Supplemental Material for the Main Submission “Specification Testing Driven by Orthogonal Series for Nonlinear Cointegration with Endogeneity”

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Abstract

In this material, Appendix C gives the proof of all the lemmas listed in this article as well as the justification of Assumption B; Appendix D provides the complete proofs of the main results, and Appendix E shows additional simulation examples to illustrate some theoretical implication of the main results in the article.

Appendix C: Proofs of Lemmas A.1-A.5 and Justification

The first section in the supplemental material gives the proofs of Lemmas A.1–A.5 of Appendix A (renamed as Lemmas C.1–C.5 in this appendix below) and the justification for Assumption B in the article. In addition, we also give the background to these lemmas in the context of the article to enhance the reading of this online supplementary material.

Without loss of generality, in what follows let $x_0 = 0$ almost surely. Recall that

$$x_t = \sum_{j=-\infty}^{t} \left(\sum_{i=\max(1,j)}^{t} \psi_{i-j} a_{t-i} \right) \epsilon_j := \sum_{j=-\infty}^{t} b_{t,j} \epsilon_j,$$

where $b_{t,j} = \sum_{i=\max(1,j)}^{t} \psi_{i-j} a_{t-i}$.

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Let $j \le t$ be fixed. Recall that

$$x_t = b_{t,j}\epsilon_j + x_{t/j}, \qquad \text{with } x_{t/j} := \sum_{\ell=-\infty,\neq j}^{t} b_{t,\ell}\epsilon_\ell, \tag{C.1}$$

where obviously $x_{t/j}$ and ϵ_j are mutually independent.

Additionally, letting $1 \le s < j \le t$, x_t also has the following decomposition:

$$x_t = x_s^* + x_{ts}, \tag{C.2}$$

where $x_s^* = x_s + \bar{x}_s$ with $\bar{x}_s = \sum_{i=1}^{s}[a_{t-i} - a_{s-i}]u_i + \sum_{j=-\infty}^{s} \epsilon_j \sum_{i=s+1}^{t} \psi_{i-j} a_{t-i}$ containing all information available up to s. Clearly, x_s^* and x_{ts} are mutually independent.

Meanwhile, for any $j : s < j \le t$, x_{ts} can be written as

$$x_{ts} = b_{t,j}\epsilon_j + x_{ts/j}, \qquad \text{with } x_{ts/j} = \sum_{\ell=s+1,\neq j}^{t} b_{t,\ell}\epsilon_\ell. \tag{C.3}$$

Evidently, $x_{ts/j}$, x_s^* and ϵ_j are independent of each other whenever $s < j \le t$.

If necessary for $s < j \neq \ell \le t$, we may write $x_t = x_s^* + b_{t,j}\epsilon_j + b_{t,\ell}\epsilon_\ell + x_{ts/j\ell}$ with $x_{ts/j\ell} = \sum_{i=s+1,\neq j,\ell}^{t} b_{t,\ell}\epsilon_i$.

Lemma C.1. *Let $d_t = (Ex_t^2)^{1/2}$. When t is large, $d_t^2 \sim \psi^2 \frac{1}{\Gamma^2(\alpha_0)} t^{2\alpha_0 - 1}$ for $\alpha_0 \in (1/2, 3/2)$. Let $d_{ts} = [E(x_{ts}^2)]^{1/2}$. When $t - s$ is large, $d_{ts}^2 \sim \psi^2 \frac{1}{\Gamma^2(\alpha_0)} (t-s)^{2\alpha_0 - 1}$.*

Proof. To calculate d_t, recall the BN decomposition of a linear process for u_t, that is, $u_t = \Psi(L;\rho_0)\epsilon_t = \Psi(1;\rho_0)\epsilon_t + [\Psi(L;\rho_0) - \Psi(1;\rho_0)]\epsilon_t = \psi\epsilon_t + \tilde{u}_{t-1} - \tilde{u}_t$, where $\psi = \Psi(1;\rho_0) = \sum_{j=0}^{\infty} \psi_j(\rho_0) \neq 0$, $\tilde{u}_t = \sum_0^\infty \tilde{\psi}_j \epsilon_{t-j}$ and $\tilde{\psi}_j = \sum_{j+1}^{\infty} \psi_k$. See, Lemma 2.1 of Phillips and Solo (1992, p. 972). Thus,

$$\begin{aligned} x_t =& \sum_{\ell=1}^{t} a_{t-\ell} u_\ell = \sum_{\ell=1}^{t} a_{t-\ell}[\psi\epsilon_\ell + \tilde{u}_{\ell-1} - \tilde{u}_\ell] \\ =& \psi \sum_{\ell=1}^{t} a_{t-\ell}\epsilon_\ell + \sum_{\ell=1}^{t} a_{t-\ell}[\tilde{u}_{\ell-1} - \tilde{u}_\ell] \\ =& \psi \sum_{\ell=1}^{t} a_{t-\ell}\epsilon_\ell + a_{t-1}\tilde{u}_0 - \sum_{\ell=1}^{t-1}[a_{t-\ell} - a_{t-\ell-1}]\tilde{u}_\ell - a_0\tilde{u}_t. \end{aligned}$$

Because $a_{t-\ell} - a_{t-\ell-1} \sim (t-\ell)^{\alpha_0 - 2}$ is one order lower than $a_{t-\ell} \sim (t-\ell)^{\alpha_0 - 1}$, we have when t is large, $d_t^2 = \psi^2 \sum_{\ell=1}^{t} a_{t-\ell}^2 (1 + o(1)) = \psi^2 \frac{1}{\Gamma^2(\alpha_0)} t^{2\alpha_0 - 1}(1 + o(1))$.

The assertion for d_{ts} follows similarly. Throughout the rest of this supplementary document, it is noted that the uniform Lipschitz condition is the same as defined in Lemma A.2 of the article.

□

Lemma C.2. *Suppose that Assumption A holds. Let t and $t-s$ be large. Let d_t and d_{ts} be defined in Lemma C.1.*

(1) $d_t^{-1}x_t$ has a uniformly bounded density $f_t(x)$ over all t and x, and it satisfies the uniform Lipschitz condition.

(2) Let $j \leq t$. $d_t^{-1} x_{t/j}$, with $x_{t/j}$ given by (C.1)*, has a uniformly bounded density $f_{t/j}(x)$ over all t, j and x, and it satisfies the uniform Lipschitz condition with some constant $C > 0$ independent of t and j.*

(3) $d_{ts}^{-1} x_{ts}$, with x_{ts} given by (C.2)*, has a uniformly bounded density $f_{ts}(x)$ over all t, s and x, and it satisfies the uniform Lipschitz condition with some constant $C > 0$ independent of t and s.*

(4) Let $1 \leq s < j \leq t$. $d_{ts}^{-1} x_{ts/j}$, where $x_{ts/j}$ given by (C.3)*, has a uniformly bounded density $f_{ts/j}(x)$ over all t, j and s, x, and it satisfies the uniform Lipschitz condition with some constant $C > 0$ independent of t, s and j.*

Proof. All the results in (1)-(4) can be shown similarly, hence we only show how to prove the results of (1). To prove the existence of the density of $d_t^{-1} x_t$, we need only to show the integrability of the characteristic function of $d_t^{-1} x_t$, i.e. $\int |\Psi_t(\lambda)| d\lambda < \infty$, where $\Psi_t(\lambda)$ is the characteristic function of $d_t^{-1} x_t$. This can be done by virtue of $x_t = \sum_{\ell=-\infty}^{t} b_{t,\ell} \epsilon_\ell$ and $\int |E e^{iu\epsilon_0}| du < \infty$ implied by the condition on ϵ_0, similar to the proof of Lemma 3.1 in Pötscher (2004, p.10) and Corollary 2.2 in Wang and Phillips (2009). Then the existence follows from the inverse formula directly.

The uniform Lipschitz condition can be shown by the similar argument. Namely, we first show that $\int |\lambda \Psi_t(\lambda)| d\lambda < \infty$ because $\int |u\, E e^{iu\epsilon_0}| dt < \infty$. It follows that the density $f_t(x)$ of $d_t^{-1} x_t$ is differentiable and has a uniformly bounded derivative, and the upper bound is independent of t (and j in the other case). Then the assertion holds immediately. $\square$

Let $\mathcal{F}_t$ be the σ–field generated by $\{\varepsilon_j : -\infty < j \leq t\}$.

Lemma C.3. *Let Assumption A hold. Let j be a fixed integer such that $j \leq t$ where t is large and let C be a generic constant. For any functions U and $f : \mathbb{R} \mapsto \mathbb{R}$ such that $\int |U(w)| dw < \infty$ and $E|\epsilon_j f(\epsilon_j)| < \infty$, we have*

(1) $E[U(x_t) f(\epsilon_j)] = E[U(x_{t/j})] E[f(\epsilon_j)] + c_{t,j} d_t^{-2}$ where $c_{t,j}$ is such that $|c_{t,j}| \leq C|b_{t,j}| \int |U(w)| dw$. In particular, if $Ef(\epsilon_j) = 0$, then $E[U(x_t) f(\epsilon_j)] = c_{t,j} d_t^{-2}$. Meanwhile, $E|U(x_t) f(\epsilon_j)| \leq C d_t^{-1} E|f(\epsilon_j)| \int |U(w)| dw$, which remains true for $f(\cdot) \equiv 1$, i.e., $E|U(x_t)| \leq C d_t^{-1} \int |U(w)| dw$.

(2) For any ℓ: $j \neq \ell \leq t$, $E[U(x_t) f(\epsilon_j) | \epsilon_\ell] = E[U(x_{t/j}) | \epsilon_\ell] E[f(\epsilon_j)] + d_t^{-2} \eta_{j\ell}$ where $\eta_{j\ell}$ is a random variable depending on ϵ_ℓ such that $|\eta_{j\ell}| \leq C|b_{t,j}|$ almost surely. If $E[f(\epsilon_j)] = 0$, then $E[U(x_t) f(\epsilon_j) | \epsilon_\ell] = d_t^{-2} \eta_{j\ell}$. Also, $E[|U(x_t) f(\epsilon_j)| | \epsilon_\ell] \leq C d_t^{-1} \int |U(w)| dw$ almost surely.

(3) For $1 \leq s < j \leq t$ and let $t - s$ be large, $E[U(x_t) f(\epsilon_j) | \mathcal{F}_s] = E[f(\epsilon_j)] E[U(x_{t/j}) | \mathcal{F}_s] + d_{ts}^{-2} \xi_{s,j}$ where $|\xi_{s,j}| \leq C|b_{t,j}| \int |U(x)| dx$ almost surely; also, $E[|U(x_t) f(\epsilon_j)| | \mathcal{F}_s] \leq C d_{ts}^{-1} \int |U(w)| dw$ a.s.

(4) Similarly, letting $1 \leq s < j \neq \ell \leq t$ where $t - s$ is large and let $g(x)$ be a function such that $E|\epsilon_\ell g(\epsilon_\ell)| < \infty$, we have $E[U(x_t) f(\epsilon_j) g(\epsilon_\ell) | \mathcal{F}_s] = E[f(\epsilon_j) g(\epsilon_\ell)] E[U(x_{t/j\ell}) | \mathcal{F}_s] + d_{ts}^{-2} \xi_{s,j\ell}$, where $x_{t/j\ell}$ is defined as before and $\xi_{s,j\ell}$ satisfies that $|\xi_{s,j\ell}| \leq C(|b_{t,j}| + |b_{t,\ell}|) \int |U(x)| dx$ almost surely. Moreover, we have $E|U(x_t) f(\epsilon_j) g(\epsilon_\ell)| \leq C d_t^{-1} \int |U(w)| dw$.

(5) *For t large and e_t defined by a linear process in A.3(i) of Assumption A,* $E[U(x_t)e_t] = q_t d_t^{-2}$ *with* $|q_t| \le C\int |U(w)|dw$ *and* $|E[U(x_t)e_t^2]| \le C d_t^{-1}\int |U(w)|dw$.

Proof. (1) Recall that $x_t = b_{t,j}\epsilon_j + x_{t/j}$, that $d_t^{-1}x_{t/j}$ has a uniformly bounded density $f_{t/j}(x)$ that satisfies the uniform Lipschitz condition and that ϵ_j has density $h_\epsilon(v)$. We have

$$
\begin{aligned}
&E[U(x_t)f(\epsilon_j)] = E[U(b_{t,j}\epsilon_j + x_{t/j})f(\epsilon_j)]\\
=&\iint U(b_{t,j}v + d_t w)f(v)h_\epsilon(v)f_{t/j}(w)dvdw\\
=&d_t^{-1}\iint U(w)f(v)h_\epsilon(v)f_{t/j}\left(\frac{w-b_{t,j}v}{d_t}\right)dvdw\\
=&d_t^{-1}\iint U(w)f(v)h_\epsilon(v)f_{t/j}\left(\frac{w}{d_t}\right)dvdw\\
&+d_t^{-1}\iint U(w)f(v)h_\epsilon(v)\left[f_{t/j}\left(\frac{w-b_{t,j}v}{d_t}\right)-f_{t/j}\left(\frac{w}{d_t}\right)\right]dvdw\\
:=&d_t^{-1}\int f(v)h_\epsilon(v)dv\int U(w)f_{t/j}\left(\frac{w}{d_t}\right)dw + c_{t,j}d_t^{-2}\\
=&E[f(\epsilon_j)]\int U(d_t w)f_{t/j}(w)dw + c_{t,j}d_t^{-2}\\
=&E[U(x_{t/j})]E[f(\epsilon_j)] + c_{t.j}d_t^{-2},
\end{aligned}
$$

where $c_{t,j} := d_t\iint U(w)f(v)h_\epsilon(v)\left[f_{t/j}\left(\frac{w-b_{t,j}v}{d_t}\right)-f_{t/j}\left(\frac{w}{d_t}\right)\right]dvdw$ satisfies

$$
\begin{aligned}
|c_{t,j}| \le& d_t\iint |f(v)|h_\epsilon(v)|U(w)|\left|f_{t/j}\left(\frac{w-b_{t,j}v}{d_t}\right)-f_{t/j}\left(\frac{w}{d_t}\right)\right|dwdv\\
\le& C_1|b_{t,j}|\iint |f(v)|h_\epsilon(v)|U(w)||v|dwdv = C|b_{t,j}|\int |U(w)|dw,
\end{aligned}
$$

using the Lipschitz condition for $f_{t/j}$ in Lemma C.2. Clearly, if $Ef(\epsilon_j)=0$, $E[U(x_t)f(\epsilon_j)] = c_{t,j}d_t^{-2}$.

In addition, it follows that

$$
\begin{aligned}
&E|U(x_t)f(\epsilon_j)| = E|U(b_{t,j}\epsilon_j + x_{t/j})f(\epsilon_j)| = \iint |U(b_{t,j}v + d_t w)f(v)|h_\epsilon(v)f_{t/j}(w)dvdw\\
&= d_t^{-1}\iint |U(w)f(v)|h_\epsilon(v)f_{t/j}\left(\frac{w-b_{t,j}v}{d_t}\right)dvdw \le C d_t^{-1}\iint |U(w)f(v)|h_\epsilon(v)dvdw\\
&= C_1 d_t^{-1}\int |f(v)|h_\epsilon(v)dv\int |U(w)|dw = C d_t^{-1}\int |U(w)|dw.
\end{aligned}
$$

The assertion for the case where $f(\cdot)\equiv 1$ holds obviously.

(2) Similar to (C.1), we have

$$
x_t = b_{t,j}\epsilon_j + b_{t,\ell}\epsilon_\ell + x_{t/j\ell}, \quad \text{with } x_{t/j\ell} = \sum_{a=-\infty,\ne j,\ell}^{t} b_{t,a}\epsilon_a.
$$

Moreover, similar to Lemma C.2, we may show that $d_t^{-1}x_{t/j\ell}$ has a uniformly bounded density $f_{t/j\ell}(x)$ that satisfies the uniform Lipschitz condition. Recalling that ϵ_j has density $h_\epsilon(v)$,

$$
E[U(x_t)f(\epsilon_j)|\epsilon_\ell] = E[U(b_{t,j}\epsilon_j + b_{t,\ell}\epsilon_\ell + x_{t/j\ell})f(\epsilon_j)|\epsilon_\ell]
$$

$$
\begin{aligned}
=&\iint U(b_{t,j}v + b_{t,\ell}\epsilon_\ell + d_t w) f(v) h_\epsilon(v) f_{t/j\ell}(w) dv dw \\
=&d_t^{-1} \iint U(w) f(v) h_\epsilon(v) f_{t/j\ell}\left(\frac{w - b_{t,j}v - b_{t,\ell}\epsilon_\ell}{d_t}\right) dv dw \\
=&d_t^{-1} \iint U(w) f(v) h_\epsilon(v) f_{t/j\ell}\left(\frac{w - b_{t,\ell}\epsilon_\ell}{d_t}\right) dv dw \\
&+ d_t^{-1} \iint U(w) f(v) h_\epsilon(v) \left[f_{t/j\ell}\left(\frac{w - b_{t,j}v - b_{t,\ell}\epsilon_\ell}{d_t}\right) - f_{t/j\ell}\left(\frac{w - b_{t,\ell}\epsilon_\ell}{d_t}\right)\right] dv dw \\
:=&E[U(x_{t/j\ell} + b_{t,\ell}\epsilon_\ell)|\epsilon_\ell] E[f(\epsilon_j)] + d_t^{-2}\eta_{j\ell} = E[U(x_{t/j})|\epsilon_\ell] E[f(\epsilon_j)] + d_t^{-2}\eta_{j\ell},
\end{aligned}
$$

and using the Lipschitz condition

$$
|\eta_{j\ell}| \le C_1 \int |f(v)| h_\epsilon(v) \int |U(w)||b_{t,j}v| dw dv = C|b_{t,j}|, \quad a.s.
$$

When $E[f(\epsilon_j)] = 0$, we shall have $E[U(x_t)f(\epsilon_j)|\epsilon_\ell] = d_t^{-2}\eta_{j\ell}$. Additionally,

$$
\begin{aligned}
E[|U(x_t)f(\epsilon_j)||\epsilon_\ell] =& \iint |U(b_{t,j}v + b_{t,\ell}\epsilon_\ell + d_t w) f(v)| h_\epsilon(v) f_{t/j\ell}(w) dv dw \\
=&d_t^{-1} \iint |U(w) f(v) h_\epsilon(v)| f_{t/j\ell}\left(\frac{w - b_{t,j}v - b_{t,\ell}\epsilon_\ell}{d_t}\right) dv dw \\
\le& C_1 d_t^{-1} \iint |U(w) f(v) h_\epsilon(v)| dv dw = C d_t^{-1},
\end{aligned}
$$

almost surely.

(3) Recalling that $x_t = x_s^* + b_{t,j}\epsilon_j + x_{ts/j}$ and $d_{ts}^{-1}x_{ts/j}$ has a uniformly bounded density $f_{ts/j}(x)$ that satisfies the uniform Lipschitz condition,

$$
\begin{aligned}
E[U(x_t)f(\epsilon_j)|\mathcal{F}_s] =& E[U(x_s^* + b_{t,j}\epsilon_j + x_{ts/j}) f(\epsilon_j)|\mathcal{F}_s] \\
=&\iint U(x_s^* + b_{t,j}v + d_{ts}x) f(v) h_\epsilon(v) f_{ts/j}(x) dx dv \\
=&d_{ts}^{-1} \iint U(x) f(v) h_\epsilon(v) f_{ts/j}\left(\frac{x - b_{t,j}v - x_s^*}{d_{ts}}\right) dv dx \\
=&d_{ts}^{-1} \iint U(x) f(v) h_\epsilon(v) f_{ts/j}\left(\frac{x - x_s^*}{d_{ts}}\right) dv dx \\
&+ d_{ts}^{-1} \iint U(x) f(v) h_\epsilon(v) \left[f_{ts/j}\left(\frac{x - b_{t,j}v - x_s^*}{d_{ts}}\right) - f_{ts/j}\left(\frac{x - x_s^*}{d_{ts}}\right)\right] dv dx \\
=&d_{ts}^{-1} \int f(v) h_\epsilon(v) dv \int U(x) f_{ts/j}\left(\frac{x - x_s^*}{d_{ts}}\right) dv dx + d_{ts}^{-2}\xi_{sj} \\
=&E[f(\epsilon_j)] E[U(x_{ts/j} + x_s^*)|\mathcal{F}_s] + d_{ts}^{-2}\xi_{sj},
\end{aligned}
$$

where

$$
\xi_{sj} = d_{ts} \iint U(x) f(v) h_\epsilon(v) \left[f_{ts/j}\left(\frac{x - b_{t,j}v - x_s^*}{d_{ts}}\right) - f_{ts/j}\left(\frac{x - x_s^*}{d_{ts}}\right)\right] dv dx, \tag{C.4}
$$

and using the Lipschitz condition,

$$
|\xi_{sj}| \le C_1 \iint |U(x) f(v)| h_\epsilon(v) |b_{t,j}v| dv dx = C|b_{t,j}| \int |U(x)| dx \quad a.s..
$$

Consequently, when $E[f(\epsilon_j)]=0$, $E[U(x_t)f(\epsilon_j)|\mathcal{F}_s]=d_{ts}^{-2}\xi_s$. Meanwhile,

$$
\begin{aligned}
E[|U(x_t)f(\epsilon_j)||\mathcal{F}_s]=&\iint |U(x_s^*+b_{t,j}v+d_{ts}x)f(v)|h_\epsilon(v)f_{ts/j}(x)dxdv\\
=&d_{ts}^{-1}\iint |U(x)f(v)|h_\epsilon(v)f_{ts/j}\left(\frac{x-b_{t,j}v-x_s^*}{d_{ts}}\right)dvdx\\
\le&C_1d_{ts}^{-1}\iint |U(x)f(v)|h_\epsilon(v)dxdv=Cd_{ts}^{-1}\int|U(x)|dx.
\end{aligned}
$$

(4) These assertions can be shown similarly to (3), and is therefore omitted for brevity.

(5) By (1) we have $E[U(x_t)e_t]=\sum_{j=-\infty}^t\phi_{t-j}E[U(x_t)\epsilon_j]=d_t^{-2}\sum_{j=-\infty}^t\phi_{t-j}c_{t,j}:=d_t^{-2}q_t$ where $|c_{t,j}|\le C|b_{t,j}|$. Thus, $|E[U(x_t)e_t]|\le d_t^{-2}|q_t|\le Cd_t^{-2}$ by the summability of $\phi_{t-j}b_{t,j}$.

In addition, $E[U(x_t)e_t^2]=\sigma_e^2E[U(x_t)]+E[U(x_t)(e_t^2-\sigma_e^2)]$, where the first term is bounded by Cd_t^{-1} and the second term has a high order of d_t^{-2}. The assertion holds. □

If e_t is defined by A.3(ii) in Assumption A, recall that for $t>m_0$,

$$x_t=x_{t-m_0}^*+x_{t-m_0,t}. \tag{C.5}$$

where $x_{t-m_0,t}=\sum_{j=t-m_0+1}^t b_{t,j}\epsilon_j$. Then, if t is large, it follows similarly from Lemma C.2 that $d_{t-m_0}^{-1}x_{t-m_0}^*$ has a density $f_{t-m_0}(w)$ that is uniformly bounded and satisfies the uniform Lipschitz condition.

If $t-s>m_0$,

$$x_t=x_s^*+x'_{s+1,t-m_0}+x_{t-m_0,t}, \tag{C.6}$$

where x_s^* is defined as before containing all the information up to s, $x'_{s+1,t-m_0}$ is defined similarly containing all information on $[s+1,t-m_0]$, and $x_{t-m_0,t}$ is the same as before. By virtue of the structure of $x'_{s+1,t-m_0}$, $d_{t-s-m_0}^{-1}x'_{s+1,t-m_0}$ has a density $f_{s+1,t-m_0}(w)$ uniformly bounded and satisfying the uniform Lipschitz condition, as $t-s$ is large.

Lemma C.4. *Let e_t be defined by A.3(ii) in Assumption A. Let $U(\cdot)$ be an integrable function on $\mathbb{R}$, i.e. $\int|U(w)|dw<\infty$, $f(\cdot)$ be a function such that $E|b(\epsilon_t,\cdots,\epsilon_{t-m_0+1})f(e_t)|<\infty$, where $b(\epsilon_t,\cdots,\epsilon_{t-m_0+1}):=x_{t-m_0,t}$. Let C be a generic constant.*

(1) For t large, $E[U(x_t)f(e_t)]=E[U(x_{t-m_0}^*)]E[f(e_t)]+d_{t-m_0}^{-2}c_U$, where $x_{t-m_0}^*$ is given by (C.5) and c_U satisfies $|c_U|\le C\int|U(w)|dwE|b(\epsilon_t\cdots\epsilon_{t-m_0+1})f(e_t)|$ almost surely. In particular, if $E[f(e_t)]=0$, $E[U(x_t)f(e_t)]=d_{t-m_0}^{-2}c_U$; $|E[U(x_t)f(e_t)]|\le Cd_{t-m_0}^{-1}E|f(e_t)|\int|U(w)|dw$, where C is an absolute constant.

(2) For $t-s$ large, $E[U(x_t)f(e_t)|\mathcal{F}_s]=E[U(x_s^*+x'_{s+1,t-m_0})|\mathcal{F}_s]E[f(e_t)]+d_{t-s-m_0}^{-2}\xi_s$ with $|\xi_s|\le CE|b(\epsilon_t\cdots\epsilon_{t-m_0+1})f(e_t)|\int|U(w)|dw$ almost surely, where x_s^* and $x'_{s+1,t-m_0}$ are defined by (C.6). If $E[f(e_t)]=0$, $E[U(x_t)f(e_t)|\mathcal{F}_s]=d_{t-s-m_0}^{-2}\xi_s$. Otherwise, we have $|E[U(x_t)f(e_t)|\mathcal{F}_s]|\le Cd_{t-s-m_0}^{-2}\int|U(w)|dwE|f(e_t)|$.

(3) For $0<t-s\le m_0$, large s and any function $g(\cdot)$ that satisfies the same condition as for $f(\cdot)$, $|E[U(x_s)f(e_t)g(e_s)]|\le Cd_{s-m_0}^{-1}\int|U(x)|dx\,E|f(e_t)g(e_s)|$.

Proof. (1). Suppose that ϵ_0 and η_0 have densities $h_\epsilon(x)$ and $h_\eta(x)$, respectively. Then,

$$\begin{aligned}
E[U(x_t)f(e_t)] =& E[U(x^*_{t-m_0} + b(\epsilon_t, \cdots, \epsilon_{t-m_0+1}))f(\wp(\epsilon_t, \cdots, \epsilon_{t-m_0+1}; \eta_t, \cdots \eta_{t-m_1+1}))] \\
=& \int \cdots \int U(d_{t-m_0}w + b(v_1, \cdots, v_{m_0}))f(\wp(v_1, \cdots, v_{m_0}; w_1, \cdots, w_{m_1})) \\
& \times f_{t-m_0}(w) \prod_{i=1}^{m_0} h_\epsilon(v_i) \prod_{i=1}^{m_1} h_\eta(w_i) dw dv_1 \cdots dv_{m_0} dw_1 \cdots dw_{m_1} \\
=& d^{-1}_{t-m_0} \int \cdots \int U(w) f(\wp(v_1, \cdots, v_{m_0}; w_1, \cdots, w_{m_1})) \\
& \times f_{t-m_0}\left(\frac{w - b(v_1, \cdots, v_{m_0})}{d_{t-m_0}}\right) \prod_{i=1}^{m_0} h_\epsilon(v_i) \prod_{i=1}^{m_1} h_\eta(w_i) dw dv_1 \cdots dw_{m_1} \\
=& d^{-1}_{t-m_0} \int \cdots \int U(w) f(\wp(v_1, \cdots, v_{m_0}; w_1, \cdots, w_{m_1})) \\
& \times f_{t-m_0}\left(\frac{w}{d_{t-m_0}}\right) \prod_{i=1}^{m_0} h_\epsilon(v_i) \prod_{i=1}^{m_1} h_\eta(w_i) dw dv_1 \cdots dw_{m_1} \\
& + d^{-1}_{t-m_0} \int \cdots \int U(w) f(\wp(v_1, \cdots, v_{m_0}; w_1, \cdots, w_{m_1})) \prod_{i=1}^{m_0} h_\epsilon(v_i) \prod_{i=1}^{m_1} h_\eta(w_i) \\
& \times \left[f_{t-m_0}\left(\frac{w - b(v_1, \cdots, v_{m_0})}{d_{t-m_0}}\right) - f_{t-m_0}\left(\frac{w}{d_{t-m_0}}\right)\right] dw dv_1 \cdots dw_{m_1} \\
:=& E[U(x^*_{t-m_0})]E[f(e_t)] + d^{-2}_{t-m_0} c_U.
\end{aligned}$$

Thus, using the Lipschitz condition for $f_{t-m_0}(\cdot)$ yields

$$|c_U| \le C \int |U(w)| dw E|b(\epsilon_t \cdots \epsilon_{t-m_0+1}) f(e_t)|.$$

In particular, if $Ef(e_t) = 0$, $E[U(x_t)f(e_t)] = d^{-2}_{t-m_0} c_U$. Meanwhile, we may derive from the above that $|E[U(x_t)f(e_t)]| \le d^{-1}_{t-m_0} E|f(e_t)| \int |U(w)| dw$.

(2). Similarly, we have

$$\begin{aligned}
& E[U(x_t)f(e_t)|\mathcal{F}_s] = E[U(x^*_s + x'_{s+1,t-m_0} + b(\epsilon_t, \cdots, \epsilon_{t-m_0+1}))f(e_t)|\mathcal{F}_s] \\
=& \int \cdots \int U(x^*_s + d_{t-s-m_0} w + b(v_1, \cdots, v_{m_0})) f(\wp(v_1, \cdots, v_{m_0}; w_1, \cdots, w_{m_1})) \\
& \times f_{s+1,t-m_0}(w) \prod_{i=1}^{m_0} h_\epsilon(v_i) \prod_{i=1}^{m_1} h_\eta(w_i) dw dv_1 \cdots dw_{m_1} \\
=& d^{-1}_{t-s-m_0} \int \cdots \int U(w) f(\wp(v_1, \cdots, v_{m_0}; w_1, \cdots, w_{m_1})) \prod_{i=1}^{m_0} h_\epsilon(v_i) \prod_{i=1}^{m_1} h_\eta(w_i) \\
& \times f_{s+1,t-m_0}\left(\frac{w - x^*_s - b(v_1, \cdots, v_{t-s-m_0})}{d_{t-s-m_0}}\right) dw dv_1 \cdots dw_{m_1} \\
=& d^{-1}_{t-s-m_0} \int \cdots \int U(w) f(\wp(v_1, \cdots, v_{m_0}; w_1, \cdots, w_{m_1})) \prod_{i=1}^{m_0} h_\epsilon(v_i) \prod_{i=1}^{m_1} h_\eta(w_i) \\
& \times f_{s+1,t-m_0}\left(\frac{w - x^*_s}{d_{t-s-m_0}}\right) dw dv_1 \cdots dw_{m_1} \\
& + d^{-1}_{t-s-m_0} \int \cdots \int U(w) f(\wp(v_1, \cdots, v_{m_0}; w_1, \cdots, w_{m_1})) \prod_{i=1}^{m_0} h_\epsilon(v_i) \prod_{i=1}^{m_1} h_\eta(w_i)
\end{aligned}$$

$$\times\left[f_{s+1,t-m_0}\left(\frac{w-x_s^*-b(v_1,\cdots,v_{m_0})}{d_{t-s-m_0}}\right)-f_{s+1,t-m_0}\left(\frac{w-x_s^*}{d_{t-s-m_0}}\right)\right]dwdv_1\cdots dw_{m_1}$$
$$=E[U(x_s^*+x'_{s+1,t-m_0})|\mathcal{F}_s]E[f(e_t)]+d_{t-s-m_0}^{-2}\xi_s,$$

where using the Lipschitz condition for $f_{s+1,t-m_0}(\cdot)$ gives

$$|\xi_s|\le CE|b(\epsilon_t\cdots\epsilon_{t-m_0+1})f(e_t)|\int|U(w)|dw,\quad a.s.,$$

in which C stands for the constant involved in the Lipschitz condition.

Clearly, if $E[f(e_t)]=0$, $E[U(x_t)f(e_t)|\mathcal{F}_s]=d_{t-s-m_0}^{-2}\xi_s$ almost surely, while $E[f(e_t)]\neq 0$, $|E[U(x_t)f(e_t)|\mathcal{F}_s]|\le Cd_{t-s-m_0}^{-1}\int|U(w)|dw\,E|f(e_t)|$.

(3) Note that $m_0<s<t<s+m_0$, implying $1\le s-m_0<t-m_0<s$. Note also that $x_s=x_{s-m_0}^*+b(\epsilon_s,\cdots,\epsilon_{s-m_0+1})$, and $d_{s-m_0}^{-1}x_{s-m_0}^*$ has a density $f_{s-m_0}(x)$. Define $\mathcal{G}_{s-m_0}^t=\sigma(\epsilon_t,\cdots,\epsilon_s,\cdots,\epsilon_{s-m_0};\eta_j,\forall j)$, so that both e_t and e_s are adapted to $\mathcal{G}_{s-m_0}^t$. Then

$$|EU(x_s)f(e_t)g(e_s)|=|E[E(U(x_{s-m_0}^*+b(\epsilon_s,\cdots,\epsilon_{s-m_0+1}))f(e_t)g(e_s)|\mathcal{G}_{s-m_0}^t)]|$$
$$=\left|E\int U(d_{s-m_0}w+b(\epsilon_s,\cdots,\epsilon_{s-m_0+1}))f(e_t)g(e_s)f_{s-m_0}(w)dw\right|$$
$$=d_{s-m_0}^{-1}\left|E\int U(w)f(e_t)g(e_s)f_{s-m_0}\left(\frac{w-b(\epsilon_s,\cdots,\epsilon_{s-m_0+1})}{d_{s-m_0}}\right)dw\right|$$
$$\le Cd_{s-m_0}^{-1}\int|U(w)|dwE|f(e_t)g(e_s)|.$$

□

Let the function sequence $\{\mathscr{T}_k(x)\}$ and the function $\mathscr{T}(x)$ be defined as follows:

$$\mathscr{T}_k(x):=\frac{1}{k}\|Z(x)\|^2=\frac{1}{k}\sum_{i=0}^{k-1}\mathscr{H}_i^2(x)\quad\text{and}\quad\mathscr{T}(x)=\begin{cases}\frac{1}{2\pi}\sqrt{4-x^2}, & \text{if } |x|\le 2,\\ 0, & \text{if } |x|>2.\end{cases}$$

Lemma C.5. *(1) The sequence $\mathscr{T}_k(x)$ converges to $\mathscr{T}(x)$ for any $x\in\mathbb{R}$ as $k\to\infty$. Moreover, as $k\to\infty$*

$$\int_{-\infty}^{\infty}|\mathscr{T}_k(x)-\mathscr{T}(x)|\,dx\to 0. \tag{C.7}$$

(2) Furthermore, $Z'(x)Z(y)=\sum_{i=0}^{k-1}\mathscr{H}_i(x)\mathscr{H}_i(y)\to\delta(x-y)$ as $k\to\infty$, where $\delta(u)$ is the Deric delta function.

Proof. (1) The convergence of $\mathscr{T}_k(x)\to\mathscr{T}(x)$ is given in Edelman and Rao (2005, p. 29), and Tao (2012, Exercise 2.6.6, Section 2.6) shows the convergence in vague topology.

Since $\mathscr{T}_k(x)$ for each $k\ge 1$ can be viewed as a probability density and $\mathscr{T}(x)$ is actually the so-called Wigner semicircle law with radius 2, namely, a probability density as well, it follows from Scheffe's Theorem of Billingsley (1968, p. 224) that (C.7) holds.

(2) The Deric delta function $\delta(u)$ is a generalized function satisfying $\delta(u)=0$ for any $u\neq 0$ and $\int\delta(u)du=1$. See Kanwal (1983, p. 5).

For any $f(x) \in L^2(R)$ sufficiently smooth, we have

$$\lim_{k\to\infty}\int f(x)[Z'(x)Z(y)]dx = \lim_{k\to\infty}\int f(x)\sum_{i=0}^{k-1}\mathscr{H}_i(x)\mathscr{H}_i(y)dx$$
$$= \lim_{k\to\infty}\sum_{i=0}^{k-1}\int f(x)\mathscr{H}_i(x)dx\ \mathscr{H}_i(y) = \lim_{k\to\infty}\sum_{i=0}^{k-1}c_i\mathscr{H}_i(y) = f(y)$$

due to the convergence of the expansion of f, where $c_i = \int f(x)\mathscr{H}_i(x)dx$ are the coefficients in the orthogonal series expansion of sufficiently smooth function f. Hence, $Z'(x)Z(y)$ is a delta sequence that converges as defined in Kanwal (1983, p. 14). Then, the assertion holds by the definition of the delta sequence. □

Justification of Assumption B. This is to justify that under the null, $\|\widehat{\theta}-\theta\| = O_P(\zeta_n)$ with $\zeta_n(n/d_n)^{1/2} = O(1)$ is achievable in the setting of this paper with $\delta_0 = 1$ (x_t is a unit root process). Consider an estimator of a scalar parameter θ from a regression model of the form:

$$y_t = \theta g(x_t) + e_t, \quad t = 1, \cdots, n,$$

where x_t and e_t satisfy Assumption A, and function $g(\cdot)$ is integrable. Then, we will show

$$(n/d_n)^{1/2}(\widehat{\theta}-\theta) = \frac{(d_n/n)^{1/2}\sum_{t=1}^n g(x_t)e_t}{(d_n/n)\sum_{t=1}^n g^2(x_t)} = O_P(1).$$

For the denominator, by Theorem 2.1 of Wang and Phillips (2009), we have $(d_n/n)\sum_{t=1}^n g^2(x_t) \to_P \int g^2(x)dx\, L_W(1,0)$, where $L_W(1,0)$ is the local time variable. For the nominator, we split

$$(d_n/n)^{1/2}\sum_{t=1}^n g(x_t)e_t = (d_n/n)^{1/2}\sum_{t=1}^{m_n} g(x_t)e_t + (d_n/n)^{1/2}\sum_{t=m_n+1}^{n} g(x_t)e_t$$

with integer m_n. Due to integrability of g function, $E|(d_n/n)^{1/2}\sum_{t=1}^{m_n} g(x_t)e_t| = o(1)$ if $m_n \to \infty$ and $m_n(d_n/n)^{1/2} \to 0$. In what follows we only consider large $t \geq m_n$. If e_t is a linear process stipulated in A.3(i) of Assumption A, using (2), (3) and (5) in Lemma C.3,

$$\begin{aligned}
\frac{d_n}{n}E\left(\sum_{t=m_n}^n g(x_t)e_t\right)^2 &= \frac{d_n}{n}\sum_{t=m_n}^n E[g^2(x_t)e_t^2] + 2\frac{d_n}{n}\sum_{t=m_n+1}^n\sum_{s=m_n}^{t-1}E[g(x_t)e_tg(x_s)e_s] \\
\leq & C\frac{d_n}{n}\sum_{t=m_n}^n\frac{1}{d_t}\int g^2(x)dx + 2\frac{d_n}{n}\sum_{t=m_n+1}^n\sum_{s=m_n}^{t-1}\sum_{j=s+1}^{t}\phi_{t-j}E[E(g(x_t)\epsilon_j|\mathcal{F}_s)g(x_s)e_s] \\
&+ 2\frac{d_n}{n}\sum_{t=m_n+1}^n\sum_{s=m_n}^{t-1}\sum_{j=-\infty}^{s}\phi_{t-j}E[E(g(x_t)|\mathcal{F}_s)\epsilon_jg(x_s)e_s] \\
\leq & C + 2\frac{d_n}{n}\sum_{t=m_n+1}^n\sum_{s=m_n}^{t-1}\sum_{j=s+1}^{t}\phi_{t-j}d_{ts}^{-2}E[\xi_sg(x_s)e_s] \\
&+ 2\frac{d_n}{n}\sum_{t=m_n+1}^n\sum_{s=m_n}^{t-1}\sum_{j=-\infty}^{s}|\phi_{t-j}|\frac{1}{d_{ts}}E|\epsilon_jg(x_s)e_s|\int|g(x)|dx = C + 2T_1 + 2T_2, \quad \text{say.}
\end{aligned}$$

Note that ξ_s is given by (C.4) with replacement U by g and $f(v) = v$, that is,

$$\xi_s = d_{ts} \iint g(x) v h_\epsilon(v) \left[f_{ts}\left(\frac{x - b_{t,j}v - x_s^*}{d_{ts}}\right) - f_{ts}\left(\frac{x - x_s^*}{d_{ts}}\right)\right] dv dx,$$

implying $|\xi_s| \le C|b_{t,j}| \int |g(x)|dx$ almost surely by the Lipschitz condition on f_{ts}.

To tackle $E[\xi_s g(x_s)e_s]$ in T_1, noting $x_s^* = x_s + \bar{x}_s$, it follows from the fact $s \ge m_n$ and Lemma 2 in Renyi (1958, p. 223) that $\frac{1}{d_s}x_s$ and $\frac{1}{d_s}x_s^*$ have the same density $f_s(x)$ asymptotically. Because of this, in the calculation of $E[\xi_s g(x_s)e_s]$ we treat x_s in the same way as for x_s^*, and emphasize this by denoting $\xi_s = \xi(x_s)$. Hence, by (5) of Lemma C.3,

$$\begin{aligned}
|T_1| =& \frac{d_n}{n} \left| \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-1} d_{ts}^{-2} \sum_{j=s+1}^{t} \phi_{t-j} E[\xi(x_s)g(x_s)e_s] \right| \\
\le& \frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-1} \sum_{j=s+1}^{t} |\phi_{t-j}| d_{ts}^{-2} d_s^{-2} |c_g| \\
\le& C\frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-1} d_{ts}^{-2} d_s^{-2} \le C\frac{d_n}{n} \ln^2(n) = o(1),
\end{aligned}$$

where c_g is such that $|c_g| \le O(1) \int |\xi(x)g(x)|dx \le O(1)|b_{t,j}| \left(\int |g(x)|dx\right)^2$ and we have used the summability of $\phi_{t-j}b_{t,j}$. Meanwhile, in view of $\phi_j = O(1)j^{-\gamma_0}$ stipulated in Assumption A,

$$\begin{aligned}
|T_2| =& \frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-1} \sum_{j=-\infty}^{s} |\phi_{t-j}| \frac{1}{d_{ts}} E|\epsilon_j g(x_s)e_s| \int |g(x)|dx \\
\le& C\frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-1} (t-s)^{-\gamma_0+1} \frac{1}{d_{ts}} \frac{1}{d_s} \le Cn^{-\gamma_0+3/2} \ln(n) = o(1),
\end{aligned}$$

due to $\gamma_0 > 3/2$, where $E|\epsilon_j g(x_s)e_s| \le \sum_{\ell=-\infty}^{s} |\phi_{s-\ell}| E|g(x_s)\epsilon_j \epsilon_\ell| \le C\frac{1}{d_s}$ by (1) and (2) of Lemma C.3.

If e_t has the functional form in A.3(ii) of Assumption A, by Lemma C.4,

$$\begin{aligned}
&\frac{d_n}{n} E\left(\sum_{t=m_n}^{n} g(x_t)e_t\right)^2 = \frac{d_n}{n} \sum_{t=m_n}^{n} E[g^2(x_t)e_t^2] + 2\frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-1} E[g(x_t)e_t g(x_s)e_s] \\
\le& \frac{d_n}{n} \sum_{t=m_n}^{n} \frac{1}{d_{t-m_0}} E[e_t^2] \int g^2(x)dx + 2\frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=t-m_0}^{t-1} E[g(x_t)e_t g(x_s)e_s] \\
&+ 2\frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_0-1} E[E(g(x_t)e_t|\mathcal{F}_s)g(x_s)e_s] \\
\le& C + \frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=t-m_0}^{t-1} E[|e_t g(x_s)e_s|] + 2\frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_0-1} d_{t-s}^{-2} E[\xi_s g(x_s)e_s] \\
\le& C + C_1\frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=t-m_0}^{t-1} \frac{1}{d_s} + C_2\frac{d_n}{n} \sum_{t=m_n+1}^{n} \sum_{s=m_n}^{t-m_0-1} d_{t-s}^{-2} d_s^{-2} \\
\le& C + C_1\frac{d_n}{n} \sum_{t=m_n+1}^{n} \frac{1}{d_{t-m_0}} + C_2\frac{d_n}{n} \ln^2(n) = C,
\end{aligned}$$

where we have used the same argument as in the first part to deal with $E[\xi_s g(x_s)e_s]$, the boundedness of g function and $E[|e_t g(x_s)e_s|] \le O(1)\frac{1}{d_s}$ by (3) of Lemma C.4.

Appendix D: Complete Proofs of the Theorems

Detailed Proof of Theorem 3.1. This proof is mainly for the case where e_t is a linear process stipulated by A.3(i) in Assumption A. Due to Lemma C.4, the proof for the case where e_t takes a functional form stipulated by A.3(ii) in Assumption A can be shown similarly. Under the hypothesis H_0, $y_t = g(x_t;\theta_0) + e_t$ for all $t = 1, \cdots, n$. Hence, it is easy to rewrite $L_n = L_{1n} + L_{2n} + L_{3n}$, where

$$L_{1n} = \sum_{t=1}^{n}\sum_{s=1}^{n} Z(x_t)'Z(x_s)e_t e_s, \tag{D.8}$$

$$L_{2n} = 2\sum_{t=1}^{n}\sum_{s=1}^{n} Z(x_t)'Z(x_s)e_t\widehat{g}(x_s), \tag{D.9}$$

$$L_{3n} = \sum_{t=1}^{n}\sum_{s=1}^{n} Z(x_t)'Z(x_s)\widehat{g}(x_t)\widehat{g}(x_s), \tag{D.10}$$

and $\widehat{g}(x) := g(x;\theta_0) - g(x;\widehat{\theta})$ for any real x.

Note that $L_{1n} = \|\sum_{t=1}^{n} Z(x_t)e_t\|^2$ and $L_{3n} = \|\sum_{t=1}^{n} Z(x_t)\widehat{g}(x_t)\|^2$, while L_{2n} is the inner product of these vectors in L_{1n} and L_{3n}, multiplied by 2. Thus, both L_{1n} and L_{3n} are nonnegative, and the Cauchy-Schwarz inequality entails $|L_{2n}| \le 2\sqrt{L_{1n}L_{3n}}$.

We first deal with L_{1n}. Observe that

$$\begin{aligned} L_{1n} =& \sum_{t=1}^{n}\sum_{s=1}^{n} Z(x_t)'Z(x_s)e_s e_t = \sum_{t=1}^{n} \|Z(x_t)\|^2 e_t^2 + 2\sum_{t=2}^{n}\sum_{s=1}^{t-1} Z(x_t)'Z(x_s)e_s e_t \\ =& \sigma_e^2 \sum_{t=1}^{n} \|Z(x_t)\|^2 + \sum_{t=1}^{n} \|Z(x_t)\|^2(e_t^2 - \sigma_e^2) + 2\sum_{t=2}^{n}\sum_{s=1}^{t-1} Z(x_t)'Z(x_s)e_s e_t \\ =& \sigma_e^2 L_{1n}' + L_{1n}'' + 2L_{1n}''', \end{aligned}$$

where $\sigma_e^2 = Ee_t^2$. We then have

$$\begin{aligned} \frac{d_n}{nk}L_{1n}' =& \frac{d_n}{nk}\sum_{t=1}^{n} \|Z(x_t)\|^2 = \frac{d_n}{n}\sum_{t=1}^{n} \frac{1}{k}\|Z(x_t)\|^2 = \frac{d_n}{n}\sum_{t=1}^{n} \mathscr{T}_k(x_t) \\ =& \frac{d_n}{n}\sum_{t=1}^{n} \mathscr{T}(x_t) + \frac{d_n}{n}\sum_{t=1}^{n} [\mathscr{T}_k(x_t) - \mathscr{T}(x_t)]. \end{aligned}$$

It follows from Wang and Phillips (2009) that, since $\int \mathscr{T}(x)dx = 1$,

$$\frac{d_n}{n}\sum_{t=1}^{n} \mathscr{T}(x_t) = \frac{d_n}{n}\sum_{t=1}^{n} \mathscr{T}(d_n x_{nt}) \to_D L_\xi(1, 0).$$

Additionally, $\frac{d_n}{n}\sum_{t=1}^{n}[\mathscr{T}_k(x_t) - \mathscr{T}(x_t)] \to_P 0$. In fact, noting that x_t/d_t has density function $f_t(x)$ that has a uniform bound, say C, over all x and t, by Lemma C.5 we have

$$E\left|\frac{d_n}{n}\sum_{t=1}^{n}[\mathscr{T}_k(x_t) - \mathscr{T}(x_t)]\right| \le \frac{d_n}{n}\sum_{t=1}^{n} E\left|\mathscr{T}_k(x_t) - \mathscr{T}(x_t)\right|$$

$$=\frac{d_n}{n}\sum_{t=1}^{n}\int|\mathscr{T}_k(d_tx)-\mathscr{T}(d_tx)|\ f_t(x)dx=\frac{d_n}{n}\sum_{t=1}^{n}\frac{1}{d_t}\int|\mathscr{T}_k(x)-\mathscr{T}(x)|\ f_t(x/d_t)dx$$
$$\leq C\frac{d_n}{n}\sum_{t=1}^{n}\frac{1}{d_t}\int|\mathscr{T}_k(x)-\mathscr{T}(x)|\,dx\leq C\int|\mathscr{T}_k(x)-\mathscr{T}(x)|\,dx\to 0.$$

Hence, $\frac{d_n}{nk}L'_{1n}\to_D L_\xi(1,0)$. To complete the proof, it suffices to show that L''_{1n}, L'''_{1n}, L_{2n} and L_{3n} after normalisation are all $o_P(1)$. We shall tackle them in the following three steps.

For convenience, let m_n be a positive integer sequence such that $m_n^4/n\to 0$ and $m_n\to\infty$ as $n\to\infty$.

Step I. For $\frac{d_n}{nk}L''_{1n}$, split the sum into two parts, $t\leq m_n$ and $t\geq m_n+1$. The first part is $o_P(1)$ due to $\|Z(x)\|^2\leq O(1)k$ uniformly in x and the stationarity of e_t. We thus only consider the second part in what follows. Note that

$$E\left(\frac{d_n}{nk}\sum_{t=m_n+1}^{n}\|Z(x_t)\|^2(e_t^2-\sigma_e^2)\right)^2$$
$$=E\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\|Z(x_t)\|^4(e_t^2-\sigma_e^2)^2$$
$$+2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2(e_t^2-\sigma_e^2)\|Z(x_s)\|^2(e_s^2-\sigma_e^2)$$
$$=\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}E[\|Z(x_t)\|^4e_t^4]-2\sigma_e^2\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}E[\|Z(x_t)\|^4e_t^2]+\sigma_e^4\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}E\|Z(x_t)\|^4$$
$$+2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2(e_t^2-\sigma_e^2)\|Z(x_s)\|^2(e_s^2-\sigma_e^2).$$

To begin with the easiest one, noting that $\|Z(x)\|^2\leq O(1)k$ uniformly in x, x_t/d_t has a uniformly bounded density $f_t(x)$ and $\int\|Z(x)\|^2dx=k$ by orthogonality,

$$\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}E\|Z(x_t)\|^4\leq\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}E\|Z(x_t)\|^2$$
$$=\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}\int\|Z(d_tx)\|^2f_t(x)dx=\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}\frac{1}{d_t}\int\|Z(x)\|^2f_t\left(\frac{x}{d_t}\right)dx$$
$$\leq C\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}\frac{1}{d_t}\int\|Z(x)\|^2dx=C\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\frac{1}{d_t}$$
$$=C\frac{d_n}{n}=o(1).$$

Moreover, by (5) of Lemma C.3,

$$\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}E[\|Z(x_t)\|^4e_t^2]\leq C\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}E[\|Z(x_t)\|^2e_t^2]$$
$$=C\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}\frac{1}{d_t}\int\|Z(x)\|^2dx=C\frac{d_n}{n}=o(1).$$

Next, to compute $E[\|Z(x_t)\|^4 e_t^4]$ in the first term, notice that

$$
\begin{aligned}
e_t^4 = \left(\sum_{j=-\infty}^{t} \phi_{t-j}\epsilon_j\right)^4 &= \sum_{j_1,j_2,j_3,j_4=-\infty}^{t} \phi_{t-j_1}\phi_{t-j_2}\phi_{t-j_3}\phi_{t-j_4}\epsilon_{j_1}\epsilon_{j_2}\epsilon_{j_3}\epsilon_{j_4} \\
&= \sum_{j=-\infty}^{t} \phi_{t-j}^4\epsilon_j^4 + 6\sum_{j=-\infty}^{t}\sum_{\ell=-\infty,\neq j}^{t} \phi_{t-\ell}^2\epsilon_j^2\phi_{t-j}^2\epsilon_\ell^2 + 4\sum_{j=-\infty}^{t}\sum_{\ell=-\infty,\neq j}^{t} \phi_{t-j}\epsilon_j\phi_{t-\ell}^3\epsilon_\ell^3 \\
&\quad + \sum_{\substack{j_1,j_2,j_3,j_4=-\infty \\ j_1\neq j_2\neq j_3\neq j_4}}^{t} \phi_{t-j_1}\phi_{t-j_2}\phi_{t-j_3}\phi_{t-j_4}\epsilon_{j_1}\epsilon_{j_2}\epsilon_{j_3}\epsilon_{j_4}.
\end{aligned}
$$

Hence, by (1) of Lemma C.3, $E[\|Z(x_t)\|^4\epsilon_j^4] \leq O(1)E[\epsilon_j^4]\frac{1}{d_t}\int\|Z(x)\|^4dx = O(1)\frac{1}{d_t}k^2$, by (2) of Lemma C.3, $E[\|Z(x_t)\|^4\epsilon_j^2\epsilon_j^2] \leq O(1)\frac{1}{d_t}k^2$, $|E[\|Z(x_t)\|^4\epsilon_j\epsilon_\ell^3]| \leq O(1)\frac{1}{d_t}k^2$, and by a similar derivation, $|E[\|Z(x_t)\|^4\epsilon_{j_1}\epsilon_{j_2}\epsilon_{j_3}\epsilon_{j_4}]| \leq O(1)\frac{1}{d_t}k^2$. It follows that

$$
\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n} E[\|Z(x_t)\|^4e_t^4] \leq O(1)\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\frac{1}{d_t}k^2 = o(1).
$$

Since $e_t^2 - \sigma_e^2 = \sum_{j=0}^{\infty}\phi_j^2(\epsilon_{t-j}^2 - 1) + \sum_{j=0}^{\infty}\sum_{j_1=0,\neq j}^{\infty}\phi_j\phi_{j_1}\epsilon_{t-j}\epsilon_{t-j_1} = \sum_{j=-\infty}^{t}\phi_{t-j}^2(\epsilon_j^2-1) + 2\sum_{j=-\infty}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}$,

$$
\begin{aligned}
&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2(e_t^2-\sigma_e^2)\|Z(x_s)\|^2(e_s^2-\sigma_e^2) \\
=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\left(\sum_{j=-\infty}^{t}\phi_{t-j}^2(\epsilon_j^2-1) + 2\sum_{j=-\infty}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}\right) \\
&\times\left(\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2(\epsilon_\ell^2-1) + 2\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1}\right) \\
=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=-\infty}^{t}\phi_{t-j}^2(\epsilon_j^2-1)\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2(\epsilon_\ell^2-1) \\
&+ 2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=-\infty}^{t}\phi_{t-j}^2(\epsilon_j^2-1)\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1} \\
&+ 2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2(\epsilon_\ell^2-1)\sum_{j=-\infty}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1} \\
&+ 4\frac{d_n^2}{n^2k^2}E\left[\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=-\infty}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}\right. \\
&\quad\left.\times\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1}\right].
\end{aligned}
$$

We shall deal with each term in the following four parts.

(1a) The first part.

$$
\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{t}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\phi_{t-j}^2E\|Z(x_t)\|^2\|Z(x_s)\|^2(\epsilon_j^2-1)(\epsilon_\ell^2-1)
$$

$$
\begin{aligned}
=&\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=s+1}^{t}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\phi_{t-j}^2E\|Z(x_t)\|^2\|Z(x_s)\|^2(\epsilon_j^2-1)(\epsilon_\ell^2-1)\\
&+\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\phi_{t-j}^2E\|Z(x_t)\|^2\|Z(x_s)\|^2(\epsilon_j^2-1)(\epsilon_\ell^2-1)\\
=&\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=s+1}^{t}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\phi_{t-j}^2E\|Z(x_t)\|^2\|Z(x_s)\|^2(\epsilon_j^2-1)(\epsilon_\ell^2-1)\\
&+\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\phi_{t-j}^2\phi_{s-j}^2E\|Z(x_t)\|^2\|Z(x_s)\|^2(\epsilon_j^2-1)^2\\
&+\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\sum_{\ell=-\infty,\neq j}^{s}\phi_{t-j}^2\phi_{s-\ell}^2E\|Z(x_t)\|^2\|Z(x_s)\|^2(\epsilon_j^2-1)(\epsilon_\ell^2-1)\\
:=&I_1+I_2+I_3.
\end{aligned}
$$

For I_1, note by (3) of Lemma C.3 that, $E[\|Z(x_t)\|^2(\epsilon_j^2-1)|\mathcal{F}_s]=d_{ts}^{-2}\xi_s$, where a.s.

$$
|\xi_s|\le O(1)|b_{t,j}|\int\|Z(x)\|^2dx=O(1)|b_{t,j}|k.
$$

Therefore, we have

$$
\begin{aligned}
|I_1|=&\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=s+1}^{t}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\phi_{t-j}^2E\|Z(x_t)\|^2\|Z(x_s)\|^2(\epsilon_j^2-1)(\epsilon_\ell^2-1)\right|\\
\le&O(1)\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=s+1}^{t}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\phi_{t-j}^2d_{ts}^{-2}E[\xi_s\|Z(x_s)\|^2(\epsilon_\ell^2-1)]\right|\\
\le&O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=s+1}^{t}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\phi_{t-j}^2d_{ts}^{-2}|b_{t,j}|E[\|Z(x_s)\|^2|\epsilon_\ell^2-1|]\\
\le&O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=s+1}^{t}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\phi_{t-j}^2d_{ts}^{-2}|b_{t,j}|d_s^{-1}\int\|Z(x)\|^2dx\\
=&O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}d_s^{-1}=O(1)n^{-(\delta_0-1/2)}=o(1),
\end{aligned}
$$

where we have used the fact that $\phi_{t-j}^2|b_{t,j}|$ is summable.

For I_2, observe by (1) of Lemma C.3 that

$$
\begin{aligned}
I_2=&\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\phi_{t-j}^2\phi_{s-j}^2E\|Z(x_t)\|^2\|Z(x_s)\|^2(\epsilon_j^2-1)^2\\
\le&O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\phi_{t-j}^2\phi_{s-j}^2E\|Z(x_t)\|^2(\epsilon_j^2-1)^2\\
\le&O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\phi_{t-j}^2\phi_{s-j}^2\frac{1}{d_t}\int\|Z(x)\|^2dx\\
\le&O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\frac{1}{d_t}\sum_{s=m_n+1}^{t-1}\phi_{t-s}^2=O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\frac{1}{d_t}=o(1)
\end{aligned}
$$

using the summability of ϕ_j in Assumption A.

As for I_3, notice by (2) of Lemma C.3 that

$$\begin{aligned}
|I_3| \leq & O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\sum_{\ell=-\infty,\neq j}^{s}\phi_{t-j}^2\phi_{s-\ell}^2E\|Z(x_t)\|^2|(\epsilon_j^2-1)(\epsilon_\ell^2-1)| \\
\leq & O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\phi_{t-j}^2\frac{1}{d_t}\int\|Z(x)\|^2dx \\
= & O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\frac{1}{d_t}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}(t-j)^{-2\gamma_0} \\
= & O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\frac{1}{d_t}\sum_{s=m_n+1}^{t-1}(t-s)^{-2\gamma_0+1} \\
= & O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\frac{1}{d_t}(1-(t-m_n)^{-2\gamma_0+2})=o(1).
\end{aligned}$$

(1b) The second part. Notice that

$$\begin{aligned}
& \frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=-\infty}^{t}\phi_{t-j}^2(\epsilon_j^2-1)\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1} \\
= & \frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=s+1}^{t}\phi_{t-j}^2(\epsilon_j^2-1)\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1} \\
& +\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=-\infty}^{s}\phi_{t-j}^2(\epsilon_j^2-1)\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1}.
\end{aligned}$$

For $j>s$, once again $E[\|Z(x_t)\|^2(\epsilon_j^2-1)|\mathcal{F}_s]=d_{ts}^{-2}\xi_s$, where $|\xi_s|\leq O(1)|b_{t,j}|k$ almost surely. Hence,

$$\begin{aligned}
& \left|\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=s+1}^{t}\phi_{t-j}^2(\epsilon_j^2-1)\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1}\right| \\
= & \left|\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}\sum_{j=s+1}^{t}\phi_{t-j}^2|b_{t,j}|\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}E[\xi_s\|Z(x_s)\|^2\epsilon_\ell\epsilon_{\ell_1}]\right| \\
\leq & O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}|\phi_{s-\ell}\phi_{s-\ell_1}|E[\|Z(x_s)\|^2|\epsilon_\ell\epsilon_{\ell_1}|] \\
\leq & O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}\frac{1}{d_s}\int\|Z(x)\|^2dx\leq O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}\frac{1}{d_s} \\
= & n^{-(\delta_0-1/2)}=o(1),
\end{aligned}$$

where we have used (2) of Lemma C.3, and $\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}|\phi_{s-\ell}\phi_{s-\ell_1}|<\infty$.

Additionally, a similar derivation to (2) of Lemma C.3 yields $E[\|Z(x_t)\|^2|\epsilon_j,\epsilon_\ell,\epsilon_{\ell_1}]\leq O(1)d_t^{-1}k$. Therefore,

$$\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\phi_{t-j}^2\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}|\phi_{s-\ell}\phi_{s-\ell_1}|E[\|Z(x_t)\|^2\|Z(x_s)\|^2|(\epsilon_j^2-1)\epsilon_\ell\epsilon_{\ell_1}|]$$

$$\leq O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\phi_{t-j}^2\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}|\phi_{s-\ell}\phi_{s-\ell_1}|E[\|Z(x_t)\|^2|(\epsilon_j^2-1)\epsilon_\ell\epsilon_{\ell_1}|]$$

$$\leq O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\phi_{t-j}^2\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}|\phi_{s-\ell}\phi_{s-\ell_1}|\frac{1}{\sqrt{t}}E[|(\epsilon_j^2-1)\epsilon_\ell\epsilon_{\ell_1}|]\int\|Z(x)\|^2dx$$

$$\leq O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s}\phi_{t-j}^2\frac{1}{d_t}=O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\frac{1}{d_t}\sum_{s=m_n}^{t-1}(t-s)^{-2\gamma_0+1}$$

$$=O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+1}^{n}\frac{1}{d_t}(1-(t-m_n)^{-2\gamma_0+2})=o(1).$$

(1c) We attempt to show the following expectation is $o(1)$,

$$\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2(\epsilon_\ell^2-1)\sum_{j=-\infty}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}$$

$$=\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2(\epsilon_\ell^2-1)$$

$$\times\left(\sum_{j=-\infty}^{s-1}\sum_{j_1=j+1}^{s}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}+\sum_{j=-\infty}^{s}\sum_{j_1=s+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}+\sum_{j=s+1}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}\right)$$

$$=\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2(\epsilon_\ell^2-1)\sum_{j=-\infty}^{s-1}\sum_{j_1=j+1}^{s}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}$$

$$+\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2(\epsilon_\ell^2-1)\sum_{j=-\infty}^{s}\sum_{j_1=s+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}$$

$$+\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2(\epsilon_\ell^2-1)\sum_{j=s+1}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}.$$

Once again, since $E[\|Z(x_t)\|^2|(\epsilon_\ell^2-1)\epsilon_j\epsilon_{j_1}|]\leq O(1)d_t^{-1}\int\|Z(x)\|^2dx=O(1)d_t^{-1}k$, the first item is dealt with as follows:

$$\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\sum_{j=-\infty}^{s-1}\sum_{j_1=j+1}^{s}|\phi_{t-j}\phi_{t-j_1}|E[\|Z(x_t)\|^2\|Z(x_s)\|^2|(\epsilon_\ell^2-1)\epsilon_j\epsilon_{j_1}|]$$

$$\leq O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\sum_{j=-\infty}^{s-1}\sum_{j_1=j+1}^{s}|\phi_{t-j}\phi_{t-j_1}|E[\|Z(x_t)\|^2|(\epsilon_\ell^2-1)\epsilon_j\epsilon_{j_1}|]$$

$$\leq O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\sum_{j=-\infty}^{s-1}\sum_{j_1=j+1}^{s}|\phi_{t-j}\phi_{t-j_1}|d_t^{-1}=O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}d_t^{-1}\sum_{s=m_n+1}^{t-1}(t-s)^{-2\gamma_0+2}$$

$$=O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}d_t^{-1}t^{-2\gamma_0+3}=o(1)$$

for $\gamma_0>3/2$, where we specify the coefficients $\phi_j=O(1)j^{-\gamma_0}$.

By virtue of (3) in Lemma C.3, $E[\|Z(x_t)\|^2\epsilon_{j_1}|\mathcal{F}_s]=d_{ts}^{-2}\xi_s$ where $|\xi_s|\leq O(1)|b_{t,j_1}|k$ a.s. Thus, for the second item,

$$\left|\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2(\epsilon_\ell^2-1)\sum_{j=-\infty}^{s}\phi_{t-j}\epsilon_j\sum_{j_1=s+1}^{t}\phi_{t-j_1}\epsilon_{j_1}\right|$$

$$=\left|\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\sum_{j=-\infty}^{s}\phi_{t-j}\sum_{j_1=s+1}^{t}\phi_{t-j_1}E[\xi_s\|Z(x_s)\|^2\epsilon_j(\epsilon_\ell^2-1)]\right|$$

$$\leq O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\sum_{j=-\infty}^{s}|\phi_{t-j}|\sum_{j_1=s+1}^{t}|\phi_{t-j_1}b_{t,j_1}|E[\|Z(x_s)\|^2|\epsilon_j(\epsilon_\ell^2-1)|]$$

$$\leq O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\sum_{j=-\infty}^{s}|\phi_{t-j}|d_s^{-1}$$

$$=O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}d_s^{-1}(t-s)^{-\gamma_0+1}=o(1).$$

To calculate the third item, using (4) of Lemma C.3, for $j\neq j_1$ and $1\leq s<j,j_1\leq t$, we have $E[\|Z(x_t)\|^2\epsilon_j\epsilon_{j_1}|\mathcal{F}_s]=d_{ts}^{-2}\xi_{s,jj_1}$, where ξ_{s,jj_1} is a random variable satisfying $|\xi_{s,jj_1}|\leq O(1)(|b_{t,j}|+|b_{t,j_1}|)k$ a.s.. Therefore, the third item can be evaluated as

$$\left|\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2(\epsilon_\ell^2-1)\sum_{j=s+1}^{t}\phi_{t-j}\epsilon_j\sum_{j_1=j+1}^{t}\phi_{t-j_1}\epsilon_{j_1}\right|$$

$$=\left|\frac{d_n^2}{n^2k^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2\sum_{j=s+1}^{t}\phi_{t-j}\sum_{j_1=j+1}^{t}\phi_{t-j_1}E[\xi_{s,jj_1}\|Z(x_s)\|^2(\epsilon_\ell^2-1)]\right|$$

$$\leq O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}\sum_{\ell=-\infty}^{s}\phi_{s-\ell}^2E|\|Z(x_s)\|^2(\epsilon_\ell^2-1)|$$

$$\times\sum_{j=s+1}^{t}|\phi_{t-j}|\sum_{j_1=j+1}^{t}|\phi_{t-j_1}|(|b_{t,j}|+|b_{t,j_1}|)$$

$$\leq O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}d_s^{-1}\int\|Z(x)\|^2dx\leq O(1)\frac{d_n^2}{n^2}\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}d_{ts}^{-2}d_s^{-1}$$

$$=o(1).$$

(1d) We start to calculate the last term. Because

$$\sum_{j=-\infty}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}=\sum_{j=-\infty}^{s}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}+\sum_{j=s+1}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}$$

$$=\sum_{j=-\infty}^{s-1}\sum_{j_1=j+1}^{s}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}+\sum_{j=-\infty}^{s}\sum_{j_1=s+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}+\sum_{j=s+1}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1},$$

we have

$$\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=-\infty}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1}$$

$$=\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=-\infty}^{s-1}\sum_{j_1=j+1}^{s}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1}$$

$$+\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=-\infty}^{s}\sum_{j_1=s+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1}$$

$$+\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+2}^{n}\sum_{s=m_n+1}^{t-1}\|Z(x_t)\|^2\|Z(x_s)\|^2\sum_{j=s+1}^{t-1}\sum_{j_1=j+1}^{t}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}\sum_{\ell=-\infty}^{s-1}\sum_{\ell_1=\ell+1}^{s}\phi_{s-\ell}\phi_{s-\ell_1}\epsilon_\ell\epsilon_{\ell_1},$$

which clearly can be shown to be $o(1)$ one by one similarly to what has been done in the first three parts. We provide herewith an outline. In the first item $\|Z(x_s)\|^2\le O(1)k$, $E\|Z(x_t)\|^2|\epsilon_j\epsilon_{j_1}\epsilon_\ell\epsilon_{\ell_1}|\le O(1)kd_t^{-1}$ and $\sum_{j=-\infty}^{s-1}\sum_{j_1=j+1}^{s}|\phi_{t-j}\phi_{t-j_1}|\le\left(\sum_{j=-\infty}^{s-1}|\phi_{t-j}|\right)^2=O(1)(t-s)^{-2\gamma_0+2}$ and all of these facts ensure that the first item is $o(1)$; while in the second item, we may use (3) of Lemma C.3 to derive $E[\|Z(x_t)\|^2\epsilon_{j_1}|\mathcal{F}_s]=d_{ts}^{-2}\xi_{s,j_1}$ and in the third item, we may use (4) of Lemma C.3 to derive $E[\|Z(x_t)\|^2\epsilon_{j_1}\epsilon_j|\mathcal{F}_s]=d_{ts}^{-2}\xi_{s,jj_1}$, which, along with a routine calculation for the remaining term, yields the desired results.

Step II. For L_{1n}''', we split it into two parts:

$$\frac{d_n}{nk}L_{1n}'''=\frac{d_n}{nk}\sum_{t=2}^{m_n}\sum_{s=1}^{t-1}Z(x_t)'Z(x_s)e_se_t+\frac{d_n}{nk}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}Z(x_t)'Z(x_s)e_se_t.$$

However, we may show the first term is $o_P(1)$ given the condition on m_n. Indeed,

$$\frac{d_n}{nk}\sum_{t=2}^{m_n}\sum_{s=1}^{t-1}E|Z(x_t)'Z(x_s)e_se_t|\le\frac{d_n}{n}\sum_{t=2}^{m_n}\sum_{s=1}^{t-1}E|e_se_t|\le O(1)\frac{d_n}{n}m_n^2=o(1),$$

since $m_n^4\frac{d_n}{n}=o(1)$. Hence, the following derivations only involve large t as follows:

$$\begin{aligned}
&E\left(\frac{d_n}{nk}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}Z(x_t)'Z(x_s)e_se_t\right)^2\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\left(\sum_{s=1}^{t-1}Z(x_t)'Z(x_s)e_se_t\right)^2\\
&+2\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\left(\sum_{s=1}^{t_1-1}Z(x_{t_1})'Z(x_s)e_se_{t_1}\right)\left(\sum_{s=1}^{t_2-1}Z(x_{t_2})'Z(x_s)e_se_{t_2}\right)\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}[Z(x_t)'Z(x_s)e_se_t]^2\\
&+2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})e_{s_1}e_t^2Z(x_t)'Z(x_{s_2})e_{s_2}\\
&+2\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=1}^{t_1-1}Z(x_{t_1})'Z(x_{s_1})e_{s_1}e_{t_1}\sum_{s_2=1}^{t_2-1}Z(x_{t_2})'Z(x_{s_2})e_{s_2}e_{t_2}\\
:=&A_1+2A_2+2A_3,\quad\text{say.}
\end{aligned}$$

Noting that $e_t=\sum_{j=-\infty}^{t}\phi_{t-j}\epsilon_j=\sum_{j=s+1}^{t}\phi_{t-j}\epsilon_j+\sum_{j=-\infty}^{s}\phi_{t-j}\epsilon_j$, we have

$$\begin{aligned}
A_1=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}[Z(x_t)'Z(x_s)e_se_t]^2\\
\le&2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}[Z(x_t)'Z(x_s)]^2e_s^2\left(\sum_{j=s+1}^{t}\phi_{t-j}\epsilon_j\right)^2
\end{aligned}$$

$$
\begin{aligned}
&+2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}[Z(x_t)'Z(x_s)]^2e_s^2\left(\sum_{j=-\infty}^{s}\phi_{t-j}\epsilon_j\right)^2\\
=&2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}[Z(x_t)'Z(x_s)]^2e_s^2\sum_{j=s+1}^{t}\phi_{t-j}^2\epsilon_j^2\\
&+4\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}[Z(x_t)'Z(x_s)]^2e_s^2\sum_{j=s+2}^{t}\sum_{j_1=s+1}^{j-1}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}\\
&+2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}[Z(x_t)'Z(x_s)]^2e_s^2\left(\sum_{j=-\infty}^{s}\phi_{t-j}\epsilon_j\right)^2\\
:=&2A_{11}+4A_{12}+2A_{13}.
\end{aligned}
$$

To tackle A_{11}, notice that

$$
E[(Z(x_t)'Z(x_s))^2\epsilon_j^2|\mathcal{F}_s]\le O(1)d_{ts}^{-1}\int(Z(x)'Z(x_s))^2dx=O(1)d_{ts}^{-1}\|Z(x_s)\|^2
$$

by (3) of Lemma C.3 and the orthogonality of the basis. Thus,

$$
\begin{aligned}
A_{11}=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}[Z(x_t)'Z(x_s)]^2\sum_{j=s+1}^{t}\phi_{t-j}^2\epsilon_j^2e_s^2\le O(1)\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}d_{ts}^{-1}E[\|Z(x_s)\|^2e_s^2]\\
\le&O(1)\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}d_{ts}^{-1}d_s^{-1}\int\|Z(x)\|^2dx=O(1)\frac{1}{k}=o(1),
\end{aligned}
$$

where due to (5) of Lemma C.3, we have $E[\|Z(x_s)\|^2e_s^2]\le O(1)d_s^{-1}\int\|Z(x)\|^2dx=O(1)d_s^{-1}k$.

Regarding A_{12}, using (4) of Lemma C.3 $E([Z(x_t)'Z(x_s)]^2\epsilon_j\epsilon_{j_1}|\mathcal{F}_s)=d_{ts}^{-2}\xi_{s,jj_1}$ where $|\xi_{s,jj_1}|\le C(|b_{t,j}|+|b_{t,j_1}|)\int[Z(x)'Z(x_s)]^2dx=C(|b_{t,j}|+|b_{t,j_1}|)\|Z(x_s)\|^2$ a.s.. Then,

$$
\begin{aligned}
|A_{12}|=&\left|\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}[Z(x_t)'Z(x_s)]^2\sum_{j=s+2}^{t}\sum_{j_1=s+1}^{j-1}\phi_{t-j}\phi_{t-j_1}\epsilon_j\epsilon_{j_1}e_s^2\right|\\
=&\left|\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}d_{ts}^{-2}\sum_{j=s+2}^{t}\sum_{j_1=s+1}^{j-1}\phi_{t-j}\phi_{t-j_1}E\xi_{s,jj_1}e_s^2\right|\\
\le&O(1)\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}d_{ts}^{-2}E[\|Z(x_s)\|^2e_s^2]\\
\le&O(1)\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}d_{ts}^{-2}d_s^{-1}\int\|Z(x)\|^2dx=O(1)\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}d_{ts}^{-2}d_s^{-1}\\
=&O(1)\frac{1}{k}n^{-(\delta_0-1/2)}=o(1),
\end{aligned}
$$

where once again we have used (5) of Lemma C.3 for $E[\|Z(x_s)\|^2e_s^2]$.

As for the last term A_{13}, notice that, given $\mathcal{F}_s$, $\frac{1}{d_{ts}}(x_t-x_s)$ has a uniformly bounded density $f_{ts}(w)$ by Lemma C.2 and Lemma 2 of Renyi (1958, p. 223). Hence,

$$
\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}[Z(x_t)'Z(x_s)]^2e_s^2\left(\sum_{j=-\infty}^{s}\phi_{t-j}\epsilon_j\right)^2
$$

$$
\begin{aligned}
=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}E[(Z(x_t)'Z(x_s))^2|\mathcal{F}_s]\left(\sum_{j=-\infty}^{s}\phi_{t-j}\epsilon_j\right)^2 e_s^2\\
=&\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}E\int[Z(d_{ts}w+x_s)'Z(x_s)]^2 f_{ts}(w)dw\left(\sum_{j=-\infty}^{s}\phi_{t-j}\epsilon_j\right)^2 e_s^2\\
\le&C\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}E\|Z(x_s)\|^2\left(\sum_{j=-\infty}^{s}\phi_{t-j}\epsilon_j\right)^2 e_s^2\\
\le&C\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}E\|Z(x_s)\|^2\left(\sum_{j=-\infty}^{s}\phi_{t-j}\epsilon_j\right)^2 e_s^2\\
\le&C\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}\frac{1}{d_s}\sum_{j=-\infty}^{s}\phi_{t-j}^2\\
&+C\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}\frac{1}{d_s}\sum_{j=-\infty}^{s}\sum_{\ell=-\infty,\neq j}^{s}|\phi_{t-j}\phi_{t-\ell}|\\
\le&C\frac{d_n^2}{n^2k}\sum_{t=m_n+1}^{n}\sum_{s=1}^{t-1}\frac{1}{d_{ts}}\frac{1}{d_s}(t-s)^{-2\gamma_0+2}=o(1).
\end{aligned}
$$

Now we move on to A_2. Note that

$$
\begin{aligned}
e_t^2=&\left(\sum_{j=-\infty}^{t}\phi_{t-j}\epsilon_j\right)^2=\left(\sum_{j=-\infty}^{s_1}\phi_{t-j}\epsilon_j+\sum_{j=s_1+1}^{t}\phi_{t-j}\epsilon_j\right)^2\\
=&\left(\sum_{j=s_1+1}^{t}\phi_{t-j}\epsilon_j\right)^2+\left(\sum_{j=-\infty}^{s_1}\phi_{t-j}\epsilon_j\right)^2+2\sum_{j=s_1+1}^{t}\phi_{t-j}\epsilon_j\sum_{j=-\infty}^{s_1}\phi_{t-j}\epsilon_j.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
A_2=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})e_t^2e_{s_1}e_{s_2}\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\left(\sum_{j=s_1+1}^{t}\phi_{t-j}\epsilon_j\right)^2e_{s_1}e_{s_2}\\
&+\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\left(\sum_{j=-\infty}^{s_1}\phi_{t-j}\epsilon_j\right)^2e_{s_1}e_{s_2}\\
&+2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\sum_{j=s_1+1}^{t}\phi_{t-j}\epsilon_j\sum_{j=-\infty}^{s_1}\phi_{t-j}\epsilon_je_{s_1}e_{s_2}\\
:=&A_{21}+A_{22}+A_{23}.
\end{aligned}
$$

It follows that

$$
A_{21}=\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\left(\sum_{j=s_1+1}^{t}\phi_{t-j}\epsilon_j\right)^2e_{s_1}e_{s_2}
$$

$$
\begin{aligned}
=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\sum_{j=s_1+1}^{t}\phi_{t-j}^2\epsilon_j^2e_{s_1}e_{s_2}\\
&+2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\sum_{j_1=s_1+2}^{t}\sum_{j_2=s_1+1}^{j_1-1}\phi_{t-j_1}\phi_{t-j_2}\epsilon_{j_1}\epsilon_{j_2}e_{s_1}e_{s_2}\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\\
&\times\sum_{j=s_1+1}^{t}\phi_{t-j}^2\epsilon_j^2\sum_{\ell_1=-\infty}^{s_1}\phi_{s_1-\ell_1}\epsilon_{\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}\epsilon_{\ell_2}\\
&+2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\sum_{j_1=s_1+2}^{t}\sum_{j_2=s_1+1}^{j_1-1}\phi_{t-j_1}\phi_{t-j_2}\epsilon_{j_1}\epsilon_{j_2}e_{s_1}e_{s_2}\\
=&\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}\\
&\times E[Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\epsilon_j^2\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=-\infty}^{s_2}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}\\
&\times E[Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\epsilon_j^2\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\sum_{j_1=s_1+2}^{t}\sum_{j_2=s_1+1}^{j_1-1}\phi_{t-j_1}\phi_{t-j_2}\epsilon_{j_1}\epsilon_{j_2}e_{s_1}e_{s_2}\\
=&\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}\\
&\times E[Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\epsilon_j^2\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell=-\infty}^{s_2}\phi_{s_1-\ell}\phi_{s_2-\ell}\\
&\times E[Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\epsilon_j^2\epsilon_\ell^2]\\
&+\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=-\infty}^{s_2}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty,\neq\ell_1}^{s_2}\phi_{s_2-\ell_2}\\
&\times E[Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\epsilon_j^2\epsilon_{\ell_1}\epsilon_{\ell_2}]\\
&+2\frac{d_n^2}{n^2k^2}E\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{t-1}\sum_{s_2=1}^{s_1-1}Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\sum_{j_1=s_1+2}^{t}\sum_{j_2=s_1+1}^{j_1-1}\phi_{t-j_1}\phi_{t-j_2}\epsilon_{j_1}\epsilon_{j_2}e_{s_1}e_{s_2}\\
:=&\sum_{i=1}^{4}A_{21}(i),\quad\text{say.}
\end{aligned}
$$

To use (2) of Lemma C.4 for the following derivations, notice that since the probabilities of $x_t = x_{s_1}$ and $x_t = x_{s_2}$ are both zero when $t \neq s_1$ and $t \neq s_2$, the expectations in the above can be computed by excluding the regions $x_t = x_{s_1}$ and $x_t = x_{s_2}$, as in the proof of Theorems 3.1 and 3.2 in the article. The detail is as follows.

$A_{21}(1)$ is dealt with first. We partition s_1 into two parts: (1) $2 \le s_1 \le \sqrt{m_n}$, and (2) $\sqrt{m_n}+1 \le s_1 \le t-1$. In situation (1), we repeatedly use (2) of Lemma C.3 to derive

$$\begin{aligned}
&\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}E[Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\epsilon_j^2\epsilon_{\ell_1}\epsilon_{\ell_2}]\right|\\
\le&\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}\frac{1}{d_{ts_1}}\sum_{\ell_1=s_2+1}^{s_1}|\phi_{s_1-\ell_1}|\sum_{\ell_2=-\infty}^{s_2}|\phi_{s_2-\ell_2}|\\
&\times\int E|Z(x)'Z(x_{s_1})Z(x)'Z(x_{s_2})\epsilon_{\ell_1}\epsilon_{\ell_2}|dx\\
\le&O(1)\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s_1=2}^{\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}\frac{1}{d_{ts_1}}\frac{1}{d_{s_1s_2}}\frac{1}{d_{s_2}}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(x)'Z(z)|dxdydz\\
=&o(1)\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(x)'Z(z)|dxdydz\to 0,
\end{aligned}$$

due to Lemma C.5 and the condition on m_n.

In situation (2), invoking (3) of Lemma C.3,

$$E[Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\epsilon_j^2|\mathcal{F}_{s_1}] = E[Z(x_{t/j})'Z(x_{s_1})Z(x_{t/j})'Z(x_{s_2})|\mathcal{F}_{s_1}] + d_{ts_1}^{-2}\xi_{s_1j}$$

where $x_{t/j} = x_{s_1}^* + x_{ts_1/j}$ and

$$\begin{aligned}
E[Z(x_{t/j})'Z(x_{s_1})Z(x_{t/j})'Z(x_{s_2})|\mathcal{F}_{s_1}] &= \frac{1}{d_{ts_1}}\int Z(x)'Z(x_{s_1})Z(x)'Z(x_{s_2})f_{ts_1/j}\left(\frac{x-x_{s_1}^*}{d_{ts_1}}\right)dx,\\
\xi_{s_1j} = d_{ts_1}&\iint Z(x)'Z(x_{s_1})Z(x)'Z(x_{s_2})v^2h_\epsilon(v)\left[f_{ts_1/j}\left(\frac{x-bv-x_{s_1}^*}{d_{ts_1}}\right)-f_{ts_1/j}\left(\frac{x-x_{s_1}^*}{d_{ts_1}}\right)\right]dxdv,
\end{aligned}$$

because $\frac{1}{d_{ts_1}}x_{ts_1/j}$ has density $f_{ts_1/j}(x)$. Thus, we have

$$\begin{aligned}
&\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+1}^{n}\sum_{s_1=\sqrt{m_n}+1}^{t}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}\right.\\
&\left.\times E[Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\epsilon_j^2\epsilon_{\ell_1}\epsilon_{\ell_2}]\right|\\
=&\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+1}^{n}\sum_{s_1=\sqrt{m_n}+1}^{t}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}\right.\\
&\left.\times E\left\{E[Z(x_{t/j})'Z(x_{s_1})Z(x_{t/j})'Z(x_{s_2})|\mathcal{F}_{s_1}]+d_{ts_1}^{-2}\xi_{s_1j}\right\}\right|\\
\le&\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+1}^{n}\sum_{s_1=\sqrt{m_n}+1}^{t}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}\right.\\
&\left.\times E\left\{E[Z(x_{t/j})'Z(x_{s_1})Z(x_{t/j})'Z(x_{s_2})|\mathcal{F}_{s_1}]\epsilon_{\ell_1}\epsilon_{\ell_2}\right\}\right|\\
&+\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+1}^{n}\sum_{s_1=\sqrt{m_n}+1}^{t}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}d_{ts_1}^{-2}E[\xi_{s_1j}\epsilon_{\ell_1}\epsilon_{\ell_2}]\right|
\end{aligned}$$

Since $s_1 > \sqrt{m_n}$ is large, it follows from Lemma 2 of Renyi (1958) that $\frac{1}{d_{s_1}}x_{s_1}^*$ and $\frac{1}{d_{s_1}}x_{s_1}$ have the same distribution asymptotically. Thus, in this part we treat x_s and x_s^* the same, or simply replace x_s^* by x_s. Then we continue to compute the first part by (3) of Lemma C.3 as follows:

$$\begin{aligned}
&\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+1}^{n}\sum_{s_1=\sqrt{m_n}+1}^{t}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}\right.\\
&\quad\left.\times E\left\{E[Z(x_{t/j})'Z(x_{s_1})Z(x_{t/j})'Z(x_{s_2})|\mathcal{F}_{s_1}]\epsilon_{\ell_1}\epsilon_{\ell_2}\right\}\right|\\
=&\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+1}^{n}\sum_{s_1=\sqrt{m_n}+1}^{t}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\frac{1}{d_{ts_1}}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}\right.\\
&\quad\left.\times E\int Z(x)'Z(x_{s_1})Z(x)'Z(x_{s_2})f_{ts_1/j}\left(\frac{x-x_{s_1}}{d_{ts_1}}\right)\epsilon_{\ell_1}\epsilon_{\ell_2}dx\right|\\
=&\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+1}^{n}\sum_{s_1=\sqrt{m_n}+1}^{t}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\frac{1}{d_{ts_1}}\frac{1}{d_{s_1s_2}^2}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}E\xi_{s_2}\epsilon_{\ell_2}\right|,
\end{aligned}$$

where

$$\begin{aligned}
\xi_{s_2}=&d_{s_1s_2}\iiint_{x\neq y}Z(x)'Z(y)Z(x)'Z(x_{s_2})vh_\epsilon(v)f_{ts_1/j}\left(\frac{x-y}{d_{ts_1}}\right)\\
&\times\left[f_{s_1s_2/\ell_1}\left(\frac{y-x_{s_2}^*-bv}{d_{s_1s_2}}\right)-f_{s_1s_2/\ell_1}\left(\frac{y-x_{s_2}^*}{d_{s_1s_2}}\right)\right]dxdydv
\end{aligned}$$

satisfies $|\xi_{s_2}|\leq O(1)\iint_{x\neq y}|Z(x)'Z(y)Z(x)'Z(x_{s_2})|dxdy$ almost surely by the Lipschitz condition. Moreover, $|E\xi_{s_2}\epsilon_{\ell_2}|\leq O(1)\frac{1}{d_{s_2}}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(x)'Z(z)|dxdydz$ by (1) of Lemma C.3. Putting all of them together yields

$$\begin{aligned}
&\frac{d_n^2}{n^2k^2}\left|\sum_{t=m_n+1}^{n}\sum_{s_1=\sqrt{m_n}+1}^{t}\sum_{s_2=1}^{s_1-1}\sum_{j=s_1+1}^{t}\phi_{t-j}^2\sum_{\ell_1=s_2+1}^{s_1}\phi_{s_1-\ell_1}\sum_{\ell_2=-\infty}^{s_2}\phi_{s_2-\ell_2}\right.\\
&\quad\left.\times E\left\{E[Z(x_{t/j})'Z(x_{s_1})Z(x_{t/j})'Z(x_{s_2})|\mathcal{F}_{s_1}]\epsilon_{\ell_1}\epsilon_{\ell_2}\right\}\right|\\
\leq&O(1)\frac{d_n^2}{n^2k^2}\sum_{t=m_n+1}^{n}\sum_{s_1=\sqrt{m_n}+1}^{t}\sum_{s_2=1}^{s_1-1}\frac{1}{d_{ts_1}}\frac{1}{d_{s_1s_2}^2}\frac{1}{d_{s_2}}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(x)'Z(z)|dxdydz\\
=&n^{-2(\delta_0-1)}k^{-2}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(x)'Z(z)|dxdydz\to 0
\end{aligned}$$

by a routine calculation, C.2 of Assumption C as well as Lemma C.5.

To finish the proof of $A_{21}(1)=o_P(1)$, we now consider the remaining part. Due to the same reason as before, we treat $x_{s_1}^*$ in ξ_{s_1} the same as x_{s_1}. Thus, by virtue of (1) of Lemma C.3, $E[\xi_{s_1}\epsilon_{\ell_1}|\mathcal{F}_{s_2}]=\frac{1}{d_{s_1s_2}}\xi_{s_2}$ and here $|\xi_{s_2}|\leq C\iint_{x\neq y}|Z(x)'Z(y)Z(x)'Z(x_{s_2})|dxdy$ almost surely. With further using (1) of Lemma C.3, $|E[\xi_{s_2}\epsilon_{\ell_2}]|\leq O(1)\frac{1}{d_{s_2}}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(x)'Z(z)|dxdydz$. Putting all the ingredients together will result in the remaining term to be $o_P(1)$ by (2) of Lemma C.5 again.

In $A_{21}(2)$, by (2) of Lemma C.3,

$$|E[Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\epsilon_j^2\epsilon_\ell^2]|\leq\frac{1}{d_{s_1s_2}}\frac{1}{d_{ts_1}}\frac{1}{d_{s_2}}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(x)'Z(z)|dxdydz,$$

which, together with $\sum_{\ell=-\infty}^{s_2}|\phi_{s_1-\ell}\phi_{s_2-\ell}|\leq O(1)(s_1-s_2)^{-\gamma_0+1}$ and $\gamma_0>3/2$ as well as (2) of Lemma C.5, yields that $A_{21}(2)=o(1)$. For the same reason, $A_{21}(3)=o(1)$.

As for $A_{21}(4)$, because of $j_1 \neq j_2$ and (4) of Lemma C.3,

$$E[Z(x_t)'Z(x_{s_1})Z(x_t)'Z(x_{s_2})\epsilon_{j_1}\epsilon_{j_2}|\mathcal{F}_{s_1}] = d_{ts_1}^{-2}\xi_{s_1,j_1j_2},$$

where $|\xi_{s_1,j_1j_2}| \leq O(1)(|b_{t,j_1}| + |b_{t,j_2}|)\int |Z(x)'Z(x_{s_1})Z(x)'Z(x_{s_2})|dx$ a.s.. Then, by a similar fashion to what has been done above, such as in the evaluation of the expectation of $\xi_{s_1,j_1j_2}e_{s_1}e_{s_2}$ by Lemma C.3, and then using Lemma C.4, we may show $A_{21}(4) = o(1)$.

Next, we shall compute A_3,

$$\begin{aligned}
A_3 =&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=1}^{t_1-1}Z(x_{t_1})'Z(x_{s_1})e_{s_1}e_{t_1}\sum_{s_2=1}^{t_2-1}Z(x_{t_2})'Z(x_{s_2})e_{s_2}e_{t_2}\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}Z(x_{t_1})'Z(x_{t_2})e_{t_2}^2e_{t_1}\sum_{s_2=1}^{t_2-1}Z(x_{t_2})'Z(x_{s_2})e_{s_2}\\
&+\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=1}^{t_2-1}Z(x_{t_1})'Z(x_{s_1})e_{s_1}e_{t_1}\sum_{s_2=1}^{t_2-1}Z(x_{t_2})'Z(x_{s_2})e_{s_2}e_{t_2}\\
&+\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=t_2+1}^{t_1-1}Z(x_{t_1})'Z(x_{s_1})e_{s_1}e_{t_1}\sum_{s_2=1}^{t_2-1}Z(x_{t_2})'Z(x_{s_2})e_{s_2}e_{t_2}\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}Z(x_{t_1})'Z(x_{t_2})e_{t_2}^2e_{t_1}\sum_{s_2=1}^{t_2-1}Z(x_{t_2})'Z(x_{s_2})e_{s_2}\\
&+\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=1}^{t_2-1}Z(x_{t_1})'Z(x_{s_1})e_{t_1}Z(x_{t_2})'Z(x_{s_1})e_{s_1}^2e_{t_2}\\
&+2\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=2}^{t_2-1}Z(x_{t_1})'Z(x_{s_1})e_{s_1}e_{t_1}\sum_{s_2=1}^{s_1-1}Z(x_{t_2})'Z(x_{s_2})e_{s_2}e_{t_2}\\
&+\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=t_2+1}^{t_1-1}Z(x_{t_1})'Z(x_{s_1})e_{s_1}e_{t_1}\sum_{s_2=1}^{t_2-1}Z(x_{t_2})'Z(x_{s_2})e_{s_2}e_{t_2}\\
:=&A_{31}+A_{32}+A_{33}+A_{34}, \quad \text{say.}
\end{aligned}$$

Let A_{31} to A_{34} be partitioned according to $t_2 = s_1$, $t_2 > s_1 = s_2$, $t_2 > s_1 > s_2$ and $s_1 > t_2 > s_2$. To begin with,

$$\begin{aligned}
A_{31} =&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_2=1}^{t_2-1}Z(x_{t_1})'Z(x_{t_2})Z(x_{t_2})'Z(x_{s_2})e_{t_1}e_{t_2}^2e_{s_2}\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_2=1}^{t_2-1}Z(x_{t_1})'Z(x_{t_2})Z(x_{t_2})'Z(x_{s_2})\sum_{j_1=-\infty}^{t_1}\phi_{t_1-j_1}\epsilon_{j_1}e_{t_2}^2e_{s_2}\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_2=1}^{t_2-1}Z(x_{t_1})'Z(x_{t_2})Z(x_{t_2})'Z(x_{s_2})\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}\epsilon_{j_1}e_{t_2}^2e_{s_2}\\
&+\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_2=1}^{t_2-1}Z(x_{t_1})'Z(x_{t_2})Z(x_{t_2})'Z(x_{s_2})\sum_{j_1=-\infty}^{t_2}\phi_{t_1-j_1}\epsilon_{j_1}e_{t_2}^2e_{s_2}.
\end{aligned}$$

Because for $j_1 > t_2$, $E[Z(x_{t_1})'Z(x_{t_2})Z(x_{t_2})'Z(x_{s_2})\epsilon_{j_1}|\mathcal{F}_{t_2}] = d_{t_1t_2}^{-2}\xi_{t_2,j_1}$ and

$$|\xi_{t_2}| \leq O(1)|b_{t_1,j_1}|\int |Z(x)'Z(x_{t_2})Z(x_{t_2})'Z(x_{s_2})|dx, \quad a.s.,$$

it follows from (5) of Lemma C.3 that

$$\begin{aligned}
&\left|\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_2=1}^{t_2-1}Z(x_{t_1})'Z(x_{t_2})Z(x_{t_2})'Z(x_{s_2})\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}\epsilon_{j_1}e_{t_2}^2e_{s_2}\right| \\
=&\left|\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_2=1}^{t_2-1}d_{t_1t_2}^{-2}\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}E[\xi_{t_2,j_1}e_{t_2}^2e_{s_2}]\right| \\
\le&O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_2=1}^{t_2-1}d_{t_1t_2}^{-2}\int E|Z(x)'Z(x_{t_2})Z(x_{t_2})'Z(x_{s_2})e_{t_2}^2e_{s_2}|dx \\
\le&O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_2=1}^{t_2-1}d_{t_1t_2}^{-2}d_{t_2s_2}^{-1}d_{s_2}^{-1}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(y)'Z(z)|dxdydz \\
=&n^{-2(\delta_0-1)}k^{-2}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(y)'Z(z)|dxdydz\to 0
\end{aligned}$$

by C.2 of Assumption C and (2) of Lemma C.5 again.

In the second term of A_{31}, in view of (3) of Lemma C.3 we have

$$\begin{aligned}
&|E[Z(x_{t_1})'Z(x_{t_2})Z(x_{t_2})'Z(x_{s_2})\epsilon_{j_1}e_{t_2}^2e_{s_2}| \\
\le&O(1)d_{t_1t_2}^{-1}d_{t_2s_2}^{-1}d_{s_2}^{-1}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(y)'Z(z)|dxdydz,
\end{aligned}$$

although ϵ_{j_1} is contained in e_{t_2} and e_{s_2}. Taking the convergence of $\sum_{j_1=-\infty}^{t_2}|\phi_{t_1-j_1}|\le O(1)(t_1-t_2)^{-\gamma_0+1}$ and $\gamma_0>3/2$ into account, the second term is not larger than the first term in absolute value. Next,

$$\begin{aligned}
A_{32}=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=1}^{t_2-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_1})e_{t_1}e_{t_2}e_{s_1}^2 \\
=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=1}^{t_2-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_1})\sum_{j_1=-\infty}^{t_1}\phi_{t_1-j_1}\epsilon_{j_1}e_{t_2}e_{s_1}^2 \\
=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=1}^{t_2-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_1})\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}\epsilon_{j_1}e_{t_2}e_{s_1}^2 \\
&+\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=1}^{t_2-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_1})\sum_{j_1=-\infty}^{t_2}\phi_{t_1-j_1}\epsilon_{j_1}e_{t_2}e_{s_1}^2.
\end{aligned}$$

In the first term, we have $E[Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_1})\epsilon_{j_1}|\mathcal{F}_{t_2}]=d_{t_1t_2}^{-2}\xi_{t_2,j_1}$ and

$$|\xi_{t_2}|\le O(1)|b_{t_1,j_1}|\int|Z(x)'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_1})|dx,\ a.s..$$

and then it follows from Lemma C.3 and the structure of e_t that

$$\begin{aligned}
&\left|\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=1}^{t_2-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_1})\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}\epsilon_{j_1}e_{t_2}e_{s_1}^2\right| \\
\le&O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=1}^{t_2-1}d_{t_1t_2}^{-2}d_{t_2s_2}^{-1}d_{s_2}^{-1}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(y)'Z(z)|dxdydz
\end{aligned}$$

$$=n^{-2(\delta_0-1)}k^{-2}\iiint_{x\neq y\neq z}|Z(x)'Z(y)Z(y)'Z(z)|dxdydz\to 0$$

by C.2 of Assumption C and (2) of Lemma C.5. Similar to A_{31}, the second term in A_{32} in absolute value is smaller than the first term, so that it is an infinitesimal as well. Moreover, we have

$$\begin{aligned}
A_{33}=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_2})e_{s_1}e_{t_1}e_{s_2}e_{t_2}\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_2})\sum_{j_1=-\infty}^{t_1}\phi_{t_1-j_1}\epsilon_{j_1}e_{s_1}e_{s_2}e_{t_2}\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_2})\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}\epsilon_{j_1}e_{s_1}e_{s_2}e_{t_2}\\
&+\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-1}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_2})\sum_{j_1=-\infty}^{t_2}\phi_{t_1-j_1}\epsilon_{j_1}e_{s_1}e_{s_2}e_{t_2}\\
:=&A_{33}(1)+A_{33}(2).
\end{aligned}$$

Notice that $A_{33}(1)$ is much tougher to be dealt with than $A_{33}(2)$, because in the latter we may use the convergence of $\sum_{j_1=-\infty}^{t_2}|\phi_{t_1-j_1}|\leq O(1)(t_1-t_2)^{-\gamma_0+1}$ and $\gamma_0>3/2$ as before. Thus, only $A_{33}(1)=o(1)$ is shown in what follows.

By (3) of Lemma C.3, $E[Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_2})\epsilon_{j_1}|\mathcal{F}_{t_2}]=d_{t_1t_2}^{-2}\xi_{t_2,j_1}$, where

$$\begin{aligned}
\xi_{t_2,j_1}=&d_{t_1t_2}\iint Z(w)'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_2})vh_\epsilon(v)\\
&\times\left[f_{t_1t_2/j_1}\left(\frac{w-bv-x^*_{t_2}}{d_{t_1t_2}}\right)-f_{t_1t_2/j_1}\left(\frac{w-x^*_{t_2}}{d_{t_1t_2}}\right)\right]dvdw,
\end{aligned}\tag{D.11}$$

in which $\frac{1}{d_{t_2}}x^*_{t_2}$ is asymptotically equal to $\frac{1}{d_{t_2}}x_{t_2}$ in distribution according to Lemma 2 of Renyi (1958, p.223) again. That is, we may focus on $t_1-t_2>\sqrt{m_n}$ in the following calculation:

$$\begin{aligned}
&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-\sqrt{m_n}}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}d_{t_1t_2}^{-2}\xi_{t_2,j_1}e_{s_1}e_{s_2}e_{t_2}\\
=&\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-\sqrt{m_n}}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}\sum_{j_2=s_1+1}^{t_2}\phi_{t_2-j_2}d_{t_1t_2}^{-2}\xi_{t_2,j_1}\epsilon_{j_2}e_{s_1}e_{s_2}\\
&+\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-\sqrt{m_n}}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}\sum_{j_2=-\infty}^{s_1}\phi_{t_2-j_2}d_{t_1t_2}^{-2}\xi_{t_2,j_1}\epsilon_{j_2}e_{s_1}e_{s_2},
\end{aligned}$$

where in ξ_2, $x^*_{t_2}$ has been replaced by x_{t_2} in (D.11). For brevity we specify $\xi_{t_2,j_1}=\xi_{t_2,j_1}(x_{t_2},x_{s_1},x_{s_2})$. By virtue of (3) of Lemma C.3 again, in the first term we derive that $E[\xi_{t_2,j_1}\epsilon_{j_2}|\mathcal{F}_{s_1}]=d_{t_2s_1}^{-2}\xi_{s_1,j_1j_2}$, where

$$\begin{aligned}
&|\xi_{s_1,j_1j_2}|\\
=&d_{t_2s_1}\left|\iint_{x\neq w}v_1h_\epsilon(v_1)\xi_{t_2,j_1}(x,x_{s_1},x_{s_2})\left[f_{t_2s_1}\left(\frac{x-x^*_{s_1}-b_{t_2,j_2}v_1}{d_{t_2s_1}}\right)-f_{t_2s_1}\left(\frac{x-x^*_{s_1}}{d_{t_2s_1}}\right)\right]dv_1dx\right|
\end{aligned}$$

$$\leq O(1)|b_{t_2,j_2}|\iint_{x\neq w}|v_1^2h_\epsilon(v_1)\xi_{t_2,j_1}(x,x_{s_1},x_{s_2})|dv_1dx$$
$$\leq O(1)|b_{t_1,j_1}||b_{t_2,j_2}|\int\cdots\int_{x\neq w}|Z(w)'Z(x_{s_1})Z(x)'Z(x_{s_2})|v^2h_\epsilon(v)dvdwv_1^2h_\epsilon(v_1)dv_1dx$$
$$=O(1)|b_{t_1,j_1}||b_{t_2,j_2}|\iint_{x\neq w}|Z(w)'Z(x_{s_1})Z(x)'Z(x_{s_2})|dwdx.$$

Then, we compute the following expectation exploiting (1) and (5) of Lemma C.3 and the structure of e_{s_1},

$$\iint_{x\neq w}E[|Z(w)'Z(x_{s_1})Z(x)'Z(x_{s_2})e_{s_1}e_{s_2}|]dwdx$$
$$\leq d_{s_1s_2}^{-1}d_{s_2}^{-1}\int\cdots\int_{x\neq w\neq y\neq z}|Z(w)'Z(y)Z(x)'Z(z)|dw\cdots dz$$
$$=d_{s_1s_2}^{-1}d_{s_2}^{-1}\left(\iint_{x\neq y}|Z(x)'Z(y)|dxdy\right)^2.$$

Therefore, we have

$$\left|\frac{d_n^2}{n^2k^2}E\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-\sqrt{m_n}}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}\sum_{j_2=s_1+1}^{t_2}\phi_{t_2-j_2}d_{t_1t_2}^{-2}\xi_{t_2}\epsilon_{j_2}e_{s_1}e_{s_2}\right|$$
$$\leq O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=m_n+1}^{t_1-\sqrt{m_n}}\sum_{s_1=2}^{t_2-1}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}d_{s_2}^{-1}\left(\iint_{x\neq y}|Z(x)'Z(y)|dxdy\right)^2$$
$$=n^{-4(\delta_0-1)}k^{-2}\left(\iint_{x\neq y}|Z(x)'Z(y)|dxdy\right)^2\to 0,$$

due to C.2 of Assumption C and (2) Lemma C.5.

Also, for $1\leq t_1-t_2\leq\sqrt{m_n}$, we partition s_1 into two parts: $1\leq s_1\leq t_2-\sqrt{m_n}$ and $t_2-\sqrt{m_n}+1\leq s_1\leq t_2-1$. In the first situation, since $t_2-s_1\geq\sqrt{m_n}$, $\frac{1}{d_{t_2s_1}}(x_{t_2}^*-x_{s_1})$ and $\frac{1}{d_{t_2s_1}}(x_{t_2}-x_{s_1})$ have the same distribution asymptotically, and therefore by (4) of Lemma C.3,

$$\frac{d_n^2}{n^2k^2}\left|E\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=2}^{t_2-\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_2})\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}\epsilon_{j_1}e_{s_1}e_{s_2}e_{t_2}\right|$$
$$=\frac{d_n^2}{n^2k^2}\left|\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=2}^{t_2-\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}d_{t_1t_2}^{-2}E[\xi_{t_2,j_1}e_{s_1}e_{s_2}e_{t_2}]\right|$$
$$\leq\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=2}^{t_2-\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-2}\sum_{j_1=t_2+1}^{t_1}|\phi_{t_1-j_1}|\left|E\xi_{t_2,j_1}\sum_{j_2=-\infty}^{t_2}\phi_{t_2-j_2}\epsilon_{j_2}e_{s_1}e_{s_2}\right|$$
$$\leq O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=2}^{t_2-\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-2}\sum_{j_1=t_2+1}^{t_1}|\phi_{t_1-j_1}|\sum_{j_2=s_1+1}^{t_2}|\phi_{t_2-j_2}||E[\xi_{t_2,j_1}\epsilon_{j_2}e_{s_1}e_{s_2}]|$$
$$+O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=2}^{t_2-\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-2}\sum_{j_1=t_2+1}^{t_1}|\phi_{t_1-j_1}|\sum_{j_2=-\infty}^{s_1}|\phi_{t_2-j_2}||E[\xi_{t_2,j_1}\epsilon_{j_2}e_{s_1}e_{s_2}]|$$
$$\leq O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=2}^{t_2-\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}\sum_{j_1=t_2+1}^{t_1}|\phi_{t_1-j_1}|\sum_{j_2=s_1+1}^{t_2}|\phi_{t_2-j_2}|E|\xi_{t_2,,j_1}\varepsilon_{j_2}e_{s_1}e_{s_2}|$$

$$
\begin{aligned}
&+O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=2}^{t_2-\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-2}\sum_{j_2=-\infty}^{s_1}|\phi_{t_2-j_2}|\\
&\times d_{t_2s_1}^{-1}d_{s_1s_2}^{-1}d_{s_2}^{-1}\int\cdots\int_{x\neq y\neq z\neq w}|Z(w)'Z(z)Z(x)'Z(y)|dxdydzdw\\
\leq &O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=2}^{t_2-\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}d_{s_2}^{-1}\\
&\times\int\cdots\int_{x\neq y\neq z\neq w}|Z(w)'Z(z)Z(x)'Z(y)|dxdydzdw\\
&+O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=2}^{t_2-\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-2}(t_2-s_1)^{-\gamma_0+1}\\
&\times d_{t_2s_1}^{-1}d_{s_1s_2}^{-1}d_{s_2}^{-1}\int\cdots\int_{x\neq y\neq z\neq w}|Z(w)'Z(z)Z(x)'Z(y)|dxdydzdw\\
\leq &O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=2}^{t_2-\sqrt{m_n}}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-2}d_{t_2s_1}^{-2}d_{s_1s_2}^{-1}d_{s_2}^{-1}\\
&\times\int\cdots\int_{x\neq y\neq z\neq w}|Z(w)'Z(z)Z(x)'Z(y)|dxdydzdw=o(1)\left(\iint_{x\neq y}|Z(x)'Z(y)|dxdy\right)^2\to 0,
\end{aligned}
$$

using (2) of Lemma C.5, where $\gamma_0>3/2$ and $E[\xi_{t_2,j_1}\epsilon_{j_2}e_{s_1}e_{s_2}]$ in the second term is evaluated by virtue of Lemma C.3 and the structure of e_s, and we also have used the summability of $\phi_{t_1-j_1}b_{t_1,j_1}$.

In the second case where $t_2-\sqrt{m_n}+1\leq s_1\leq t_2-1$, simply using (1) and (3) of Lemma C.3 as well as the structure of e_s, we have

$$
\begin{aligned}
&\frac{d_n^2}{n^2k^2}\left|E\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=t_2-\sqrt{m_n}+1}^{t_2-1}\sum_{s_2=1}^{s_1-1}Z(x_{t_1})'Z(x_{s_1})Z(x_{t_2})'Z(x_{s_2})\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}\epsilon_{j_1}e_{s_1}e_{s_2}e_{t_2}\right|\\
&=\frac{d_n^2}{n^2k^2}\left|\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=t_2-\sqrt{m_n}+1}^{t_2-1}\sum_{s_2=1}^{s_1-1}\sum_{j_1=t_2+1}^{t_1}\phi_{t_1-j_1}d_{t_1t_2}^{-2}E[\xi_{t_2,j_1}e_{s_1}e_{s_2}e_{t_2}]\right|\\
&\leq O(1)\frac{d_n^2}{n^2k^2}\sum_{t_1=m_n+2}^{n}\sum_{t_2=t_1-\sqrt{m_n}+1}^{t_1-1}\sum_{s_1=t_2-\sqrt{m_n}+1}^{t_2-1}\sum_{s_2=1}^{s_1-1}d_{t_1t_2}^{-2}d_{t_2s_1}^{-1}d_{s_1s_2}^{-1}d_{s_2}^{-1}\\
&\qquad\times\left(\iint_{x\neq y}|Z(x)'Z(y)|dxdy\right)^2=o(1)\left(\iint_{x\neq y}|Z(x)'Z(y)|dxdy\right)^2\to 0,
\end{aligned}
$$

due to the choice of m_n. Thus, $A_{33}=o(1)$ is completed.

A_{34} has all different indices, and the only difference between A_{34} and A_{33} is the interchange of t_2 and s_1. Hence similarly, $A_{34}=o(1)$. Therefore, $\frac{d_n}{nk}L_{1n}'''=o_P(1)$.

Step III. Next, we shall prove $\frac{d_n}{nk}L_{3n}=o_P(1)$. As $\|\widehat{\theta}-\theta_0\|=O_P(\zeta_n)$ assumed in Assumption B, for any $\epsilon>0$, there exists a fixed $M>0$ such that $P(\|\widehat{\theta}-\theta_0\|>\zeta_nM)<\epsilon$. Then, for any $\delta>0$,

$$
\begin{aligned}
P\left(\frac{d_n}{nk}L_{3n}>\delta\right)\leq &P(\|\widehat{\theta}-\theta_0\|>\zeta_nM)+P\left(\frac{d_n}{nk}L_{3n}>\delta,\|\widehat{\theta}-\theta_0\|\leq\zeta_nM\right)\\
\leq &\epsilon+\frac{d_n}{nk\delta}E[L_{3n}I(\|\widehat{\theta}-\theta_0\|\leq\zeta_nM)],
\end{aligned}
$$

by virtue of Markov's inequality and noting that L_{3n} is nonnegative, where $I(\cdot)$ stands for the conventional indicator function.

Using a Taylor expansion for $g(x;\theta)$ with respect to θ in a neighborhood of θ_0, we have

$$\widehat{g}(x_t) = g(x_t;\theta_0) - g(x_t;\widehat{\theta}) = l_1(x_t;\theta_0)'(\theta_0 - \widehat{\theta}) - \frac{1}{2}(\theta_0 - \widehat{\theta})' l_2(x_t;\bar{\theta})(\theta_0 - \widehat{\theta})$$

where $\bar{\theta}$ is between θ_0 and $\widehat{\theta}$. In view of this, it follows that

$$\begin{aligned}
&\frac{d_n}{nk}E(|L_{3n}|I(\|\widehat{\theta}-\theta_0\| \le \zeta_n M)) = \frac{d_n}{nk}E\sum_{t=1}^{n}\sum_{s=1}^{n} Z(x_t)'Z(x_s)\widehat{g}(x_t)\widehat{g}(x_s)I(\|\widehat{\theta}-\theta_0\| \le \zeta_n M)\\
=&\frac{d_n}{nk}E\sum_{t=1}^{n}\sum_{s=1}^{n} Z(x_t)'Z(x_s)l_1(x_t;\theta_0)'(\theta_0-\widehat{\theta})l_1(x_s;\theta_0)'(\theta_0-\widehat{\theta})I(\|\widehat{\theta}-\theta_0\| \le \zeta_n M)\\
&-\frac{d_n}{nk}E\sum_{t=1}^{n}\sum_{s=1}^{n} Z(x_t)'Z(x_s)l_1(x_t;\theta_0)'(\theta_0-\widehat{\theta})(\theta_0-\widehat{\theta})'l_2(x_s;\bar{\theta})(\theta_0-\widehat{\theta})I(\|\widehat{\theta}-\theta_0\| \le \zeta_n M)\\
&+\frac{d_n}{4nk}E\sum_{t=1}^{n}\sum_{s=1}^{n} Z(x_t)'Z(x_s)(\theta_0-\widehat{\theta})'l_2(x_t;\bar{\theta})(\theta_0-\widehat{\theta})(\theta_0-\widehat{\theta})'l_2(x_s;\bar{\theta})(\theta_0-\widehat{\theta})I(\|\widehat{\theta}-\theta_0\| \le \zeta_n M)\\
=&T_1 + T_2 + T_3, \qquad \text{say.}
\end{aligned}$$

Notice that

$$\begin{aligned}
T_1 =&\frac{d_n}{nk}E\sum_{t=1}^{n}\sum_{s=1}^{n} Z(x_t)'Z(x_s)l_1(x_t;\theta_0)'(\theta_0-\widehat{\theta})l_1(x_s;\theta_0)'(\theta_0-\widehat{\theta})I(\|\widehat{\theta}-\theta_0\| \le \zeta_n M)\\
=&\frac{d_n}{nk}E\sum_{t=1}^{n}\|Z(x_t)\|^2[l_1(x_t;\theta_0)'(\theta_0-\widehat{\theta})]^2 I(\|\widehat{\theta}-\theta_0\| \le \zeta_n M)\\
&+2\frac{d_n}{nk}E\sum_{t=2}^{n}\sum_{s=1}^{t-1} Z(x_t)'Z(x_s)l_1(x_t;\theta_0)'(\theta_0-\widehat{\theta})l_1(x_s;\theta_0)'(\theta_0-\widehat{\theta})I(\|\widehat{\theta}-\theta_0\| \le \zeta_n M)\\
:=&T_{11} + T_{12}, \quad \text{say.}
\end{aligned}$$

Using the Cauchy-Schwarz inequality, $\|Z(x)\|^2 \le O(1)k$ uniformly in x, and the boundedness of the densities $f_t(x)$ for $d_t^{-1}x_t$, we have

$$T_{11} \le M^2\zeta_n^2\frac{d_n}{n}E\sum_{t=1}^{n}\|l_1(x_t;\theta_0)\|^2 \le C\zeta_n^2\frac{d_n}{n}\sum_{t=1}^{n}d_t^{-1}\int\|l_1(x;\theta_0)\|^2dx \le C\zeta_n^2 = o(1).$$

For T_{12}, since $t > s$, the probability of $x_t = x_s$ is equal to zero, and hence in the calculation of the expectations in T_{12}, we shall exclude these regions. Let $f_{ts}(x)$ and $f_s(y)$ stand for the densities of $d_{ts}^{-1}x_{ts}$ and $d_s^{-1}x_s$, respectively. Using the uniform boundedness of $f_{ts}(x)$ and $f_s(y)$ over x, y, t and s, and Lemma A.4,

$$\begin{aligned}
|T_{12}| \le&C\frac{d_n}{nk}\zeta_n^2\sum_{t=2}^{n}\sum_{s=1}^{t-1}E[|Z(x_t)'Z(x_s)|\|l_1(x_t;\theta_0)\|\|l_1(x_s;\theta_0)\|]\\
=&C\frac{d_n}{nk}\zeta_n^2\sum_{t=2}^{n}\sum_{s=1}^{t-1}d_{ts}^{-1}d_s^{-1}\iint_{x\neq y}|Z(x)'Z(y)|\|l_1(x;\theta_0)\|\|l_1(y;\theta_0)\|dxdy\\
\le&C\frac{n}{d_nk}\zeta_n^2\iint_{x\neq y}|Z(x)'Z(y)|dxdy = O(1)\iint_{x\neq y}|Z(x)'Z(y)|dxdy \to 0,
\end{aligned}$$

since $(n/d_n k)\zeta_n^2 = O(1)$ using Assumption B.

Similarly, using Assumption B again, we have as $n \to \infty$

$$\begin{aligned}
0 \le T_3 \le & C\zeta_n^4 \frac{d_n}{n} E \sum_{t=1}^{n} \|l_2(x_t;\bar{\theta})\|^2 + C\zeta_n^4 \frac{d_n}{n} E \sum_{t=2}^{n}\sum_{s=1}^{t-1} \|l_2(x_t;\bar{\theta})\| \|l_2(x_s;\bar{\theta})\| \\
\le & O(1)\zeta_n^4 \frac{d_n}{n} \sum_{t=1}^{n} d_t^{-1} \int l^2(x)dx + O(1)\zeta_n^4 \frac{d_n}{n} \sum_{t=2}^{n}\sum_{s=1}^{t-1} d_s^{-1} d_{ts}^{-1} \iint l(x)l(y)dxdy \\
\le & O(1)\zeta_n^4 + O(1)\zeta_n^4 n/d_n \to 0.
\end{aligned}$$

Notice that $T_1 = o_P(1)$ and $T_3 = o_P(1)$ imply $T_2 = o_P(1)$. Hence, it follows that $\frac{d_n}{nk} L_{3n} \to_P 0$ and then from the Cauchy-Schwarz inequality that $\frac{d_n}{nk} L_{2n} \to_P 0$ as well. The proof of Theorem 3.1 is finished. □

Detailed Proof of Lemma 3.1. It follows from Hualde and Robinson (2011) that $\widehat{\rho}$ is consistent, $\widehat{\rho} \to_P \rho_0$. Meanwhile, the series $\sum_{j=0}^{\infty} \psi_j(\widehat{\rho})$ converges uniformly in $\widehat{\rho}$ because of A.2 in Assumption A and the compactness of Ξ. The uniform convergence gives the continuity of $\widehat{\psi}$ on $\widehat{\rho}$. Hence $\widehat{\psi} \to_P \psi$ and $\widehat{d}_n/d_n \to_P 1$ by the continuous mapping theorem.

Next, we shall show $\widehat{\sigma}_e^2 \to_P \sigma_e^2$. Under the null,

$$\begin{aligned}
\widehat{\sigma}^2 = & \frac{1}{n}\sum_{t=1}^{n} \widehat{e}_t^2 = \frac{1}{n}\sum_{t=1}^{n} (y_t - g(x_t,\widehat{\theta}))^2 = \frac{1}{n}\sum_{t=1}^{n} (e_t + g(x_t,\theta) - g(x_t,\widehat{\theta}))^2 \\
= & \frac{1}{n}\sum_{t=1}^{n} (e_t + \widehat{g}(x_t))^2 = \frac{1}{n}\sum_{t=1}^{n} e_t^2 + \frac{1}{n}\sum_{t=1}^{n} \widehat{g}^2(x_t) + 2\frac{1}{n}\sum_{t=1}^{n} e_t \widehat{g}(x_t),
\end{aligned}$$

where $\widehat{g}(x) := g(x,\theta) - g(x,\widehat{\theta})$ for any real x.

To begin with, we shall show that $\frac{1}{n}\sum_{t=1}^{n} e_t^2 \to_P \sigma_e^2$. First, suppose $e_t = \sum_{j=0}^{\infty} \phi_j \epsilon_{t-j}$ stipulated in A.3(i) of Assumption A. The assertion holds due to Theorem 3.7 of Phillips and Solo (1992).

Second, suppose $e_t = \wp(\epsilon_t, \cdots, \epsilon_{t-m_0+1}; \eta_t, \cdots, \eta_{t-m_1+1})$ stipulated in A.3(ii) of Assumption A. Let $\bar{m} = \max(m_0, m_1)$. Notice that whenever $|t-s| > \bar{m}$, e_t and e_s are independent. Thus,

$$\begin{aligned}
E\left(\frac{1}{n}\sum_{t=1}^{n} e_t^2 - \sigma_e^2\right)^2 = & \frac{1}{n^2}\sum_{t=1}^{n} E(e_t^2 - \sigma_e^2)^2 + 2\frac{1}{n^2} E\sum_{t=2}^{n}\sum_{s=1}^{t-1} (e_t^2 - \sigma_e^2)(e_s^2 - \sigma_e^2) \\
= \frac{1}{n} Var(e_1^2) + 2\frac{1}{n^2} E \sum_{t=\bar{m}+1}^{n} \sum_{s=t-\bar{m}}^{t-1} & (e_t^2 - \sigma_e^2)(e_s^2 - \sigma_e^2) \le o(1) + 2\frac{1}{n^2} n\bar{m} Var(e_1^2) = o(1).
\end{aligned}$$

Next, we shall show $\frac{1}{n}\sum_{t=1}^{n} \widehat{g}^2(x_t) = o_P(1)$. For convenience, denote $\widehat{G}_n = \frac{1}{n}\sum_{t=1}^{n} \widehat{g}^2(x_t)$. Moreover, as $\|\widehat{\theta} - \theta_0\| = O_P(\zeta_n)$ assumed in Assumption B, for any $\epsilon > 0$, there exists a fixed $M > 0$ such that $P(\|\widehat{\theta} - \theta_0\| > \zeta_n M) < \epsilon$. Hence,

$$\begin{aligned}
P(\widehat{G}_n > \delta) \le & P(\|\widehat{\theta} - \theta\| > M\zeta_n) + P(\widehat{G}_n > \delta, \|\widehat{\theta} - \theta\| \le M\zeta_n) \\
\le & \epsilon + \frac{1}{\delta} E[\widehat{G}_n I(\|\widehat{\theta} - \theta\| \le M\zeta_n)],
\end{aligned}$$

by virtue of the Markov's inequality.

Using a Taylor expansion for $g(x;\theta)$ with respect to θ in a neighbourhood of θ_0, we have

$$\widehat{g}(x_t) = g(x_t;\theta_0) - g(x_t;\widehat{\theta}) = l_1(x_t;\theta_0)'(\theta_0 - \widehat{\theta}) - \frac{1}{2}(\theta_0 - \widehat{\theta})' l_2(x_t;\bar{\theta})(\theta_0 - \widehat{\theta})$$

where $\bar{\theta}$ is between θ_0 and $\widehat{\theta}$. It follows that

$$\begin{aligned}
&E[\widehat{G}_n I(\|\widehat{\theta} - \theta\| \le \zeta_n M)] = \frac{1}{n}\sum_{t=1}^{n} E[\widehat{g}^2(x_t) I(\|\widehat{\theta} - \theta\| \le \zeta_n M)] \\
=&\frac{1}{n}\sum_{t=1}^{n} E[l_1(x_t;\theta_0)'(\theta_0 - \widehat{\theta}) - \frac{1}{2}(\theta_0 - \widehat{\theta})' l_2(x_t;\bar{\theta})(\theta_0 - \widehat{\theta})]^2 I(\|\widehat{\theta} - \theta\| \le \zeta_n M) \\
\le&2\frac{1}{n}\sum_{t=1}^{n} E[l_1(x_t;\theta_0)'(\theta_0 - \widehat{\theta})]^2 I(\|\widehat{\theta} - \theta\| \le \zeta_n M) \\
&+ \frac{1}{2n}\sum_{t=1}^{n} E[(\theta_0 - \widehat{\theta})' l_2(x_t;\bar{\theta})(\theta_0 - \widehat{\theta})]^2 I(\|\widehat{\theta} - \theta\| \le \zeta_n M) \\
\le&2\zeta_n^2 M^2 \frac{1}{n}\sum_{t=1}^{n} E\|l_1(x_t;\theta_0)\|^2 + \zeta_n^4 M^4 \frac{1}{2n}\sum_{t=1}^{n} E\|l_2(x_t;\bar{\theta})\|^2 \\
\le&O(1)\zeta_n^2 \frac{1}{n}\sum_{t=1}^{n} d_t^{-1}\int \|l_1(x;\theta_0)\|^2 dx + O(1)\zeta_n^4 \frac{1}{n}\sum_{t=1}^{n} d_t^{-1}\int \|l(x)\|^2 dx \\
\le&O(1)\zeta_n^2 \frac{1}{\sqrt{n}} + O(1)\zeta_n^4 \frac{1}{\sqrt{n}} = o(1),
\end{aligned}$$

which, together with the convergence of $\frac{1}{n}\sum_{t=1}^{n} e_t^2$, implies $\frac{1}{n}\sum_{t=1}^{n} e_t \widehat{g}(x_t) = o_P(1)$. The assertion is proved. □

Proof of Theorem 3.3. This follows from Lemma 3.1, Theorem 3.1 and Theorem 3.2. □

Proof of Theorem 3.4. The proof is almost the same as that of Theorem 3.1 except

$$\begin{aligned}
&\frac{d_n}{nk}\sum_{t=1}^{n} \|Z(x_t)\|^2 \varphi^2(x_t) = \frac{d_n}{n}\sum_{t=1}^{n} \mathscr{T}_k(x_t)\varphi^2(x_t) \\
=&\frac{d_n}{n}\sum_{t=1}^{n} \mathscr{T}(x_t)\varphi^2(x_t) + \frac{d_n}{n}\sum_{t=1}^{n} [\mathscr{T}_k(x_t) - \mathscr{T}(x_t)]\varphi^2(x_t) \\
\to_D&\int \mathscr{T}(x)\varphi^2(x) dx L_\xi(1,0)
\end{aligned}$$

by Lemma C.5 and Wang and Phillips (2009). Thus, the limit is different from that of Theorem 3.1. □

Proof of Theorem 3.5. The proof can be completed in the same way as for that of Theorem 3.2. □

Appendix E: Additional simulation examples

Four simulation examples are given in this section to illustrate some points. First, given that L_n and Π_n are proposed to test regression functions in $L^2(\mathbb{R})$ and $L^2(\mathbb{R}, \exp(-x^2))$, respectively, and that mathematically $L^2(\mathbb{R})$ is a subspace of $L^2(\mathbb{R}, \exp(-x^2))$, i.e. $L^2(\mathbb{R}) \subset L^2(\mathbb{R}, \exp(-x^2))$, one may wonder whether researchers can always use Π_n ignoring which function space the regression function belongs to. From both the mathematical point of view and the derivation of Theorems 3.1, 3.2 and 3.4, it is true that Π_n can be used even if the regression function is in $L^2(\mathbb{R})$. We shall show one example below.

Second, we shall demonstrate how Π_n performs by an example when under H_0 the regression function $m(x) = g(x;\theta_0)$, where $g(x;\theta_0)$ is a polynomial function.

Thirdly, several cases are given to illustrate the performance of the test Π_n when the coefficient α_0 is big in absolute, say $\alpha_0 = 0.5$ and $\alpha_0 = 0.8$.

Lastly, in a special case where the departure function $\Delta(x) = xI(x > c)$ with $c > 0$, we investigate the performance of the power of Π_n in a finite sample situation with different c.

To simplify the discussion, we only consider the second case where the data generating process is the same as in Section 4. That is, suppose $\{(\epsilon_j, \eta_j), j \in \mathbb{Z}\}$ is a sequence of iid $N(0, \sigma^2 I_2)$. Let $e_t = a\epsilon_t + b\eta_t$, $u_t = \alpha_0 u_{t-1} + \epsilon_t$ with $|\alpha_0| < 1$ and $x_t = x_{t-1} + u_t$ with $x_0 = O_P(1)$.

In addition, the truncation parameter is selected by $k = [c \cdot n^\kappa]$ with $\kappa = 1/4.5$ and $c = 2.2$, respectively. We choose $\triangle_n(x) = \delta_n \frac{1}{1+x^2}$, in which $\delta_n = \frac{1}{2}\sqrt{k\log(n)/\sqrt{n}}$.

Example E.1. This example examines Π_n using an integrable regression model. The model for simulation is $y_t = m(x_t) + e_t$, and the null hypothesis is H_0: $P(m(x_t) = \theta_0/(1 + x_t^4)) = 1$ with $\theta_0 = 1$; and the alternative is H_1: $P(m(x_t) = \theta_1/(1 + x_t^4)) + \triangle_n(x_t)) = 1$ with $\theta_1 = 1$ and $\triangle_n(x) = \delta_n/(1 + x^2)$.

Table 1: Size: $m(x) = \theta_0/(1 + x^4)$, $\theta_0 = 1$

	Nominal size 1%	Nominal size 5%	Nominal size 10%
n		$\kappa = 1/4.5$	
	$\alpha_0 = 0.05$,	$a = 0.2$,	$b = 0.9$
300	0.0120	0.0640	0.1230
1200	0.0085	0.0485	0.1030
	$\alpha_0 = 0.01$,	$a = -0.05$,	$b = -0.1$
300	0.0110	0.0515	0.1110
1200	0.0100	0.0500	0.1050

$u_t = \alpha_0 u_{t-1} + \epsilon_t$, $e_t = a\epsilon_t + b\eta_t$, $(\epsilon_t, \eta_t) \sim iiN(0, \sigma^2 I_2)$ and $\sigma = 0.6$.

It can be seen clearly from Tables 1 and 2 that the proposed statistic test Π_n for the regression functions in $L^2(\mathbb{R}, \exp(-x^2))$ can work well for the regression functions in $L^2(\mathbb{R})$. The sizes approach the nominal level when the sample size increases, and the power performs well for the choice of $\triangle_n(x) = \delta_n/(1 + x^2)$, that is possibly because the function $1/(1 + x^2)$ has support that covers a neighbourhood of zero. See also Example E.4 below.

Example E.2. This example is to verify the performance of the Π_n when a regression function is a polynomial. We consider H_0: $P(m(x_t) = \theta_0(1 + x_t^2)) = 1$ with $\theta_0 = 1$ versus H_1: $P(m(x_t) = \theta_1(1 + x_t^2)) + \triangle_n(x_t)) = 1$ with $\theta_1 = 1$.

Table 2: Power: $m(x) = \theta_1/(1+x^4) + \triangle_n(x)$, $\theta_1 = 1$

	Nominal size 1%	Nominal size 5%	Nominal size 10%
n		$\kappa = 1/4.5$	
		$\alpha_0 = 0.05$, $a = 0.2$, $b = 0.9$	
300	1.0000	0.9995	1.0000
1200	1.0000	1.0000	1.0000
		$\alpha_0 = 0.01$, $a = -0.05$, $b = -0.1$	
300	0.9865	0.9955	0.9985
1200	0.9915	0.9965	0.9985

$u_t = \alpha_0 u_{t-1} + \epsilon_t$, $e_t = a\epsilon_t + b\eta_t$, $(\epsilon_t, \eta_t) \sim iiN(0, \sigma^2 I_2)$ and $\sigma = 0.6$.

Table 3: Size: $m(x) = \theta_0(1+x^2)$, $\theta_0 = 1$

	Nominal size 1%	Nominal size 5%	Nominal size 10%
n		$\kappa = 1/4.5$	
		$\alpha_0 = 0.05$, $a = 0.2$, $b = 0.9$	
300	0.0060	0.0350	0.0850
1200	0.0085	0.0445	0.1025
		$\alpha_0 = 0.1$, $a = -0.05$, $b = 0.3$	
300	0.0085	0.0400	0.0940
1200	0.0095	0.0455	0.1005

$u_t = \alpha_0 u_{t-1} + \epsilon_t$, $e_t = a\epsilon_t + b\eta_t$, $(\epsilon_t, \eta_t) \sim iiN(0, \sigma^2 I_2)$ and $\sigma = 0.6$.

Table 4: Power: $m(x) = \theta_1(1+x^2) + \triangle_n(x)$, $\theta_1 = 1$

	Nominal size 1%	Nominal size 5%	Nominal size 10%
n		$\kappa = 1/4.5$	
		$\alpha_0 = 0.05$, $a = 0.2$, $b = 0.9$	
300	0.9250	0.9505	0.9630
1200	0.9595	0.9765	0.9820
		$\alpha_0 = 0.1$, $a = -0.05$, $b = 0.3$	
300	0.9980	0.9985	0.9985
1200	0.9995	0.9995	0.9990

$u_t = \alpha_0 u_{t-1} + \epsilon_t$, $e_t = a\epsilon_t + b\eta_t$, $(\epsilon_t, \eta_t) \sim iiN(0, \sigma^2 I_2)$ and $\sigma = 0.6$.

Under H_0, $m(x)$ is a polynomial function of order 2. Thus, we may have an accurate expansion $m(x) = c_0 H_0(x) + c_1 H_1(x) + c_2 H_2(x)$, since in the infinite expansion $m(x) = \sum_{i=0}^{\infty} c_i H_i(x)$, for all $i \geq 3$, $c_i = 0$. This is because $H_i(x)$ for all $i \geq 3$ are orthogonal with $H_j(x)$ for $j \leq 2$, and are therefore orthogonal with $m(x)$. Therefore, we do not have any bias term for the approximation in the derivation of the test Π_n. It is obvious that the test performs very well in both size and power.

Example E.3. (1) We consider H_0: $P(m(x_t) = \theta_0/(1+x_t^6)) = 1$ with $\theta_0 = 1$ versus H_1: $P(m(x_t) = \theta_1/(1+x_t^6) + \triangle_n(x_t)) = 1$ with $\theta_1 = 1$. This example will compare the effect of the coefficient α_0 in the AR(1) process for the test L_n.

The results of the size and power are reported in Tables 5 and 6, respectively. In each of the cases of both $\alpha_0 = 0.5$ and $\alpha_0 = 0.1$, the size approaches the significant level and the power is very close to one as the sample size increases from 300 to 1200.

Table 5: Size: $m(x) = \theta_0/(x^6+1)$, $\theta_0 = 1$

L_n	Nominal size 5%	Nominal size 10%
n	$\kappa = 1/4.5$	
	$\alpha_0 = 0.5$, $a = 0.2$, $b = 0.7$	
300	0.0460	0.0860
1200	0.0520	0.0990
	$\alpha_0 = 0.1$, $a = -0.05$, $b = 0.3$	
300	0.0420	0.1110
1200	0.0490	0.0990

$u_t = \alpha_0 u_{t-1} + \epsilon_t$, $e_t = a\epsilon_t + b\epsilon_{t-1}$, $\epsilon_t \sim iiN(0, \sigma^2)$ and $\sigma = 0.6$.

Table 6: Power: $m(x) = \theta_1/(x^6+1)$, $\theta_1 = 1$

L_n	Nominal size 5%	Nominal size 10%
n	$\kappa = 1/4.5$	
	$\alpha_0 = 0.5$, $a = 0.2$, $b = 0.7$	
300	0.9820	0.9870
1200	0.9990	0.9930
	$\alpha_0 = 0.1$, $a = -0.05$, $b = 0.3$	
300	0.9970	0.9990
1200	1.0000	1.0000

$u_t = \alpha_0 u_{t-1} + \epsilon_t$, $e_t = a\epsilon_t + b\epsilon_{t-1}$, $\epsilon_t \sim iiN(0, \sigma^2)$ and $\sigma = 0.6$.

(2) In addition, we consider H_0: $P(m(x_t) = \theta_0(1+x_t^2)) = 1$ with $\theta_0 = 1$ versus H_1: $P(m(x_t) =$

$\theta_1(1+x_t^2)) + \triangle_n(x_t)) = 1$ with $\theta_1 = 1$. This example mainly illustrates how the test statistic Π_n performs when the coefficient α_0 in the AR(1) process is relatively large, say 0.8.

Table 7: Size: $m(x) = \theta_0(x^2+1),\ \ \theta_0 = 1$

Π_n	Nominal size 1%	Nominal size 10%
n	$\kappa = 1/4.5$	
	$\alpha_0 = 0.8, \quad a = 0.2, \quad b = 1$	
280	0.0010	0.0840
1300	0.0130	0.0870
	$\alpha_0 = 0.8, \quad a = 0, \quad b = 1$	
280	0.0070	0.0870
1300	0.0100	0.1030

$u_t = \alpha_0 u_{t-1} + \epsilon_t$, $e_t = a\epsilon_t + b\epsilon_{t-1}$, $\epsilon_t \sim iiN(0,\sigma^2)$ and $\sigma = 0.3$.

Table 8: Power: $m(x) = \theta_0(x^2+1),\ \ \theta_0 = 1$

Π_n	Nominal size 1%	Nominal size 10%
n	$\kappa = 1/4.5$	
	$\alpha_0 = 0.8, \quad a = 0.2, \quad b = 1$	
280	0.9910	0.9920
1300	0.9920	0.9980
	$\alpha_0 = 0.8, \quad a = 0, \quad b = 1$	
280	0.9870	0.9910
1300	0.9950	0.9950

$u_t = \alpha_0 u_{t-1} + \epsilon_t$, $e_t = a\epsilon_t + b\epsilon_{t-1}$, $\epsilon_t \sim iiN(0,\sigma^2)$ and $\sigma = 0.3$.

The results reported in Tables 7 and 8 indicate, even for relatively large α_0, the test Π_n still has stable size and robust power.

Example E.4. Here, we consider an interesting example provided by the co-editor. Let $g(x;\theta_1) = \theta_1 x^2 + \delta_n \Delta(x)$ with $\theta_1 = 1$, $\Delta(x) = xI(x > c)$ and $c > 0$. We investigate the finite–sample performance of the power of Π_n for different c.

Though it is clear from the proof of Theorem 3.4 that the power of Π_n always has order $O_P(\delta_n^2 n/kd_n)$ theoretically, its performance in the finite sample situation may be affected by other factors, such as the local time $L_B(1,0)$ and the integral $\int \Delta(x)e^{-x^2/2}\mathscr{H}_i(x)dx$ for $i = 0,\cdots,k-1$.

We use the same data generating process, $x_t = x_{t-1} + u_t$, $u_t = \alpha_0 u_{t-1} + \epsilon_t$, $e_t = a\epsilon_t + b\epsilon_{t-1}$, $\epsilon_t \sim iiN(0,\sigma^2)$ and $\sigma = 0.3$. Let $k = [2.2n^\kappa]$ with $\kappa = 1/4$. Take $\delta_n = 2\sqrt{k\log(n)/\sqrt{n}}$.

Table 9: Power at nominal size 1%: $\Delta(x) = xI(x > c)$

	$c = 0.5$			$c = 1.5$			$c = 2.5$		
n	280	600	1300	280	600	1300	280	600	1300
				$a = 0.1, \quad b = 0.7$					
	0.8650	0.8890	0.9320	0.6430	0.7610	0.8210	0.3440	0.4920	0.6500
				$a = 0.2, \quad b = 1$					
	0.8540	0.9120	0.9190	0.6310	0.7460	0.8310	0.2890	0.4190	0.5700
				$a = 0, \quad b = 1$					
	0.8310	0.8810	0.9310	0.6030	0.7470	0.8230	0.3000	0.4300	0.5490

Table 10: Power at nominal size 5%: $\Delta(x) = xI(x > c)$

	$c = 0.5$			$c = 1.5$			$c = 2.5$		
n	280	600	1300	280	600	1300	280	600	1300
				$a = 0.1, \quad b = 0.7$					
	0.8820	0.8960	0.9340	0.6640	0.7730	0.8310	0.3990	0.5480	0.6900
				$a = 0.2, \quad b = 1$					
	0.8600	0.9180	0.9210	0.6530	0.7640	0.8370	0.3560	0.4830	0.6150
				$a = 0, \quad b = 1$					
	0.8500	0.8920	0.9380	0.6230	0.7630	0.8300	0.3620	0.4920	0.6070

Table 11: Power at nominal size 10%: $\Delta(x) = xI(x > c)$

	$c = 0.5$			$c = 1.5$			$c = 2.5$		
n	280	600	1300	280	600	1300	280	600	1300
				$a = 0.1, \quad b = 0.7$					
	0.8870	0.9000	0.9380	0.6840	0.7860	0.8370	0.4350	0.5860	0.7220
				$a = 0.2, \quad b = 1$					
	0.8700	0.9240	0.9260	0.6790	0.7780	0.8470	0.4170	0.5320	0.6440
				$a = 0, \quad b = 1$					
	0.8580	0.9000	0.9440	0.6390	0.7810	0.8380	0.4210	0.5310	0.6370

At the nominal levels 1%, 5% and 10%, we calculate the powers, respectively, which are reported in the Tables 9-11. Although $\Delta(x) = 0$ for $x < c$, in the case where either $c = 0.5$ or $c = 1.5$, the power of Π_n is robust for all sample sizes. However, when $c = 2.5$, the powers are reduced, particularly in the case where the sample size is small.

From these experiments, the test Π_n appears to have robust power in finite samples when the deviation function $\Delta(x)$ has its support covering some interval that contains zero, or even part of such an interval (like the case $c = 0.5$). This ensures that the integral $\int \Delta(x)e^{-x^2/2}\mathscr{H}_i(x)dx$ is non–negligible. However, beyond some interval containing zero the integrand and hence the integral will be close to zero.

References

Billingsley, P. (1968). *Convergence of Probability Measures.* John Wiley & Sons, Inc., New York.

Edelman, A. and Rao, N. R. (2005). Random matrix theory. *Acta Numerica*, 14(1):233–297.

Hualde, J. and Robinson, P. M. (2011). Gaussian pseudo-maximum likelihood estimation of fractional time series models. *Annals of Statistics*, 39:3152–3181.

Kanwal, R. P. (1983). *Generalized Functions: Theory and Technique.* Academic Press, New York.

Phillips, P. C. B. and Solo, V. (1992). Asymptotics for linear processes. *Annals of Statistics*, 20(2):971–1001.

Pötscher, B. M. (2004). Nonlinear functions and convergence to Brownian motion: beyond the continuous mapping theorem. *Econometric Theory*, 20:1–22.

Renyi, A. (1958). On mixing sequences of sets. *Acta Mathematica Hungarica*, 9:215–228.

Tao, T. (2012). *Topics in Random Matrix Theory.* Graduate studies in Mathematics, V-132. American Mathematical Society.

Wang, Q. and Phillips, P. C. B. (2009). Asymptotic theory for local time density estimation and nonparametric cointegreting regression. *Econometric Theory*, 25:710–738.