Supplementary Online Appendix

to

Semi-Parametric Seasonal Unit Root Tests

by

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S.1 Introduction

This supplement contains supporting material for our paper "Semi-Parametric Seasonal Unit Root Tests". Equation references (S.n) for $n \geq 1$ refer to equations in this supplement and other equation references are to the main paper.

The supplement is organised as follows. Proofs of the main theoretical results in the paper can be found in section S.2. A more detailed outline of the augmented HEGY seasonal unit root tests are given in section S.3. Section S.4 details the limiting distributions of the lag un-augmented HEGY seasonal unit root tests which obtain from (2.4) with p^* set to zero. These are shown in Theorem S.1 to be non-pivotal depending on any (un-modelled) serial correlation present in u_{Sn+s} of (2.1b). Seasonal implementations of the PP unit root tests are outlined in section S.5 and their limiting distributions are given in Theorem S.2 in section S.6. The proofs of Theorems S.1 and S.2 are provided in section S.7. Additional Monte Carlo results relating to size unadjusted finite sample power results are reported in section S.8. All additional references are included at the end of the supplement.

S.2 Proofs of Main Results

S.2.1 Preliminary Results

Before providing the proofs of the main results given in the paper, a number of preliminary results are needed first. To that end, we first note that under (2.3), x_{Sn+s} in (2.1b) can be written as,

$$\Delta_0^{c_0}\Delta_{S/2}^{c_{S/2}}\prod\nolimits_{k=1}^{S^*}\Delta_k^{c_k}x_{Sn+s} = u_{Sn+s} \tag{S.1}$$

where $\Delta_0^{c_0} := 1-\alpha_0 L = 1-\left(1+\frac{c_0}{SN}\right)L$, $\Delta_{S/2}^{c_{S/2}} := 1+\alpha_{S/2}L = 1+\left(1+\frac{c_{S/2}}{SN}\right)L$, and $\Delta_k^{c_k} := 1-2\cos\left[\omega_k\right]\alpha_k L+\alpha_k^2L^2 = 1-2\cos\left[\omega_k\right]\left(1+\frac{c_k}{SN}\right)L+\left(1+\frac{c_k}{SN}\right)^2L^2$, for $k=1,...,S^*$. Consequently, (S.1) can be equivalently written as,

$$x_{Sn+s} = \left[S_{0,c_0}\left(Sn+s\right)\right]\left[S_{S/2,c_{S/2}}\left(Sn+s\right)\right]\left[\prod\nolimits_{k=1}^{S^*}S_{k,c_k}\left(Sn+s\right)\right]u_{Sn+s} \tag{S.2}$$

where, for $\omega_0=0$ and $\omega_{S/2}=\pi$,

$$S_{i,c_i}\left(Sn+s\right) := \sum_{j=1}^{Sn+s}\cos\left[\left(\left(Sn+s\right)-j\right)\omega_i\right]\alpha_i^{Sn+s-j}L^{Sn+s-j},\ i=0,S/2$$

and, for $\omega_k=(2\pi k)/S, k=1,...,S^*$,

$$\begin{aligned} S_{k,c_k}\left(Sn+s\right) &:= \sin\left[\omega_k\right]^{-1}\sum_{j=0}^{Sn+s-1}\sin\left[\left(\left(Sn+s\right)+1-j\right)\omega_k\right]\alpha_k^{Sn+s-j}L^{Sn+s-j} \\ &= \sin\left[\omega_k\right]^{-1}\Big(\sin\left[\left(\left(Sn+s\right)+1\right)\omega_k\right]S_{k,c_k}^{\alpha}\left(Sn+s\right) \\ &\qquad -\cos\left[\left(\left(Sn+s\right)+1\right)\omega_k\right]S_{k,c_k}^{\beta}\left(Sn+s\right)\Big) \end{aligned}$$

with

$$S^{\alpha}_{k,c_k}\left(Sn+s\right) := \sum_{j=1}^{Sn+s} \cos\left[j\omega_k\right] \alpha_k^{Sn+s-j} L^{Sn+s-j}$$

$$S^{\beta}_{k,c_k}\left(Sn+s\right) := \sum_{j=1}^{Sn+s} \sin\left[j\omega_k\right] \alpha_k^{Sn+s-j} L^{Sn+s-j}.$$

In view of the foregoing, the identities given in Gregoir (1999, p. 463) can be extended to the terms in (2.3) as follows,

$$\frac{\Delta_0^{c_0}}{2} + \frac{\Delta_{S/2}^{c_{S/2}}}{2} = 1 + \frac{1}{2}\left(\frac{c_{S/2} - c_0}{SN}\right) L = 1 + O\left(1/N\right) \tag{S.3}$$

$$\begin{aligned} \frac{\Delta_k^{c_k} + \left(1 - 2\cos\left[\omega_k\right] + L\right)\Delta_0^{c_0}}{2\kappa_0(\omega_k)} &= 1 - \frac{c_0}{2\kappa_0(\omega_k)SN} L - \frac{2\cos\left[\omega_k\right]}{2\kappa_0(\omega_k)} \frac{\left(c_k - c_0\right)}{SN} L \\ &\quad + \frac{\left(2c_k - c_0\right)}{2\kappa_0(\omega_k)SN} L^2 + \frac{c_k^2}{2\kappa_0(\omega_k)\left(SN\right)^2} L^2 \\ &= 1 - O\left(\frac{1}{N}\right) - O\left(\frac{1}{N}\right) + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^2}\right) \end{aligned} \tag{S.4}$$

$$\begin{aligned} \frac{\Delta_k^{c_k} + \left(1 + 2\cos\left[\omega_k\right] - L\right)\Delta_{S/2}^{c_{S/2}}}{2\kappa_{S/2}(\omega_k)} &= 1 + \frac{c_{S/2}}{2\kappa_{S/2}(\omega_k)SN} L + \frac{2\cos\left[\omega_k\right]}{2\kappa_{S/2}(\omega_k)} \frac{\left(c_{S/2} - c_k\right)}{SN} L \\ &\quad + \frac{\left(2c_k - c_{S/2}\right)}{2\kappa_{S/2}(\omega_k)SN} L^2 + \frac{c_k^2}{2\kappa_{S/2}(\omega_k)\left(SN\right)^2} L^2 \\ &= 1 + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^2}\right) \end{aligned} \tag{S.5}$$

and

$$\begin{aligned} &\frac{2\cos\left[\omega_k\right] - L}{2\kappa(\omega_{kj})}\Delta_j^{c_j} + \frac{2\cos\left[\omega_j\right] - L}{2\kappa(\omega_{jk})}\Delta_k^{c_k} \\ &= 1 - \frac{4\cos\left[\omega_k\right]\cos\left[\omega_j\right]}{2\kappa(\omega_{kj})} \frac{\left(c_j - c_k\right)}{SN} L + \frac{2\left[\cos\left[\omega_j\right]\frac{c_j}{SN} - \cos\left[\omega_k\right]\frac{c_k}{SN}\right]}{2\kappa(\omega_{kj})} L^2 \\ &\quad + \frac{4\left[\cos\left[\omega_k\right]\frac{c_j}{SN} - \cos\left[\omega_j\right]\frac{c_k}{SN}\right]}{2\kappa(\omega_{kj})} L^2 - \frac{2}{2\kappa(\omega_{kj})} \frac{\left(c_j - c_k\right)}{SN} L^3 \\ &\quad + \frac{2\left[\cos\left[\omega_k\right]\left(\frac{c_j}{SN}\right)^2 - \cos\left[\omega_j\right]\left(\frac{c_k}{SN}\right)^2\right]}{2\kappa(\omega_{kj})} L^2 - \frac{1}{2\kappa(\omega_{kj})} \frac{\left(c_j^2 - c_k^2\right)}{\left(SN\right)^2} L^3 \\ &= 1 - O\left(\frac{1}{N}\right) + O\left(\frac{1}{N}\right) + O\left(\frac{1}{N}\right) - O\left(\frac{1}{N}\right) + O\left(\frac{1}{N^2}\right) - O\left(\frac{1}{N^2}\right) \end{aligned} \tag{S.6}$$

where $\kappa_0(\omega_k) := 1 - \cos\left[\omega_k\right]$, $\kappa_{S/2}(\omega_k) := 1 + \cos\left[\omega_k\right]$ and $\kappa(\omega_{kj}) := \cos\left[\omega_k\right] - \cos\left[\omega_j\right]$, $j, k = 1, ..., S^*$.

Consequently, noting that $\Delta_k^{c_k} S_{k,c_k}\left(Sn+s\right) = 1$ and using (S.3)-(S.6), it follows from (S.2) after some tedious algebra and using the standard trigonometric identities, $\sin\left[\left(\left(Sn+s\right)+1\right)\omega_k\right]$

$\equiv \cos[\omega_k]\sin[(Sn+s)\omega_k]+\sin[\omega_k]\cos[(Sn+s)\omega_k]$ and $\cos[((Sn+s)+1)\omega_k] \equiv \cos[\omega_k]\cos[(Sn+s)\omega_k]$ $-\sin[\omega_k]\sin[(Sn+s)\omega_k]$, that x_{Sn+s} can be decomposed into the sum of frequency specific partial sums plus an asymptotically negligible term (see also Gregoir, 1999); that is,

$$\begin{aligned} x_{Sn+s} &= \frac{1}{S}S_{0,c_0}(Sn+s)u_{Sn+s} + \frac{1}{S}S_{S/2,c_{S/2}}(Sn+s)u_{Sn+s} \\ &+\frac{2}{S}\sum_{k=1}^{S^*}\Big[\cos[(Sn+s)\omega_k]S^{\alpha}_{k,c_k}(Sn+s)u_{Sn+s} \\ &\qquad + \sin[(Sn+s)\omega_k]S^{\beta}_{k,c_k}(Sn+s)u_{Sn+s}\Big] + o_p(1). \end{aligned} \tag{S.7}$$

Defining $X_n := [x_{Sn-(S-1)}, x_{Sn-(S-2)}, ..., x_{Sn}]'$, $n = 0, ..., N$, and $U_n := [u_{Sn-(S-1)}, u_{Sn-(S-2)}, ..., u_{Sn}]'$, $n = 1, ..., N$, and noting that $\sum_{j=1}^{n}\exp\left(\frac{c_k}{SN}\right)^{S(n-j)}U_j = \sum_{j=1}^{n}\exp\left(\frac{c_k}{N}\right)^{n-j}U_j$, it will prove convenient, for the analysis that follows, to re-write (S.7) in the so-called vector-of-seasons representation:

$$X_n = \sum_{k=0}^{\lfloor S/2 \rfloor}\left(\frac{1+\delta_k}{S}\right)C_k\sum_{i=1}^{n}\exp\left(\frac{c_k}{N}\right)^{n-i}U_i + o_p(1) \tag{S.8}$$

where $\delta_k := 0$ for $k = 0$ and $k = S/2$ and $\delta_k := 1$ otherwise, and where $C_i := \mathsf{Circ}[\cos[0], \cos[\omega_i], \cos[2\omega_i], \dots, \cos[(S-1)\omega_i]]$, $i = 0, \dots, \lfloor S/2 \rfloor$, such that C_0 and $C_{S/2}$ are $S \times S$ circulant matrices of rank 1, while for $\omega_i = 2\pi i/S$ with $i = 1, \dots, S^*$, C_i are $S \times S$ circulant matrices of rank 2. For further details on these circulant matrices see, for example, Osborn and Rodrigues (2002) and Smith *et al.* (2009).

Remark S.1: In order to relate (S.8) to (S.7) we have made use of the fact that the circulant matrices involved can be written as $C_0 = \mathbf{v}_0\mathbf{v}_0{}'$, where $\mathbf{v}_0{}' := [1, 1, 1, ..., 1]$, $C_{S/2} = \mathbf{v}_{S/2}\mathbf{v}_{S/2}{}'$, where $\mathbf{v}_{S/2}{}' := [-1, 1, -1, ..., 1]$, and for $j = 1, ..., S^*$, $C_j = \mathbf{v}_j\mathbf{v}_j'$ and finally the matrix $\overline{C}_j := \mathsf{Circ}[\sin[0], \sin[(S-1)\omega_j], \sin[(S-2)\omega_j], \dots, \sin[\omega_j]]$, with $\overline{C}_j = \mathbf{v}_j\mathbf{v}_j^{*\prime}$, which will be used later in lemma S.1 where

$$\mathbf{v}_j' := \begin{bmatrix} \cos[\omega_j(1-S)] & \cos[\omega_j(2-S)] & \cdots & \cos[0] \\ \sin[\omega_j(1-S)] & \sin[\omega_j(2-S)] & \cdots & \sin[0] \end{bmatrix} =: \begin{bmatrix} \mathbf{h}_j' \\ \mathbf{h}_j^{*\prime} \end{bmatrix}$$

and

$$\mathbf{v}_j^{*\prime} := \begin{bmatrix} -\sin[\omega_j(1-S)] & -\sin[\omega_j(2-S)] & \cdots & -\sin[0] \\ \cos[\omega_j(1-S)] & \cos[\omega_j(2-S)] & \cdots & \cos[0] \end{bmatrix} =: \begin{bmatrix} -\mathbf{h}_j^{*\prime} \\ \mathbf{h}_j' \end{bmatrix},$$

$j = 1, \dots, S^*$. □

Remark S.2: As shown in Burridge and Taylor (2001), the error process, U_n, defined above (S.8) satisfies the vector $MA(\infty)$ representation

$$U_n = \sum_{j=0}^{\infty}\mathbf{\Psi}_j E_{n-j} \tag{S.9}$$

where $E_n := [\varepsilon_{Sn-(S-1)}, \varepsilon_{Sn-(S-2)}, ..., \varepsilon_{Sn}]'$ is a vector of IID errors, and the $S \times S$ matrices $\mathbf{\Psi}_0$, $\mathbf{\Psi}_j$, $j = 1, 2, \ldots$, are given by

$$\mathbf{\Psi}_0 := \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \psi_1 & 1 & 0 & 0 & \cdots & 0 \\ \psi_2 & \psi_1 & 1 & 0 & \cdots & 0 \\ \psi_3 & \psi_2 & \psi_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{S-1} & \psi_{S-2} & \psi_{S-3} & \psi_{S-4} & \cdots & 1 \end{bmatrix}$$

and

$$\mathbf{\Psi}_j := \begin{bmatrix} \psi_{jS} & \psi_{jS-1} & \psi_{jS-2} & \psi_{jS-3} & \cdots & \psi_{jS-(S-1)} \\ \psi_{jS+1} & \psi_{jS} & \psi_{jS-1} & \psi_{jS-2} & \cdots & \psi_{jS-(S-2)} \\ \psi_{jS+2} & \psi_{jS+1} & \psi_{jS} & \psi_{jS-1} & \cdots & \psi_{jS-(S-3)} \\ \psi_{jS+3} & \psi_{jS+2} & \psi_{jS+1} & \psi_{jS} & \cdots & \psi_{jS-(S-4)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_{jS+S-1} & \psi_{jS+S-2} & \psi_{jS+S-3} & \psi_{jS+S-4} & \cdots & \psi_{jS} \end{bmatrix}, \quad j \geq 1.$$

□

Next in Lemma S.1 we provide a multivariate invariance principle for $Y_n^\xi := [y^\xi_{Sn-(S-1)}, y^\xi_{Sn-(S-2)}, ..., y^\xi_{Sn}]'$, where $y^\xi_{Sn+s} := x_{Sn+s} - \hat{\delta}' z_{Sn+s,\xi}$, and where it is recalled that the parameter $\xi \in \{1, 2, 3\}$ denotes the deterministic Case of interest.

Lemma S.1. *Let the conditions of Theorem 4.1 hold. Then, as $N \to \infty$,*

$$\begin{aligned} N^{-1/2} Y^\xi_{\lfloor rN \rfloor} &\Rightarrow \frac{\sigma_\varepsilon}{S} \sum_{i=0}^{\lfloor S/2 \rfloor} (1+\delta_i) \left(C_i \mathbf{\Psi}(1) \mathbf{J}^\xi_{c_i}(r) \right), \ r \in [0,1] \\ &= \frac{\sigma_\varepsilon}{S} \left[\psi(1) C_0 \mathbf{J}^\xi_{c_0}(r) + \psi(-1) C_{S/2} \mathbf{J}^\xi_{c_{S/2}}(r) + 2 \sum_{i=1}^{S^*} \left(b_i C_i \mathbf{J}^\xi_{c_i}(r) + a_i \overline{C}_i \mathbf{J}^\xi_{c_i}(r) \right) \right] \end{aligned} \quad \text{(S.10)}$$

where $\{\delta_i\}_{i=0}^{\lfloor S/2 \rfloor}$, are as defined below (S.8)*; $\mathbf{J}^\xi_{c_k}(r) := [J^\xi_{c_k,1-S}(r), J^\xi_{c_k,2-S}(r), ..., J^\xi_{c_k,0}(r)]'$ is an $S \times 1$ vector OU process such that $d\mathbf{J}^\xi_{c_k}(r) = \mathbf{c}\mathbf{J}^\xi_{c_k}(r)\,dr + d\mathbf{W}^\xi(r)$ and $\mathbf{W}^\xi(r)$ is an $S \times 1$ vector Brownian motion process; $a_i := \mathcal{I}m(\psi[\exp(\mathrm{i}\omega_i)])$ and $b_i := \mathcal{R}e(\psi[\exp(\mathrm{i}\omega_i)])$, $i = 1, ..., S^*$, with $\mathcal{R}e(\cdot)$ and $\mathcal{I}m(\cdot)$ denoting the real and imaginary parts of their arguments, respectively; and C_0, $C_{S/2}$, C_i and $\overline{C}_i$, $i = 1, \ldots, S^*$, are the $S \times S$ circulant matrices defined in Remark S.1. Finally, with OLS de-trending:*

$$\begin{aligned} J^1_{c_k,s}(r) &:= J_{c_k,s}(r) - \int_0^1 J_{c_k,s}(r)\,dr \\ J^2_{c_k,s}(r) &:= J^1_{c_k,s}(r) - 12\left(r - \frac{1}{2}\right) \int_0^1 \left(r - \frac{1}{2}\right) \left[\frac{1}{S} \sum_{s=1-S}^{0} J^1_{c_k,s}(r) \right] dr \\ J^3_{c_k,s}(r) &:= J^1_{c_k,s}(r) - 12\left(r - \frac{1}{2}\right) \int_0^1 \left(r - \frac{1}{2}\right) J^1_{c_k,s}(r)\,dr \end{aligned}$$

and with local GLS de-trending:

$$\begin{aligned} J^1_{c_k,s}(r) &:= J_{c_k,s}(r) \\ J^2_{c_k,s}(r) &:= J_{c_k,s}(r) - r\left[\frac{1}{S}\sum_{s=1-S}^{0}\left(\lambda J_{c_k,s}(1) + 3(1-\lambda)\int_0^1 hJ_{c_k,s}(h)\,dh\right)\right] \\ J^3_{c_k,s}(r) &:= J_{c_k,s}(r) - r\left[\lambda J_{c_k,s}(1) + 3(1-\lambda)\int_0^1 hJ_{c_k,s}(h)\,dh\right] \end{aligned}$$

with $\lambda := (1-\bar{c})/\left(1+\bar{c}+\bar{c}^2/3\right)$*, in all cases for the indices* $s = 1-S, ..., 0$ *and* $k = 0, ..., \lfloor S/2 \rfloor$.

Proof of Lemma S.1: Following along the same lines as for the proof of Lemma 1 in del Barrio Castro, Osborn and Taylor (2012) and Phillips (1988) it follows that, as $N \to \infty$,

$$\frac{\sigma_\varepsilon^{-1}}{\sqrt{N}}\sum_{i=1}^{\lfloor rN \rfloor}\exp\left(\frac{c_k}{N}\right)^{\lfloor rN \rfloor - i} E_i \Rightarrow \mathbf{J}_{c_k}(r), \quad r \in [0,1] \tag{S.11}$$

$$\begin{aligned} \frac{\sigma_\varepsilon^{-1}}{\sqrt{N}}\sum_{i=1}^{\lfloor rN \rfloor}\exp\left(\frac{c_k}{N}\right)^{\lfloor rN \rfloor - i} U_i &= \frac{\sigma_\varepsilon^{-1}\mathbf{\Psi}(1)}{\sqrt{N}}\sum_{i=1}^{\lfloor rN \rfloor}\exp\left(\frac{c_k}{N}\right)^{\lfloor rN \rfloor - i} E_i + o_p(1) \\ &\Rightarrow \mathbf{\Psi}(1)\mathbf{J}_{c_k}(r), \quad r \in [0,1] \end{aligned} \tag{S.12}$$

where E_i and U_i are as previously defined, $d\mathbf{J}_{c_k}(r) = c_k\mathbf{J}_{c_k}(r)\,dr + d\mathbf{W}(r)$, $\mathbf{W}(r)$ is an $S \times 1$ vector standard Brownian motion and $\mathbf{J}_{c_k}(r)$ is an $S \times 1$ vector standard OU process. Next observe from (S.8) and (S.9), that

$$\begin{aligned} N^{-1/2}X_{\lfloor rN \rfloor} &= \sum_{k=0}^{\lfloor S/2 \rfloor}\left(\frac{1+\delta_k}{S}\right)C_k N^{-1/2}\sum_{i=1}^{\lfloor rN \rfloor}\exp\left(\frac{c_k}{N}\right)^{\lfloor rN \rfloor - i} U_i + o_p(1) \\ &= \sum_{k=0}^{\lfloor S/2 \rfloor}\left(\frac{1+\delta_k}{S}\right)C_k\mathbf{\Psi}(1)N^{-1/2}\sum_{i=1}^{\lfloor rN \rfloor}\exp\left(\frac{c_k}{N}\right)^{\lfloor rN \rfloor - i} E_i + o_p(1) \end{aligned} \tag{S.13}$$

where $\{\delta_k\}_{k=0}^{\lfloor S/2 \rfloor}$, are as defined below (S.8), and the approximation in (S.13) follows from (S.12) and using similar arguments to those used in Boswijk and Franses (1996, p.238). From (S.11), (S.13) and the continuous mapping theorem [CMT] the result in (S.10) follows immediately. Noting that $\mathbf{\Psi}(1)$ is also a circulant matrix, then by the properties of products of circulant matrices it can be shown that $C_0\mathbf{\Psi}(1) = \psi(1)C_0$, $C_{S/2}\mathbf{\Psi}(1) = \psi(-1)C_{S/2}$, $C_j\mathbf{\Psi}(1) = b_jC_j + a_j\overline{C}_j$ and $\overline{C}_j\mathbf{\Psi}(1) = -a_jC_j + b_j\overline{C}_j$ for $j = 1, \ldots S^*$; see, *inter alia*, Davis (1979, Theorem 3.2.4), Gray (2006, Theorem 3.1) and Smith *et al.* (2009) for further details. The stated result then follows immediately. □

Remark S.3: Note that the circulant matrices C_0 and $C_{S/2}$ are associated with the auxiliary variables $y^\xi_{0,Sn+s}$ and $y^\xi_{S/2,Sn+s}$, respectively. Moreover, the circulant matrices C^k, $k = 1, ..., S^*$ (see Remark 2 in Smith, Taylor and del Barrio Castro, 2009) defined as:

$$\begin{aligned} C^k &:= \mathsf{Circ}\left[\frac{\sin[\omega_k]}{\sin[\omega_k]}, \frac{\sin[S\omega_k]}{\sin[\omega_k]}, \frac{\sin[(S-1)\omega_k]}{\sin[\omega_k]}, \ldots, \frac{\sin[2\omega_k]}{\sin[\omega_k]}\right] \\ &= C_k + \frac{\cos[\omega_k]}{\sin[\omega_k]}\overline{C}_k, \quad k = 1, ..., S^* \end{aligned} \tag{S.14}$$

where C_k and $\overline{C}_k$, $k = 1, ..., S^*$, are as defined in Remark S.1 and are associated with the filter $\Delta_k^0(L) = \sin[\omega_k]^{-1}\left(\sum_{j=0}^{S-1}\sin[(j+1)\omega_k]L^j\right)$ in (3.11). Finally the circulant matrices $D_{\omega_k}^+$ and $D_{\omega_k}^-$, $k = 1, ..., S^*$, defined as, $D_{\omega_k}^+ := \mathsf{Circ}\left[1, 0, 0, \cdots, 0, e^{\mathrm{i}\omega_k}\right]$ and $D_{\omega_k}^- := \mathsf{Circ}\left[1, 0, 0, \cdots, 0, e^{-\mathrm{i}\omega_k}\right]$ are associated with the filters $\left(1 - e^{\mathrm{i}\omega_k}L\right)$ and $\left(1 - e^{-\mathrm{i}\omega_k}L\right)$, respectively. □

In order to obtain results for the asymptotic distributions of the test statistics discussed in this paper, the limiting results collected together in the following Lemma will prove useful.

Lemma S.2. *Let the conditions of Lemma S.1 hold. Then, as $N \to \infty$,*

$$N^{-1/2}C_0Y_{\lfloor rN\rfloor}^{\xi} \Rightarrow \sigma_\varepsilon\,\psi(1)\,C_0\mathbf{J}_{c_0}^{\xi}(r) \tag{S.15}$$

$$N^{-1/2}C_{S/2}Y_{\lfloor rN\rfloor}^{\xi} \Rightarrow \sigma_\varepsilon\,\psi(-1)\,\mathbf{J}_{c_{S/2}}^{\xi}(r) \tag{S.16}$$

$$N^{-1/2}C^kY_{\lfloor rN\rfloor}^{\xi} \Rightarrow \sigma_\varepsilon\left(C_k + \frac{\cos[\omega_k]}{\sin[\omega_k]}\overline{C}_k\right)\mathbf{\Psi}(1)\,\mathbf{J}_{c_k}^{\xi}(r), k = 1, \ldots, S^* \tag{S.17}$$

$$\frac{1}{\sqrt{N}}D_{\omega_k}^+C^kY_{\lfloor rN\rfloor}^{\xi} \Rightarrow \sigma_\varepsilon C_k^-\Psi(1)\,\mathbf{J}_{c_k}^{\xi}(r) = \sigma_\varepsilon\psi\left(e^{\mathrm{i}\omega_k}\right)\mathcal{E}_{1,k}^-\mathcal{E}_{2,k}^{-\prime}\mathbf{J}_{c_k}^{\xi}(r), k = 1, \ldots, S^* \tag{S.18}$$

$$\frac{1}{\sqrt{N}}D_{\omega_k}^-C^kY_{\lfloor rN\rfloor}^{\xi} \Rightarrow \sigma_\varepsilon C_k^+\Psi(1)\,\mathbf{J}_{c_k}^{\xi}(r) = \sigma_\varepsilon\psi\left(e^{-\mathrm{i}\omega_k}\right)\mathcal{E}_{1,k}^+\mathcal{E}_{2,k}^{+\prime}\mathbf{J}_{c_k}^{\xi}(r), k = 1, \ldots, S^* \tag{S.19}$$

where the vector OU processes, $\mathbf{J}_{c_i}^{\xi}(r)$, $i = 0, \ldots, \lfloor S/2\rfloor$, and the circulant matrices, C_i, $i = 0, \ldots, \lfloor S/2\rfloor$ and $\overline{C}_i$, $i = 1, \ldots, S^*$, are defined in Lemma S.1, C^k is defined in (S.14), $D_{\omega_k}^+ :=$ $\mathsf{Circ}\left[1, 0, 0, \cdots, 0, e^{\mathrm{i}\omega_k}\right]$, $D_{\omega_k}^- := \mathsf{Circ}\left[1, 0, 0, \cdots, 0, e^{-\mathrm{i}\omega_k}\right]$, $C_k^- := \mathsf{Circ}\left[1, e^{-\mathrm{i}(S-1)\omega_k}, e^{-\mathrm{i}(S-2)\omega_k}, \cdots, e^{-\mathrm{i}\omega_k}\right]$, $C_k^+ := \mathsf{Circ}\left[1, e^{\mathrm{i}(S-1)\omega_k}, e^{\mathrm{i}(S-2)\omega_k}, \cdots, e^{\mathrm{i}\omega_k}\right]$, $k = 1, \ldots, S^*$, $\mathcal{E}_{1,k}^- := [1, e^{-\mathrm{i}\omega_k}, e^{-\mathrm{i}2\omega_k}, ..., e^{-\mathrm{i}(S-1)\omega_k}]'$, $\mathcal{E}_{2,k}^- := [1, e^{-\mathrm{i}(S-1)\omega_k}, e^{-\mathrm{i}(S-2)\omega_k}, ..., e^{-\mathrm{i}\omega_k}]'$, $\mathcal{E}_{1,k}^+ := [1, e^{\mathrm{i}\omega_k}, e^{\mathrm{i}2\omega_k}, ..., e^{\mathrm{i}(S-1)\omega_k}]'$ and $\mathcal{E}_{2,k}^+ := [1, e^{\mathrm{i}(S-1)\omega_k}, e^{\mathrm{i}(S-2)\omega_k}, ..., e^{\mathrm{i}\omega_k}]'$.

Proof of Lemma S.2: The results in (S.15) to (S.17) follow immediately from Lemma S.1 using the following identities: $C_0C_0 \equiv SC_0$,$C_{S/2}C_{S/2} \equiv SC_{S/2}$, $C_kC_k \equiv \frac{S}{2}C_k$ and $\overline{C}_kC_k \equiv \frac{S}{2}\overline{C}_k$, recalling that the matrix products between C_0, $C_{S/2}$, C_j and $\overline{C}_j$, $j = 1, \ldots, S^*$ are all zero matrices, and that multiplication between circulant matrices is commutative, and finally that $C^k := \left(C_k + \frac{\cos[\omega_k]}{\sin[\omega_k]}\overline{C}_k\right)$. Consider next the results in (S.18) and (S.19). We first note, using Property 1.3 and expression (2) in Gregoir (2006), that

$$C^k = \frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k} - e^{\mathrm{i}\omega_k}}C_k^- + \frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k} - e^{-\mathrm{i}\omega_k}}C_k^+ \tag{S.20}$$

with $C_k^- := \mathsf{Circ}\left[1, e^{-\mathrm{i}(S-1)\omega_k}, e^{-\mathrm{i}(S-2)\omega_k}, \cdots, e^{-\mathrm{i}\omega_k}\right]$ and $C_k^+ := \mathsf{Circ}\left[1, e^{\mathrm{i}(S-1)\omega_k}, e^{\mathrm{i}(S-2)\omega_k}, \cdots, e^{\mathrm{i}\omega_k}\right]$. Moreover, $D_{\omega_k}^-C_k^- = D_{\omega_k}^+C_k^+ = 0$, $\frac{e^{-\mathrm{i}\omega_k}}{e^{-\mathrm{i}\omega_k}-e^{\mathrm{i}\omega_k}}D_{\omega_k}^+C_k^- = C_k^-$, and $\frac{e^{\mathrm{i}\omega_k}}{e^{\mathrm{i}\omega_k}-e^{-\mathrm{i}\omega_k}}D_{\omega_k}^-C_k^+ = C_k^+$, each of which follows from the properties of the product of circulant matrices. Also, because $\Psi(1)$ is a circulant matrix, by the properties of products of circulant matrices it further holds that $C_k^-\Psi(1) = \psi\left(e^{\mathrm{i}\omega_k}\right)C_k^-$ and $C_k^+\Psi(1) = \psi\left(e^{-\mathrm{i}\omega_k}\right)C_k^+$. Finally as both C_k^- and C_k^+ are $S \times S$ circulant matrices of rank 1 we can write $C_k^- = \mathcal{E}_{1,k}^-\mathcal{E}_{2,k}^{-\prime}$ and $C_k^+ = \mathcal{E}_{1,k}^+\mathcal{E}_{2,k}^{+\prime}$. The stated results then follow immediately. □

S.2.2 Proof of Theorem 4.1

Using the results that C_0 and $C_{S/2}$ are symmetric and orthogonal both to one another and to C_i and $\overline{C}_i$, $i = 1, ..., S^*$, and the fact that $C_j C_j C_j \equiv S^2 C_j$ for $j = 0, S/2$, then appealing to the multivariate invariance principle in (S.10) and using an application of the CMT we have that

$$\begin{aligned} T^{-2}\sum_{n=1}^{N}\sum_{s=1-S}^{0}\left(y_{j,Sn+s-1}^{\xi}\right)^2 &= T^{-2}\sum_{n=1}^{N} S\left(Y_{n-1}^{\xi\prime} C_j Y_{n-1}^{\xi}\right) + o_p(1) \\ &\Rightarrow \frac{\sigma_\varepsilon^2}{S^2}\frac{\psi\left(\cos[\omega_j]\right)^2}{S}\int_0^1 \mathbf{J}_{c_j}^{\xi}(r)' C_j' C_j C_j \mathbf{J}_{c_j}^{\xi}(r)\, dr \\ &= \sigma_\varepsilon^2 \psi\left(\cos[\omega_j]\right)^2 \int_0^1 \mathbf{J}_{c_j}^{\xi *}(r)' C_j \mathbf{J}_{c_j}^{\xi *}(r)\, dr, j = 0, S/2 \quad \text{(S.21)} \end{aligned}$$

where $\omega_0 = 0$, $\omega_{S/2} = \pi$ and $\mathbf{J}_{c_j}^{\xi *}(r) := \frac{1}{\sqrt{S}}\mathbf{J}_{c_j}^{\xi}(r)$ for $j = 0, S/2$.

Using Remark S.1, together with the results in (S.15) and (S.16), for the zero and Nyquist frequencies, applications of the multivariate FCLT and CMT establish that, as $N \to \infty$,

$$\begin{aligned} N^{-1/2} y_{0,S\lfloor rN\rfloor+s}^{\xi} &\Rightarrow \sigma_\varepsilon\sqrt{S}\psi(1)\,\mathbf{v}_1' \frac{1}{\sqrt{S}}\mathbf{J}_{c_0}^{\xi}(r) =: \sigma_\varepsilon\sqrt{S}\psi(1)\,\mathbf{v}_1'\mathbf{J}_{c_0}^{\xi *}(r) \\ &=: \sigma_\varepsilon\sqrt{S}\psi(1)\, J_{0,c_0}^{\zeta}(r) \quad \text{(S.22)} \\ N^{-1/2} y_{S/2,S\lfloor rN\rfloor+s}^{\xi} &\Rightarrow \sigma_\varepsilon\sqrt{S}\psi(-1)(-1)^s\,\mathbf{v}_{S/2}' \frac{1}{\sqrt{S}}\mathbf{J}_{c_{S/2}}^{\xi}(r) =: \sigma_\varepsilon\sqrt{S}\psi(-1)(-1)^s\,\mathbf{v}_{S/2}'\mathbf{J}_{c_{S/2}}^{\xi *}(r) \\ &=: \sigma_\varepsilon\sqrt{S}\psi(-1)(-1)^s J_{S/2,c_{S/2}}^{\zeta}(r) \quad \text{(S.23)} \end{aligned}$$

where $\mathbf{v}_1'$ and $\mathbf{v}_{S/2}'$ are defined in Remark S.1, and $J_{0,c_0}^{\zeta}(r)$ and $J_{S/2,c_{S/2}}^{\zeta}(r)$ are as defined in Theorem 4.1. Consequently, for the $\mathcal{MZ}_k$, $k = 0, S/2$ tests we obtain from (S.22) and (S.23) that,

$$\begin{aligned} (SN)^{-1/2} y_{0,SN}^{\xi} &\Rightarrow \sigma_\varepsilon\psi(1)\, J_{0,c_0}^{\zeta}(1) \quad \text{(S.24)} \\ (SN)^{-1/2} y_{S/2,SN}^{\xi} &\Rightarrow \sigma_\varepsilon\psi(-1)(-1)^S J_{S/2,c_{S/2}}^{\zeta}(1). \quad \text{(S.25)} \end{aligned}$$

Using the results in (S.24), (S.25) and (S.21) and the fact that $\hat{\lambda}_0^2 \xrightarrow{p} \sigma_\varepsilon^2\psi(1)^2$ and $\hat{\lambda}_{S/2}^2 \xrightarrow{p} \sigma_\varepsilon^2\psi(-1)^2$, it therefore follows that,

$$\mathcal{MZ}_k \Rightarrow \frac{\sigma_\varepsilon^2\psi\left(\cos[\omega_k]\right)^2 J_{k,c_k}^{\zeta}(1)^2 - \sigma_\varepsilon^2\psi\left(\cos[\omega_k]\right)^2}{2\sigma_\varepsilon^2\psi\left(\cos[\omega_k]\right)^2\int_0^1\left[J_{k,c_k}^{\zeta}(r)\right]^2 dr} = \frac{\left[J_{k,c_k}^{\zeta}(1)\right]^2 - 1}{2\int_0^1\left[J_{k,c_k}^{\zeta}(r)\right]^2 dr},\ k = 0, S/2 \quad \text{(S.26)}$$

where $\omega_0 = 0$ and $\omega_{S/2} = \pi$. The results for the $\mathcal{MSB}_k$, $k = 0, S/2$, statistics are obtained straightforwardly from (S.21). Combining the results for $\mathcal{MSB}_k$ with (S.26), the limit of $\mathcal{MZ}_{t_k}$ then follows straightforwardly.

Turning to the harmonic frequency statistics, note first that the vector of seasons representations of (3.9) and (3.10) with $Y_{k,n}^{\xi,\mathtt{D}h} := \left[y_{k,Sn-(S-1)}^{\xi,\mathtt{D}h},\ y_{k,Sn-(S-2)}^{\xi,\mathtt{D}h}, \cdots, y_{k,Sn}^{\xi,\mathtt{D}h}\right]'$, $h \in \{a, b\}$, based on (S.18) and (S.19) are such that, for $k = 1, ..., S^*$,

$$\begin{aligned}
\frac{1}{\sqrt{SN}} Y^{\xi,\mathtt{D}a}_{k,\lfloor rN \rfloor} &\Rightarrow \frac{\sigma_\varepsilon}{\sqrt{S}} \psi\left(e^{\mathrm{i}\omega_k}\right)\left(e^{\mathrm{i}\omega_k}\mathbf{1}\right)\mathcal{E}^{-\prime}_{2,k}\mathbf{J}^{\xi}_{c_k}(r) = \frac{\sigma_\varepsilon}{\sqrt{S}} \psi\left(e^{\mathrm{i}\omega_k}\right)\mathbf{1}e^{\mathrm{i}\omega_k}\mathcal{E}^{+\prime}_{1,k}\mathbf{J}^{\xi}_{c_k}(r) \\
&= \frac{\sigma_\varepsilon}{\sqrt{2}} \psi\left(e^{\mathrm{i}\omega_k}\right)\mathbf{1}\left[\mathbf{h}'_k \frac{1}{\sqrt{S/2}}\mathbf{J}^{\xi}_{c_k}(r) + \mathrm{i}\mathbf{h}^{*\prime}_k \frac{1}{\sqrt{S/2}}\mathbf{J}^{\xi}_{c_k}(r)\right] \\
&= \frac{\sigma_\varepsilon}{\sqrt{2}} \psi\left(e^{\mathrm{i}\omega_k}\right)\mathbf{1}\left[\mathbf{h}'_k\mathbf{J}^{\xi\dagger}_{c_k}(r) + \mathrm{i}\mathbf{h}^{*\prime}_k\mathbf{J}^{\xi\dagger}_{c_k}(r)\right] \\
&= \frac{\sigma_\varepsilon}{\sqrt{2}} \psi\left(e^{\mathrm{i}\omega_k}\right)\mathbf{1}\left[J^{\zeta}_{k,c_k}(r) + \mathrm{i}J^{\zeta*}_{k,c_k}(r)\right] \qquad \text{(S.27)}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\sqrt{SN}} Y^{\xi,\mathtt{D}b}_{k,\lfloor rN \rfloor} &\Rightarrow \frac{\sigma_\varepsilon}{\sqrt{S}} \psi\left(e^{-\mathrm{i}\omega_k}\right)\left(e^{-\mathrm{i}\omega_k}\mathbf{1}\right)\mathcal{E}^{+\prime}_{2,k}\mathbf{J}^{\xi}_{c_k}(r) = \frac{\sigma}{\sqrt{S}} \psi\left(e^{-\mathrm{i}\omega_k}\right)\mathbf{1}e^{-\mathrm{i}\omega_k}\mathcal{E}^{-\prime}_{1,k}\mathbf{J}^{\xi}_{c_k}(r) \\
&= \frac{\sigma_\varepsilon}{\sqrt{2}} \psi\left(e^{-\mathrm{i}\omega_k}\right)\mathbf{1}\left[\mathbf{h}'_k \frac{1}{\sqrt{S/2}}\mathbf{J}^{\xi}_{c_k}(r) - \mathrm{i}\mathbf{h}^{*\prime}_k \frac{1}{\sqrt{S/2}}\mathbf{J}^{\xi}_{c_k}(r)\right] \\
&= \frac{\sigma_\varepsilon}{\sqrt{2}} \psi\left(e^{-\mathrm{i}\omega_k}\right)\mathbf{1}\left[\mathbf{h}'_k\mathbf{J}^{\xi\dagger}_{c_k}(r) - \mathrm{i}\mathbf{h}^{*\prime}_k\mathbf{J}^{\xi\dagger}_{c_k}(r)\right] \\
&= \frac{\sigma_\varepsilon}{\sqrt{2}} \psi\left(e^{-\mathrm{i}\omega_k}\right)\mathbf{1}\left[J^{\zeta}_{k,c_k}(r) - \mathrm{i}J^{\zeta*}_{k,c_k}(r)\right], \qquad \text{(S.28)}
\end{aligned}$$

respectively, where $\mathbf{1}$ is an $S \times 1$ vector of ones, $\mathbf{h}_k$ and $\mathbf{h}^*_k$, are defined in Remark S.1, $\mathbf{J}^{\xi}_{c_k}(r)$ and $\mathbf{J}^{\xi\dagger}_{c_k}(r)$ are defined in Lemma S.1, and where $J^{\zeta}_{k,c_k}(r)$ and $J^{\zeta*}_{k,c_k}(r)$ are as defined in Theorem 4.1.

Using the consistency of the estimators $\breve{\lambda}_{k,AR} := s_e\{1 - [\widehat{\phi}(e^{\mathrm{i}\omega_k})]\}^{-1}$ and $\breve{\lambda}^*_{k,AR} := s_e\{1 - [\widehat{\phi}(e^{-\mathrm{i}\omega_k})]\}^{-1}$ of $\sigma_\varepsilon\psi\left(e^{\mathrm{i}\omega_k}\right)$ and $\sigma_\varepsilon\psi\left(e^{-\mathrm{i}\omega_k}\right)$, respectively, $k = 1, ..., S^*$, it is then possible to show that, in each case for $k = 1, ..., S^*$,

$$(\breve{\lambda}^2_{k,AR}T)^{-1/2} y^{\xi,\mathtt{D}a}_{k,S\lfloor rN \rfloor + s} \Rightarrow \frac{1}{\sqrt{2}}\left[J^{\zeta}_{k,c_k}(r) + \mathrm{i}J^{\zeta*}_{k,c_k}(r)\right] =: \frac{1}{\sqrt{2}}\mathbb{J}_{k,c_k}(r)$$

$$(\breve{\lambda}^{*2}_{k,AR}T)^{-1/2} y^{\xi,\mathtt{D}b}_{k,S\lfloor rN \rfloor + s} \Rightarrow \frac{1}{\sqrt{2}}\left[J^{\zeta}_{k,c_k}(r) - \mathrm{i}J^{\zeta*}_{k,c_k}(r)\right] =: \frac{1}{\sqrt{2}}\overline{\mathbb{J}_{k,c_k}(r)}.$$

Noting that the auxiliary variables $y^{\mathcal{R}e,\xi}_{k,Sn+s}$ and $y^{\mathcal{I}m,\xi}_{k,Sn+s}$ defined in (3.14) and (3.15) are free from nuisance parameters, it is then straightforward to obtain the representations given for the asymptotic distributions of the $\mathcal{K}$-$\mathcal{MZ}_k$, $\mathcal{K}$-$\mathcal{MSB}_k$ and $\mathcal{K}$-$\mathcal{MZ}_{t_k}$ statistics in (4.4), (4.5) and (4.6), together with the results for the joint frequency statistics from section 3.3 given in Corollary 4.1 □

Remark S.4: Note that the deterministic kernels considered for the de-meaning and de-trending of the variables, have different impacts on the frequency specific OU processes. These set of processes at each frequency for each case are summarised for convenience as follows,

Case 1 ($\xi = 1$) : $J^1_{0,c_0}(r)$, $J^1_{S/2,c_{S/2}}(r)$, $J^1_{i,c_i}(r)$, $J^{1*}_{i,c_i}(r)$, $i = 1, ..., S^*$
Case 2 ($\xi = 2$) : $J^2_{0,c_0}(r)$, $J^1_{S/2,c_{S/2}}(r)$, $J^1_{i,c_i}(r)$, $J^{1*}_{i,c_i}(r)$, $i = 1, ..., S^*$
Case 3 ($\xi = 3$) : $J^2_{0,c_0}(r)$, $J^2_{S/2,c_{S/2}}(r)$, $J^2_{i,c_i}(r)$, $J^{2*}_{i,c_i}(r)$, $i = 1, ..., S^*$

where it is to be recalled that $\zeta = 1$ and $\zeta = 2$ correspond to de-meaned and de-trended OU processes, respectively. These are defined as: $J^{\zeta}_{0,c_0}(r) := \mathbf{v}_1' \mathbf{J}^{\xi *}_{c_0}(r)$, $J^{\zeta}_{S/2,c_{S/2}}(r) := \mathbf{v}_{S/2}' \mathbf{J}^{\xi *}_{c_{S/2}}(r)$, $J^{\zeta}_{k,c_k}(r) := \mathbf{h}_k' \mathbf{J}^{\xi \dagger}_{c_k}(r)$ and $J^{\zeta *}_{k,c_k}(r) := \mathbf{h}^{*\prime}_k \mathbf{J}^{\xi \dagger}_{c_k}(r)$ for $k = 1, \ldots, S^*$. □

S.3 Augmented HEGY Seasonal Unit Root Tests

Unit roots at the zero, Nyquist and harmonic seasonal frequencies imply that $\pi_0 = 0$, $\pi_{S/2} = 0$ and $\pi_k = \pi_k^* = 0$, $k = 1, ..., S^*$, respectively, in (2.4); see Smith *et al.* (2009). Consequently, tests for the presence or otherwise of a unit root at the zero and Nyquist frequencies are conventional lower tailed regression t-tests, denoted t_0 and $t_{S/2}$, for the exclusion of $y^{\xi}_{0,Sn+s-1}$ and $y^{\xi}_{S/2,Sn+s-1}$, respectively, from (2.4). Notice that for $S = 1$, t_0 is the standard non-seasonal ADF unit root test statistic. Similarly, the hypothesis of a pair of complex unit roots at the kth harmonic seasonal frequency may be tested by the lower-tailed t_k and two-tailed t_k^* regression t-tests from (2.4) for the exclusion of $y^{\xi}_{k,Sn+s-1}$ and $y^{*\xi}_{k,Sn+s-1}$, respectively, or by the (upper-tailed) regression F-test, denoted F_k, for the exclusion of both $y^{\xi}_{k,Sn+s-1}$ and $y^{*\xi}_{k,Sn+s-1}$ from (2.4). Ghysels *et al.* (1994) also consider the joint frequency (upper-tail) regression F-tests from (2.4), $F_{1\ldots\lfloor S/2\rfloor}$ for the exclusion of $y^{\xi}_{S/2,Sn+s-1}$, $\{y^{\xi}_{j,Sn+s-1}\}_{j=1}^{S^*}$ and $\{y^{*\xi}_{k,Sn+s-1}\}_{k=1}^{S^*}$, and $F_{0\ldots\lfloor S/2\rfloor}$ for the exclusion of $y^{\xi}_{0,Sn+s-1}$, $y^{\xi}_{S/2,Sn+s-1}$, $\{y^{\xi}_{j,Sn+s-1}\}_{j=1}^{S^*}$ and $\{y^{*\xi}_{k,Sn+s-1}\}_{k=1}^{S^*}$. The former tests the null hypothesis of unit roots at all of the seasonal frequencies, defined as $H_{0,\text{seas}} := \cap_{k=1}^{\lfloor S/2\rfloor} H_{0,k}$, while the latter tests the null hypothesis of unit roots at the zero and all of the seasonal frequencies, defined as $H_0 := \cap_{k=0}^{\lfloor S/2\rfloor} H_{0,k}$. Observe that $\alpha(L) = \Delta_S$ under H_0.

The limiting null distributions of the OLS de-trended HEGY statistics are given for the case where $\psi(z) = 1$ in (2.1b) and accordingly $p^* = 0$ in (2.4) by Smith and Taylor (1998). In the case where $\psi(z)$ is invertible with (unique) inverse $\phi(z)$, with $\phi(z)$ a pth order, $0 \le p < \infty$, lag polynomial, Burridge and Taylor (2001) and Smith *et al.* (2009) show that the limiting null distributions of the OLS de-trended t_0, $t_{S/2}$ and F_k, $k = 1, ..., S^*$, statistics from (2.4) are as for $p = 0$, provided $p^* \ge p$ in (2.4). They show that this is not true, however, for the t_k and t_k^*, $k = 1, ..., S^*$, statistics whose limit distributions depend on functions of the parameters characterising the serial dependence in u_{Sn+s} in (2.1b). Representations for the corresponding limiting distributions under near seasonally integrated alternatives are given in Rodrigues and Taylor (2004) and again shown to be free of nuisance parameters with the exception of the t_k and t_k^*, $k = 1, ..., S^*$, statistics. Corresponding results for the local GLS de-trended HEGY-type statistic are given in Rodrigues and Taylor (2007) and here it is also the case that the harmonic frequency t-statistics depend on nuisance parameters arising from the serial correlation in u_{Sn+s}. Where $\phi(z)$ is (potentially) infinite-ordered, del Barrio Castro *et al.* (2012) show that provided the lag length p^* in (2.4) is such that $1/p^* + (p^*)^3/T \to 0$, as $T \to \infty$, then limiting distributions of the OLS and local GLS de-trended HEGY statistics will be of the same form as derived for those statistics under finite p.

S.4 Limiting Distributions of the Lag Un-augmented HEGY Statistics

In Theorem S.1 we now provide representations for the limiting distributions of the normalised OLS estimates together with the corresponding regression t- and F-statistics computed from the un-augmented HEGY regression given by (2.4) with the lag augmentation length, p^*, set to zero. These representations are again indexed by the parameter ζ which has exactly the same meaning as was given prior to Theorem 4.1.

Theorem S.1. *Let y_{Sn+s} be generated by (2.1) under $H_{1,\mathbf{c}}$ and let Assumption 1 hold. Then the HEGY-type statistics computed from (2.4) with $p^* = 0$ are such that, as $T \to \infty$,*

$$T\hat{\pi}_k \Rightarrow \frac{\int_0^1 J_{k,c_k}^{\zeta}(r)dJ_{k,c_k}^{\zeta}(r) + \mathcal{D}_k \int_0^1 J_{k,c_k}^{\zeta*}(r)dJ_{k,c_k}^{\zeta*}(r) + \frac{\lambda_k^2-\gamma_0}{2\lambda_k^2}}{\frac{(2-\mathcal{D}_k)}{2}\left\{\int_0^1 \left[J_{k,c_k}^{\zeta}(r)\right]^2 dr + \mathcal{D}_k \int_0^1 \left[J_{k,c_k}^{\zeta*}(r)\right]^2 dr\right\}}, \quad k = 0, ..., \lfloor S/2 \rfloor \quad \text{(S.29)}$$

$$T\hat{\pi}_k^* \Rightarrow \frac{\int_0^1 J_{k,c_k}^{\zeta*}(r)dJ_{k,c_k}^{\zeta}(r) - \int_0^1 J_{k,c_k}^{\zeta}(r)dJ_{k,c_k}^{\zeta*}(r) + \frac{\lambda_k^{*2}-\gamma_0}{2\lambda_k^2}}{\frac{1}{2}\left\{\int_0^1 \left[J_{k,c_k}^{\zeta}(r)\right]^2 dr + \int_0^1 \left[J_{k,c_k}^{\zeta*}(r)\right]^2 dr\right\}}, \quad k = 1, ..., S^* \quad \text{(S.30)}$$

and

$$t_k \Rightarrow \frac{\lambda_k}{\gamma_0^{1/2}} \frac{\int_0^1 J_{k,c_k}^{\zeta}(r)dJ_{k,c_k}^{\zeta}(r) + \mathcal{D}_k \int_0^1 J_{k,c_k}^{\zeta*}(r)dJ_{k,c_k}^{\zeta*}(r) + \frac{\lambda_k^2-\gamma_0}{2\lambda_k^2}}{\left\{\int_0^1 \left[J_{k,c_k}^{\zeta}(r)\right]^2 dr + \mathcal{D}_k \int_0^1 \left[J_{k,c_k}^{\zeta*}(r)\right]^2 dr\right\}^{1/2}} =: \Upsilon_k^{\zeta}, \ k = 0, ..., \lfloor S/2 \rfloor \quad \text{(S.31)}$$

$$t_k^* \Rightarrow \frac{\lambda_k}{\gamma_0^{1/2}} \frac{\int_0^1 J_{k,c_k}^{\zeta*}(r)dJ_{k,c_k}^{\zeta}(r) - \int_0^1 J_{k,c_k}^{\zeta}(r)dJ_{k,c_k}^{\zeta*}(r) + \frac{(\lambda_k^{*2}-\gamma_0)}{2\lambda_k^2}}{\left\{\int_0^1 \left[J_{k,c_k}^{\zeta}(r)\right]^2 dr + \int_0^1 \left[J_{k,c_k}^{\zeta*}(r)\right]^2 dr\right\}^{1/2}} =: \Upsilon_k^{*\zeta}, \ \ k = 1, ..., S^* \quad \text{(S.32)}$$

where $\mathcal{D}_k := 0$, for $k = 0$, $S/2$ and $\mathcal{D}_k := 1$, for $k = 1, ..., S^*$, $\lambda_k^{*2} := \gamma_0 + 2\sum_{i=1}^{\infty} \sin(\omega_k i)\gamma_k$, $k = 1, ..., S^*$, and where the limiting processes, $J_{0,c_0}^{\zeta}(r)$, $J_{S/2,c_{S/2}}^{\zeta}(r)$, $J_{k,c_k}^{\zeta}(r)$ and $J_{k,c_k}^{\zeta*}(r)$, $k = 1, \ldots, S^*$, are as defined in Theorem 4.1.

Remark S.5. Representations for the limiting distributions of the corresponding joint F statistics, F_k, $k = 1, ..., S^*$, $F_{1...\lfloor S/2 \rfloor}$ and $F_{0...\lfloor S/2 \rfloor}$ are given by the average of the squares of the limiting distributions for the t-statistics involved in their formulation given in Theorem S.1. So that, for example, $F_k \Rightarrow \frac{1}{2}\left[(\Upsilon_k^{\zeta})^2 + (\Upsilon_k^{*\zeta})^2\right]$, $k = 1, ..., S^*$. □

Remark S.6. The results in Theorem S.1 (and consequently also in Remark S.5) show that the limiting distributions (under both null and local alternatives) of the uncorrected un-augmented HEGY tests depend on nuisance parameters which arise when u_{Sn+s} is weakly dependent. When u_{Sn+s} is IID, which occurs where $\psi(z) = 1$, then the true lag order in (2.4) is $p^* = 0$, and the representations in (S.29)-(S.32) are pivotal because here $\lambda_k^2 = \gamma_0$, $k = 0, \ldots, \lfloor S/2 \rfloor$, and $\lambda_k^{*2} = \gamma_0$, $k = 1, \ldots, S^*$. Indeed, these pivotal forms, for the statistics at the zero and Nyquist frequencies and for all of the F-type tests coincide with those which obtain from the appropriately augmented HEGY tests discussed in section S.3. Relative to these pivotal distributions, we see that in the presence of weak dependence in u_{Sn+s} the un-augmented HEGY statistics have limiting distributions whose numerator includes an additional term arising from the difference between the short run variance of u_{Sn+s} and the long run variance(s) of u_{Sn+s} at the frequency component relating to that statistic and, in the case of the t-statistics (and, hence, the F-statistics), are also scaled by the ratio of the long and short run variances of u_{Sn+s} at that frequency. □

The representations given for the limiting distributions of the un-augmented HEGY statistics in Theorem S.1 are useful because they enable us to see immediately how, given consistent estimators for γ_0, λ_k^2, $k = 0, \ldots, \lfloor S/2 \rfloor$, and λ_k^{*2}, $k = 1, \ldots, S^*$, these statistics can be transformed to obtain modified statistics whose limiting distributions coincide with those which obtain in the case where $\psi(z) = 1$. To that end in section S.5 we now propose seasonal analogues of the non-seasonal PP tests.

S.5 Phillips-Perron-Type Seasonal Unit Root Tests

The finite sample size control of seasonal Phillips-Perron type tests under weak dependence was found to be very poor relative to both augmented HEGY tests and the seasonal $\mathcal{M}$ tests; see the accompanying working paper, del Barrio Castro, Rodrigues and Taylor (2015).

Computation of seasonal versions of the non-seasonal PP unit root tests will require consistent estimators of the nuisance parameters which feature in the limit distributions, given in Theorem S.1, of the un-augmented HEGY statistics which obtain from estimating (2.4) with p^* set to zero. Consistent sums-of-covariances and ASD estimators for λ_k^2, $k = 0, \ldots, \lfloor S/2 \rfloor$, were discussed in section 3.2. Corresponding estimators for λ_k^{*2}, $k = 1, \ldots, S^*$, which are also consistent under the conditions given in section 3.2, can be defined as follows, where notation is the same as used in section 3.2. First, the sum-of-covariances estimators

$$\hat{\lambda}_{k,WA}^{*2} := \sum_{j=-T+1}^{T-1} \kappa(j/m)\hat{\gamma}_j \cos(\pi/2 + \omega_k j), \qquad k = 1, \ldots, S^*. \tag{S.33}$$

Second the corresponding ASD estimators

$$\hat{\lambda}_{k,AR}^{*2} := \frac{s_e^2}{\left\{1 - \sum_{j=1}^{p^*} \hat{\phi}_j^* \cos\left(\left[j\omega_k + \frac{\pi}{2}\right]\right)\right\}^2 + \left\{\sum_{j=1}^{p^*} \hat{\phi}_j^* \sin\left(\left[j\omega_k + \frac{\pi}{2}\right]\right)\right\}^2}, \ k = 1, \ldots, S^*. \tag{S.34}$$

Based on the estimators $\hat{\lambda}_{0,h}^2$, $\hat{\lambda}_{S/2,h}^2$, $\hat{\lambda}_{k,h}^2$ and $\hat{\lambda}_{k,h}^{*2}$, $h = WA$, AR, $k = 1, \ldots, S^*$, defined in (3.3), (S.33), (3.4), (3.5) and (S.34), seasonal analogues of the non-seasonal PP unit root statistics can be derived from the functional forms of the limit distributions of the un-augmented

HEGY statistics given in Theorem S.1, as follows:

$$Z_k := \quad T\hat{\pi}_k - \frac{\left(\hat{\lambda}_{k,h}^2 - \hat{\gamma}_0\right)}{2}\left[\frac{1}{T^2}\sum_{Sn+s=1}^{T}\left(y_{k,Sn+s-1}^{\xi}\right)^2\right]^{-1}, \quad k = 0, ..., \lfloor S/2 \rfloor \quad \text{(S.35)}$$

$$Z_k^* := \quad T\hat{\pi}_k^* - \frac{\left(\hat{\lambda}_{k,h}^{*2} - \hat{\gamma}_0\right)}{2}\left[\frac{1}{T^2}\sum_{Sn+s=1}^{T}\left(y_{k,Sn+s-1}^{*\xi}\right)^2\right]^{-1}, \quad k = 1, \ldots, S^* \quad \text{(S.36)}$$

and

$$Z_{t_k} := \quad \frac{\hat{\gamma}_0^{1/2}}{\hat{\lambda}_{k,h}}t_k - \frac{\left(\hat{\lambda}_{k,h}^2 - \hat{\gamma}_0\right)}{2}\left[\frac{\hat{\lambda}_{k,h}^2}{T^2}\sum_{Sn+s=1}^{T}\left(y_{k,Sn+s-1}^{\xi}\right)^2\right]^{-1/2}, \quad k = 0, ..., \lfloor S/2 \rfloor \text{(S.37)}$$

$$Z_{t_k}^* := \quad \frac{\hat{\gamma}_0^{1/2}}{\hat{\lambda}_{k,h}}t_k^* - \frac{\left(\hat{\lambda}_{k,h}^{*2} - \hat{\gamma}_0\right)}{2}\left[\frac{\hat{\lambda}_{k,h}^2}{T^2}\sum_{Sn+s=1}^{T}\left(y_{k,Sn+s-1}^{*\xi}\right)^2\right]^{-1/2}, \quad k = 1, \ldots, S^* \quad \text{(S.38)}$$

where $\hat{\gamma}_0$ is the OLS residual variance estimate from estimating (2.4) with p^* set to zero.

Remark S.7. Notice that for $S = 1$, Z_0 in (S.35) and Z_{t_0} in (S.37) reduce to the non-seasonal unit root tests proposed in PP and defined in section 3.1. □

Remark S.8. PP-type analogues of the F-type statistics F_k, $k = 1, ..., S^*$, $F_{1,...,\lfloor S/2 \rfloor}$ and $F_{0,...,\lfloor S/2 \rfloor}$ discussed in section S.3 can also be constructed using the corrected normalised coefficient estimate statistics in (S.35) and (S.36). With an obvious notation we will denote these statistics as $F_{PP,k}$, $k = 1, \ldots, S^*$, $F_{PP,1\ldots\lfloor S/2 \rfloor}$, and $F_{PP,0\ldots\lfloor S/2 \rfloor}$. These statistics can be defined generically as follows:

$$F_{PP} := \frac{1}{\upsilon}(R\mathbf{Z})' \left[R\Lambda\mathbf{Y}'\mathbf{Y}R'\right](R\mathbf{Z}) \quad \text{(S.39)}$$

where υ denotes the number of restrictions being tested; $Z := [Z_0, Z_1, Z_1^*, Z_2, Z_2^*, \ldots, Z_{S^*}, Z_{S^*}^*, Z_{S/2}]'$ is $S \times 1$; $Y := [\mathbf{y}_0|\mathbf{y}_1|\mathbf{y}_1^*|\mathbf{y}_2|\mathbf{y}_2^*| \ldots |\mathbf{y}_{S^*}|\mathbf{y}_{S^*}^*|\mathbf{y}_{S/2}]$ is a $T \times S$ matrix where $\mathbf{y}_i$, $i = 0, S/2$, are $T \times 1$ vectors with generic element $y_{i,Sn+s-1}^{\xi}$, and $\mathbf{y}_i$ and $\mathbf{y}_i^*$, $i = 1, ..., S^*$ are $T \times 1$ vectors with generic elements $y_{i,Sn+s-1}^{\xi}$ and $y_{i,Sn+s-1}^{*\xi}$, respectively; Λ is an $S \times S$ diagonal matrix such that, $\Lambda := T^{-2}diag\left\{1/\hat{\lambda}_{0,h}^2, 1/\hat{\lambda}_{1,h}^2, 1/\hat{\lambda}_{1,h}^2, 1/\hat{\lambda}_{2,h}^2, 1/\hat{\lambda}_{2,h}^2 \ldots, 1/\hat{\lambda}_{S^*,h}^2, 1/\hat{\lambda}_{S^*,h}^2, 1/\hat{\lambda}_{S/2,h}^2\right\}$, and finally R is the relevant $\upsilon \times S$ selection matrix; for example, setting

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \end{bmatrix},$$

yields the $F_{PP,1}$ statistic, whilst setting $R = I_S$, where I_q denotes the $q \times q$ identity matrix for any positive integer q, results in $F_{PP,0\ldots\lfloor S/2 \rfloor}$. □

S.6 Asymptotic Results for the Seasonal PP Tests

In Theorem S.2 we now present the large sample distributions of the seasonal PP-type unit root test statistics proposed in section S.5. In particular, we show that these have pivotal limiting distributions whose form coincides with those which obtain in the case where the shocks are serially uncorrelated.

Theorem S.2. *Let the conditions of Theorem 4.1 hold. Then, as $T \to \infty$, the PP-type coefficient statistics introduced in section S.2 and Remark S.4 satisfy,*

$$Z_k \Rightarrow \frac{(1+\mathcal{D}_k)\left[\int_0^1 J^{\zeta}_{k,c_k}(r)dJ^{\zeta}_{k,c_k}(r) + \mathcal{D}_k \int_0^1 J^{\zeta *}_{k,c_k}(r)dJ^{\zeta *}_{k,c_k}(r)\right]}{\int_0^1 \left[J^{\zeta}_{k,c_k}(r)\right]^2 dr + \mathcal{D}_k \int_0^1 \left[J^{\zeta *}_{k,c_k}(r)\right]^2 dr}, \; k = 0, \ldots, \lfloor S/2 \rfloor \quad \text{(S.40)}$$

$$Z^*_k \Rightarrow \frac{2\left[\int_0^1 J^{\zeta *}_{k,c_k}(r)dJ^{\zeta}_{k,c_k}(r) - \int_0^1 J^{\zeta}_{k,c_k}(r)dJ^{\zeta *}_{k,c_k}(r)\right]}{\int_0^1 \left[J^{\zeta}_{k,c_k}(r)\right]^2 dr + \int_0^1 \left[J^{\zeta *}_{k,c_k}(r)\right]^2 dr}, \quad k = 1, \ldots, S^* \quad \text{(S.41)}$$

while the corresponding t- and F−type statistics satisfy

$$Z_{t_k} \Rightarrow \frac{\int_0^1 J^{\zeta}_{k,c_k}(r)dJ^{\zeta}_{k,c_k}(r) + \mathcal{D}_k \int_0^1 J^{\zeta *}_{k,c_k}(r)dJ^{\zeta *}_{k,c_k}(r)}{\left\{\int_0^1 \left[J^{\zeta}_{k,c_k}(r)\right]^2 dr + \mathcal{D}_k \int_0^1 \left[J^{\zeta *}_{k,c_k}(r)\right]^2 dr\right\}^{1/2}} =: \mathcal{T}^{\zeta}_k, \quad k = 0, ..., \lfloor S/2 \rfloor \quad \text{(S.42)}$$

$$Z^*_{t_k} \Rightarrow \frac{\int_0^1 J^{\zeta *}_{k,c_k}(r)dJ^{\zeta}_{k,c_k}(r) - \int_0^1 J^{\zeta}_{k,c_k}(r)dJ^{\zeta *}_{k,c_k}(r)}{\left\{\int_0^1 \left[J^{\zeta}_{k,c_k}(r)\right]^2 dr + \int_0^1 \left[J^{\zeta *}_{k,c_k}(r)\right]^2 dr\right\}^{1/2}} =: \mathcal{T}^{*\zeta}_k, \quad k = 1, \ldots, S^* \quad \text{(S.43)}$$

$$F_{PP,k} \Rightarrow \frac{1}{2}\left[\left(\mathcal{T}^{\zeta}_k\right)^2 + \left(\mathcal{T}^{*\zeta}_k\right)^2\right], \quad k = 1, \ldots, S^* \quad \text{(S.44)}$$

$$F_{PP,j\ldots\lfloor S/2 \rfloor} \Rightarrow \frac{1}{S-j}\left[\sum_{i=j}^{\lfloor S/2 \rfloor}\left(\mathcal{T}^{\zeta}_i\right)^2 + \sum_{k=1}^{S^*}\left(\mathcal{T}^{*\zeta}_k\right)^2\right], \quad j = 0, 1 \quad \text{(S.45)}$$

where $\mathcal{D}_k = 0$, for $k = 0, S/2$ and $\mathcal{D}_k = 1$, for $k = 1, ..., S^*$, and the limiting processes, $J^{\zeta}_{0,c_0}(r)$, $J^{\zeta}_{S/2,c_{S/2}}(r)$, $J^{\zeta}_{k,c_k}(r)$ and $J^{\zeta *}_{k,c_k}(r)$, $k = 1, \ldots, S^*$, are as defined in Theorem 4.1.

Remark S.9: The limiting null distributions of the PP-type statistics from section S.5 are obtained on setting $c_k = 0$ (so that, correspondingly, $H_{0,k}$ holds) in the representations given in Theorem S.2. These limiting null distributions coincide with those reported in Smith *et al.* (2009) and Rodrigues and Taylor (2007), for OLS and local GLS de-trending respectively, for the corresponding HEGY statistics from (2.4) in the case where u_{Sn+s} is serially uncorrelated. Notice also that, contrary to what is shown in, *inter alia*, Burridge and Taylor (2001) and del Barrio Castro, Osborn and Taylor (2012), for the corresponding t_k and t^*_k augmented HEGY statistics from (2.4), when u_{Sn+s} is serially correlated the limiting null distributions of the harmonic frequency PP-type test statistics Z_k, Z_{t_k}, Z^*_k and $Z^*_{t_k}$, $k = 1, ..., S^*$, are free from nuisance parameters. Indeed, the asymptotic null distributions of Z^*_k and $Z^*_{t_k}$ coincide with those reported for the augmented HEGY t_k and t^*_k statistics, $k = 1, ..., S^*$, in Burridge and Taylor (2001) and del Barrio Castro, Osborn and Taylor (2012) for the case where $a_k = 0$ and $b_k = 1$; that is, in the absence of serial correlation in u_{Sn+s}. The foregoing asymptotic equivalence results between the HEGY and corresponding PP-type statistics also hold under the local alternative, $H_{1,\mathbf{c}}$. □

Remark S.10: Selected critical values for tests based on the statistics in (S.40)-(S.43) and (S.44)-(S.45) (for the quarterly, $S = 4$, and monthly, $S = 12$, cases) are provided for the case of OLS de-trended tests in HEGY, Ghysels *et al.* (1994) and Smith and Taylor (1998), and for GLS de-trended tests in Rodrigues and Taylor (2007). Notice that the limiting null distribution in (S.40) for both $k = 0$ and $k = \lfloor S/2 \rfloor$ coincides with the limiting null distribution of the standard normalised bias statistic of Dickey and Fuller (1979), with relevant critical values provided in Fuller (1996). Furthermore, the limiting null distribution in (S.40), for $k = 1, ..., S^*$, coincides with the limiting null distribution of the Dickey *et al.* (1984) unit root test statistic, from where relevant critical values can be obtained. □

S.7 Proofs of Theorems S.1 and S.2

First re-write (2.4) with p^* set to zero in vector form, *viz*, $\mathbf{y} = \mathbf{Y}\beta_0 + \mathbf{u}$, where $\mathbf{y}$ is a $T \times 1$ vector with generic element $\Delta_S y^{\xi}_{Sn+s}$; $\mathbf{Y} := [\mathbf{y}_0|\mathbf{y}_1|\mathbf{y}^*_1|\mathbf{y}_2|\mathbf{y}^*_2| \ldots |\mathbf{y}_{S^*}|\mathbf{y}^*_{S^*}|\mathbf{y}_{S/2}]$ is a $T \times S$ matrix where $\mathbf{y}_i$, $i = 0$, ..., $\lfloor S/2 \rfloor$ are $T \times 1$ vectors with generic elements $y^{\xi}_{i,Sn+s-1}$, and $\mathbf{y}^*_i$, $i = 1, ..., S^*$ are $T \times 1$ vectors with generic elements $y^{*\xi}_{i,Sn+s-1}$, respectively, and $\beta_0 := [\pi_0, \pi_1\pi^*_1, \pi_2, \pi^*_2, \ldots, \pi_{S^*}, \pi^*_{S^*}, \pi_{S/2,}]'$. The OLS estimator from the un-augmented form of (2.4), may then be defined via,

$$T\hat{\beta}_0 := \left[T^{-2}\mathbf{Y}'\mathbf{Y}\right]^{-1}\left[T^{-1}\mathbf{Y}'\mathbf{y}\right]. \tag{S.46}$$

Because $T^{-2}\mathbf{Y}'\mathbf{Y}$ weakly converges to an $S \times S$ diagonal matrix, this as a consequence of the asymptotic orthogonality of the HEGY auxiliary variables discussed previously, we may therefore separately derive the large sample behavior of the OLS estimators of π_j, $j = 0, ..., \lfloor S/2 \rfloor$, and π^*_i, $i = 1, ..., S^*$. To that end, the so-called *normalised bias* statistics then satisfy the following,

$$T\widehat{\pi}_j = \frac{T^{-1}\mathbf{y}'_j\mathbf{y}}{T^{-2}\mathbf{y}'_j\mathbf{y}_j} + o_p(1) = \frac{T^{-1}\sum_{n=1}^{N}\sum_{s=1-S}^{0} y^{\xi}_{j,Sn+s-1}\Delta_S y^{\xi}_{Sn+s}}{T^{-2}\sum_{n=1}^{N}\sum_{s=1-S}^{0}\left(y^{\xi}_{j,Sn+s-1}\right)^2} + o_p(1), \quad j = 0, ..., \lfloor S/2 \rfloor \tag{S.47}$$

and

$$T\widehat{\pi}^*_i = \frac{T^{-1}\mathbf{y}^{*\prime}_i\mathbf{y}}{T^{-2}\mathbf{y}^{*\prime}_i\mathbf{y}^*_i} + o_p(1) = \frac{T^{-1}\sum_{n=1}^{N}\sum_{s=1-S}^{0} y^{*\xi}_{i,Sn+s-1}\Delta_S y^{\xi}_{Sn+s}}{T^{-2}\sum_{n=1}^{N}\sum_{s=1-S}^{0}\left(y^{*\xi}_{i,Sn+s-1}\right)^2} + o_p(1), \quad i = 1, ..., S^*. \tag{S.48}$$

Consider first the numerators of (S.47) and (S.48). For (S.47) observe first that,

$$T^{-1}\sum_{n=1}^{N}\sum_{s=1-S}^{0} y^{\xi}_{j,Sn+s-1}\Delta_S y^{\xi}_{Sn+s} = T^{-1}\sum_{n=1}^{N} Y^{\xi\prime}_{n-1}C_j\Delta_S Y^{\xi}_n + \mathbf{A}_j + o_p(1), \quad j = 0, S/2 \tag{S.49}$$

where $\mathbf{A}_j := S^{-1}\sum_{i=1}^{S-1}(S-i)\cos[i\omega_j]\, N^{-1}\sum_{n=1}^{N}\left(u^{\xi}_{S-i,n}u^{\xi}_{Sn}\right)$, and where $\Delta_S Y^{\xi}_n := [\Delta_S y^{\xi}_{Sn-(S-1)}, \Delta_S\ y^{\xi}_{Sn-(S-2)}, ..., \Delta_S\ y^{\xi}_{Sn}]'$. Notice then that $\mathbf{A}_j \to \Psi_j := S^{-1}\sum_{i=1}^{S-1}(S-i)\cos[i\omega_j]\,\gamma_i$ for

$\omega_j = \frac{2\pi j}{S}$, $j = 0, S/2$. Similarly, for $j = 1, \ldots, S^*$, we have that

$$T^{-1}\sum_{n=1}^{N}\sum_{s=1-S}^{0} y^{\xi}_{j,Sn+s-1}\Delta_S y^{\xi}_{Sn+s} = T^{-1}\sum_{n=1}^{N} Y^{\xi\prime}_{n-1} C_j \Delta_S Y^{\xi}_n + \mathbf{A}_j + o_p(1) \qquad \text{(S.50)}$$

$$T^{-1}\sum_{n=1}^{N}\sum_{s=1-S}^{0} y^{*\xi}_{j,Sn+s-1}\Delta_S y^{\xi}_{Sn+s} = T^{-1}\sum_{n=1}^{N} Y^{\xi\prime}_{n-1} \overline{C}_j \Delta_S Y^{\xi}_n + \overline{\mathbf{A}}_j + o_p(1) \qquad \text{(S.51)}$$

where $\mathbf{A}_j := S^{-1}\sum_{i=1}^{S-1}(S-i)\cos[i\omega_j]\, N^{-1}\sum_{n=1}^{N}\left(u^{\xi}_{S-i,n}u^{\xi}_{Sn}\right)$ and $\overline{\mathbf{A}}_j := -S^{-1}\sum_{i=1}^{S-1}(S-i)\sin[i\omega_j]$ $N^{-1}\sum_{n=1}^{N}\left(u^{\xi}_{S-i,n}u^{\xi}_{Sn}\right)$. We observe that $\mathbf{A}_j \to \Psi^1_j := S^{-1}\sum_{i=1}^{S-1}(S-i)\cos[i\omega_j]\gamma_i$ and $\overline{\mathbf{A}}_j \to \Psi^2_j := -S^{-1}\sum_{i=1}^{S-1}(S-i)\sin[i\omega_j]\gamma_i$ for $\omega_j = \frac{2\pi j}{S}$, $\quad j = 1, \ldots, S^*$.

Again using (S.10), applications of the CMT, the identities $C_k C_k C_k \equiv S^2 C_k$ for $k = 0, S/2$, and $C_j' C_j C_j \equiv \left(\frac{S}{2}\right)^2 C_j$, $C_j' C_j \overline{C}_j \equiv \left(\frac{S}{2}\right)^2 \overline{C}_j$, $\overline{C}_j'\ C_j C_j \equiv -\left(\frac{S}{2}\right)^2 \overline{C}_j$, $\overline{C}_j'\ C_j \overline{C}_j \equiv \left(\frac{S}{2}\right)^2 C_j$, $C_j' \overline{C}_j C_j \equiv \left(\frac{S}{2}\right)^2 \overline{C}_j$, $C_j' \overline{C}_j \overline{C}_j \equiv -\left(\frac{S}{2}\right)^2 C_j$, $\overline{C}_j' \overline{C}_j C_j \equiv \left(\frac{S}{2}\right)^2 C_j$ and $\overline{C}_j' \overline{C}_j \overline{C}_j \equiv \left(\frac{S}{2}\right)^2 \overline{C}_j$ for $j = 1, ..., S^*$, the orthogonality between the circulant matrices and Theorem 2.6 in Phillips (1988), the following results are obtained:

i) For the zero and Nyquist frequencies ($k = 0, S/2$),

$$\begin{aligned}
T^{-1}\sum_{n=1}^{N} Y^{\xi\prime}_{n-1} C_k \Delta_S Y^{\xi}_n &\Rightarrow \frac{\sigma^2_\varepsilon}{S}\frac{\psi(\cos[\omega_k])}{S^2}\int_0^1 \mathbf{J}^{\xi}_{c_k}(r)' C_k' C_k C_k \boldsymbol{\Psi}(1)\, d\mathbf{J}^{\xi}_{c_k}(r) + \frac{1}{S}\sum_{j=2}^{\infty} E\left(U^{\xi\prime}_1 C_k U^{\xi}_j\right) \\
&= \frac{\sigma^2_\varepsilon}{S}\psi(\cos[\omega_k])^2\int_0^1 \mathbf{J}^{\xi}_{c_k}(r)' C_k d\mathbf{J}^{\xi}_{c_k}(r) + \frac{1}{S}\sum_{j=2}^{\infty} E\left(U^{\xi\prime}_1 C_k U^{\xi}_j\right) \\
&= \sigma^2_\varepsilon\psi(\cos[\omega_k])^2\int_0^1 \mathbf{J}^{\xi*}_{c_k}(r)' C_k d\mathbf{J}^{\xi*}_{c_k}(r) + \frac{1}{S}\sum_{j=2}^{\infty} E\left(U^{\xi\prime}_1 C_k U^{\xi}_j\right) \qquad \text{(S.52)}
\end{aligned}$$

where $\omega_0 = 0$ and $\omega_{S/2} = \pi$.

ii) For the harmonic frequencies ($j = 1, ..., S^*$),

$$\begin{aligned}
T^{-1}\sum_{n=1}^{N} Y^{\xi\prime}_{n-1} C_j \Delta_S Y^{\xi}_n &\Rightarrow \frac{\sigma^2_\varepsilon}{S}\left(\frac{2}{S}\right)^2 b_j\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)' C_j' C_j\left(b_j C_j + a_j\overline{C}_j\right) d\mathbf{J}^{\xi}_{c_j}(r) \\
&\quad + \frac{\sigma^2_\varepsilon}{S}\left(\frac{2}{S}\right)^2 a_j\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)' \overline{C}_j' C_j\left(b_j C_j + a_j\overline{C}_j\right) d\mathbf{J}^{\xi}_{c_j}(r) + \frac{1}{S}\sum_{k=2}^{\infty} E\left(U^{\xi\prime}_1 C_j U^{\xi}_k\right) \\
&= \frac{\sigma^2_\varepsilon}{S} b_j^2\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)' C_j d\mathbf{J}^{\xi}_{c_j}(r) + \frac{\sigma^2_\varepsilon}{S} a_j b_j\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)' \overline{C}_j d\mathbf{J}^{\xi}_{c_j}(r) \\
&\quad + \frac{\sigma^2_\varepsilon}{S} a_j^2\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)' C_j d\mathbf{J}^{\xi}_{c_j}(r) - \frac{\sigma^2_\varepsilon}{S} a_j b_j\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)' \overline{C}_j d\mathbf{J}^{\xi}_{c_j}(r) \\
&\quad + \frac{1}{S}\sum_{k=2}^{\infty} E\left(U^{\xi\prime}_1 C_j U^{\xi}_k\right) \\
&= \frac{\sigma^2_\varepsilon\left(a_j^2 + b_j^2\right)}{2}\int_0^1 \mathbf{J}^{\xi\dagger}_{c_j}(r)' C_j d\mathbf{J}^{\xi\dagger}_{c_j}(r) + \frac{1}{S}\sum_{k=2}^{\infty} E\left(U^{\xi\prime}_1 C_j U^{\xi}_k\right), \qquad \text{(S.53)}
\end{aligned}$$

$$
\begin{aligned}
T^{-1}\sum_{n=1}^{N} Y^{\xi\prime}_{n-1}\overline{C}_j\Delta_S Y^{\xi}_n \;\Rightarrow\;& \frac{\sigma_\varepsilon^2}{S}\left(\frac{2}{S}\right)^2 b_j\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)' C_j'\overline{C}_j\left(b_jC_j+a_j\overline{C}_j\right)d\mathbf{J}^{\xi}_{c_j}(r)\\
&+\frac{\sigma_\varepsilon^2}{S}\left(\frac{2}{S}\right)^2 a_j\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)'\overline{C}_j'\overline{C}_j\left(b_jC_j+a_j\overline{C}_j\right)d\mathbf{J}^{\xi}_{c_j}(r)+\frac{1}{S}\sum_{k=2}^{\infty}E\left(U_1^{\xi\prime}\overline{C}_jU_k^{\xi}\right)\\
=\;& \frac{\sigma_\varepsilon^2}{S}b_j^2\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)'\overline{C}_j d\mathbf{J}^{\xi}_{c_j}(r)-\frac{\sigma_\varepsilon^2}{S}b_ja_j\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)' C_j d\mathbf{J}^{\xi}_{c_j}(r)\\
&+\frac{\sigma_\varepsilon^2}{S}a_jb_j\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)' C_j d\mathbf{J}^{\xi}_{c_j}(r)+\frac{\sigma_\varepsilon^2}{S}a_j^2\int_0^1 \mathbf{J}^{\xi}_{c_j}(r)'\overline{C}_j d\mathbf{J}^{\xi}_{c_j}(r)\\
&+\frac{1}{S}\sum_{k=2}^{\infty}E\left(U_1^{\xi\prime}C_jU_k^{\xi}\right)\\
=\;& \frac{\sigma_\varepsilon^2\left(a_j^2+b_j^2\right)}{2}\int_0^1 \mathbf{J}^{\xi\dagger}_{c_j}(r)'\overline{C}_j d\mathbf{J}^{\xi\dagger}_{c_j}(r)+\frac{1}{S}\sum_{k=2}^{\infty}E\left(U_1^{\xi\prime}\overline{C}_jU_k^{\xi}\right) \qquad \text{(S.54)}
\end{aligned}
$$

where $\mathbf{J}^{\xi\dagger}_{c_j}(r) := \frac{1}{\sqrt{S/2}}\mathbf{J}^{\xi}_{c_j}(r)$.

Moreover, for $k = 0$ and $k = S/2$,

$$
\frac{1}{S}\sum_{j=2}^{\infty}E\left(U_1^{\xi\prime}C_kU_j^{\xi}\right)+\Psi_k=\sum_{i=1}^{\infty}\cos\left[i\omega_k\right]\gamma_i=\frac{1}{2}(\lambda_k^2-\gamma_k) \qquad \text{(S.55)}
$$

and for $j = 1, 2, \ldots, S^*$,

$$
\begin{aligned}
\frac{1}{S}\sum_{k=2}^{\infty}E\left(U_1^{\xi\prime}C_jU_k^{\xi}\right)+\Psi_j^1 &= \sum_{i=1}^{\infty}\cos\left[(S-i)\,\omega_j\right]\gamma_i=\frac{1}{4}(\lambda_j^2-\gamma_0) && \text{(S.56)}\\
\frac{1}{S}\sum_{k=2}^{\infty}E\left(U_1^{\xi\prime}\overline{C}_jU_k^{\xi}\right)+\Psi_j^2 &= -\sum_{i=1}^{\infty}\sin\left[(S-i)\,\omega_j\right]\gamma_i=\frac{1}{4}(\lambda_j^{*2}-\gamma_0) && \text{(S.57)}
\end{aligned}
$$

with $\omega_j = \frac{2\pi j}{S}$.

In the case of the denominator of (S.47) the required results for $j = 0$ and $j = S/2$ are collected in (S.21). Consider next the denominators of (S.47) and (S.48) over the values $1, ..., S^*$ of the index parameters j and i, respectively. Here we have the results that C_i, $i = 1, ..., S^*$, is symmetric and that $\overline{C}_i' = -\overline{C}_i$, and noting also that C_i and $\overline{C}_i$ are orthogonal to C_0 and $C_{S/2}$ and that $C_iC_iC_i \equiv \left(\frac{S}{2}\right)^2 C_i$, $C_iC_i\overline{C}_i \equiv \left(\frac{S}{2}\right)^2 \overline{C}_i$, $\overline{C}_i'C_iC_i \equiv -\left(\frac{S}{2}\right)^2 \overline{C}_i$ and $\overline{C}_i'C_i\overline{C}_i \equiv \left(\frac{S}{2}\right)^2 C_i$. Using these results we have that,

$$
T^{-2}\sum_{n=1}^{N}\sum_{s=1-S}^{0}\left(y^{\xi}_{i,Sn+s-1}\right)^2=T^{-2}\sum_{n=1}^{N}\left(\frac{S}{2}\right)\left(Y^{\xi\prime}_{n-1}C_i\,Y^{\xi}_{n-1}\right)+o_p(1)
$$

$$
T^{-2}\sum_{n=1}^{N}\sum_{s=1-S}^{0}\left(y^{*\xi}_{i,Sn+s-1}\right)^2=T^{-2}\sum_{n=1}^{N}\left(\frac{S}{2}\right)\left(Y^{\xi\prime}_{n-1}C_i\,Y^{\xi}_{n-1}\right)+o_p(1)
$$

$$T^{-2}\sum_{n=1}^{N}\left(\frac{S}{2}\right)\left(Y_{n-1}^{\xi\prime}C_iY_{n-1}^{\xi}\right)\Rightarrow\quad \frac{\sigma_\varepsilon^2}{S^2}\left(\frac{S}{2}\right)b_i^2\left(\frac{2}{S}\right)^2\int_0^1\mathbf{J}_{c_i}^{\xi}(r)'C_iC_iC_i\mathbf{J}_{c_i}^{\xi}(r)\,dr+$$
$$\frac{\sigma_\varepsilon^2}{S^2}\left(\frac{S}{2}\right)b_ia_i\left(\frac{2}{S}\right)^2\int_0^1\mathbf{J}_{c_i}^{\xi}(r)'C_iC_i\overline{C}_i\mathbf{J}_{c_i}^{\xi}(r)\,dr+$$
$$\frac{\sigma_\varepsilon^2}{S^2}\left(\frac{S}{2}\right)b_ia_i\left(\frac{2}{S}\right)^2\int_0^1\mathbf{J}_{c_i}^{\xi}(r)'\overline{C}_i'C_iC_i\mathbf{J}_{c_i}^{\xi}(r)\,dr+$$
$$\frac{\sigma_\varepsilon^2}{S^2}\left(\frac{S}{2}\right)a_i^2\left(\frac{2}{S}\right)^2\int_0^1\mathbf{J}_{c_i}^{\xi}(r)'\overline{C}_i'C_i\overline{C}_i\mathbf{J}_{c_i}^{\xi}(r)\,dr$$
$$=\quad\frac{\sigma_\varepsilon^2\left(a_i^2+b_i^2\right)}{4}\int_0^1\mathbf{J}_{c_i}^{\xi\dagger}(r)'C_i\mathbf{J}_{c_i}^{\xi\dagger}(r)\,dr \tag{S.58}$$

where $i=1,\ldots,S^*$ and $\mathbf{J}_{c_i}^{\xi\dagger}(r):=\frac{1}{\sqrt{S/2}}\mathbf{J}_{c_i}^{\xi}(r)$.

Combining the results in (S.49)-(S.57) with (S.21) and (S.58) we establish that for $k=0$ ($\omega_0=0$) and $k=S/2$ ($\omega_{S/2}=\pi$),

$$T\widehat{\pi}_k\Rightarrow\frac{\int_0^1\mathbf{J}_{c_k}^{\xi*}(r)'C_kd\mathbf{J}_{c_k}^{\xi*}(r)+\left(\sum_{i=1}^{\infty}\cos\left[i\omega_k\right]\gamma_i\right)/\sigma_\varepsilon^2\left[\psi\left(\cos[\omega_k]\right)\right]^2}{\int_0^1\mathbf{J}_{c_k}^{\xi*}(r)'C_k\mathbf{J}_{c_k}^{\xi*}(r)\,dr} \tag{S.59}$$

and for $j=1,...,S^*$ that,

$$T\widehat{\pi}_j\quad\Rightarrow\quad\frac{\frac{\sigma_\varepsilon^2\left(a_j^2+b_j^2\right)}{2}\int_0^1\mathbf{J}_{c_j}^{\xi\dagger}(r)'C_jd\mathbf{J}_{c_j}^{\xi\dagger}(r)+\left(\sum_{i=1}^{\infty}\cos\left[(S-i)\,\omega_j\right]\gamma_i\right)}{\frac{\sigma_\varepsilon^2\left(a_j^2+b_j^2\right)}{4}\int_0^1\mathbf{J}_{c_j}^{\xi\dagger}(r)'C_j\mathbf{J}_{c_j}^{\xi\dagger}(r)\,dr} \tag{S.60}$$
$$T\widehat{\pi}_j^*\quad\Rightarrow\quad\frac{\frac{\sigma_\varepsilon^2\left(a_j^2+b_j^2\right)}{2}\int_0^1\mathbf{J}_{c_j}^{\xi\dagger}(r)'\overline{C}_jd\mathbf{J}_{c_j}^{\xi\dagger}(r)+\left(\sum_{i=1}^{\infty}\sin\left[(S-i)\,\omega_j\right]\gamma_i\right)}{\frac{\sigma_\varepsilon^2\left(a_j^2+b_j^2\right)}{4}\int_0^1\mathbf{J}_{c_j}^{\xi\dagger}(r)'C_j\mathbf{J}_{c_j}^{\xi\dagger}(r)\,dr}. \tag{S.61}$$

Next observe that the corresponding t-statistics from the un-augmented form of (2.4) can be written as

$$t_k\quad=\quad\widehat{\gamma}_0^{-1/2}T\widehat{\pi}_k\left[T^{-2}\sum_{n=1}^{N}\sum_{s=1-S}^{0}\left(y_{k,Sn+s}^{\xi}\right)^2\right]^{1/2}+o_p(1),\quad k=0,...,\lfloor S/2\rfloor \tag{S.62}$$
$$t_i^*\quad=\quad\widehat{\gamma}_0^{-1/2}T\widehat{\pi}_i^*\left[T^{-2}\sum_{n=1}^{N}\sum_{s=1-S}^{0}\left(y_{i,Sn+s}^{*\xi}\right)^2\right]^{1/2}+o_p(1),\quad i=1,\ldots,S^* \tag{S.63}$$

where $\widehat{\gamma}_0$ is the usual OLS variance estimator from the un-augmented form of (2.4); that is, $\widehat{\gamma}_0:=T^{-1}\sum_{n=1}^{N}\sum_{s=1-S}^{0}(\hat{u}_{Sn+s}^{\xi})^2$. Observe from the results in (S.59)-(S.61) that $\widehat{\pi}_j=o_p(1)$ and $\widehat{\pi}_j^*=o_p(1)$, and hence $\widehat{\gamma}_0:=T^{-1}\sum_{n=1}^{N}\sum_{s=1-S}^{0}(\Delta_S y_{Sn+s}^{\xi})^2+o_p(1)$ so that $\widehat{\gamma}_0\xrightarrow{p}\sigma_\varepsilon^2\left(1+\sum_{j=1}^{\infty}\psi_j^2\right)$.

Substituting the result that $\widehat{\gamma}_0\xrightarrow{p}\sigma_\varepsilon^2\left(1+\sum_{j=1}^{\infty}\psi_j^2\right)$, the results in Remark S.1, and the results in (S.59)-(S.61), (S.21) and (S.58) into (S.62)-(S.63) and using applications of the CMT, after some simple manipulations, we finally obtain the stated results in Theorem S.1, where we have defined the independent standard OU processes $J_{i,c_i}^{\zeta}(r):=\mathbf{v}_i'\mathbf{J}_{c_i}^{\xi*}(r)$, $i=0,S/2$, $J_{j,c_j}^{\zeta}(r):=\mathbf{h}_j'\mathbf{J}_{c_j}^{\xi\dagger}(r)$ and $J_{j,c_j}^{\zeta*}(r):=\mathbf{h}_j^{*\prime}\mathbf{J}_{c_j}^{\xi\dagger}(r)$ where $\mathbf{h}_j'$ and $\mathbf{h}_j^{*\prime}$ are the first and second rows of $\mathbf{v}_j'$, respectively, for $j=1,\ldots,S^*$ (see Remarks S.1 and S.3). The proof of Theorem S.2 then follows directly from these results and the consistency properties of the long and short run variance estimators used in the construction of the PP-type statistics. □

S.8 Additional Monte Carlo Results

Figures S.1-S.4 report complementary finite sample local power figures to those given in Figures 3-6 in the main text for the case where the tests are not size-adjusted but rather were run using the relevant asymptotic critical values (obtained from the sources given in Remarks 4.2 and 4.3). The Monte Carlo DGP and set-up of these experiments were otherwise exactly as detailed in Section 5.2.

Additional References

Boswijk, H.P., and P.H. Franses, 1996, Unit roots in periodic autoregressions, Journal of Time Series Analysis 17, 221-245.

Burridge, P. and A.M.R.Taylor, 2001, On the properties of regression-based tests for seasonal unit roots in the presence of higher-order serial correlation, Journal of Business and Economic Statistics 19, 374-379.

del Barrio Castro, T., D. R. Osborn and A.M.R. Taylor, 2012, On augmented HEGY tests For seasonal unit roots, Econometric Theory 28, 1121-1143.

Davis, P.J., 1979, Circulant Matrices. Wiley-Interscience: New York.

Dickey, D.A., D.P. Hasza and W.A. Fuller, 1984, Testing for unit roots in seasonal time series, Journal of the American Statistical Association 79, 355-367.

Dickey, D.A. and W.A. Fuller, 1979, Distribution of the estimators for autoregressive time series with a unit root, Journal of the American Statistical Association 74, 427-431.

Fuller, W.A., 1996, Introduction to Statistical Time Series, Second Edition, Wiley: New York.

Ghysels, E., H.S. Lee and J. Noh, 1994, Testing for unit roots in seasonal time series: some theoretical extensions and a Monte Carlo investigation, Journal of Econometrics 62, 415-442.

Gray, R. M., 2006, Toeplitz and Circulant Matrices, A Review. Foundation and Trends(R) in Communications and Information Theory: Now Publishers Inc.

Osborn, D.R. and P.M.M. Rodrigues, 2002, The asymptotic distributions of seasonal unit root tests: a unifying approach, Econometric Reviews 21, 221-241.

Phillips, P.C.B., 1988, Regression theory for near-integrated time series, Econometrica 56, 1021-43.

Rodrigues, P.M.M. and A.M.R. Taylor, 2004, Asymptotic distributions for regression-based seasonal unit root test statistics in a near-integrated model, Econometric Theory 20, 645-670.

Said, S.E. and D.A. Dickey, 1984, Test for unit roots in autoreressive-moving average models of unknown order, Biometrika 71, 599-609.

Smith, R.J. and A.M.R. Taylor, 1998, Additional critical values and asymptotic representations for seasonal unit root tests, Journal of Econometrics 85, 269-288.

Figure S.1: Finite sample size-unadjusted power functions of zero frequency unit root tests (quarterly case, $S = 4$)

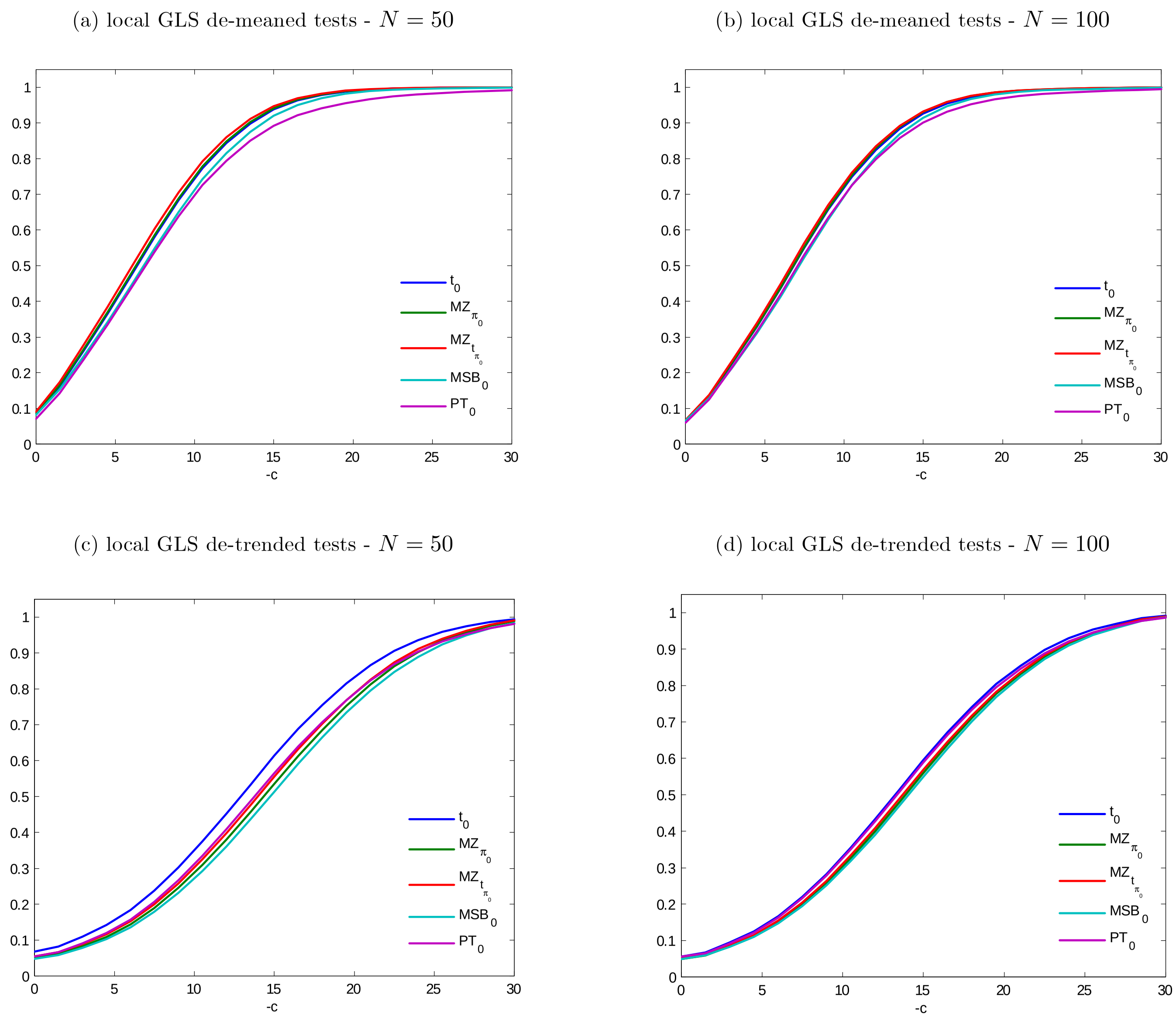
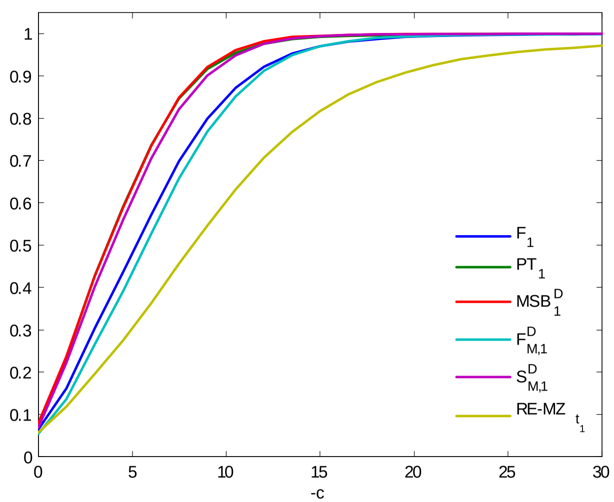
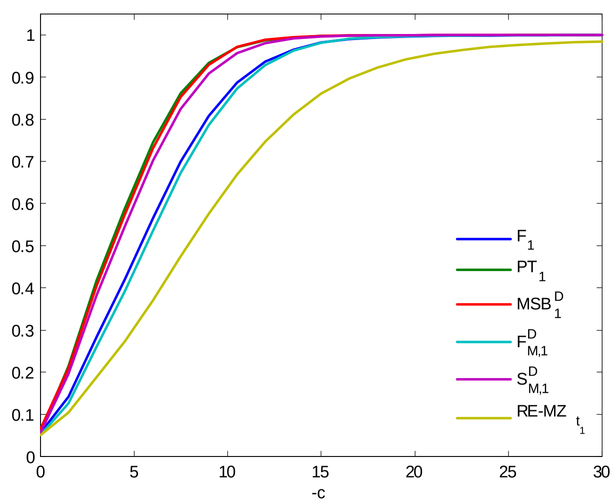


Figure S.2: Finite sample size-unadjusted power functions of harmonic frequency unit root tests (quarterly case, $S = 4$)

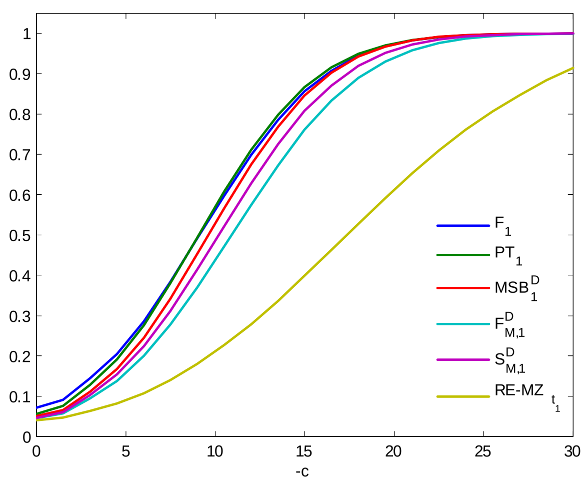
(a) local GLS de-meaned tests - $N = 50$



(b) local GLS de-meaned tests - $N = 100$



(c) local GLS de-trended tests - $N = 50$



(d) local GLS de-trended tests - $N = 100$

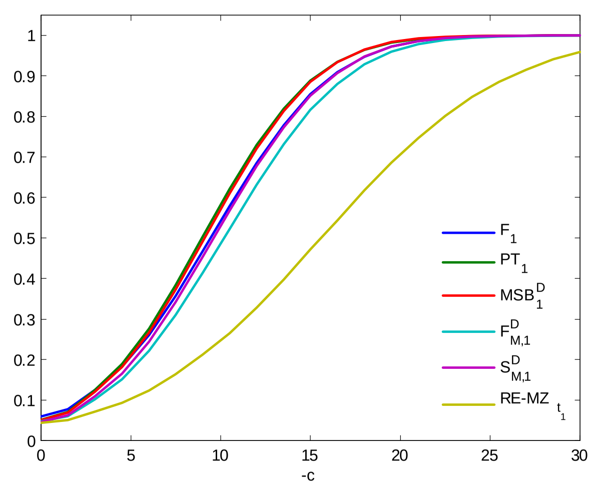
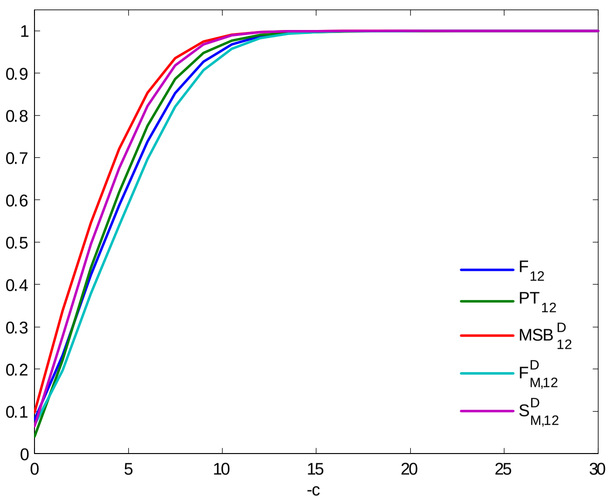
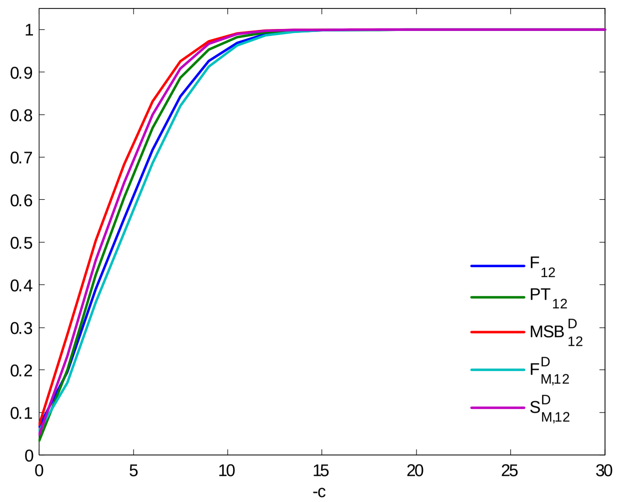


Figure S.3: Finite sample size-unadjusted power functions of joint seasonal frequency tests (quarterly case, $S = 4$)

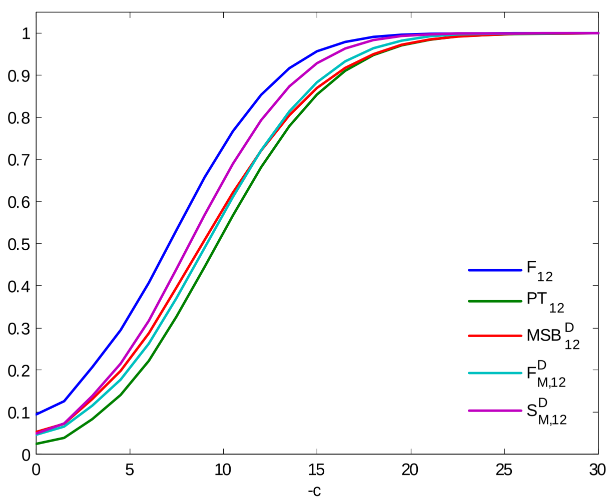
(a) local GLS de-meaned tests - $N = 50$



(b) local GLS de-meaned tests - $N = 100$



(c) local GLS de-trended tests - $N = 50$



(d) local GLS de-trended tests - $N = 100$

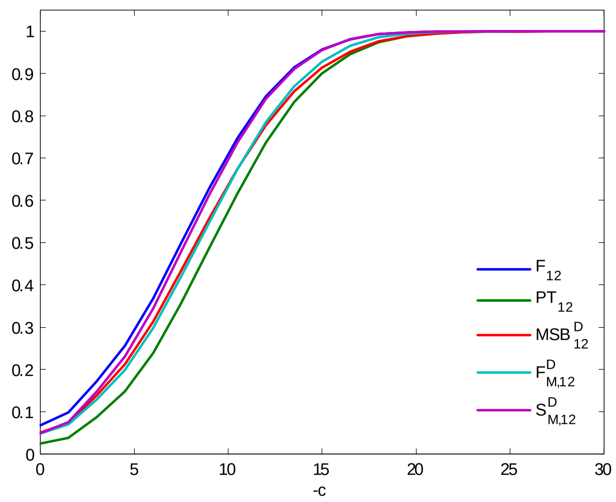
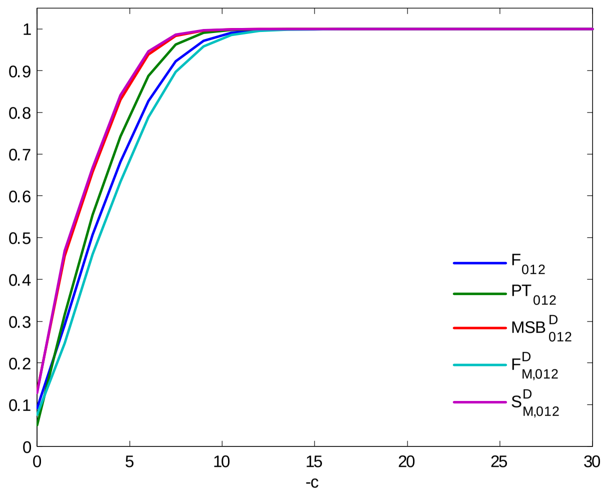
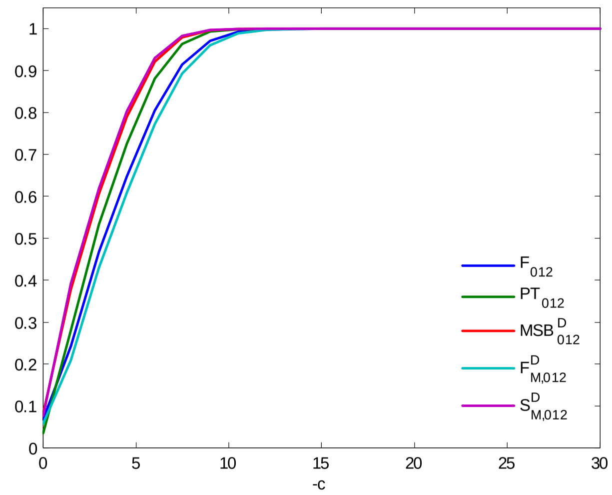


Figure S.4: Finite sample size-unadjusted power functions of joint zero and seasonal frequency tests (quarterly case, $S = 4$)

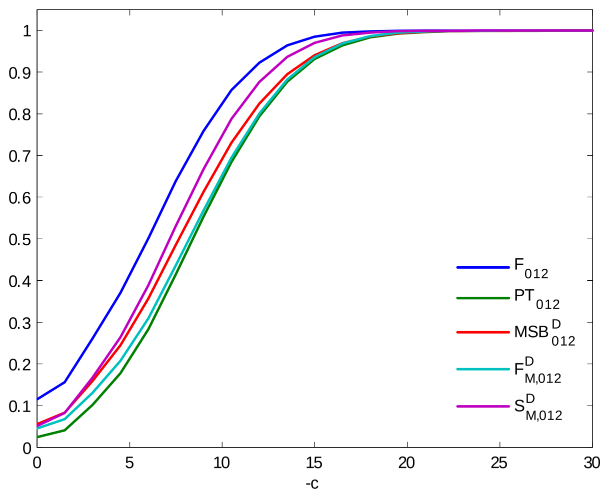
(a) local GLS de-meaned tests - $N = 50$



(b) local GLS de-meaned tests - $N = 100$



(c) local GLS de-trended tests - $N = 50$



(d) local GLS de-trended tests - $N = 100$