Online Supplementary Material to "A local Gaussian bootstrap method for realized volatility and realized beta"

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This document contains supplementary results to be published online alongside the main paper, "A local Gaussian bootstrap method for realized volatility and realized beta". All references to section numbers are references to sections in the main paper. Note that there is no overlap between the labelling of equations below, asymptotic results as well as equations in the main text. We organized this online appendix as follows. First, in Appendix C1, we state some auxiliary lemmas useful for the proof of results in Sections 3 and 5.1. In particular, we state Lemmas C1.1 and C1.2 and their proofs, which are useful for establishing Theorem 5.1 in the main text. These results are used to obtain the formal Edgeworth expansions through order $O(h)$ for realized volatility. In Appendix C1, we also state Lemma C1.3, which is utilized when deriving the mean squared error of the bootstrap variance estimator that appears in Section 3. Second, in Appendix C2, we provide key lemmas for the proofs of results in Theorem 5.2. Third, in Appendix C3, we provide key marginal limit results for the proofs of the results appearing in Section 6.2.

Appendix C1: Key lemmas for the proofs of results in Sections 3 and 5.1

To make for greater comparability, and in order to use some existing results, we have kept the notation from Gonçalves and Meddahi (2009) whenever possible. We introduce some notation, recall that, for any $q > 0$, $\overline{\sigma^q} \equiv \int_0^1 \sigma_u^q du$, and let $\bar{\sigma}_h^q \equiv h \sum_{i=1}^{1/h} \left(\frac{\sigma_i^2}{h}\right)^{q/2}$, where $\sigma_i^2 \equiv \int_{(i-1)h}^{ih} \sigma_u^2 du$. We let $\sigma_{q,p} \equiv \frac{\overline{\sigma^q}}{\left(\overline{\sigma^p}\right)^{q/p}}$, when $\overline{\sigma^q}$ is replaced with $\overline{\sigma_h^q}$ we write $\sigma_{q,p,h}$, and $R_q \equiv Mh \sum_{j=1}^{1/Mh} \left(\frac{RV_{j,M}}{Mh}\right)^{q/2}$, where $RV_{j,M} = \sum_{i=1}^{M} y_{i+(j-1)M}^2$. We also let $R_{q,p} \equiv \frac{R_q}{(R_p)^{q/p}}$. Let $\mu_q = E\left|\eta\right|^q$ where $\eta \sim N(0,1)$, with $q > 0$, and note that $\mu_2 = 1$, $\mu_4 = 3$, $\mu_6 = 15$, and $\mu_8 = 105$. Recall that $c_{M,q} \equiv E\left(\left(\frac{\chi_M^2}{M}\right)^{q/2}\right)$ with χ_M^2 the standard χ^2 distribution with M

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degrees of freedom. Note that $c_{M,2}=1$, $c_{M,4}=\frac{M+2}{M}$, $c_{M,6}=\frac{(M+2)(M+4)}{M^2}$ and $c_{M,8}=\frac{(M+2)(M+4)(M+6)}{M^3}$. This follows by using the definition of $c_{M,q}$ gives in equation (7) in the main text and the fact that, for all $x>0$, $\Gamma(x+1)=x\Gamma(x)$. Recall the definition of $T_{\bar{\sigma}^2,h}$ given in (27) in the main text and $\hat{V}_{\bar{\sigma}^2,h}=\frac{(\mu_4-\mu_2^2)}{\mu_4}h^{-1}\sum_{i=1}^{h^{-1}}y_i^4$.

We follow Gonçalves and Meddahi (2009) and we write

$$T^*_{\bar{\sigma}^2,h,M}=\frac{\sqrt{h^{-1}}\left(R_2^*-\mu_2R_2\right)}{\sqrt{\hat{V}^*_{\bar{\sigma}^2,h}}},$$

where

$$\hat{V}^*_{\bar{\sigma}^2,h,M}=\frac{\left(\mu_4-\mu_2^2\right)}{\mu_4}h^{-1}\sum_{i=1}^{h^{-1}}y_i^{*4}=\frac{\left(\mu_4-\mu_2^2\right)}{\mu_4}Mh\sum_{i=1}^{1/Mh}\left(\frac{RV_{j,M}}{Mh}\right)^2\frac{1}{M}\sum_{i=1}^{M}\left(\eta^{*4}_{i+(j-1)M}-\mu_4\right).$$

We can write

$$T^*_{\bar{\sigma}^2,h,M}=S^*_{\bar{\sigma}^2,h,M}\left(\frac{\hat{V}^*_{\bar{\sigma}^2,h,M}}{V^*_{\bar{\sigma}^2,h,M}}\right)^{-1/2}=S^*_{\bar{\sigma}^2,h,M}\left(1+\sqrt{h}U^*_{\bar{\sigma}^2,h,M}\right)^{-1/2},$$

where

$$S^*_{\bar{\sigma}^2,h,M}=\frac{\sqrt{h^{-1}}\left(R_2^*-\mu_2R_2\right)}{\sqrt{V^*_{\bar{\sigma}^2,h,M}}},\ U^*_{\bar{\sigma}^2,h,M}\equiv\frac{\sqrt{h^{-1}}\left(\hat{V}^*_{\bar{\sigma}^2,h}-V^*_{\bar{\sigma}^2,h,M}\right)}{V^*_{\bar{\sigma}^2,h,M}},$$

and $V^*_{\bar{\sigma}^2,h,M}=Var^*\left(n^{1/2}R_2^*\right)=\left(\mu_4-\mu_2^2\right)\cdot Mh\sum_{i=1}^{1/Mh}\left(\frac{RV_{j,M}}{Mh}\right)^2$.

Note that for any $q>0$, $\left|\frac{RV_{j,M}}{Mh}\right|^{q/2}\frac{h^{q/2}}{M}\sum_{i=1}^{M}\left(\left|\eta^*_{i+(j-1)M}\right|^q-\mu_q\right)$ are conditionally on σ independent with zero mean since $\eta^*_{i+(j-1)M}\sim i.i.d.\ N(0,1)$. We rewrite $R_2^*-\mu_2R_2$ and $\hat{V}^*_{\bar{\sigma}^2,h,M}-V^*_{\bar{\sigma}^2,h,M}$ as follows

$$R_2^*-\mu_2R_2=Mh\sum_{j=1}^{1/Mh}\left|\frac{RV_{j,M}}{Mh}\right|\frac{1}{M}\sum_{i=1}^{M}\left(\left|\eta^*_{i+(j-1)M}\right|^2-\mu_2\right),$$

$$\hat{V}^*_{\bar{\sigma}^2,h,M}-V^*_{\bar{\sigma}^2,h,M}=\frac{\left(\mu_4-\mu_2^2\right)}{\mu_4}Mh\sum_{j=1}^{1/Mh}\left|\frac{RV_{j,M}}{Mh}\right|^2\frac{1}{M}\sum_{i=1}^{M}\left(\left|\eta^*_{i+(j-1)M}\right|^4-\mu_4\right).$$

Lemma C1.1. *Suppose (1), (2) and (3) hold. Let $M\geq 1$ such that $M\approx ch^{-\alpha}$ with $\alpha\in[0,1)$ for any $q>0$, we have*

(a1) $E^*\left(\left|y^*_{i+(j-1)M}\right|^q\right)=\mu_q\left|\frac{RV_{j,M}}{Mh}\right|^{q/2}h^{q/2}$, *for $i=1,\ldots,M$ and $j=1,\ldots,\frac{1}{Mh}$,*

(a2) $V^*_{\bar{\sigma}^2,h,M}\equiv Var^*\left(\sqrt{h^{-1}}R_2^*\right)=\left(\mu_4-\mu_2^2\right)R_4$,

(a3) $E^*\left[\left(R_2^*-\mu_2R_2\right)^3\right]=h^2\left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)R_6$,

(a4) $E^*\left[\left(R_2^* - \mu_2 R_2\right)^4\right] = 3h^2\left(\mu_4 - \mu_2^2\right)^2 (R_4)^2 + h^3\left(\mu_8 - 4\mu_2\mu_6 + 12\mu_2^2\mu_4 - 6\mu_2^4 - 3\mu_4^2\right) R_8,$

(a5) $E^*\left[\left(R_2^* - \mu_2 R_2\right)\left(\hat{V}^*_{\bar{\sigma}^2,h,M} - V^*_{\bar{\sigma}^2,h,M}\right)\right] = \left(\frac{\mu_4 - \mu_2^2}{\mu_4}\right)\left(\mu_6 - \mu_2\mu_4\right) h R_6,$

(a6) $E^*\left[\left(R_2^* - \mu_2 R_2\right)^2\left(\hat{V}^*_{\bar{\sigma}^2,h,M} - V^*_{\bar{\sigma}^2,h,M}\right)\right] = \left(\frac{\mu_4 - \mu_2^2}{\mu_4}\right) h^2\left(\mu_8 - \mu_4^2 - 2\mu_2\mu_6 + \mu_2^2\mu_4\right) R_8,$

(a7) $E\left[\left(R_2^* - \mu_2 R_2\right)^3\left(\hat{V}^*_{\bar{\sigma}^2,h,M} - V^*_{\bar{\sigma}^2,h,M}\right)\right] = 3h^2\frac{\left(\mu_4 - \mu_2^2\right)^2\left(\mu_6 - \mu_2\mu_4\right)}{\mu_4} R_4 R_6 + O_P\left(h^3\right)$ *as* $h \to 0,$

(a8) $E^*\left[\left(R_2^* - \mu_2 R_2\right)^4\left(\hat{V}^*_{\bar{\sigma}^2,h,M} - V^*_{\bar{\sigma}^2,h,M}\right)\right]$

$$= h^3\frac{\mu_4 - \mu_2^2}{\mu_4}\left[\begin{array}{c} 4\left(\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3\right)\left(\mu_6 - \mu_2\mu_4\right) R_6^2 \\ +6\left(\mu_8 - \mu_4^2 - 2\mu_2\mu_6 + 2\mu_2^2\mu_4\right)\left(\mu_4 - \mu_2^2\right) R_4 R_8 \end{array}\right] + O_P\left(h^4\right) \text{ as } h \to 0,$$

(a9) $E^*\left[\left(R_2^* - \mu_2 R_2\right)\left(\hat{V}^*_{\bar{\sigma}^2,h,M} - V^*_{\bar{\sigma}^2,h,M}\right)^2\right] = O_P\left(h^2\right)$ *as* $h \to 0,$

(a10) $E^*\left[\left(R_2^* - \mu_2 R_2\right)^2\left(\hat{V}^*_{\bar{\sigma}^2,h,M} - V^*_{\bar{\sigma}^2,h,M}\right)^2\right]$

$$= h^2\frac{\left(\mu_4 - \mu_2^2\right)^2}{\mu_4^2}\left(\left(\mu_4 - \mu_2^2\right)\left(\mu_8 - \mu_4^2\right) R_4 R_8 + 2\left(\mu_6 - \mu_2\mu_4\right)^2 R_6^2\right) + O_P\left(h^3\right) \text{ as } h \to 0,$$

(a11) $E^*\left[\left(R_2^* - \mu_2 R_2\right)^3\left(\hat{V}^*_{\bar{\sigma}^2,h,M} - V^*_{\bar{\sigma}^2,h,M}\right)^2\right] = O_P\left(h^3\right)$ *as* $h \to 0,$

(a12) $E^*\left[\left(R_2^* - \mu_2 R_2\right)^4\left(\hat{V}^*_{\bar{\sigma}^2,h,M} - V^*_{\bar{\sigma}^2,h,M}\right)^2\right]$

$$= h^3\frac{\left(\mu_4 - \mu_2^2\right)^2}{\mu_4^2}\left[3\left(\mu_4 - \mu_2^2\right)^2\left(\mu_8 - \mu_4^2\right) R_4^2 R_8 + 12\left(\mu_4 - \mu_2^2\right)\left(\mu_6 - \mu_2\mu_4\right) R_6^2 R_4\right] + O_P\left(h^4\right) \text{ as } h \to 0.$$

Lemma C1.2. *Suppose (1), (2) and (3) hold. Let* $M \geq 1$ *such that* $M \approx ch^{-\alpha}$ *with* $\alpha \in [0,1)$ *for any* $q > 0$*, we have*

$$\begin{aligned} E^*\left(S^*_{\bar{\sigma}^2,h,M}\right) &= 0, \\ E^*\left(S^{*2}_{\bar{\sigma}^2,h,M}\right) &= 1, \\ E^*\left(S^{*3}_{\bar{\sigma}^2,h,M}\right) &= \sqrt{h} B_1 R_{6,4}, \\ E^*\left(S^{*4}_{\bar{\sigma}^2,h,M}\right) &= 3 + h B_2 R_{8,4}, \\ E^*\left(S^*_{\bar{\sigma}^2,h,M} U^*_{\bar{\sigma}^2,h,M}\right) &= A_1 R_{6,4}, \\ E^*\left(S^{*2}_{\bar{\sigma}^2,h,M} U^*_{\bar{\sigma}^2,h,M}\right) &= \sqrt{h} A_2 R_{8,4}, \end{aligned}$$

and as $h \to 0$ we have,

$$
\begin{aligned}
E^*\left(S_{h,M}^3 U_{\bar{\sigma}^2,h,M}^*\right) &= A_3 R_{6,4} + O_P\left(h\right), \\
E^*\left(S_{h,M}^4 U_{\bar{\sigma}^2,h,M}^*\right) &= \sqrt{h}\left(D_1 R_{8,4} + D_2 R_{6,4}^2\right) + O_P\left(h^{3/2}\right), \\
E^*\left(S_{\bar{\sigma}^2,h,M}^* U_{\bar{\sigma}^2,h,M}^{*2}\right) &= O_P\left(h^{1/2}\right), \\
E^*\left(S_{\bar{\sigma}^2,h,M}^{*3} U_{\bar{\sigma}^2,h,M}^{*2}\right) &= O_P\left(h^{1/2}\right), \\
E^*\left(S_{\bar{\sigma}^2,h,M}^{*2} U_{\bar{\sigma}^2,h,M}^{*2}\right) &= C_1 R_{8,4} + C_2 R_{6,4}^2 + O_P\left(h\right), \\
E^*\left(S_{\bar{\sigma}^2,h,M}^{*4} U_{\bar{\sigma}^2,h,M}^{*2}\right) &= E_1 R_{8,4} + E_2 R_{6,4}^2 + O_P\left(h\right).
\end{aligned}
$$

The constant A_1, B_1, A_2, B_2 and C_1 are defined in the text, and we have $A_3 = 3A_1$, $C_2 = 2A_1^2$, $D_1 = 6A_2$, $D_2 = 4A_1B_1$, $E_1 = 3C_1$ and $E_2 = 12A_1^2$.

Proof of Lemma C1.1 (a1) follows from $\left|y_{i+(j-1)M}^*\right|^q = \left|\frac{RV_{j,M}}{Mh}\right|^{q/2} h^{q/2}\left|\eta_{i+(j-1)M}^*\right|^q$, for $i = 1,\ldots,M$ and $j = 1,\ldots,\frac{1}{Mh}$, where $\eta_{i+(j-1)M}^* \sim i.i.d.\ N(0,1)$. For (a2) recall the definition of R_2^* and remark that given the result in (a1) we have $E^*\left(R_2^*\right) = \mu_2 R_2$, then note that we can write

$$
R_2^* - \mu_2 R_2 = Mh \sum_{j=1}^{1/Mh} \left|\frac{RV_{j,M}}{Mh}\right| \frac{1}{M} \sum_{i=1}^{M} \left(\left|\eta_{i+(j-1)M}^*\right|^2 - \mu_2\right).
$$

It follows that,

$$
\begin{aligned}
Var^*\left(\sqrt{h^{-1}} R_2^*\right) &= h^{-1} E^*\left[R_2^* - \mu_2 R_2\right]^2 \\
&= h^{-1}\left(Mh\right)^2 \sum_{j=1}^{1/Mh} \left|\frac{RV_{j,M}}{Mh}\right|^2 \frac{1}{M^2} \sum_{i=1}^{M} E^*\left(\left|\eta_{i+(j-1)M}^*\right|^2 - \mu_2\right)^2 \\
&= \left(\mu_4 - \mu_2^2\right) Mh \sum_{j=1}^{1/Mh} \left|\frac{RV_{j,M}}{Mh}\right|^2 \\
&= \left(\mu_4 - \mu_2^2\right) R_4.
\end{aligned}
$$

For (a3), we have

$$
\begin{aligned}
E^*\left[R_2^* - \mu_2 R_2\right]^3 &= E^*\left[M \sum_{j=1}^{1/Mh} \left|\frac{RV_{j,M}}{Mh}\right| \frac{h}{M} \sum_{i=1}^{M} \left(\left|\eta_{i+(j-1)M}^*\right|^2 - \mu_2\right)\right]^3 \\
&= h^3 \sum_{j_1=1}^{1/Mh} \sum_{j_2=1}^{1/Mh} \sum_{j_3=1}^{1/Mh} \left|\frac{RV_{j_1,M}}{Mh}\right| \left|\frac{RV_{j_2,M}}{Mh}\right| \left|\frac{RV_{j_3,M}}{Mh}\right| \sum_{i_1=1}^{M} \sum_{i_2=1}^{M} \sum_{i_3=1}^{M} I_{i_1,j_1,i_2,j_2,i_3,j_3}^*,
\end{aligned}
$$

where $I_{i_1,j_1,i_2,j_2,i_3,j_3}^* = E^*\left(\left|\eta_{i_1+(j_1-1)M}^*\right|^2 - \mu_2\right)\left(\left|\eta_{i_2+(j_2-1)M}^*\right|^2 - \mu_2\right)\left(\left|\eta_{i_3+(j_3-1)M}^*\right|^2 - \mu_2\right)$. It is easy to see that the only nonzero contribution to $I_{i_1,j_1,i_2,j_2,i_3,j_3}^*$ is when $(i_1,j_1) = (i_2,j_2) = (i_3,j_3)$, in which case we obtain

$$I^*_{i_1,j_1,i_2,j_2,i_3,j_3} = \sum_{i=1}^{M} \left(\left| \eta^*_{i+(j-1)M} \right|^2 - \mu_2 \right)^3 = M \left(\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3 \right).$$

Hence,

$$\begin{aligned} E^* \left[R_2^* - \mu_2 R_2 \right]^3 &= h^3 \sum_{j=1}^{1/Mh} \left| \frac{RV_{j,M}}{Mh} \right|^3 M \left(\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3 \right) \\ &= h^2 \left(\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3 \right) R_6. \end{aligned}$$

To prove the remaining results, we follow the same structure of proofs as Gonçalves and Meddahi (2009). Here $\left| \frac{RV_{j,M}}{Mh} \right|^{q/2} \frac{h^{q/2}}{M} \sum_{i=1}^{M} \left(\left| \eta^*_{i+(j-1)M} \right|^q - \mu_q \right)$ plays the role of $|r_i^*|^q - \mu_q |r_i|^q$, where r_i^* denotes the wild bootstrap returns in Gonçalves and Meddahi (2009).

Proof of Lemma C1.2 Results follow immediately by using Lemma C1.1 given the definitions of $S^*_{\bar{\sigma}^2,h,M}$ and $U^*_{\bar{\sigma}^2,h,M}$.

Lemma C1.3. *Consider the simplified model $dX_t = \sigma dW_t$ (i.e. the constant volatility model without drift), then we have*

a1) $Bias\left(V^*_{\bar{\sigma}^2,h,M} \right) = \frac{4\sigma^2}{M}$,

a2) $Var\left(V^*_{\bar{\sigma}^2,h,M} \right) = 4Mh \frac{(M+2)\left(M^2+9M+24\right)}{M^3} \sigma^4$

Proof of Lemma C1.3 part a1). Given that $dX_t = \sigma dW_t$, for a given frequency h of the observations, we can write $y_i = X_{ih} - X_{(i-1)h} =^d \sigma h^{1/2} \nu_i$, with $\nu_i \sim$ i.i.d. $N(0,1)$. It follows that

$$\begin{aligned} RV_{j,M} &= \sigma^2 h \left(\sum_{i=1}^{M} \nu^2_{i+(j-1)M} \right) = \sigma^2 h \cdot \chi^2_{M,j}, \\ R_4 &= (Mh)^{-1} \sum_{j=1}^{1/Mh} RV^2_{j,M} = \sigma^4 h^2 (Mh)^{-1} \sum_{j=1}^{1/Mh} \left(\chi^2_{M,j} \right)^2, \\ V^*_{\bar{\sigma}^2,h,M} &= M \left(c_{M,4} - c^2_{M,2} \right) R_4 = 2\sigma^4 \frac{h}{M} \sum_{j=1}^{1/Mh} \left(\chi^2_{M,j} \right)^2, \text{ and } V_{\bar{\sigma}^2} = 2\sigma^4. \end{aligned}$$

where $\chi^2_{M,j}$ is i.i.d.$\sim \chi^2_M$. Thus,

$$\begin{aligned} Bias\left(V^*_{\bar{\sigma}^2,h,M} \right) &= E\left(V^*_{\bar{\sigma}^2,h,M} \right) - V_{\bar{\sigma}^2} \\ &= 2\sigma^4 \frac{h}{M} \sum_{j=1}^{1/Mh} E\left(\chi^2_{M,j} \right)^2 - 2\sigma^4 \\ &= 2\sigma^4 \frac{h}{M} (Mh)^{-1} \left(2M + M^2 \right) - 2\sigma^4 = \frac{4\sigma^2}{M}. \end{aligned}$$

Proof of Lemma C1.3 part a2). Given that $V^*_{\bar{\sigma}^2,h,M} = 2\sigma^4 \frac{h}{M} \sum_{j=1}^{1/Mh} \left(\chi^2_{M,j}\right)^2$ we can write

$$
\begin{aligned}
Var\left(V^*_{\bar{\sigma}^2,h,M}\right) &= 4\sigma^8 \frac{h^2}{M^2} \sum_{j=1}^{1/Mh} Var\left(\chi^2_{M,j}\right)^2 \\
&= 4\sigma^8 \frac{h^2}{M^2} (Mh)^{-1} \left[E\left(\chi^2_{M,j}\right)^4 - \left(E\left(\chi^2_{M,j}\right)^2\right)^2\right] \\
&= 4\sigma^8 \frac{h^2}{M^2} (Mh)^{-1} M^4 \left[E\left(\frac{\chi^2_{M,j}}{M}\right)^4 - \left(E\left(\frac{\chi^2_{M,j}}{M}\right)^2\right)^2\right] \\
&= 4\sigma^8 Mh \left(c_{M,8} - c^2_{M,4}\right) \\
&= 4\sigma^8 Mh \left[\frac{(M+2)(M+4)(M+6)}{M^3} - \frac{(M+2)^2}{M^2}\right] \\
&= 4Mh \frac{(M+2)\left(M^2+9M+24\right)}{M^3} \sigma^4.
\end{aligned}
$$

Appendix C2: Key lemmas for the proofs of results in Section 5.2: Barndorff-Nielsen and Shephard's (2004) type estimator

This section concerns the multivariate case. In particular, we provide key marginal limit results for the proofs of Theorem 5.2. We start by introducing some notations. For $j = 1, \ldots, 1/Mh$, and $i = 1, \ldots, M$, let $\epsilon^*_{i+(j-1)M} = y^*_{l,i+(j-1)M} - \hat{\beta}_{lk} y^*_{k,i+(j-1)M}$, and let $\hat{\epsilon}^*_{i+(j-1)M} = y^*_{l,i+(j-1)M} - \hat{\beta}^*_{lk} y^*_{k,i+(j-1)M}$ be the bootstrap OLS residuals. We can write

$$
T^*_{\beta,h,M} \equiv \frac{\sqrt{h^{-1}}\left(\hat{\beta}^*_{lk} - \hat{\beta}_{lk}\right)}{\sqrt{\left(\sum_{i=1}^{1/h} y^{*2}_{k,i}\right)^{-2} \hat{B}^*_{h,M}}} = \frac{\sqrt{h^{-1}}\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right)}{\sqrt{\hat{B}^*_{h,M}}} = S^*_{\beta,h,M}\left(1 + \sqrt{h} U^*_{\beta,h,M}\right)^{-1/2}, \qquad \text{(C2.1)}
$$

where

$$
S^*_{\beta,h,M} = \frac{\sqrt{h^{-1}}\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right)}{\sqrt{B^*_{h,M}}} \text{ and } U^*_{\beta,h,M} \equiv \frac{\sqrt{h^{-1}}\left(\hat{B}^*_{h,M} - B^*_{h,M}\right)}{B^*_{h,M}},
$$

such that

$$
B^*_{h,M} = Var^*\left(\sqrt{h^{-1}}\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right)\right) = Mh \sum_{j=1}^{1/Mh}\left(\hat{\Gamma}^2_{kl(j)} + \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - 4\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} + 2\hat{\beta}^2_{lk}\hat{\Gamma}^2_{kk(j)}\right),
$$

and

$$
\hat{B}^*_{h,M} = Mh \sum_{j=1}^{1/Mh}\left[\begin{array}{c} \hat{\Gamma}^{*2}_{kl(j)} + \hat{\Gamma}^*_{kk(j)}\hat{\Gamma}^*_{ll(j)} + \frac{1}{M}\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} + 3\hat{\Gamma}^2_{kl(j)}\right) \\ -4\hat{\beta}^*_{lk}\left(\hat{\Gamma}^*_{kk(j)}\hat{\Gamma}^*_{kl(j)} + \frac{2}{M}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\right) + 2\hat{\beta}^{*2}_{lk}\left(\hat{\Gamma}^{*2}_{kk(j)} + 2\frac{\hat{\Gamma}^2_{kk(j)}}{M}\right) \end{array}\right].
$$

Recall that $c_{M,q} \equiv E\left(\left(\frac{\chi^2_M}{M}\right)^{q/2}\right)$ with χ^2_M the standard χ^2 distribution with M degrees of freedom. Note that $c_{M,2} = 1$, $c_{M,4} = \frac{M+2}{M}$, $c_{M,6} = \frac{(M+2)(M+4)}{M^2}$ and $c_{M,8} = \frac{(M+2)(M+4)(M+6)}{M^3}$.

Lemma C2.4. *Suppose (1), (2) and (3) hold. We have that, for any $q_1, q_2 \geq 0$ such that $q_1 + q_2 > 0$, and for any $k, l, k', l' = 1, \ldots, d$,*

$$Mh \sum_{j=1}^{1/Mh} \left|\hat{\Gamma}_{kl(j)}\right|^{q_1} \left|\hat{\Gamma}_{k'l'(j)}\right|^{q_2} = O_P(1),$$

where $\hat{\Gamma}_{kl(j)} = \frac{1}{Mh} \sum_{i=1}^{M} y_{k,i+(j-1)M} y_{l,i+(j-1)M}$, for $j = 1, \ldots, 1/Mh$.

Proof of Lemma C2.4. Apply Theorem 3 of Li, Todorov and Tauchen (2017).

Lemma C2.5. *Suppose (1), (2) and (3) hold. We have that, for any $q_1, q_2 \geq 0$ such that $q_1 + q_2 > 0$, and for any $k, l, k', l' = 1, \ldots, d$, as soon as $M \to \infty$ such that $Mh \to 0$, as $h \to 0$.*

(a1) $Mh \sum_{j=1}^{1/Mh} \hat{\Gamma}^*_{kl(j)} - Mh \sum_{j=1}^{1/Mh} \hat{\Gamma}_{kl(j)} \xrightarrow{P^*} 0$, *in probability-$P$.*

(a2) $Mh \sum_{j=1}^{1/Mh} \hat{\Gamma}^*_{kl(j)} \hat{\Gamma}^*_{k'l'(j)} - Mh \sum_{j=1}^{1/Mh} \hat{\Gamma}_{kl(j)} \hat{\Gamma}_{k'l'(j)} \xrightarrow{P^*} 0$, *in probability-$P$.*

Proof of Lemma C2.5. We show that the results hold in quadratic mean with respect to P^*, with probability approaching one. This ensures that the bootstrap convergence also holds in probability under P.

For part (a1). Recall (18) in the main text, and notice that we can write

$$x^*_{i+(j-1)M} = \begin{pmatrix} y^{*2}_{k,i+(j-1)M} & y^*_{k,i+(j-1)M} y^*_{l,i+(j-1)M} & y^{*2}_{l,i+(j-1)M} \end{pmatrix}',$$

such that

$$\begin{aligned}
y^{*2}_{k,i+(j-1)M} &= \hat{\Gamma}_{kk(j)} \eta^{*2}_{k,i+(j-1)M}, \\
y^*_{k,i+(j-1)M} y^*_{l,i+(j-1)M} &= \hat{\Gamma}_{kl(j)} \eta^{2*}_{k,i+(j-1)M} + \sqrt{\hat{\Gamma}_{kk(j)} \hat{\Gamma}_{ll(j)} - \hat{\Gamma}^2_{kl(j)}} \eta^*_{k,i+(j-1)M} \eta^*_{l,i+(j-1)M}, \\
y^{*2}_{l,i+(j-1)M} &= \frac{\hat{\Gamma}^2_{kl(j)}}{\hat{\Gamma}_{kk(j)}} \eta^{2*}_{k,i+(j-1)M} + 2 \frac{\hat{\Gamma}_{kl(j)}}{\hat{\Gamma}_{kk(j)}} \sqrt{\hat{\Gamma}_{kk(j)} \hat{\Gamma}_{ll(j)} - \hat{\Gamma}^2_{kl(j)}} \eta^*_{k,i+(j-1)M} \eta^*_{l,i+(j-1)M} \\
&\quad + \left(\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}^2_{kl(j)}}{\hat{\Gamma}_{kk(j)}}\right) \eta^{2*}_{l,i+(j-1)M},
\end{aligned}$$

where $\begin{pmatrix} \eta^*_{k,i+(j-1)M} \\ \eta^*_{l,i+(j-1)M} \end{pmatrix} \sim i.i.d.N(0, I_2)$. It follows that

$$
\begin{aligned}
E^*\left(Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^*_{kl(j)}\right) &= E^*\left[Mh\sum_{j=1}^{1/Mh}\left(\frac{1}{Mh}\sum_{i=1}^{M}y^*_{k,i+(j-1)M}y^*_{l,i+(j-1)M}\right)\right] \\
&= \sum_{j=1}^{1/Mh}\sum_{i=1}^{M}E^*\left(y^*_{k,i+(j-1)M}y^*_{l,i+(j-1)M}\right) \\
&= h\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}E^*\left(\hat{\Gamma}_{kl(j)}\eta^{2*}_{k,i+(j-1)M} + \sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^2_{kl(j)}}\eta^*_{k,i+(j-1)M}\eta^*_{l,i+(j-1)M}\right) \\
&= Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kl(j)} \xrightarrow{P} \int_0^1 \Sigma_{kk,s}ds.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
Var^*\left(Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^*_{kl(j)}\right) &= Var^*\left[\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}E^*\left(y^*_{k,i+(j-1)M}y^*_{l,i+(j-1)M}\right)\right] \\
&= h^2\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}Var^*\left(\hat{\Gamma}_{kl(j)}\eta^{2*}_{k,i+(j-1)M} + \sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^2_{kl(j)}}\eta^*_{k,i+(j-1)M}\eta^*_{l,i+(j-1)M}\right) \\
&= h^2\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}E^*\left(\hat{\Gamma}_{kl(j)}\eta^{2*}_{k,i+(j-1)M} + \sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^2_{kl(j)}}\eta^*_{k,i+(j-1)M}\eta^*_{l,i+(j-1)M}\right)^2 \\
&\quad -h^2\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\left[E^*\left(\hat{\Gamma}_{kl(j)}\eta^{2*}_{k,i+(j-1)M} + \sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^2_{kl(j)}}\eta^*_{k,i+(j-1)M}\eta^*_{l,i+(j-1)M}\right)\right]^2 \\
&= Mh^2\sum_{j=1}^{1/Mh}\left(2\hat{\Gamma}^2_{kl(j)} + \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}\right) - Mh^2\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\hat{\Gamma}^2_{kl(j)} \\
&= h\underbrace{\left[Mh\sum_{j=1}^{1/Mh}\left(\hat{\Gamma}^2_{kl(j)} + \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}\right)\right]}_{=O_P(1)} = O_P\left(h\right) = o_P\left(1\right).
\end{aligned}
$$

The proof of part (a2) follows similarly and therefore we omit the details.

Lemma C2.6. *Suppose (1), (2) and (3) hold. We have that,*

(a1) $E^*\left(S^*_{h,M}\right) = 0$,

(a2) $\Pi^*_{h,M} \equiv Var^*\left(S^*_{h,M}\right) = Mh\sum_{j=1}^{1/Mh}\begin{pmatrix} 2\hat{\Gamma}^2_{kk(j)} & 2\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & 2\hat{\Gamma}^2_{kl(j)} \\ 2\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} + \hat{\Gamma}^2_{kl(j)} & 2\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\ 2\hat{\Gamma}^2_{kl(j)} & 2\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} & 2\hat{\Gamma}^2_{ll(j)} \end{pmatrix}$.

Proof of Lemma C2.6 part (a1). Given the definitions of $S^*_{h,M}$, x^*_i, and x_i we have

$$\begin{aligned} E^*\left(S^*_{h,M}\right) &= \sqrt{h^{-1}}\sum_{j=1}^{1/Mh}\sum_{i=1}^{M} E^*\left(x^*_{i+(j-1)M} - x_{i+(j-1)M}\right) \\ &= \sqrt{h^{-1}}\sum_{j=1}^{1/Mh}\sum_{i=1}^{M} E^*\left(x^*_{i+(j-1)M}\right) - \sqrt{h^{-1}}\sum_{j=1}^{1/Mh}\sum_{i=1}^{M} x_{i+(j-1)M} = 0, \end{aligned}$$

where results follow, since we have $\sum_{j=1}^{1/Mh}\sum_{i=1}^{M} E^*\left(y^*_{k,i+(j-1)M}y^*_{l,i+(j-1)M}\right) = \sum_{i=1}^{h^{-1}} y_{k,i}y_{l,i}$.

Proof of Lemma C2.6 part (a2). Given the definition of $\Pi^*_{M,h}$, we have

$$\begin{aligned} \Pi^*_{h,M} &= Var^*\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h}\left(x^*_i - x_i\right)\right) \\ &= h^{-1}\sum_{i=1}^{1/h} Var^*\left(x^*_i\right) \\ &= h^{-1}\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\left[E^*\left(x^*_{i+(j-1)M}x^{*\prime}_{i+(j-1)M}\right) - E^*\left(x^*_{i+(j-1)M}\right)E^*\left(x^*_{i+(j-1)M}\right)^{\prime}\right] \\ &= h\sum_{j=1}^{1/Mh}\sum_{i=1}^{M} h^{-2}\left[E^*\left(x^*_{i+(j-1)M}x^{*\prime}_{i+(j-1)M}\right) - E^*\left(x^*_{i+(j-1)M}\right)E^*\left(x^*_{i+(j-1)M}\right)^{\prime}\right] \quad \text{(C2.2)} \end{aligned}$$

Let

$$h^{-2}x^*_{i+(j-1)M}x^{*\prime}_{i+(j-1)M} \equiv \left(a^*_{i_1,i_2}\right)_{1\le i_1,i_2\le 3}.$$

It follows that

$$\begin{aligned} a^*_{1,1} &= \hat{\Gamma}^2_{kk(j)}\eta^{*4}_{k,i+(j-1)M}, \\ a^*_{2,1} &= \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\eta^{*4}_{k,i+(j-1)M} + \sqrt{\hat{\Gamma}^3_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^2_{kk(j)}\hat{\Gamma}^2_{kl(j)}}\,\eta^{*3}_{k,i+(j-1)M}\eta^*_{l,i+(j-1)M}, \\ a^*_{3,1} &= \hat{\Gamma}^2_{kl(j)}\eta^{4*}_{k,i+(j-1)M} + 2\hat{\Gamma}_{kl(j)}\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^2_{kl(j)}}\,\eta^{*3}_{k,i+(j-1)M}\eta^*_{l,i+(j-1)M} \\ &\quad + \left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^2_{kl(j)}\right)\eta^{*2}_{k,i+(j-1)M}\eta^{2*}_{l,i+(j-1)M}, \end{aligned}$$

$$
\begin{aligned}
a_{1,2}^{*} &= a_{2,1}^{*}, \\
a_{2,2}^{*} &= \left[\hat{\Gamma}_{kl(j)}\eta_{k,i+(j-1)M}^{2*} + \sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}_{kl(j)}^{2}}\,\eta_{k,i+(j-1)M}^{*}\eta_{l,i+(j-1)M}^{*}\right]^{2}, \\
a_{3,2}^{*} &= \frac{\hat{\Gamma}_{kl(j)}^{3}}{\hat{\Gamma}_{kk(j)}}\eta_{k,i+(j-1)M}^{4*} + 2\frac{\hat{\Gamma}_{kl(j)}^{2}}{\sqrt{\hat{\Gamma}_{kk(j)}}}\sqrt{\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}_{kl(j)}^{2}}{\hat{\Gamma}_{kk(j)}}}\,\eta_{k,i+(j-1)M}^{3*}\eta_{l,i+(j-1)M}^{*} \\
&\quad + \left(\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}_{kl(j)}^{3}}{\hat{\Gamma}_{kk(j)}}\right)\eta_{k,i+(j-1)M}^{2*}\eta_{l,i+(j-1)M}^{2*} \\
&\quad + \frac{\hat{\Gamma}_{kl(j)}^{2}}{\hat{\Gamma}_{kk(j)}}\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}_{kl(j)}^{2}}\,\eta_{k,i+(j-1)M}^{3*}\eta_{l,i+(j-1)M}^{*} \\
&\quad + 2\hat{\Gamma}_{kl(j)}\left(\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}_{kl(j)}^{2}}{\hat{\Gamma}_{kk(j)}}\right)\eta_{k,i+(j-1)M}^{2*}\eta_{l,i+(j-1)M}^{2*} \\
&\quad + \left(\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}_{kl(j)}^{2}}{\hat{\Gamma}_{kk(j)}}\right)\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}_{kl(j)}^{2}}\,\eta_{k,i+(j-1)M}^{*}\eta_{l,i+(j-1)M}^{3*},
\end{aligned}
$$

$$
\begin{aligned}
a_{1,3}^{*} &= a_{3,1}^{*}, \\
a_{2,3}^{*} &= a_{3,2}^{*}, \\
a_{3,3}^{*} &= \left[\frac{\hat{\Gamma}_{kl(j)}^{2}}{\hat{\Gamma}_{kk(j)}}\eta_{k,i+(j-1)M}^{2*} + 2\frac{\hat{\Gamma}_{kl(j)}}{\sqrt{\hat{\Gamma}_{kk(j)}}}\sqrt{\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}_{kl(j)}^{2}}{\hat{\Gamma}_{kk(j)}}}\,\eta_{k,i+(j-1)M}^{*}\eta_{l,i+(j-1)M}^{*} + \left(\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}_{kl(j)}^{2}}{\hat{\Gamma}_{kk(j)}}\right)\eta_{l,i+(j-1)M}^{2*}\right]^{2}.
\end{aligned}
$$

Hence, we have

$$
h^{-2}E^{*}\left(x_{i+(j-1)M}^{*}x_{i+(j-1)M}^{*\prime}\right) = \begin{pmatrix} 3\hat{\Gamma}_{kk(j)}^{2} & 3\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & 2\hat{\Gamma}_{kl(j)}^{2} + \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} \\ 3\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & 2\hat{\Gamma}_{kl(j)}^{2} + \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} & 3\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\ 2\hat{\Gamma}_{kl(j)}^{2} + \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} & 3\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} & 3\hat{\Gamma}_{ll(j)}^{2} \end{pmatrix}. \tag{C2.3}
$$

Next, remark that given the definition of $x_{i+(j-1)M}^{*}$, and by using the linearization property of $E^{*}(\cdot)$, we have

$$
E^{*}\left(x_{i+(j-1)M}^{*}\right) = h\begin{pmatrix} \hat{\Gamma}_{kk(j)} & \hat{\Gamma}_{kl(j)} & \hat{\Gamma}_{ll(j)} \end{pmatrix}'.
$$

Thus, we can write

$$
h^{-2}E^{*}\left(x_{i+(j-1)M}^{*}\right)E^{*}\left(x_{i+(j-1)M}^{*}\right)' = \begin{pmatrix} \hat{\Gamma}_{kk(j)}^{2} & \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} \\ \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & \hat{\Gamma}_{kl(j)}^{2} & \hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\ \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} & \hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} & \hat{\Gamma}_{ll(j)}^{2} \end{pmatrix}. \tag{C2.4}
$$

Given (C2.3) and (C2.4), we have

$$h^{-2}\left[E^*\left(x^*_{i+(j-1)M}x^{*\prime}_{i+(j-1)M}\right)-E^*\left(x^*_{i+(j-1)M}\right)E^*\left(x^*_{i+(j-1)M}\right)'\right]$$
$$=\left(\begin{array}{ccc} 2\hat{\Gamma}^2_{kk(j)} & 2\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & 2\hat{\Gamma}^2_{kl(j)} \\ 2\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)} & \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}+\hat{\Gamma}^2_{kl(j)} & 2\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\ 2\hat{\Gamma}^2_{kl(j)} & 2\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} & 2\hat{\Gamma}^2_{ll(j)} \end{array}\right).$$

Result follows by using (C2.2).

Lemma C2.7. *Suppose (1), (2) and (3) hold. We have that,*

(a1) $E^*\left(\hat{\Gamma}^{*3}_{kl(j)}\right)=\frac{(M+2)(M+1)}{M^2}\hat{\Gamma}^3_{kl(j)}+\frac{3(M+2)}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}$;

(a2) $E^*\left(\hat{\Gamma}^*_{kk(j)}\hat{\Gamma}^*_{kl(j)}\hat{\Gamma}^*_{ll(j)}\right)=\frac{(M+2)^2}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}+2\frac{(M+2)}{M^2}\hat{\Gamma}^3_{kl(j)}$;

(a3) $E^*\left(\hat{\Gamma}^*_{kk(j)}\hat{\Gamma}^{*2}_{kl(j)}\right)=\frac{(M+2)(M+3)}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^2_{kl(j)}+\frac{(M+2)}{M^2}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{ll(j)}$;

(a4) $E^*\left(\hat{\Gamma}^{*2}_{kk(j)}\hat{\Gamma}^*_{ll(j)}\right)=\frac{M(M+2)}{M^2}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{ll(j)}+\frac{4(M+2)}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^2_{kl(j)}$;

(a5) $E^*\left(\hat{\Gamma}^{*2}_{kk(j)}\hat{\Gamma}^*_{kl(j)}\right)=\frac{M(M+2)(M+4)}{M^3}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{kl(j)}$;

(a6) $E^*\left(\hat{\Gamma}^{*2}_{kk(j)}\hat{\Gamma}^{*2}_{kl(j)}\right)=\frac{M^3+11M^2+42M+36}{M^3}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}^2_{kl(j)}+\frac{M^2+2M+12}{M^3}\hat{\Gamma}^3_{kk(j)}\hat{\Gamma}_{ll(j)}$;

(a7) $E^*\left(\hat{\Gamma}^{*3}_{kk(j)}\hat{\Gamma}^*_{kl(j)}\right)=\frac{M(M+2)(M+4)(M+6)}{M^3}\hat{\Gamma}^3_{kk(j)}\hat{\Gamma}_{kl(j)}$;

(a8) $E^*\left(\hat{\Gamma}^*_{kk(j)}\hat{\Gamma}^*_{kl(j)}\right)=\frac{M(M+2)}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}$;

(a9) $E^*\left(\hat{\Gamma}^{*2}_{kl(j)}\right)=\frac{M+1}{M}\hat{\Gamma}^2_{kl(j)}+\frac{1}{M}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}$;

(a10) $E^*\left(\hat{\Gamma}^{*2}_{ll(j)}\right)=\frac{M+2}{M}\hat{\Gamma}^2_{ll(j)}$;

(a11) $E^*\left(\hat{\Gamma}^{*2}_{kk(j)}\right)=\frac{M+2}{M}\hat{\Gamma}^2_{kk(j)}$.

Proof of Lemma C2.7. Note that given the definitions of $\hat{\Gamma}^*_{kk(j)}$, $\hat{\Gamma}^*_{kl(j)}$ $\hat{\Gamma}^*_{ll(j)}$ and (18), we have

$$\hat{\Gamma}^*_{kk(j)}=\frac{1}{Mh}\sum_{i=1}^{M}y^{*2}_{k,i+(j-1)M}=\hat{\Gamma}_{kk(j)}\left[\frac{1}{M}\sum_{i=1}^{M}\eta^{*2}_{k,i+(j-1)M}\right],$$

$$\begin{aligned}\hat{\Gamma}^*_{kl(j)} &= \frac{1}{Mh}\sum_{i=1}^{M}y^*_{k,i+(j-1)M}y^*_{l,i+(j-1)M}\\ &= \hat{\Gamma}_{kl(j)}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)+\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^2_{kl(j)}}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^*_{k,i+(j-1)M}\eta^*_{l,i+(j-1)M}\right)\end{aligned}$$

and

$$\begin{aligned}
\hat{\Gamma}^{*}_{ll(j)} &= \frac{1}{Mh}\sum_{i=1}^{M} y^{*2}_{l,i+(j-1)M} \\
&= \frac{\hat{\Gamma}^{2}_{kl(j)}}{\hat{\Gamma}_{kk(j)}}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right) + 2\frac{\hat{\Gamma}_{kl(j)}}{\hat{\Gamma}_{kk(j)}}\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^{2}_{kl(j)}}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right) \\
&\quad + \left(\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}^{2}_{kl(j)}}{\hat{\Gamma}_{kk(j)}}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{l,i+(j-1)M}\right),
\end{aligned}$$

where $\begin{pmatrix}\eta^{*}_{k,i+(j-1)M}\\ \eta^{*}_{l,i+(j-1)M}\end{pmatrix} \sim i.i.d.N(0, I_2)$. By tedious but simple algebra, all results follow from the normality and i.i.d properties of $\eta^{*}_{k,i+(j-1)M}$, $\eta^{*}_{l,i+(j-1)M}$ across (i, j). For instance, for (a1)

$$\begin{aligned}
E^{*}\left(\hat{\Gamma}^{*3}_{kl(j)}\right) &= E^{*}\left(\hat{\Gamma}_{kl(j)}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right) + \sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^{2}_{kl(j)}}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right)\right)^{3} \\
&= E^{*}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)\hat{\Gamma}^{3}_{kl(j)} \\
&\quad +3\hat{\Gamma}^{2}_{kl(j)}\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^{2}_{kl(j)}}E^{*}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{2}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right) \\
&\quad +3\hat{\Gamma}_{kl(j)}\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^{2}_{kl(j)}\right)E^{*}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right)^{2} \\
&\quad +\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^{2}_{kl(j)}\right)^{3/2}E^{*}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right)^{3} \\
&= c_{M,6}\hat{\Gamma}^{3}_{kl(j)} + 0 \\
&\quad +\frac{3}{M^{3}}\hat{\Gamma}_{kl(j)}\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^{2}_{kl(j)}\right)E^{*}\left(\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)\left(\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right)^{2} + 0 \\
&= \frac{M(M+2)(M+4)}{M^{3}}\hat{\Gamma}^{3}_{kl(j)} + \frac{3M(M+2)}{M^{3}}\hat{\Gamma}_{kl(j)}\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)} - \hat{\Gamma}^{2}_{kl(j)}\right) \\
&= \frac{(M+2)(M+1)}{M^{2}}\hat{\Gamma}^{3}_{kl(j)} + \frac{3(M+2)}{M^{2}}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}.
\end{aligned}$$

For (a2), note that

$$
\begin{aligned}
& \hat{\Gamma}^{*}_{kk(j)}\hat{\Gamma}^{*}_{kl(j)}\hat{\Gamma}^{*}_{ll(j)} \\
= \; & \hat{\Gamma}^{3}_{kl(j)}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{3} \\
& +2\hat{\Gamma}^{2}_{kl(j)}\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{2}_{kl(j)}}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{2}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right) \\
& +\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{3}_{kl(j)}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{2}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{l,i+(j-1)M}\right) \\
& +\hat{\Gamma}^{2}_{kl(j)}\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{2}_{kl(j)}}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{2}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right) \\
& +2\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{3}_{kl(j)}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right)^{2} \\
& +\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{2}_{kl(j)}\right)^{3/2}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{l,i+(j-1)M}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
E^{*}\left(\hat{\Gamma}^{*}_{kk(j)}\hat{\Gamma}^{*}_{kl(j)}\hat{\Gamma}^{*}_{ll(j)}\right) &= \frac{M\left(M+2\right)\left(M+4\right)}{M^{3}}\hat{\Gamma}^{3}_{kl(j)}+0+\frac{M^{2}\left(M+2\right)}{M^{3}}\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{3}_{kl(j)}\right) \\
&\quad +0+2\frac{M\left(M+2\right)}{M^{3}}\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{3}_{kl(j)}\right)+0 \\
&= \frac{\left(M+2\right)^{2}}{M^{2}}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}+2\frac{\left(M+2\right)}{M^{2}}\hat{\Gamma}^{3}_{kl(j)}.
\end{aligned}
$$

For (a3), note that

$$
\begin{aligned}
\hat{\Gamma}^{*}_{kk(j)}\hat{\Gamma}^{*2}_{kl(j)} &= \hat{\Gamma}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{3} \\
&\quad +2\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{2}_{kl(j)}}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{2}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right) \\
&\quad +\left(\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right)^{2},
\end{aligned}
$$

then we have

$$
\begin{aligned}
E^{*}\left(\hat{\Gamma}^{*}_{kk(j)}\hat{\Gamma}^{*2}_{kl(j)}\right) &= \frac{M\left(M+2\right)\left(M+4\right)}{M^{3}}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}+0+\frac{M\left(M+2\right)}{M^{3}}\left(\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}\right) \\
&= \frac{\left(M+2\right)\left(M+3\right)}{M^{2}}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}+\frac{\left(M+2\right)}{M^{2}}\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}_{ll(j)}.
\end{aligned}
$$

For (a4), note that

$$\begin{aligned}
\hat{\Gamma}^{*2}_{kk(j)}\hat{\Gamma}^{*}_{ll(j)} &= \hat{\Gamma}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{3} \\
&\quad +2\hat{\Gamma}^{2}_{kk(j)}\frac{\hat{\Gamma}_{kl(j)}}{\hat{\Gamma}_{kk(j)}}\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{2}_{kl(j)}}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{2}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right) \\
&\quad +\left(\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{l,i+(j-1)M}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{2},
\end{aligned}$$

then we can write

$$\begin{aligned}
E^{*}\left(\hat{\Gamma}^{*2}_{kk(j)}\hat{\Gamma}^{*}_{ll(j)}\right) &= \frac{M\,(M+2)\,(M+4)}{M^{3}}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}+0+\frac{M^{2}\,(M+2)}{M^{3}}\left(\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}\right) \\
&= \frac{M\,(M+2)}{M^{2}}\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}_{ll(j)}+\frac{4\,(M+2)}{M^{2}}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}.
\end{aligned}$$

For (a5), note that

$$\begin{aligned}
\hat{\Gamma}^{*2}_{kk(j)}\hat{\Gamma}^{*2}_{kl(j)} &= \hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{4} \\
&\quad +2\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}_{kl(j)}\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{2}_{kl(j)}}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{3}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right) \\
&\quad +\left(\hat{\Gamma}^{3}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}\right)\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{2}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right)^{2},
\end{aligned}$$

then we have

$$\begin{aligned}
E^{*}\left(\hat{\Gamma}^{*2}_{kk(j)}\hat{\Gamma}^{*2}_{kl(j)}\right) &= \frac{M\,(M+2)\,(M+4)\,(M+6)}{M^{4}}\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}+0 \\
&\quad +\frac{15M+3\left(M^{2}-M\right)+\left(M^{3}-M^{2}\right)}{M^{4}}\left(\hat{\Gamma}^{3}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}\right) \\
&= \frac{M^{3}+11M^{2}+42M+36}{M^{3}}\hat{\Gamma}^{2}_{kk(j)}\hat{\Gamma}^{2}_{kl(j)}+\frac{M^{2}+2M+12}{M^{3}}\hat{\Gamma}^{3}_{kk(j)}\hat{\Gamma}_{ll(j)}.
\end{aligned}$$

Result follows for (a6), by noting that

$$\begin{aligned}
\hat{\Gamma}^{*3}_{kk(j)}\hat{\Gamma}^{*}_{kl(j)} &= \hat{\Gamma}^{3}_{kk(j)}\hat{\Gamma}_{kl(j)}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{4} \\
&\quad +\hat{\Gamma}^{3}_{kk(j)}\sqrt{\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}-\hat{\Gamma}^{2}_{kl(j)}}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{2*}_{k,i+(j-1)M}\right)^{3}\left(\frac{1}{M}\sum_{i=1}^{M}\eta^{*}_{k,i+(j-1)M}\eta^{*}_{l,i+(j-1)M}\right),
\end{aligned}$$

and

$$E^*\left(\left(\frac{1}{M}\sum_{i=1}^{M}\eta_{k,i+(j-1)M}^{2*}\right)^4\right) = c_{M,8} = \frac{M(M+2)(M+4)(M+6)}{M^4}$$

$$E^*\left(\frac{1}{M}\sum_{i=1}^{M}\eta_{k,i+(j-1)M}^{2*}\right)^3\left(\frac{1}{M}\sum_{i=1}^{M}\eta_{k,i+(j-1)M}^{*}\eta_{l,i+(j-1)M}^{*}\right) = 0.$$

The remaining results follow similarly and therefore we omit the details.

Lemma C2.8. *Suppose (1), (2) and (3) hold. We have that,*

(a1) $E^*\left(\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\right) = 0;$

(a2) $E^*\left(\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\right)^2 = h\tilde{B}_{h,M}^*;$

(a3) $E^*\left(\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\right)^3 = h^2\tilde{A}_{1,h,M}^*;$

(a4) $E^*\left[\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kl(j)}^{*2} + \hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{ll(j)}^* - E^*\left(\hat{\Gamma}_{kl(j)}^{*2} + \hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{ll(j)}^*\right)\right]\right]$
$= \frac{2(M+3)}{M^2}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kl(j)}^3 + \frac{6M+10}{M^2}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}$
$- \frac{(6M+14)}{M^2}\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}^2 - \frac{2(M+1)}{M^2}\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}^2\hat{\Gamma}_{ll(j)};$

(a5) $E^*\left[\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{kl(j)}^* - E^*\left(\hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{kl(j)}^*\right)\right]\right]$
$= \frac{(M+2)}{M^2}\left[3Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}^2 + Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}^2\hat{\Gamma}_{ll(j)} - 4\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}^2\hat{\Gamma}_{kl(j)}\right];$

(a6) $E^*\left[\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kk(j)}^{*2} - E^*\left(\hat{\Gamma}_{kk(j)}^{*2}\right)\right]\right] = \frac{4(M+2)}{M^2}\left[Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}^2\hat{\Gamma}_{kl(j)} - \hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kl(j)}^3\right];$

(a7) $E^*\left(\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\right)^4 = 3h^2\tilde{B}_{h,M}^{*2} + O_P(h)$, *as* $h \to 0;$

(a8) $E^*\left[\left(\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\right)^2\left(\begin{array}{c}\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kl(j)}^{*2} + \hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{ll(j)}^* - E^*\left(\hat{\Gamma}_{kl(j)}^{*2} + \hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{ll(j)}^*\right)\right]\\ +\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{kl(j)}^* - E^*\left(\hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{kl(j)}^*\right) + \hat{\Gamma}_{kk(j)}^{*2} - E^*\left(\hat{\Gamma}_{kk(j)}^{*2}\right)\right]\end{array}\right)\right] =$
$O_P\left(h^2\right),$
as $h \to 0;$

(a9) $E^*\left[\left(\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\right)^3\left(\begin{array}{c}\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kl(j)}^{*2} + \hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{ll(j)}^* - E^*\left(\hat{\Gamma}_{kl(j)}^{*2} + \hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{ll(j)}^*\right)\right]\\ +\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{kl(j)}^* - E^*\left(\hat{\Gamma}_{kk(j)}^*\hat{\Gamma}_{kl(j)}^*\right) + \hat{\Gamma}_{kk(j)}^{*2} - E^*\left(\hat{\Gamma}_{kk(j)}^{*2}\right)\right]\end{array}\right)\right]$
$= 3h^2\tilde{B}_{h,M}^*\tilde{A}_{1,h,M}^{*2} + O_P(h)$, *as* $h \to 0.$

Proof of Lemma C2.8. In the following, recall that by definition for $j = 1, \ldots, 1/Mh$, and $i =$

$1,\ldots,M$, $\epsilon^*_{i+(j-1)M} = y^*_{l,i+(j-1)M} - \hat{\beta}_{lk} y^*_{k,i+(j-1)M}$, and (18) in the main text. For (a1), we can write

$$\begin{aligned}
E^*\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right) &= E^*\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right) \\
&= \sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\left[E^*\left(y^*_{k,i+(j-1)M} y^*_{l,i+(j-1)M}\right) - \hat{\beta}_{lk} E^*\left(y^{2*}_{k,i+(j-1)M}\right)\right] \\
&= Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kl(j)} - \hat{\beta}_{lk} Mh \sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)} \\
&= Mh\sum_{j=1}^{1/Mh}\left(\frac{1}{Mh}\sum_{i=1}^{M} y_{k,i+(j-1)M} y_{l,i+(j-1)M}\right) - \frac{\sum_{i=1}^{1/h} y_{k,i}y_{l,i}}{\sum_{i=1}^{1/h} y^2_{k,i}} Mh \sum_{j=1}^{1/Mh}\left(\frac{1}{Mh}\sum_{i=1}^{M} y^2_{k,i+(j-1)M}\right) \\
&= 0.
\end{aligned}$$

For (a2), note that by definition

$$E^*\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right)^2 = Var^*\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right) + E^*\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right).$$

Then, result follows given the definition of $\tilde{B}^*_{h,M}$, and the fact that by (a1) $E^*\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right) = 0$. For the remaining results, write $z^*_i = y^*_{k,i}\epsilon^*_i - E^*\left(y^*_{k,i}\epsilon^*_i\right)$ and note that by definition, the $z^{*\prime}_i$s are conditionally independent with $E^*\left(z^*_i\right) = 0$. Next, note also that $\sum_{i=1}^{1/h} z^*_i = \sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i$, since $E^*\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right) = 0$. For (a3), note that

$$E^*\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right)^3 = E^*\left(\sum_{i=1}^{1/h} z^*_i\right)^3 = \sum_{i=1}^{1/h} E^*\left(z^{*3}_i\right) = \sum_{j=1}^{1/Mh}\sum_{i=1}^{M} E^*\left(z^{*3}_{i+(j-1)M}\right).$$

Thus, we can write

$$\begin{aligned}
& E^*\left(\sum_{i=1}^{1/h} y^*_{k,i}\epsilon^*_i\right)^3 \\
=& \sum_{j=1}^{1/Mh}\sum_{i=1}^{M} E^*\left[y^*_{k,i+(j-1)M}\epsilon^*_{i+(j-1)M} - E^*\left(y^*_{k,i+(j-1)M}\epsilon^*_{i+(j-1)M}\right)\right]^3 \\
=& \sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\left[\begin{array}{c} E^*\left(y^{*3}_{k,i+(j-1)M}\epsilon^{*3}_{i+(j-1)M}\right) - 3E^*\left(y^{*2}_{k,i+(j-1)M}\epsilon^{*2}_{i+(j-1)M}\right)E^*\left(y^*_{k,i+(j-1)M}\epsilon^*_{i+(j-1)M}\right) \\ +2\left[E^*\left(y^*_{k,i+(j-1)M}\epsilon^*_{i+(j-1)M}\right)\right]^3 \end{array}\right].
\end{aligned}$$

Next, to arrive at the desired expression for $E^*\left(\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\right)^3$, note that

$$\begin{aligned}
&\sum_{j=1}^{1/Mh}\sum_{i=1}^{M} E^*\left(y_{k,i+(j-1)M}^{*3}\epsilon_{i+(j-1)M}^{*3}\right) \\
=&\sum_{j=1}^{1/Mh}\sum_{i=1}^{M} E^*\left(y_{k,i+(j-1)M}^{*3}\left(y_{l,i+(j-1)M}^{*}-\hat{\beta}_{lk}y_{k,i+(j-1)M}^{*}\right)^3\right) \\
=& h^3\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\left(\begin{array}{c} 6\hat{\Gamma}_{kl(j)}^3+9\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}-36\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}^2\hat{\beta}_{lk} \\ -9\hat{\Gamma}_{kk(j)}^2\hat{\Gamma}_{ll(j)}\hat{\beta}_{lk}+45\hat{\beta}_{lk}^2\hat{\Gamma}_{kk(j)}^2\hat{\Gamma}_{kl(j)}-15\hat{\beta}_{lk}^3\hat{\Gamma}_{kk(j)}^3\end{array}\right),
\end{aligned}$$

$$\begin{aligned}
&-3\sum_{j=1}^{1/Mh}\sum_{i=1}^{M} E^*\left(y_{k,i+(j-1)M}^{*2}\epsilon_{i+(j-1)M}^{*2}\right)E^*\left(y_{k,i+(j-1)M}^{*}\epsilon_{i+(j-1)M}^{*}\right) \\
=&-3h\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\left(2\hat{\Gamma}_{kl(j)}^2+\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}-6\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}+3\hat{\beta}_{lk}^2\hat{\Gamma}_{kk(j)}^2\right)\left(\hat{\Gamma}_{kl(j)}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\right) \\
=& h\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\left(\begin{array}{c} -6\hat{\Gamma}_{kl(j)}^3-3\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}+18\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}^2-9\hat{\beta}_{lk}^2\hat{\Gamma}_{kk(j)}^2\hat{\Gamma}_{kl(j)} \\ +6\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}^2+3\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^2\hat{\Gamma}_{ll(j)}-186\hat{\beta}_{lk}^2\hat{\Gamma}_{kk(j)}^2\hat{\Gamma}_{kl(j)}+9\hat{\beta}_{lk}^3\hat{\Gamma}_{kk(j)}^3\end{array}\right),
\end{aligned}$$

and

$$\begin{aligned}
&2\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\left[E^*\left(y_{k,i+(j-1)M}^{*}\left(y_{l,i+(j-1)M}^{*}-\hat{\beta}_{lk}y_{k,i+(j-1)M}^{*}\right)\right)\right]^3 \\
=& 2h^3\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\left(\hat{\Gamma}_{kl(j)}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\right)^3 \\
=& h^3\sum_{j=1}^{1/Mh}\sum_{i=1}^{M}\left(2\hat{\Gamma}_{kl(j)}^3-6\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}^2+6\hat{\beta}_{lk}^2\hat{\Gamma}_{kk(j)}^2\hat{\Gamma}_{kl(j)}-2\hat{\beta}_{lk}^3\hat{\Gamma}_{kk(j)}^3\right).
\end{aligned}$$

For (a4), note that

$$\begin{aligned}
&E^*\left[\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kl(j)}^{*2}+\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}-E^*\left(\hat{\Gamma}_{kl(j)}^{*2}+\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}\right)\right]\right] \\
=& E^*\left[\sum_{i=1}^{1/h} z_i^*\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kl(j)}^{*2}+\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}-E^*\left(\hat{\Gamma}_{kl(j)}^{*2}+\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}\right)\right]\right] \\
=& Mh\sum_{j=1}^{1/Mh}\left[\begin{array}{c} E^*\left(\hat{\Gamma}_{kl(j)}^{*}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^{*}\left(\hat{\Gamma}_{kl(j)}^{*2}+\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}\right)\right) \\ -\left(\hat{\Gamma}_{kl(j)}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\right)\left(\hat{\Gamma}_{kl(j)}^2+\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}+\frac{1}{M}\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}+3\hat{\Gamma}_{kl(j)}^2\right)\right)\end{array}\right] \\
=& Mh\sum_{j=1}^{1/Mh}\left[\begin{array}{c} E^*\left(\hat{\Gamma}_{kl(j)}^{*3}+\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*2}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^{*2}\hat{\Gamma}_{ll(j)}^{*}\right) \\ -\left(\hat{\Gamma}_{kl(j)}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\right)\left(\hat{\Gamma}_{kl(j)}^2+\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}+\frac{1}{M}\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}+3\hat{\Gamma}_{kl(j)}^2\right)\right)\end{array}\right].
\end{aligned}$$

Using Lemma C2.7 we deduce that

$$
\begin{aligned}
& E^*\left(\hat{\Gamma}_{kl(j)}^{*3}+\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*2}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^{*2}\hat{\Gamma}_{ll(j)}^{*}\right) \\
= & E^*\left(\hat{\Gamma}_{kl(j)}^{*3}\right)+E^*\left(\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}\right)-\hat{\beta}_{lk}E^*\left(\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*2}\right)-\hat{\beta}_{lk}E^*\left(\hat{\Gamma}_{kk(j)}^{*2}\hat{\Gamma}_{ll(j)}^{*}\right) \\
= & \frac{(M+2)(M+1)}{M^2}\hat{\Gamma}_{kl(j)}^{3}+\frac{3(M+2)}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\
& +\frac{(M+2)^2}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)}+2\frac{(M+2)}{M^2}\hat{\Gamma}_{kl(j)}^{3} \\
& -\hat{\beta}_{lk}\left[\frac{(M+2)(M+3)}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}^{2}+\frac{(M+2)}{M^2}\hat{\Gamma}_{kk(j)}^{2}\hat{\Gamma}_{ll(j)}\right] \\
& -\hat{\beta}_{lk}\left[\frac{M(M+2)}{M^2}\hat{\Gamma}_{kk(j)}^{2}\hat{\Gamma}_{ll(j)}+\frac{4(M+2)}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}^{2}\right] \\
= & \frac{(M+2)(M+3)}{M^2}\hat{\Gamma}_{kl(j)}^{3}+\frac{(M+2)(M+5)}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\
& -\hat{\beta}_{lk}\left[\frac{(M+2)(M+7)}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}^{2}+\frac{(M+2)(M+1)}{M^2}\hat{\Gamma}_{kk(j)}^{2}\hat{\Gamma}_{ll(j)}\right].
\end{aligned}
$$

Adding and subtracting appropriately gives result for (a4).

For (a5), note that

$$
\begin{aligned}
& E^*\left[\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*}-E^*\left(\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*}\right)\right]\right] \\
= & E^*\left[\sum_{i=1}^{1/h} z_i^*\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*}-E^*\left(\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*}\right)\right]\right] \\
= & Mh\sum_{j=1}^{1/Mh}\left[E^*\left(\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*2}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^{*2}\hat{\Gamma}_{kl(j)}^{*}\right)-\left(\hat{\Gamma}_{kl(j)}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\right)\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}+\frac{2}{M}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\right)\right],
\end{aligned}
$$

then use Lemma C2.7 and remark that

$$
\begin{aligned}
& E^*\left(\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*2}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^{*2}\hat{\Gamma}_{kl(j)}^{*}\right) \\
= & \frac{(M+2)(M+3)}{M^2}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}^{2}+\frac{(M+2)}{M^2}\hat{\Gamma}_{kk(j)}^{2}\hat{\Gamma}_{ll(j)}-\hat{\beta}_{lk}\frac{(M+2)(M+4)}{M^2}\hat{\Gamma}_{kk(j)}^{2}\hat{\Gamma}_{kl(j)}.
\end{aligned}
$$

Adding and subtracting appropriately gives result for (a5).

For (a6), note that

$$
\begin{aligned}
& E^*\left[\sum_{i=1}^{1/h} y_{k,i}^*\epsilon_i^*\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kk(j)}^{*2}-E^*\left(\hat{\Gamma}_{kk(j)}^{*2}\right)\right]\right] \\
= & Mh\sum_{j=1}^{1/Mh}\left[E^*\left(\hat{\Gamma}_{kk(j)}^{*2}\hat{\Gamma}_{kl(j)}^{*}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^{*3}\right)-\left(\hat{\Gamma}_{kl(j)}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\right)\left(\hat{\Gamma}_{kk(j)}^{2}+\frac{2}{M}\hat{\Gamma}_{kk(j)}^{2}\right)\right].
\end{aligned}
$$

Next use Lemma C2.7 and compute

$$
E^*\left(\hat{\Gamma}_{kk(j)}^{*2}\hat{\Gamma}_{kl(j)}^{*}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^{*3}\right)=\frac{M(M+2)(M+4)}{M^3}\left(\hat{\Gamma}_{kk(j)}^{2}\hat{\Gamma}_{kl(j)}-\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}^{3}\right).
$$

Adding and subtracting appropriately gives result for (a6). The remaining results follow similarly and therefore we omit the details.

Lemma C2.9. *Suppose (1), (2) and (3) hold. We have that,*

(a1) $E^*\left(S^*_{\beta,h,M}\right)=0;$

(a2) $E^*\left(S^*_{\beta,h,M}\right)^2=1;$

(a3) $E^*\left(S^*_{\beta,h,M}\right)^3=\sqrt{h}\frac{\tilde{A}^*_{1,h,M}}{\tilde{B}^{*3/2}_{h,M}};$

(a4) $E^*\left(S^*_{\beta,h,M}U^*_{1,\beta,h,M}\right)=\frac{1}{\tilde{B}^{*3/2}_{h,M}}\left[\begin{array}{c} 2Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^3_{kl(j)}+6Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\ -6\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^2_{kl(j)}-2\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{ll(j)} \end{array}\right]$
$+\frac{1}{\tilde{B}^{*3/2}_{h,M}M}\left[\begin{array}{c} 6Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^3_{kl(j)}+10Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}\hat{\Gamma}_{ll(j)} \\ -14\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^2_{kl(j)}-2\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{ll(j)} \end{array}\right];$

(a5) $E^*\left(S^*_{\beta,h,M}U^*_{2,\beta,h,M}\right)$
$=-4\frac{\hat{\beta}_{lk}}{\tilde{B}^{*3/2}_{h,M}}\left(1+\frac{2}{M}\right)\left[3Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}_{kk(j)}\hat{\Gamma}^2_{kl(j)}+Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{ll(j)}-4\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{kl(j)}\right];$

(a6) $E^*\left(S^*_{\beta,h,M}U^*_{3,\beta,h,M}\right)=8\frac{\hat{\beta}^2_{lk}}{\tilde{B}^{*3/2}_{h,M}}\left(1+\frac{2}{M}\right)\left[Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^2_{kk(j)}\hat{\Gamma}_{kl(j)}-\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\hat{\Gamma}^3_{kl(j)}\right];$

(a7) $E^*\left(S^*_{\beta,h,M}U^*_{4,\beta,h,M}\right)=-4\frac{\tilde{A}^*_{0,h,M}}{\tilde{B}^{*1/2}_{h,M}\sum_{i=1}^{1/h}y^2_{k,i}};$

(a8) $E^*\left(S^*_{\beta,h,M}\right)^4=3+O_P\left(h\right),$ *as* $h\to 0;$

(a9) $E^*\left[S^{*2}_{\beta,h,M}\check{U}^*_{\beta,h,M}\right]=O_P\left(\sqrt{h}\right),$ *as* $h\to 0;$

(a10) $E^*\left[S^{*3}_{\beta,h,M}\check{U}^*_{\beta,h,M}\right]=3\frac{\tilde{A}^*_{1,h,M}}{\tilde{B}^{*3/2}_{h,M}}+O_P\left(h\right),$ *as* $h\to 0.$

Proof of Lemma C2.9. We apply Lemma C2.8, results follow directly given the definitions of $S^*_{\beta,h,M}$, $U^*_{1,\beta,h,M}$, $U^*_{2,\beta,h,M}$, $U^*_{3,\beta,h,M}$, $U^*_{4,\beta,h,M}$ and $\check{U}^*_{\beta,h,M}$.

Lemma C2.10. *Suppose (1), (2) and (3) hold. We have that,*

$$U^*_{\beta,h,M}=\check{U}^*_{\beta,h,M}+U^*_{4,\beta,h,M}+O_{P^*}\left(\sqrt{h}\right),$$

in probability, where

$$\begin{aligned}
\check{U}^*_{\beta,h,M} &= \frac{M\sqrt{h}}{\tilde{B}^*_{h,M}} \sum_{j=1}^{1/Mh} \left[\hat{\Gamma}^{*2}_{kl(j)} + \hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{ll(j)} - E^* \left(\hat{\Gamma}^{*2}_{kl(j)} + \hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{ll(j)} \right) \right] \\
&\quad -4 \frac{\hat{\beta}_{lk}}{\tilde{B}^*_{h,M}} M\sqrt{h} \sum_{j=1}^{1/Mh} \left[\hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{kl(j)} - E^* \left(\hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{kl(j)} \right) \right] \\
&\quad +2 \frac{\hat{\beta}^2_{lk}}{\tilde{B}^*_{h,M}} M\sqrt{h} \sum_{j=1}^{1/Mh} \left[\hat{\Gamma}^{*2}_{kk(j)} - E^* \left(\hat{\Gamma}^{*2}_{kk(j)} \right) \right] \\
&\equiv U^*_{1,\beta,h,M} + U^*_{2,\beta,h,M} + U^*_{3,\beta,h,M},
\end{aligned}$$

and

$$U^*_{4,\beta,h,M} = -4 \frac{\tilde{A}^*_{0,h,M}}{\sqrt{\tilde{B}^*_{h,M} \sum_{i=1}^{1/h} y^2_{k,i}}} S^*_{\beta,h,M}.$$

Proof of Lemma C2.10. Using the definition of $\hat{B}^*_{h,M}$, by adding and subtracting appropriately, we can write

$$\hat{B}^*_{h,M} = Mh \sum_{j=1}^{1/Mh} \left[\begin{array}{c} \hat{\Gamma}^{*2}_{kl(j)} + \hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{ll(j)} - \frac{1}{M} \left(\hat{\Gamma}_{kk(j)} \hat{\Gamma}_{ll(j)} + 3\hat{\Gamma}^2_{kl(j)} \right) \\ -4 \left(\hat{\beta}^*_{lk} - \hat{\beta}_{lk} \right) \left(\hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{kl(j)} - \frac{2}{M} \hat{\Gamma}_{kk(j)} \hat{\Gamma}_{kl(j)} \right) + 4\hat{\beta}_{lk} \left(\hat{\beta}^*_{lk} - \hat{\beta}_{lk} \right) \left(\hat{\Gamma}^{*2}_{kk(j)} - 2\frac{\hat{\Gamma}^2_{kk(j)}}{M} \right) \\ -4\hat{\beta}_{lk} \left(\hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{kl(j)} - \frac{2}{M} \hat{\Gamma}_{kk(j)} \hat{\Gamma}_{kl(j)} \right) \\ +2\hat{\beta}^2_{lk} \left(\hat{\Gamma}^{*2}_{kk(j)} - 2\frac{\hat{\Gamma}^2_{kk(j)}}{M} \right) + 2 \left(\hat{\beta}^*_{lk} - \hat{\beta}_{lk} \right)^2 \left(\hat{\Gamma}^{*2}_{kk(j)} - 2\frac{\hat{\Gamma}^2_{kk(j)}}{M} \right) \end{array} \right].$$

Since $\hat{\beta}^*_{lk} - \hat{\beta}_{lk} = O_{P^*}\left(h^{1/2}\right)$ and $Mh \sum_{j=1}^{1/Mh} \left(\hat{\Gamma}^{*2}_{kk(j)} - 2\frac{\hat{\Gamma}^2_{kk(j)}}{M} \right) = O_{P^*}(1)$, in probability, then

$$\left(\hat{\beta}^*_{lk} - \hat{\beta}_{lk} \right)^2 Mh \sum_{j=1}^{1/Mh} \left(\hat{\Gamma}^{*2}_{kk(j)} - 2\frac{\hat{\Gamma}^2_{kk(j)}}{M} \right) = O_{P^*}(h),$$

in probability. Next, note that

$$Mh \sum_{j=1}^{1/Mh} \left(\hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{kl(j)} - \frac{2}{M} \hat{\Gamma}_{kk(j)} \hat{\Gamma}_{kl(j)} \right) = Mh \sum_{j=1}^{1/Mh} \hat{\Gamma}_{kk(j)} \hat{\Gamma}_{kl(j)} + O_{P^*}\left(h^{1/2}\right),$$

in probability, which together with $\hat{\beta}^*_{lk} - \hat{\beta}_{lk} = O_{P^*}\left(h^{1/2}\right)$, and $\hat{\beta}_{lk} = O_{P^*}(1)$ implies that

$$\begin{aligned}
\left(\hat{\beta}^*_{lk} - \hat{\beta}_{lk} \right) \cdot Mh \sum_{j=1}^{1/Mh} \left(\hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{kl(j)} - \frac{2}{M} \hat{\Gamma}_{kk(j)} \hat{\Gamma}_{kl(j)} \right) &= \left(\hat{\beta}^*_{lk} - \hat{\beta}_{lk} \right) \cdot Mh \sum_{j=1}^{1/Mh} \hat{\Gamma}_{kk(j)} \hat{\Gamma}_{kl(j)} + O_{P^*}(h) \\
\hat{\beta}_{lk} \left(\hat{\beta}^*_{lk} - \hat{\beta}_{lk} \right) \cdot Mh \sum_{j=1}^{1/Mh} \left(\hat{\Gamma}^{*2}_{kk(j)} - 2\frac{\hat{\Gamma}^2_{kk(j)}}{M} \right) &= \hat{\beta}_{lk} \left(\hat{\beta}^*_{lk} - \hat{\beta}_{lk} \right) \cdot Mh \sum_{j=1}^{1/Mh} \hat{\Gamma}^2_{kk(j)} + O_{P^*}(h).
\end{aligned}$$

Note also that

$$
\begin{aligned}
E^{*}\left(\hat{\Gamma}_{kl(j)}^{*2}+\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}\right) &= \hat{\Gamma}_{kl(j)}^{2}+\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}+\frac{1}{M}\left(\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}+3\hat{\Gamma}_{kl(j)}^{2}\right), \\
E^{*}\left(\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*}\right) &= \hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}+\frac{2}{M}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}, \\
E^{*}\left(\hat{\Gamma}_{kk(j)}^{*2}\right) &= \hat{\Gamma}_{kk(j)}^{2}+2\frac{\hat{\Gamma}_{kk(j)}^{2}}{M},
\end{aligned}
$$

and recall that by definition

$$
\tilde{B}_{h,M}^{*}=Mh\sum_{j=1}^{1/Mh}\left(\hat{\Gamma}_{kl(j)}^{2}+\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{ll(j)}-4\hat{\beta}_{lk}\hat{\Gamma}_{kk(j)}\hat{\Gamma}_{kl(j)}+2\hat{\beta}_{lk}^{2}\hat{\Gamma}_{kk(j)}^{2}\right).
$$

Hence, we obtain

$$
\begin{aligned}
\hat{B}_{h,M}^{*} = {} & \tilde{B}_{h,M}^{*}+Mh\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kl(j)}^{*2}+\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}-E^{*}\left(\hat{\Gamma}_{kl(j)}^{*2}+\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{ll(j)}^{*}\right)\right] \\
& -4\hat{\beta}_{lk}Mh\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*}-E^{*}\left(\hat{\Gamma}_{kk(j)}^{*}\hat{\Gamma}_{kl(j)}^{*}\right)\right]+2\hat{\beta}_{lk}^{2}Mh\sum_{j=1}^{1/Mh}\left[\hat{\Gamma}_{kk(j)}^{*2}-E^{*}\left(\hat{\Gamma}_{kk(j)}^{*2}\right)\right] \\
& -4\tilde{A}_{0,h,M}^{*}\left(\hat{\beta}_{lk}^{*}-\hat{\beta}_{lk}\right)+O_{P^{*}}(h).
\end{aligned}
$$

Next, we can use $\hat{\beta}_{lk}^{*}-\hat{\beta}_{lk}=\frac{\sum_{i=1}^{1/h}y_{k,i}^{*}\epsilon_{i}^{*}}{\sum_{i=1}^{1/h}y_{k,i}^{*2}}$ and the definition of $S_{\beta,h,M}^{*}=\frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h}y_{k,i}^{*}\epsilon_{i}^{*}}{\sqrt{B_{h,M}^{*}}}$ to write

$$
\begin{aligned}
\tilde{A}_{0,h,M}^{*}\left(\hat{\beta}_{lk}^{*}-\hat{\beta}_{lk}\right) &= \sqrt{h}\tilde{A}_{0,h,M}^{*}S_{\beta,h,M}^{*}\frac{\sqrt{\tilde{B}_{h,M}^{*}}}{\sum_{i=1}^{1/h}y_{k,i}^{2}}\left(\frac{\sum_{i=1}^{1/h}y_{k,i}^{*2}}{\sum_{i=1}^{1/h}y_{k,i}^{2}}\right)^{-1} \\
&= \sqrt{h}\tilde{A}_{0,h,M}^{*}\frac{\sqrt{\tilde{B}_{h,M}^{*}}}{\sum_{i=1}^{1/h}y_{k,i}^{2}}S_{\beta,h,M}^{*}\left(1+\frac{\sum_{i=1}^{1/h}y_{k,i}^{*2}-\sum_{i=1}^{1/h}y_{k,i}^{2}}{\sum_{i=1}^{1/h}y_{k,i}^{2}}+O_{P^{*}}(h)\right) \\
&= \sqrt{h}\frac{\tilde{A}_{0,h,M}^{*}\sqrt{\tilde{B}_{h,M}^{*}}}{\sum_{i=1}^{1/h}y_{k,i}^{2}}S_{\beta,h,M}^{*}+\underbrace{\sqrt{h}\frac{\tilde{A}_{0,h,M}^{*}\sqrt{\tilde{B}_{h,M}^{*}}}{\sum_{i=1}^{1/h}y_{k,i}^{2}}S_{\beta,h,M}^{*}\left(\frac{\sum_{i=1}^{1/h}y_{k,i}^{*2}-\sum_{i=1}^{1/h}y_{k,i}^{2}}{\sum_{i=1}^{1/h}y_{k,i}^{2}}\right)}_{=O_{P^{*}}(h)}+O_{P^{*}}(h),
\end{aligned}
$$

where we have used the fact that $S_{\beta,h,M}^{*}=O_{P^{*}}(1)$ and $\sum_{i=1}^{1/h}y_{k,i}^{*2}-\sum_{i=1}^{1/h}y_{k,i}^{2}=O_{P^{*}}\left(\sqrt{h}\right)$, in probability-

P. It follows that

$$\begin{aligned}
U^*_{\beta,h,M} &= \frac{M\sqrt{h}}{\tilde{B}^*_{h,M}} \sum_{j=1}^{1/Mh} \left[\hat{\Gamma}^{*2}_{kl(j)} + \hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{ll(j)} - E^* \left(\hat{\Gamma}^{*2}_{kl(j)} + \hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{ll(j)} \right) \right] \\
&\quad -4 \frac{\hat{\beta}_{lk}}{\tilde{B}^*_{h,M}} M\sqrt{h} \sum_{j=1}^{1/Mh} \left[\hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{kl(j)} - E^* \left(\hat{\Gamma}^*_{kk(j)} \hat{\Gamma}^*_{kl(j)} \right) \right] \\
&\quad +2 \frac{\hat{\beta}^2_{lk}}{\tilde{B}^*_{h,M}} M\sqrt{h} \sum_{j=1}^{1/Mh} \left[\hat{\Gamma}^{*2}_{kk(j)} - E^* \left(\hat{\Gamma}^{*2}_{kk(j)} \right) \right] - 4 \frac{\tilde{A}^*_{0,h,M}}{\sqrt{\tilde{B}^*_{h,M}} \sum_{i=1}^{1/h} y^2_{k,i}} S^*_{\beta,h,M} + O_{P^*}(h) \\
&\equiv \underbrace{U^*_{1,\beta,h,M} + U^*_{2,\beta,h,M} + U^*_{3,\beta,h,M}}_{\equiv \breve{U}^*_{\beta,h,M}} + U^*_{4,\beta,h,M} + O_{P^*}\left(\sqrt{h}\right).
\end{aligned}$$

Appendix C3: Key lemmas for the proofs of results in Section 6.2: Mykland and Zhang's (2009) realized beta-type estimator

We introduce some notation. We let

$$\begin{aligned}
V_{1,(j)} &= \frac{M^2 h}{M-1} \left(\sum_{i=1}^{M} y^2_{k,i+(j-1)M} \right)^{-1} \left(\sum_{i=1}^{M} u^2_{i+(j-1)M} \right), \\
V_{2,(j)} &= \frac{M^2 h}{M-1} \left(\sum_{i=1}^{M} y^2_{k,i+(j-1)M} \right)^{-2} \left(\sum_{i=1}^{M} y_{k,i+(j-1)M} u_{i+(j-1)M} \right)^2.
\end{aligned}$$

We denote by $y_{k(j)} = \left(y_{k,1+(j-1)M}, \cdots, y_{k,Mj} \right)'$, the M returns of asset k observed within the block j. Similarly for the bootstrap, we let

$$\begin{aligned}
V^*_{1,(j)} &\equiv \frac{M^2 h}{M-1} \left(\sum_{i=1}^{M} y^{*2}_{k,i+(j-1)M} \right)^{-1} \left(\sum_{i=1}^{M} u^{*2}_{i+(j-1)M} \right), \\
V^*_{2,(j)} &\equiv \frac{M^2 h}{M-1} \left(\sum_{i=1}^{M} y^{*2}_{k,i+(j-1)M} \right)^{-2} \left(\sum_{i=1}^{M} y^*_{k,i+(j-1)M} u^*_{i+(j-1)M} \right)^2,
\end{aligned}$$

and $y^*_{k(j)} = \left(y^*_{k,1+(j-1)M}, \cdots, y^*_{k,Mj} \right)'$.

Recall that $\hat{C}_{(j)} \equiv \begin{pmatrix} \hat{C}_{kk(j)} & 0 \\ \hat{C}_{lk(j)} & \hat{C}_{ll(j)} \end{pmatrix} = \begin{pmatrix} \sqrt{\hat{\Gamma}_{kk(j)}} & 0 \\ \frac{\hat{\Gamma}_{kl(j)}}{\sqrt{\hat{\Gamma}_{kk(j)}}} & \sqrt{\hat{\Gamma}_{ll(j)} - \frac{\hat{\Gamma}^2_{kl(j)}}{\hat{\Gamma}_{kk(j)}}} \end{pmatrix}$. For any $q > M$, let $R_{\beta,q} \equiv Mh \sum_{j=1}^{1/Mh} \left(\frac{M}{M-1} \right)^{\frac{q}{2}} \frac{1}{b_{M,q} c_{M-1,q}} \left(\frac{\widehat{C}_{ll(j)}}{\widehat{C}_{kk(j)}} \right)^q$, where the definition of $c_{M,q}$ is given in equation (7) in the main paper, and for any $q > M$, we have $b_{M,q} \equiv E\left(\left(\frac{M}{\chi^2_M} \right)^{\frac{q}{2}} \right) = \left(\frac{M}{2} \right)^{\frac{q}{2}} \frac{\Gamma\left(\frac{M}{2} - \frac{q}{2} \right)}{\Gamma\left(\frac{M}{2} \right)}$, where χ^2_M is the standard χ^2 distribution with M degrees of freedom. Note that $b_{M,2} = \frac{M}{M-2}$, $b_{M,4} = \frac{M^2}{(M-2)(M-4)}$, and $b_{M,6} = \frac{M^3}{(M-2)(M-4)(M-6)}$. It follows by using the definition of $b_{M,q}$ and this property of the Gamma function, for all $x > 0$, $\Gamma(x+1) = x\Gamma(x)$.

Lemma C3.11. *Suppose (1) and (2) hold. Then, we have that*

$$\hat{V}_{\check{\beta},h,M} = \sum_{j=1}^{1/Mh} V_{1,(j)} - \sum_{j=1}^{1/Mh} V_{2,(j)}.$$

Lemma C3.12. *Suppose (1) and (2) hold with W independent of σ. Assume that $M = O(1)$, then conditionally on σ and under $Q_{h,M}$, the following hold*

(a1) $E\left(V_{1,(j)}\right) = \frac{M^3 h}{(M-1)(M-2)}\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^2$, *for* $M > 2$;

(a2) $E\left(V_{1,(j)}^2\right) = \frac{M^5(M+2)}{(M-1)^2(M-2)(M-4)}h^2\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^4$, *for* $M > 4$;

(a3) $E\left(V_{2,(j)}\right) = \frac{M^2 h}{(M-1)(M-2)}\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^2$, *for* $M > 2$;

(a4) $E\left(V_{2,(j)}^2\right) = \frac{3M^4}{(M-1)^2(M-2)(M-4)}h^2\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^4$, *for* $M > 4$;

(a5) $E\left(V_{1,(j)}V_{2,(j)}\right) = \frac{M^4(M+2)}{(M-1)^2(M-2)(M-4)}h^2\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^4$, *for* $M > 4$;

(a6) $Var\left(V_{1,(j)}\right) = \frac{4M^5}{(M-1)(M-2)^2(M-4)}h^2\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^4$, *for* $M > 4$;

(a7) $Var\left(V_{2,(j)}\right) = \frac{2M^4}{(M-1)(M-2)^2(M-4)}h^2\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^4$, *for* $M > 4$;

(a8) $Cov\left(V_{1,(j)}, V_{2,(j)}\right) = \frac{4M^4}{(M-1)(M-2)^2(M-4)}h^2\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^4$, *for* $M > 4$;

(a9) $Var\left(V_{1,(j)} - V_{2,(j)}\right) = \frac{2M^5(2M-3)}{(M-1)(M-2)^2(M-4)}h^2\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^4$, *for* $M > 4$.

Lemma C3.13. *Suppose (1) and (2) hold with W independent of σ. Assume that $M = O(1)$, then conditionally on σ and under $Q_{h,M}$, for any $M > 4$, the following hold*

(a1) $E\left(\hat{V}_{\check{\beta},h,M}\right) = V_{\check{\beta},h,M}$;

(a2) $Var\left(\hat{V}_{\check{\beta},h,M}\right) = \frac{2M^4(2M-3)}{(M-1)(M-2)^2(M-4)}h \cdot Mh\sum_{j=1}^{1/Mh}\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^4$;

(a3) $\hat{V}_{\check{\beta},h,M} - V_{\check{\beta},h,M} \to 0$ *in probability;*

(a4) $V_{\check{\beta},h,M} \to V_{\check{\beta}}$.

Lemma C3.14. *Suppose (1) and (2) hold with W independent of σ. Assume that $M = O(1)$, then conditionally on σ and under $Q_{h,M}$, the following hold*

(a1) $(\hat{\Gamma}_{kk(j)})^{-1}\sum_{i=1}^{M}\hat{u}_{i+(j-1)M}^2 = \left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^2$;

(a2) $E\left(\frac{\widehat{C}_{ll(j)}}{\widehat{C}_{kk(j)}}\right)^q = \left(\frac{M-1}{M}\right)^{\frac{q}{2}} b_{M,q}c_{M-1,q}\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^q$, *for* $M > q$;

(a3) $R_{\beta,q} - Mh\sum_{j=1}^{1/Mh}\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^{q} \to 0$ *in probability under* $Q_{h,M}$ *and* P*, for any* $M > q\,(1{+}\delta)$*, for some* $\delta > 0$*;*

(a4) $\hat{V}_{\check{\beta},h,M} - V_{\check{\beta},h,M} \to 0$ *in probability under* $Q_{h,M}$ *and* P*, for any* $M > 2\,(1{+}\delta)$*, for some* $\delta > 0$*.*

Proof of Lemma C3.11. Given the definition of $\hat{V}_{\beta,h,M}$ in the main text (see Equation (38)), and the definition of $\hat{u}_{i+(j-1)M} = y_{l,i+(j-1)M} - \check{\beta}_{lk(j)}y_{k,i+(j-1)M}$, we can write

$$\begin{aligned}\hat{V}_{\check{\beta},h,M} &= M^2h\sum_{j=1}^{1/Mh}\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\left(\frac{1}{M-1}\sum_{i=1}^{M}\left(y_{l,i+(j-1)M} - \check{\beta}_{lk(j)}y_{k,i+(j-1)M}\right)^2\right)\\ &= \frac{M^2h}{M-1}\sum_{j=1}^{1/Mh}\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\left(\sum_{i=1}^{M}\left(u_{i+(j-1)M} - \left(\check{\beta}_{lk(j)} - \beta_{lk(j)}\right)y_{k,i+(j-1)M}\right)^2\right),\end{aligned}$$

where we used the definition of $y_{l,i+(j-1)M}$ see equation (38). Adding and subtracting appropriately, it follows that

$$\begin{aligned}\hat{V}_{\check{\beta},h,M} &= \frac{M^2h}{M-1}\sum_{j=1}^{1/Mh}\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\left(\left(\sum_{i=1}^{M}u_{i+(j-1)M}^2\right) + \left(\check{\beta}_{lk(j)} - \beta_{lk(j)}\right)^2\right)\\ &\quad -2\frac{M^2h}{M-1}\sum_{j=1}^{1/Mh}\left(\check{\beta}_{lk(j)} - \beta_{lk(j)}\right)\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}u_{i+(j-1)M}\right)\\ &= \frac{M^2h}{M-1}\sum_{j=1}^{1/Mh}\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\left(\sum_{i=1}^{M}u_{i+(j-1)M}^2\right)\\ &\quad -\frac{M^2h}{M-1}\sum_{j=1}^{1/Mh}\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-2}\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}u_{i+(j-1)M}\right)^2\\ &= \sum_{j=1}^{1/Mh}V_{1,(j)} - \sum_{j=1}^{1/Mh}V_{2,(j)},\end{aligned}$$

where we used $\left(\check{\beta}_{lk(j)} - \beta_{lk(j)}\right) = \left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}u_{i+(j-1)M}\right)$.

Proof of Lemma C3.12 part (a1). Given the definition of $V_{1,(j)}$, the law of iterated expectations and the fact that $u_{i+(j-1)M}|y_{k(j)} \sim i.i.d.N\left(0, V_{(j)}\right)$, we can write

$$\begin{aligned}E\left(V_{1,(j)}\right) &= E\left(E\left(V_{1,(j)}|y_{k(j)}\right)\right)\\ &= \frac{M^2h}{M-1}E\left(E\left(\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\left(\sum_{i=1}^{M}u_{i+(j-1)M}^2\right)\right)|y_{k(j)}\right)\\ &= \frac{M^2h}{M-1}E\left(\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\left(\sum_{i=1}^{M}E\left(u_{i+(j-1)M}^2|y_{k(j)}\right)\right)\right)\\ &= \frac{M^3h}{M-1}V_{(j)}E\left(\left(\sum_{i=1}^{M}y_{k,i+(j-1)M}^2\right)^{-1}\right),\end{aligned}$$

then given equation (37) in the text and by replacing $V_{(j)}$ by $hC_{ll(j)}^2$, we deduce

$$E\left(V_{1,(j)}\right)=\frac{M^3h}{(M-1)(M-2)}\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^2.$$

Proof of Lemma C3.12 part (a2). Given the definition of $V_{1,(j)}$ and the law of iterated expectations, we can write

$$\begin{aligned}
E\left(V_{1,(j)}^2\right) &= E\left(E\left(V_{1,(j)}^2|y_{k(j)}\right)\right)\\
&= \frac{M^4h^2}{(M-1)^2}E\left(E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-2}\left(\sum_{i=1}^M u_{i+(j-1)M}^2\right)^2\right)|y_{k(j)}\right)\\
&= \frac{M^4h}{(M-1)^2}V_{(j)}^2E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-2}E\left(\sum_{i=1}^M\left(\frac{u_{i+(j-1)M}}{\sqrt{V_{(j)}}}\right)^2|y_{k(j)}\right)^2\right).
\end{aligned}$$

Note that since $u_{i+(j-1)M}|y_{k(j)}\sim i.i.d.N\left(0,V_{(j)}\right)$, $E\left(\sum_{i=1}^M\left(\frac{u_{i+(j-1)M}}{\sqrt{V_{(j)}}}\right)^2|y_{k(j)}\right)^2=E\left(\chi_{j,M}^2\right)^2=M(M+2)$ where $\chi_{j,M}^2$ follow the standard χ^2 distribution with M degrees of freedom. Then we have

$$E\left(V_{1,(j)}^2\right)=\frac{M^5(M+2)h}{(M-1)^2}V_{(j)}^2E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-2}\right),$$

then given the fact that $\sum_{i=1}^M y_{k,i+(j-1)M}^2\overset{d}{=}hC_{kk(j)}^2\chi_{j,M}^2$, where '$\overset{d}{=}$' denotes equivalence in distribution, by using the second moment of an inverse of χ^2 distribution, we have $E\left(\frac{1}{\chi_{j,M}^2}\right)^2=\frac{1}{(M-2)(M-4)}$, and by replacing $V_{(j)}$ by $hC_{ll(j)}^2$ it follows that

$$E\left(V_{1,(j)}^2\right)=\frac{M^5(M+2)}{(M-1)^2(M-2)(M-4)}h^2\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^4.$$

Proof of Lemma C3.12 part (a3). Given the definition of $V_{1,(j)}$, the law of iterated expectations and the fact that $u_{i+(j-1)M}|y_{k(j)}\sim i.i.d.N\left(0,V_{(j)}\right)$, we can write

$$\begin{aligned}
E\left(V_{2,(j)}\right) &= E\left(E\left(V_{2,(j)}|y_{k(j)}\right)\right)\\
&= \frac{M^2h}{M-1}E\left(E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-2}\left(\sum_{i=1}^M y_{k,i+(j-1)M}u_{i+(j-1)M}\right)^2\right)|y_{k(j)}\right)\\
&= \frac{M^2h}{M-1}E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-2}\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2E\left(u_{i+(j-1)M}^2|y_{k(j)}\right)\right)\right)\\
&= \frac{M^2h}{M-1}V_{(j)}E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-1}\right),
\end{aligned}$$

then using equation (37) in the text and replacing $V_{(j)}$ by $hC_{ll(j)}^2$ yields

$$E\left(V_{2,(j)}\right) = \frac{M^2 h}{(M-1)(M-2)}\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^2.$$

Proof of Lemma C3.12 part (a4). Given the definition of $V_{2,(j)}$ and the law of iterated expectations, we can write

$$\begin{aligned}
E\left(V_{2,(j)}^2\right) &= E\left(E\left(V_{2,(j)}^2|y_{k(j)}\right)\right) \\
&= \frac{M^4 h^2}{(M-1)^2} E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-4} E\left(\sum_{i=1}^M y_{k,i+(j-1)M}u_{i+(j-1)M}\right)^4 |y_{k(j)}\right) \\
&\equiv \frac{M^4 h^2}{(M-1)^2} E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-4} \Xi\right).
\end{aligned}$$

Then using the conditional independence and mean zero property of $y_{k,i+(j-1)M}u_{i+(j-1)M}$ we have that

$$\begin{aligned}
\Xi &\equiv E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}u_{i+(j-1)M}\right)^4 |y_{k(j)}\right) \\
&= \sum_{i=1}^M E\left(y_{k,i+(j-1)M}^4 u_{i+(j-1)M}^4 |y_{k(j)}\right) \\
&\quad +3\sum_{i\neq s} E\left(y_{k,i+(j-1)M}^2 u_{i+(j-1)M}^2 |y_{k(j)}\right) E\left(y_{k,s+(j-1)M}^2 u_{s+(j-1)M}^2 |y_{k(j)}\right) \\
&= 3V_{(j)}^2\left(\sum_{i=1}^M y_{k,i+(j-1)M}^4 + \sum_{i\neq s} y_{k,i+(j-1)M}^2 y_{k,s+(j-1)M}^2\right) = 3V_{(j)}^2\left(\sum_{i=1}^M y_{k,i+(j-1)M}^4\right)^2,
\end{aligned}$$

thus we can write

$$E\left(V_{2,(j)}^2\right) = \frac{M^4 h^2}{(M-1)^2} 3V_{(j)}^2 E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-2}\right),$$

result follows similarly where we use the same arguments as in the proof of Lemma C3.12 part (a2).

Proof of Lemma C3.12 part (a5). The proof follows similarly as parts (a2) and (a4) of Lemma C3.12 and therefore we omit the details.

Proof of Lemma C3.13 part (a1). Given the definitions of $\hat{V}_{\beta,h,M}$, $V_{1,(j)}$, $V_{2,(j)}$ and by using Lemma

C3.11 and part (a1) of Lemma C3.12, we can write

$$
\begin{aligned}
E\left(\hat{V}_{\check{\beta},h,M}\right) &= E\left(\sum_{j=1}^{1/Mh} V_{1,(j)}\right) - E\left(\sum_{j=1}^{1/Mh} V_{2,(j)}\right) \\
&= \sum_{j=1}^{1/Mh} E\left(V_{1,(j)}\right) - \sum_{j=1}^{1/Mh} E\left(V_{1,(j)}\right) \\
&= \frac{M^3 h}{(M-1)(M-2)} \sum_{j=1}^{1/Mh} \left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^2 - \frac{M^2 h}{(M-1)(M-2)} \sum_{j=1}^{1/Mh} \left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^2 \\
&= \frac{M}{M-1} V_{\check{\beta},h,M} - \frac{1}{M-1} V_{\check{\beta},h,M} = V_{\check{\beta},h,M}.
\end{aligned}
$$

Proof of Lemma C3.13 part (a2). Given the definitions of $\hat{V}_{\check{\beta},h,M}$, $V_{1,(j)}$, $V_{2,(j)}$ and Lemma C3.11, we can write

$$
Var\left(\hat{V}_{\check{\beta},h,M}\right) = Var\left(\sum_{j=1}^{1/Mh} V_{1,(j)}\right) + Var\left(\sum_{j=1}^{1/Mh} V_{2,(j)}\right) - 2Cov\left(\sum_{j=1}^{1/Mh} V_{1,(j)}, \sum_{j=1}^{1/Mh} V_{2,(j)}\right),
$$

given the fact that $u_{i+(j-1)M} | y_{k(j)} \sim i.i.d.N\left(0, V_{(j)}\right)$, we have $V_{1,(j)}$ and $V_{2,(j)}$ are conditionally independent given $y_{k(j)}$, $V_{1,(j)}$ and $V_{2,(t)}$ are conditionally independent for all $t \neq j$ given $y_{k(j)}$. It follows that

$$
\begin{aligned}
Var\left(\hat{V}_{\check{\beta},h,M}\right) &= \sum_{j=1}^{1/Mh} \left(E\left(V_{1,(j)}^2\right) - E\left(V_{1,(j)}\right)^2\right) + \left(E\left(V_{2,(j)}^2\right) - E\left(V_{2,(j)}\right)^2\right) \\
&\quad -2 \sum_{j=1}^{1/Mh} \left(E\left(V_{2,(j)} V_{2,(j)}\right) - E\left(V_{1,(j)}\right) E\left(V_{2,(j)}\right)\right),
\end{aligned}
$$

finally results follow given Lemma C3.12.

Proof of Lemma C3.13 part (a3). Results follow directly given Lemma C3.12 parts a1) and a2) since $E\left(\hat{V}_{\check{\beta},h,M} - V_{\check{\beta},h,M}\right) = 0$ and $Var\left(\hat{V}_{\check{\beta},h,M} - V_{\check{\beta},h,M}\right) \to 0$ as $h \to 0$.

Proof of Lemma C3.13 part (a4). This result follows from the boundedness of $\Sigma_{kk,s}$, $\Sigma_{ll,s}$ and the Riemann integrable of $\Sigma_{kl,s}$ for any $k, l = 1, \cdots, d$.

Proof of Lemma C3.14 part (a1). Given the definition of $\hat{u}_{i+(j-1)M}$, we can write

$$
\begin{aligned}
(\hat{\Gamma}_{kk(j)})^{-1} \sum_{i=1}^{M} \hat{u}_{i+(j-1)M}^2 &= \frac{1}{\hat{\Gamma}_{kk(j)}} \sum_{i=1}^{M} \left(y_{l,i+(j-1)M} - \check{\beta}_{lk(j)} y_{k,i+(j-1)M}\right)^2 \\
&= \frac{1}{\hat{\Gamma}_{kk(j)}} \sum_{i=1}^{M} \left(y_{l,i+(j-1)M}^2 - 2\check{\beta}_{lk(j)} y_{l,i+(j-1)M} y_{k,i+(j-1)M} + \check{\beta}_{lk(j)}^2 y_{k,i+(j-1)M}^2\right) \\
&= \frac{1}{\hat{\Gamma}_{kk(j)}} \left(\sum_{i=1}^{M} y_{l,i+(j-1)M}^2 - 2\check{\beta}_{lk(j)} \sum_{i=1}^{M} y_{l,i+(j-1)M} y_{k,i+(j-1)M} + \check{\beta}_{lk(j)}^2 \sum_{i=1}^{M} y_{k,i+(j-1)M}^2\right),
\end{aligned}
$$

thus results follow by replacing $\check{\beta}_{lk(j)} = \frac{\hat{\Gamma}_{lk(j)}}{\hat{\Gamma}_{kk(j)}}$.

Proof of Lemma C3.14 part (a2). Given the definitions of $\hat{\Gamma}_{ll(j)}$, $\hat{\Gamma}_{kl(j)}$ and $\hat{\Gamma}_{lk(j)}$ and using part (a1) of Lemma C3.14, we can write

$$\begin{aligned} E\left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^q &= E\left(\frac{\sum_{i=1}^M \hat{u}_{i+(j-1)M}^2}{\hat{\Gamma}_{kk(j)}}\right)^{\frac{q}{2}} \\ &= E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-\frac{q}{2}} E\left(\sum_{i=1}^M \hat{u}_{i+(j-1)M}^2\right)^{\frac{q}{2}} |y_{k(j)}\right) \\ &= E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-\frac{q}{2}} V_{(j)}^{\frac{q}{2}} E\left(\left(\sum_{i=1}^M \frac{\hat{u}_{i+(j-1)M}^2}{V_{(j)}}\right)^{\frac{q}{2}} |y_{k(j)}\right)\right), \end{aligned}$$

where we use the law of iterated expectations and the fact that $u_{i+(j-1)M}|y_{k(j)} \sim i.i.d.N\left(0, V_{(j)}\right)$. Then given the definition of $c_{M,q}$, we can write

$$\begin{aligned} E\left(\left(\sum_{i=1}^M \frac{\hat{u}_{i+(j-1)M}^2}{V_{(j)}}\right)^{\frac{q}{2}} |y_{k(j)}\right) &= E\left(\left(\chi_{j,M}^2\right)^{\frac{q}{2}}\right) \\ &= (M-1)^{\frac{q}{2}} c_{M-1,q}, \end{aligned}$$

it follows that

$$\begin{aligned} E\left(\frac{\widehat{C}_{ll(j)}}{\widehat{C}_{kk(j)}}\right)^q &= E\left(\frac{\sum_{i=1}^M \hat{u}_{i+(j-1)M}^2}{\hat{\Gamma}_{k(j)}}\right)^{\frac{q}{2}} \\ &= (M-1)^{\frac{q}{2}} c_{M-1,q} V_{(j)}^{\frac{q}{2}} E\left(\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-\frac{q}{2}}\right) \\ &= (M-1)^{\frac{q}{2}} c_{M-1,q} V_{(j)}^{\frac{q}{2}} \Gamma_{k(j)}^{-\frac{q}{2}} E\left(\left(\frac{M}{\chi_{j,M}^2}\right)^{\frac{q}{2}}\right) \\ &= \left(\frac{M-1}{M}\right)^{\frac{q}{2}} b_{M,q} c_{M-1,q} \left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^q; \end{aligned}$$

where $b_{M,q} = E\left(\left(\frac{M}{\chi_{j,M}^2}\right)^{\frac{q}{2}}\right)$, for $M > q$.

Proof of Lemma C3.14 part (a3). We verify the moments conditions of the weak law of large numbers for independent and nonidentically distributed on $z_j \equiv \frac{M^{\frac{q}{2}}}{(M-1)^{\frac{q}{2}} b_{M,q} c_{M-1,q}} \left(\frac{\widehat{C}_{ll(j)}}{\widehat{C}_{kk(j)}}\right)^q$, $j = 1, \ldots, \frac{1}{Mh}$. By using part (a2) of Lemma C3.14, for any $\delta > 0$, and conditionally on σ, we can write

$$E\left|z_j\right|^{1+\delta} = \left(\frac{M-1}{M}\right)^{\frac{\delta q}{2}} \frac{b_{M,(1+\delta)q}}{b_{M,q}} \frac{c_{M-1,(1+\delta)q}}{c_{M-1,q}} \left(\frac{C_{ll(j)}}{C_{kk(j)}}\right)^{(1+\delta)q} < \infty$$

since Σ is an adapted càdlàg spot covolatility matrix and locally bounded and invertible (in particular, $C_{kk(j)}^2 > 0$).

Proof of Lemma C3.14 part (a4). Result follows directly given the definition of $\hat{V}_{\check{\beta},h,M}, V_{\check{\beta},h,M}$ and part (a3) of Lemma C3.14, where we let $q=2$.

Remark 1. *The bootstrap analogue of Lemma C3.11 and C3.12 replace* $V_{1(j)}$ *with* $V_{1(j)}^*$, $V_{2(j)}$ *with* $V_{2(j)}^*$. *The bootstrap analogue of Lemma C3.13 replaces* $\hat{V}_{\check{\beta},h,M}$ *with* $\hat{V}_{\check{\beta},h,M}^*$, $V_{\check{\beta},h,M}$ *with* $V_{\check{\beta},h,M}^*$ *and* $\frac{C_{ll(j)}}{C_{kk(j)}}$ *with* $\frac{\widehat{C}_{ll(j)}}{\widehat{C}_{kk(j)}}$.

Lemma C3.15. *Suppose (1) and (2) hold with* W *independent of* σ. *Assume that* $M=O(1)$, *then conditionally on* σ *and under* $Q_{h,M}$, *for some small* $\delta>0$, *the following hold*

(a1) $E^*\left(\sum_{i=1}^M y_{k,i+(j-1)M}^{2*}\right)^{-2(2+\delta)} = b_{M,4(2+\delta)}\hat{\Gamma}_{k(j)}^{-2(2+\delta)}$, *for* $M>4(2+\delta)$;

(a2) $E^*\left(\left|\sum_{i=1}^M y_{k,i+(j-1)M}^{*}u_{i+(j-1)M}^*\right|^{2(2+\delta)}\right) \le \mu_{2(2+\delta)}^2 M^{2+\delta}\hat{\Gamma}_{k(j)}^{2+\delta}\hat{V}_{(j)}^{2+\delta}$;

Proof of Lemma C3.15 part (a1). Given the definition of $y_{k,i+(j-1)M}^*$, we can write $\sum_{i=1}^M y_{k,i+(j-1)M}^{2*} \overset{d}{=} h\widehat{C}_{kk(j)}^2 \sum_{i=1}^M v_{i+(j-1)M}^2 = h\widehat{C}_{kk(j)}^2\chi_{j,M}^2$, where $v_{i+(j-1)M}\sim i.i.d.N(0,1)$, and $\chi_{j,M}^2$ follow the standard χ^2 distribution with M degrees of freedom. Then for any integer $M>4(2+\delta)$, we have that,

$$E\left(\sum_{i=1}^M y_{k,i+(j-1)M}^2\right)^{-2(2+\delta)} = E\left(\frac{M}{\chi_{j,M}^2}\right)^{2(2+\delta)}\hat{\Gamma}_{kk(j)}^{-2(2+\delta)} = b_{M,4(2+\delta)}\hat{\Gamma}_{kk(j)}^{-2(2+\delta)}.$$

Proof of Lemma C3.15 part (a2). Indeed by using the C_r inequality, the law of iterated expectations and the fact that $u_{i+(j-1)M}^*|y_{k(j)}^*\sim i.i.d.N\left(0,\hat{V}_{(j)}\right)$, we can write for any $\delta>0$,

$$\begin{aligned} E^*\left(\left|\sum_{i=1}^M y_{k,i+(j-1)M}^*u_{i+(j-1)M}^*\right|^{2(2+\delta)}\right) &\le M^{3+2\delta}\sum_{i=1}^M E^*\left|y_{k,i+(j-1)M}^*u_{i+(j-1)M}^*\right|^{2(2+\delta)} \\ &= M^{3+2\delta}\sum_{i=1}^M E^*\left(y_{k,i+(j-1)M}^{*2(2+\delta)}E^*\left(u_{i+(j-1)M}^{*2(2+\delta)}|y_{k(j)}^*\right)\right) \\ &= \mu_{2(2+\delta)}^2 M^{2+\delta}\hat{\Gamma}_{kk(j)}^{2+\delta}\hat{V}_{(j)}^{2+\delta}, \end{aligned}$$

where the last equality follows since $y_{k,i+(j-1)M}^{2*}\overset{d}{=}h\widehat{C}_{kk(j)}^2 v_{i+(j-1)M}^2$, where $v_{i+(j-1)M}\sim i.i.d.N(0,1)$ and $\mu_{2(2+\delta)}=E|v|^{2(2+\delta)}$.

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