Online supplementary material to "Structural change in non-stationary AR(1) models"*

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Proof of Lemma A.1. The proofs of (a) and (b) can be completed by following the proof of Theorem 3.2 in Phillips and Magdalinos (2007a) and by using the truncation technique as shown in (A.1). The details are omitted. To prove part (c), it suffices to note that

$$\begin{aligned} \frac{y_{[rT]}}{\sqrt{k_T l(\eta_T)}} &= \frac{1}{\sqrt{k_T l(\eta_T)}}\Big((1-\frac{c}{k_T})^{[rT]}y_0 + \sum_{i=0}^{[rT]-1}(1-\frac{c}{k_T})^i \varepsilon_{[rT]-i}\Big) \\ &= \sum_{i=0}^{[rT]-1}\big((1-\frac{c}{k_T})^{k_T}\big)^{i/k_T}\frac{\varepsilon_{[rT]-i}}{\sqrt{k_T l(\eta_T)}} + o_p(1) \\ &\Rightarrow \int_0^\infty \exp(-cs)dW(s) \end{aligned}$$

by Theorem 1 in Csörgő et al. (2003), (3.4) in Giné et al. (1997) and assumptions C2 and C3. □

Proof of Lemma A.2. To prove part (a), we first note that

$$y_{t-1}\varepsilon_t = \frac{1}{2}(y_t^2 - y_{t-1}^2 - \varepsilon_t^2), \quad t = [\tau_0 T]+1, \cdots, T,$$

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which implies that

$$\frac{1}{Tl(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t = \frac{1}{2Tl(\eta_T)}\Big(y_T^2 - y_{[\tau_0 T]}^2 - \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t^2\Big). \tag{S.1}$$

Note also that

$$\frac{y_{[\tau_0 T]}}{\sqrt{Tl(\eta_T)}} = o_p(1) \tag{S.2}$$

by part (c) of Lemma A.1. We let

$$S_T(\tau_0, s) := \frac{1}{\sqrt{Tl(\eta_T)}} \sum_{t=[\tau_0 T]+1}^{[sT]} \varepsilon_t.$$

Then

$$S_T(\tau_0, s) \Rightarrow \overline{W}(s) - \overline{W}(\tau_0) \tag{S.3}$$

for any $\tau_0 \le s \le 1$ by the functional central limit theorem for the i.i.d. random variables from the DAN (see Theorem 1 in Csörgő et al. (2003) and (3.4) in Giné et al. (1997)). Combining this result with (S.2) immediately leads to

$$\frac{y_T^2 - y_{[\tau_0 T]}^2}{2Tl(\eta_T)} = \frac{(y_{[\tau_0 T]} + \sum_{i=[\tau_0 T]+1}^{T} \varepsilon_i)^2 - y_{[\tau_0 T]}^2}{2Tl(\eta_T)} \Rightarrow \frac{1}{2}(\overline{W}(1) - \overline{W}(\tau_0))^2. \tag{S.4}$$

In addition, it is trivial that

$$\frac{\sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t^2}{2Tl(\eta_T)} \xrightarrow{p} \frac{1-\tau_0}{2} \tag{S.5}$$

when $\varepsilon_t' s$ are i.i.d. random variables in the DAN. Now, it follows from (S.1), (S.4) and (S.5) that part (a) holds.

To prove part (b), using (S.2) and (S.3), we have

$$\begin{aligned}
& \frac{1}{T^2 l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 \\
= \ & \frac{1}{T^2 l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \Big(y_{[\tau_0 T]} + \sum_{i=[\tau_0 T]+1}^{t-1} \varepsilon_i\Big)^2 \\
= \ & \frac{1}{T} \sum_{t=[\tau_0 T]+1}^{T} \Big(\frac{y_{[\tau_0 T]}}{\sqrt{Tl(\eta_T)}} + S_T(\tau_0, \frac{t-1}{T})\Big)^2 \\
= \ & \sum_{t=[\tau_0 T]+1}^{T} \int_{\frac{t-1}{T}}^{\frac{t}{T}} \Big(\frac{y_{[\tau_0 T]}}{\sqrt{Tl(\eta_T)}} + S_T(\tau_0, \frac{t-1}{T})\Big)^2 ds \\
= \ & \sum_{t=[\tau_0 T]+1}^{T} \int_{\frac{t-1}{T}}^{\frac{t}{T}} \Big(\frac{y_{[\tau_0 T]}}{\sqrt{Tl(\eta_T)}} + S_T(\tau_0, s)\Big)^2 ds \cdot (1 + o_p(1)) \\
= \ & \int_{\tau_0}^{1} \Big(\frac{y_{[\tau_0 T]}}{\sqrt{Tl(\eta_T)}} + S_T(\tau_0, s)\Big)^2 ds \cdot (1 + o_p(1)) \\
\Rightarrow \ & \int_{\tau_0}^{1} (\overline{W}(s) - \overline{W}(\tau_0))^2 ds.
\end{aligned} \tag{S.6}$$

Finally, since part (a) and part (b) hold jointly, this completes our proof. $\square$

Proof of Lemma A.3. Noting that the fact $|\hat{\tau}_T - \tau_0| = O_p(k_T/T)$ is proved in the proof of part (a) of Theorem 1.1 and the limiting distribution in Lemma A.1(c) is irrespective of the constant r, it suffices to study the magnitude of $\sum_{t=[\tau_0 T-k_T]+1}^{[\tau_0 T]} y_{t-1}^2$ in order to obtain the magnitude of $\sum_{t=[\hat{\tau}_T T]+1}^{[\tau_0 T]} y_{t-1}^2$. Therefore, when $\hat{\tau}_T \leq \tau_0$, Lemma A.1(c) implies that

$$\sum_{t=[\tau_0 T-k_T]+1}^{[\tau_0 T]} y_{t-1}^2 = O_p(k_T l(\eta_T)) \cdot k_T = O_p(k_T^2 l(\eta_T)), \tag{S.7}$$

which yields

$$\sum_{t=[\hat{\tau}_T T]+1}^{[\tau_0 T]} y_{t-1}^2 = O_p(k_T^2 l(\eta_T)). \tag{S.8}$$

In addition, to obtain the magnitude of $\sum_{t=[\hat{\tau}_T T]+1}^{[\tau_0 T]} y_{t-1}\varepsilon_t$, it suffices to study the magnitude of $\sum_{t=[\tau_0 T-k_T]}^{[\tau_0 T]} y_{t-1}\varepsilon_t$. By squaring $y_t = \beta_{1T} y_{t-1} + \varepsilon_t$ and summing over $t \in \{[\tau_0 T - k_T] + 1, \cdots, [\tau_0 T]\}$ we obtain

$$\begin{aligned}\sum_{t=[\tau_0 T-k_T]+1}^{[\tau_0 T]} y_{t-1}\varepsilon_t &= \frac{1-\beta_{1T}^2}{2\beta_{1T}} \sum_{t=[\tau_0 T-k_T]+1}^{[\tau_0 T]} y_{t-1}^2 - \frac{\beta_{1T}}{2}(y_{[\tau_0 T-k_T]}^2 - y_{[\tau_0 T]}^2) \\ &\quad -\frac{1}{2\beta_{1T}} \sum_{t=[\tau_0 T-k_T]+1}^{[\tau_0 T]} \varepsilon_t^2. \end{aligned}\tag{S.9}$$

It follows from (S.7) and Lemma A.1(c) respectively that

$$\frac{1-\beta_{1T}^2}{2\beta_{1T}} \sum_{t=[\tau_0 T-k_T]+1}^{[\tau_0 T]} y_{t-1}^2 = O_p(k_T l(\eta_T)) \tag{S.10}$$

and

$$\frac{\beta_{1T}}{2}(y_{[\tau_0 T-k_T]}^2 - y_{[\tau_0 T]}^2) = \frac{\beta_{1T}}{2}(y_{[\tau_0 T-k_T]} - y_{[\tau_0 T]})(y_{[\tau_0 T-k_T]} + y_{[\tau_0 T]}) = O_p(k_T l(\eta_T)). \tag{S.11}$$

Moreover, by applying the Law of Large Numbers, we have

$$\frac{1}{2\beta_{1T}} \sum_{t=[\tau_0 T-k_T]+1}^{[\tau_0 T]} \varepsilon_t^2 = O_p(k_T l(\eta_T)). \tag{S.12}$$

Combining (S.9)-(S.12) leads to

$$\sum_{t=[\tau_0 T-k_T]+1}^{[\tau_0 T]} y_{t-1}\varepsilon_t = O_p(k_T l(\eta_T)),$$

yielding

$$\sum_{t=[\hat{\tau}_T T]+1}^{[\tau_0 T]} y_{t-1}\varepsilon_t = O_p(k_T l(\eta_T)). \tag{S.13}$$

Similarly, if $\hat{\tau}_T > \tau_0$, we have

$$\begin{aligned}
\sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T]} y_{t-1}^2 &= \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T]} \Big(y_{[\tau_0 T]} + \sum_{i=1}^{t-1-[\tau_0 T]} \varepsilon_{[\tau_0 T]+i} \Big)^2 \\
&\leq 2 \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T]} y_{[\tau_0 T]}^2 + 2 \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T]} \Big(\sum_{i=1}^{t-1-[\tau_0 T]} \varepsilon_{[\tau_0 T]+i} \Big)^2 \\
&= O_p(k_T^2 l(\eta_T)) + O_p(k_T^2 l(\eta_T)) \\
&= O_p(k_T^2 l(\eta_T)),
\end{aligned}$$

leading to

$$\sum_{t=[\tau_0 T]+1}^{[\hat{\tau}_T T]} y_{t-1}^2 = O_p(k_T^2 l(\eta_T)). \tag{S.14}$$

In addition, by squaring $y_t = y_{t-1} + \varepsilon_t$ and summing over $t \in \{[\tau_0 T]+1, \cdots, [\tau_0 T + k_T]\}$ we obtain

$$\begin{aligned}
\sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T]} y_{t-1}\varepsilon_t &= \frac{1}{2}\Big(y_{[\tau_0 T+k_T]}^2 - y_{[\tau_0 T]}^2 \Big) - \frac{1}{2} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T]} \varepsilon_t^2 \\
&= \frac{1}{2}\left(2 y_{[\tau_0 T]} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T]} \varepsilon_t + \Big(\sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T]} \varepsilon_t \Big)^2 \right) - \frac{1}{2} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T]} \varepsilon_t^2 .
\end{aligned}$$

Similarly to the derivation of (S.13), we have

$$\sum_{t=[\tau_0 T]+1}^{[\hat{\tau}_T T]} y_{t-1}\varepsilon_t = O_p(k_T l(\eta_T)). \tag{S.15}$$

□

Proof of Lemma B.1. This lemma can easily be proved by using the standard arguments in the unit root model with finite variance and by applying the truncation technique (A.1). The details are omitted. □

Proof of Lemma B.2. The proof can be completed by following the proof of part (c) of Lemma A.1. The details are therefore omitted. □

Proof of Lemma B.3. To prove part (a), we note that the following decomposition holds,

$$
\begin{aligned}
& \frac{1}{\sqrt{Tk_T}l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t \\
= & \frac{1}{\sqrt{Tk_T}l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \Big(\beta_{2T}^{t-[\tau_0 T]-1} y_{[\tau_0 T]} + \sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-i-1}\varepsilon_i\Big)\varepsilon_t \\
= & \frac{1}{\sqrt{Tk_T}l(\eta_T)} \Big(\Big(y_0 + \sum_{j=1}^{[\tau_0 T]} \varepsilon_j\Big) \sum_{t=[\tau_0 T]+1}^{T} \beta_{2T}^{t-[\tau_0 T]-1}\varepsilon_t + \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t \sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-i-1}\varepsilon_i\Big) \\
:= & I_1 + I_2, \qquad\qquad (S.16)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \frac{1}{\sqrt{Tk_T}l(\eta_T)} \Big(y_0 + \sum_{j=1}^{[\tau_0 T]} \varepsilon_j\Big) \sum_{t=[\tau_0 T]+1}^{T} \beta_{2T}^{t-[\tau_0 T]-1}\varepsilon_t \\
&= \frac{1+o_p(1)}{\sqrt{Tk_T}l(\eta_T)} \sum_{j=1}^{[\tau_0 T]} \varepsilon_j \sum_{t=[\tau_0 T]+1}^{T} \beta_{2T}^{t-[\tau_0 T]-1}\varepsilon_t \\
&= \frac{1+o_p(1)}{\sqrt{Tk_T}l(\eta_T)} \sum_{j=1}^{[\tau_0 T]} \varepsilon_j^{(1)} \sum_{t=[\tau_0 T]+1}^{T} \beta_{2T}^{t-[\tau_0 T]-1}\varepsilon_t^{(1)}
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \frac{1}{\sqrt{Tk_T}l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t \sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-i-1}\varepsilon_i \\
&= \frac{1+o_p(1)}{\sqrt{Tk_T}l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t^{(1)} \sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-i-1}\varepsilon_i^{(1)}
\end{aligned}
$$

by assumptions C1-C3 and Lemma 1 in Csörgő et al. (2003).

For the term I_1, it is obvious that

$$
\phi_j = \frac{1}{\sqrt{Tk_T}} \frac{\varepsilon_j^{(1)}}{\sqrt{l(\eta_T)}} \sum_{t=[\tau_0 T]+1}^{T} \beta_{2T}^{t-[\tau_0 T]-1} \frac{\varepsilon_t^{(1)}}{\sqrt{l(\eta_T)}}, \quad j = 1, \cdots, [\tau_0 T] \qquad (S.17)
$$

is a sequence of martingale difference with respect to the filtration $\mathfrak{F}_j = \sigma(\varepsilon_1, \cdots, \varepsilon_j)$ with

$$
\sum_{j=1}^{[\tau_0 T]} E(\phi_j^2 | \mathfrak{F}_{j-1}) = \frac{1}{Tk_T} \sum_{t=[\tau_0 T]+1}^{T} \beta_{2T}^{2(t-[\tau_0 T]-1)} \cdot (1+o(1)) = \frac{\tau_0}{2c} \cdot (1+o(1)).
$$

In addition, for any $\delta > 0$, we have

$$
\begin{aligned}
\sum_{j=1}^{[\tau_0 T]} E(\phi_j^2 I\{|\phi_j| > \delta\} | \mathfrak{F}_{j-1}) &= [\tau_0 T] E(\phi_1^2 I\{|\phi_1| > \delta\}) \\
&= \frac{[\tau_0 T]}{T} E\left((\sqrt{T}\phi_1)^2 I\{|\sqrt{T}\phi_1| > \delta\sqrt{T}\}\right) \\
&\to 0
\end{aligned}
$$

since $E(\sqrt{T}\phi_1)^2 < \infty$. Hence, the Lindeberg condition is verified. Then, applying the central limit theorem for martingale difference sequences, we have

$$I_1 \Rightarrow N(0, \frac{\tau_0}{2c}). \tag{S.18}$$

For the term I_2, it is easy to see that

$$\varphi_t = \frac{1}{\sqrt{Tk_T}} \frac{\varepsilon_t^{(1)}}{\sqrt{l(\eta_T)}} \sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-i-1} \frac{\varepsilon_i^{(1)}}{\sqrt{l(\eta_T)}}, \quad t = [\tau_0 T]+1, \cdots, T \tag{S.19}$$

is a sequence of martingale difference with respect to the filtration $\mathfrak{F}_t = \sigma(\varepsilon_{[\tau_0 T]}, \cdots, \varepsilon_t)$ with

$$\sum_{t=[\tau_0 T]+1}^{T} E(\varphi_t^2 | \mathfrak{F}_{t-1}) = \frac{1-\tau_0}{2c}(1 + o_p(1)).$$

In addition, by the similar arguments in pages 203-204 in Huang et al. (2014), it is easy to verify that the Lindeberg condition

$$\sum_{t=[\tau_0 T]+1}^{T} E(\varphi_t^2 I\{|\varphi_t| > \delta\} | \mathfrak{F}_{t-1}) = o_p(1), \quad \text{for any } \delta > 0$$

holds. Hence, applying the central limit theorem for martingale difference sequences leads to

$$I_2 \Rightarrow N(0, \frac{1-\tau_0}{2c}). \tag{S.20}$$

Note that the two sequences (S.17) and (S.19) are independent. The proof of part (a) is then completed by combining (S.16), (S.18) and (S.20).

To prove part (b), by squaring $y_t = (1 - c/k_T)y_{t-1} + \varepsilon_t$ and summing over $t \in \{[\tau_0 T] + 1, \cdots, T\}$, we obtain

$$\sum_{t=[\tau_0 T]+1}^{T} y_t^2 = (1 - \frac{c}{k_T})^2 \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 + \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t^2 + 2(1 - \frac{c}{k_T}) \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t.$$

This implies that

$$(\frac{2c}{k_T} - \frac{c^2}{k_T^2}) \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 = y_{[\tau_0 T]}^2 - y_T^2 + \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t^2 + 2(1 - \frac{c}{k_T}) \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t. \tag{S.21}$$

Note that

$$\frac{y_{[\tau_0 T]}}{\sqrt{Tl(\eta_T)}} \Rightarrow W(\tau_0) \tag{S.22}$$

since $\beta_1 = 1$, it follows from Lemma B.2 and assumption C2 that

$$\frac{y_T}{\sqrt{Tl(\eta_T)}} = \frac{\beta_{2T}^{T-[\tau_0 T]} y_{[\tau_0 T]} + \sum_{i=[\tau_0 T]+1}^{T} \beta_{2T}^{T-i} \varepsilon_i}{\sqrt{Tl(\eta_T)}} = o_p(1). \tag{S.23}$$

In addition,

$$2(1-\frac{c}{k_T})\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t = o_p(Tl(\eta_T)) \tag{S.24}$$

by part (a). Then, combining (S.21)-(S.24) and (S.5) together yields

$$\frac{1}{Tk_T l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 \Rightarrow \frac{1}{2c}(W^2(\tau_0)+1-\tau_0).$$

Note that the random part (i.e., $W^2(\tau_0)$) of the limiting distribution of $\frac{1}{Tk_T l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2$ is due to the limiting behavior of $(\sum_{i=1}^{[\tau_0 T]}\varepsilon_i)^2$, while the limiting distribution of $\frac{1}{\sqrt{Tk_T}l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t$ is determined by the limiting behavior of

$$\sum_{j=1}^{[\tau_0 T]}\varepsilon_j\sum_{t=[\tau_0 T]+1}^{T}\beta_{2T}^{t-[\tau_0 T]-1}\varepsilon_t \quad\text{and}\quad \sum_{t=[\tau_0 T]+1}^{T}\varepsilon_t\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i.$$

Note also that $\sum_{i=1}^{[\tau_0 T]}\varepsilon_i$ and $\sum_{t=[\tau_0 T]+1}^{T}\varepsilon_t\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i$ are independent and $\sum_{i=1}^{[\tau_0 T]}\varepsilon_i$ and $\sum_{j=1}^{[\tau_0 T]}\varepsilon_j\sum_{t=[\tau_0 T]+1}^{T}\beta_{2T}^{t-[\tau_0 T]-1}\varepsilon_t$ are uncorrelated. Hence, the two limiting distributions in part (a) and part (b) are independent. Therefore, part (a) and part (b) hold jointly. □

Proof of Lemma B.4. By applying Lemmas B.1 and B.3, it is trivial that $A_1 = O_p(1/\sqrt{Tk_T}) = o_p(1/k_T)$ and $A_4 = O_p(1/T) = o_p(1/k_T)$. To find the order of the term A_2, note that since $\beta_1 = 1$, it is not difficult to see that

$$\begin{aligned}
\sum_{t=m+1}^{[\tau_0 T]} y_{t-1}\varepsilon_t &= y_0\sum_{t=m+1}^{[\tau_0 T]}\varepsilon_t + \sum_{t=m+1}^{[\tau_0 T]}\left(\sum_{i=1}^{t-1}\varepsilon_i\right)\varepsilon_t \\
&= \left(y_0\sum_{t=m+1}^{[\tau_0 T]}\varepsilon_t^{(1)} + \sum_{t=m+1}^{[\tau_0 T]}\left(\sum_{i=1}^{t-1}\varepsilon_i^{(1)}\right)\varepsilon_t^{(1)}\right)\cdot(1+o_p(1)) \\
&= o_p\left(\sqrt{T([\tau_0 T]-m)l(\eta_T)}\right) + O_p\left(\sqrt{\sum_{t=m+1}^{[\tau_0 T]} E\left(\sum_{i=1}^{t-1}\varepsilon_i^{(1)}\right)^2 l(\eta_T)}\right) \\
&= o_p\left(\sqrt{T([\tau_0 T]-m)l(\eta_T)}\right) + O_p\left(\sqrt{([\tau_0 T]-m)([\tau_0 T]+m)l(\eta_T)}\right) \\
&= O_p\left(\sqrt{([\tau_0 T]-m)([\tau_0 T]+m)l(\eta_T)}\right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{t=m+1}^{[\tau_0 T]} y_{t-1}^2 &= T([\tau_0 T]-m)l(\eta_T)\cdot\frac{1}{[\tau_0 T]-m}\sum_{t=m+1}^{[\tau_0 T]}\left(\frac{y_{t-1}}{\sqrt{Tl(\eta_T)}}\right)^2 \\
&= O_p(T([\tau_0 T]-m)l(\eta_T)).
\end{aligned}$$

Consequently, we have

$$A_2 = O_p\left(\sup_{m\in D_{1T}} \frac{\sqrt{([\tau_0 T]-m)([\tau_0 T]+m)}}{T([\tau_0 T]-m)}\right) \le O_p\left(\frac{1}{\sqrt{TM_T}}\right) = o_p(1/k_T). \tag{S.25}$$

To find the order of the term A_5, note that for any small $0<\delta<1$, there exist two large constants $N_1 = N_1(\delta)$ such that $\beta_{2T}^{k_T/N_1} \to e^{-c/N_1} > 1-\delta$ and $N_2 = N_2(\delta)$ such that $\beta_{2T}^{N_2 k_T} \to e^{-cN_2} < \delta$. We then divide the set $\{M_T+1,\cdots,T-[\tau_0 T]\}$, which is the domain of $m-[\tau_0 T]$ when $m \in D_{2T}$, into three subsets: $\{M_T+1,\cdots,[k_T/N_1]\}$, $\{[k_T/N_1]+1,\cdots,[N_2 k_T]\}$ and $\{[N_2 k_T]+1,\cdots,T-[\tau_0 T]\}$. Note that

$$\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t = \sum_{t=[\tau_0 T]+1}^{m}\left(\beta_{2T}^{t-1-[\tau_0 T]} y_{[\tau_0 T]} + \sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-1-i}\varepsilon_i\right)\varepsilon_t,$$

$$\begin{aligned}
& y_{[\tau_0 T]}\sum_{t=[\tau_0 T]+1}^{m}\beta_{2T}^{t-1-[\tau_0 T]}\varepsilon_t \\
= \; & O_p(\sqrt{Tl(\eta_T)})\cdot O_p\left(\sqrt{k_T(1-\beta_{2T}^{2(m-[\tau_0 T])})l(\eta_T)}\right) \\
= \; & O_p\left(\sqrt{Tk_T(1-\beta_{2T}^{2(m-[\tau_0 T])})l(\eta_T)}\right) \\
= \; & \begin{cases} O_p\left(\sqrt{T(m-[\tau_0 T])}l(\eta_T)\right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p\left(\sqrt{Tk_T}l(\eta_T)\right), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p\left(\sqrt{Tk_T}l(\eta_T)\right), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases}
\end{aligned}$$

since

$$(1-(1-c/k_T)^{a_T}) = O(a_T/k_T) \;\text{ if }\; a_T = o(k_T)$$

and

$$\begin{aligned}
& \sum_{t=[\tau_0 T]+1}^{m}\left(\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-1-i}\varepsilon_i\right)\varepsilon_t \\
= \; & O_p\left(\sqrt{\sum_{t=[\tau_0 T]+1}^{m}\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{2(t-1-i)}l(\eta_T)}\right) \\
= \; & O_p\left(\sqrt{k_T(m-[\tau_0 T]) - k_T^2(1-\beta_{2T}^{2(m-[\tau_0 T])})l(\eta_T)}\right) \\
= \; & \begin{cases} O_p\left(\sqrt{k_T(m-[\tau_0 T])}l(\eta_T)\right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p\left(k_T l(\eta_T)\right), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p\left(\sqrt{k_T(m-[\tau_0 T])}l(\eta_T)\right), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases} .
\end{aligned}$$

We then have

$$
\begin{aligned}
&\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t \\
= &\begin{cases} O_p\left(\sqrt{T(m-[\tau_0 T])}l(\eta_T)\right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p\left(\sqrt{Tk_T}l(\eta_T)\right), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p\left(\sqrt{Tk_T}l(\eta_T)\right), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases}.
\end{aligned} \tag{S.26}
$$

Moreover, we also have

$$
\begin{aligned}
&\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2 \\
= &-\frac{1}{1-\beta_{2T}^2}(y_m^2 - y_{[\tau_0 T]}^2) + \frac{1}{1-\beta_{2T}^2}\sum_{t=[\tau_0 T]+1}^{m}\varepsilon_t^2 + \frac{2\beta_{2T}}{1-\beta_{2T}^2}\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t,
\end{aligned} \tag{S.27}
$$

$$
\begin{aligned}
&\frac{2\beta_{2T}}{1-\beta_{2T}^2}\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t \\
= &\begin{cases} O_p\left(k_T\sqrt{T(m-[\tau_0 T])}l(\eta_T)\right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p\left(\sqrt{T}k_T^{3/2}l(\eta_T)\right), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p\left(\sqrt{T}k_T^{3/2}l(\eta_T)\right), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases}
\end{aligned} \tag{S.28}
$$

by (S.26),

$$
\begin{aligned}
&\frac{1}{1-\beta_{2T}^2}\sum_{t=[\tau_0 T]+1}^{m}\varepsilon_t^2 \\
= &\; O_p\left(k_T(m-[\tau_0 T])l(\eta_T)\right) \\
= &\begin{cases} O_p\left(k_T(m-[\tau_0 T])l(\eta_T)\right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p\left(k_T^2 l(\eta_T)\right), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p\left(k_T(m-[\tau_0 T])l(\eta_T)\right), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases}
\end{aligned} \tag{S.29}
$$

and

$$
\begin{aligned}
&-\frac{1}{1-\beta_{2T}^2}(y_m^2 - y_{[\tau_0 T]}^2) \\
= &-\frac{1}{1-\beta_{2T}^2}(y_m + y_{[\tau_0 T]})(y_m - y_{[\tau_0 T]}) \\
= &-\frac{1}{1-\beta_{2T}^2}\left(\sum_{i=[\tau_0 T]+1}^{m}\beta_{2T}^{m-i}\varepsilon_i + \left(\beta_{2T}^{m-[\tau_0 T]}+1\right)y_{[\tau_0 T]}\right) \\
&\cdot\left(\sum_{i=[\tau_0 T]+1}^{m}\beta_{2T}^{m-i}\varepsilon_i + \left(\beta_{2T}^{m-[\tau_0 T]}-1\right)y_{[\tau_0 T]}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= k_T \cdot \left(O_p\left(\sqrt{k_T(1-\beta_{2T}^{2(m-[\tau_0 T])})l(\eta_T)}\right) + O_p(\sqrt{Tl(\eta_T)})\right) \\
&\quad \cdot \left(O_p\left(\sqrt{k_T(1-\beta_{2T}^{2(m-[\tau_0 T])})l(\eta_T)}\right) + O_p\left((1-\beta_{2T}^{m-[\tau_0 T]})\sqrt{Tl(\eta_T)}\right)\right) \\
&= O_p(k_T\sqrt{Tl(\eta_T)}) \cdot \left(O_p\left(\sqrt{k_T(1-\beta_{2T}^{2(m-[\tau_0 T])})l(\eta_T)}\right) + O_p\left((1-\beta_{2T}^{m-[\tau_0 T]})\sqrt{Tl(\eta_T)}\right)\right) \\
&= \begin{cases} O_p(k_T\sqrt{Tl(\eta_T)}) \cdot O_p\left(\frac{m-[\tau_0 T]}{k_T}\sqrt{Tl(\eta_T)}\right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p(k_T\sqrt{Tl(\eta_T)}) \cdot O_p(\sqrt{Tl(\eta_T)}), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p(k_T\sqrt{Tl(\eta_T)}) \cdot O_p(\sqrt{Tl(\eta_T)}), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases} \\
&= \begin{cases} O_p\left(T(m-[\tau_0 T])l(\eta_T)\right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p(Tk_T l(\eta_T)), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p(Tk_T l(\eta_T)), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases}.
\end{aligned} \tag{S.30}
$$

Combining (S.27)-(S.30), we have

$$
\begin{aligned}
&\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2 \\
&= \begin{cases} O_p\left(\max\left\{k_T\sqrt{T(m-[\tau_0 T])}l(\eta_T), T(m-[\tau_0 T])l(\eta_T)\right\}\right), & \\ & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p(Tk_T l(\eta_T)), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p\left(\max\left\{k_T(m-[\tau_0 T])l(\eta_T), Tk_T l(\eta_T)\right\}\right), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases} \\
&= \begin{cases} O_p\left(T(m-[\tau_0 T])l(\eta_T)\right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p(Tk_T l(\eta_T)), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p(Tk_T l(\eta_T)), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases}.
\end{aligned} \tag{S.31}
$$

Hence, it follows from (S.26) and (S.31) that, when $M_T < m-[\tau_0 T] \le [k_T/N_1]$, we have

$$
\frac{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2} = O_p\left(\frac{\sqrt{T(m-[\tau_0 T])}l(\eta_T)}{T(m-[\tau_0 T])l(\eta_T)}\right) \le O_p\left(\frac{1}{\sqrt{TM_T}}\right) = o_p(1/k_T)
$$

since $k_T = O(\sqrt{T})$ and $M_T \to \infty$. When $[k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T]$ or $[N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T]$, we have

$$
\frac{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2} = O_p\left(\frac{\sqrt{Tk_T}l(\eta_T)}{Tk_T l(\eta_T)}\right) = o_p(1/k_T).
$$

Using the above arguments, we have

$$
A_5 = \sup_{m\in D_{2T}} \frac{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2} = o_p(1/k_T).
$$

In addition, we have

$$
\begin{aligned}
A_3 &= \sup_{m\in D_{1T}} \Big| \frac{\sum_{t=m+1}^{T} y_{t-1}^2}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 \sum_{t=m+1}^{[\tau_0 T]} y_{t-1}^2} \Lambda_T(\frac{m}{T}) \Big| \\
&\le \Big(\frac{1}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2} + \frac{1}{\sum_{t=[\tau_0 T]-M_T}^{[\tau_0 T]} y_{t-1}^2} \Big) \Big(\sup_{m\in D_{1T}} \Big| \frac{(\sum_{t=1}^{[\tau_0 T]} y_{t-1}\varepsilon_t)^2}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} - \frac{(\sum_{t=1}^{m} y_{t-1}\varepsilon_t)^2}{\sum_{t=1}^{m} y_{t-1}^2} \Big| \\
&\qquad + \sup_{m\in D_{1T}} \Big| \frac{(\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t)^2}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=m+1}^{T} y_{t-1}\varepsilon_t)^2}{\sum_{t=m+1}^{T} y_{t-1}^2} \Big| \Big) \\
&= \Big(O_p(\frac{1}{Tk_T l(\eta_T)}) + O_p(\frac{1}{TM_T l(\eta_T)}) \Big) \cdot O_p(l(\eta_T)) \\
&= o_p(1/k_T^2),
\end{aligned}
$$

and

$$
\begin{aligned}
A_6 &= \sup_{m\in D_{2T}} \Big| \frac{\sum_{t=1}^{m} y_{t-1}^2}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2 \sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} \Lambda_T(\frac{m}{T}) \Big| \\
&\le \Big(\frac{1}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} + \frac{1}{\sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+M_T} y_{t-1}^2} \Big) \Big(\sup_{m\in D_{2T}} \Big| \frac{(\sum_{t=1}^{[\tau_0 T]} y_{t-1}\varepsilon_t)^2}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} - \frac{(\sum_{t=1}^{m} y_{t-1}\varepsilon_t)^2}{\sum_{t=1}^{m} y_{t-1}^2} \Big| \\
&\qquad + \sup_{m\in D_{2T}} \Big| \frac{(\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t)^2}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=m+1}^{T} y_{t-1}\varepsilon_t)^2}{\sum_{t=m+1}^{T} y_{t-1}^2} \Big| \Big) \\
&\le \Big(O_p(\frac{1}{T^2 l(\eta_T)}) + O_p(\frac{1}{TM_T l(\eta_T)}) \Big) \cdot O_p(l(\eta_T)) \\
&= o_p(1/k_T^2).
\end{aligned}
$$

The proofs are complete. □

Proof of Lemma B.5. To prove part (a), using equation (B.2) in Chong (2001), we have

$$
\begin{aligned}
& RSS_T(\tau_0 - \frac{m}{T}) - RSS_T(\tau_0) \\
=\ & 2(\beta_{2T} - \beta_1)\Big(\frac{\sum_{t=[\tau_0 T]-m+1}^{[\tau_0 T]} y_{t-1}^2 \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}^2} - \frac{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 \sum_{t=[\tau_0 T]-m+1}^{[\tau_0 T]} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}^2} \Big) \\
& + (\beta_{2T} - \beta_1)^2 \frac{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 \sum_{t=[\tau_0 T]-m+1}^{[\tau_0 T]} y_{t-1}^2}{\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}^2} + \Lambda_T(\tau_0 - \frac{m}{T})
\end{aligned}
$$

where

$$
\begin{aligned}
& \Lambda_T(\tau_0 - \frac{m}{T}) \\
=\ & \frac{\Big(\sum_{t=1}^{[\tau_0 T]} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} - \frac{\Big(\sum_{t=1}^{[\tau_0 T]-m} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=1}^{[\tau_0 T]-m} y_{t-1}^2} + \frac{\Big(\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2} - \frac{\Big(\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}^2}.
\end{aligned}
$$

Obviously, $\Lambda_T(\tau_0 - \frac{m}{T}) = O_p(l(\eta_T))$. Then it follows that

$$
\begin{aligned}
& \frac{k_T^2}{Tl(\eta_T)}\Big(RSS_T(\tau_0 - \frac{m}{T}) - RSS_T(\tau_0)\Big) \\
= \ & \frac{k_T^2}{Tl(\eta_T)} \cdot \Big(-\frac{2c}{k_T}\Big) \cdot \Big(\frac{O_p(Tl(\eta_T))O_p(\sqrt{Tk_T}l(\eta_T))}{O_p(Tl(\eta_T)) + O_p(Tk_Tl(\eta_T))} + \frac{O_p(Tk_Tl(\eta_T))O_p(\sqrt{T}l(\eta_T))}{O_p(Tl(\eta_T)) + O_p(Tk_Tl(\eta_T))}\Big) \\
& + \frac{k_T^2}{Tl(\eta_T)} \cdot \frac{c^2}{k_T^2} \cdot \frac{O_p(Tk_Tl(\eta_T))}{O_p(Tl(\eta_T)) + O_p(Tk_Tl(\eta_T))} \sum_{[\tau_0T]-m+1}^{[\tau_0T]} y_{t-1}^2 + o_p(1) \\
= \ & \frac{c^2(1+o_p(1))}{Tl(\eta_T)} \sum_{[\tau_0T]-m+1}^{[\tau_0T]} y_{t-1}^2 + o_p(1) \\
\Rightarrow \ & c^2 m W^2(\tau_0)
\end{aligned}
$$

by Lemma B.3, $k_T = o(\sqrt{T})$ and the observation

$$
\Big|\sum_{t=[\tau_0T]-m+1}^{[\tau_0T]} y_{t-1}\varepsilon_t\Big| \leq \sqrt{\sum_{t=[\tau_0T]-m+1}^{[\tau_0T]} \varepsilon_t^2} \cdot \sqrt{\sum_{t=[\tau_0T]-m+1}^{[\tau_0T]} y_{t-1}^2} = O_p(\sqrt{T}l(\eta_T)).
$$

To prove part (b), using equation (B.4) in Chong (2001), we have

$$
\begin{aligned}
& RSS_T(\tau_0 + \frac{m}{T}) - RSS_T(\tau_0) \\
= \ & 2(\beta_{2T} - \beta_1)\Big(\frac{\sum_{t=1}^{[\tau_0T]} y_{t-1}^2 \sum_{t=[\tau_0T]+1}^{[\tau_0T]+m} y_{t-1}\varepsilon_t}{\sum_{t=1}^{[\tau_0T]+m} y_{t-1}^2} - \frac{\sum_{t=[\tau_0T]+1}^{[\tau_0T]+m} y_{t-1}^2 \sum_{t=1}^{[\tau_0T]} y_{t-1}\varepsilon_t}{\sum_{t=1}^{[\tau_0T]+m} y_{t-1}^2}\Big) \\
& + (\beta_{2T} - \beta_1)^2 \frac{\sum_{t=[\tau_0T]+1}^{[\tau_0T]+m} y_{t-1}^2 \sum_{t=1}^{[\tau_0T]} y_{t-1}^2}{\sum_{t=1}^{[\tau_0T]+m} y_{t-1}^2} + \Lambda_T(\tau_0 + \frac{m}{T})
\end{aligned}
$$

where

$$
\begin{aligned}
& \Lambda_T(\tau_0 + \frac{m}{T}) \\
= \ & \frac{\Big(\sum_{t=1}^{[\tau_0T]} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=1}^{[\tau_0T]} y_{t-1}^2} - \frac{\Big(\sum_{t=1}^{[\tau_0T]+m} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=1}^{[\tau_0T]+m} y_{t-1}^2} + \frac{\Big(\sum_{t=[\tau_0T]+1}^{T} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=[\tau_0T]+1}^{T} y_{t-1}^2} - \frac{\Big(\sum_{t=[\tau_0T]+m+1}^{T} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=[\tau_0T]+m+1}^{T} y_{t-1}^2}.
\end{aligned}
$$

Since $\Lambda_T(\tau_0 + \frac{m}{T}) = O_p(l(\eta_T))$ and

$$
\Big|\sum_{t=[\tau_0T]+1}^{[\tau_0T]+m} y_{t-1}\varepsilon_t\Big| \leq \sqrt{\sum_{t=[\tau_0T]+1}^{[\tau_0T]+m} \varepsilon_t^2} \cdot \sqrt{\sum_{t=[\tau_0T]+1}^{[\tau_0T]+m} y_{t-1}^2} = O_p(\sqrt{T}l(\eta_T)),
$$

we have

$$
\begin{aligned}
&\frac{k_T^2}{Tl(\eta_T)}\Big(RSS_T(\tau_0+\frac{m}{T})-RSS_T(\tau_0)\Big)\\
=\;&\frac{k_T^2}{Tl(\eta_T)}\cdot\Big(-\frac{2c}{k_T}\Big)\cdot\Big(\frac{O_p(T^2l(\eta_T))O_p(\sqrt{T}l(\eta_T))}{O_p(T^2l(\eta_T))+O_p(Tl(\eta_T))}+\frac{O_p(Tl(\eta_T))O_p(Tl(\eta_T))}{O_p(T^2l(\eta_T))+O_p(Tl(\eta_T))}\Big)\\
&+\frac{k_T^2}{Tl(\eta_T)}\cdot\frac{c^2}{k_T^2}\cdot\frac{O_p(T^2l(\eta_T))}{O_p(T^2l(\eta_T))+O_p(Tl(\eta_T))}\sum_{t=[\tau_0T]+1}^{[\tau_0T]+m}y_{t-1}^2+o_p(1)\\
=\;&\frac{c^2(1+o_p(1))}{Tl(\eta_T)}\sum_{t=[\tau_0T]+1}^{[\tau_0T]+m}y_{t-1}^2+o_p(1)\\
\Rightarrow\;&c^2mW^2(\tau_0)
\end{aligned}
$$

by Lemma B.1 and $k_T=o(\sqrt{T})$. □

Proof of Lemma B.6. Note that we have proven that $|\hat{\tau}_T-\tau_0|=O_p(k_T^2/T^2)$ when $\sqrt{T}=o(k_T)$ in part (a) of Theorem 1.2. Then, for $\hat{\tau}_T\le\tau_0$, we have

$$
\begin{aligned}
\sum_{t=[\tau_0T-k_T^2/T]+1}^{[\tau_0T]}y_{t-1}^2 &= \sum_{t=[\tau_0T-k_T^2/T]+1}^{[\tau_0T]}\Big(y_{[\tau_0T]}-\sum_{i=t}^{[\tau_0T]}\varepsilon_i\Big)^2\\
&= \sum_{t=[\tau_0T-k_T^2/T]+1}^{[\tau_0T]}y_{[\tau_0T]}^2\cdot(1+o_p(1))\\
&= O_p(k_T^2l(\eta_T)), \qquad\qquad (S.32)
\end{aligned}
$$

implying

$$\sum_{t=[\hat{\tau}_TT]+1}^{[\tau_0T]}y_{t-1}^2=O_p(k_T^2l(\eta_T)). \qquad\qquad (S.33)$$

In addition, squaring $y_t=y_{t-1}+\varepsilon_t$ and summing over $t\in\{[\tau_0T-k_T^2/T]+1,\cdots,[\tau_0T]\}$, we have

$$
\begin{aligned}
&\sum_{t=[\tau_0T-k_T^2/T]+1}^{[\tau_0T]}y_{t-1}\varepsilon_t\\
=\;&\frac{1}{2}\left(y_{[\tau_0T]}^2-y_{[\tau_0T-k_T^2/T]}^2-\sum_{t=[\tau_0T-k_T^2/T]+1}^{[\tau_0T]}\varepsilon_t^2\right)\\
=\;&\frac{1}{2}\Big(y_{[\tau_0T]}+y_{[\tau_0T-k_T^2/T]}\Big)\Big(y_{[\tau_0T]}-y_{[\tau_0T-k_T^2/T]}\Big)-\frac{1}{2}\sum_{t=[\tau_0T-k_T^2/T]+1}^{[\tau_0T]}\varepsilon_t^2\\
=\;&O_p(\sqrt{Tl(\eta_T)})O_p(\sqrt{k_T^2l(\eta_T)/T})+O_p(k_T^2l(\eta_T)/T)\\
=\;&O_p(k_Tl(\eta_T)).
\end{aligned}
$$

As a result,

$$\sum_{t=[\hat{\tau}_T T]+1}^{[\tau_0 T]} y_{t-1}\varepsilon_t = O_p(k_T l(\eta_T)). \tag{S.34}$$

Similarly, if $\hat{\tau}_T > \tau_0$, it follows from Lemma B.2 that

$$\begin{aligned}
\sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} y_{t-1}^2 &= \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} \Big(\beta_{2T}^{t-1-[\tau_0 T]} y_{[\tau_0 T]} + \sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-1-i}\varepsilon_i\Big)^2 \\
&= \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} \beta_{2T}^{2(t-1-[\tau_0 T])} y_{[\tau_0 T]}^2 \cdot (1+o_p(1)) \\
&= k_T^2/T \cdot O_p(Tl(\eta_T)) \\
&= O_p(k_T^2 l(\eta_T)),
\end{aligned} \tag{S.35}$$

implying

$$\sum_{t=[\tau_0 T]+1}^{[\hat{\tau}_T T]} y_{t-1}^2 = O_p(k_T^2 l(\eta_T)). \tag{S.36}$$

By squaring $y_t = \beta_{2T} y_{t-1} + \varepsilon_t$ and summing over $t \in \{[\tau_0 T]+1, \cdots, [\tau_0 T + k_T^2/T]\}$, we obtain

$$\sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} y_{t-1}\varepsilon_t = \frac{1-\beta_{2T}^2}{2\beta_{2T}} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} y_{t-1}^2 + \frac{y_{[\tau_0 T+k_T^2/T]}^2 - y_{[\tau_0 T]}^2}{2\beta_{2T}} - \frac{1}{2\beta_{2T}} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} \varepsilon_t^2. \tag{S.37}$$

First, applying (S.35) immediately yields

$$\frac{1-\beta_{2T}^2}{2\beta_{2T}} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} y_{t-1}^2 = O_p(k_T l(\eta_T)). \tag{S.38}$$

Secondly, since $\sqrt{T} = o(k_T)$, we have

$$\frac{1}{2\beta_{2T}} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} \varepsilon_t^2 = O_p(k_T^2 l(\eta_T)/T). \tag{S.39}$$

Thirdly, since $|\beta_{2T}| < 1$, we have

$$\sum_{i=0}^{[k_T^2/T]-1} \beta_{2T}^{2i} = O(k_T^2/T)$$

which implies that

$$\sum_{i=0}^{[k_T^2/T]-1} \beta_{2T}^{i} \varepsilon_{[\tau_0 T+k_T^2/T]-i} = O_p(k_T\sqrt{l(\eta_T)}/\sqrt{T}).$$

Then, we have

$$
\begin{aligned}
& y^2_{[\tau_0 T + k_T^2/T]} - y^2_{[\tau_0 T]} \\
= & \left(y_{[\tau_0 T + k_T^2/T]} + y_{[\tau_0 T]} \right) \left(y_{[\tau_0 T + k_T^2/T]} - y_{[\tau_0 T]} \right) \\
= & \left(\sum_{i=0}^{[k_T^2/T]-1} \beta_{2T}^i \varepsilon_{[\tau_0 T + k_T^2/T]-i} + (\beta_{2T}^{[k_T^2/T]} + 1) y_{[\tau_0 T]} \right) \\
& \cdot \left(\sum_{i=0}^{[k_T^2/T]-1} \beta_{2T}^i \varepsilon_{[\tau_0 T + k_T^2/T]-i} + (\beta_{2T}^{[k_T^2/T]} - 1) y_{[\tau_0 T]} \right) \\
= & \left(O_p(k_T \sqrt{l(\eta_T)}/\sqrt{T}) + O_p(\sqrt{T l(\eta_T)}) \right) \left(O_p(k_T \sqrt{l(\eta_T)}/\sqrt{T}) + \frac{k_T^2/T}{k_T} O_p(\sqrt{T l(\eta_T)} \right) \\
= & O_p(\sqrt{T l(\eta_T)}) O_p(k_T \sqrt{l(\eta_T)}/\sqrt{T}) \\
= & O_p(k_T l(\eta_T)). \qquad \text{(S.40)}
\end{aligned}
$$

Combining (S.37)-(S.40) together leads to

$$
\sum_{t=[\tau_0 T]+1}^{[\tau_0 T + k_T^2/T]} y_{t-1} \varepsilon_t = O_p(k_T l(\eta_T)), \qquad \text{(S.41)}
$$

which implies

$$
\sum_{t=[\tau_0 T]+1}^{[\hat{\tau}_T T]} y_{t-1} \varepsilon_t = O_p(k_T l(\eta_T)). \qquad \text{(S.42)}
$$

□

Proof of Lemma C.1. The proof can be completed by following the proof of Lemma 4.2 in Phillips and Magdalinos (2007a) and the truncation technique (A.1). Thus the details are omitted. □

Proof of Lemma C.2. To prove part (a), we first note that the following decomposition holds:

$$
\begin{aligned}
& \frac{1}{\beta_{2T}^{T-[\tau_0 T]} \sqrt{T k_T} l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} y_{t-1} \varepsilon_t \\
= & \frac{1}{\beta_{2T}^{T-[\tau_0 T]} \sqrt{T k_T} l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \Big(\beta_{2T}^{t-[\tau_0 T]-1} y_{[\tau_0 T]} + \sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-i-1} \varepsilon_i \Big) \varepsilon_t \\
= & \frac{1}{\beta_{2T}^{T-[\tau_0 T]} \sqrt{T k_T} l(\eta_T)} \left(\Big(y_0 + \sum_{j=1}^{[\tau_0 T]} \varepsilon_j \Big) \sum_{t=[\tau_0 T]+1}^{T} \beta_{2T}^{t-[\tau_0 T]-1} \varepsilon_t + \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t \sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-i-1} \varepsilon_i \right) \\
:= & II_1 + II_2, \qquad \text{(S.43)}
\end{aligned}
$$

where

$$\begin{aligned} II_1 &= \frac{1}{\beta_{2T}^{T-[\tau_0 T]}\sqrt{Tk_T}l(\eta_T)}\Big(y_0+\sum_{j=1}^{[\tau_0 T]}\varepsilon_j\Big)\sum_{t=[\tau_0 T]+1}^{T}\beta_{2T}^{t-[\tau_0 T]-1}\varepsilon_t \\ &= \frac{1}{\beta_{2T}^{T-[\tau_0 T]}\sqrt{Tk_T}l(\eta_T)}\sum_{j=1}^{[\tau_0 T]}\varepsilon_j\sum_{t=[\tau_0 T]+1}^{T}\beta_{2T}^{t-[\tau_0 T]-1}\varepsilon_t\cdot(1+o_p(1)) \end{aligned}$$

and

$$\begin{aligned} II_2 &= \frac{1}{\beta_{2T}^{T-[\tau_0 T]}\sqrt{Tk_T}l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\varepsilon_t\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i \\ &= \frac{1}{\beta_{2T}^{T-[\tau_0 T]}\sqrt{Tk_T}l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\varepsilon_t^{(1)}\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i^{(1)}\cdot(1+o_p(1)) \end{aligned}$$

by the truncation technique (A.1) and some simple calculus.

For the term II_1, applying part (a) of Lemma C.1 and observing the fact $\frac{\sum_{t=1}^{[\tau_0 T]}\varepsilon_t}{\sqrt{Tl(\eta_T)}}\Rightarrow W(\tau_0)$, it can be shown that

$$II_1 \Rightarrow XW(\tau_0) \tag{S.44}$$

by noting that

$$\begin{aligned} &\frac{1}{\beta_{2T}^{T-[\tau_0 T]}\sqrt{k_T l(\eta_T)}}\sum_{t=[\tau_0 T]+1}^{T}\beta_{2T}^{t-[\tau_0 T]-1}\varepsilon_t \\ =\ &\frac{1}{\beta_{2T}^{T-[\tau_0 T]}\sqrt{k_T l(\eta_T)}}\sum_{s=1}^{T-[\tau_0 T]}\beta_{2T}^{s-1}\varepsilon_{[\tau_0 T]+s} \\ =\ &\frac{1}{\sqrt{k_T l(\eta_T)}}\sum_{s=1}^{[(1-\tau_0)T]}\beta_{2T}^{s-1-[(1-\tau_0)T]}\varepsilon_{[\tau_0 T]+s}\cdot(1+o_p(1)) \end{aligned}$$

and

$$\frac{1}{\sqrt{k_T l(\eta_T)}}\sum_{s=1}^{[(1-\tau_0)T]}\beta_{2T}^{s-1-[(1-\tau_0)T]}\varepsilon_{[\tau_0 T]+s}\overset{d}{=}\frac{1}{\sqrt{k_T l(\eta_T)}}\sum_{s=1}^{[(1-\tau_0)T]}\beta_{2T}^{s-1-[(1-\tau_0)T]}\varepsilon_s\Rightarrow X.$$

For the term II_2, it is trivial that

$$E\Big(\frac{1}{\beta_{2T}^{T-[\tau_0 T]}\sqrt{Tk_T}l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\varepsilon_t^{(1)}\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i^{(1)}\Big)=0$$

and

$$
\begin{aligned}
& Var\Big(\frac{1}{\beta_{2T}^{T-[\tau_0 T]}\sqrt{Tk_T}l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\varepsilon_t^{(1)}\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i^{(1)}\Big) \\
= & \frac{1}{\beta_{2T}^{2(T-[\tau_0 T])}Tk_T}\sum_{t=[\tau_0 T]+1}^{T}\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{2(t-i-1)}\cdot(1+o(1)) \\
= & \frac{1}{\beta_{2T}^{2(T-[\tau_0 T])}T}\cdot\frac{1}{k_T(\beta_{2T}^2-1)}\cdot\Big(\frac{\beta_{2T}^{2(T-[\tau_0 T])}-1}{\beta_{2T}^2-1}-(T-[\tau_0 T])\Big)\cdot(1+o(1)) \\
= & o(1) \qquad\qquad (S.45)
\end{aligned}
$$

by $k_T = o(T)$. Consequently,

$$
II_2 = o_p(1). \qquad (S.46)
$$

Obviously, combining (S.43), (S.44) and (S.46) yields part (a).

To prove part (b), we write

$$
\begin{aligned}
& \frac{1}{\beta_{2T}^{2(T-[\tau_0 T])}Tk_T l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 \\
= & \frac{1}{\beta_{2T}^{2(T-[\tau_0 T])}Tk_T l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\Big(\beta_{2T}^{t-[\tau_0 T]-1}y_{[\tau_0 T]}+\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i\Big)^2 \\
= & \frac{y_{[\tau_0 T]}^2}{\beta_{2T}^{2(T-[\tau_0 T])}Tk_T l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\beta_{2T}^{2(t-[\tau_0 T]-1)}+\frac{1}{\beta_{2T}^{2(T-[\tau_0 T])}Tk_T l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\Big(\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i\Big)^2 \\
& +\frac{2y_{[\tau_0 T]}}{\beta_{2T}^{2(T-[\tau_0 T])}Tk_T l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\Big(\beta_{2T}^{t-[\tau_0 T]-1}\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i\Big) \\
:= & III_1+III_2+III_3, \qquad\qquad (S.47)
\end{aligned}
$$

where

$$
III_1 = \frac{y_{[\tau_0 T]}^2}{\beta_{2T}^{2(T-[\tau_0 T])}Tk_T l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\beta_{2T}^{2(t-[\tau_0 T]-1)},
$$

$$
III_2 = \frac{1}{\beta_{2T}^{2(T-[\tau_0 T])}Tk_T l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\Big(\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i\Big)^2
$$

and

$$
III_3 = \frac{2y_{[\tau_0 T]}}{\beta_{2T}^{2(T-[\tau_0 T])}Tk_T l(\eta_T)}\sum_{t=[\tau_0 T]+1}^{T}\Big(\beta_{2T}^{t-[\tau_0 T]-1}\sum_{i=[\tau_0 T]+1}^{t-1}\beta_{2T}^{t-i-1}\varepsilon_i\Big).
$$

Obviously,

$$
III_1 \Rightarrow \frac{1}{2c}W^2(\tau_0). \qquad (S.48)
$$

For the term III_2, it is true that

$$III_2 = \frac{1}{\beta_{2T}^{2(T-[\tau_0 T])} T k_T l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \Big(\sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-i-1} \varepsilon_i^{(1)} \Big)^2 \cdot (1 + o_p(1))$$

and

$$\begin{aligned} & E\left(\frac{1}{\beta_{2T}^{2(T-[\tau_0 T])} T k_T l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \Big(\sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-i-1} \varepsilon_i^{(1)} \Big)^2 \right) \\ = & \frac{1}{\beta_{2T}^{2(T-[\tau_0 T])} T k_T} \sum_{t=[\tau_0 T]+1}^{T} \sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{2(t-i-1)} \cdot (1 + o(1)) \\ = & o(1) \end{aligned}$$

by similar arguments as in (S.45). Hence, we have

$$III_2 = o_p(1). \tag{S.49}$$

For the term III_3, using (S.48) and (S.49) and applying Cauchy-Schwarz inequality immediately leads to

$$III_3 = o_p(1). \tag{S.50}$$

Combining (S.47)-(S.50), we show part (b).

Moreover, by checking the above proofs carefully, one can find that parts (a) and (b) hold jointly and X and $W(\tau_0)$ are mutually independent. □

Proof of Lemma C.3. First, it is trivial that

$$A_1 = o_p(1/k_T), \quad A_4 = o_p(1/k_T) \tag{S.51}$$

by Lemma B.1 and Lemma C.2. Secondly, note that the same reasoning in (S.25) also implies

$$A_2 = o_p(1/k_T). \tag{S.52}$$

For the term A_5, following the proof of Theorem 1.2, we have

$$\begin{aligned} & y_{[\tau_0 T]} \sum_{t=[\tau_0 T]+1}^{m} \beta_{2T}^{t-1-[\tau_0 T]} \varepsilon_t \\ = & O_p\left(\sqrt{T k_T (\beta_{2T}^{2(m-[\tau_0 T])} - 1) l(\eta_T)} \right) \\ = & \begin{cases} O_p\left(\sqrt{T(m-[\tau_0 T])} l(\eta_T) \right), & \text{if } M_T < m - [\tau_0 T] \le [k_T/N_1] \\ O_p\left(\sqrt{T k_T} l(\eta_T) \right), & \text{if } [k_T/N_1] < m - [\tau_0 T] \le [N_2 k_T] \\ O_p\left(\sqrt{T k_T} \beta_{2T}^{(m-[\tau_0 T])} l(\eta_T) \right), & \text{if } [N_2 k_T] < m - [\tau_0 T] \le T - [\tau_0 T] \end{cases} \end{aligned}$$

and

$$
\begin{aligned}
& \sum_{t=[\tau_0 T]+1}^{m} \left(\sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-1-i} \varepsilon_i \right) \varepsilon_t \\
= & \; O_p\left(\sqrt{k_T(m-[\tau_0 T]) + k_T^2(\beta_{2T}^{2(m-[\tau_0 T])} - 1)} l(\eta_T) \right) \\
= & \begin{cases} O_p\left(\sqrt{k_T(m-[\tau_0 T])} l(\eta_T) \right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p\left(k_T l(\eta_T) \right), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p\left(k_T \beta_{2T}^{m-[\tau_0 T]} l(\eta_T) \right), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases} .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& \sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t \\
= & \; y_{[\tau_0 T]} \sum_{t=[\tau_0 T]+1}^{m} \beta_{2T}^{t-1-[\tau_0 T]} \varepsilon_t + \sum_{t=[\tau_0 T]+1}^{m} \left(\sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-1-i} \varepsilon_i \right) \varepsilon_t \\
= & \begin{cases} O_p\left(\sqrt{T(m-[\tau_0 T])} l(\eta_T) \right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p\left(\sqrt{T k_T} l(\eta_T) \right), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p\left(\sqrt{T k_T} \beta_{2T}^{(m-[\tau_0 T])} l(\eta_T) \right), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases} . \qquad \text{(S.53)}
\end{aligned}
$$

Moreover, we also have

$$
\begin{aligned}
& \frac{2\beta_{2T}}{1-\beta_{2T}^2} \sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t \\
= & \begin{cases} O_p\left(k_T \sqrt{T(m-[\tau_0 T])} l(\eta_T) \right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p\left(\sqrt{T} k_T^{3/2} l(\eta_T) \right), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p\left(\sqrt{T} k_T^{3/2} \beta_{2T}^{(m-[\tau_0 T])} l(\eta_T) \right), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases} ,
\end{aligned}
$$

$$
\begin{aligned}
& \frac{1}{1-\beta_{2T}^2} \sum_{t=[\tau_0 T]+1}^{m} \varepsilon_t^2 \\
= & \; O_p\left(k_T(m-[\tau_0 T]) l(\eta_T) \right) \\
= & \begin{cases} O_p\left(k_T(m-[\tau_0 T]) l(\eta_T) \right), & \text{if } M_T < m-[\tau_0 T] \le [k_T/N_1] \\ O_p\left(k_T^2 l(\eta_T) \right), & \text{if } [k_T/N_1] < m-[\tau_0 T] \le [N_2 k_T] \\ O_p\left(k_T(m-[\tau_0 T]) l(\eta_T) \right), & \text{if } [N_2 k_T] < m-[\tau_0 T] \le T-[\tau_0 T] \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
& -\frac{1}{1-\beta_{2T}^2}(y_m^2 - y_{[\tau_0 T]}^2) \\
= & -\frac{1}{1-\beta_{2T}^2}(y_m + y_{[\tau_0 T]})(y_m - y_{[\tau_0 T]}) \\
= & -\frac{1}{1-\beta_{2T}^2}\left(\sum_{i=[\tau_0 T]+1}^{m} \beta_{2T}^{m-i}\varepsilon_i + \left(\beta_{2T}^{m-[\tau_0 T]}+1\right) y_{[\tau_0 T]}\right) \\
& \cdot \left(\sum_{i=[\tau_0 T]+1}^{m} \beta_{2T}^{m-i}\varepsilon_i + \left(\beta_{2T}^{m-[\tau_0 T]}-1\right) y_{[\tau_0 T]}\right) \\
= & k_T \cdot \left(O_p\left(\sqrt{k_T(\beta_{2T}^{2(m-[\tau_0 T])}-1)l(\eta_T)}\right) + O_p(\beta_{2T}^{m-[\tau_0 T]}\sqrt{Tl(\eta_T)})\right) \\
& \cdot \left(O_p\left(\sqrt{k_T(\beta_{2T}^{2(m-[\tau_0 T])}-1)l(\eta_T)}\right) + O_p\left((\beta_{2T}^{m-[\tau_0 T]}-1)\sqrt{Tl(\eta_T)}\right)\right) \\
= & \begin{cases} O_p(k_T\sqrt{Tl(\eta_T)}) \cdot O_p\left(\frac{m-[\tau_0 T]}{k_T}\sqrt{Tl(\eta_T)}\right), & \text{if } M_T < m-[\tau_0 T] \leq [k_T/N_1] \\ O_p(k_T\sqrt{Tl(\eta_T)}) \cdot O_p(\sqrt{Tl(\eta_T)}), & \text{if } [k_T/N_1] < m-[\tau_0 T] \leq [N_2 k_T] \\ O_p(k_T\beta_{2T}^{m-[\tau_0 T]}\sqrt{Tl(\eta_T)}) \cdot O_p(\beta_{2T}^{m-[\tau_0 T]}\sqrt{Tl(\eta_T)}), & \text{if } [N_2 k_T] < m-[\tau_0 T] \leq T-[\tau_0 T] \end{cases} \\
= & \begin{cases} O_p\left(T(m-[\tau_0 T])l(\eta_T)\right), & \text{if } M_T < m-[\tau_0 T] \leq [k_T/N_1] \\ O_p(Tk_T l(\eta_T)), & \text{if } [k_T/N_1] < m-[\tau_0 T] \leq [N_2 k_T] \\ O_p(Tk_T\beta_{2T}^{2(m-[\tau_0 T])}l(\eta_T)), & \text{if } [N_2 k_T] < m-[\tau_0 T] \leq T-[\tau_0 T] \end{cases}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2 \\
= & -\frac{1}{1-\beta_{2T}^2}(y_m^2 - y_{[\tau_0 T]}^2) + \frac{1}{1-\beta_{2T}^2}\sum_{t=[\tau_0 T]+1}^{m}\varepsilon_t^2 + \frac{2\beta_{2T}}{1-\beta_{2T}^2}\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t \\
= & \begin{cases} O_p\left(T(m-[\tau_0 T])l(\eta_T)\right), & \text{if } M_T < m-[\tau_0 T] \leq [k_T/N_1] \\ O_p(Tk_T l(\eta_T)), & \text{if } [k_T/N_1] < m-[\tau_0 T] \leq [N_2 k_T] \\ O_p(Tk_T\beta_{2T}^{2(m-[\tau_0 T])}l(\eta_T)), & \text{if } [N_2 k_T] < m-[\tau_0 T] \leq T-[\tau_0 T] \end{cases}. \qquad \text{(S.54)}
\end{aligned}
$$

Consequently, it follows from (S.53) and (S.54) that

$$
\frac{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2} = O_p\left(\frac{\sqrt{T(m-[\tau_0 T])}l(\eta_T)}{T(m-[\tau_0 T])l(\eta_T)}\right) \leq O_p(1/\sqrt{TM_T}) = o_p(1/k_T)
$$

by $k_T = O(\sqrt{T})$ and $M_T \to \infty$ when $M_T < m-[\tau_0 T] \leq [k_T/N_1]$,

$$
\frac{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2} = O_p\left(\frac{\sqrt{Tk_T}l(\eta_T)}{Tk_T l(\eta_T)}\right) = o_p(1/k_T)
$$

when $[k_T/N_1] < m - [\tau_0 T] \le [N_2 k_T]$ and

$$\frac{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2} = O_p\left(\frac{\sqrt{Tk_T}\beta_{2T}^{m-[\tau_0 T]} l(\eta_T)}{Tk_T\beta_{2T}^{2(m-[\tau_0 T])} l(\eta_T)}\right) = o_p(1/k_T)$$

when $[N_2 k_T] < m - [\tau_0 T] \le T - [\tau_0 T]$. These imply

$$A_5 = \sup_{m\in D_{2T}} \frac{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2} = o_p(1/k_T). \tag{S.55}$$

Thirdly, it is clear that

$$A_3 \le \Big(O_p(\frac{1}{Tk_T\beta_{2T}^{2(T-[\tau_0 T])} l(\eta_T)}) + O_p(\frac{1}{TM_T l(\eta_T)})\Big) \cdot O_p(l(\eta_T)) = o_p(1/k_T^2) \tag{S.56}$$

and

$$A_6 \le \Big(O_p(\frac{1}{T^2 l(\eta_T)}) + O_p(\frac{1}{TM_T l(\eta_T)})\Big) \cdot O_p(l(\eta_T)) = o_p(1/k_T^2). \tag{S.57}$$

The proofs are complete. □

Proof of Lemma C.4. Note that $|\hat\tau_T - \tau_0| = O_p(k_T^2/T^2)$ when $\sqrt{T} = o(k_T)$. Then, for $\hat\tau_T \le \tau_0$, we have

$$\sum_{t=[\hat\tau_T T]+1}^{[\tau_0 T]} y_{t-1}^2 = O_p(k_T^2 l(\eta_T)) \tag{S.58}$$

and

$$\sum_{t=[\hat\tau_T T]+1}^{[\tau_0 T]} y_{t-1}\varepsilon_t = O_p(k_T l(\eta_T)) \tag{S.59}$$

by (S.33) and (S.34). For $\hat\tau_T > \tau_0$, it follows from the similar arguments in the proofs of (S.35) and (S.41) respectively that

$$\begin{aligned}
\sum_{t=[\tau_0 T]+1}^{[\tau_0 T + k_T^2/T]} y_{t-1}^2 &= \sum_{t=[\tau_0 T]+1}^{[\tau_0 T + k_T^2/T]} \Big(\beta_{2T}^{t-1-[\tau_0 T]} y_{[\tau_0 T]} + \sum_{i=[\tau_0 T]+1}^{t-1} \beta_{2T}^{t-1-i}\varepsilon_i\Big)^2 \\
&= \sum_{t=[\tau_0 T]+1}^{[\tau_0 T + k_T^2/T]} \Big(\beta_{2T}^{t-1-[\tau_0 T]} y_{[\tau_0 T]} + O_p(\sqrt{k_T\beta_{2T}^{t-1-[\tau_0 T]} l(\eta_T)})\Big)^2 \\
&= \sum_{t=[\tau_0 T]+1}^{[\tau_0 T + k_T^2/T]} \beta_{2T}^{2(t-1-[\tau_0 T])} y_{[\tau_0 T]}^2 \cdot (1 + o_p(1)) \\
&= k_T^2/T \cdot O_p(T l(\eta_T)) \\
&= O_p(k_T^2 l(\eta_T)),
\end{aligned}$$

yielding

$$\sum_{t=[\tau_0 T]+1}^{[\hat\tau_T T]} y_{t-1}^2 = O_p(k_T^2 l(\eta_T)) \tag{S.60}$$

and

$$\begin{aligned}\sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} y_{t-1}\varepsilon_t &= \frac{1-\beta_{2T}^2}{2\beta_{2T}} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} y_{t-1}^2 + \frac{y_{[\tau_0 T+k_T^2/T]}^2 - y_{[\tau_0 T]}^2}{2\beta_{2T}} - \frac{1}{2\beta_{2T}} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T+k_T^2/T]} \varepsilon_t^2 \\ &= O_p(k_T l(\eta_T)) + O_p(\sqrt{Tl(\eta_T)}) \cdot O_p(k_T \sqrt{l(\eta_T)}/\sqrt{T}) + O_p(k_T^2 l(\eta_T)/T) \\ &= O_p(k_T l(\eta_T)),\end{aligned}$$

yielding

$$\sum_{t=[\tau_0 T]+1}^{[\hat{\tau}_T T]} y_{t-1}\varepsilon_t = O_p(k_T l(\eta_T)). \tag{S.61}$$

□

Proof of Lemma C.5. The proof is similar to that of Lemma B.5, with the use of the result in (C.1). The details are omitted. □

Proof of Lemma D.1. The results can be proved by following the proof of Theorem 4.3 in Phillips and Magdalinos (2007a) and applying the truncation technique as shown in (A.1). The details are omitted. □

Proof of Lemma D.2. To prove part (a), we write

$$\begin{aligned}&\frac{\beta_{1T}^{-[\tau_0 T]}}{\sqrt{Tk_T}l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t \\ =\ &\frac{\beta_{1T}^{-[\tau_0 T]}}{\sqrt{Tk_T}l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \Big(y_{[\tau_0 T]} + \sum_{i=[\tau_0 T]+1}^{t-1} \varepsilon_i\Big)\varepsilon_t \\ =\ &\frac{\beta_{1T}^{-[\tau_0 T]}}{\sqrt{Tk_T}l(\eta_T)} \left(\Big(\beta_{1T}^{[\tau_0 T]} y_0 + \sum_{j=1}^{[\tau_0 T]} \beta_{1T}^{[\tau_0 T]-j}\varepsilon_j\Big) \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t + \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t \sum_{i=[\tau_0 T]+1}^{t-1} \varepsilon_i\right) \\ :=\ &IV_1 + IV_2,\end{aligned} \tag{S.62}$$

where

$$IV_1 = \frac{\beta_{1T}^{-[\tau_0 T]}}{\sqrt{Tk_T}l(\eta_T)} \Big(\beta_{1T}^{[\tau_0 T]} y_0 + \sum_{j=1}^{[\tau_0 T]} \beta_{1T}^{[\tau_0 T]-j}\varepsilon_j\Big) \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t$$

and

$$IV_2 = \frac{\beta_{1T}^{-[\tau_0 T]}}{\sqrt{Tk_T}l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t \sum_{i=[\tau_0 T]+1}^{t-1} \varepsilon_i.$$

Applying part (b) of Lemma C.1 and assumption C3, it can be shown that

$$IV_1 = \frac{1}{\sqrt{k_T l(\eta_T)}} \sum_{j=1}^{[\tau_0 T]} \beta_{1T}^{-j} \varepsilon_j \cdot \frac{1}{\sqrt{T l(\eta_T)}} \sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t + o_p(1) \Rightarrow Y(\overline{W}(1) - \overline{W}(\tau_0)). \quad \text{(S.63)}$$

In addition, it is not difficult to show that

$$IV_2 = O_p\Big(\frac{\beta_{1T}^{-[\tau_0 T]}}{\sqrt{T k_T} l(\eta_T)} \cdot T l(\eta_T)\Big) = o_p(1) \quad \text{(S.64)}$$

by (C.1).

Combining (S.62), (S.63) and (S.64) leads to part (a).

To prove part (b), we write

$$\begin{aligned}
& \frac{\beta_{1T}^{-2[\tau_0 T]}}{T k_T l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 \\
= & \frac{\beta_{1T}^{-2[\tau_0 T]}}{T k_T l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \Big(y_{[\tau_0 T]} + \sum_{i=[\tau_0 T]+1}^{t-1} \varepsilon_i \Big)^2 \\
= & \frac{\beta_{1T}^{-2[\tau_0 T]}}{T k_T l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \Big(\sum_{j=1}^{[\tau_0 T]} \beta_{1T}^{[\tau_0 T]-j} \varepsilon_j + \sum_{i=[\tau_0 T]+1}^{t-1} \varepsilon_i \Big)^2 \cdot (1 + o_p(1)) \\
:= & (V_1 + V_2 + V_3) \cdot (1 + o_p(1)), \quad \text{(S.65)}
\end{aligned}$$

where

$$V_1 = \frac{1}{T k_T l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \Big(\sum_{j=1}^{[\tau_0 T]} \beta_{1T}^{-j} \varepsilon_j \Big)^2,$$

$$V_2 = \frac{\beta_{1T}^{-2[\tau_0 T]}}{T k_T l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{T} \Big(\sum_{i=[\tau_0 T]+1}^{t-1} \varepsilon_i \Big)^2$$

and

$$V_3 = \frac{2\beta_{1T}^{-2[\tau_0 T]}}{T k_T l(\eta_T)} \cdot \sum_{j=1}^{[\tau_0 T]} \beta_{1T}^{[\tau_0 T]-j} \varepsilon_j \cdot \sum_{t=[\tau_0 T]+1}^{T} \sum_{i=[\tau_0 T]+1}^{t-1} \varepsilon_i$$

Obviously,

$$V_1 \Rightarrow (1 - \tau_0) Y^2 \quad \text{(S.66)}$$

by applying part (b) of Lemma C.1.

For the term V_2, we have

$$V_2 = O_p\Big(\frac{\beta_{1T}^{-2[\tau_0 T]}}{T k_T l(\eta_T)} \cdot T^2 l(\eta_T)\Big) = o_p(1) \quad \text{(S.67)}$$

by making use of (C.1).

Finally, applying Cauchy-Schwarz inequality, it follows from (S.66) and (S.67) that

$$V_3 = o_p(1). \tag{S.68}$$

It is ready to see that part (b) is proved by combining (S.65)-(S.68).

One can also verfiy that parts (a) and (b) hold jointly and Y and $\overline{W}(1) - \overline{W}(\tau_0)$ are mutually independent. □

Proof of Lemma D.3. First, by applying Lemmas D.1 and D.2, it can be shown that

$$A_1 = o_p(1/k_T) \text{ and } A_4 = o_p(1/k_T).$$

Secondly, we investigate the magnitude of the terms A_2 and A_5 in probability. For A_2, note that Cauchy-Schwarz inequality implies

$$\sup_{m\in D_{1T}} \left| \frac{\sum_{t=m+1}^{[\tau_0 T]} y_{t-1}\varepsilon_t}{\sum_{t=m+1}^{[\tau_0 T]} y_{t-1}^2} \right| \leq \sup_{m\in D_{1T}} \sqrt{\frac{\sum_{t=m+1}^{[\tau_0 T]} \varepsilon_t^2}{\sum_{t=m+1}^{[\tau_0 T]} y_{t-1}^2}},$$

so it suffices to prove that the magnitude of the right hand side of the above inequality is $o_p(1/k_T)$. Applying Lemma C.1, we immediately have

$$\sup_{m\in D_{1T}} \sqrt{\frac{\sum_{t=m+1}^{[\tau_0 T]} \varepsilon_t^2}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2}} \leq \sqrt{\frac{\sum_{t=1}^{[\tau_0 T]} \varepsilon_t^2}{y_{[\tau_0 T]-1}^2}} \leq O_p\left(\sqrt{\frac{Tl(\eta_T)}{k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)}}\right) = o_p(1/k_T)$$

by (C.1), which implies that

$$A_2 \leq \sup_{m\in D_{1T}} \sqrt{\frac{\sum_{t=m+1}^{[\tau_0 T]} \varepsilon_t^2}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2}} = o_p(1/k_T).$$

Similarly, for the term A_5, we have

$$\begin{aligned}
\sup_{m\in D_{2T}} \left| \frac{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2} \right| &\leq \sup_{m\in D_{2T}} \sqrt{\frac{\sum_{t=[\tau_0 T]+1}^{m} \varepsilon_t^2}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2}} \\
&\leq \sqrt{\frac{\sum_{t=[\tau_0 T]+1}^{T} \varepsilon_t^2}{y_{[\tau_0 T]}^2}} \\
&= O_p\left(\sqrt{\frac{Tl(\eta_T)}{k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)}}\right) \\
&= o_p(1/k_T),
\end{aligned}$$

which implies that

$$A_5 = o_p(1/k_T).$$

Thirdly, note also that

$$
\begin{aligned}
A_3 &= \sup_{m\in D_{1T}} \Big| \frac{\sum_{t=m+1}^{T} y_{t-1}^2}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 \sum_{t=m+1}^{[\tau_0 T]} y_{t-1}^2} \Lambda_T(\frac{m}{T}) \Big| \\
&\le \Big(\frac{1}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2} + \frac{1}{\sum_{t=[\tau_0 T]-M_T}^{[\tau_0 T]} y_{t-1}^2} \Big) \Big(\sup_{m\in D_{1T}} \Big| \frac{(\sum_{t=1}^{[\tau_0 T]} y_{t-1}\varepsilon_t)^2}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} - \frac{(\sum_{t=1}^{m} y_{t-1}\varepsilon_t)^2}{\sum_{t=1}^{m} y_{t-1}^2} \Big| \\
&\qquad + \sup_{m\in D_{1T}} \Big| \frac{(\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t)^2}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=m+1}^{T} y_{t-1}\varepsilon_t)^2}{\sum_{t=m+1}^{T} y_{t-1}^2} \Big| \Big) \\
&\le \Big(O_p(\frac{1}{Tk_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)}) + O_p(\frac{1}{k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)}) \Big) \cdot O_p(l(\eta_T)) \\
&= o_p(1/k_T^2),
\end{aligned}
$$

and

$$
\begin{aligned}
A_6 &= \sup_{m\in D_{2T}} \Big| \frac{\sum_{t=1}^{m} y_{t-1}^2}{\sum_{t=[\tau_0 T]+1}^{m} y_{t-1}^2 \sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} \Lambda_T(\frac{m}{T}) \Big| \\
&\le \Big(\frac{1}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} + \frac{1}{\sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+M_T} y_{t-1}^2} \Big) \Big(\sup_{m\in D_{2T}} \Big| \frac{(\sum_{t=1}^{[\tau_0 T]} y_{t-1}\varepsilon_t)^2}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} - \frac{(\sum_{t=1}^{m} y_{t-1}\varepsilon_t)^2}{\sum_{t=1}^{m} y_{t-1}^2} \Big| \\
&\qquad + \sup_{m\in D_{2T}} \Big| \frac{(\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t)^2}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2} - \frac{(\sum_{t=m+1}^{T} y_{t-1}\varepsilon_t)^2}{\sum_{t=m+1}^{T} y_{t-1}^2} \Big| \Big) \\
&\le \Big(O_p(\frac{1}{k_T^2 \beta_{1T}^{2[\tau_0 T]} l(\eta_T)}) + O_p(\frac{1}{k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)}) \Big) \cdot O_p(l(\eta_T)) \\
&= o_p(1/k_T^2).
\end{aligned}
$$

These complete the proofs. □

Proof of Lemma D.4. To prove part (a), applying equation (B.2) in Chong (2001), we have

$$
\begin{aligned}
& RSS_T(\tau_0 - \frac{m}{T}) - RSS_T(\tau_0) \\
=\;& 2(\beta_2 - \beta_{1T}) \Big(\frac{\sum_{t=[\tau_0 T]-m+1}^{[\tau_0 T]} y_{t-1}^2 \sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}^2} - \frac{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 \sum_{t=[\tau_0 T]-m+1}^{[\tau_0 T]} y_{t-1}\varepsilon_t}{\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}^2} \Big) \\
& + (\beta_2 - \beta_{1T})^2 \frac{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2 \sum_{t=[\tau_0 T]-m+1}^{[\tau_0 T]} y_{t-1}^2}{\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}^2} + \Lambda_T(\tau_0 - \frac{m}{T}),
\end{aligned}
$$

where

$$
\begin{aligned}
& \Lambda_T(\tau_0 - \frac{m}{T}) \\
=\;& \frac{\Big(\sum_{t=1}^{[\tau_0 T]} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} - \frac{\Big(\sum_{t=1}^{[\tau_0 T]-m} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=1}^{[\tau_0 T]-m} y_{t-1}^2} + \frac{\Big(\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2} - \frac{\Big(\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}\varepsilon_t\Big)^2}{\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}^2}.
\end{aligned}
$$

Obviously, $\Lambda_T(\tau_0 - \frac{m}{T}) = O_p(l(\eta_T))$. It follows that

$$
\begin{aligned}
& \frac{k_T}{\beta_{1T}^{2[\tau_0 T]} l(\eta_T)} \Big(RSS_T(\tau_0 - \frac{m}{T}) - RSS_T(\tau_0) \Big) \\
= \; & \frac{k_T}{\beta_{1T}^{2[\tau_0 T]} l(\eta_T)} \cdot \Big(-\frac{2c}{k_T} \Big) \Big(\frac{O_p(m k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)) O_p(\sqrt{T k_T} \beta_{1T}^{[\tau_0 T]} l(\eta_T))}{O_p(m k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)) + O_p(T k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T))} + \frac{O_p(T k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T))}{O_p(\sqrt{k_T} \beta_{1T}^{[\tau_0 T]})} \Big) \\
& + \frac{c^2}{k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)} \cdot \frac{O_p(T k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T))}{O_p(m k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)) + O_p(T k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T))} \sum_{[\tau_0 T]-m+1}^{[\tau_0 T]} y_{t-1}^2 + o_p(1) \\
= \; & \frac{c^2(1 + o_p(1))}{k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)} \sum_{[\tau_0 T]-m+1}^{[\tau_0 T]} y_{t-1}^2 + o_p(1) \\
\Rightarrow \; & c^2 m Y^2
\end{aligned}
$$

by Lemmas C.1 and D.2 and the observations

$$
\Bigg| \frac{\sum_{t=[\tau_0 T]-m+1}^{[\tau_0 T]} y_{t-1} \varepsilon_t}{\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}^2} \Bigg| \le \frac{\sqrt{\sum_{t=[\tau_0 T]-m+1}^{[\tau_0 T]} \varepsilon_t^2}}{\sqrt{\sum_{t=[\tau_0 T]-m+1}^{T} y_{t-1}^2}} = O_p(\frac{1}{\sqrt{k_T} \beta_{1T}^{[\tau_0 T]}})
$$

and

$$
\frac{\sum_{t=[\tau_0 T]-m+1}^{[\tau_0 T]} y_{t-1}^2}{m y_{[\tau_0 T]}^2} \xrightarrow{p} 1.
$$

To prove part (b), applying equation (B.4) in Chong (2001), we have

$$
\begin{aligned}
& RSS_T(\tau_0 + \frac{m}{T}) - RSS_T(\tau_0) \\
= \; & 2(\beta_2 - \beta_{1T}) \Big(\frac{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2 \sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+m} y_{t-1} \varepsilon_t}{\sum_{t=1}^{[\tau_0 T]+m} y_{t-1}^2} - \frac{\sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+m} y_{t-1}^2 \sum_{t=1}^{[\tau_0 T]} y_{t-1} \varepsilon_t}{\sum_{t=1}^{[\tau_0 T]+m} y_{t-1}^2} \Big) \\
& + (\beta_2 - \beta_{1T})^2 \frac{\sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+m} y_{t-1}^2 \sum_{t=1}^{[\tau_0 T]} y_{t-1}^2}{\sum_{t=1}^{[\tau_0 T]+m} y_{t-1}^2} + \Lambda_T(\tau_0 + \frac{m}{T})
\end{aligned}
$$

where

$$
\begin{aligned}
& \Lambda_T(\tau_0 + \frac{m}{T}) \\
= \; & \frac{\Big(\sum_{t=1}^{[\tau_0 T]} y_{t-1} \varepsilon_t \Big)^2}{\sum_{t=1}^{[\tau_0 T]} y_{t-1}^2} - \frac{\Big(\sum_{t=1}^{[\tau_0 T]+m} y_{t-1} \varepsilon_t \Big)^2}{\sum_{t=1}^{[\tau_0 T]+m} y_{t-1}^2} + \frac{\Big(\sum_{t=[\tau_0 T]+1}^{T} y_{t-1} \varepsilon_t \Big)^2}{\sum_{t=[\tau_0 T]+1}^{T} y_{t-1}^2} - \frac{\Big(\sum_{t=[\tau_0 T]+m+1}^{T} y_{t-1} \varepsilon_t \Big)^2}{\sum_{t=[\tau_0 T]+m+1}^{T} y_{t-1}^2}.
\end{aligned}
$$

Then, using the facts $\Lambda_T(\tau_0 + \frac{m}{T}) = O_p(l(\eta_T))$,

$$
\Bigg| \frac{\sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+m} y_{t-1} \varepsilon_t}{\sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+m} y_{t-1}^2} \Bigg| \le \frac{\sqrt{\sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+m} \varepsilon_t^2}}{\sqrt{\sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+m} y_{t-1}^2}} = O_p(\frac{1}{\sqrt{k_T} \beta_{1T}^{[\tau_0 T]}}),
$$

and

$$
\frac{\sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+m} y_{t-1}^2}{m y_{[\tau_0 T]}^2} \xrightarrow{p} 1,
$$

we have

$$
\begin{aligned}
& \frac{k_T}{\beta_{1T}^{2[\tau_0 T]} l(\eta_T)} \Big(RSS_T(\tau_0 + \frac{m}{T}) - RSS_T(\tau_0) \Big) \\
= \quad & \frac{k_T}{\beta_{1T}^{2[\tau_0 T]} l(\eta_T)} \cdot \Big(- \frac{2c}{k_T} \Big) \Big(\frac{O_p(k_T^2 \beta_{1T}^{2[\tau_0 T]} l(\eta_T))}{\sqrt{k_T} \beta_{1T}^{[\tau_0 T]}} + \frac{O_p(m k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)) O_p(k_T \beta_{1T}^{[\tau_0 T]} l(\eta_T))}{O_p(k_T^2 \beta_{1T}^{2[\tau_0 T]} l(\eta_T)) + O_p(m k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T))} \Big) \\
& + \frac{c^2}{k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)} \cdot \frac{O_p(k_T^2 \beta_{1T}^{2[\tau_0 T]} l(\eta_T))}{O_p(k_T^2 \beta_{1T}^{2[\tau_0 T]} l(\eta_T)) + O_p(m k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T))} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+m} y_{t-1}^2 + o_p(1) \\
= \quad & \frac{c^2(1 + o_p(1))}{k_T \beta_{1T}^{2[\tau_0 T]} l(\eta_T)} \sum_{t=[\tau_0 T]+1}^{[\tau_0 T]+m} y_{t-1}^2 + o_p(1) \\
\Rightarrow \quad & c^2 m Y^2
\end{aligned}
$$

by Lemmas C.1 and D.1. □

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