Online Supplement to Nonparametric Stochastic Volatility

Federico M. Bandi
Johns Hopkins University and Edhec-Risk Institute

Roberto Renò
Università di Verona

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Abstract

This document provides proofs for selected lemmas and theorems of the paper "Nonparametric Stochastic Volatility".

Below, in order not to burden the notation, we will work with $f(\sigma^2) = \sigma^2$ and, with the exception of the variance (resp. price) jump component, we will dispense with the superscript/subscript σ^2 (resp. r). The case of a general variance transformation $f(.)$ satisfying Assumption 5 (a.4) is immediate given the following treatment and the discussion in the main Appendix. We will be explicit about the left limit of the variance process and write σ^2_{t-} only when needed for clarity.

Proof of Lemma 2. We prove the result working with

$$\frac{1}{h_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)m(\sigma_s^2)-\mathbf{K}\left(\frac{\sigma^2_{(i-1)\Delta_{n,T}}-x}{h_{n,T}}\right)m\left(\sigma^2_{(i-1)\Delta_{n,T}}\right)\right]\mathrm{d}s.$$

In other words, we include the first term in the summation but notice that

$$\frac{\Delta_{n,T}}{h_{n,T}}\mathbf{K}\left(\frac{\sigma_0^2-x}{h_{n,T}}\right)m\left(\sigma_0^2\right)\leq\frac{\Delta_{n,T}}{h_{n,T}}\sup_{s\leq T}\left|\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\right|\sup_{s\leq T}\left|m(\sigma_s^2)\right|\overset{p}{\sim}\frac{\Delta_{n,T}}{h_{n,T}}C_{\mathbf{K}}\mathrm{E}(\mathcal{M}(m(\sigma_s^2)))\leq\frac{\Delta^*_{n,T}}{h_{n,T}}C_{\mathbf{K}},$$

for some constant $C_{\mathbf{K}}$ deriving from the boundedness of the kernel function (c.f., Assumption 3). The notation $\mathcal{M}(.)$ was defined in the statement of Assumption 5.
Using Itô's lemma, we obtain

$$\begin{aligned}
&\frac{1}{h_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)m(\sigma_s^2)-\mathbf{K}\left(\frac{\sigma^2_{(i-1)\Delta_{n,T}}-x}{h_{n,T}}\right)m\left(\sigma^2_{(i-1)\Delta_{n,T}}\right)\right]\mathrm{d}s=\\
&\frac{1}{h^2_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}'\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m^2(\sigma_v^2)\mathrm{d}v\right]\mathrm{d}s\\
&+\frac{1}{2h^3_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}''\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m(\sigma_v^2)\Lambda^2(\sigma_v^2)\mathrm{d}v\right]\mathrm{d}s\\
&+\frac{1}{h^2_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}'\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m(\sigma_v^2)\Lambda(\sigma_v^2)\mathrm{d}W_v^{\sigma}\right]\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{h_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\sum_{\Delta\sigma_v^2\neq 0,\ (i-1)\Delta_{n,T}\leq v<s}\left(\overline{\mathbf{K}}\left(\frac{\sigma_{v-}^2+\Delta\sigma_v^2-x}{h_{n,T}}\right)-\overline{\mathbf{K}}\left(\frac{\sigma_{v-}^2-x}{h_{n,T}}\right)\right)\right]\mathrm{d}s\\
&+\frac{1}{h_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m'(\sigma_v^2)m(\sigma_v^2)\mathrm{d}v\right]\mathrm{d}s\\
&+\frac{1}{2h_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m''(\sigma_v^2)\Lambda^2(\sigma_v^2)\mathrm{d}v\right]\mathrm{d}s\\
&+\frac{1}{h_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m'(\sigma_v^2)\Lambda(\sigma_v^2)\mathrm{d}W_v^\sigma\right]\mathrm{d}s\\
&+\frac{1}{h_{n,T}^2}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}'\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m'(\sigma_v^2)\Lambda^2(\sigma_v^2)\mathrm{d}v\right]\mathrm{d}s\\
&=\Phi_{1,n,T}+\Phi_{2,n,T}+\Phi_{3,n,T}+\Phi_{4,n,T}+\overline{\Phi}_{1,n,T}+\overline{\Phi}_{2,n,T}+\overline{\Phi}_{3,n,T}+\overline{\Phi}_{4,n,T},
\end{aligned}
$$

where we employed the notation $\overline{\mathbf{K}}=\mathbf{K}m$.
We begin with $\Phi_{1,n,T}$. Integrating by parts (see, e.g., Protter, 2004, Corollary 2, page 68), we have

$$
\begin{aligned}
&\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}'\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m^2(\sigma_v^2)\mathrm{d}v\right]\mathrm{d}s\\
&=\left(\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)m^2(\sigma_s^2)\mathrm{d}s\right)\left(\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\mathrm{d}s\right)-\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathrm{d}v\right]\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)m^2(\sigma_s^2)\mathrm{d}s\\
&=\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{s}^{i\Delta_{n,T}}\mathrm{d}v\right]\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)m^2(\sigma_s^2)\mathrm{d}s,
\end{aligned}
$$

so that

$$
\begin{aligned}
\Phi_{1,n,T}&=\frac{1}{h_{n,T}^2}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}'\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m^2(\sigma_v^2)\mathrm{d}v\right]\mathrm{d}s\\
&=\frac{1}{h_{n,T}^2}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}(i\Delta_{n,T}-s)\,\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)m^2(\sigma_s^2)\mathrm{d}s\\
&\overset{p}{\lesssim}\frac{\Delta_{n,T}}{h_{n,T}^2}\int_0^T\left|\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\right|m^2(\sigma_s^2)\mathrm{d}s\\
&\overset{p}{\lesssim}\frac{\Delta_{n,T}\mathcal{M}(m^2)}{h_{n,T}^2}\int_0^T\left|\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\right|\mathrm{d}s.
\end{aligned}
$$

Note that, by Lemma 1, setting $\mathcal{S}=|\mathbf{K}'|$, we obtain $\frac{1}{h_{n,T}}\int_0^T\left|\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\right|\mathrm{d}s=O_p(v(T))$. Hence, $\Phi_{1,n,T}=O_p\left(\frac{\Delta_{n,T}^*v(T)}{h_{n,T}}\right)$. We now turn to $\Phi_{2,n,T}$. Integrating by parts again,

$$
\Phi_{2,n,T}=\frac{1}{2h_{n,T}^3}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}''\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m(\sigma_v^2)\Lambda^2(\sigma_v^2)\mathrm{d}v\right]\mathrm{d}s
$$

$$
\begin{aligned}
&= \frac{1}{2h_{n,T}^3}\sum_{i=1}^n\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left(i\Delta_{n,T}-s\right)\mathbf{K}''\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)m(\sigma_s^2)\Lambda^2(\sigma_s^2)\mathrm{d}s\\
&\overset{p}{\lesssim} \frac{\Delta_{n,T}}{h_{n,T}^3}\int_0^T\left|\mathbf{K}''\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\right|\left|m(\sigma_s^2)\right|\Lambda^2(\sigma_s^2)\mathrm{d}s\\
&\overset{p}{\lesssim} \frac{\Delta_{n,T}\mathcal{M}(|m|\Lambda^2)}{h_{n,T}^3}\int_0^T\left|\mathbf{K}''\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\right|\mathrm{d}s.
\end{aligned}
$$

By Lemma 1, once more, after setting $\mathcal{S}=|\mathbf{K}''|$, we have $\Phi_{2,n,T}=O_p\left(\frac{\Delta_{n,T}^{*}v(T)}{h_{n,T}^2}\right)$. Next, write

$$
\begin{aligned}
\Phi_{3,n,T} &= \frac{1}{h_{n,T}^2}\sum_{i=1}^n\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\mathbf{K}'\left(\frac{\sigma_v^2-x}{h_{n,T}}\right)m(\sigma_v^2)\Lambda(\sigma_v^2)\mathrm{d}W_v^\sigma\right]\mathrm{d}s\\
&= \frac{1}{h_{n,T}^2}\sum_{i=1}^n\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left(i\Delta_{n,T}-s\right)\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)m(\sigma_s^2)\Lambda(\sigma_s^2)\mathrm{d}W_s^\sigma.
\end{aligned}
$$

The quadratic variation of $\Phi_{3,n,T}$ can be expressed as

$$
\begin{aligned}
[\Phi_{3,n,T}] &= \frac{1}{h_{n,T}^4}\sum_{i=1}^n\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left(i\Delta_{n,T}-s\right)^2\left(\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\right)^2 m^2(\sigma_s^2)\Lambda^2(\sigma_s^2)\mathrm{d}s\\
&\leq \frac{\Delta_{n,T}^2}{h_{n,T}^4}\int_0^T\left(\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\right)^2 m^2(\sigma_s^2)\Lambda^2(\sigma_s^2)\mathrm{d}s\\
&\leq \frac{\Delta_{n,T}^2\mathcal{M}(m^2\Lambda^2)}{h_{n,T}^4}\int_0^T\left(\mathbf{K}'\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\right)^2\mathrm{d}s\\
&= O_p\left(\frac{\Delta_{n,T}^{*2}v(T)}{h_{n,T}^3}\right),
\end{aligned}
$$

where the probability order term derives, again, from Lemma 1 (setting $\mathcal{S}=(\mathbf{K}')^2$). Hence, $\Phi_{3,n,T}=O_p\left(\frac{\Delta_{n,T}^{*}v^{1/2}(T)}{h_{n,T}^{3/2}}\right)$. Finally,

$$
\begin{aligned}
&\Phi_{4,n,T} = \frac{1}{h_{n,T}}\sum_{i=1}^n\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\sum_{\Delta\sigma_v^2\neq0,\ (i-1)\Delta_{n,T}\leq v<s}\left(\overline{\mathbf{K}}\left(\frac{\sigma_{v-}^2+\Delta\sigma_v^2-x}{h_{n,T}}\right)-\overline{\mathbf{K}}\left(\frac{\sigma_{v-}^2-x}{h_{n,T}}\right)\right)\right]\mathrm{d}s\\
&= \frac{1}{h_{n,T}}\sum_{i=1}^n\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\int_\xi\left(\overline{\mathbf{K}}\left(\frac{\sigma_{u-}^2+\xi^\sigma-x}{h_{n,T}}\right)-\overline{\mathbf{K}}\left(\frac{\sigma_{u-}^2-x}{h_{n,T}}\right)\right)v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\right]\mathrm{d}s\\
&= \frac{1}{h_{n,T}}\sum_{i=1}^n\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\int_\xi\left(\mathbf{K}\left(\frac{\sigma_{u-}^2+\xi^\sigma-x}{h_{n,T}}\right)-\mathbf{K}\left(\frac{\sigma_{u-}^2-x}{h_{n,T}}\right)\right)m(\sigma_{u-}^2+\xi^\sigma)v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\right]\mathrm{d}s\\
&\quad+\frac{1}{h_{n,T}}\sum_{i=1}^n\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\int_\xi\left(m\left(\sigma_{u-}^2+\xi^\sigma\right)-m\left(\sigma_{u-}^2\right)\right)\mathbf{K}\left(\frac{\sigma_{u-}^2-x}{h_{n,T}}\right)v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\right]\mathrm{d}s\\
&= \frac{1}{h_{n,T}}\sum_{i=1}^n\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\int_\xi\left(\mathbf{K}\left(\frac{\sigma_{u-}^2+\xi^\sigma-x}{h_{n,T}}\right)-\mathbf{K}\left(\frac{\sigma_{u-}^2-x}{h_{n,T}}\right)\right)\times\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times(m(\sigma^2_{u-}+\xi^\sigma)-m(\sigma^2_{u-}))v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\big]\,\mathrm{d}s\\
&+\frac{1}{h_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\int_\xi\left(\mathbf{K}\left(\frac{\sigma^2_{u-}+\xi^\sigma-x}{h_{n,T}}\right)-\mathbf{K}\left(\frac{\sigma^2_{u-}-x}{h_{n,T}}\right)\right)m(\sigma^2_{u-})v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\right]\mathrm{d}s\\
&+\frac{1}{h_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\int_\xi\left(m\left(\sigma^2_{u-}+\xi^\sigma\right)-m\left(\sigma^2_{u-}\right)\right)\mathbf{K}\left(\frac{\sigma^2_{u-}-x}{h_{n,T}}\right)v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\right]\mathrm{d}s.
\end{aligned}
$$

It is easy to show that the first and the second term after the last "equal" sign have the same probability order (i.e., $O_p\left(\frac{\Delta^*_{n,T}v(T)}{h^2_{n,T}}\right)$) and dominate the third term. Here, we therefore only focus on the first term for brevity. Write

$$
\begin{aligned}
&\frac{1}{h_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\int_\xi\left(\mathbf{K}\left(\frac{\sigma^2_{u-}+\xi^\sigma-x}{h_{n,T}}\right)-\mathbf{K}\left(\frac{\sigma^2_{u-}-x}{h_{n,T}}\right)\right)\times\right.\\
&\times(m(\sigma^2_{u-}+\xi^\sigma)-m(\sigma^2_{u-}))v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\big]\,\mathrm{d}s\\
\le\;&\frac{1}{h_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\int_\xi\left|\mathbf{K}\left(\frac{\sigma^2_{u-}+\xi^\sigma-x}{h_{n,T}}\right)-\mathbf{K}\left(\frac{\sigma^2_{u-}-x}{h_{n,T}}\right)\right|\times\right.\\
&\times|m(\sigma^2_{u-}+\xi^\sigma)-m(\sigma^2_{u-})|v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\big]\,\mathrm{d}s\\
\le\;&\frac{C_{\mathbf{K}'}C_m}{h^2_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\int_\xi(\xi^\sigma)^2v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\right]\mathrm{d}s,
\end{aligned}
$$

where the last inequality derives from the Lipschitz property of the kernel function (whose first derivative is bounded, see Assumption 3) and that of the function m (from Assumption 5 (a.0)). Ignoring constants and integrating by parts, we have

$$
\frac{1}{h^2_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\left[\int_{(i-1)\Delta_{n,T}}^{s}\int_\xi(\xi^\sigma)^2v_\sigma(\mathrm{d}u,\mathrm{d}\xi^\sigma)\right]\mathrm{d}s=\frac{1}{h^2_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}(i\Delta_{n,T}-s)\int_\xi(\xi^\sigma)^2v_\sigma(\mathrm{d}s,\mathrm{d}\xi^\sigma).
$$

Let us now compensate the random measure $v_\sigma(\mathrm{d}s,\mathrm{d}\xi^\sigma)$ to obtain

$$
\begin{aligned}
&\frac{1}{h^2_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}(i\Delta_{n,T}-s)\int_\xi(\xi^\sigma)^2\overline{v}_\sigma(\mathrm{d}s,\mathrm{d}\xi^\sigma)+\frac{1}{h^2_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}(i\Delta_{n,T}-s)\mathrm{E}_\xi((\xi^\sigma)^2)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s\\
=\;&\Phi^a_{4,n,T}+\Phi^b_{4,n,T}.
\end{aligned}
$$

We begin with $\Phi^b_{4,n,T}$. Write

$$
\Phi^b_{4,n,T}\;\le\;\frac{\overline{\mathrm{E}}_\xi((\xi^\sigma)^2)\Delta_{n,T}}{h^2_{n,T}}\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s=O_p\left(\frac{\Delta^*_{n,T}v(T)}{h^2_{n,T}}\right),
$$

where $\overline{\mathrm{E}}((\xi^\sigma)^2)$ is the uniform bound on the second moment of the variance jumps (c.f., Assumption 5 (a.3)). Given the integrability of the jump intensity with respect to the variance invariant density (c.f., again, Assumption 5 (a.3)), the probability order, i.e., $v(T)$, of the additive functional $\sum_{i=1}^n\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s=\int_0^T\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s$ derives from a simple application of the Darling-Kac theorem (see, e.g., Höpfner and Löcherbach, 2003, point (ii), page 3).

As for $\Phi^a_{4,n,T}$, its variance can be written as follows:

$$\begin{aligned}
\mathrm{V}(\Phi^a_{4,n,T}) &= \frac{1}{h^4_{n,T}}\mathrm{E}\left(\sum_{i=1}^{n}\mathrm{E}\left(\left(\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}(i\Delta_{n,T}-s)\int_{\xi}(\xi^{\sigma})^2\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})\right)^2|\mathcal{F}_{(i-1)\Delta_{n,T}}\right)\right)\\
&\leq \frac{\Delta^2_{n,T}}{h^4_{n,T}}\mathrm{E}\left(\sum_{i=1}^{n}\int_{(i-1)\Delta_{n,T}}^{i\Delta_{n,T}}\mathrm{E}_{\xi}\left((\xi^{\sigma})^4\right)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s\right)\\
&\sim \frac{\Delta^2_{n,T}}{h^4_{n,T}}\mathrm{E}[N_T]\mathrm{E}\left(\int_{R_m}^{R_{m+1}}\mathrm{E}_{\xi}\left((\xi^{\sigma})^4\right)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s\right),
\end{aligned}$$

where the first equality depends on the uncorrelatedness of the increments of the discontinuous portion of the variance process and the expression following the "asymptotic equivalence" sign hinges on Nummelin splitting, see Lemma 1 of Bandi and Moloche (2017). We recall (from Lemma 1 in this paper) that the notation $\{R_m : m \geq 1\}$ denotes the regeneration times of the variance process. As in the main Appendix, the symbol N_T is used to represent the random number of regenerations up to time T. Since, $\mathrm{E}[N_T] \sim v(T)$, then $\mathrm{V}(\Phi^a_{4,n,T}) \sim \frac{\Delta^{*2}_{n,T}v(T)}{h^4_{n,T}}$ and $\Phi^a_{4,n,T} = O_p\left(\frac{\Delta^*_{n,T}v^{1/2}(T)}{h^2_{n,T}}\right)$.

Finally, using the same method of proof, it is immediate to show that $\overline{\Phi}_{1,n,T} = O_p(h_{n,T}\Phi_{1,n,T})$, $\overline{\Phi}_{2,n,T} = O_p(h^2_{n,T}\Phi_{2,n,T})$, $\overline{\Phi}_{3,n,T} = O_p(h_{n,T}\Phi_{3,n,T})$ and $\overline{\Phi}_{4,n,T} = O_p(\Phi_{1,n,T})$. Thus,

$$\begin{aligned}
&\frac{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)m\left(\sigma^2_{i\Delta_{n,T}}\right)}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s} - \frac{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)m(\sigma^2_s)\mathrm{d}s}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s}\\
&= \frac{\frac{1}{v(T)}\left(\Phi_{1,n,T}+\Phi_{2,n,T}+\Phi_{3,n,T}+\Phi_{4,n,T}+\overline{\Phi}_{1,n,T}+\overline{\Phi}_{2,n,T}+\overline{\Phi}_{3,n,T}+\overline{\Phi}_{4,n,T}\right)}{\frac{1}{h_{n,T}v(T)}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s}\\
&= O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right).
\end{aligned}$$

The result follows from the fact that $\frac{1}{h_{n,T}v(T)}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s = O_p(1)$ (see Remark 11). □

Proof of Lemma 4. Consider $u_{i\Delta_{n,T},(i+1)\Delta_{n,T}} = \frac{1}{\sqrt{h_{n,T}v(T)}}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_{\xi}\xi^{\sigma}\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})$. The objects $\left\{u_{i\Delta_{n,T},(i+1)\Delta_{n,T}},\mathcal{F}_{i\Delta_{n,T}}, 1 \leq i \leq n-1, n \geq 2\right\}$ constitute a martingale difference array. Write

$$\mathbf{U}_{n,T} = \frac{1}{\sqrt{h_{n,T}v(T)}}\sum_{i=1}^{n-1}\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{\xi}\xi^{\sigma}\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma}) = \sum_{i=1}^{n-1}u_{i\Delta_{n,T},(i+1)\Delta_{n,T}}.$$

The sum of the conditional variances of the $u_{i\Delta_{n,T},(i+1)\Delta_{n,T}}$s is

$$\mathbf{V}_{n,T} = \frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\mathrm{E}_{i\Delta_{n,T}}\left(\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_{\xi}\xi^{\sigma}\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})\right)^2\right)$$

$$
\begin{aligned}
&= \frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{E}_{\xi}((\xi^{\sigma})^2)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s\right)\\
&= \frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\left[\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{E}_{\xi}((\xi^{\sigma})^2)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s\right)-\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{E}_{\xi}((\xi^{\sigma})^2)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s\right]\\
&\quad+\frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{E}_{\xi}((\xi^{\sigma})^2)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s\\
&= \mathbf{V}_{1,n,T}+\mathbf{V}_{2,n,T},
\end{aligned}
$$

with $\mathrm{E}_{i\Delta_{n,T}}[.]=\mathrm{E}[.|\mathcal{F}_{i\Delta_{n,T}}]$, as in the main Appendix. Now, write $\mathbf{V}_{1,n,T}=\sum_{i=1}^{n-1}w_{i\Delta_{n,T},(i+1)\Delta_{n,T}}$. We have $\mathrm{E}_{i\Delta_{n,T}}\left[w_{i\Delta_{n,T},(i+1)\Delta_{n,T}}\right]=0$. Clearly,

$$
\begin{aligned}
\mathrm{Cov}(w_{i\Delta_{n,T},(i+1)\Delta_{n,T}},w_{j\Delta_{n,T},(j+1)\Delta_{n,T}}) =&\mathrm{E}(w_{i\Delta_{n,T},(i+1)\Delta_{n,T}}\times w_{j\Delta_{n,T},(j+1)\Delta_{n,T}})\\
=&\mathrm{E}(w_{j\Delta_{n,T},(j+1)\Delta_{n,T}}\mathrm{E}_{i\Delta_{n,T}}(w_{i\Delta_{n,T},(i+1)\Delta_{n,T}}))\\
=&0,
\end{aligned}
$$

for all $j<i$. Thus, the variance of $\mathbf{V}_{1,n,T}$ can be expressed as follows:

$$
\begin{aligned}
\mathrm{V}(\mathbf{V}_{1,n,T}) &= \sum_{i=1}^{n-1}\mathrm{V}\left[w_{i\Delta_{n,T},(i+1)\Delta_{n,T}}\right]\\
&= \sum_{i=1}^{n-1}\mathrm{E}\left[w^2_{i\Delta_{n,T},(i+1)\Delta_{n,T}}\right]\\
&= \mathrm{E}\left(\sum_{i=1}^{n-1}\mathrm{E}_{i\Delta_{n,T}}\left[w^2_{i\Delta_{n,T},(i+1)\Delta_{n,T}}\right]\right)\\
&= \mathrm{E}\left(\frac{1}{h^2_{n,T}v^2(T)}\sum_{i=1}^{n-1}\mathbf{K}^4\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{E}_{\xi}((\xi^{\sigma})^2)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s\right)^2\right)\\
&\quad-\mathrm{E}\left(\frac{1}{h^2_{n,T}v^2(T)}\sum_{i=1}^{n-1}\mathbf{K}^4\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\left(\mathrm{E}_{i\Delta_{n,T}}\left[\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{E}_{\xi}((\xi^{\sigma})^2)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s\right]\right)^2\right)\\
&\leq \mathrm{E}\left(\frac{1}{h^2_{n,T}v^2(T)}\sum_{i=1}^{n-1}\mathbf{K}^4\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{E}_{\xi}((\xi^{\sigma})^2)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s\right)^2\right)\\
&\leq \mathrm{E}\left(\frac{\Delta_{n,T}}{h^2_{n,T}v^2(T)}\sum_{i=1}^{n-1}\mathbf{K}^4\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\mathrm{E}_{\xi}((\xi^{\sigma})^2)\right)^2\left(\lambda^{\sigma^2}(\sigma^2_s)\right)^2\mathrm{d}s\right)\right)\\
&= C\frac{\Delta_{n,T}}{h_{n,T}v(T)}\mathrm{E}\left(\frac{\Delta_{n,T}}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^4\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\right)\\
&= O_p\left(\frac{\Delta_{n,T}}{h_{n,T}v(T)}\right), \qquad (1)
\end{aligned}
$$

where the last inequality is Jensen's and the order term derives from the boundedness of the jump intensity and the second moment of the jumps (Assumption 5 (a.3)). The term $\mathbf{V}_{2,n,T}$ is the dominating term in

$\mathbf{V}_{n,T}$. We turn to it. Write

$$\begin{aligned}
\mathbf{V}_{2,n,T} &= \frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}(\mathrm{E}_{\xi,\sigma^2_s}((\xi^\sigma)^2)\lambda^{\sigma^2}(\sigma^2_s)-\mathrm{E}_{\xi,\sigma^2_{i\Delta_{n,T}}}((\xi^\sigma)^2)\lambda^{\sigma^2}(\sigma^2_{i\Delta_{n,T}}))\mathrm{d}s \\
&+ \frac{\Delta_{n,T}}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\mathrm{E}_{\xi,\sigma^2_{i\Delta_{n,T}}}((\xi^\sigma)^2)\lambda^{\sigma^2}(\sigma^2_{i\Delta_{n,T}}) \\
&= \mathbf{V}^a_{2,n,T}+\mathbf{V}^b_{2,n,T},
\end{aligned}$$

where the subscripts in $\mathrm{E}_{\xi,\sigma^2_s}((\xi^\sigma)^2)$ and $\mathrm{E}_{\xi,\sigma^2_{i\Delta_{n,T}}}((\xi^\sigma)^2)$ make the (possible) dependence between the moments of the jumps and the variance level explicit (c.f., again, Assumption 5 (a.3)). Now, in order to simplify the notation, we write $\lambda_{\mathrm{E}}(\sigma^2_s)=\mathrm{E}_{\xi,\sigma^2_s}((\xi^\sigma)^2)\lambda^{\sigma^2}(\sigma^2_s)$ and $\lambda_{\mathrm{E}}(\sigma^2_{i\Delta_{n,T}})=\mathrm{E}_{\xi,\sigma^2_{i\Delta_{n,T}}}((\xi^\sigma)^2)\lambda^{\sigma^2}(\sigma^2_{i\Delta_{n,T}})$. By the twice differentiability of the jump intensity and the moments of the jump sizes (Assumption 5 (a.3)), we have

$$\begin{aligned}
\mathbf{V}^a_{2,n,T} &= \frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\lambda_{\mathrm{E}}(\sigma^2_s)-\lambda_{\mathrm{E}}(\sigma^2_{i\Delta_{n,T}})\right)\mathrm{d}s \\
&= \frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\lambda'_{\mathrm{E}}(\sigma^2_v)m(\sigma^2_v)\mathrm{d}v\right)\mathrm{d}s \\
&+\frac{1}{2h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\lambda''_{\mathrm{E}}(\sigma^2_v)\Lambda^2(\sigma^2_v)\mathrm{d}v\right)\mathrm{d}s \\
&+\frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\lambda'_{\mathrm{E}}(\sigma^2_v)\Lambda(\sigma^2_v)\mathrm{d}W^\sigma_v\right)\mathrm{d}s \\
&+\frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sum_{\Delta\sigma^2_v\neq 0,\ i\Delta_{n,T}\leq v<s}\left(\lambda_{\mathrm{E}}\left(\sigma^2_{v-}+\Delta\sigma^2_v\right)-\lambda_{\mathrm{E}}\left(\sigma^2_{v-}\right)\right)\right)\mathrm{d}s \\
&= \Omega_{1,n,T}+\Omega_{2,n,T}+\Omega_{3,n,T}+\Omega_{4,n,T}.
\end{aligned}$$

We note that $\Omega_{1,n,T}=O_p(\Delta^*_{n,T})$ and $\Omega_{2,n,T}=O_p(\Delta^*_{n,T})$ using the same method of proof as for term $\Phi_{1,n,T}$ in Lemma 2. Similarly, $\Omega_{3,n,T}=O_p\left(\frac{\Delta^*_{n,T}}{h^{1/2}_{n,T}v^{1/2}(T)}\right)$ and $\Omega_{4,n,T}=O_p\left(\frac{\Delta^*_{n,T}}{h^{1/2}_{n,T}v^{1/2}(T)}\right)$ using the same method of proof as for term $\Phi_{3,n,T}$ in Lemma 2. Turning to $\mathbf{V}^b_{2,n,T}$, by Lemma 2, after replacing m with λ_{E}, we may write

$$\begin{aligned}
\mathbf{V}^b_{2,n,T} &= \frac{\Delta_{n,T}}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\lambda_{\mathrm{E}}(\sigma^2_{i\Delta_{n,T}}) \\
&= \frac{1}{h_{n,T}v(T)}\int_0^T\mathbf{K}^2\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\lambda_{\mathrm{E}}(\sigma^2_s)\mathrm{d}s+O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right). \qquad (2)
\end{aligned}$$

The sum of the conditional fourth moments of the $u_{i\Delta_{n,T},(i+1)\Delta_{n,T}}$s is

$$\mathbf{Q}_{n,T}=\frac{1}{h^2_{n,T}v^2(T)}\sum_{i=1}^{n-1}\mathbf{K}^4\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\mathrm{E}_{i\Delta_{n,T}}\left(\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_\xi\xi^\sigma\overline{v}_\sigma(\mathrm{d}s,\mathrm{d}\xi^\sigma)\right)^4\right)$$

$$
\begin{aligned}
&= \frac{1}{h_{n,T}^2 v^2(T)} \sum_{i=1}^{n-1} \mathbf{K}^4 \left(\frac{\sigma^2_{i\Delta_{n,T}} - x}{h_{n,T}} \right) \mathrm{E}_{i\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \mathrm{E}_{\xi}((\xi^{\sigma})^4) \lambda^{\sigma^2}(\sigma^2_s) \mathrm{d}s \right) \\
&= \frac{1}{h_{n,T}^2 v^2(T)} \sum_{i=1}^{n-1} \mathbf{K}^4 \left(\frac{\sigma^2_{i\Delta_{n,T}} - x}{h_{n,T}} \right) \left[\mathrm{E}_{i\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \mathrm{E}_{\xi}((\xi^{\sigma})^4) \lambda^{\sigma^2}(\sigma^2_s) \mathrm{d}s \right) - \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \mathrm{E}_{\xi}((\xi^{\sigma})^4) \lambda^{\sigma^2}(\sigma^2_s) \mathrm{d}s \right] \\
&\quad + \frac{1}{h_{n,T}^2 v^2(T)} \sum_{i=1}^{n-1} \mathbf{K}^4 \left(\frac{\sigma^2_{i\Delta_{n,T}} - x}{h_{n,T}} \right) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \mathrm{E}_{\xi}((\xi^{\sigma})^4) \lambda^{\sigma^2}(\sigma^2_s) \mathrm{d}s \\
&= \mathbf{Q}_{1,n,T} + \mathbf{Q}_{2,n,T}.
\end{aligned}
$$

Notice that $\mathbf{Q}_{1,n,T}$ and $\mathbf{Q}_{2,n,T}$ can be treated as $\mathbf{V}_{1,n,T}$ and $\mathbf{V}_{2,n,T}$, respectively. In particular, $\mathbf{Q}_{2,n,T} = O_p\left(\frac{1}{h_{n,T}v(T)}\right) = o_p(1)$, since $h_{n,T}v(T) \to \infty$. As in the case of $\mathbf{V}_{1,n,T}$, which is such that $\mathbf{V}_{1,n,T} = o_p\left(\mathbf{V}_{2,n,T}\right)$, we have $\mathbf{Q}_{1,n,T} = o_p\left(\mathbf{Q}_{2,n,T}\right)$.
Now, by Theorem 1 in Touati (1992), as $n, T \to \infty$ (*jointly*, in such a way that $\Delta_{n,T} \to 0$ and $h_{n,T} \to 0$), we have stable convergence of pairs so that

$$
\begin{aligned}
&(\mathbf{V}_{n,T}, \mathbf{U}_{n,T}) \\
&\underset{stably}{\Rightarrow} \left(C_{\sigma^2} \mathrm{E}_{\xi}((\xi^{\sigma})^2) \lambda^{\sigma^2}(x) s(x) \left(\int_{\mathrm{S}} \mathbf{K}^2(u) \mathrm{d}u \right) g_{\alpha}, \sqrt{C_{\sigma^2} \mathrm{E}_{\xi}((\xi^{\sigma})^2) \lambda^{\sigma^2}(x) s(x) \left(\int_{\mathrm{S}} \mathbf{K}^2(u) \mathrm{d}u \right)} \mathrm{N} \circ g_{\alpha} \right),
\end{aligned} \tag{3}
$$

where $\mathrm{N} \circ g_{\alpha}$ is a Gaussian mixture with variance g_{α} and C_{σ^2} is a process-specific constant. We note that

$$
\mathbf{V}_{n,T} \Rightarrow C_{\sigma^2} \mathrm{E}_{\xi}((\xi^{\sigma})^2) \lambda^{\sigma^2}(x) s(x) \left(\int_{\mathrm{S}} \mathbf{K}^2(u) \mathrm{d}u \right) g_{\alpha}
$$

is also a consequence of Lemma 1, given Eq. (2).
Consistent with the statement of Lemma 4 in the main Appendix, Eq. (3) can, also, be written as follows:

$$
\left(\frac{\mathbf{V}_{n,T}}{\frac{\Delta_{n,T}}{h_{n,T}v(T)} \sum_{i=1}^n \mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}} - x}{h_{n,T}} \right)}, \frac{\mathbf{U}_{n,T}}{\sqrt{\frac{\Delta_{n,T}}{h_{n,T}v(T)} \sum_{i=1}^n \mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}} - x}{h_{n,T}} \right)}} \right) \Rightarrow \left(\widetilde{\mathbf{V}}, \left(\sqrt{\widetilde{\mathbf{V}}} \right) \mathrm{N} \right).
$$

In fact, given Eq. (2), we have

$$
\begin{aligned}
\frac{\mathbf{V}_{n,T}}{\frac{\Delta_{n,T}}{h_{n,T}v(T)} \sum_{i=1}^n \mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}} - x}{h_{n,T}} \right)} &= \frac{\frac{1}{h_{n,T}v(T)} \int_0^T \mathbf{K}^2\left(\frac{\sigma^2_s - x}{h_{n,T}} \right) \mathrm{E}_{\xi}((\xi^{\sigma})^2) \lambda^{\sigma^2}(\sigma^2_s) \mathrm{d}s + O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}} \right)}{\frac{1}{h_{n,T}v(T)} \int_0^T \mathbf{K}\left(\frac{\sigma^2_s - x}{h_{n,T}} \right) \mathrm{d}s + O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}} \right)} \\
&\xrightarrow{p} \mathrm{E}_{\xi}((\xi^{\sigma})^2) \lambda^{\sigma^2}(x) \int_{\mathrm{S}} \mathbf{K}^2(u) \mathrm{d}u = \widetilde{\mathbf{V}},
\end{aligned}
$$

where convergence in probability is a consequence of the same arguments as in the proof of Theorem 4 of Bandi and Moloche (2017). This concludes the proof of point *1* in the statement of Lemma 4. Point *2* can be proved in just the same way. $\square$

Proof of Lemma 5. Define, as in the proof of Lemma 4, the martingale difference arrays

$$u_{i\Delta_{n,T},(i+1)\Delta_{n,T}} = \frac{1}{\sqrt{v(T)h_{n,T}}}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\left[\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_{\xi}\xi^{\sigma}\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})\right]$$

and

$$u'_{i\Delta_{n,T},(i+1)\Delta_{n,T}} = \frac{1}{\sqrt{v(T)h_{n,T}}}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\left[\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\Lambda(\sigma^2_s)\mathrm{d}W^{\sigma}_s\right].$$

Lemma 4 yields the result once we take into account that the covariance between the $u_{i\Delta_{n,T},(i+1)\Delta_{n,T}}$s and the $u'_{j\Delta_{n,T},(j+1)\Delta_{n,T}}$s is zero for all i and j by the assumed independence between the jumps and the driving Brownian shocks.

□

Proof of Lemma 6. Write

$$\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right)}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}-\theta_{\sigma^2,1}(x) = \frac{\dfrac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right)}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s}}{\dfrac{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s}}-\theta_{\sigma^2,1}(x).$$

Next, we compensate the measure $v_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})$ and write $\overline{m}(\sigma^2_s) = m(\sigma^2_s)+\mathrm{E}_{\xi}(\xi^{\sigma})\lambda^{\sigma^2}(\sigma^2_s)$. We also employ Lemma 3 to express the denominator in the previous expression as

$$\frac{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s} = 1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right).$$

Therefore,

$$\frac{\dfrac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right)}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s}}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}-\frac{\theta_{\sigma^2,1}(x)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}-\frac{\theta_{\sigma^2,1}(x)O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}$$

$$= \frac{\dfrac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\overline{m}(\sigma^2_s)\mathrm{d}s}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s}}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}+\frac{\dfrac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\Lambda(\sigma^2_s)\mathrm{d}W^{\sigma}_s}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s}}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}$$

$$+\frac{\dfrac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_{\xi}\xi^{\sigma}\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s}}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}-\frac{\theta_{\sigma^2,1}(x)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}-\frac{\theta_{\sigma^2,1}(x)O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}$$

$$= \left[\frac{\frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \overline{m}(\sigma_s^2) \mathrm{d}s}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s}}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} - \frac{\theta_{\sigma^2,1}(x)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} \right] + \frac{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}}\right) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \Lambda(\sigma_s^2) \mathrm{d}W_s^\sigma}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s}}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}$$

$$+ \frac{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}}\right) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \int_\xi \xi^\sigma \overline{v}_\sigma(\mathrm{d}s, \mathrm{d}\xi^\sigma)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s}}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} + \frac{O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}$$

$$= \Psi_{1,n,T} + \Psi_{2,n,T} + \Psi_{3,n,T} + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right),$$

where the order symbol $\frac{O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}$ derives from the result

$$\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}}\right) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \overline{m}(\sigma_s^2) \mathrm{d}s}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s} - \frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \overline{m}(\sigma_s^2) \mathrm{d}s}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s} = O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right), \tag{4}$$

which we prove next. Notice that

$$\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}}\right) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \overline{m}(\sigma_s^2) \mathrm{d}s}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s} - \frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \overline{m}(\sigma_s^2) \mathrm{d}s}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s}$$

$$= \frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}}\right) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \overline{m}(\sigma_s^2) \mathrm{d}s}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s} - \frac{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}}\right) \overline{m}\left(\sigma_{i\Delta_{n,T}}^2\right)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s} \tag{5}$$

$$+ \frac{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}}\right) \overline{m}\left(\sigma_{i\Delta_{n,T}}^2\right)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s} - \frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \overline{m}(\sigma_s^2) \mathrm{d}s}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s}. \tag{6}$$

However, by Lemma 2, Eq. (6) is such that

$$\frac{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}\left(\frac{\sigma_{i\Delta_{n,T}}^2 - x}{h_{n,T}}\right) \overline{m}\left(\sigma_{i\Delta_{n,T}}^2\right)}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s} - \frac{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \overline{m}(\sigma_s^2) \mathrm{d}s}{\frac{1}{h_{n,T}} \int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s} = O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right).$$

Hence, one only has to show that Eq. (5) is of smaller (or equal) asymptotic order than $O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)$. After standardizing by $v(T)$, write

$$\frac{\frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\overline{m}(\sigma_s^2)\mathrm{d}s}{\frac{1}{h_{n,T}v(T)}\int_0^T\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s}-\frac{\frac{\Delta_{n,T}}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\overline{m}\left(\sigma^2_{i\Delta_{n,T}}\right)}{\frac{1}{h_{n,T}v(T)}\int_0^T\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s}.$$

The numerator can be expressed as

$$\begin{aligned}
&\frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\overline{m}(\sigma_s^2)-\overline{m}(\sigma^2_{i\Delta_{n,T}})\right)\mathrm{d}s\\
&=\frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\overline{m}'(\sigma_v^2)m(\sigma_v^2)\mathrm{d}v\right)\mathrm{d}s\\
&+\frac{1}{2h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\overline{m}''(\sigma_v^2)\Lambda^2(\sigma_v^2)\mathrm{d}v\right)\mathrm{d}s\\
&+\frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\overline{m}'(\sigma_v^2)\Lambda(\sigma_v^2)\mathrm{d}W_v^\sigma\right)\mathrm{d}s\\
&+\frac{1}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sum_{\Delta\sigma_v^2\neq 0,\ i\Delta_{n,T}\leq v<s}\left(\overline{m}\left(\sigma^2_{v-}+\Delta\sigma_v^2\right)-\overline{m}\left(\sigma^2_{v-}\right)\right)\right)\mathrm{d}s\\
&=\Upsilon_{1,n,T}+\Upsilon_{2,n,T}+\Upsilon_{3,n,T}+\Upsilon_{4,n,T},
\end{aligned}$$

where the asymptotic orders of the terms $\Upsilon_{q,n,T}$, with $q=1,...,4$, are the same as those of the terms $\Omega_{q,n,T}$, with $q=1,...,4$, in the proof of Lemma 3. Hence, $\Upsilon_{1,n,T}=\Upsilon_{2,n,T}=O_p(\Delta^*_{n,T})$ and $\Upsilon_{3,n,T}=\Upsilon_{4,n,T}=O_p\left(\frac{\Delta^*_{n,T}}{h^{1/2}_{n,T}v^{1/2}(T)}\right)$. Eq. (5) is, therefore, of smaller order than $O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right)$ (since its denominator is $O_p(1)$, given Lemma 1).

Now, as in the proof of Theorem 4 of Bandi and Moloche (2017), we have

$$\begin{aligned}
\Psi_{1,n,T}&=\frac{\dfrac{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\overline{m}(\sigma_s^2)\mathrm{d}s}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s}-\theta_{\sigma^2,1}(x)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}\\
&=\frac{h^2_{n,T}\left(\int_{\mathrm{S}}s^2\mathbf{K}(s)\mathrm{d}s\right)\left(\frac{\partial\theta_{\sigma^2,1}(x)}{\partial x}\frac{\partial s(x)/\partial x}{s(x)}+\frac{1}{2}\frac{\partial^2\theta_{\sigma^2,1}(x)}{\partial x\partial x}\right)+o_p(h^2_{n,T})}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}.
\end{aligned}\tag{7}$$

Also, by Lemmas 4 and 5,

$$\Psi_{2,n,T}=\frac{\dfrac{\frac{1}{v(T)h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\Lambda(\sigma_s^2)\mathrm{d}W_s^\sigma}{\frac{1}{v(T)h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s}}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}=\frac{O_p\left(\frac{1}{\sqrt{v(T)h_{n,T}}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)},$$

and

$$\Psi_{3,n,T} = \frac{\frac{\frac{1}{v(T)h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_{\xi}\xi^{\sigma}\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})}{\frac{1}{v(T)h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s}}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)} = \frac{O_p\left(\frac{1}{\sqrt{v(T)h_{n,T}}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}.$$

Thus,

$$\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right)}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}-\theta_{\sigma^2,1}(x) = O_p(h^2_{n,T})+O_p\left(\frac{1}{\sqrt{h_{n,T}v(T)}}\right)+O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right).$$

The last expression can be written as follows

$$O_p(h^2_{n,T})+O_p\left(\frac{1}{\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}}\right)+O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right),$$

since, when $\frac{\Delta_{n,T}}{h^2_{n,T}}\to 0$, which is implied by $\frac{\Delta^*_{n,T}}{h^2_{n,T}}\to 0$, $\widehat{L}_{\sigma^2}(T,x)$ and $\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s$ are asymptotically equivalent (Lemma 3). Finally, recall that $\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s = O_p(v(T))$ by Remark 11, a result which was used repeatedly above. □

Proof of Lemma 7. As in the proof of Lemma 6, write

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}\left(\frac{\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\overline{m}(\sigma_s^2)\mathrm{d}s}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s}}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}+\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\Lambda(\sigma_s^2)\mathrm{d}W_s^{\sigma}}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}\right.$$

$$\left.+\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_{\xi}\xi^{\sigma}\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}-\frac{\theta_{\sigma^2,1}(x)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}-\frac{\theta_{\sigma^2,1}(x)O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}\right)$$

$$=\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}\left(\frac{\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\overline{m}(\sigma_s^2)\mathrm{d}s}{\frac{1}{h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s}-\theta_{\sigma^2,1}(x)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}-\frac{\theta_{\sigma^2,1}(x)O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}\right)$$

$$+\frac{\frac{1}{\sqrt{h_{n,T}}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\Lambda(\sigma_s^2)\mathrm{d}W_s^{\sigma}}{\sqrt{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}}+\frac{\frac{1}{\sqrt{h_{n,T}}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_{\xi}\xi^{\sigma}\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})}{\sqrt{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}}$$

$$= \sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \left(\frac{\frac{\frac{1}{h_{n,T}}\int_0^T \mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\overline{m}(\sigma_s^2)\mathrm{d}s}{\frac{1}{h_{n,T}}\int_0^T \mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s} - \theta_{\sigma^2,1}(x)}{1+O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} + \frac{O_p\left(\frac{\Delta^*_{n,T}}{h_{n,T}^2}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} \right)$$

$$+ \frac{\frac{1}{\sqrt{h_{n,T}}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\Lambda(\sigma_s^2)\mathrm{d}W_s^\sigma}{\sqrt{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}} + \frac{\frac{1}{\sqrt{h_{n,T}}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_\xi \xi^\sigma\overline{v}_\sigma(\mathrm{d}s,\mathrm{d}\xi^\sigma)}{\sqrt{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}},$$

where the order term in the denominators was derived in Lemma 3 and the order term inside the parenthesis derives from Eq. (4). If $h_{n,T}^5\widehat{L}_{\sigma^2}(T,x)\xrightarrow{p}C$, $h_{n,T}\widehat{L}_{\sigma^2}(T,x)\xrightarrow{p}\infty$ and $\frac{\Delta^*_{n,T}}{h^2_{n,T}}\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}\xrightarrow{p}0$, the result now follows from Lemma 5 and Eq. (7). □

Proof of Lemma 8. Using Itô's lemma,

$$\begin{aligned}
\left(\sigma^2_{(i+1)\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right)^R &= R\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sigma_s^2-\sigma^2_{i\Delta_{n,T}}\right)^{R-1}m(\sigma_s^2)\mathrm{d}s \\
&\quad +R\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sigma_s^2-\sigma^2_{i\Delta_{n,T}}\right)^{R-1}\Lambda(\sigma_s^2)\mathrm{d}W_s^\sigma \\
&\quad +\frac{1}{2}R(R-1)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sigma_s^2-\sigma^2_{i\Delta_{n,T}}\right)^{R-2}\Lambda^2(\sigma_s^2)\mathrm{d}s \\
&\quad +\sum_{\Delta\sigma_s^2\neq 0,\ i\Delta_{n,T}\le s<(i+1)\Delta_{n,T}}\left[\left(\sigma^2_{s-}+\Delta\sigma_s^2-\sigma^2_{i\Delta_{n,T}}\right)^R-\left(\sigma^2_{s-}-\sigma^2_{i\Delta_{n,T}}\right)^R\right].
\end{aligned}$$

Also, by a straightforward application of the binomial identity,

$$\begin{aligned}
&\sum_{\Delta\sigma_s^2\neq 0\ i\Delta_{n,T}\le s<(i+1)\Delta_{n,T}}\left[\left(\sigma^2_{s-}+\Delta\sigma_s^2-\sigma^2_{i\Delta_{n,T}}\right)^R-\left(\sigma^2_{s-}-\sigma^2_{i\Delta_{n,T}}\right)^R\right] \\
&= \sum_{k=0}^{R-1}\binom{R}{k}\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_\xi\left(\sigma_s^2-\sigma^2_{i\Delta_{n,T}}\right)^k(\xi^\sigma)^{(R-k)}v_\sigma(\mathrm{d}s,\mathrm{d}\xi^\sigma).
\end{aligned}$$

Next, breaking up the summation above into the first term, i.e., $k=0$, and the summation $\sum_{k=1}^{R-1}$, we can write

$$\begin{aligned}
&\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right)^R}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)} \\
&= \frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)R\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sigma_s^2-\sigma^2_{i\Delta_{n,T}}\right)^{R-1}m(\sigma_s^2)\mathrm{d}s}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)R\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sigma^2_s-\sigma^2_{i\Delta_{n,T}}\right)^{R-1}\Lambda(\sigma^2_s)\mathrm{d}W^{\sigma}_s}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}\\
&+\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\frac{1}{2}R(R-1)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sigma^2_s-\sigma^2_{i\Delta_{n,T}}\right)^{R-2}\Lambda^2(\sigma^2_s)\mathrm{d}s}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}\\
&+\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\sum_{k=1}^{R-1}\left(\begin{array}{c}R\\k\end{array}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sigma^2_s-\sigma^2_{i\Delta_{n,T}}\right)^{k}\int_{\xi}(\xi^{\sigma})^{R-k}v_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}\\
&+\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_{\xi}(\xi^{\sigma})^{R}v_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}\\
=\;&\Xi_{1,R,n,T}+\Xi_{2,R,n,T}+\Xi_{3,R,n,T}+\Xi_{4,R,n,T}+\Xi_{5,R,n,T}.
\end{aligned}
$$

Now, notice that $\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\sigma^2_s-\sigma^2_{i\Delta_{n,T}}\right|^R=O_p\left(\sqrt{\Delta^*_{n,T}}\right)$ for all $R\geq 1$. We begin with $R=1$. By triangle inequality,

$$
\begin{aligned}
\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\sigma^2_s-\sigma^2_{i\Delta_{n,T}}\right| \;\leq\;& \sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\int_{i\Delta_{n,T}}^{s}(m(\sigma^2_u)+\mathrm{E}_{\xi}(\xi^{\sigma})\lambda^{\sigma^2}(\sigma^2_u))\mathrm{d}u\right|\\
&+\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\int_{i\Delta_{n,T}}^{s}\Lambda(\sigma^2_u)\mathrm{d}W^{\sigma}_u\right|\\
&+\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\int_{i\Delta_{n,T}}^{s}\int_{\xi}\xi^{\sigma}\overline{v}_{\sigma}(\mathrm{d}u,\mathrm{d}\xi^{\sigma})\right|.
\end{aligned}
$$

But,

$$
\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\int_{i\Delta_{n,T}}^{s}\overline{m}(\sigma^2_u)\mathrm{d}u\right|\leq\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left|\overline{m}(\sigma^2_u)\right|\mathrm{d}u=O_p(\mathrm{E}(\mathcal{M}(|\overline{m}|))\Delta_{n,T})=O_p(\Delta^*_{n,T}),
$$

where, again, $\overline{m}(\sigma^2_u)=m(\sigma^2_u)+\mathrm{E}_{\xi}(\xi^{\sigma})\lambda^{\sigma^2}(\sigma^2_u)$. Also,

$$
\begin{aligned}
\mathrm{E}\left(\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\int_{i\Delta_{n,T}}^{s}\Lambda(\sigma^2_u)\mathrm{d}W^{\sigma}_u\right|\right)&\leq C\mathrm{E}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\Lambda^2(\sigma^2_u)\mathrm{d}u\right)^{1/2}\\
&\leq C\left(\Delta_{n,T}\mathrm{E}\left(\sup_{u\leq T}\Lambda^2(\sigma^2_u)\right)\right)^{1/2}\\
&\sim\left(\Delta^*_{n,T}\right)^{1/2},
\end{aligned}
$$

where the first inequality is BDG's and the second is Jensen's. Thus, by Markov's inequality, the second term is $O_p\left(\sqrt{\Delta^*_{n,T}}\right)$. The remaining term can be treated as follows:

$$\begin{aligned}
\mathrm{E}\left(\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\int_{i\Delta_{n,T}}^{s}\int_{\xi}\xi^{\sigma}\overline{v}_{\sigma}(\mathrm{d}u,\mathrm{d}\xi^{\sigma})\right|\right) &\leq C\mathrm{E}\left(\sum_{i\Delta_{n,T}\leq u\leq(i+1)\Delta_{n,T}}\left(\Delta\sigma^2_u\right)^2\right)^{1/2}\\
&\leq C\left(\mathrm{E}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{E}_{\xi}((\xi^{\sigma})^2)\lambda^{\sigma^2}(\sigma^2_u)\mathrm{d}u\right)\right)^{1/2}\\
&\sim \left(\Delta_{n,T}\right)^{1/2},
\end{aligned}$$

where the first inequality is Meyer's version of BDG's inequality for possibly discontinuous local martingales, the second is Jensen's and the order term depends on the boundedness of the jump intensities (Assumption 5 (a.3)). We conclude that $\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\sigma^2_s-\sigma^2_{i\Delta_{n,T}}\right| = O_p\left(\sqrt{\Delta^*_{n,T}}\right)$.

Next, consider $\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\sigma^2_s-\sigma^2_{i\Delta_{n,T}}\right|^2$. Using Itô's lemma, once more, we have

$$\begin{aligned}
&\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\sigma^2_s-\sigma^2_{i\Delta_{n,T}}\right|^2\\
=&\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left(\left|2\int_{i\Delta_{n,T}}^{s}(\sigma^2_u-\sigma^2_{i\Delta_{n,T}})m(\sigma^2_u)\mathrm{d}u+2\int_{i\Delta_{n,T}}^{s}(\sigma^2_u-\sigma^2_{i\Delta_{n,T}})\Lambda(\sigma^2_u)\mathrm{d}W^{\sigma}_u+\int_{i\Delta_{n,T}}^{s}\Lambda^2(\sigma^2_u)\mathrm{d}u\right.\right.\\
&\left.\left.+\int_{i\Delta_{n,T}}^{s}\int_{\xi}(\xi^{\sigma})^2v_{\sigma}(\mathrm{d}u,\mathrm{d}\xi^{\sigma})+2\int_{i\Delta_{n,T}}^{s}\int_{\xi}(\sigma^2_u-\sigma^2_{i\Delta_{n,T}})\xi^{\sigma}v_{\sigma}(\mathrm{d}u,\mathrm{d}\xi^{\sigma})\right|\right),
\end{aligned}$$

which is, again, $O_p\left(\sqrt{\Delta^*_{n,T}}\right)$ since the leading order term is $\int_{i\Delta_{n,T}}^{s}\int_{\xi}(\xi^{\sigma})^2v_{\sigma}(\mathrm{d}u,\mathrm{d}\xi^{\sigma})$. By induction, we obtain the same result for $\sup_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left|\sigma^2_s-\sigma^2_{i\Delta_{n,T}}\right|^R$ with $R>2$.

It is, therefore, clear that - when $R>2$ - the probability limit of the infinitesimal moment estimator is given by the compensation of the term $\Xi_{5,R,n,T}$. If $R=2$, instead, the limit is given by $\Xi_{3,2,n,T}$ along with the compensation of the term $\Xi_{5,2,n,T}$.

After compensating,

$$\begin{aligned}
\Xi_{5,R,n,T} &= \frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_{\xi}\left(\xi^{\sigma}\right)^R\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}\\
&+\frac{\frac{1}{h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{E}_{\xi}\left(\left(\xi^{\sigma}\right)^R\right)\lambda^{\sigma^2}(\sigma^2_s)\mathrm{d}s}{\frac{\Delta_{n,T}}{h_{n,T}}\sum_{i=1}^{n}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)}\\
&= \frac{\dfrac{\frac{1}{v(T)h_{n,T}}\sum_{i=1}^{n-1}\mathbf{K}\left(\frac{\sigma^2_{i\Delta_{n,T}}-x}{h_{n,T}}\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_{\xi}(\xi^{\sigma})^R\overline{v}_{\sigma}(\mathrm{d}s,\mathrm{d}\xi^{\sigma})}{\frac{1}{v(T)h_{n,T}}\int_0^T\mathbf{K}\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\mathrm{d}s}}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}
\end{aligned}$$

$$
\begin{aligned}
& + \frac{\dfrac{\frac{1}{h_{n,T}}\int_0^T \mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{E}_\xi\left((\xi^\sigma)^R\right)\lambda^{\sigma^2}(\sigma_s^2)\mathrm{d}s}{\frac{1}{h_{n,T}}\int_0^T \mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s} + O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)} \\
= & \frac{O_p\left(\frac{1}{\sqrt{v(T)h_{n,T}}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)} + \frac{\dfrac{\frac{1}{h_{n,T}}\int_0^T \mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{E}_\xi\left((\xi^\sigma)^R\right)\lambda^{\sigma^2}(\sigma_s^2)\mathrm{d}s}{\frac{1}{h_{n,T}}\int_0^T \mathbf{K}\left(\frac{\sigma_s^2-x}{h_{n,T}}\right)\mathrm{d}s} + O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)} \\
\xrightarrow{p} & \ \mathrm{E}_\xi\left(\left(\xi^\sigma\right)^R\right)\lambda^{\sigma^2}(x),
\end{aligned}
$$

where the second "equal" sign derives from the same method of proof leading to Eq. (4) and the order term $O_p\left(\frac{1}{\sqrt{v(T)h_{n,T}}}\right)$ derives from the same method of proof leading to the findings in Lemma 4. Finally, $\mathrm{E}_\xi\left(\left(\xi^\sigma\right)^R\right)\lambda^{\sigma^2}(x) = \theta_{\sigma^2,R}(x)$ if $R > 2$. If $R = 2$, instead, $\Xi_{3,2,n,T} \xrightarrow{p} \Lambda^2(x)$ by the arguments in Lemma 2 and $\Xi_{3,2,n,T} + \Xi_{5,2,n,T} \xrightarrow{p} \Lambda^2(x) + \mathrm{E}_\xi\left(\left(\xi^\sigma\right)^2\right)\lambda^{\sigma^2}(x) = \theta_{\sigma^2,2}(x)$. □

Proof of Lemma 9. The result follows from the same argument as in the proof of Lemma 7 after recognizing that, for $R \geq 2$, the term driving the limiting distribution is the compensated term $\Xi_{5,R,n,T}$, which was defined in Lemma 8. The limiting distribution of the compensated term $\Xi_{5,R,n,T}$ follows from the same method of proof as for Lemma 4. The jump compensation of $\Xi_{5,R,n,T}$ (along with $\Xi_{3,R,n,T}$ - which is also defined in Lemma 8 - when $R = 2$) drives the bias term.
It is, then, sufficient to notice that, as in the proof of Lemma 7, the condition

$$
\frac{\Delta^*_{n,T}}{h^2_{n,T}}\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} = \frac{\Delta^*_{n,T}\sqrt{\widehat{L}_{\sigma^2}(T,x)}}{h^{3/2}_{n,T}} \xrightarrow{p} 0
$$

eliminates the discretization error asymptotically. The condition

$$
h^2_{n,T}\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \xrightarrow{p} C
$$

or, equivalently,

$$
h^5_{n,T}\widehat{L}_{\sigma^2}(T,x) \xrightarrow{p} C
$$

leads to an MSE-optimal rate by preserving the bias term (of order $h^2_{n,T}$) in the limiting distribution. □

Proof of Theorems 3 and 4. For conciseness, we focus on the two functions in points *1* and *2* of the statement of Theorem 3 (i.e., $\mu_\xi(x)$ and $\lambda^{\sigma^2}(x)$). The other functions can be treated analogously. The estimation error induced by the spot variance estimates is handled as in proof of Theorem 2. By the Cramér-Wold theorem, the limiting multivariate normal distribution of the infinitesimal moment estimates is derived immediately. By the delta method, write

$$
\widehat{\mu}_\xi(x) - \mu_\xi(x) \overset{d}{=} \frac{\left(\widehat{\theta}_{\sigma^2,4}(x) - \theta_{\sigma^2,4}(x)\right)}{4\theta_{\sigma^2,3}(x)} - \frac{\theta_{\sigma^2,4}(x)\left(\widehat{\theta}_{\sigma^2,3}(x) - \theta_{\sigma^2,3}(x)\right)}{4\left(\theta_{\sigma^2,3}(x)\right)^2}.
$$

Thus,

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}\left\{\widehat{\mu}_{\xi}(x)-\mu_{\xi}(x)\right\}$$
$$\Rightarrow \mathrm{N}\left(0, \frac{\theta_{\sigma^2,8}(x)}{16\left(\theta_{\sigma^2,3}(x)\right)^2}+\frac{\left(\theta_{\sigma^2,4}(x)\right)^2\theta_{\sigma^2,6}(x)}{16\left(\theta_{\sigma^2,3}(x)\right)^4}-2\frac{\theta_{\sigma^2,4}(x)\theta_{\sigma^2,7}(x)}{16\theta_{\sigma^2,3}(x)\left(\theta_{\sigma^2,3}(x)\right)^2}\right).$$

Similarly,

$$\begin{aligned}
\widehat{\lambda}^{\sigma^2}(x)-\lambda^{\sigma^2}(x) &\overset{d}{=}\frac{\left(\widehat{\theta}_{\sigma^2,4}(x)-\theta_{\sigma^2,4}(x)\right)}{24\mu_{\xi}^4(x)}-\frac{4\theta_{\sigma^2,4}(x)\left(\widehat{\mu}_{\xi}(x)-\mu_{\xi}(x)\right)}{24\mu_{\xi}^5(x)}\\
&=\frac{\left(\widehat{\theta}_{\sigma^2,4}(x)-\theta_{\sigma^2,4}(x)\right)}{24\mu_{\xi}^4(x)}\\
&\quad-\frac{4\theta_{\sigma^2,4}(x)}{24\mu_{\xi}^5(x)}\left\{\frac{\left(\widehat{\theta}_{\sigma^2,4}(x)-\theta_{\sigma^2,4}(x)\right)}{4\theta_{\sigma^2,3}(x)}-\frac{\theta_{\sigma^2,4}(x)\left(\widehat{\theta}_{\sigma^2,3}(x)-\theta_{\sigma^2,3}(x)\right)}{4\left(\theta_{\sigma^2,3}(x)\right)^2}\right\}\\
&=\left(\frac{4(24)\mu_{\xi}^5(x)\theta_{\sigma^2,3}(x)-4(24)\mu_{\xi}^4(x)\theta_{\sigma^2,4}(x)}{4(24)^2\mu_{\xi}^9(x)\theta_{\sigma^2,3}(x)}\right)\left(\widehat{\theta}_{\sigma^2,4}(x)-\theta_{\sigma^2,4}(x)\right)\\
&\quad+\frac{4\left(\theta_{\sigma^2,4}(x)\right)^2}{24\mu_{\xi}^5(x)4\left(\theta_{\sigma^2,3}(x)\right)^2}\left(\widehat{\theta}_{\sigma^2,3}(x)-\theta_{\sigma^2,3}(x)\right).
\end{aligned}$$

Therefore,

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}\left\{\widehat{\lambda}^{\sigma^2}(x)-\lambda^{\sigma^2}(x)\right\}$$
$$\begin{aligned}
\Rightarrow\quad &\mathrm{N}\left(0,\left(\frac{4(24)\mu_{\xi}^5(x)\theta_{\sigma^2,3}(x)-4(24)\mu_{\xi}^4(x)\theta_{\sigma^2,4}(x)}{4(24)^2\mu_{\xi}^9(x)\theta_{\sigma^2,3}(x)}\right)^2\theta_{\sigma^2,8}(x)+\left(\frac{4\left(\theta_{\sigma^2,4}(x)\right)^2}{24\mu_{\xi}^5(x)4\left(\theta_{\sigma^2,3}(x)\right)^2}\right)^2\theta_{\sigma^2,6}(x)\right.\\
&\left.+2\left(\frac{4(24)\mu_{\xi}^5(x)\theta_{\sigma^2,3}(x)-4(24)\mu_{\xi}^4(x)\theta_{\sigma^2,4}(x)}{4(24)^2\mu_{\xi}^9(x)\theta_{\sigma^2,3}(x)}\right)\left(\frac{4\left(\theta_{\sigma^2,4}(x)\right)^2}{24\mu_{\xi}^5(x)4\left(\theta_{\sigma^2,3}(x)\right)^2}\right)\theta_{\sigma^2,7}(x)\right).
\end{aligned}$$

Replacing the infinitesimal moments in the limiting variances with their closed-form expressions leads to the stated results for point *1* and point *2* of Theorem 3. Point *3* and point *4* follow similarly. The proof of Theorem 4 is identical. □

Proof of Theorem 6. For consistency with $f(\sigma^2)=\sigma^2$, which is the case explicitly discussed in this Supplement, we assume $\xi^{\sigma}\overset{\mathrm{d}}{=}\exp(\mu_{\xi})$ as above. The case $f(\sigma^2)=\log\sigma^2$, with $\xi^{\sigma}\overset{\mathrm{d}}{=}\mathrm{N}(0,\sigma_{\xi}^2)$, can be handled analogously.
Define the feasible estimator of the infinitesimal covariance ($C(x)$) as

$$\widehat{C}(x)=\frac{\sum_{i=1}^{n-1}\widehat{\mathbf{K}}_{i\Delta_{n,T}}\left(\log p_{(i+1)\Delta_{n,T}}-\log p_{i\Delta_{n,T}}\right)\left(\widehat{\sigma}^2_{(i+1)\Delta_{n,T}}-\widehat{\sigma}^2_{i\Delta_{n,T}}\right)}{\Delta_{n,T}\sum_{i=1}^{n}\widehat{\mathbf{K}}_{i\Delta_{n,T}}},$$

where the notation $\mathbf{K}_{i\Delta_{n,T}}$ and $\widehat{\mathbf{K}}_{i\Delta_{n,T}}$ was introduced in the proof of Theorem 2 in the main Appendix. The symbol $C^{*}(x)$ denotes the same (infeasible) estimator computed with true variances.
As in the proof of Theorem 2, we begin by separating the measurement error and write, using the abbreviation $\Delta \log p_{i\Delta n,T} = \log p_{(i+1)\Delta_{n,T}} - \log p_{i\Delta_{n,T}}$,

$$
\begin{aligned}
& \widehat{C}(x) - C^{*}(x) \\
= \; & \frac{\sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \left(\Delta \log p_{i\Delta_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}} - \sigma^2_{i\Delta_{n,T}}\right)}{\Delta_{n,T} \sum_{i=1}^{n} \widehat{\mathbf{K}}_{i\Delta_{n,T}}} - \frac{\sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \left(\Delta \log p_{i\Delta_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}} - \sigma^2_{i\Delta_{n,T}}\right)}{\Delta_{n,T} \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}} \\
+ \; & \frac{\sum_{i=1}^{n-1} \widehat{\mathbf{K}}_{i\Delta_{n,T}} \left(\Delta \log p_{i\Delta_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}} - \sigma^2_{i\Delta_{n,T}}\right)}{\Delta_{n,T} \sum_{i=1}^{n} \widehat{\mathbf{K}}_{i\Delta_{n,T}}} - \frac{\sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \left(\Delta \log p_{i\Delta_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}} - \sigma^2_{i\Delta_{n,T}}\right)}{\Delta_{n,T} \sum_{i=1}^{n} \widehat{\mathbf{K}}_{i\Delta_{n,T}}} \\
+ \; & \frac{\sum_{i=1}^{n-1} \widehat{\mathbf{K}}_{i\Delta_{n,T}} \left(\Delta \log p_{i\Delta_{n,T}}\right)\left(\widehat{\sigma}^2_{(i+1)\Delta_{n,T}} - \widehat{\sigma}^2_{i\Delta_{n,T}}\right)}{\Delta_{n,T} \sum_{i=1}^{n} \widehat{\mathbf{K}}_{i\Delta_{n,T}}} - \frac{\sum_{i=1}^{n-1} \widehat{\mathbf{K}}_{i\Delta_{n,T}} \left(\Delta \log p_{i\Delta_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}} - \sigma^2_{i\Delta_{n,T}}\right)}{\Delta_{n,T} \sum_{i=1}^{n} \widehat{\mathbf{K}}_{i\Delta_{n,T}}} \\
= \; & \Pi_{1,n,T,k,\phi} + \Pi_{2,n,T,k,\phi} + \Pi_{3,n,T,k,\phi}.
\end{aligned}
$$

We focus on $\Pi_{1,n,T,k,\phi}$ first. Using the mean-value theorem as in the proofs of Theorem 1 and Theorem 2, we have

$$
\begin{aligned}
\Pi_{1,n,T,k,\phi} &= -\frac{\sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \left(\Delta \log p_{i\Delta_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}} - \sigma^2_{i\Delta_{n,T}}\right)}{\Delta_{n,T} \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}} \times \frac{\Delta_{n,T}\left(\sum_{i=1}^{n} \widehat{\mathbf{K}}_{i\Delta_{n,T}} - \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}\right)}{\Delta_{n,T} \sum_{i=1}^{n} \widehat{\mathbf{K}}_{i\Delta_{n,T}}} \\
&\leq |C^{*}(x)| \frac{\max_{1 \leq i \leq n} \left| \frac{\widehat{\sigma}^2_{i\Delta_{n,T}} - \sigma^2_{i\Delta_{n,T}}}{h_{n,T}} \right| \frac{\Delta_{n,T}}{v(T) h_{n,T}} \sum_{i=1}^{n} \left| \mathbf{K}'_{i\Delta_{n,T}} \left(\frac{\sigma^2_{i\Delta_{n,T}} + O_p(\mathbf{M}_{n,k,T,\phi}) - x}{h_{n,T}} \right) \right|}{\frac{\Delta_{n,T}}{v(T) h_{n,T}} \sum_{i=1}^{n} \widehat{\mathbf{K}}_{i\Delta_{n,T}}} \\
&= O_p\left(\frac{\mathbf{M}_{n,k,T,\phi}}{h_{n,T}}\right).
\end{aligned}
$$

Itô's lemma, now, yields

$$
\begin{aligned}
& \left(\log p_{(i+1)\Delta_{n,T}} - \log p_{i\Delta_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}} - \sigma^2_{i\Delta_{n,T}}\right) \\
= \; & \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\sigma_s^2 - \sigma^2_{i\Delta_{n,T}}\right) \mu(\sigma_s^2) \mathrm{d}s + \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\sigma_s^2 - \sigma^2_{i\Delta_{n,T}}\right) \sigma_s \mathrm{d}W_s^r \\
& + \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \int_{\psi} \left(\sigma_s^2 - \sigma^2_{i\Delta_{n,T}}\right) \psi^r v_r(\mathrm{d}s, \mathrm{d}\psi^r) + \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\log p_s - \log p_{i\Delta_{n,T}}\right) m(\sigma_s^2) \mathrm{d}s \\
& + \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\log p_s - \log p_{i\Delta_{n,T}}\right) \Lambda(\sigma_s^2) \mathrm{d}W_s^{\sigma} + \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \int_{\xi} \left(\log p_s - \log p_{i\Delta_{n,T}}\right) \xi^{\sigma} v_{\sigma}(\mathrm{d}s, \mathrm{d}\xi^{\sigma})
\end{aligned}
$$

$$+\sum_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left(\Delta\log p_s\Delta\sigma_s^2\right)+\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\rho(\sigma_s^2)\sigma_s\Lambda(\sigma_s^2)\mathrm{d}s, \tag{8}$$

where $\Delta\log p_s=\log p_s-\log p_{s-}$ and $\Delta\sigma_s^2=\sigma_s^2-\sigma_{s-}^2$. Thus,

$$\begin{aligned}
\Pi_{2,n,T,k,\phi} &= \frac{\sum_{i=1}^{n-1}\left(\widehat{\mathbf{K}}_{i\Delta_{n,T}}-\mathbf{K}_{i\Delta_{n,T}}\right)\left(\Delta\log p_{i\Delta_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right)}{\Delta_{n,T}\sum_{i=1}^{n}\widehat{\mathbf{K}}_{i\Delta_{n,T}}}\\
&\leq \frac{\max_{1\leq i\leq n}\left|\frac{\widehat{\sigma}^2_{i\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}}{h_{n,T}}\right|\frac{1}{v(T)h_{n,T}}\sum_{i=1}^{n-1}\left|\mathbf{K}'_{i\Delta_{n,T}}\left(\frac{\sigma^2_{i\Delta_{n,T}}+O_p(\mathbf{M}_{n,k,T,\phi})-x}{h_{n,T}}\right)\right|\left|\left(\Delta\log p_{i\Delta_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right)\right|}{\frac{\Delta_{n,T}}{v(T)h_{n,T}}\sum_{i=1}^{n}\widehat{\mathbf{K}}_{i\Delta_{n,T}}}\\
&= O_p\left(\frac{\mathbf{M}_{n,k,T,\phi}\Delta^*_{n,T}}{h_{n,T}\Delta_{n,T}}\right).
\end{aligned}$$

Using Eq. (8), the probability order follows from the same steps leading to the result in Theorem 2 after recognizing that there are no co-jumps, by the assumed independence between price and variance jumps, with probability one (e.g., Cont and Tankov, 2004, Proposition 5.3).
Finally, turning to $\Pi_{3,n,T,k,\phi}$, we have

$$\Pi_{3,n,T,k,\phi}\leq\frac{\sum_{i=1}^{n-1}\widehat{\mathbf{K}}_{i\Delta_{n,T}}\left|\Delta\log p_{i\Delta_{n,T}}\right|\left|(\widehat{\sigma}^2_{(i+1)\Delta_{n,T}}-\widehat{\sigma}^2_{i\Delta_{n,T}})-(\sigma^2_{(i+1)\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}})\right|}{\Delta_{n,T}\sum_{i=1}^{n}\widehat{\mathbf{K}}_{i\Delta_{n,T}}}=O_p\left(\frac{\mathbf{M}_{n,k,T,\phi}\Delta^*_{n,T}}{\Delta^{3/2}_{n,T}}\right).$$

In sum, given Assumption 4.2 in the main text, the estimation error for the infinitesimal covariance estimator has the following order:

$$\widehat{C}(x)-C^*(x)=\Pi_{1,n,T,k,\phi}+\Pi_{2,n,T,k,\phi}+\Pi_{3,n,T,k,\phi}=O_p\left(\frac{\mathbf{M}_{n,k,T,\phi}\Delta^*_{n,T}}{\Delta^{3/2}_{n,T}}\right).$$

We now focus on the estimator based on the true variance process, i.e., $C^*(x)$. Compensating the random measures $v_r(\mathrm{d}s,\mathrm{d}\psi^r)$ and $v_\sigma(\mathrm{d}s,\mathrm{d}\xi^\sigma)$ and writing, as earlier, $\overline{m}(\sigma_s^2)=m(\sigma_s^2)+\mathrm{E}_\xi(\xi^\sigma)\lambda^{\sigma^2}(\sigma_s^2)$ in place of $m(\sigma_s^2)$ and $\overline{\mu}(\sigma_s^2)=\mu(\sigma_s^2)+\mathrm{E}_\psi(\psi^r)\lambda^r(\sigma_s^2)$ in place of $\mu(\sigma_s^2)$, we have

$$\begin{aligned}
&\left(\log p_{(i+1)\Delta_{n,T}}-\log p_{i\Delta_{n,T}}\right)\left(\sigma^2_{(i+1)\Delta_{n,T}}-\sigma^2_{i\Delta_{n,T}}\right)\\
={}&\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sigma_s^2-\sigma^2_{i\Delta_{n,T}}\right)\overline{\mu}(\sigma_s^2)\mathrm{d}s+\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\sigma_s^2-\sigma^2_{i\Delta_{n,T}}\right)\sigma_s\mathrm{d}W_s^r\\
&+\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_\psi\left(\sigma_s^2-\sigma^2_{i\Delta_{n,T}}\right)\psi^r\overline{v}_r(\mathrm{d}s,\mathrm{d}\psi^r)+\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\log p_s-\log p_{i\Delta_{n,T}}\right)\overline{m}(\sigma_s^2)\mathrm{d}s\\
&+\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\log p_s-\log p_{i\Delta_{n,T}}\right)\Lambda(\sigma_s^2)\mathrm{d}W_s^\sigma+\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_\xi\left(\log p_s-\log p_{i\Delta_{n,T}}\right)\xi^\sigma\overline{v}_\sigma(\mathrm{d}s,\mathrm{d}\xi^\sigma)\\
&+\sum_{i\Delta_{n,T}\leq s\leq(i+1)\Delta_{n,T}}\left(\Delta\log p_s\Delta\sigma_s^2\right)+\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\rho(\sigma_s^2)\sigma_s\Lambda(\sigma_s^2)\mathrm{d}s
\end{aligned}$$

$$= \quad \Theta_{i,1,n,T} + \Theta_{i,2,n,T} + \Theta_{i,3,n,T} + \Theta_{i,4,n,T} + \Theta_{i,5,n,T} + \Theta_{i,6,n,T} + \Theta_{i,7,n,T} + \Theta_{i,8,n,T}. \tag{9}$$

Next, we consider all terms one by one. Using the same method of proof as for Lemma 2, we have

$$\frac{\sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \Theta_{i,8,n,T}}{\Delta_{n,T} \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}} = \frac{\frac{\int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \rho(\sigma_s^2)\sigma_s \Lambda(\sigma_s^2) \mathrm{d}s}{\int_0^T \mathbf{K}\left(\frac{\sigma_s^2 - x}{h_{n,T}}\right) \mathrm{d}s} + O_p\left(\frac{\Delta_{n,T}^*}{h_{n,T}^2}\right)}{1 + O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}.$$

Clearly, $\frac{\sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \Theta_{i,1,n,T}}{\Delta_{n,T} \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}}$ and $\frac{\sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \Theta_{i,4,n,T}}{\Delta_{n,T} \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}}$ are of smaller order. Notice also that $\frac{\sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \Theta_{i,7,n,T}}{\Delta_{n,T} \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}} = 0$, almost surely, once more, by the independence of the price and variance jumps.

Now, write

$$\mathbf{C}_{n,T} = \underbrace{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \Theta_{i,2,n,T}}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}}}_{\mathbf{C}_{1,n,T}} + \underbrace{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \Theta_{i,5,n,T}}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}}}_{\mathbf{C}_{2,n,T}}$$

and

$$\mathbf{J}_{n,T} = \underbrace{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \Theta_{i,3,n,T}}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}}}_{\mathbf{J}_{1,n,T}} + \underbrace{\frac{\frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \Theta_{i,6,n,T}}{\frac{\Delta_{n,T}}{h_{n,T}} \sum_{i=1}^{n} \mathbf{K}_{i\Delta_{n,T}}}}_{\mathbf{J}_{2,n,T}}.$$

We begin with $\mathbf{C}_{2,n,T}$. The quantities

$$\left\{ \left(\sqrt{\frac{h_{n,T}}{\Delta_{n,T}}}\right) \frac{1}{h_{n,T}} \mathbf{K}_{i\Delta_{n,T}} \Theta_{i,5,n,T}, \mathcal{F}_{i\Delta_{n,T}}, 1 \leq i \leq n-1, n \geq 2 \right\}$$

constitute a martingale difference array. Express the sum of the conditional variances of the standardized (by $\sqrt{\frac{h_{n,T}}{\Delta_{n,T}}}$) numerator of $\mathbf{C}_{2,n,T}$, i.e., $\left(\sqrt{\frac{h_{n,T}}{\Delta_{n,T}}}\right) \frac{1}{h_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}} \Theta_{i,5,n,T}$, as

$$\mathbf{V}_{C_{2,n,T}^{num}} = \frac{1}{h_{n,T}\Delta_{n,T}} \sum_{i=1}^{n-1} \mathbf{K}_{i\Delta_{n,T}}^2 \mathrm{E}_{i\Delta_{n,T}} \left\{ \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\log p_s - \log p_{i\Delta_{n,T}}\right)^2 \Lambda^2(\sigma_s^2) \mathrm{d}s \right\}.$$

Using the notation $\Delta_i \log p_s = \log p_s - \log p_{i\Delta_{n,T}}$, Itô's lemma gives

$$(\Delta_i \log p_s)^2 = 2\int_{i\Delta_{n,T}}^{s} (\Delta_i \log p_u)\, \overline{\mu}(\sigma_u^2)\mathrm{d}u + 2\int_{i\Delta_{n,T}}^{s} (\Delta_i \log p_u)\, \sigma_u \mathrm{d}W_u^r + \int_{i\Delta_{n,T}}^{s} \int_{\psi} (\psi^r)^2 \overline{v}_r(\mathrm{d}u, \mathrm{d}\psi^r)$$
$$+ 2\int_{i\Delta_{n,T}}^{s} \int_{\psi} (\Delta_i \log p_u)\psi^r \overline{v}_r(\mathrm{d}u, \mathrm{d}\psi^r) + \int_{i\Delta_{n,T}}^{s} \sigma_u^2 \mathrm{d}u + \int_{i\Delta_{n,T}}^{s} \mathrm{E}_{\psi}((\psi^r)^2)\lambda^r(\sigma_u^2)\mathrm{d}u,$$

and, as a consequence,

$$
\begin{aligned}
&\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\log p_s - \log p_{i\Delta_{n,T}}\right)^2 \mathrm{d}s = 2\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \left(\Delta_i \log p_u\right) \overline{\mu}(\sigma_u^2)\mathrm{d}u\right) \mathrm{d}s \\
&+2\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \left(\Delta_i \log p_u\right) \sigma_u \mathrm{d}W_u^r\right) \mathrm{d}s + \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \int_\psi (\psi^r)^2 \overline{v}_r(\mathrm{d}u, \mathrm{d}\psi^r)\right) \mathrm{d}s \\
&+2\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \int_\psi (\Delta_i \log p_u) \psi^r \overline{v}_r(\mathrm{d}u, \mathrm{d}\psi^r)\right) \mathrm{d}s \\
&+\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \left\{(\sigma_u^2 + \mathrm{E}_\psi((\psi^r)^2)\lambda^r(\sigma_u^2)) - (\sigma_{i\Delta_{n,T}}^2 + \mathrm{E}_\psi((\psi^r)^2)\lambda^r(\sigma_{i\Delta_{n,T}}^2))\right\} \mathrm{d}u\right) \mathrm{d}s \\
&+\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \mathrm{d}u\right) \mathrm{d}s\right) (\sigma_{i\Delta_{n,T}}^2 + \mathrm{E}_\psi((\psi^r)^2)\lambda^r(\sigma_{i\Delta_{n,T}}^2)).
\end{aligned}
$$

Noting that

$$
\begin{aligned}
\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\log p_s - \log p_{i\Delta_{n,T}}\right)^2 \Lambda^2(\sigma_s^2)\mathrm{d}s \quad &= \quad \Lambda^2(\sigma_{i\Delta_{n,T}}^2) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\log p_s - \log p_{i\Delta_{n,T}}\right)^2 \mathrm{d}s \\
&\quad + \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\log p_s - \log p_{i\Delta_{n,T}}\right)^2 (\Lambda^2(\sigma_s^2) - \Lambda^2(\sigma_{i\Delta_{n,T}}^2))\mathrm{d}s,
\end{aligned}
$$

we have, because $\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \mathrm{d}u\right) \mathrm{d}s\right) = \frac{\Delta_{n,T}^2}{2}$, that

$$
\begin{aligned}
&\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\log p_s - \log p_{i\Delta_{n,T}}\right)^2 \Lambda^2(\sigma_s^2)\mathrm{d}s \\
= \quad & 2\Lambda^2(\sigma_{i\Delta_{n,T}}^2) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \left(\Delta_i \log p_u\right) \overline{\mu}(\sigma_u^2)\mathrm{d}u\right) \mathrm{d}s \\
& +2\Lambda^2(\sigma_{i\Delta_{n,T}}^2) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \left(\Delta_i \log p_u\right) \sigma_u \mathrm{d}W_u^r\right) \mathrm{d}s \\
& +\Lambda^2(\sigma_{i\Delta_{n,T}}^2) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \int_\psi (\psi^r)^2 \overline{v}_r(\mathrm{d}u, \mathrm{d}\psi^r)\right) \mathrm{d}s \\
& +2\Lambda^2(\sigma_{i\Delta_{n,T}}^2) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \int_\psi (\Delta_i \log p_u) \psi^r \overline{v}_r(\mathrm{d}u, \mathrm{d}\psi^r)\right) \mathrm{d}s \\
& +\Lambda^2(\sigma_{i\Delta_{n,T}}^2) \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\int_{i\Delta_{n,T}}^{s} \left\{(\sigma_u^2 + \mathrm{E}_\psi((\psi^r)^2)\lambda^r(\sigma_u^2)) - (\sigma_{i\Delta_{n,T}}^2 + \mathrm{E}_\psi((\psi^r)^2)\lambda^r(\sigma_{i\Delta_{n,T}}^2))\right\} \mathrm{d}u\right) \mathrm{d}s \\
& +\frac{1}{2}\Delta_{n,T}^2(\sigma_{i\Delta_{n,T}}^2 + \mathrm{E}_\psi((\psi^r)^2)\lambda^r(\sigma_{i\Delta_{n,T}}^2))\Lambda^2(\sigma_{i\Delta_{n,T}}^2) \\
& +\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}} \left(\left(\log p_s - \log p_{i\Delta_{n,T}}\right)^2 (\Lambda^2(\sigma_s^2) - \Lambda^2(\sigma_{i\Delta_{n,T}}^2))\right) \mathrm{d}s \\
= \quad & \Sigma_{i,1,n,T} + \Sigma_{i,2,n,T} + \Sigma_{i,3,n,T} + \Sigma_{i,4,n,T} + \Sigma_{i,5,n,T} + \Sigma_{i,6,n,T} + \Sigma_{i,7,n,T}.
\end{aligned}
$$

Now, we recognize that

$$
\begin{aligned}
&\frac{1}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\mathrm{E}_{i\Delta_{n,T}}\left\{\Sigma_{i,6,n,T}\right\}\\
&=\frac{1}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\mathrm{E}_{i\Delta_{n,T}}\left\{\frac{1}{2}\Delta^2_{n,T}(\sigma^2_{i\Delta_{n,T}}+\mathrm{E}_\psi((\psi^r)^2)\lambda^r(\sigma^2_{i\Delta_{n,T}}))\Lambda^2(\sigma^2_{i\Delta_{n,T}})\right\}\\
&=\frac{\Delta_{n,T}}{h_{n,T}v(T)}\sum_{i=1}^{n-1}\frac{1}{2}\mathbf{K}^2_{i\Delta_{n,T}}(\sigma^2_{i\Delta_{n,T}}+\mathrm{E}_\psi((\psi^r)^2)\lambda^r(\sigma^2_{i\Delta_{n,T}}))\Lambda^2(\sigma^2_{i\Delta_{n,T}})\\
&=\frac{1}{2}\frac{1}{h_{n,T}v(T)}\int_0^T\mathbf{K}^2\left(\frac{\sigma^2_s-x}{h_{n,T}}\right)\Lambda^2(\sigma^2_s)\left[\sigma^2_s+\mathrm{E}_\psi((\psi^r)^2)\lambda^r(\sigma^2_s)\right]\mathrm{d}s+O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right),
\end{aligned}\tag{10}
$$

using the same method of proof leading to the result in Lemma 2. Next, we turn to $\Sigma_{i,3,n,T}$. Integrating by parts, we have

$$
\begin{aligned}
\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\int_\psi(\psi^r)^2\overline{v}_r(\mathrm{d}u,\mathrm{d}\psi^r)\right)\mathrm{d}s &= -\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\mathrm{d}u\right)\int_\psi(\psi^r)^2\overline{v}_r(\mathrm{d}s,\mathrm{d}\psi^r)\\
&\quad+\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{d}u\right)\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\int_\psi(\psi^r)^2\overline{v}_r(\mathrm{d}s,\mathrm{d}\psi)\\
&= \int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}((i+1)\Delta_{n,T}-s)\int_\psi(\psi^r)^2\overline{v}_r(\mathrm{d}s,\mathrm{d}\psi).
\end{aligned}
$$

Hence, $\mathrm{E}_{i\Delta_{n,T}}\left\{\Sigma_{i,3,n,T}\right\}=0$. Similarly, $\mathrm{E}_{i\Delta_{n,T}}\left\{\Sigma_{i,2,n,T}\right\}=\mathrm{E}_{i\Delta_{n,T}}\left\{\Sigma_{i,4,n,T}\right\}=0$. Consider, now,

$$
\begin{aligned}
\frac{1}{2}\Sigma_{i,1,n,T} &= \Lambda^2(\sigma^2_{i\Delta_{n,T}})\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}(\Delta_i\log p_u)\,\overline{\mu}(\sigma^2_u)\mathrm{d}u\right)\mathrm{d}s\\
&= \Lambda^2(\sigma^2_{i\Delta_{n,T}})\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\left(\int_{i\Delta_{n,T}}^{u}\overline{\mu}(\sigma^2_v)\mathrm{d}v\right)\overline{\mu}(\sigma^2_u)\mathrm{d}u\right)\mathrm{d}s\\
&\quad+\Lambda^2(\sigma^2_{i\Delta_{n,T}})\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\left(\int_{i\Delta_{n,T}}^{u}\sigma_v\mathrm{d}W^r_v\right)\overline{\mu}(\sigma^2_u)\mathrm{d}u\right)\mathrm{d}s\\
&\quad+\Lambda^2(\sigma^2_{i\Delta_{n,T}})\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\left(\int_{i\Delta_{n,T}}^{u}\int_\psi\psi^r\overline{v}_r(\mathrm{d}v,\mathrm{d}\psi^r)\right)\overline{\mu}(\sigma^2_u)\mathrm{d}u\right)\mathrm{d}s\\
&= \Sigma_{i,1,1,n,T}+\Sigma_{i,1,2,n,T}+\Sigma_{i,1,3,n,T}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\mathrm{E}_{i\Delta_{n,T}}(\Sigma_{i,1,1,n,T}) &\le \mathrm{E}_{i\Delta_{n,T}}\left(\Lambda^2(\sigma^2_{i\Delta_{n,T}})\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\left(\int_{i\Delta_{n,T}}^{u}|\overline{\mu}(\sigma^2_v)|\mathrm{d}v\right)|\overline{\mu}(\sigma^2_u)|\mathrm{d}u\right)\mathrm{d}s\right)\\
&\le \mathrm{E}_{i\Delta_{n,T}}\left(\Lambda^2(\sigma^2_{i\Delta_{n,T}})\Delta_{n,T}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}|\overline{\mu}(\sigma^2_s)|\mathrm{d}s\right)^2\right)\\
&\le \mathrm{E}_{i\Delta_{n,T}}\left(\Delta^2_{n,T}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\overline{\mu}^2(\sigma^2_s)\mathrm{d}s\right)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{1}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\mathrm{E}_{i\Delta_{n,T}}\left\{\Sigma_{i,1,1,n,T}\right\}\\
&\leq \frac{\Delta^2_{n,T}}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\overline{\mu}^2(\sigma^2_s)\mathrm{d}s\right)\\
&\leq \frac{\Delta^2_{n,T}}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left(\overline{\mu}^2(\sigma^2_s)-\overline{\mu}^2(\sigma^2_{i\Delta_{n,T}})\right)\mathrm{d}s\right)\\
&+\frac{\Delta^3_{n,T}}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\overline{\mu}^2(\sigma^2_{i\Delta_{n,T}}).
\end{aligned}
$$

Using standard, by now, arguments (see, e.g., the proof of Lemma 2), the dominating term is the second one, for which we have

$$
\frac{\Delta^3_{n,T}}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\overline{\mu}^2(\sigma^2_{i\Delta_{n,T}})=O_p(\Delta^*_{n,T})+O_p\left(\frac{\Delta^{*2}_{n,T}}{h^2_{n,T}}\right).
$$

By Holder's inequality, Jensen's inequality, and BDG's inequality, we obtain

$$
\begin{aligned}
&\mathrm{E}_{i\Delta_{n,T}}(\Sigma_{i,1,2,n,T})\\
\leq\;&\Lambda^2(\sigma^2_{i\Delta_{n,T}})\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\mathrm{E}_{i\Delta_{n,T}}\left(\left(\int_{i\Delta_{n,T}}^{s}\left|\overline{\mu}(\sigma^2_u)\right|\mathrm{d}u\right)\left(\sup_{u\leq s}\left|\int_{i\Delta_{n,T}}^{u}\sigma_v\mathrm{d}W^r_v\right|\right)\right)\mathrm{d}s\\
\leq\;&\Lambda^2(\sigma^2_{i\Delta_{n,T}})\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\sqrt{\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{s}\left|\overline{\mu}(\sigma^2_u)\right|\mathrm{d}u\right)^2\mathrm{E}_{i\Delta_{n,T}}\left(\sup_{u\leq s}\left|\int_{i\Delta_{n,T}}^{u}\sigma_v\mathrm{d}W^r_v\right|\right)^2}\mathrm{d}s\\
\leq\;&C\Lambda^2(\sigma^2_{i\Delta_{n,T}})\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\sqrt{\Delta_{n,T}\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left|\overline{\mu}(\sigma^2_u)\right|^2\mathrm{d}u\right)\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\sigma^2_v\mathrm{d}v\right)}\mathrm{d}s\\
\leq\;&C\Lambda^2(\sigma^2_{i\Delta_{n,T}})\Delta^{3/2}_{n,T}\left(\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left|\overline{\mu}(\sigma^2_u)\right|^2\mathrm{d}u\right)\right)^{1/2}\left(\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\sigma^2_v\mathrm{d}v\right)\right)^{1/2}.
\end{aligned}
$$

Employing the bound $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ twice, we have

$$
\begin{aligned}
&\frac{1}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\mathrm{E}_{i\Delta_{n,T}}\left\{\Sigma_{i,1,2,n,T}\right\}\leq\\
&\frac{C}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\Delta^{3/2}_{n,T}\left(\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\left|\overline{\mu}(\sigma^2_u)\right|^2\mathrm{d}u\right)\right)^{1/2}\left(\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\sigma^2_v\mathrm{d}v\right)\right)^{1/2}\\
&\leq\frac{C}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\Delta^{3/2}_{n,T}\left(\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}(\overline{\mu}^2(\sigma^2_u)-\overline{\mu}^2(\sigma^2_{i\Delta_{n,T}}))\mathrm{d}u\right)\right)^{1/2}\times\\
&\times\left(\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\sigma^2_v\mathrm{d}v\right)\right)^{1/2}
\end{aligned}
$$

$$+\frac{C}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\Delta^2_{n,T}\overline{\mu}(\sigma^2_{i\Delta_{n,T}})\left(\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\sigma_v^2\mathrm{d}v\right)\right)^{1/2}$$

$$\leq\frac{C}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\Delta^{3/2}_{n,T}\left(\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}(\overline{\mu}^2(\sigma^2_u)-\overline{\mu}^2(\sigma^2_{i\Delta_{n,T}}))\mathrm{d}u\right)\right)^{1/2}\times$$

$$\times\left(\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}\sigma_v^2\mathrm{d}v\right)\right)^{1/2}$$

$$+\frac{C}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\Delta^2_{n,T}\overline{\mu}(\sigma^2_{i\Delta_{n,T}})\left(\mathrm{E}_{i\Delta_{n,T}}\left(\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}(\sigma_v^2-\sigma^2_{i\Delta_{n,T}})\mathrm{d}v\right)\right)^{1/2}$$

$$+\frac{C}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\Delta^{5/2}_{n,T}\overline{\mu}(\sigma^2_{i\Delta_{n,T}})\sigma_{i\Delta_{n,T}}.$$

The dominating term is the last one, giving

$$\frac{C}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\Lambda^2(\sigma^2_{i\Delta_{n,T}})\Delta^{5/2}_{n,T}\overline{\mu}(\sigma^2_{i\Delta_{n,T}})\sigma_{i\Delta_{n,T}}=O_p\left(\Delta^{1/2}_{n,T}\frac{\Delta^*_{n,T}}{\Delta_{n,T}}\right)+O_p\left(\frac{\Delta^{*3/2}_{n,T}}{h^2_{n,T}}\right).$$

Recall that $\frac{\Delta^*_{n,T}}{\Delta^{1/2}_{n,T}}=\mathrm{E}\left(\mathcal{M}(F)\Delta^{1/2}_{n,T}\right)\to 0$, as implied by Assumption 5 (a.2). Thus, $O_p\left(\Delta^{1/2}_{n,T}\frac{\Delta^*_{n,T}}{\Delta_{n,T}}\right)=o_p(1)$.

Next, the term $\Sigma_{1,3,i,n,T}$ can be handled in just the same way as $\Sigma_{1,2,i,n,T}$. As for $\Sigma_{i,5,n,T}$, it is clear that

$$\frac{1}{h_{n,T}\Delta_{n,T}v(T)}\sum_{i=1}^{n-1}\mathbf{K}^2_{i\Delta_{n,T}}\mathrm{E}_{i\Delta_{n,T}}\left\{\Sigma_{i,5,n,T}\right\}$$

is dominated by Eq. (10).

The sum of the conditional fourth moments of the terms $\left(\sqrt{\frac{h_{n,T}}{\Delta_{n,T}}}\right)\frac{1}{h_{n,T}}\mathbf{K}_{i\Delta_{n,T}}\Theta_{i,5,n,T}$ can be treated like the term $\mathbf{V}_{C^{num}_{2,n,T}}$ above and it is easily shown to vanish asymptotically. Using the same method of proof as for Lemma 4, as $n,T\to\infty$ and $\Delta_{n,T},h_{n,T}\to 0$ *jointly*, along with $\frac{\Delta^*_{n,T}}{h^2_{n,T}}\to 0$, we therefore have

$$\left(\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\right)\frac{\mathbf{C}^{num}_{2,n,T}}{\widehat{L}_{\sigma^2}(T,x)}\Rightarrow\left(\sqrt{\mathbf{K}_2\frac{1}{2}\left(x+\mathrm{E}_\psi((\psi^r)^2)\lambda^r(x)\right)\Lambda^2(x)}\right)\mathrm{N}(0,1),$$

and, similarly,

$$\left(\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\right)\frac{\mathbf{C}^{num}_{1,n,T}}{\widehat{L}_{\sigma^2}(T,x)}\Rightarrow\left(\sqrt{\mathbf{K}_2\frac{1}{2}\left(\Lambda^2(x)+\mathrm{E}_\xi((\xi^\sigma)^2)\lambda^{\sigma^2}(x)\right)x}\right)\mathrm{N}(0,1).$$

In addition, the asymptotic covariance term can be expressed as

$$\mathrm{Cov}\left(\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\frac{\mathbf{C}^{num}_{1,n,T}}{\widehat{L}_{\sigma^2}(T,x)},\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\frac{\mathbf{C}^{num}_{2,n,T}}{\widehat{L}_{\sigma^2}(T,x)}\right)$$

$$\xrightarrow[T,n\to\infty,\quad \Delta_{n,T},h_{n,T}\to 0]{p} \frac{1}{2}\mathbf{K}_2\rho^2(x)x\Lambda^2(x).$$

We, then, have

$$\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\mathbf{C}_{n,T} \Rightarrow \left(\sqrt{\mathbf{K}_2\left(\begin{array}{l}\frac{1}{2}\lambda^{\sigma^2}(x)\mathrm{E}_{\xi}((\xi^{\sigma})^2)x\\ +\frac{1}{2}\lambda^{r}(x)\mathrm{E}_{\psi}((\psi^{r})^2)\Lambda^2(x)\\ +\left(1+\rho^2(x)\right)x\Lambda^2(x)\end{array}\right)}\right)\mathrm{N}(0,1).$$

By the same reasoning (combined with the independence of the jumps), we obtain

$$\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\mathbf{J}_{n,T} \Rightarrow \left(\sqrt{\frac{\mathbf{K}_2}{2}\left(\begin{array}{l}(\Lambda^2(x)+\lambda^{\sigma^2}(x)\mathrm{E}_{\xi}((\xi^{\sigma})^2))\lambda^{r}(x)\mathrm{E}_{\psi}((\psi^{r})^2)\\ +\left(x+\lambda^{r}(x)\mathrm{E}_{\psi}((\psi^{r})^2)\right)\lambda^{\sigma^2}(x)\mathrm{E}_{\xi}((\xi^{\sigma})^2)\end{array}\right)}\right)\mathrm{N}(0,1).$$

Now, write the (standardized) estimation error decomposition of $\widehat{C}(x)$, excluding the bias term, as

$$\begin{aligned}
&\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\left\{\frac{\sum_{i=1}^{n-1}\mathbf{K}_{i\Delta_{n,T}}\Theta_{i,1,n,T}}{\Delta_{n,T}\sum_{i=1}^{n}\mathbf{K}_{i\Delta_{n,T}}}+\frac{\sum_{i=1}^{n-1}\mathbf{K}_{i\Delta_{n,T}}\Theta_{i,4,n,T}}{\Delta_{n,T}\sum_{i=1}^{n}\mathbf{K}_{i\Delta_{n,T}}}+\frac{\sum_{i=1}^{n-1}\mathbf{K}_{i\Delta_{n,T}}\Theta_{i,7,n,T}}{\Delta_{n,T}\sum_{i=1}^{n}\mathbf{K}_{i\Delta_{n,T}}}\right.\\
&\left.+\mathbf{C}_{n,T}+\mathbf{J}_{n,T}+O_p\left(\frac{\mathbf{M}_{n,k,T,\phi}\Delta^{*}_{n,T}}{\Delta^{3/2}_{n,T}}\right)\right\}\\
=\;&\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\left\{O_p(\Delta^{*1/2}_{n,T})\left(1+O_p\left(\frac{\Delta^{*}_{n,T}}{h^2_{n,T}}\right)\right)+\mathbf{C}_{n,T}+\mathbf{J}_{n,T}+O_p\left(\frac{\mathbf{M}_{n,k,T,\phi}\Delta^{*}_{n,T}}{\Delta^{3/2}_{n,T}}\right)\right\}\\
=\;&\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}O_p(\Delta^{*1/2}_{n,T})\left(1+O_p\left(\frac{\Delta^{*}_{n,T}}{h^2_{n,T}}\right)\right)+\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\left\{\mathbf{C}_{n,T}+\mathbf{J}_{n,T}\right\}\\
&+O_p\left(\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\left(\frac{\mathbf{M}_{n,k,T,\phi}\Delta^{*}_{n,T}}{\Delta^{3/2}_{n,T}}\right)\right).
\end{aligned}$$

Finally, consider the bias term

$$\begin{aligned}
&\frac{\sum_{i=1}^{n-1}\mathbf{K}_{i\Delta_{n,T}}\Theta_{i,8,n,T}}{\Delta_{n,T}\sum_{i=1}^{n}\mathbf{K}_{i\Delta_{n,T}}}-\rho(x)\sqrt{x}\Lambda(x)\\
=\;&\frac{\sum_{i=1}^{n-1}\mathbf{K}_{i\Delta_{n,T}}\int_{i\Delta_{n,T}}^{(i+1)\Delta_{n,T}}(\rho(\sigma_s^2)\sigma_s\Lambda(\sigma_s^2))\mathrm{d}s}{\Delta_{n,T}\sum_{i=1}^{n}\mathbf{K}_{i\Delta_{n,T}}}-\rho(x)\sqrt{x}\Lambda(x)\\
=\;&\frac{\frac{\int_0^T\mathbf{K}_s\rho(\sigma_s^2)\sigma_s\Lambda(\sigma_s^2)\mathrm{d}s}{\int_0^T\mathbf{K}_s\mathrm{d}s}-\rho(x)\sqrt{x}\Lambda(x)+O_p\left(\frac{\Delta^{*}_{n,T}}{h^2_{n,T}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}-\frac{(\rho(x)\sqrt{x}\Lambda(x))\,O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h^2_{n,T}}\right)}
\end{aligned}$$

$$= \frac{\Gamma_C(x)}{1+O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)} + \frac{O_p\left(\frac{\Delta^*_{n,T}}{h_{n,T}^2}\right)}{1+O_p\left(\frac{\Delta_{n,T}}{h_{n,T}^2}\right)}.$$

Naturally, $C(.) = \rho(.)\sqrt{.}\Lambda(.)$. As in the proof of Theorem 4 of Bandi and Moloche (2017), we have

$$\Gamma_C(x) = h_{n,T}^2 \mathbf{K}_1 \left(\frac{\partial C(x)}{\partial x}\frac{\partial s(x)/\partial x}{s(x)} + \frac{1}{2}\frac{\partial^2 C(x)}{\partial x \partial x}\right) + o_p(h_{n,T}^2).$$

Thus,

$$\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)/\Delta_{n,T}}\left\{\widehat{C}(x) - C(x) - \Gamma_C(x)\right\} \Rightarrow \mathrm{N}\left(0, \mathbf{K}_2\theta_C(x)\right),$$

with

$$\begin{aligned}\theta_C(x) &= \Lambda^2(x)x(1+\rho^2(x)) + x\lambda^{\sigma^2}(x)\mathrm{E}_\xi((\xi^\sigma)^2) \\ &\quad + \Lambda^2(x)\lambda^r(x)\mathrm{E}_\psi((\psi^r)^2) + \lambda^{\sigma^2}(x)\mathrm{E}_\xi((\xi^\sigma)^2)\lambda^r(x)\mathrm{E}_\psi((\psi^r)^2),\end{aligned}$$

if $h_{n,T}\widehat{L}_{\sigma^2}(T,x)\frac{\Delta^*_{n,T}}{\Delta_{n,T}} \xrightarrow{p} 0$, $\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}} \xrightarrow{p} \infty$, $\frac{h^5_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}} \xrightarrow{p} C$, $\frac{\Delta^*_{n,T}}{h^2_{n,T}}\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}} \xrightarrow{p} 0$, $\frac{\Delta^*_{n,T}}{h^2_{n,T}} \to 0$ and $\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}}\left(\frac{g(n,T,k,\phi)h_{n,T}\Delta^*_{n,T}}{\Delta^{3/2}_{n,T}}\right) \xrightarrow{p} 0$.

Given $h_{n,T}\widehat{L}_{\sigma^2}(T,x)\frac{\Delta^*_{n,T}}{\Delta_{n,T}} \xrightarrow{p} 0$, however, the condition $\frac{h^5_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}} = \frac{h^5_{n,T}\widehat{L}_{\sigma^2}(T,x)\Delta^*_{n,T}}{\Delta_{n,T}\Delta^*_{n,T}} = O_p(1)$ implies $\frac{h^4_{n,T}}{\Delta^*_{n,T}} \to \infty$. Thus, $\frac{\Delta^*_{n,T}}{h^4_{n,T}} \to 0$ and the requirement

$$\frac{\Delta^*_{n,T}}{h^2_{n,T}}\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}{\Delta_{n,T}}} = \frac{\Delta^*_{n,T}}{h^2_{n,T}\Delta^{*1/2}_{n,T}}\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)\Delta^*_{n,T}}{\Delta_{n,T}}} = \underbrace{\frac{\Delta^{*1/2}_{n,T}}{h^2_{n,T}}}_{\to 0}\underbrace{\sqrt{\frac{h_{n,T}\widehat{L}_{\sigma^2}(T,x)\Delta^*_{n,T}}{\Delta_{n,T}}}}_{\xrightarrow{p} 0} \xrightarrow{p} 0$$

is satisfied. Next, by the delta method, after defining $(\Lambda^*(x))^2$ as the infeasible estimator of $\Lambda^2(x)$, we obtain

$$\begin{aligned}
&\widehat{\rho}(x) - \rho(x) \\
&= \frac{\widehat{C}(x)}{\sqrt{x}\widehat{\Lambda}(x)} - \frac{C(x)}{\sqrt{x}\Lambda(x)} \\
&\overset{p}{\sim} \frac{C^*(x) - C(x)}{\sqrt{x}\Lambda(x)} - \frac{C(x)\left((\Lambda^*(x))^2 - \Lambda^2(x)\right)}{2\sqrt{x}\Lambda^3(x)} + O_p\left(\frac{\mathbf{M}_{n,k,T,\phi}\Delta^*_{n,T}}{\Delta^{3/2}_{n,T}}\right) + O_p\left(\frac{\mathbf{M}_{n,k,T,\phi}\Delta^*_{n,T}}{\Delta^{3/2}_{n,T}h_{n,T}}\right) \\
&\overset{p}{\sim} \frac{O_p(h^2_{n,T}) + O_p\left(\sqrt{\frac{\Delta_{n,T}}{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}}\right) + O_p(\Delta^{*1/2}_{n,T})\left(1 + O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right)\right) + O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right)}{\sqrt{x}\Lambda(x)} \\
&- \rho(x)\frac{O_p(h^2_{n,T}) + O_p\left(\sqrt{\frac{1}{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}}\right) + O_p\left(\frac{\Delta^*_{n,T}}{h^2_{n,T}}\right)}{2\Lambda^2(x)} + O_p\left(\frac{\mathbf{M}_{n,k,T,\phi}\Delta^*_{n,T}}{\Delta^{3/2}_{n,T}h_{n,T}}\right).
\end{aligned}$$

Hence, if $h_{n,T}\widehat{L}_{\sigma^2}(T,x) \xrightarrow{p} \infty$, $h_{n,T}\widehat{L}_{\sigma^2}(T,x)\Delta^*_{n,T} \xrightarrow{p} 0$ and $\frac{\Delta^*_{n,T}}{h^2_{n,T}}\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} \xrightarrow{p} 0$, the asymptotic distribution of the leverage estimator is driven by $(\Lambda^*(x))^2 - \Lambda^2(x)$ and has a limiting variance given by

$$\widetilde{\mathrm{V}}(\widehat{\rho}(x)) = \frac{\rho^2(x)}{4\Lambda^4(x)}\left(\widetilde{\mathrm{V}}\left(\widehat{\Lambda}^2(x)\right)\right) = \frac{\rho^2(x)}{4\Lambda^4(x)}\left(\frac{40\mathbf{K}_2\lambda^{\sigma^2}(x)\mu^2_{\xi}(x)}{h_{n,T}\widehat{L}_{\sigma^2}(T,x)}\right).$$

Also, if $h^5_{n,T}\widehat{L}_{\sigma^2}(T,x) \xrightarrow{p} 0$, $\widehat{\rho}(x)-\rho(x)$ has a vanishing asymptotic bias. Now, notice that the requirement

$$\frac{\Delta^*_{n,T}}{h^2_{n,T}}\sqrt{h_{n,T}\widehat{L}_{\sigma^2}(T,x)} = \frac{\Delta^*_{n,T}\sqrt{\widehat{L}_{\sigma^2}(T,x)}}{h^{3/2}_{n,T}} \xrightarrow{p} 0$$

can be re-written as follows

$$\frac{\Delta^*_{n,T}\widehat{L}_{\sigma^2}(T,x)h_{n,T}}{h^{5/2}_{n,T}\widehat{L}^{1/2}_{\sigma^2}(T,x)} \xrightarrow{p} 0.$$

Thus, the condition for a vanishing bias, i.e., $h^5_{n,T}\widehat{L}_{\sigma^2}(T,x) \xrightarrow{p} 0$, implies that $\Delta^*_{n,T}\widehat{L}_{\sigma^2}(T,x)h_{n,T} \xrightarrow{p} 0$. We can therefore dispense with $h_{n,T}\widehat{L}_{\sigma^2}(T,x)\Delta^*_{n,T} \xrightarrow{p} 0$. $\square$

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