Online Supplementary Appendix to “On Nonparametric Inference in the Regression Discontinuity Design”

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Abstract

This document provides a proof to Theorem 4.2 in the author's paper “On Nonparametric Inference in the Regression Discontinuity Design”.

KEYWORDS: Regression discontinuity design, uniform testing.

JEL classification codes: C12, C14.

A Additional Notation

Let $Z^{(n)} = \{Z_i : 1 \leq i \leq n\}$ denote the observed sample of the random variable Z. Let $a_n \precsim b_n$ denote $a_n \leq A b_n$, where a_n and b_n are deterministic sequences and A is a positive constant uniform in $\mathbf{P}$. Let $|\cdot|$ denote the Euclidean matrix norm. As we use the notion of convergence in probability under the sequence of distributions P_n, let $A_n = o_{P_n}(1)$ denote

$$P_n(|A_n| > \epsilon) \to 0 \text{ as } n \to \infty \ ,$$

for a sequences of random variables $A_n \sim P_n$, where ϵ is any constant such that $\epsilon > 0$. Further, in Table 1 below, we introduce additional notation to keep our arguments concise.

$H(h_n)$	$\text{diag}(1,h_n^{-1},h_n^{-2})$
$r(Z_i/h_n)$	$(1, Z_i/h_n, (Z_i/h_n)^2)'$
$Z_n(h_n)$	$(r(Z_1/h_n), \ldots, r(Z_n/h_n))'$
$k(u)$	$(1-u)1\{0 \leq u \leq 1\}$
$K(u)$	$k(-u)1\{u < 0\} + k(u)1\{u \geq 0\}$
$K_{h_n}(u)$	$K(u/h_n)/h_n$

$W_{+,n}(h_n)$	$\text{diag}(1\{Z_1 \geq 0\}K_{h_n}(Z_1), \ldots, 1\{Z_n \geq 0\}K_{h_n}(Z_n))$
$\Gamma_{+,n}(h_n)$	$Z_n(h_n)'W_{+,n}(h_n)Z_n(h_n)/n$
$S_n(h_n)$	$((Z_1/h_n)^3, \ldots, (Z_n/h_n)^3)'$
$\nu_{+,n}$	$Z_n(h_n)'W_{+,n}(h_n)S_n(h_n)/n$
e	$(1, 0, 0)'$
$\mu(z, P)$	$E_P[Y\|Z = z]$
$\mu_+(P)$	$\lim_{z\to 0^+} \mu(z, P)$
$\mu_-(P)$	$\lim_{z\to 0^-} \mu(z, P)$
$\mu^v(z, P)$	$d^v\mu(z, P)/dz^v$
$\mu^v_+(P)$	$\lim_{z\to 0^+} \mu^v(z, P)$
$\sigma^2(z, P)$	$Var_P[Y\|Z = z]$
$\Sigma_n(P)$	$\text{diag}(\sigma^2(Z_1, P), \ldots, \sigma^2(Z_n, P))$
$\Psi_{+,n}(h_n, P)$	$Z_n(h_n)'W_{+,n}(h_n)\Sigma_n(P)W_{+,n}(h_n)Z_n(h_n)/n$
$\mathbf{Y}_n$	$(Y_1, \ldots, Y_n)'$
$\hat{\beta}_{+,n}$	$H(h_n)\Gamma^{-1}_{+,n}(h_n)Z_n(h_n)'W_{+,n}(h_n)\mathbf{Y}_n/n$

Table 1: Important Notation

Next, we provide an extended description of the test statistic used. For our null hypothesis as stated in the paper, the test statistic can be rewritten as

$$T_n^{CCT} = \frac{\hat{\mu}_{+,n} + \hat{\mu}_{-,n} - (\mu_+(P) - \mu_-(P))}{\hat{S}_n} \ , \tag{A-1}$$

where $\mu_+(P) - \mu_-(P) = \theta_0$, $\hat{\mu}_{+,n}$ and $\hat{\mu}_{-,n}$ are bias corrected local linear estimates of $\mu_+(P)$ and $\mu_-(P)$, and

$$\hat{S}_n = \sqrt{\hat{V}_{+,n} + \hat{V}_{-,n}} \ ,$$

where $\hat{V}_{+,n}$ and $\hat{V}_{+,n}$ are plug-in estimates conditional on $Z^{(n)}$ of the variances of $\hat{\mu}_{+,n}$ and $\hat{\mu}_{-,n}$; see (C-13) for the plug-in estimator used. The bias of both estimates are estimated using local quadratic estimators. Furthermore, for all estimates, we use the triangular kernel, i.e. $k(u)$ in Table 1, and a deterministic sequence of bandwidth choices denoted by h_n. Throughout this document, we provide results for quantities with subscript $(+)$ as arguments for those with subscript $(-)$ follow symmetrically. In addition, as noted in Calonico et al. (2014a, Remark 7), we exploit the fact that in our simple version of the test statistic the estimates are numerically equivalent to those from a non-bias-corrected local quadratic estimator. In turn, we can write

$$\hat{\mu}_{+,n} = e'\hat{\beta}_{+,n} \ , \tag{A-2}$$

which reduces the length of the proof presented below. Further, as stated in the paper, note that

$$\mathbf{Q} = \{Q \in \mathbf{Q}_{\mathcal{W}} : Q \text{ satisfies Assumption 4.1}\} \ , \tag{A-3}$$

and that

$$\mathbf{P} = \{QM^{-1} : Q \in \mathbf{Q}\} \ , \tag{A-4}$$

where $\mathbf{Q}_{\mathcal{W}}$, M^{-1} and Assumption 4.1 are as defined in the paper.

B Auxiliary Lemmas

Lemma B.1. *Let* $\mathbf{Q}$ *be defined as in* (A-3), $\mathbf{P}$ *be as in* (A-4) *and* $P_n \in \mathbf{P}$ *for all* $n \geq 1$. *If* $nh_n \to \infty$ *and* $h_n \to 0$, *then*

(i) $\Gamma_{+,n}(h_n) = \tilde{\Gamma}_{+,n}(h_n) + o_{P_n}(1)$, *where* $\tilde{\Gamma}_{+,n}(h_n) = \int_0^\infty K(u)r(u)r(u)' f_{P_n}(uh_n)du \in [\Gamma_L, \Gamma_U]$.

(ii) $\nu_{+,n}(h_n) = \tilde{\nu}_{+,n}(h_n) + o_{P_n}(1)$, *where* $\tilde{\nu}_{+,n}(h_n) = \int_0^\infty K(u)r(u)u^2 f_{P_n}(uh_n)du \in [\nu_L, \nu_U]$.

(iii) $h_n \Psi_{+,n}(h_n, P_n) = \tilde{\Psi}_{+,n}(h_n) + o_{P_n}(1)$, *where* $\tilde{\Psi}_{+,n}(h_n) = \int_0^\infty K(u)^2 r(u)r(u)'\sigma^2_{P_n}(uh_n) f_{P_n}(uh_n)du \in [\Psi_L, \Psi_U]$.

Proof. For (i), a change of variables gives us

$$
\begin{aligned}
E_{P_n^n}[\Gamma_{+,n}(h_n)] &= E_{P_n}\left[\frac{1}{nh_n}\sum_{i=1}^n 1\{Z_i \geq 0\}K(Z_i/h_n)r(Z_i/h_n)r(Z_i/h_n)'\right] \\
&= \frac{1}{h_n}\int_0^\infty K(z/h_n)r(z/h_n)r(z/h_n)' f_{P_n}(z)dz \\
&= \int_0^\infty K(u)r(u)r(u)' f_{P_n}(uh_n) \equiv \tilde{\Gamma}_{+,n}(h_n) \ .
\end{aligned}
$$

Further, since $h_n < \tilde{\kappa}$ for large enough n, we have that $\tilde{L} \leq f_{P_n}(z) \leq \tilde{U}$ by Assumption 4.1, which implies that

$$
\Gamma_L \equiv \tilde{L}\int_0^\infty K(u)r(u)r(u)'du \leq \tilde{\Gamma}_{+,n}(h_n) \leq \tilde{U}\int_0^\infty K(u)r(u)r(u)'du \equiv \Gamma_U \ ,
$$

and that

$$
\begin{aligned}
E_{P_n^n}[|\Gamma_{+,n}(h_n) - E_{P_n}[\Gamma_{+,n}(h_n)]|^2] &\leq \frac{1}{h_n^2}E_{P_n}\left[|1\{Z_i \geq 0\}K(Z_i/h_n)r(Z_i/h_n)r(Z_i/h_n)'|^2\right] \\
&= \frac{1}{nh_n}\int_0^\infty K(u)^2|r(u)|^4 f_{P_n}(uh_n)du \\
&\leq \frac{\tilde{U}}{nh_n}\int_0^\infty K(u)^2|r(u)|^4 du \\
&= O(n^{-1}h_n^{-1}) = o(1) \ .
\end{aligned}
$$

It then follows by Markov's Inequality that $\Gamma_{+,n}(h_n) = \tilde{\Gamma}_{+,n}(h_n) + o_{P_n}(1)$. Analogously, closely following Calonico et al. (2014b, Lemma S.A.1), we can show Lemma B.1(ii)-(iii). ■

Lemma B.2. *Let* $\mathbf{Q}$ *be defined as in* (A-3), $\mathbf{P}$ *be as in* (A-4) *and* $P_n \in \mathbf{P}$ *for all* $n \geq 1$. *If* $nh_n \to \infty$ *and* $h_n \to 0$, *then*

(i) $E_{P_n^n}[\hat{\mu}_{+,n}|Z^{(n)}] = \mu_+(P_n) + O_{P_n}(h_n^3)$.

(ii) $V_{P_n^n}[\hat{\mu}_{+,n}|Z^{(n)}] = n^{-1}e'\Gamma_{+,n}^{-1}(h_n)\Psi_{+,n}(h_n, P_n)\Gamma_{+,n}^{-1}(h_n)e \equiv V_{+,n}(h_n, P_n)$.

(iii) $(V_{+,n}(h_n, P_n))^{-1/2}\left(\hat{\mu}_{+,n} - E_{P_n^n}[\hat{\mu}_{+,n}|Z^{(n)}]\right) \xrightarrow{d} \mathcal{N}(0, 1)$.

Proof. For (i), by taking the conditional on $Z^{(n)}$ expectation, we have

$$E_{P_n^n}[\hat{\mu}_{+,n}|Z^{(n)}] = e'H(h_n)\Gamma_{+,n}^{-1}(h_n)Z_n(h_n)'W_{+,n}(h_n)E_{P_n^n}[\mathbf{Y}_n|Z^{(n)}]/n \ .$$

Further, as $h_n < \tilde{\kappa}$ for large enough n, we have by the required differentiability in Assumption 4.1 and a Taylor expansion for $0 < Z < h_n$ that

$$E_{P_n}[Y|Z] = \mu_+(P_n) + Z\mu_+^1(P_n) + (Z/2)^2\mu_+^2(P_n) + O_{P_n}(h_n^3) \ .$$

It then follows from Lemma B.1 and the previous two expressions that

$$E_{P_n^n}[\hat{\mu}_+|Z^{(n)}] = \mu_+(P_n) + O_{P_n}(h_n^3) \ .$$

For (ii), a simple calculation gives us

$$\begin{aligned} V_{P_n^n}[\hat{\mu}_+(h_n)|Z^{(n)}] &= e'H(h_n)\Gamma_{+,n}^{-1}(h_n)Z_n(H_n)'W_{+,n}(h_n)\Sigma_n(P_n)W_{+,n}(h_n)Z_n(h_n)\Gamma_{+,n}^{-1}(h_n)H(h_n)e/n^2 \\ &= n^{-1}e'\Gamma_{+,n}^{-1}(h_n)\Psi_{+,n}(h_n,P_n)\Gamma_{+,n}^{-1}(h_n)e \equiv V_{+,n}(h_n,P_n) \ . \end{aligned}$$

For (iii), first note that from Lemma B.1 we have $V_{+,n}(h_n,P_n) = \tilde{V}_{+,n}(h_n) + o_{P_n}(1)$, where

$$\tilde{V}_{+,n}(h_n) = (nh_n)^{-1}e'\tilde{\Gamma}_{+,n}^{-1}(h_n)\tilde{\Psi}_{+,n}(h_n)\tilde{\Gamma}_{+,n}^{-1}(h_n)e \ .$$

Then rewrite as follows

$$\frac{\hat{\mu}_{+,n} - E_{P_n^n}[\hat{\mu}_{+,n}|Z^{(n)}]}{\sqrt{V_{+,n}(h_n,P_n)}} = \left(\frac{\tilde{V}_{+,n}(h_n,P_n)}{V_{+,n}(h_n,P_n)}\right)^{1/2}\left(\tilde{V}_{+,n}(h_n)\right)^{-1/2}e'\Gamma_{+,n}^{-1}(h_n)\tilde{\Gamma}_{+,n}(h_n)\tilde{A}_n^{1/2}\xi_n \ , \tag{B-5}$$

where

$$\begin{aligned} \xi_n &= \sum_{i=1}^{n}\omega_{n,i}\epsilon_{n,i} \ , \\ \epsilon_{n,i} &= Y_i - E_{P_n}[Y_i|Z_i] \ , \\ \tilde{A}_n &= (nh_n)^{-1}\tilde{\Gamma}_{+,n}^{-1}(h_n)\tilde{\Psi}_{+,n}(h_n)\tilde{\Gamma}_{+,n}^{-1}(h_n) \ , \text{ and} \\ \omega_{n,i} &= \tilde{A}_n^{-1/2}\tilde{\Gamma}_{+,n}^{-1}(h_n)K_{h_n}(Z_i/h_n)r(Z_i/h_n)/n \ . \end{aligned}$$

Next note that for any $a \in \mathbf{R}^3$ we have that $\{a'\omega_{n,i}\epsilon_{n,i} : 1 \le i \le n\}$ is a triangular array of independent random variables with $E_{P_n^n}[a'\xi_n] = 0$ and $V_{P_n^n}[a'\xi_n] = a'a$. Further, this triangular array satisfies the Lindeberg condition. To see why, first note that by Lemma B.1 we have

$$|\tilde{A}_n| \ge (nh_n)^{-1}|\tilde{A}_L| \ , \tag{B-6}$$

for some value $\tilde{A}_L \in \mathbf{R}$, which is uniform in $\mathbf{P}$. We then have by Lemma B.1 and a change of variables that

$$\begin{aligned} \sum_{1=1}^{n}E_{P_n}[|a'\omega_{n,i}\epsilon_i|^4] &\precsim (nh_n)^2\sum_{1=1}^{n}E_{P_n}\left[|a'K_{h_n}(Z/h_n)r(Z/h_n)/n|^4\right] \\ &\precsim (nh_n)^2 n^{-3}h_n^{-4}\int_0^\infty |a'K(z/h_n)r(z/h_n)|^4 f_{P_n}(z)dz \\ &\precsim (nh_n)^2 n^{-3}h_n^{-3} = O\left((nh_n)^{-1}\right) = o(1) \end{aligned}$$

and hence, using the Lindeberg-Feller CLT, we have that $a'\xi_n \xrightarrow{d} \mathcal{N}(0, a'a)$ as $n \to \infty$. Since this holds for any $a \in \mathbf{R}^3$, the Cramér-Wold theorem implies that $\xi_n \xrightarrow{d} \mathcal{N}(0, I_3)$ as $n \to \infty$, where I_3 denotes the identity matrix of size three. Furthermore, analogous to $V_+(h_n, P_n) = \tilde{V}_+(h_n) + o_{P_n}(1)$, we can show that

$$\frac{V_{+,n}(h_n, P_n)}{\tilde{V}_{+,n}(h_n)} = 1 + o_{P_n}(1) \ . \tag{B-7}$$

Further, by Lemma B.1 we have that

$$\Gamma_{+,n}^{-1}(h_n)\tilde{\Gamma}_{+,n}(h_n) = I_3 + o_{P_n}(1) \ . \tag{B-8}$$

Substituting the above results in (B-5) concludes the proof. ■

C Proof of Theorem 4.2

Here we show only that

$$\frac{\hat{\mu}_{+,n} - \mu_+(P_n)}{\sqrt{\hat{V}_{+,n}}} \xrightarrow{d} \mathcal{N}(0,1) \ ,$$

since under similar arguments it will follow that

$$\frac{\hat{\mu}_{n,-} - \mu_-(P_n)}{\sqrt{\hat{V}_{n,-}}} \xrightarrow{d} \mathcal{N}(0,1) \ ,$$

and then due to independence we can conclude that $T_n^{CCT} \xrightarrow{d} \mathcal{N}(0,1)$ as $n \to \infty$. To this end, first rewrite

$$\frac{\hat{\mu}_{+,n} - \mu_+(P_n)}{\sqrt{\hat{V}_{+,n}}} = \frac{\hat{\mu}_{+,n} - \mu_+(P_n)}{\sqrt{V_{+,n}(h_n, P_n)}} \cdot \sqrt{\frac{V_{+,n}(h_n, P_n)}{\hat{V}_{+,n}}} \ .$$

Step 1. We show that

$$\frac{\hat{\mu}_{+,n} - \mu_+(P_n)}{\sqrt{V_{+,n}(h_n, P_n)}} \xrightarrow{d} \mathcal{N}(0,1) \ . \tag{C-9}$$

To begin, first rewrite the above as

$$\frac{\hat{\mu}_{+,n} - E_{P_n^n}[\hat{\mu}_{+,n}|Z^{(n)}]}{\sqrt{V_{+,n}(h_n, P_n)}} + \left(\frac{\tilde{V}_{+,n}(h_n)}{V_{+,n}(h_n, P_n)}\right)^{1/2} \frac{E_{P_n^n}[\hat{\mu}_{+,n}|Z^{(n)}] - \mu_+(P_n)}{\sqrt{\tilde{V}_{+,n}(h_n)}} \ .$$

In Lemma B.2 (iii), we showed that

$$\frac{\hat{\mu}_{+,n} - E_{P_n^n}[\hat{\mu}_{+,n}|Z^{(n)}]}{\sqrt{V_{+,n}(h_n, P_n)}} \xrightarrow{d} \mathcal{N}(0,1) \ \text{ and } \ \frac{\tilde{V}_{+,n}(h_n)}{V_{+,n}(h_n, P_n)} = 1 + o_{P_n}(1) \ .$$

It then remains to show that

$$\frac{E_{P_n^n}[\hat{\mu}_{+,n}|Z^{(n)}] - \mu_+(P_n)}{\sqrt{\tilde{V}_{+,n}(h_n)}} = o_{P_n}(1) \ ,$$

to conclude. To this end, note that by Lemma B.2 and (B-6), it follows that

$$\frac{\left|E_{P_n^n}[\hat{\mu}_{+,n}|Z^{(n)}] - \mu_+(P_n)\right|}{\sqrt{\tilde{V}_{+,n}(h_n)}} = O\left((nh_n)^{1/2}\right) O_{P_n}(h_n^3) = O_{P_n}\left((nh_n^7)^{1/2}\right) = o_{P_n}(1)$$

as $h_n \to 0$, $nh_n \to \infty$ and $nh_n^7 \to 0$.

Step 2. We show that

$$\frac{V_{+,n}(h_n, P_n)}{\hat{V}_{+,n}} = 1 + o_{P_n}(1) \ . \tag{C-10}$$

To begin note that

$$nh_n\left(V_{+,n}(h_n, P_n) - \hat{V}_{+,n}\right) = e'\Gamma_{+,n}^{-1}(h_n) \cdot h_n\left(\Psi_{+,n}(h_n, P_n) - \hat{\Psi}_{+,n}(h_n)\right) \cdot \Gamma_{+,n}^{-1}(h_n)e \ , \tag{C-11}$$

where

$$h_n\left(\Psi_{+,n}(h_n, P_n) - \hat{\Psi}_{+,n}(h_n)\right) = h_n Z_n(h_n)' W_{+,n}(h_n)\left(\Sigma_n(P_n) - \hat{\Sigma}_n\right) W_{+,n}(h_n) Z_n(h_n)/n \ , \tag{C-12}$$

and

$$\hat{\Sigma}_{+,n} = \text{diag}(\hat{\epsilon}^2_{+,n,1}, \ldots, \hat{\epsilon}^2_{+,n,n}) \ , \tag{C-13}$$

such that $\hat{\epsilon}_{+,n,i} = Y_i - \hat{\mu}_{+,n}$. Further, note that by construction, we can write

$$Y_i = \mu(Z_i, P_n) + \epsilon_{n,i} \ , \tag{C-14}$$

such that $E_{P_n}[\epsilon_{n,i}] = 0$ and $Var_{P_n}[\epsilon_{n,i}|Z = z] = \sigma^2(z, P_n)$. This in turn implies

$$\hat{\epsilon}_{+,n,i} = \epsilon_{n,i} + \mu(Z_i, P_n) - \mu_+(P_n) + \mu_+(P_n) - \hat{\mu}_{+,n} \ . \tag{C-15}$$

We can then expand (C-12) to get the following

$$\begin{aligned}
h_n\left(\Psi_{+,n}(h_n, P_n) - \hat{\Psi}_{+,n}(h_n)\right) = &\underbrace{h_n \sum_{i=1}^n 1\{Z_i \geq 0\}(\sigma^2(Z_i, P_n) - \epsilon^2_{n,i}) K_{h_n}(Z_i)^2 r(Z_i/h_n) r(Z_i/h_n)'/n}_{\equiv B_{1,n} \ , \ (a)} \\
&\underbrace{- h_n \sum_{i=1}^n 1\{Z_i \geq 0\}(\mu(Z_i, P_n) - \hat{\mu}_{+,n})^2 K_{h_n}(Z_i)^2 r(Z_i/h_n) r(Z_i/h_n)'/n}_{\equiv B_{2,n} \ , \ (b)} \\
&\underbrace{+ 2\, h_n \sum_{i=1}^n 1\{Z_i \geq 0\}\epsilon_{n,i}(\mu(Z_i, P_n) - \hat{\mu}_{+,n}) K_{h_n}(Z_i)^2 r(Z_i/h_n) r(Z_i/h_n)'/n}_{\equiv B_{3,n} \ , \ (c)} \ .
\end{aligned}$$

For quantity (a), since Assumption 2.1 (i), Assumption 2.1 (ii) and Assumption 2.1 (iv) are satisfied with the required uniform constants, we have by a change of variables that

$$\begin{aligned}
E_{P_n}\left[|B_{1,n}|^2\right] &\lesssim (nh_n)^{-1} \int_0^\infty K(u)^4 |r(u)|^4 du \\
&= O((nh_n)^{-1}) = o(1) \ ,
\end{aligned}$$

which implies by Markov's Inequality that $B_{n,1} = o_{P_n}(1)$. For quantity (b), note that first we can rewrite it as

$$B_{n,2} = \underbrace{h_n \sum_{i=1}^n 1\{Z_i \geq 0\}(\mu(Z_i, P_n) - \mu_+(P_n))^2 K_{h_n}(Z_i)^2 r(Z_i/h_n) r(Z_i/h_n)'/n}_{\equiv B_{n,21}}$$
$$+ (\mu_+(P_n) - \hat{\mu}_{+,n})^2 \cdot \underbrace{h_n \sum_{i=1}^n 1\{Z_i \geq 0\} K_{h_n}(Z_i)^2 r(Z_i/h_n) r(Z_i/h_n)'/n}_{\equiv B_{n,22}}$$
$$+ 2(\mu_+(P_n) - \hat{\mu}_{+,n}) \cdot \underbrace{h_n \sum_{i=1}^n 1\{Z_i \geq 0\}(\mu(Z_i, P_n) - \mu_+(P_n)) K_{h_n}(Z_i)^2 r(Z_i/h_n) r(Z_i/h_n)'/n}_{\equiv B_{n,23}} ,$$

Next, since Assumption 2.1 (i) and Assumption 2.1 (iii) are satisfied with the required uniform constants, we have by a Taylor approximation and a change of variables that

$$\begin{aligned} E_{P_n}[|B_{n,21}|^2] &\precsim n^{-1} h_n^3 \int_0^\infty K(u)^4 |r(u)|^4 du \\ &= O(n^{-1} h_n^3) = o(1) , \end{aligned}$$

which implies by Markov's inequality that $B_{n,21} = o_{P_n}(1)$. Further, since Assumption 2.1 (i) is satisfied with the required uniform constants, we have by a change of variables that

$$\begin{aligned} E_{P_n}[|B_{n,22}|^2] &\precsim (n h_n)^{-1} \int_0^\infty K(u)^4 |r(u)|^4 du \\ &= O((n h_n)^{-1}) = o(1) , \end{aligned}$$

which implies by Markov's inequality that $B_{n,22} = o_{P_n}(1)$. Finally, since Assumption 2.1 (i) and Assumption 2.1 (iii) are satisfied with the required uniform constants, we have by a Taylor approximation and a change of variables that

$$\begin{aligned} E_{P_n}\left[|B_{n,23}|^2\right] &\precsim (n)^{-1} h_n \int_0^\infty K(u)^4 |r(u)|^4 du \\ &= O(n^{-1} h_n) = o(1) , \end{aligned}$$

which implies by Markov's inequality that $B_{n,23} = o_{P_n}(1)$. Since $\mu_+(P_n) - \hat{\mu}_{+,n} = o_{P_n}(1)$ by (C-9), we can conclude for quantity (b) that $B_{n,2} = o_{P_n}(1)$. For quantity (c), using analogous arguments, we can conclude that $B_{n,3} = o_{P_n}(1)$, and hence

$$h_n \left(\Psi_{+,n}(h_n, P_n) - \hat{\Psi}_{+,n}(h_n) \right) = o_{P_n}(1) . \tag{C-16}$$

In addition, since from Lemma B.1 we have that $\Gamma_{+,n}^{-1}(h_n) = \tilde{\Gamma}_{+,n}^{-1}(h_n)$, it then follows that

$$n h_n \left(V_{+,n}(h_n, P_n) - \hat{V}_{+,n} \right) = o_{P_n}(1) . \tag{C-17}$$

To conclude, first rewrite (C-17) as

$$\frac{V_{+,n}(h_n, P_n) - \hat{V}_{+,n}}{\tilde{V}_{+,n}(h_n)} = o_{P_n}(1) ,$$

and our result then follows from (B-7).

References

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