligned}$$

so because $|\gamma-\gamma_0|\geq g$ and by application of Lemma S.15, noting that $\gamma_0-\delta\geq -1/2+g>-1/2$, $\mu_0\neq 0$, and (7), (S.101) for $j=1,2$ holds on showing

$$\lim_{T\to\infty}\inf_{\gamma_0-\delta\geq -1/2+g}\frac{1}{T^{2(\gamma_0-\delta)+1}}\sum_{t=1}^{T}b_t^2\left(\gamma_0,\delta\right)>\epsilon, \tag{S.102}$$

$$\sup_{\{\gamma_0-\delta\geq -1/2+g\}\cap\Phi_j}\frac{1}{T^{2(\gamma_0-\delta)+1}}\left(\sum_{t=1}^{T}c_t\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right)^2=o\left(1\right). \tag{S.103}$$

First, (S.102) follows by almost identical arguments to those in the proof of (S.198).

Next, we show (S.103) for $j=1$. By (S.197) of Lemma S.15, the left-hand side of (S.103) is bounded by

$$\sup_{\{\gamma_0-\delta\geq -1/2+g\}\cap\Phi_1}\left(T^{-1/2-(\gamma_0-\delta)}\sum_{t=1}^{T}c_t\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)c_t\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right)^2. \tag{S.104}$$

By Lemma S.14, (S.104) is $O\left(T^{-2\alpha}\right)=o\left(1\right)$ to conclude the proof of (S.103), and therefore that of (S.101), for $j=1$. Regarding $j=2$, the left-hand side of (S.103) is bounded by

$$\frac{\left(\sup_{\{\gamma_0-\delta\geq -1/2+g\}\cap\Phi_2}T^{-(\gamma_0-2\delta+\gamma+1)}\sum_{t=1}^{T}c_t\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)c_t\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right)^2}{\inf_{\Phi_2}T^{-2(\gamma-\delta)-1}\sum_{t=1}^{T}c_t^2\left(\gamma,\delta,\boldsymbol{\varphi}\right)}, \tag{S.105}$$

where the denominator can be made arbitrarily large by setting α close enough to zero, see (S.198) of Lemma S.15. By (S.193) of Lemma S.14 the square-root of the numerator of

(S.105) is $O(T^{-g+\alpha}\sum_{t=1}^{T}t^{g-1-\alpha})=O(1)$. This completes the proof of (S.103), and hence that of (S.101), for $j=2$.

Finally we show (S.101) for $j=3$. By very similar steps to those in the proofs of (S.195), (S.196) in Lemma S.15, noting that $\gamma_0-\delta\geq -1/2+g$, it is straightforward to show that

$$\frac{1}{T^{2(\gamma_0-\delta)+1}}\sum_{t=1}^{T}d_t^2(\boldsymbol{\vartheta})=\frac{\mu_0^2\phi^2(1;\boldsymbol{\varphi})}{T^{2(\gamma_0-\delta)+1}}\left(\sum_{t=1}^{T}b_t^2(\gamma_0,\delta)-\frac{\left(\sum_{t=1}^{T}b_t(\gamma_0,\delta)\,b_t(\gamma,\delta)\right)^2}{\sum_{t=1}^{T}b_t^2(\gamma,\delta)}\right)+q_{1T}(\gamma_0,\gamma,\delta,\boldsymbol{\varphi}),\tag{S.106}$$

where $\sup_{\{\gamma_0-\delta\geq-1/2+g\}\cap\Phi_3}|q_{1T}(\gamma_0,\gamma,\delta,\boldsymbol{\varphi})|=o(1)$. Next, by almost identical steps as in the proofs of (S.184) and (S.198), it can be easily shown that the first term on the right-hand side of (S.106) equals

$$\frac{\mu_0^2\phi^2(1;\boldsymbol{\varphi})\Gamma^2(\gamma_0+1)}{T^{2(\gamma_0-\delta)+1}}\left(\sum_{t=1}^{T}\pi_t^2(\gamma_0+1-\delta)-\frac{\left(\sum_{t=1}^{T}\pi_t(\gamma_0+1-\delta)\,\pi_t(\gamma+1-\delta)\right)^2}{\sum_{t=1}^{T}\pi_t^2(\gamma+1-\delta)}\right)\tag{S.107}$$

$$+q_{2T}(\gamma_0,\gamma,\delta,\boldsymbol{\varphi}),$$

where $\sup_{\{\gamma_0-\delta\geq-1/2+g\}\cap\Phi_3}|q_{2T}(\gamma_0,\gamma,\delta,\boldsymbol{\varphi})|=o(1)$. Approximating sums by integrals, by (7) and given that $\Gamma^2(\gamma_0+1)>0$, the first term on (S.107) is bounded from below by

$$\begin{aligned}&\epsilon\inf_{\{\gamma_0-\delta\geq-1/2+g\}\cap\Phi_3}\frac{1}{\Gamma^2(\gamma_0-\delta+1)}\left(\frac{1}{T^{2(\gamma_0-\delta)+1}}\sum_{t=1}^{T}t^{2(\gamma_0-\delta)}-\frac{\left(\frac{1}{T^{\gamma_0+\gamma-2\delta+1}}\sum_{t=1}^{T}t^{\gamma_0+\gamma-2\delta}\right)^2}{\frac{1}{T^{2(\gamma-\delta)+1}}\sum_{t=1}^{T}t^{2(\gamma-\delta)}}\right)\\&=\epsilon\inf_{\{\gamma_0-\delta\geq-1/2+g\}\cap\Phi_3}\frac{(\gamma_0-\gamma)^2}{\Gamma^2(\gamma_0-\delta+1)(2(\gamma_0-\delta)+1)(\gamma_0+\gamma-2\delta+1)^2}-o(1)\\&\geq\epsilon\inf_{\gamma_0-\delta\geq-1/2+g}\frac{g^2}{\Gamma^2(\gamma_0-\delta+1)\,2g\,(\alpha+g)^2}-o(1),\end{aligned}$$

which is positive and bounded away from zero, to complete the proof of (S.101) for $j=3$. ■

Lemma S.3 *Under Assumptions A1 and A3, for $i=1,\ldots,4$,*

$$\frac{1}{T^{\gamma_0-\delta+1}}\frac{\partial\overline{d}_T(\boldsymbol{\vartheta})}{\partial\gamma}=\frac{\mu_0\phi(1;\boldsymbol{\varphi})\Gamma(\gamma_0+1)}{\Gamma(\gamma_0-\delta+1)}\frac{2(\gamma-\delta)^2+2(\gamma-\delta)-(\gamma_0-\delta)}{(\gamma-\delta+1)^2(\gamma_0+\gamma-2\delta+1)^2}+g_T(\boldsymbol{\vartheta}),\tag{S.108}$$

where $\sup_{\overline{\mathcal{H}}_i}|g_T(\boldsymbol{\vartheta})|=o(1)$, and for an arbitrarily large K (setting ξ large enough),

$$\lim_{T\to\infty}\inf_{\overline{\mathcal{H}}_i}\frac{T^{2\varkappa_i}}{T^{2(\gamma_0-\delta)+2}}\overline{d}_T^2(\boldsymbol{\vartheta})>K.\tag{S.109}$$

Proof. First, $\partial \overline{d}_T(\boldsymbol{\vartheta})/\partial\gamma$ equals

$$-\frac{\mu_0 \sum_{t=1}^T c_t^{(1)}(\gamma,\delta,\boldsymbol{\varphi}) \sum_{j=1}^T c_j(\gamma_0,\delta,\boldsymbol{\varphi}) c_j(\gamma,\delta,\boldsymbol{\varphi})}{\sum_{j=1}^T c_j^2(\gamma,\delta,\boldsymbol{\varphi})}$$
$$-\frac{\mu_0 \sum_{t=1}^T c_t(\gamma,\delta,\boldsymbol{\varphi}) \sum_{j=1}^T c_j(\gamma_0,\delta,\boldsymbol{\varphi}) c_j^{(1)}(\gamma,\delta,\boldsymbol{\varphi})}{\sum_{j=1}^T c_j^2(\gamma,\delta,\boldsymbol{\varphi})}$$
$$+\frac{2\mu_0 \sum_{t=1}^T c_t(\gamma,\delta,\boldsymbol{\varphi}) \sum_{j=1}^T c_j(\gamma_0,\delta,\boldsymbol{\varphi}) c_j(\gamma,\delta,\boldsymbol{\varphi}) \sum_{k=1}^T c_k(\gamma,\delta,\boldsymbol{\varphi}) c_k^{(1)}(\gamma,\delta,\boldsymbol{\varphi})}{\left(\sum_{j=1}^T c_j^2(\gamma,\delta,\boldsymbol{\varphi})\right)^2}.$$

Noting that in $\cup_{i=1}^4 \overline{\mathcal{H}}_i$, $\gamma_0-\delta \geq \eta$ and $\gamma-\delta \geq \eta-\varrho$, proceeding as in the proof of Lemma S.15,

$$\begin{aligned}\frac{1}{T^{\gamma_0-\delta+1}}\frac{\partial \overline{d}_T(\gamma,\boldsymbol{\tau})}{\partial\gamma} = &-\frac{\mu_0\phi(1;\boldsymbol{\varphi})}{T^{\gamma_0-\delta+1}}\left(\frac{\sum_{t=1}^T b_t^{(1)}(\gamma,\delta)\sum_{j=1}^T b_j(\gamma_0,\delta) b_j(\gamma,\delta)}{\sum_{j=1}^T b_j^2(\gamma,\delta)}\right.\\ &+\frac{\sum_{t=1}^T b_t(\gamma,\delta)\sum_{j=1}^T b_j(\gamma_0,\delta) b_j^{(1)}(\gamma,\delta)}{\sum_{j=1}^T b_j^2(\gamma,\delta)}\\ &\left.-\frac{2\sum_{t=1}^T b_t(\gamma,\delta)\sum_{j=1}^T b_j(\gamma_0,\delta) b_j(\gamma,\delta)\sum_{k=1}^T b_k(\gamma,\delta) b_k^{(1)}(\gamma,\delta)}{(\sum_{j=1}^T b_j^2(\gamma,\delta))^2}\right)\\ &+g_{1T}(\boldsymbol{\vartheta}),\end{aligned} \tag{S.110}$$

where $\sup_{\overline{\mathcal{H}}_i}|g_{1T}(\boldsymbol{\vartheta})| = o(1)$. Now, substituting (S.185) into (S.110),

$$\begin{aligned}\frac{1}{T^{\gamma_0-\delta+1}}\frac{\partial \overline{d}_T(\gamma,\boldsymbol{\tau})}{\partial\gamma} = &-\frac{\mu_0\phi(1;\boldsymbol{\varphi})}{T^{\gamma_0-\delta+1}}\left(\frac{\sum_{t=1}^T \log(t/T) b_t(\gamma,\delta)\sum_{j=1}^T b_j(\gamma_0,\delta) b_j(\gamma,\delta)}{\sum_{j=1}^T b_j^2(\gamma,\delta)}\right.\\ &+\frac{\sum_{t=1}^T b_t(\gamma,\delta)\sum_{j=1}^T \log(j/T) b_j(\gamma_0,\delta) b_j(\gamma,\delta)}{\sum_{j=1}^T b_j^2(\gamma,\delta)}\\ &\left.-\frac{2\sum_{t=1}^T b_t(\gamma,\delta)\sum_{j=1}^T b_j(\gamma_0,\delta) b_j(\gamma,\delta)\sum_{k=1}^T \log(k/T) b_k^2(\gamma,\delta)}{(\sum_{j=1}^T b_j^2(\gamma,\delta))^2}\right)+g_{2T}(\boldsymbol{\vartheta}),\end{aligned} \tag{S.111}$$

where $\sup_{\overline{\mathcal{H}}_i}|g_{2T}(\boldsymbol{\vartheta})| = o(1)$, noting that the contribution of the second term on the right hand side of (S.185) and the $\log T$ terms cancel. Hence, using (S.184), it can be easily shown that

$$\begin{aligned}\frac{1}{T^{\gamma_0-\delta+1}}\frac{\partial \overline{d}_T(\gamma,\boldsymbol{\tau})}{\partial\gamma} = &-\frac{\mu_0\phi(1;\boldsymbol{\varphi})\Gamma(\gamma_0+1)}{T^{\gamma_0-\delta+1}\Gamma(\gamma_0+1-\delta)}\left(\frac{\sum_{t=1}^T \log(t/T) t^{\gamma-\delta}\sum_{j=1}^T j^{\gamma_0+\gamma-2\delta}}{\sum_{j=1}^T j^{2\gamma-2\delta}}\right.\\ &+\frac{\sum_{t=1}^T t^{\gamma-\delta}\sum_{j=1}^T \log(j/T) j^{\gamma_0+\gamma-2\delta}}{\sum_{j=1}^T j^{2\gamma-2\delta}}\\ &\left.-\frac{2\sum_{t=1}^T t^{\gamma-\delta}\sum_{j=1}^T j^{\gamma_0+\gamma-2\delta}\sum_{k=1}^T \log(k/T) k^{2\gamma-2\delta}}{(\sum_{j=1}^T j^{2\gamma-2\delta})^2}\right)+g_{3T}(\boldsymbol{\vartheta}),\end{aligned}$$

where $\sup_{\overline{\mathcal{H}}_i} |g_{3T}(\boldsymbol{\vartheta})| = o(1)$. Finally, (S.108) then follows by approximating sums by integrals, see Lemma S.10.

Next, because $\overline{d}_T(\gamma_0, \boldsymbol{\tau}) = 0$, the mean value theorem yields $\overline{d}_T(\gamma, \boldsymbol{\tau}) = (\gamma - \gamma_0)\,\partial \overline{d}_T(\overline{\gamma}, \boldsymbol{\tau})/\partial\gamma$, where $|\overline{\gamma} - \gamma_0| \leq |\gamma - \gamma_0|$, so the left-hand side of (S.109) can be bounded from below by

$$\begin{aligned}
&\lim_{T\to\infty} \inf_{\overline{\mathcal{H}}_i} T^{2\varkappa_i} (\gamma - \gamma_0)^2 \left(\frac{1}{T^{\gamma_0-\delta+1}} \frac{\partial \overline{d}_T(\overline{\gamma}, \boldsymbol{\tau})}{\partial\gamma} \right)^2 \\
&\geq \lim_{T\to\infty} \inf_{\overline{\mathcal{H}}_i} T^{2\varkappa_i} (\gamma - \gamma_0)^2 \inf_{\overline{\mathcal{H}}_i} \left(\frac{1}{T^{\gamma_0-\delta+1}} \frac{\partial \overline{d}_T(\gamma, \boldsymbol{\tau})}{\partial\gamma} \right)^2 \\
&= \xi^2 \lim_{T\to\infty} \inf_{\overline{\mathcal{H}}_i} \left(\frac{1}{T^{\gamma_0-\delta+1}} \frac{\partial \overline{d}_T(\gamma, \boldsymbol{\tau})}{\partial\gamma} \right)^2 .
\end{aligned}$$

Thus, setting ξ large enough, (S.109) follows if

$$\lim_{T\to\infty} \inf_{\overline{\mathcal{H}}_i} \left(\frac{1}{T^{\gamma_0-\delta+1}} \frac{\partial \overline{d}_T(\boldsymbol{\tau}, \gamma)}{\partial\gamma} \right)^2 > \epsilon, \tag{S.112}$$

which, noting that $\gamma_0 - \delta \geq \eta$, is a consequence of (7) and (S.108) because

$$\begin{aligned}
\inf_{\overline{\mathcal{H}}_i} \left(2(\gamma-\delta)^2 + 2(\gamma-\delta) - (\gamma_0 - \delta) \right) &= \inf_{\overline{\mathcal{H}}_i} \left(2(\gamma-\delta)^2 + (\gamma-\delta) - (\gamma_0 - \gamma) \right) \\
&\geq 2(\eta - \varrho)^2 + \eta - 2\varrho > 0.
\end{aligned}$$

■

Lemma S.4 *Under the conditions of Theorem 1(ii) it holds that*

$$\Pr\left(\inf_{\boldsymbol{\vartheta}\in\overline{M}_\varepsilon} R_T(\boldsymbol{\vartheta}) \leq \sigma_0^2 + \epsilon \right) \to 0 \textit{ as } T \to \infty.$$

Proof. Recall the intervals $\mathcal{I}_i$ and define $\mathcal{W}_i = \{\boldsymbol{\vartheta} \in \overline{M}_\varepsilon : \delta \in \mathcal{I}_i\}$ for $i = 1, 2, 3$, and $\mathcal{W}_4 = \{\boldsymbol{\vartheta} \in \overline{M}_\varepsilon : \delta \in \mathcal{I}_4 \cup \mathcal{I}_5\}$. Then the result follows on showing

$$\Pr\left(\inf_{\mathcal{W}_i} R_T(\boldsymbol{\vartheta}) \leq \sigma_0^2 + \epsilon \right) \to 0 \text{ as } T \to \infty \tag{S.113}$$

for $i = 1, \ldots, 4$, noting that

$$R_T(\boldsymbol{\vartheta}) = \frac{1}{T} \left(\sum_{t=1}^{T} \left(\phi(L; \boldsymbol{\varphi}) \Delta_+^\delta x_t \right)^2 - \left(\sum_{t=1}^{T} \phi(L; \boldsymbol{\varphi}) \Delta_+^\delta x_t h_{t,T}(\gamma, \delta, \boldsymbol{\varphi}) \right)^2 \right). \tag{S.114}$$

Proof of (S.113) for $i = 4$. Given (S.82), we first apply the bound

$$\frac{1}{T} \sum_{t=1}^{T} \left(\phi(L; \boldsymbol{\varphi}) \Delta_+^\delta x_t \right)^2 \geq \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t^2(\boldsymbol{\tau}) - \frac{2|\mu_0|}{T} \left| \sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) c_t(\gamma_0, \delta, \boldsymbol{\varphi}) \right| \tag{S.115}$$

and note that $\delta_0 - \delta \leq 1/2 - \eta$ when $\delta \in \mathcal{I}_4 \cup \mathcal{I}_5$, so $\Delta_+^{\delta-\delta_0} u_t$ is asymptotically stationary. In view of (S.114) and (S.115), the proof of (S.113) for $i = 4$ then follows by Hualde and Robinson (2011) (see the proof of their (2.7) for $i = 4$) by showing

$$\sup_{\mathcal{W}_4} \frac{1}{T} \left| \sum_{t=1}^{T} \varepsilon_t (\boldsymbol{\tau}) c_t (\gamma_0, \delta, \boldsymbol{\varphi}) \right| = o_p (1), \tag{S.116}$$

$$\sup_{\mathcal{W}_4} \frac{1}{T} \left(\sum_{t=1}^{T} \phi (L; \boldsymbol{\varphi}) \Delta_+^{\delta} x_t h_{t,T} (\gamma, \delta, \boldsymbol{\varphi}) \right)^2 = o_p (1). \tag{S.117}$$

First, noting that $\delta \in \mathcal{I}_4 \cup \mathcal{I}_5$ implies $\delta_0 - \delta \leq 1/2 - \eta$ and $\gamma_0 - \delta \leq 1/2 + \gamma_0 - \delta_0 - \eta$, (S.235) of Lemma S.21 implies that the left-hand side of (S.116) is $O_p(T^{\gamma_0-\delta_0+1/2-2\eta} + T^{-\varsigma-\eta} + T^{-1} \log T) = o_p (1)$.

Next, using (S.82), (S.117) follows by showing

$$\sup_{\mathcal{W}_4} T^{-1/2} \sum_{t=1}^{T} \varepsilon_t (\boldsymbol{\tau}) h_{t,T} (\gamma, \delta, \boldsymbol{\varphi}) = o_p (1), \tag{S.118}$$

$$\sup_{\mathcal{W}_4} T^{-1/2} \sum_{t=1}^{T} c_t (\gamma_0, \delta, \boldsymbol{\varphi}) h_{t,T} (\gamma, \delta, \boldsymbol{\varphi}) = o (1). \tag{S.119}$$

Here, (S.222) of Lemma S.18 shows that the left-hand side of (S.118) is $O_p(T^{\theta-1/2} + T^{-\eta}) = o_p (1)$ by choosing $\theta < 1/2$, while (S.232) of Lemma S.20 shows that the left-hand side of (S.119) $O(\left(T^{\theta-1/2} + T^{\gamma_0+1/2-\delta_0-\eta}\right) \log T) = o (1)$, to conclude the proof of (S.113) for $i = 4$.

Proof of (S.113) for $i = 3$. Noting (S.35), (S.114), and (S.115), the proof follows on showing

$$\sup_{\mathcal{W}_3} \frac{1}{T} \left| \sum_{t=1}^{T} \varepsilon_t (\boldsymbol{\tau}) c_t (\gamma_0, \delta, \boldsymbol{\varphi}) \right| = O_p (1), \tag{S.120}$$

$$\sup_{\mathcal{W}_3} \frac{1}{T} \left(\sum_{t=1}^{T} \phi (L; \boldsymbol{\varphi}) \Delta_+^{\delta} x_t h_{t,T} (\gamma, \delta, \boldsymbol{\varphi}) \right)^2 = O_p (1). \tag{S.121}$$

Both (S.120) and (S.121) follow straightforwardly by identical steps as those given in the proofs of (S.116) and (S.117) just replacing η by 0.

Proof of (S.113) for $i = 2$. Clearly,

$$\begin{aligned} \Pr \left(\inf_{\mathcal{W}_2} R_T (\boldsymbol{\vartheta}) \leq \sigma_0^2 + \epsilon \right) &\leq \Pr \left(\inf_{\mathcal{W}_2} \frac{T^{2(\delta_0-\delta)}}{T} \inf_{\mathcal{W}_2} \frac{T}{T^{2(\delta_0-\delta)}} R_T (\boldsymbol{\vartheta}) \leq \sigma_0^2 + \epsilon \right) \\ &= \Pr \left(\inf_{\mathcal{W}_2} \frac{T}{T^{2(\delta_0-\delta)}} R_T (\boldsymbol{\vartheta}) \leq \sigma_0^2 + \epsilon \right). \end{aligned} \tag{S.122}$$

Thus, in view of (S.43), (S.114), and (S.115), (S.113) for $i = 2$ follows on showing

$$\sup_{\mathcal{W}_2} \frac{1}{T^{2(\delta_0-\delta)}} \left| \sum_{t=1}^{T} \varepsilon_t (\boldsymbol{\tau}) c_t (\gamma_0, \delta, \boldsymbol{\varphi}) \right| = O_p (1), \tag{S.123}$$

$$\sup_{\mathcal{W}_2} \frac{1}{T^{2(\delta_0-\delta)}} \left(\sum_{t=1}^{T} \phi (L; \boldsymbol{\varphi}) \Delta_+^{\delta} x_t h_{t,T} (\gamma, \delta, \boldsymbol{\varphi}) \right)^2 = O_p (1). \tag{S.124}$$

The proofs of (S.123) and (S.124) are almost identical to those of (S.116) and (S.117), taking into account the different normalization, which implies using (S.233) instead of (S.232) in Lemma S.20, (S.236) instead of (S.235) in Lemma S.21, and (S.223) instead of (S.222) in Lemma S.18.

Proof of (S.113) for $i = 1$. Following identical steps to those given in (S.122),

$$\Pr\left(\inf_{\mathcal{W}_1} R_T(\boldsymbol{\vartheta}) \leq \sigma_0^2 + \epsilon\right) \leq \Pr\left(\inf_{\mathcal{W}_1} \frac{T}{T^{2(\delta_0-\delta)}} R_T(\boldsymbol{\vartheta}) \leq \frac{\sigma_0^2 + \epsilon}{T^{2\eta}}\right).$$

Letting $\alpha > 0$ be arbitrarily small (in particular $\alpha < (\varsigma - 1/2)/3$) and defining the sets

$$\Phi_1 = \{\boldsymbol{\vartheta} \in \Xi : \gamma - \delta \leq -1/2 - \alpha\}, \quad \Phi_2 = \{\boldsymbol{\vartheta} \in \Xi : -1/2 - \alpha \leq \gamma - \delta \leq -1/2 + \alpha\},$$
$$\Phi_3 = \{\boldsymbol{\vartheta} \in \Xi : \gamma - \delta \geq -1/2 + \alpha\},$$

the required result follows on showing

$$\Pr\left(\inf_{\mathcal{W}_1 \cap \Phi_j} \frac{T}{T^{2(\delta_0-\delta)}} R_T(\boldsymbol{\vartheta}) > \epsilon\right) \to 1 \text{ as } T \to \infty \tag{S.125}$$

for $j = 1, 2, 3$ and $\epsilon > 0$ arbitrarily small. In the proof of their (2.7) for $i = 1$, Hualde and Robinson (2011) showed that

$$\Pr\left(\inf_{\|\boldsymbol{\tau}-\boldsymbol{\tau}_0\| \geq \varepsilon, \boldsymbol{\tau} \in \mathcal{T}_1} \frac{1}{T^{2(\delta_0-\delta)}} \sum_{t=1}^{T} \varepsilon_t^2(\boldsymbol{\tau}) > \epsilon\right) \to 1 \text{ as } T \to \infty, \tag{S.126}$$

although their proof based on the Cauchy-Schwarz inequality does not appear sufficient; see the discussion just below (S.52). We shortly prove (S.126), so in view of (S.82), (S.114), and (S.115), (S.125) for $j = 1, 2$ holds if we also show that

$$\sup_{\mathcal{W}_1} \frac{1}{T^{2(\delta_0-\delta)}} \sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) c_t(\gamma_0, \delta, \boldsymbol{\varphi}) = o_p(1), \tag{S.127}$$

$$\sup_{\mathcal{W}_1} \frac{1}{T^{\delta_0-\delta}} \sum_{t=1}^{T} c_t(\gamma_0, \delta, \boldsymbol{\varphi}) h_{t,T}(\gamma, \delta, \boldsymbol{\varphi}) = o(1), \tag{S.128}$$

$$\sup_{\mathcal{W}_1 \cap \Phi_j} \frac{1}{T^{2(\delta_0-\delta)}} \left(\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) h_{t,T}(\gamma, \delta, \boldsymbol{\varphi})\right)^2 = o_p(1). \tag{S.129}$$

First, noting that $\delta_0 - \delta \geq 1/2 + \eta$ and $\gamma_0 - \delta \geq 1/2 + \eta + \gamma_0 - \delta_0$, by (S.236) of Lemma S.21 and (S.233) of Lemma S.20 with $\theta < 1/2 + \eta$, the left-hand sides of (S.127) and (S.128) are $O_p(T^{\max\{\gamma_0-\delta_0+1/2, -\varsigma-\eta\}} + T^{-1-2\eta} \log T) = o_p(1)$ and $O(T^{\gamma_0-\delta_0+1/2} + T^{-1/2-\eta+\theta} \log T) = o(1)$, respectively. Next, for $j = 1$, by (S.197) of Lemma S.15, the left-hand side of (S.129) is

$$\sup_{\mathcal{W}_1 \cap \Phi_1} T^{2(\delta-\delta_0)} \left(\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) c_t(\gamma, \delta, \boldsymbol{\varphi})\right)^2,$$

which is easily shown to be $O_p(T^{-2\alpha}) = o_p(1)$ by (S.192) of Lemma S.14 and (S.219) of Lemma S.17. For $j = 2$, we use (S.3) to bound the left-hand side of (S.129) by

$$\frac{\sup_{\mathcal{W}_1 \cap \Phi_2} T^{-1} \left(\sum_{t=1}^{T} T^{-(\delta_0-\delta)} \varepsilon_t(\boldsymbol{\tau}) T^{-(\gamma-\delta)} c_t(\gamma, \delta, \boldsymbol{\varphi})\right)^2}{\inf_{\mathcal{W}_1 \cap \Phi_2} T^{-2(\gamma-\delta)-1} \sum_{j=1}^{T} c_j^2(\gamma, \delta, \boldsymbol{\varphi})},$$

where the denominator can be made arbitrarily large by (S.198) of Lemma S.15 and the numerator is easily seen to be $O_p(1)$ by direct application of (S.193) of Lemma S.14 and (S.219) of Lemma S.17.

Thus, to prove (S.125) for $j = 1, 2$ it only remains to prove (S.126). By application of the bound (S.181) in Lemma S.12,

$$\frac{1}{T^{2(\delta_0-\delta)}}\sum_{t=1}^{T}\varepsilon_t^2(\boldsymbol{\tau}) \geq \left(1+O(T^{-1})\right)\frac{\pi^2/4}{T^{2(\delta_0-\delta)+2}}\sum_{t=1}^{T}\varepsilon_t^2(\delta_0-\delta+1,\boldsymbol{\varphi}),$$

where $\varepsilon_t(\delta_0-\delta+1,\boldsymbol{\varphi}) = \sum_{j=1}^{t}\varepsilon_t(\boldsymbol{\tau})$ is defined in (S.27) and the $O(T^{-1})$ term does not depend on the parameters. Proceeding as in the proof of (S.52), we obtain the weak convergence

$$T^{\delta-\delta_0-1/2}\varepsilon_{[Tr]}(\delta-1,\boldsymbol{\varphi}) \Rightarrow \phi(1;\boldsymbol{\varphi})\,\omega(1;\boldsymbol{\varphi}_0)\,W(r;1+\delta_0-\delta), \tag{S.130}$$

so by the continuous mapping theorem,

$$\frac{\pi^2/4}{T^{2(\delta_0-\delta)+2}}\sum_{t=1}^{T}\varepsilon_t^2(\delta-1,\boldsymbol{\varphi}) \Rightarrow \frac{\pi^2}{4}\phi^2(1;\boldsymbol{\varphi})\,\omega^2(1;\boldsymbol{\varphi}_0)\int_0^1 W(r;1+\delta_0-\delta)^2\,dr.$$

It follows that the probability in (S.126) is bounded from below by

$$\begin{aligned}&\Pr\left(\left(1+O(T^{-1})\right)\inf_{\|\boldsymbol{\tau}-\boldsymbol{\tau}_0\|\geq\varepsilon,\boldsymbol{\tau}\in\mathcal{T}_1}\frac{\pi^2/4}{T^{2(\delta_0-\delta)+2}}\sum_{t=1}^{T}\varepsilon_t^2(\delta-1,\boldsymbol{\varphi}) > \epsilon\right)\\ &\quad\to \Pr\left(\inf_{\|\boldsymbol{\tau}-\boldsymbol{\tau}_0\|\geq\varepsilon,\boldsymbol{\tau}\in\mathcal{T}_1}\frac{\pi^2}{4}\phi^2(1;\boldsymbol{\varphi})\,\omega^2(1;\boldsymbol{\varphi}_0)\int_0^1 W(r;1+\delta_0-\delta)^2\,dr > \epsilon\right),\end{aligned}$$

and (S.126) follows because $\epsilon > 0$ is arbitrarily small. For additional details, see Lemma 3 of Johansen and Nielsen (2018) for the same argument.

We finally show (S.125) for $j = 3$. Using (S.15), applying (S.127) and (S.128) together with Lemmas S.16, S.17, we have

$$T^{1-2(\delta_0-\delta)}R_T(\boldsymbol{\vartheta}) \geq T^{-2(\delta_0-\delta)}\sum_{t=1}^{T}s_t^2(\boldsymbol{\vartheta}) + q_{1T}(\boldsymbol{\vartheta}),$$

where $\sup_{\mathcal{W}_1\cap\Phi_3}|q_{1T}(\boldsymbol{\vartheta})| = o_p(1)$. Thus, (S.125) for $j = 3$ holds on showing

$$\Pr\left(\inf_{\mathcal{W}_1\cap\Phi_3}\frac{1}{T^{2(\delta_0-\delta)}}\sum_{t=1}^{T}s_t^2(\boldsymbol{\vartheta}) > \epsilon\right) \to 1 \text{ as } T\to\infty. \tag{S.131}$$

The proof of (S.131) is essentially identical to the main steps of the proof of (S.51). The only relevant difference is that we now need to establish a convergence result on a larger set where $\gamma-\delta \geq -1/2+\alpha$ (instead of $\gamma-\delta\geq\alpha$). However, this does not lead to any relevant changes in the proof because for fixed $\boldsymbol{\vartheta}$ such that $\gamma-\delta\geq -1/2+\alpha$ and $1+\delta_0-\delta \geq 3/2+\eta$ with $\alpha>0, \eta>0$, the integral $\int_0^1 u^{\gamma-\delta-1}W(u;1+\delta_0-\delta)du$ is well defined. This completes the proof of (S.125) for $j=3$ and therefore that of (S.113) for $i=1$. ■

Lemma S.5 *Under the conditions of Theorem 2(i) it holds that:*
(a) The first-order derivatives satisfy

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T}\left(s_t\left(\boldsymbol{\vartheta}_0\right)-\varepsilon_t\right)\frac{\partial s_{1t}\left(\boldsymbol{\tau}_0\right)}{\partial\boldsymbol{\tau}}=o_p\left(1\right),\ \frac{1}{T^{\gamma_0-\delta_0+1/2}}\sum_{t=1}^{T}\left(s_t\left(\boldsymbol{\vartheta}_0\right)-\varepsilon_t\right)\frac{\partial d_t\left(\boldsymbol{\vartheta}_0\right)}{\partial\gamma}=o_p\left(1\right), \tag{S.132}$$

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T}s_t\left(\boldsymbol{\vartheta}_0\right)\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\boldsymbol{\tau}}=o_p\left(1\right),\ \frac{1}{T^{\gamma_0-\delta_0+1/2}}\sum_{t=1}^{T}s_t\left(\boldsymbol{\vartheta}_0\right)\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\gamma}=o_p\left(1\right). \tag{S.133}$$

(b) The second-order derivatives satisfy

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\partial s_{1t}\left(\boldsymbol{\tau}_0\right)}{\partial\boldsymbol{\tau}}\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\boldsymbol{\tau}'}=o_p\left(1\right),\ \frac{1}{T^{\gamma_0-\delta_0+1}}\sum_{t=1}^{T}\frac{\partial s_{1t}\left(\boldsymbol{\tau}_0\right)}{\partial\boldsymbol{\tau}}\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\gamma}=o_p\left(1\right),$$

$$\frac{1}{T^{\gamma_0-\delta_0+1}}\sum_{t=1}^{T}\frac{\partial s_{1t}\left(\boldsymbol{\tau}_0\right)}{\partial\boldsymbol{\tau}}\frac{\partial d_t\left(\boldsymbol{\vartheta}_0\right)}{\partial\gamma}=o_p\left(1\right),\ \frac{1}{T}\sum_{t=1}^{T}\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\boldsymbol{\tau}}\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\boldsymbol{\tau}'}=o_p\left(1\right),$$

$$\frac{1}{T^{\gamma_0-\delta_0+1}}\sum_{t=1}^{T}\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\boldsymbol{\tau}}\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\gamma}=o_p\left(1\right),\ \frac{1}{T^{\gamma_0-\delta_0+1}}\sum_{t=1}^{T}\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\boldsymbol{\tau}}\frac{\partial d_t\left(\boldsymbol{\vartheta}_0\right)}{\partial\gamma}=o_p\left(1\right),$$

$$\frac{1}{T^{2(\gamma_0-\delta_0)+1}}\sum_{t=1}^{T}\left(\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\gamma}\right)^2=o_p\left(1\right),\ \frac{1}{T^{2(\gamma_0-\delta_0)+1}}\sum_{t=1}^{T}\frac{\partial s_{2t}\left(\boldsymbol{\vartheta}_0\right)}{\partial\gamma}\frac{\partial d_t\left(\boldsymbol{\vartheta}_0\right)}{\partial\gamma}=o_p\left(1\right),$$

$$\frac{1}{T}\sum_{t=1}^{T}s_t(\boldsymbol{\vartheta}_0)\frac{\partial^2 s_t(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}'}=o_p\left(1\right),\ \frac{1}{T^{\gamma_0-\delta_0+1}}\sum_{t=1}^{T}s_t(\boldsymbol{\vartheta}_0)\frac{\partial^2 s_{2t}(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\tau}\partial\gamma}=o_p\left(1\right),$$

$$\frac{1}{T^{2(\gamma_0-\delta_0)+1}}\sum_{t=1}^{T}s_t(\boldsymbol{\vartheta}_0)\frac{\partial^2 s_{2t}(\boldsymbol{\vartheta}_0)}{\partial\gamma^2}=o_p\left(1\right),\ \frac{1}{T^{\gamma_0-\delta_0+1}}\sum_{t=1}^{T}s_t(\boldsymbol{\vartheta}_0)\frac{\partial^2 d_t(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\tau}\partial\gamma}=o_p\left(1\right),$$

$$\frac{1}{T^{2(\gamma_0-\delta_0)+1}}\sum_{t=1}^{T}s_t(\boldsymbol{\vartheta}_0)\frac{\partial^2 d_t(\boldsymbol{\vartheta}_0)}{\partial\gamma^2}=o_p\left(1\right).$$

Proof. First we show the first equality in (S.132). By definition of $s_t\left(\boldsymbol{\vartheta}\right)$,

$$s_t\left(\boldsymbol{\vartheta}_0\right)-\varepsilon_t=-\sum_{j=t}^{\infty}\phi_j(\boldsymbol{\varphi}_0)u_{t-j}-h_{t,T}\left(\gamma_0,\delta_0,\boldsymbol{\varphi}_0\right)\sum_{j=1}^{T}s_{1j}\left(\boldsymbol{\tau}_0\right)h_{j,T}\left(\gamma_0,\delta_0,\boldsymbol{\varphi}_0\right), \tag{S.134}$$

where $s_{1j}\left(\boldsymbol{\tau}_0\right)=\varepsilon_j-\sum_{k=j}^{\infty}\phi_k(\boldsymbol{\varphi}_0)u_{j-k}$, so the result holds if

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T}\sum_{j=t}^{\infty}\phi_j(\boldsymbol{\varphi}_0)u_{t-j}\frac{\partial s_{1t}\left(\boldsymbol{\tau}_0\right)}{\partial\boldsymbol{\tau}}=o_p\left(1\right), \tag{S.135}$$

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T}h_{t,T}\left(\gamma_0,\delta_0,\boldsymbol{\varphi}_0\right)\frac{\partial s_{1t}\left(\boldsymbol{\tau}_0\right)}{\partial\boldsymbol{\tau}}\sum_{j=1}^{T}s_{1j}\left(\boldsymbol{\tau}_0\right)h_{j,T}\left(\gamma_0,\delta_0,\boldsymbol{\varphi}_0\right)=o_p\left(1\right). \tag{S.136}$$

First, for $t \geq 2$,

$$\frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial \delta} = -\sum_{j=1}^{t-1}\sum_{k=0}^{j-1}\phi_k(\boldsymbol{\varphi}_0)(j-k)^{-1}u_{t-j}, \tag{S.137}$$

so, noting (6) and applying Lemma S.9,

$$E\left|\frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial \delta}\right| \leq K\sum_{j=1}^{t-1}\sum_{k=1}^{j-1}k^{-1-\varsigma}(j-k)^{-1} \leq K\sum_{j=1}^{t-1}j^{-1}\log j \leq K\log^2 t.$$

Similarly, by (14),

$$E\left\|\frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial \boldsymbol{\varphi}}\right\| = E\left\|\sum_{j=1}^{t-1}\frac{\partial \phi_j(\boldsymbol{\varphi}_0)}{\partial \boldsymbol{\varphi}}u_{t-j}\right\| = O(1), \tag{S.138}$$

so that

$$E\left\|\frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial \boldsymbol{\tau}}\right\| = O\left(\log^2 t\right).$$

Thus, noting (S.93),

$$E\left\|\frac{1}{T^{1/2}}\sum_{t=1}^{T}\sum_{j=t}^{\infty}\phi_j(\boldsymbol{\varphi}_0)u_{t-j}\frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial \boldsymbol{\tau}}\right\| \leq \frac{K}{T^{1/2}}\sum_{t=1}^{T}t^{-1/2-\varsigma}\log^2 t \leq KT^{-1/2} = o(1)$$

because $\varsigma > 1/2$, which proves (S.135). Next, by (S.93) and (S.209) of Lemma S.16, it is straightforward to show that

$$\begin{aligned}&\sum_{j=1}^{T}s_{1j}(\boldsymbol{\tau}_0)\,h_{j,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0)\\ &\quad = \sum_{j=1}^{T}(\varepsilon_j - \sum_{k=j}^{\infty}\phi_k(\boldsymbol{\varphi}_0)u_{j-k})h_{j,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0) = O_p(T^{\theta}),\end{aligned} \tag{S.139}$$

so (S.136) holds on showing that

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T}h_{t,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0)\frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial \boldsymbol{\tau}} = O_p\left(T^{\theta-1/2}\log T\right) \tag{S.140}$$

and setting $\theta < 1/4$. Noting (S.137), by simple calculations,

$$\frac{\partial s_{1t}(\boldsymbol{\tau}_0)}{\partial \delta} = -\sum_{j=1}^{t-1}\frac{1}{j}\varepsilon_{t-j} + \sum_{j=1}^{t-1}\frac{1}{j}\sum_{k=t-j}^{\infty}\phi_k(\boldsymbol{\varphi}_0)u_{t-j-k}. \tag{S.141}$$

The contribution of the first term on the right-hand side of (S.141) to the left-hand side of (S.140) is, by (S.209) of Lemma S.16,

$$\begin{aligned}-\frac{1}{T^{1/2}}\sum_{t=2}^{T}\sum_{j=1}^{t-1}j^{-1}\varepsilon_{t-j}h_{t,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0) &= -\frac{1}{T^{1/2}}\sum_{t=1}^{T}\varepsilon_t\sum_{j=1}^{T-t}j^{-1}h_{t+j,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0)\\ &= O_p\left(\frac{T^{\theta}}{T^{1/2}}\left(\sum_{t=1}^{T}\left(\sum_{j=1}^{T-t}\frac{1}{j}(t+j)^{-1/2-\theta}\right)^2\right)^{1/2}\right)\\ &= O_p(T^{\theta-1/2}\log T)\end{aligned} \tag{S.142}$$

because

$$\sum_{j=1}^{T-t} j^{-1}(t+j)^{-1/2-\theta} \leq t^{-1/2-\theta}\sum_{j=1}^{T} j^{-1} \leq Kt^{-1/2-\theta}\log T.$$

Similarly, the contribution of the second term on the right-hand side of (S.141) to the left-hand side of (S.140) is

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T} h_{t,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0)\sum_{j=1}^{t-1} j^{-1}\sum_{k=t-j}^{\infty}\phi_k(\boldsymbol{\varphi}_0)u_{t-j-k},$$

which can be easily shown to be $O_p\left(T^{\theta-1/2}\log T\right)$ by (S.93), Lemma S.9 and (S.209) of Lemma S.16. Next, the contribution of $\partial s_{1t}(\boldsymbol{\tau}_0)/\partial\boldsymbol{\varphi}$ to the left-hand side of (S.140) is

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T} u_t\sum_{j=1}^{T-t}\frac{\partial\phi_j(\boldsymbol{\varphi}_0)}{\partial\boldsymbol{\varphi}}h_{t+j,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0),$$

which, by very similar arguments to (S.142), can easily be shown to be $O_p\left(T^{\theta-1/2}\right)$ by (14) and (S.209) of Lemma S.16, to conclude the proof of (S.140) and hence of the first equality in (S.132).

Next, because $\sum_{t=1}^{T} c_t(\gamma_0,\delta_0,\boldsymbol{\varphi}_0)\,\partial d_t(\boldsymbol{\vartheta}_0)/\partial\gamma = 0$, the proof of the second equality in (S.132) follows by showing that

$$\frac{1}{T^{\gamma_0-\delta_0+1/2}}\sum_{t=1}^{T}\sum_{j=t}^{\infty}\phi_j(\boldsymbol{\varphi}_0)u_{t-j}\frac{\partial d_t(\boldsymbol{\vartheta}_0)}{\partial\gamma} = o_p(1),$$

which, noting the proof of (S.69) and (S.93), follows easily by previous arguments.

The proofs of the two equalities in (S.133) are almost identical, but the second is simpler, so we show only the first. By (S.134), the first equality in (S.133) holds if

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T} s_{1t}(\boldsymbol{\tau}_0)\frac{\partial s_{2t}(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\tau}} = o_p(1), \tag{S.143}$$

$$\frac{1}{T^{1/2}}\sum_{t=1}^{T} h_{t,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0)\frac{\partial s_{2t}(\boldsymbol{\vartheta}_0)}{\partial\boldsymbol{\tau}}\sum_{j=1}^{T} s_{1j}(\boldsymbol{\tau}_0)h_{j,T}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0) = o_p(1). \tag{S.144}$$

As defined before, $s_{2t}(\boldsymbol{\vartheta}) = h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\sum_{j=1}^{T} s_{1j}(\boldsymbol{\tau})h_{j,T}(\gamma,\delta,\boldsymbol{\varphi})$ so that

$$\begin{aligned}\frac{\partial s_{2t}(\boldsymbol{\vartheta})}{\partial\boldsymbol{\tau}} =& \frac{\partial h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})}{\partial\boldsymbol{\tau}}\sum_{j=1}^{T} s_{1j}(\boldsymbol{\tau})h_{j,T}(\gamma,\delta,\boldsymbol{\varphi}) \\ &+ h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\sum_{j=1}^{T}\frac{\partial s_{1j}(\boldsymbol{\tau})}{\partial\boldsymbol{\tau}}h_{j,T}(\gamma,\delta,\boldsymbol{\varphi}) \\ &+ h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\sum_{j=1}^{T} s_{1j}(\boldsymbol{\tau})\frac{\partial h_{j,T}(\gamma,\delta,\boldsymbol{\varphi})}{\partial\boldsymbol{\tau}}.\end{aligned} \tag{S.145}$$

First, given that $\gamma_0 + 1/2 > \delta_0$, setting $\theta < \gamma_0 - \delta_0 + 1/2$, by a simple modification of the proof of (S.210) of Lemma S.16,

$$\left\| \frac{\partial h_{t,T}(\gamma_0, \delta_0, \boldsymbol{\varphi}_0)}{\partial \boldsymbol{\tau}} \right\| = O\left(t^{-1/2} \left(\frac{T}{t} \right)^{\theta} \log T \right). \tag{S.146}$$

Then noting (S.139), (S.140), and by application of Lemma S.16, it follows that

$$\frac{\partial s_{2t}(\boldsymbol{\vartheta}_0)}{\partial \boldsymbol{\tau}} = O_p(t^{-1/2-\theta} T^{2\theta} \log T). \tag{S.147}$$

By (S.93) and (S.147), it follows that the left-hand side of (S.143) is $O_p(T^{2\theta-1/2} \log T) = o_p(1)$ by setting $\theta < 1/4$. Similarly, by (S.139), (S.147) and (S.209) of Lemma S.16, the left-hand side of (S.144) is $O_p(T^{4\theta-1/2} \log T) = o_p(1)$ by setting $\theta < 1/8$. This concludes the proof of the first equality in (S.133).

Finally, the proofs for the results in part (b) are heavily based on the arguments employed in the proofs of (S.132) and (S.133), and are therefore omitted. ■

Lemma S.6 *Under the conditions of Theorem 2(i), for some fixed* $\varkappa > 0$, $T^{\varkappa}(\widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0) \to_p 0$.

Proof. As in the proof of Theorem 1(i), noting (S.1), (S.2), (S.4), (S.62), the result holds on establishing that

$$\Pr\left(\inf_{\boldsymbol{\vartheta} \in \overline{M}^*_{\varepsilon}} S_T(\boldsymbol{\vartheta}) \le 0 \right) \to 0 \text{ as } T \to \infty, \tag{S.148}$$

$$\Pr\left(\widehat{\boldsymbol{\vartheta}} \in \overline{N}^*_{\varepsilon} \cap M^*_{\varepsilon}, \inf_{\overline{N}^*_{\varepsilon} \cap M^*_{\varepsilon}} R_T(\widehat{\boldsymbol{\tau}}, \gamma) - R_T(\widehat{\boldsymbol{\tau}}, \gamma_0) \le 0 \right) \to 0 \text{ as } T \to \infty, \tag{S.149}$$

where

$$M^*_{\varepsilon} = \left\{ \boldsymbol{\vartheta} \in \Xi : \|\boldsymbol{\tau} - \boldsymbol{\tau}_0\| < \varepsilon T^{-\varkappa} \right\}, \quad \overline{M}^*_{\varepsilon} = \{ \boldsymbol{\vartheta} \in \Xi : \varepsilon T^{-\varkappa} \le \|\boldsymbol{\tau} - \boldsymbol{\tau}_0\| < \varepsilon \},$$
$$N^*_{\varepsilon} = \left\{ \boldsymbol{\vartheta} \in \Xi : |\gamma - \gamma_0| < \varepsilon T^{-\varkappa} \right\}, \quad \overline{N}^*_{\varepsilon} = \{ \boldsymbol{\vartheta} \in \Xi : \varepsilon T^{-\varkappa} \le |\gamma - \gamma_0| < \varepsilon \}.$$

We first prove (S.148), which, defining $\mathcal{J}_i = \{ \boldsymbol{\vartheta} \in \overline{M}^*_{\varepsilon} : \delta \in \mathcal{I}_i \}$, $i = 4, 5$, holds if

$$\Pr\left(\inf_{\mathcal{J}_i} S_T(\boldsymbol{\vartheta}) \le 0 \right) \to 0 \text{ as } T \to \infty \tag{S.150}$$

for $i = 4, 5$. Note here that $\boldsymbol{\vartheta} \in \overline{M}^*_{\varepsilon}$ implies $\|\boldsymbol{\tau} - \boldsymbol{\tau}_0\| < \varepsilon$, so necessarily $\delta \in \mathcal{I}_4 \cup \mathcal{I}_5$ and there is no need to consider the intervals $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$. Clearly, (S.150) for $i = 5$ would hold if

$$\Pr\left(\inf_{\mathcal{J}_5} T^{2\varkappa} S_T(\boldsymbol{\vartheta}) \le 0 \right) \to 0 \text{ as } T \to \infty. \tag{S.151}$$

Proceeding as in the proof of (S.5)–(S.7) for $i = 5$, (S.151) holds if

$$\inf_{\mathcal{J}_5} T^{2\varkappa} U(\boldsymbol{\tau}) > \epsilon, \tag{S.152}$$

$$\frac{1}{T^{1-2\varkappa}} \sum_{t=1}^{T} \left((\phi(L;\boldsymbol{\varphi}_0)\{u_t \mathbb{I}(t \geq 1)\})^2 - \sigma_0^2 \right) = o_p(1), \tag{S.153}$$

$$\sup_{\mathcal{J}_5} \frac{1}{T^{1-2\varkappa}} \sum_{t=1}^{T} \left(\varepsilon_t^2(\boldsymbol{\tau}) - E\left(\left(\phi(L;\boldsymbol{\varphi})\Delta^{\delta-\delta_0} u_t \right)^2 \right) \right) = o_p(1), \tag{S.154}$$

$$\sup_{\mathcal{J}_5} \frac{1}{T^{1-2\varkappa}} \left| \sum_{t=1}^{T} d_t(\boldsymbol{\vartheta}) s_t(\boldsymbol{\vartheta}) \right| = o_p(1), \tag{S.155}$$

$$\sup_{\mathcal{J}_5} \frac{1}{T^{1-2\varkappa}} \left(\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) h_{t,T}(\gamma,\delta,\boldsymbol{\varphi}) \right)^2 = o_p(1). \tag{S.156}$$

First, we justify (S.152). Clearly

$$U(\boldsymbol{\tau}) = \sigma_0^2 \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\left|\phi\left(e^{i\lambda};\boldsymbol{\varphi}\right)\right|^2}{\left|\phi\left(e^{i\lambda};\boldsymbol{\varphi}_0\right)\right|^2} \left|1 - e^{i\lambda}\right|^{2(\delta-\delta_0)} d\lambda - 1 \right),$$

and we show that $U(\boldsymbol{\tau})$ is a strictly convex function at $\boldsymbol{\tau}_0$ with a strict local minimum at $\boldsymbol{\tau} = \boldsymbol{\tau}_0$. Noting that

$$\int_{-\pi}^{\pi} \frac{e^{iq\lambda}}{\phi\left(e^{i\lambda};\boldsymbol{\varphi}_0\right)} d\lambda = 0 \text{ for any } q = \pm 1, \pm 2, \ldots, \tag{S.157}$$

and $\int_{-\pi}^{\pi} \log(2 - 2\cos\lambda) d\lambda = 0$, it is straightforward to show that $\partial U(\boldsymbol{\tau}_0)/\partial\boldsymbol{\tau} = 0$. Similarly, using again (S.157),

$$\begin{aligned}\frac{\partial^2 U(\boldsymbol{\tau}_0)}{\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}'} &= \begin{pmatrix} \int_{-\pi}^{\pi} \log^2(2-2\cos\lambda) d\lambda & 2\int_{-\pi}^{\pi} \frac{\partial\phi(e^{i\lambda};\boldsymbol{\varphi}_0)/\partial\boldsymbol{\varphi}'}{\phi(e^{i\lambda};\boldsymbol{\varphi}_0)} \log(2-2\cos\lambda) d\lambda \\ 2\int_{-\pi}^{\pi} \frac{\partial\phi(e^{i\lambda};\boldsymbol{\varphi}_0)/\partial\boldsymbol{\varphi}}{\phi(e^{i\lambda};\boldsymbol{\varphi}_0)} \log(2-2\cos\lambda) d\lambda & 2\int_{-\pi}^{\pi} \frac{\partial\phi(e^{i\lambda};\boldsymbol{\varphi}_0)/\partial\boldsymbol{\varphi}\partial\phi(e^{i\lambda};\boldsymbol{\varphi}_0)/\partial\boldsymbol{\varphi}'}{\left|\phi(e^{i\lambda};\boldsymbol{\varphi}_0)\right|^2} d\lambda \end{pmatrix} \\ &= \begin{pmatrix} 2\pi^3/3 & -4\pi\sum_{j=1}^{\infty} \boldsymbol{b}_j'(\boldsymbol{\varphi}_0)/j \\ -4\pi\sum_{j=1}^{\infty} \boldsymbol{b}_j(\boldsymbol{\varphi}_0)/j & 4\pi\sum_{j=1}^{\infty} \boldsymbol{b}_j(\boldsymbol{\varphi}_0)\boldsymbol{b}_j'(\boldsymbol{\varphi}_0) \end{pmatrix},\end{aligned}$$

which by A4(iii) is positive definite, to complete the proof of strict convexity of $U(\boldsymbol{\tau})$ at $\boldsymbol{\tau}_0$. Thus, by continuity there exists a point $\boldsymbol{\tau}^*$ such that $\|\boldsymbol{\tau}_0 - \boldsymbol{\tau}^*\| = \varepsilon T^{-\varkappa}$ and $\inf_{\mathcal{J}_5} U(\boldsymbol{\tau}) = U(\boldsymbol{\tau}^*)$. Then, noting that $U(\boldsymbol{\tau}_0) = 0$ and $\partial U(\boldsymbol{\tau}_0)/\partial\boldsymbol{\tau} = 0$, by Taylor's expansion,

$$U(\boldsymbol{\tau}^*) \geq \frac{1}{2}(\boldsymbol{\tau}^* - \boldsymbol{\tau}_0)' \frac{\partial^2 U(\boldsymbol{\tau}_0)}{\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}'}(\boldsymbol{\tau}^* - \boldsymbol{\tau}_0) - |w_T|, \tag{S.158}$$

where it can be shown that $w_T = O(T^{-3\varkappa})$. Here, the main issue is to justify that the third derivative of $U(\boldsymbol{\tau})$ evaluated at an arbitrarily small neighborhood of $\boldsymbol{\tau}_0$ is bounded, but this follows straightforwardly from A4(ii). Additionally,

$$(\boldsymbol{\tau}^* - \boldsymbol{\tau}_0)' \frac{\partial^2 U(\boldsymbol{\tau}_0)}{\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}'}(\boldsymbol{\tau}^* - \boldsymbol{\tau}_0) \geq \underline{\lambda} \|\boldsymbol{\tau}^* - \boldsymbol{\tau}_0\|^2,$$

where $\underline{\lambda}$ denotes the minimum eigenvalue of the matrix $\partial^2 U(\boldsymbol{\tau}_0)/\partial\boldsymbol{\tau}\partial\boldsymbol{\tau}'$, so by (S.158) and noting that $\underline{\lambda}$ is strictly positive, for a sufficiently small $\epsilon > 0$,

$$U(\boldsymbol{\tau}^*) > \frac{\epsilon}{\varepsilon^2}\|\boldsymbol{\tau}^* - \boldsymbol{\tau}_0\|^2,$$

which justifies (S.152). The proofs of (S.153)–(S.156) are omitted as, for small enough $\varkappa$, they follow by almost identical arguments to those of (S.10)–(S.13).

Next, the proof of (S.150) for $i = 4$ is omitted because it is basically identical to those of (S.5)–(S.7) for $i = 4$. The only difference is that now $\varepsilon T^{-\varkappa} \le \|\boldsymbol{\tau} - \boldsymbol{\tau}_0\| \le \varepsilon$ instead of $\|\boldsymbol{\tau} - \boldsymbol{\tau}_0\| \ge \varepsilon$, but this does not make any difference. This completes the justification of (S.148).

Finally, we prove (S.149). For the same reason as in the proof of (S.62), we need to prove that

$$\lim_{T\to\infty} \inf_{\boldsymbol{\vartheta}\in\overline{N}^*_\varepsilon\cap M^*_\varepsilon} \frac{T^{2\varkappa}}{T^{2(\gamma_0-\delta)+1}} \sum_{t=1}^{T} d_t^2(\boldsymbol{\vartheta}) > \epsilon, \tag{S.159}$$

$$\sup_{\boldsymbol{\vartheta}\in\overline{N}^*_\varepsilon\cap M^*_\varepsilon} \frac{T^{2\varkappa}}{T^{2(\gamma_0-\delta)+1}} \left|\sum_{t=1}^{T} d_t(\boldsymbol{\vartheta})\, s_{1t}(\boldsymbol{\tau})\right| = o_p(1), \tag{S.160}$$

$$\sup_{\boldsymbol{\vartheta}\in\overline{N}^*_\varepsilon\cap M^*_\varepsilon} \frac{T^{2\varkappa}}{T^{2(\gamma_0-\delta)+1}} \left(\sum_{j=1}^{T} s_{1j}(\boldsymbol{\tau})\, h_{j,T}(\gamma,\delta,\boldsymbol{\varphi})\right)^2 = o_p(1). \tag{S.161}$$

As in (S.20), the proof of (S.159) follows by Lemma S.3, whereas the proofs of (S.160) and (S.161) hold as in (S.64) and (S.65) for $\varkappa > 0$ sufficiently small. ■

Lemma S.7 *Let $\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau}_0 = O_p(T^{-\varkappa})$, $\widehat{\gamma} - \gamma_0 = O_p(T^{-\varkappa})$ for $\varkappa > 0$. Then, under Assumptions A1–A4,*

$$c_t(\widehat{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}) = c_t(\gamma_0,\delta_0,\boldsymbol{\varphi}_0) + O_p\left(T^{-\varkappa} t^{\max\{\gamma_0-\delta_0,-1-\varsigma\}}\log^2 t\right), \tag{S.162}$$

$$c_t^{(1)}(\widehat{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}) = c_t^{(1)}(\gamma_0,\delta_0,\boldsymbol{\varphi}_0) + O_p\left(T^{-\varkappa} t^{\max\{\gamma_0-\delta_0,-1-\varsigma\}}\log^3 t\right), \tag{S.163}$$

and, uniformly in $t = 1,\ldots,T$,

$$\varepsilon_t(\widehat{\boldsymbol{\tau}}) = \sum_{j=0}^{t-1} a_j\left(\delta_0 - \widehat{\delta}, \widehat{\boldsymbol{\varphi}}\right) u_{t-j} = \sum_{j=0}^{t-1} \phi_j(\boldsymbol{\varphi}_0)\, u_{t-j} + O_p\left(T^{-\varkappa}\right). \tag{S.164}$$

Proof. First we show (S.162). Clearly

$$\begin{aligned} c_t(\widehat{\gamma},\widehat{\delta},\widehat{\boldsymbol{\varphi}}) - c_t(\gamma_0,\delta_0,\boldsymbol{\varphi}_0) &= \sum_{j=1}^{t} b_j(\gamma_0,\delta_0)\left(\phi_{t-j}(\widehat{\boldsymbol{\varphi}}) - \phi_{t-j}(\boldsymbol{\varphi}_0)\right) \\ &\quad + \sum_{j=1}^{t}\left(b_j\left(\widehat{\gamma},\widehat{\delta}\right) - b_j(\gamma_0,\delta_0)\right)\phi_{t-j}(\boldsymbol{\varphi}_0) \\ &\quad + \sum_{j=1}^{t}\left(b_j\left(\widehat{\gamma},\widehat{\delta}\right) - b_j(\gamma_0,\delta_0)\right)\left(\phi_{t-j}(\widehat{\boldsymbol{\varphi}}) - \phi_{t-j}(\boldsymbol{\varphi}_0)\right). \end{aligned} \tag{S.165}$$

Fix $\epsilon < 1/2$. Then

$$\phi_j(\widehat{\boldsymbol{\varphi}}) - \phi_j(\boldsymbol{\varphi}_0) = (\phi_j(\widehat{\boldsymbol{\varphi}}) - \phi_j(\boldsymbol{\varphi}_0))\left(\mathbb{I}\left(\|\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}_0\| < \epsilon\right) + \mathbb{I}\left(\|\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}_0\| \geq \epsilon\right)\right), \tag{S.166}$$

so by the mean value theorem the left-hand side of (S.166) is bounded by

$$\sup_{\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_0\|<\epsilon} \left\| \frac{\partial \phi_j(\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi}} \right\| \|\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}_0\| + K \sup_{\boldsymbol{\varphi}\in\Psi} |\phi_j(\boldsymbol{\varphi})| \frac{\|\widehat{\boldsymbol{\varphi}} - \boldsymbol{\varphi}_0\|^N}{\epsilon^N}, \tag{S.167}$$

for any arbitrarily large fixed number N. Then by (6) and the $T^{-\varkappa}$-consistency of $\widehat{\boldsymbol{\tau}}$, the second term in (S.167) is of smaller order, whereas by (14), the first one is $O_p(T^{-\varkappa}j^{-1-\varsigma})$. This implies that the first term on the right-hand side of (S.165) is $O_p(T^{-\varkappa}t^{\max\{\gamma_0-\delta_0,-1-\varsigma\}}\log t)$ by (S.184) of Lemma S.13, using also Lemmas S.9 and S.11.

Next, we show that

$$b_j(\widehat{\gamma},\widehat{\delta}) = b_j(\gamma_0,\delta_0) + O_p(T^{-\varkappa}j^{\gamma_0-\delta_0}\log j). \tag{S.168}$$

Using a N'th-order Taylor expansion,

$$b_j(\widehat{\gamma},\widehat{\delta}) = b_j(\gamma_0,\delta_0) + (\widehat{\gamma}-\gamma_0,\widehat{\delta}-\delta_0)(b_j^{(1)}(\gamma_0,\delta_0), b_j^{(2)}(\gamma_0,\delta_0))' + p_{j,N}(\widehat{\gamma},\widehat{\delta},\gamma_0,\delta_0),$$

where $b_j^{(i)}(\cdot,\cdot)$ denotes derivative of $b_j(\cdot,\cdot)$ with respect to the i'th argument and $p_{j,N}(\widehat{\gamma},\widehat{\delta},\gamma_0,\delta_0)$ collects derivatives of $b_j(\cdot,\cdot)$ evaluated at (γ_0,δ_0) (whose order of magnitude is given by (S.186)), powers of $(\widehat{\gamma}-\gamma_0)$, $(\widehat{\delta}-\delta_0)$, and a last term which involves, $(\widehat{\gamma}-\gamma_0)^N$, $(\widehat{\delta}-\delta_0)^N$ and N'th derivatives of $b_j(\cdot,\cdot)$ evaluated at intermediate points. This last term is bounded by $KT^{-N\varkappa}\log^N T \sup_{\gamma\in[\square_1,\square_2],\delta\in[\nabla_1,\nabla_2]} \sum_{k=1}^{j-1} j^{-\delta-1}(j-k)^{\gamma}$, which can be easily shown to be $o_p(T^{-\varkappa}j^{\gamma_0-\delta_0}\log j)$ for N large enough. Then, (S.168) follows by (S.186), and using also (6), Lemma S.9 and (S.184), the second term on the right-hand side of (S.165) is

$$O_p(T^{-\varkappa}t^{\max\{\gamma_0-\delta_0,-1-\varsigma\}}\log^2 t).$$

Finally, combining the arguments for the first two terms, the third term on the right-hand side of (S.165) is of smaller order, to conclude the proof of (S.162).

The proof of (S.163) is omitted because it is almost identical to that of (S.162) with the only difference that the coefficients $b_j^{(1)}(\cdot,\cdot)$ instead of $b_j(\cdot,\cdot)$ lead to an extra $\log t$ factor, see Lemma S.13.

Finally, we show (S.164). Clearly

$$\begin{aligned} a_j(\delta_0-\widehat{\delta},\widehat{\boldsymbol{\varphi}}) =& \phi_j(\boldsymbol{\varphi}_0) + \sum_{k=0}^{j}(\phi_k(\widehat{\boldsymbol{\varphi}}) - \phi_k(\boldsymbol{\varphi}_0))\pi_{j-k}(0) + \sum_{k=0}^{j}\phi_k(\boldsymbol{\varphi}_0)(\pi_{j-k}(\delta_0-\widehat{\delta}) - \pi_{j-k}(0)) \\ &+ \sum_{k=0}^{j}(\phi_k(\widehat{\boldsymbol{\varphi}}) - \phi_k(\boldsymbol{\varphi}_0))(\pi_{j-k}(\delta_0-\widehat{\delta}) - \pi_{j-k}(0)), \end{aligned}$$

so that

$$\begin{aligned}\varepsilon_t(\widehat{\boldsymbol{\tau}}) =& \sum_{j=0}^{t-1}\phi_j(\boldsymbol{\varphi}_0)u_{t-j} + \sum_{j=0}^{t-1}(\phi_j(\widehat{\boldsymbol{\varphi}}) - \phi_j(\boldsymbol{\varphi}_0))u_{t-j} \\ &+ \sum_{j=0}^{t-1}\sum_{k=0}^{j}\phi_k(\boldsymbol{\varphi}_0)(\pi_{j-k}(\delta_0 - \widehat{\delta}) - \pi_{j-k}(0))u_{t-j} \\ &+ \sum_{j=0}^{t-1}\sum_{k=0}^{j}(\phi_k(\widehat{\boldsymbol{\varphi}}) - \phi_k(\boldsymbol{\varphi}_0))(\pi_{j-k}(\delta_0 - \widehat{\delta}) - \pi_{j-k}(0))u_{t-j}. \qquad \text{(S.169)}\end{aligned}$$

Using the mean value theorem as in (S.166) and (S.167) and summation by parts, it can be shown that the second term on the right-hand side of (S.169) is $O_p(T^{-\varkappa})$. Similarly, by Lemma C.5 of Robinson and Hualde (2003) and (6), the third and fourth terms on the right-hand side of (S.169) are also $O_p(T^{-\varkappa})$, to conclude the proof of (S.164). ■

Lemma S.8 *Under the conditions of Theorem 2(ii),* $T^{1/2}\partial S_T(\widehat{\boldsymbol{\vartheta}})/\partial\boldsymbol{\tau} = o_p(1)$.

Proof. First, for any $\epsilon > 0$, clearly

$$\begin{aligned}\Pr\left(\left\|T^{1/2}\partial S_T(\widehat{\boldsymbol{\vartheta}})/\partial\boldsymbol{\tau}\right\| \geq \epsilon\right) =& \Pr\left(\left\|T^{1/2}\partial S_T(\widehat{\boldsymbol{\vartheta}})/\partial\boldsymbol{\tau}\right\| \geq \epsilon, \|\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau}_0\| < \varepsilon\right) \\ &+ \Pr\left(\left\|T^{1/2}\partial S_T(\widehat{\boldsymbol{\vartheta}})/\partial\boldsymbol{\tau}\right\| \geq \epsilon, \|\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau}_0\| \geq \varepsilon\right) \\ \leq& \Pr\left(\left\|T^{1/2}\partial S_T(\widehat{\boldsymbol{\vartheta}})/\partial\boldsymbol{\tau}\right\| \geq \epsilon, \|\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau}_0\| < \varepsilon\right) \\ &+ \Pr\left(\|\widehat{\boldsymbol{\tau}} - \boldsymbol{\tau}_0\| \geq \varepsilon\right),\end{aligned}$$

so, in view of Theorem 1(ii) and (S.95), the result holds on showing

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon}\frac{1}{T^{1/2}}\sum_{t=1}^{T}(d_t(\boldsymbol{\vartheta}) - s_{2t}(\boldsymbol{\vartheta}))\left(\frac{\partial d_t(\boldsymbol{\vartheta})}{\partial\boldsymbol{\tau}} - \frac{\partial s_{2t}(\boldsymbol{\vartheta})}{\partial\boldsymbol{\tau}}\right) = o_p(1), \qquad \text{(S.170)}$$

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon}\frac{1}{T^{1/2}}\sum_{t=1}^{T}s_{1t}(\boldsymbol{\tau})\left(\frac{\partial d_t(\boldsymbol{\vartheta})}{\partial\boldsymbol{\tau}} - \frac{\partial s_{2t}(\boldsymbol{\vartheta})}{\partial\boldsymbol{\tau}}\right) = o_p(1), \qquad \text{(S.171)}$$

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon}\frac{1}{T^{1/2}}\sum_{t=1}^{T}\frac{\partial s_{1t}(\boldsymbol{\tau})}{\partial\boldsymbol{\tau}}(d_t(\boldsymbol{\vartheta}) - s_{2t}(\boldsymbol{\vartheta})) = o_p(1). \qquad \text{(S.172)}$$

The proof of (S.170) follows upon showing that, for any $\theta > 0$ and ε such that $0 < \varepsilon < \theta$,

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon}|d_t(\boldsymbol{\vartheta})| = O(t^{\max\{\gamma_0-\delta_0+\varepsilon,-1-\varsigma\}} + T^{2\theta}t^{-1/2-\theta}), \qquad \text{(S.173)}$$

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon}|s_{2t}(\boldsymbol{\vartheta})| = O_p(T^{2\theta}t^{-1/2-\theta}), \qquad \text{(S.174)}$$

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon}\left\|\frac{\partial d_t(\boldsymbol{\vartheta})}{\partial\boldsymbol{\tau}}\right\| = O(t^{\max\{\gamma_0-\delta_0+\varepsilon,-1-\varsigma\}}\log t + T^{4\theta}t^{-1/2-\theta}), \qquad \text{(S.175)}$$

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon}\left\|\frac{\partial s_{2t}(\boldsymbol{\vartheta})}{\partial\boldsymbol{\tau}}\right\| = O_p(T^{4\theta}t^{-1/2-\theta}), \qquad \text{(S.176)}$$

and then letting θ be sufficiently small. We only show (S.175) and (S.176) because the proofs for (S.173) and (S.174) are very similar but simpler. First, by (S.192) of Lemma S.14,

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon} |c_t(\gamma_0,\delta,\boldsymbol{\varphi})| = O(t^{\max\{\gamma_0-\delta_0+\varepsilon,-1-\varsigma\}}), \tag{S.177}$$

and by a simple modification of that result,

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon} \left\| \frac{\partial c_t(\gamma_0,\delta,\boldsymbol{\varphi})}{\partial \boldsymbol{\tau}} \right\| = O(t^{\max\{\gamma_0-\delta_0+\varepsilon,-1-\varsigma\}}\log t). \tag{S.178}$$

Then (S.175) follows by direct application of (S.177) and (S.178) and (S.209) of Lemma S.16, noting that the bound in (S.209) also applies if the derivative is taken with respect to $\boldsymbol{\tau}$.

To prove (S.176) we apply (S.145), where the $\sup_{\boldsymbol{\vartheta}\in M_\varepsilon}$ of the absolute values of the first and third terms on the right-hand side are $O_p(T^{4\theta}t^{-1/2-\theta})$ by direct application of (S.209) of Lemma S.16 and (S.222) of Lemma S.18, noting that $\delta_0-\delta\le\varepsilon$ and that these bounds also apply if the derivatives are taken with respect to $\boldsymbol{\tau}$. For the second term on the right-hand side of (S.145), noting that $\sum_{l=1}^{t} s_{1l}(\boldsymbol{\tau}) = \sum_{j=0}^{t-1} a_j(\delta_0-\delta,\boldsymbol{\varphi})\sum_{l=1}^{t-j} u_l$, it is straightforward to show that, by (S.192) of Lemma S.14,

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon} \left\| \sum_{j=1}^{t} \frac{\partial s_{1j}(\boldsymbol{\tau})}{\partial \boldsymbol{\tau}} \right\| = O_p(t^{1/2+\varepsilon}\log t). \tag{S.179}$$

Therefore, by (S.209) of Lemma S.16 and using summation by parts as in the proof of Lemma S.18, the $\sup_{\boldsymbol{\vartheta}\in M_\varepsilon}$ of the absolute value of the second term on the right-hand side of (S.145) is $O_p(T^{2\theta}t^{-1/2-\theta})$, to justify (S.176) and hence (S.170).

Finally, (S.171) and (S.172) can be established by using summation by parts followed by direct application of the results in (S.173), (S.175), (S.179), and Lemma S.17, noting also that by previous arguments it can be easily shown that

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon} |d_{t+1}(\boldsymbol{\vartheta}) - d_t(\boldsymbol{\vartheta})| = O(t^{\max\{\gamma_0-\delta_0+\varepsilon-1,-1-\varsigma\}} + T^{2\theta}t^{-3/2-\theta}),$$

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon} \left\| \frac{\partial d_{t+1}(\boldsymbol{\vartheta})}{\partial \boldsymbol{\tau}} - \frac{\partial d_t(\boldsymbol{\vartheta})}{\partial \boldsymbol{\tau}} \right\| = O(t^{\max\{\gamma_0-\delta_0+\varepsilon-1,-1-\varsigma\}}\log t + T^{4\theta}t^{-3/2-\theta}),$$

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon} |s_{2t+1}(\boldsymbol{\vartheta}) - s_{2t}(\boldsymbol{\vartheta})| = O_p(T^{2\theta}t^{-3/2-\theta}),$$

$$\sup_{\boldsymbol{\vartheta}\in M_\varepsilon} \left\| \frac{\partial s_{2t+1}(\boldsymbol{\vartheta})}{\partial \boldsymbol{\tau}} - \frac{\partial s_{2t}(\boldsymbol{\vartheta})}{\partial \boldsymbol{\tau}} \right\| = O_p(T^{4\theta}t^{-3/2-\theta}).$$

■

S.4 Technical lemmas

Lemma S.9 *Uniformly for* $\max\{|\alpha|,|\beta|\}\le a_0$, $\sum_{j=1}^{t-1} j^{\alpha-1}(t-j)^{\beta-1} \le K(\log t)t^{\max\{\alpha+\beta-1,\alpha-1,\beta-1\}}$.

Proof. The proof of Lemma S.9 is given in Lemma B.4 of Johansen and Nielsen (2010). ■

Lemma S.10 *For any $d > -1$ and any fixed $a \geq 0$, as $T \to \infty$,*

$$\frac{1}{T^{d+1}} \sum_{t=1}^{T} t^d \to \frac{1}{d+1}, \quad \frac{1}{T^{d+1}} \sum_{t=1}^{T} \log\left(\frac{t+a}{T}\right) t^d \to -\frac{1}{(d+1)^2},$$

$$\frac{1}{T^{d+1}} \sum_{t=1}^{T} \log^2\left(\frac{t+a}{T}\right) t^d \to \frac{2}{(d+1)^3}.$$

Proof. The proof of the first result is straightforward by approximating the sum by an integral. Next, by the mean value theorem, it is simple to show that

$$\frac{1}{T^{d+1}} \sum_{t=1}^{T} \log\left(\frac{t+a}{T}\right) t^d = \frac{1}{T^{d+1}} \sum_{t=1}^{T} \log\left(\frac{t}{T}\right) t^d + o(1).$$

Approximating the sum by an integral we find

$$\frac{1}{T} \sum_{t=1}^{T} \log\left(\frac{t}{T}\right)\left(\frac{t}{T}\right)^d \sim \int_0^1 \log(x)\, x^d dx = B(d+1,1)\left(\psi(d+1) - \psi(d+2)\right),$$

see p. 535 of Gradshteyn and Ryzhik (2000), where $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function and $\psi(\cdot)$ is the digamma function. Thus, the second result follows by the recurrence formulae for the gamma and digamma functions, see pp. 256 and 258 of Abramowitz and Stegun (1970). Similarly, $T^{-d-1}\sum_{t=1}^{T} \log^2((t+a)/T) t^d$ can be approximated by

$$\begin{aligned}\frac{1}{T} \sum_{t=1}^{T} \log^2\left(\frac{t}{T}\right)\left(\frac{t}{T}\right)^d &\sim \int_0^1 \log^2(x)\, x^d dx \\ &= B(d+1,1)\left(\left(\psi(d+1) - \psi(d+2)\right)^2 + \psi'(d+1) - \psi'(d+2)\right),\end{aligned}$$

see p. 538 of Gradshteyn and Ryzhik (2000), where $\psi'(\cdot)$ is the trigamma function. Then the third result follows by the recurrence formulae for the gamma, digamma, and trigamma functions, see pp. 256, 258, and 260 of Abramowitz and Stegun (1970). ■

Lemma S.11 *Let $j \geq 1$ and $\mathbb{K}$ denote any compact subset of $\mathbb{R}\backslash\mathbb{N}_0$. Then*

$$\pi_j(-v) = \frac{1}{\Gamma(-v)} j^{-v-1}(1 + \epsilon_j(v)), \tag{S.180}$$

where $\max_{v\in\mathbb{K}} |\epsilon_j(v)| \to 0$ as $j \to \infty$. Thus, uniformly in $j \geq 1, m \geq 0$,

(i) $\pi_j(-v) \geq K j^{-v-1}$ *uniformly in $v \in \mathbb{K}$,*

(ii) $|\frac{\partial^m}{\partial u^m}\pi_j(u)| \leq K(1+\log j)^m j^{u-1}$ *uniformly in $|u| \leq u_0$,*

(iii) $|\frac{\partial^m}{\partial u^m} T^{-u}\pi_j(u)| \leq K T^{-u}(1 + |\log(j/T)|)^m j^{u-1}$ *uniformly in $|u| \leq u_0$.*

Proof. The proof of Lemma S.11 is given in Lemma B.3 of Johansen and Nielsen (2010) and Lemma A.5 of Johansen and Nielsen (2012). ■

Lemma S.12 *Let $Z_t, t = 1, \ldots, T$, be arbitrary. Then*

$$\sum_{t=1}^{T} Z_t^2 \geq \left(\frac{\pi^2}{4} T^{-2} + O(T^{-3}) \right) \sum_{t=1}^{T} \left(\Delta_+^{-1} Z_t \right)^2, \tag{S.181}$$

where the $O(T^{-3})$ term does not depend on any parameters.

Proof. Following the proof of Lemma 2 of Johansen and Nielsen (2018), we let $Z = (Z_1, \ldots, Z_T)'$ and define the cumulation matrix

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}$$

such that $(\Delta_+^{-1} Z_1, \ldots, \Delta_+^{-1} Z_T)' = CZ$. Then, using $X = CZ$,

$$\frac{\sum_{t=1}^{T} Z_t^2}{\sum_{t=1}^{T} \left(\Delta_+^{-1} Z_t \right)^2} = \frac{Z'Z}{Z'C'CZ} = \frac{X'C'^{-1}C^{-1}X}{X'X} \geq \lambda_{\min} \left(C'^{-1} C^{-1} \right), \tag{S.182}$$

where the inequality follows from, e.g., Horn and Johnson (2013, p. 258). From Rutherford (1948), see also Tanaka (1996, eqn. (1.4)), we find the eigenvalues

$$\lambda_t \left(C'^{-1} C^{-1} \right) = 4 \sin^2 \left(\frac{\pi}{2} \frac{2t-1}{2T+1} \right), \quad t = 1, \ldots, T,$$

such that, in particular,

$$\lambda_{\min} \left(C'^{-1} C^{-1} \right) = 4 \sin^2 \left(\frac{\pi}{2} \frac{1}{2T+1} \right) = \frac{\pi^2}{4} T^{-2} + O(T^{-3}). \tag{S.183}$$

The bound (S.181) follows by combining (S.182) and (S.183). ■

Lemma S.13 *For any real numbers $\diamondsuit > -1$, $\overline{d}_1$, $\underline{d}_2$, $\overline{d}_2$, and $\diamondsuit \leq d_1 \leq \overline{d}_1$, $\underline{d}_2 \leq d_2 \leq \overline{d}_2$, $m \geq 1$, $0 \leq p \leq m$, denoting by $\psi(\cdot)$ the digamma function,*

$$b_t(d_1, d_2) = \Gamma(d_1 + 1) \pi_t(d_1 + 1 - d_2) + s_{1t}(d_1, d_2), \tag{S.184}$$

$$\begin{aligned} \frac{\partial b_t(d_1, d_2)}{\partial d_1} =& \log(t) b_t(d_1, d_2) \\ &+ \left(\psi(d_1 + 1) - \psi(d_1 + 1 - d_2) \right) \Gamma(d_1 + 1) \pi_t(d_1 + 1 - d_2) + s_{2t}(d_1, d_2), \end{aligned} \tag{S.185}$$

$$\frac{\partial^m b_t(d_1, d_2)}{\partial d_1^p \partial d_2^{m-p}} = O\left(t^{d_1 - d_2} (\log t)^m \right), \tag{S.186}$$

where $|s_{1t}(d_1, d_2)| \leq K(t^{d_1 - d_2 - 1} + t^{-d_2 - 1})$ and $|s_{2t}(d_1, d_2)| \leq K((t^{d_1 - d_2 - 1} + t^{-d_2 - 1}) \log t)$.

Proof. First we show (S.184). For integer d_2, it can be easily shown that (S.184) holds with $|r_{1t}(d_1,d_2)| \leq Kt^{d_1-d_2-1}$ using standard results for summations and the Taylor's theorem. Next, when d_1 is integer and d_2 is noninteger, by (3) and standard properties of the gamma function, it is simple to show that for integer $k \geq 1$,

$$j^k \pi_j(-d_2) = \frac{\Gamma(k-d_2)}{\Gamma(-d_2)} \pi_{j-1}(k-d_2) + \sum_{l=1}^{k-1} g_l(d_2) \pi_{j-1}(l-d_2), \tag{S.187}$$

where $g_l(\cdot)$ are polynomials and the second term on the right-hand side of (S.187) is 0 when $k=1$. Then

$$\sum_{j=0}^{t-1} j^k \pi_j(-d_2) = \frac{\Gamma(k-d_2)}{\Gamma(-d_2)} \pi_{t-2}(k+1-d_2) + O\left(t^{k-d_2-1}\right)$$

using Lemma S.11. Then, by (11) it is simple, but tedious, to show that (S.184) holds with $|r_{1t}(d_1,d_2)| \leq Kt^{d_1-d_2-1}$.

Next, we deal with the case where neither d_1 nor d_2 are integers. From Abramowitz and Stegun (1970, p. 257, eqn. 6.1.47), for a fixed integer $p \geq 1$ and any $k = -1, 0, 1, \ldots, p-1$,

$$\frac{\Gamma(t+d_1-k)}{\Gamma(t+1)} = t^{d_1-k-1} + \lambda_1(d_1-k) t^{d_1-k-2} + \cdots + \lambda_{p-k-1}(d_1-k) t^{d_1-p} + r_{p-k-1,t}(d_1-k),$$

for $t \neq k-d_1, k-d_1-1, k-d_1-2, \ldots,$ (which does not hold because d_1 is not an integer), where for any l, k, $\sup_{\Diamond \leq d_1 \leq \bar{d}_1} |\lambda_l(d_1-k)| \leq K$ and $|r_{p-k-1,t}(d_1-k)| \leq Kt^{d_1-p-1}$. Then by recursive substitution it can easily be shown that

$$t^{d_1} = \frac{\Gamma(t+d_1+1)}{\Gamma(t+1)} + \nu_1(d_1) \frac{\Gamma(t+d_1)}{\Gamma(t+1)} + \cdots + \nu_p(d_1) \frac{\Gamma(t+d_1-p+1)}{\Gamma(t+1)} + q_{p,t}(d_1),$$

where the $\nu_l(d_1)$'s are complicated combinations of the $\lambda_l(\cdot)$'s, and $q_{p,t}(d_1)$ is a weighted sum of the $r_{p-k-1,t}(\cdot)$'s with coefficients which depend in a complicated manner on the $\lambda_l(\cdot)$'s. Given that p is fixed it can be shown that for any l, $\sup_{\Diamond \leq d_1 \leq \bar{d}_1} |\nu_l(d_1)| \leq K$ and $|q_{p,t}(d_1)| \leq Kt^{d_1-p-1}$. Therefore, noting (3), it is immediately shown that

$$t^{d_1} = \Gamma(d_1+1) \pi_t(d_1+1) + \bar{\nu}_1(d_1) \pi_t(d_1) + \cdots + \bar{\nu}_p(d_1) \pi_t(d_1-p+1) + \bar{q}_{p,t}(d_1),$$

where $\bar{\nu}_k(d_1) = \Gamma(d_1+1)\Gamma(d_1-k+1)\nu_k(d_1)$, $\bar{q}_{p,t}(d_1) = \Gamma(d_1+1) q_{p,t}(d_1)$. Then, given that $\Delta^d \pi_t(c) = \pi_t(c-d)$,

$$\begin{aligned} b_t(d_1,d_2) =& \Gamma(d_1+1) \pi_t(d_1+1-d_2) + \bar{\nu}_1(d_1) \pi_t(d_1-d_2) + \cdots + \bar{\nu}_p(d_1) \pi_t(d_1-p+1-d_2) \\ &+ \sum_{j=0}^{t-1} \pi_j(-d_2) \bar{q}_{p,t-j}(d_1). \end{aligned} \tag{S.188}$$

Clearly, by Lemma S.11,

$$\left| \sum_{j=0}^{t-1} \pi_j(-d_2) \bar{q}_{p,t-j}(d_1) \right| \leq K \sum_{j=1}^{t-1} j^{-d_2-1} (t-j)^{d_1-p-1} \leq Kt^{-d_2-1},$$

for p large enough, so (S.184) immediately follows by Lemma S.11.

Next we show (S.185). When d_1 and/or d_2 are integers the proof follows from relatively simple (but cumbersome) arguments very similar to those employed in the proof of (S.184). Thus we focus on the case where d_1 and d_2 are not integers. By (S.188),

$$\frac{\partial b_t(d_1,d_2)}{\partial d_1} = \Gamma^{(1)}(d_1+1)\pi_t(d_1+1-d_2) + \Gamma(d_1+1)\pi_t^{(1)}(d_1+1-d_2) + \sum_{k=1}^{p}\left(\overline{\nu}_k^{(1)}(d_1)\pi_t(d_1-d_2-k+1) + \overline{\nu}_k(d_1)\pi_t^{(1)}(d_1-d_2-k+1)\right) + \sum_{j=0}^{t-1}\pi_j(-d_2)\overline{q}_{p,t-j}^{(1)}(d_1), \quad \text{(S.189)}$$

where superscript (1) denotes the first derivative. Noting that

$$\pi_j^{(1)}(d) = (\psi(d+j) - \psi(d))\pi_j(d) \quad \text{(S.190)}$$

and that for a fixed a,

$$\psi(t+a) = \log t + O\left(t^{-1}\right), \quad \text{(S.191)}$$

see Abramowitz and Stegun (1970, p. 259, eqn. 6.3.18), it can be easily shown that $|\overline{q}_{p,j}^{(1)}(d_1)| \leq Kj^{d_1-p-1}\log j$, so, for p large enough, the last term on the right-hand side of (S.189) is $O\left(t^{-d_2-1}\right)$. Then by (S.184), (S.189), and noting that $\Gamma^{(1)}(d_1+1) = \Gamma(d_1+1)\psi(d_1+1)$,

$$\begin{aligned} &\frac{\partial b_t(d_1,d_2)}{\partial d_1} \\ &= \log(t)\Gamma(d_1+1)\pi_t(d_1+1-d_2) \\ &\quad + \Gamma(d_1+1)(\psi(t+d_1+1-d_2) - \psi(d_1+1-d_2) + \psi(d_1+1) - \log t)\pi_t(d_1+1-d_2) \\ &\quad + O\left(t^{d_1-d_2-1}\log t + t^{-d_2-1}\right), \end{aligned}$$

so (S.185) is justified by (S.184) and (S.191).

Finally, the proof of (S.186) follows by taking appropriate derivatives on $b_t(d_1,d_2)$, as in (S.189), and using standard bounds for the derivatives of the digamma function given by

$$\left|\frac{\partial^l \psi(z)}{\partial z^l}\right| \leq Kz^{-l}, \quad l = 1,2,3,\ldots$$

■

Lemma S.14 *Under Assumptions A1, A3, uniformly in* $t = 1,\ldots,T$ *and* $T \geq 1$, *for any* *real numbers* $\lozenge > -1$, $\overline{d}_1$, $\underline{d}_2$, $\overline{d}_2$, *and* $\lozenge \leq d_1 \leq \overline{d}_1$, $\underline{d}_2 \leq d_2 \leq \overline{d}_2$, $m \geq 0$, $0 \leq p \leq m$,

$$\sup_{d_1 \geq \lozenge, d_1 - d_2 \leq g, \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial^m c_t(d_1, d_2, \boldsymbol{\varphi})}{\partial d_1^p \partial d_2^{m-p}} \right| = O(t^{\max\{g, -1-\varsigma\}} (\log t)^m), \tag{S.192}$$

$$\sup_{d_1 \geq \lozenge, d_1 - d_2 \geq g, \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial^m}{\partial d_1^p \partial d_2^{m-p}} T^{-(d_1 - d_2)} c_t(d_1, d_2, \boldsymbol{\varphi}) \right| = O(T^{-g} t^{\max\{g, -1-\varsigma\}} (1 + |\log(t/T)|)^m). \tag{S.193}$$

$$\sup_{d \leq g, \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial^m a_t(d, \boldsymbol{\varphi})}{\partial d^m} \right| = O(t^{\max\{g-1, -1-\varsigma\}} (\log t)^m), \tag{S.194}$$

$$\sup_{d \geq g, \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial^m}{\partial d^m} T^{-d} a_t(d, \boldsymbol{\varphi}) \right| = O(T^{-g} t^{\max\{g-1, -1-\varsigma\}} (1 + |\log(t/T)|)^m).$$

Proof. Using Lemma S.13, the proof of Lemma S.14 is almost identical to that of Lemma 1 of Hualde and Robinson (2011) and is therefore omitted. ■

Lemma S.15 *Under Assumptions A1 and A3, for any real numbers* $\lozenge > -1$, $\overline{d}_1$, $\underline{d}_2$, $\overline{d}_2$, *and* $\lozenge \leq d_1 \leq \overline{d}_1$, $\underline{d}_2 \leq d_2 \leq \overline{d}_2$,

$$\frac{1}{T^{2(d_1-d_2)+1}} \sum_{t=1}^{T} c_t^2(d_1, d_2, \boldsymbol{\varphi}) \geq \frac{\phi^2(1; \boldsymbol{\varphi})}{T^{2(d_1-d_2)+1}} \sum_{t=1}^{T} b_t^2(d_1, d_2) - |r_{1T}(d_1, d_2, \boldsymbol{\varphi})|, \tag{S.195}$$

$$\frac{1}{T^{2(d_1-d_2)+1}} \sum_{t=1}^{T} c_t^2(d_1, d_2, \boldsymbol{\varphi}) \leq \frac{\phi^2(1; \boldsymbol{\varphi})}{T^{2(d_1-d_2)+1}} \sum_{t=1}^{T} b_t^2(d_1, d_2) + |r_{2T}(d_1, d_2, \boldsymbol{\varphi})|, \tag{S.196}$$

where, for any $\eta > 0$, $\sup_{d_1 \geq \lozenge, d_1 - d_2 \geq -1/2+\eta, \boldsymbol{\varphi} \in \Psi} |r_{iT}(d_1, d_2, \boldsymbol{\varphi})| = o(1)$, $i = 1, 2$. *Furthermore, for any* α *such that* $0 < \alpha < \min\{(\varsigma - 1/2)/3, (1 + \lozenge)/2\}$,

$$\inf_{d_1, d_2, \boldsymbol{\varphi} \in \Psi} \sum_{t=1}^{T} c_t^2(d_1, d_2, \boldsymbol{\varphi}) \geq 1, \tag{S.197}$$

$$\inf_{d_1 \geq \lozenge, -1/2-\alpha \leq d_1 - d_2 \leq -1/2+\alpha, \boldsymbol{\varphi} \in \Psi} \frac{1}{T^{2(d_1-d_2)+1}} \sum_{t=1}^{T} c_t^2(d_1, d_2, \boldsymbol{\varphi}) \geq \frac{\epsilon}{\alpha} + o(1), \tag{S.198}$$

$$\inf_{d_1 \geq \lozenge, d_1 - d_2 \geq -1/2-\alpha, \boldsymbol{\varphi} \in \Psi} \frac{1}{T^{2(d_1-d_2)+1}} \sum_{t=1}^{T} c_t^2(d_1, d_2, \boldsymbol{\varphi}) \geq \epsilon, \tag{S.199}$$

for some $\epsilon > 0$, *which does not depend on* α *or* T.

Proof. First we show (S.195). By summation by parts

$$c_t(d_1, d_2, \boldsymbol{\varphi}) = \sum_{j=1}^{t} b_j(d_1, d_2) \phi_{t-j}(\boldsymbol{\varphi}) = b_t(d_1, d_2) \sum_{j=0}^{t-1} \phi_j(\boldsymbol{\varphi}) - \sum_{j=1}^{t-1} (b_{j+1}(d_1, d_2) - b_j(d_1, d_2)) \sum_{l=1}^{j} \phi_{t-l}(\boldsymbol{\varphi}).$$

Using (S.28),

$$b_{j+1}(d_1,d_2) - b_j(d_1,d_2) = \sum_{k=0}^{j} \pi_k(-d_2-1)(j+1-k)^{d_1} = b_{j+1}(d_1,d_2+1)$$

and

$$\begin{aligned} c_t(d_1,d_2,\boldsymbol{\varphi}) =& \phi(1;\boldsymbol{\varphi})\, b_t(d_1,d_2) - b_t(d_1,d_2)\sum_{k=t}^{\infty}\phi_k(\boldsymbol{\varphi}) \\ & - \sum_{j=1}^{t-1} b_{j+1}(d_1,d_2+1)\sum_{l=1}^{j}\phi_{t-l}(\boldsymbol{\varphi}). \end{aligned} \tag{S.200}$$

Then

$$\begin{aligned} \sum_{t=1}^{T} c_t^2(d_1,d_2,\boldsymbol{\varphi}) \geq& \phi^2(1;\boldsymbol{\varphi})\sum_{t=1}^{T} b_t^2(d_1,d_2) - 2\phi(1;\boldsymbol{\varphi})\sum_{t=1}^{T} b_t^2(d_1,d_2)\sum_{j=t}^{\infty}\phi_j(\boldsymbol{\varphi}) \\ & - 2\phi(1;\boldsymbol{\varphi})\sum_{t=1}^{T} b_t(d_1,d_2)\sum_{j=1}^{t-1} b_{j+1}(d_1,d_2+1)\sum_{l=1}^{j}\phi_{t-l}(\boldsymbol{\varphi}) \\ & + 2\sum_{t=1}^{T} b_t(d_1,d_2)\sum_{j=t}^{\infty}\phi_j(\boldsymbol{\varphi})\sum_{k=1}^{t-1} b_{k+1}(d_1,d_2+1)\sum_{l=1}^{k}\phi_{t-l}(\boldsymbol{\varphi}). \end{aligned} \tag{S.201}$$

Noting (6), the fourth term on the right-hand side of (S.201) is of smaller order than the third term. Then the proof of (S.195) follows on showing

$$\sup_{d_1\geq\Diamond,\, d_1-d_2\geq -1/2+\eta,\, \boldsymbol{\varphi}\in\Psi} \frac{1}{T^{2(d_1-d_2)+1}}\left|\sum_{t=1}^{T} b_t^2(d_1,d_2)\sum_{j=t}^{\infty}\phi_j(\boldsymbol{\varphi})\right| = o(1), \tag{S.202}$$

$$\sup_{d_1\geq\Diamond,\, d_1-d_2\geq -1/2+\eta,\, \boldsymbol{\varphi}\in\Psi} \frac{1}{T^{2(d_1-d_2)+1}}\left|\sum_{t=1}^{T} b_t(d_1,d_2)\sum_{j=1}^{t-1} b_{j+1}(d_1,d_2+1)\sum_{l=1}^{j}\phi_{t-l}(\boldsymbol{\varphi})\right| = o(1). \tag{S.203}$$

First, by (6), Lemma S.9 and (S.184) of Lemma S.13, the left-hand side of (S.202) is bounded by

$$\begin{aligned} K \sup_{d_1-d_2\geq -1/2+\eta} \frac{1}{T}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^{2(d_1-d_2)} t^{-\varsigma} &\leq K\frac{1}{T}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^{-1+2\eta} t^{-\varsigma} \\ &\leq K\frac{1}{T^{2\eta}}\sum_{t=1}^{T} t^{-1+2\eta-\varsigma} = o(1), \end{aligned}$$

so (S.202) holds. Similarly, the left-hand side of (S.203) is bounded by

$$
\begin{aligned}
K \sup_{d_1-d_2\geq -1/2+\eta} \frac{1}{T}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^{d_1-d_2}\sum_{j=1}^{t-1}\left(\frac{j}{T}\right)^{d_1-d_2} j^{-1}\left(t-j\right)^{-\varsigma} \\
&\leq K\frac{1}{T}\sum_{t=1}^{T}\left(\frac{t}{T}\right)^{-1/2+\eta}\sum_{j=1}^{t-1}\left(\frac{j}{T}\right)^{-1/2+\eta} j^{-1}\left(t-j\right)^{-\varsigma} \\
&\leq K\frac{1}{T^{2\eta}}\sum_{t=1}^{T} t^{-1/2+\eta}\sum_{j=1}^{t-1} j^{-3/2+\eta}\left(t-j\right)^{-\varsigma} \\
&\leq K\frac{1}{T^{2\eta}}\sum_{t=1}^{T}\left(t^{-1/2+\eta-\varsigma}+t^{-1+2\eta-\varsigma}\right)\left(1+\log t\right)=o\left(1\right),
\end{aligned}
$$

noting that $\varsigma > 1/2$, where the third inequality is due to Lemma S.9, to conclude the proof of (S.195).

Next, in view of (S.200), the proof of (S.196) follows by (S.202), (S.203) and

$$
\sup_{d_1\geq\Diamond, d_1-d_2\geq -1/2+\eta, \boldsymbol{\varphi}\in\Psi} \frac{1}{T^{2(d_1-d_2)+1}}\sum_{t=1}^{T} b_t^2\left(d_1,d_2\right)\left(\sum_{j=t}^{\infty}\phi_j\left(\boldsymbol{\varphi}\right)\right)^2 = o\left(1\right),
$$

$$
\sup_{d_1\geq\Diamond, d_1-d_2\geq -1/2+\eta, \boldsymbol{\varphi}\in\Psi} \frac{1}{T^{2(d_1-d_2)+1}}\sum_{t=1}^{T}\left(\sum_{j=1}^{t-1} b_{j+1}\left(d_1,d_2+1\right)\sum_{l=1}^{j}\phi_{t-l}\left(\boldsymbol{\varphi}\right)\right)^2 = o\left(1\right),
$$

which follow by straightforward arguments using (6), Lemma S.9 and (S.184).

The proof of (S.197) is immediate because

$$
\inf_{d_1,d_2,\boldsymbol{\varphi}\in\Psi}\sum_{t=1}^{T} c_t^2\left(d_1,d_2,\boldsymbol{\varphi}\right)\geq \inf_{d_1,d_2,\boldsymbol{\varphi}\in\Psi} c_1^2\left(d_1,d_2,\boldsymbol{\varphi}\right)=1.
$$

For the proof of (S.198), denoting by $[\cdot]$ the integer part of the argument and given that

$$
\frac{1}{T^{2(d_1-d_2)+1}}\sum_{t=1}^{T} c_t^2\left(d_1,d_2,\boldsymbol{\varphi}\right)\geq \frac{1}{T^{2(d_1-d_2)+1}}\sum_{t=[T^{1/2}]}^{T} c_t^2\left(d_1,d_2,\boldsymbol{\varphi}\right), \tag{S.204}
$$

as in the proof of (S.195), the right-hand side of (S.204) is bounded from below by

$$
\epsilon \inf_{d_1\geq\Diamond, -1/2-\alpha\leq d_1-d_2\leq -1/2+\alpha} \frac{1}{T^{2(d_1-d_2)+1}}\sum_{t=[T^{1/2}]}^{T} b_t^2\left(d_1,d_2\right) \tag{S.205}
$$

$$
-\sup_{d_1\geq\Diamond, -1/2-\alpha\leq d_1-d_2\leq -1/2+\alpha, \boldsymbol{\varphi}\in\Psi} \frac{K}{T^{2(d_1-d_2)+1}}\left|\sum_{t=[T^{1/2}]}^{T} b_t^2\left(d_1,d_2\right)\sum_{j=t}^{\infty}\phi_j\left(\boldsymbol{\varphi}\right)\right| \tag{S.206}
$$

$$
-\sup_{d_1\geq\Diamond, -1/2-\alpha\leq d_1-d_2\leq -1/2+\alpha, \boldsymbol{\varphi}\in\Psi} \frac{K}{T^{2(d_1-d_2)+1}}\left|\sum_{t=[T^{1/2}]}^{T} b_t\left(d_1,d_2\right)\sum_{j=1}^{t-1} b_{j+1}\left(d_1,d_2+1\right)\sum_{l=1}^{j}\phi_{t-l}\left(\boldsymbol{\varphi}\right)\right|. \tag{S.207}
$$

First, by (6) and (S.184), (S.206) is bounded by

$$\sup_{-1/2-\alpha\leq d_1-d_2\leq -1/2+\alpha}\frac{K}{T}\sum_{t=[T^{1/2}]}^{T}\left(\frac{t}{T}\right)^{2(d_1-d_2)}t^{-\varsigma}\leq \frac{K}{T}\sum_{t=[T^{1/2}]}^{T}\left(\frac{t}{T}\right)^{-1-2\alpha}t^{-\varsigma}=O\left(T^{\alpha-\varsigma/2}\right)=o\left(1\right),$$

because $\alpha<(\varsigma-1/2)/3<\varsigma/2$. Similarly, (S.207) is bounded by

$$\begin{aligned}KT^{2\alpha}\sum_{t=[T^{1/2}]}^{T}t^{-1/2-\alpha}\sum_{j=1}^{t-1}j^{-3/2-\alpha}\left(t-j\right)^{-\varsigma}&\leq KT^{2\alpha}\log T\sum_{t=[T^{1/2}]}^{T}t^{-1/2-\alpha-\varsigma}\\&\leq KT^{3\alpha/2+1/4-\varsigma/2}\log T,\end{aligned}$$

which is $o(1)$ because $\alpha<(\varsigma-1/2)/3$.

Finally, by (S.184), (S.205) is bounded from below by

$$\begin{aligned}&\epsilon\inf_{d_1\geq\Diamond,-1/2-\alpha\leq d_1-d_2\leq -1/2+\alpha}\frac{\Gamma^2\left(d_1+1\right)}{T^{2(d_1-d_2)+1}}\sum_{t=\left[T^{1/2}\right]}^{T}\pi_t^2\left(d_1+1-d_2\right)\\&-K\sup_{d_1\geq\Diamond,-1/2-\alpha\leq d_1-d_2\leq -1/2+\alpha}\frac{1}{T^{2(d_1-d_2)+1}}\sum_{t=\left[T^{1/2}\right]}^{T}\left|\pi_t\left(d_1+1-d_2\right)\right|\left|s_{1t}\left(d_1,d_2\right)\right|.\end{aligned}\tag{S.208}$$

By (S.184) and Lemma S.11 the second term on (S.208) is bounded by

$$\begin{aligned}&K\sup_{d_1\geq\Diamond,-1/2-\alpha\leq d_1-d_2\leq -1/2+\alpha}\frac{1}{T}\sum_{t=\left[T^{1/2}\right]}^{T}\left(\frac{t}{T}\right)^{2(d_1-d_2)}\left(t^{-1}+t^{-d_1-1}\right)\\&\leq KT^{2\alpha}\sum_{t=\left[T^{1/2}\right]}^{T}t^{-1-2\alpha}\left(t^{-1}+t^{-\Diamond-1}\right)\leq K\left(T^{\alpha-1/2}+T^{\alpha-1/2-\Diamond/2}\right)=o\left(1\right),\end{aligned}$$

because $\alpha<\min\left\{(\varsigma-1/2)/3,(1+\Diamond)/2\right\}$. Next using Lemma S.11 the first term on (S.208) is bounded from below by

$$\begin{aligned}\epsilon\inf_{-1/2-\alpha\leq d_1-d_2\leq -1/2+\alpha}\frac{1}{T}\sum_{t=\left[T^{1/2}\right]}^{T}\left(\frac{t}{T}\right)^{2(d_1-d_2)}&\geq\epsilon\frac{1}{T}\sum_{t=\left[T^{1/2}\right]}^{T}\left(\frac{t}{T}\right)^{-1+2\alpha}\geq\epsilon\int_{[T^{1/2}]/T}^{1}x^{2\alpha-1}dx\\&=\epsilon\frac{1-([T^{1/2}]/T)^{2\alpha}}{2\alpha}=\frac{\epsilon}{2\alpha}-O\left(T^{-\alpha}\right)\end{aligned}$$

In view of (S.204), (S.206), and (S.207), this proves (S.198).

Finally, the proof for (S.199) is almost identical to that for (S.198) with the only difference of the treatment of the first term on (S.208). Here, noting that $d_1-d_2\leq\overline{d}_1-\underline{d}_2$, defining $g_T=T^{-\frac{1}{2}\left(2\left(\overline{d}_1-\underline{d}_2\right)+1\right)}\mathbb{I}\left(\overline{d}_1-\underline{d}_2<-1/2\right)+\log T\,\mathbb{I}\left(\overline{d}_1-\underline{d}_2=-1/2\right)+\mathbb{I}\left(\overline{d}_1-\underline{d}_2>-1/2\right)$,

$$\epsilon\inf_{d_1-d_2\geq -1/2-\alpha}\frac{1}{T}\sum_{t=\left[T^{1/2}\right]}^{T}\left(\frac{t}{T}\right)^{2(d_1-d_2)}\geq\epsilon\frac{1}{T}\sum_{t=\left[T^{1/2}\right]}^{T}\left(\frac{t}{T}\right)^{2\left(\overline{d}_1-\underline{d}_2\right)}\geq\epsilon g_T\geq\epsilon,$$

to conclude the proof of Lemma S.15. ∎

Lemma S.16 *Let θ be an arbitrary number such that $0 < \theta < \varsigma - 1/2$. Then, under Assumptions A1 and A3, for any real numbers $\lozenge > -1$, $D_1 < -1/2 - \theta$ and $D_2 > -1/2 + \theta$, $m = 0, 1$, $0 \leq p \leq m$, uniformly in $t = 1, \ldots, T$ and $T \geq 1$,*

$$\sup_{d_1 \geq \lozenge, d_1 - d_2 \in [D_1, D_2], \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial^m h_{t,T}(d_1, d_2, \boldsymbol{\varphi})}{\partial d_1^p \partial d_2^{m-p}} \right| = O(t^{-1/2-\theta} T^{\theta + 2\theta m}), \quad \text{(S.209)}$$

$$\sup_{d_1 \geq \lozenge, d_1 - d_2 \geq -1/2+\theta, \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial h_{t,T}(d_1, d_2, \boldsymbol{\varphi})}{\partial d_1} \right| = O\left(t^{-1/2+\theta} T^{-\theta} \left(1 + \left|\log(t/T)\right|\right)\right), \quad \text{(S.210)}$$

$$\sup_{d_1 \geq \lozenge, d_1 - d_2 \in [D_1, D_2], \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial^m (h_{t+1,T}(d_1, d_2, \boldsymbol{\varphi}) - h_{t,T}(d_1, d_2, \boldsymbol{\varphi}))}{\partial d_1^p \partial d_2^{m-p}} \right| = O(t^{-3/2-\theta} T^{\theta + 2\theta m}), \quad \text{(S.211)}$$

$$\sup_{d_1 \geq \lozenge, d_1 - d_2 \geq -1/2+\theta, \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial (h_{t+1,T}(d_1, d_2, \boldsymbol{\varphi}) - h_{t,T}(d_1, d_2, \boldsymbol{\varphi}))}{\partial d_1} \right| = O\left(t^{-3/2+\theta} T^{-\theta} \left(1 + \left|\log(t/T)\right|\right)\right), \quad \text{(S.212)}$$

$$\sup_{d_1 \geq \lozenge, d_1 - d_2 \in [D_1, D_2], \boldsymbol{\varphi} \in \Psi} \left| \sum_{t=1}^{T} h_{t,T}(d_1, d_2, \boldsymbol{\varphi}) \right| = O(T^{1/2}). \quad \text{(S.213)}$$

Proof. The left-hand side of (S.209) is bounded by

$$\sup_{d_1 \geq \lozenge, D_1 \leq d_1 - d_2 \leq -1/2-\theta, \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial^m}{\partial d_1^p \partial d_2^{m-p}} h_{t,T}(d_1, d_2, \boldsymbol{\varphi}) \right| + \sup_{d_1 \geq \lozenge, -1/2-\theta \leq d_1 - d_2 \leq D_2, \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial^m}{\partial d_1^p \partial d_2^{m-p}} h_{t,T}(d_1, d_2, \boldsymbol{\varphi}) \right|. \quad \text{(S.214)}$$

Suppose first that $m = 0$. Using the definition (S.3) and applying (S.192) of Lemma S.14 and (S.197) of Lemma S.15, the first term of (S.214) is bounded by

$$\frac{\sup_{d_1 \geq \lozenge, D_1 \leq d_1 - d_2 \leq -1/2-\theta, \boldsymbol{\varphi} \in \Psi} |c_t(d_1, d_2, \boldsymbol{\varphi})|}{\inf_{d_1 \geq \lozenge, D_1 \leq d_1 - d_2 \leq -1/2-\theta, \boldsymbol{\varphi} \in \Psi} \left(\sum_{j=1}^{T} c_j^2(d_1, d_2, \boldsymbol{\varphi}) \right)^{1/2}} \leq \sup_{d_1 \geq \lozenge, D_1 \leq d_1 - d_2 \leq -1/2-\theta, \boldsymbol{\varphi} \in \Psi} |c_t(d_1, d_2, \boldsymbol{\varphi})| = O(t^{-1/2-\theta}),$$

so the bound in (S.209) applies to the first term of (S.214) (although it is not tight). Next, the second term of (S.214) is bounded by

$$\frac{\sup_{d_1 \geq \lozenge, -1/2-\theta \leq d_1 - d_2 \leq D_2, \boldsymbol{\varphi} \in \Psi} T^{-(d_1 - d_2)} |c_t(d_1, d_2, \boldsymbol{\varphi})|}{\inf_{d_1 \geq \lozenge, -1/2-\theta \leq d_1 - d_2 \leq D_2, \boldsymbol{\varphi} \in \Psi} \left(T^{-2(d_1 - d_2)} \sum_{j=1}^{T} c_j^2(d_1, d_2, \boldsymbol{\varphi}) \right)^{1/2}}.$$

By (S.193) of Lemma S.14 the numerator is $O(t^{-1/2-\theta} T^{1/2+\theta})$ and by (S.199) of Lemma S.15 the denominator is bounded from below by $\epsilon T^{1/2}$. Thus (S.209) for $m = 0$ follows.

Next, for the derivative we find

$$\frac{\partial}{\partial d_1^p \partial d_2^{1-p}} h_{t,T}(d_1, d_2, \boldsymbol{\varphi}) = \frac{\partial c_t(d_1, d_2, \boldsymbol{\varphi}) / \partial d_1^p \partial d_2^{1-p}}{\left(\sum_{j=1}^T c_j^2(d_1, d_2, \boldsymbol{\varphi})\right)^{1/2}} - \frac{h_{t,T}(d_1, d_2, \boldsymbol{\varphi}) \sum_{j=1}^T c_j(d_1, d_2, \boldsymbol{\varphi}) \partial c_j(d_1, d_2, \boldsymbol{\varphi}) / \partial d_1^p \partial d_2^{1-p}}{\sum_{j=1}^T c_j^2(d_1, d_2, \boldsymbol{\varphi})}. \tag{S.215}$$

First we show (S.209). Proceeding as in the proof for $m = 0$, taking into account the extra log-term arising from (S.192) in Lemma S.14, the first term of (S.215) is $O(t^{-1/2}(T/t)^\theta \log T)$, so the bound in (S.209) applies. Next, using again (S.192) in Lemma S.14 and also (S.209) for $m = 0$, the second term of (S.215) is $O(t^{-1/2}(T/t)^\theta T^{2\theta} \sum_{j=1}^T j^{-1-2\theta} \log j)$, so the bound in (S.209) applies for $m = 1$.

Next we show (S.210). Clearly

$$\frac{\partial h_{t,T}(d_1, d_2, \boldsymbol{\varphi})}{\partial d_1} = \frac{\partial T^{-(d_1-d_2)} c_t(d_1, d_2, \boldsymbol{\varphi}) / \partial d_1}{\left(\sum_{j=1}^T T^{-2(d_1-d_2)} c_j^2(d_1, d_2, \boldsymbol{\varphi})\right)^{1/2}} - \frac{T^{-(d_1-d_2)} c_t(d_1, d_2, \boldsymbol{\varphi}) \sum_{j=1}^T T^{-(d_1-d_2)} c_j(d_1, d_2, \boldsymbol{\varphi}) \partial T^{-(d_1-d_2)} c_j(d_1, d_2, \boldsymbol{\varphi}) / \partial d_1}{\left(\sum_{j=1}^T T^{-2(d_1-d_2)} c_j^2(d_1, d_2, \boldsymbol{\varphi})\right)^{3/2}}.$$

First,

$$\begin{aligned}
&\sup_{d_1 \geq \Diamond, d_1-d_2 \geq -1/2+\theta, \boldsymbol{\varphi} \in \Psi} \left| \frac{\partial T^{-(d_1-d_2)} c_t(d_1, d_2, \boldsymbol{\varphi}) / \partial d_1}{\left(\sum_{j=1}^T T^{-2(d_1-d_2)} c_j^2(d_1, d_2, \boldsymbol{\varphi})\right)^{1/2}} \right| \\
&\leq \frac{\sup_{d_1 \geq \Diamond, d_1-d_2 \geq -1/2+\theta, \boldsymbol{\varphi} \in \Psi} \left| \partial T^{-(d_1-d_2)} c_t(d_1, d_2, \boldsymbol{\varphi}) / \partial d_1 \right|}{\left(\inf_{d_1 \geq \Diamond, d_1-d_2 \geq -1/2+\theta, \boldsymbol{\varphi} \in \Psi} \sum_{j=1}^T T^{-2(d_1-d_2)} c_j^2(d_1, d_2, \boldsymbol{\varphi})\right)^{1/2}} \\
&= O\left(t^{-1/2}(t/T)^\theta (1 + |\log(t/T)|)\right)
\end{aligned} \tag{S.216}$$

by (S.193) of Lemma S.14 and (S.199) of Lemma S.15. Similarly, like in (S.216),

$$\sup_{d_1 \geq \Diamond, d_1-d_2 \geq -1/2+\theta, \boldsymbol{\varphi} \in \Psi} \left| \frac{T^{-(d_1-d_2)} c_t(d_1, d_2, \boldsymbol{\varphi})}{\left(\sum_{j=1}^T T^{-2(d_1-d_2)} c_j^2(d_1, d_2, \boldsymbol{\varphi})\right)^{1/2}} \right| = O\left(t^{-1/2}(t/T)^\theta\right), \tag{S.217}$$

so, by (S.216) and (S.217), it is straightforward to show that

$$\sup_{d_1 \geq \Diamond, d_1-d_2 \geq -1/2+\theta, \boldsymbol{\varphi} \in \Psi} \left| \frac{\frac{1}{T^{d_1-d_2}} c_t(d_1, d_2, \boldsymbol{\varphi}) \sum_{j=1}^T \frac{1}{T^{d_1-d_2}} c_j(d_1, d_2, \boldsymbol{\varphi}) \partial \frac{1}{T^{d_1-d_2}} c_j(d_1, d_2, \boldsymbol{\varphi}) / \partial d_1}{\left(\sum_{j=1}^T T^{-2(d_1-d_2)} c_j^2(d_1, d_2, \boldsymbol{\varphi})\right)^{3/2}} \right|$$

$$= O\left(t^{-1/2}(t/T)^\theta\right),$$

to conclude the proof of (S.210).

The proofs of (S.211)–(S.213) are omitted because they follow by identical arguments, noting that

$$h_{t,T}(d_1,d_2,\boldsymbol{\varphi}) - h_{t-1,T}(d_1,d_2,\boldsymbol{\varphi}) = \frac{c_t(d_1,d_2+1,\boldsymbol{\varphi})}{\left(\sum_{j=1}^{T} c_j^2(d_1,d_2,\boldsymbol{\varphi})\right)^{1/2}}.$$

■

Lemma S.17 *Under Assumptions A1–A3, uniformly in $t = 1,\ldots,T$, $T \geq 1$, and $\boldsymbol{\varphi} \in \Psi$,*

$$\sup_{d\leq g} |\phi(L;\boldsymbol{\varphi})\Delta^{-d}\{u_t \mathbb{I}(t\geq 1)\}| = O_p(t^{g-1/2} + \log t\mathbb{I}(g=1/2) + \mathbb{I}(g<1/2)), \quad \text{(S.218)}$$

$$\sup_{d\geq g} |T^{-d}\phi(L;\boldsymbol{\varphi})\Delta^{-d}\{u_t \mathbb{I}(t\geq 1)\}| = O_p(T^{-g}(t^{g-1/2} + \log t\mathbb{I}(g=1/2) + \mathbb{I}(g<1/2))). \quad \text{(S.219)}$$

Proof. First we show (S.218). Write $\phi(L;\boldsymbol{\varphi})\Delta^{-d}\{u_t\mathbb{I}(t\geq 1)\} = \sum_{j=0}^{t-1} a_j(d,\boldsymbol{\varphi}) u_{t-j}$ and apply summation by parts,

$$\sum_{j=0}^{t-1} a_j(d,\boldsymbol{\varphi}) u_{t-j} = a_{t-1}(d,\boldsymbol{\varphi}) \sum_{j=0}^{t-1} u_{t-j} - \sum_{j=0}^{t-2} (a_{j+1}(d,\boldsymbol{\varphi}) - a_j(d,\boldsymbol{\varphi})) \sum_{l=0}^{j} u_{t-l}. \quad \text{(S.220)}$$

Noting that $a_{j+1}(d,\boldsymbol{\varphi}) - a_j(d,\boldsymbol{\varphi}) = a_{j+1}(d-1,\boldsymbol{\varphi})$, the right-hand side of (S.220) is bounded by

$$|a_{t-1}(d,\boldsymbol{\varphi})| \left|\sum_{j=0}^{t-1} u_{t-j}\right| + \sum_{j=0}^{t-2} |a_{j+1}(d-1,\boldsymbol{\varphi})| \left|\sum_{l=0}^{j} u_{t-l}\right|. \quad \text{(S.221)}$$

Under our conditions, $E\left|\sum_{l=1}^{t} u_l\right| = O\left(t^{1/2}\right)$, so, in view of (S.194) of Lemma S.14, the expectation of the left-hand side of (S.218) is bounded by

$$Kt^{\max\{g-1/2,-1/2-\varsigma\}} + K\sum_{j=1}^{t} j^{\max\{g-3/2,-1/2-\varsigma\}} \leq K(t^{g-1/2} + \log t\mathbb{I}(g=1/2) + \mathbb{I}(g<1/2))$$

to conclude the proof of (S.218). The proof of (S.219) is omitted because it is almost identical to that for (S.218). ■

Lemma S.18 *Let θ be an arbitrary number such that $0 < \theta < \varsigma - 1/2$. Then, under Assumptions A1–A3, for $m = 0,1$, and uniformly in $\boldsymbol{\vartheta} \in \Xi$,*

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\leq g} \left|\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) \frac{\partial^m}{\partial\gamma^m} h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\right| = O_p(T^{\max\{\theta,g\}+2\theta m}), \quad \text{(S.222)}$$

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\geq g} \frac{1}{T^{\delta_0-\delta}} \left|\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) \frac{\partial^m}{\partial\gamma^m} h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\right| = O_p(T^{\max\{\theta,g\}-g+2\theta m}), \quad \text{(S.223)}$$

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\leq g,\gamma-\delta\geq -1/2+\theta} \left|\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) \frac{\partial}{\partial\gamma} h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\right| = O_p(T^{\max\{\theta,g\}}), \quad \text{(S.224)}$$

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\geq g,\gamma-\delta\geq -1/2+\theta} \frac{1}{T^{\delta_0-\delta}} \left|\sum_{t=1}^{T} \varepsilon_t(\boldsymbol{\tau}) \frac{\partial}{\partial\gamma} h_{t,T}(\gamma,\delta,\boldsymbol{\varphi})\right| = O_p(T^{\max\{\theta,g\}-g}). \quad \text{(S.225)}$$

Proof. By summation by parts as in (S.237), we find

$$\left|\sum_{t=1}^{T}\varepsilon_t(\boldsymbol{\tau})h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right| \leq \left|h_{T,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right|\left|\varepsilon_T(\delta_0-\delta+1,\boldsymbol{\varphi})\right| + \sum_{t=1}^{T-1}\left|h_{t+1,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)-h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right|\left|\varepsilon_t(\delta_0-\delta+1,\boldsymbol{\varphi})\right|,$$

noting (S.27). First, application of (S.209), (S.211) of Lemma S.16 together with (S.218), (S.219) of Lemma S.17 implies (S.222) and (S.223). Next, (S.224) and (S.225) follow from (S.210), (S.212) of Lemma S.16 and (S.218), (S.219) of Lemma S.17. ■

Lemma S.19 *Under Assumptions A1–A3, for any $g_2 > -1/2$ and for any arbitrary θ such that $0 < \theta < \min\{\varsigma - 1/2, g_2 + 1/2\}$,*

$$\left|\sum_{t=1}^{T} d_t\left(\boldsymbol{\vartheta}\right) s_t\left(\boldsymbol{\vartheta}\right)\right| \leq \left|\gamma-\gamma_0\right|\left|M_T\left(\boldsymbol{\vartheta}\right)\right|, \tag{S.226}$$

where, uniformly in $\boldsymbol{\vartheta} \in \Xi$,

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\leq g_1,\gamma_0-\delta\leq g_2}\left|M_T\left(\boldsymbol{\vartheta}\right)\right| = O_p(T^{\max\{\theta,g_1\}+2\theta+g_2+1/2}), \tag{S.227}$$

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\leq g_1,\gamma_0-\delta\geq g_2} T^{\delta-\gamma_0}\left|M_T\left(\boldsymbol{\vartheta}\right)\right| = O_p(T^{\max\{\theta,g_1\}+2\theta+1/2}), \tag{S.228}$$

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\geq g_1,\gamma_0-\delta\geq g_2} T^{2\delta-\delta_0-\gamma_0}\left|M_T\left(\boldsymbol{\vartheta}\right)\right| = O_p(T^{\max\{\theta,g_1\}+2\theta-g_1+1/2}), \tag{S.229}$$

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\leq g_1,\gamma_0-\delta\geq g_2,\gamma-\delta\geq -1/2+\theta} T^{\delta-\gamma_0}\left|M_T\left(\boldsymbol{\vartheta}\right)\right| = O_p(T^{\max\{\theta,g_1\}+1/2}), \tag{S.230}$$

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\geq g_1,\gamma_0-\delta\geq g_2,\gamma-\delta\geq -1/2+\theta} T^{2\delta-\delta_0-\gamma_0}\left|M_T\left(\boldsymbol{\vartheta}\right)\right| = O_p(T^{\max\{\theta,g_1\}-g_1+1/2}). \tag{S.231}$$

Proof. Letting $d_t\left(\boldsymbol{\tau},\gamma\right) = d_t\left(\boldsymbol{\vartheta}\right)$, noting (S.15) and that $d_t\left(\boldsymbol{\tau},\gamma_0\right) = 0$, by the mean value theorem,

$$\left|\sum_{t=1}^{T} d_t\left(\boldsymbol{\vartheta}\right) s_t\left(\boldsymbol{\vartheta}\right)\right| \leq \left|\gamma-\gamma_0\right|\left|\frac{\partial}{\partial\gamma}\sum_{t=1}^{T} d_t\left(\boldsymbol{\tau},\overline{\gamma}\right)\varepsilon_t(\boldsymbol{\tau})\right|,$$

where $\left|\overline{\gamma}-\gamma_0\right| \leq \left|\gamma-\gamma_0\right|$. Then we find the bound

$$\left|\frac{\partial}{\partial\gamma}\sum_{t=1}^{T} d_t\left(\boldsymbol{\tau},\gamma\right)\varepsilon_t(\boldsymbol{\tau})\right| \leq \left|\sum_{t=1}^{T}\varepsilon_t(\boldsymbol{\tau})\frac{\partial h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)}{\partial\gamma}\sum_{j=1}^{T} c_j\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)h_{j,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right| + \left|\sum_{t=1}^{T}\varepsilon_t(\boldsymbol{\tau})h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\sum_{j=1}^{T} c_j\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)\frac{\partial h_{j,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)}{\partial\gamma}\right|.$$

The results (S.227)–(S.229) now all follow by direct application of (S.232), (S.233) of Lemma S.20 with $\theta < g + 1/2$ and (S.222), (S.223) of Lemma S.18. Results (S.230) and (S.231) are derived straightforwardly from (S.224), (S.225) of Lemma S.18 and (S.234) of Lemma S.20. ■

Lemma S.20 *Let θ be an arbitrary number such that $0 < \theta < \varsigma - 1/2$. Then, under Assumptions A1 and A3, for $m = 0, 1$ and uniformly in $\boldsymbol{\vartheta} \in \Xi$,*

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\gamma_0-\delta\leq g}\left|\sum_{t=1}^{T} c_t\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)\frac{\partial^m}{\partial\gamma^m}h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right| = O(T^{\max\{\theta,g+1/2\}+2\theta m}\log T), \quad \text{(S.232)}$$

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\gamma_0-\delta\geq g}\frac{1}{T^{\gamma_0-\delta}}\left|\sum_{t=1}^{T} c_t\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)\frac{\partial^m}{\partial\gamma^m}h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right| = O(T^{\max\{\theta,g+1/2\}-g+2\theta m}\log T), \quad \text{(S.233)}$$

and for $g > -1/2$, uniformly in $\boldsymbol{\vartheta} \in \Xi$,

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\gamma_0-\delta\geq g,\gamma-\delta\geq -1/2+\theta}\frac{1}{T^{\gamma_0-\delta}}\left|\sum_{t=1}^{T} c_t\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)\frac{\partial}{\partial\gamma}h_{t,T}\left(\gamma,\delta,\boldsymbol{\varphi}\right)\right| = O(T^{1/2}). \quad \text{(S.234)}$$

Proof. The results follow by direct application of (S.209), (S.210) in Lemma S.16 and (S.192), (S.193) of Lemma S.14. ■

Lemma S.21 *Under Assumptions A1–A3, uniformly in $\boldsymbol{\vartheta} \in \Xi$,*

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\leq g_1,\gamma_0-\delta\leq g_2}\frac{1}{T}\left|\sum_{t=1}^{T}\varepsilon_t(\boldsymbol{\tau})c_t\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)\right|$$
$$= O_p(T^{g_1+g_2-1/2} + T^{-1}\log T + T^{g_1-1/2-\varsigma} + T^{g_2-1}\log^2 T\mathbb{I}\left(g_1\leq -1/2\right)), \quad \text{(S.235)}$$

$$\sup_{\boldsymbol{\vartheta}\in\Xi,\delta_0-\delta\geq g_1,\gamma_0-\delta\geq g_2}\frac{1}{T^{\gamma_0+\delta_0-2\delta}}\left|\sum_{t=1}^{T}\varepsilon_t(\boldsymbol{\tau})c_t\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)\right|$$
$$= O_p(T^{1/2} + T^{1/2-g_2-\varsigma} + T^{-g_1-g_2}\log T + T^{-g_1}\log^2 T\mathbb{I}\left(g_1\leq -1/2\right)). \quad \text{(S.236)}$$

Proof. By summation by parts and (S.53) we find

$$\left|\sum_{t=1}^{T}\varepsilon_t(\boldsymbol{\tau})c_t\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)\right| \leq \left|c_T\left(\gamma_0,\delta,\boldsymbol{\varphi}\right)\right|\left|\varepsilon_T(\delta_0-\delta+1,\boldsymbol{\varphi})\right|$$
$$+\left|\sum_{t=1}^{T-1}c_{t+1}\left(\gamma_0,\delta+1,\boldsymbol{\varphi}\right)\varepsilon_t(\delta_0-\delta+1,\boldsymbol{\varphi})\right|, \quad \text{(S.237)}$$

see (S.27). The result (S.235) then follows by application of (S.218) of Lemma S.17 and (S.192) of Lemma S.14, while the result (S.236) follows by application of (S.219) of Lemma S.17 and (S.193) of Lemma S.14. ■

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