Supplementary Appendix to

"Closed-Form Identification of Dynamic Discrete Choice Models with Proxies for Unobserved State Variables"

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This supplementary appendix presents proofs for the identification results under extended models (Sections B.1–B.4) and discuss the consistency of the analog estimator for the law of state transition (Section C).

B Additional Proofs

B.1 Proof of Theorem 2

Our closed-form identification includes four steps.

Step 1: Closed-form identification of the transition rule $f_3\left(x_t^*|d_{t-1},w_{t-1},x_{t-1}^*\right)$**:** First, we show the identification of the parameters and the distributions in transition of x_t^*. Since

$$
\begin{aligned}
x_t &= x_t^* + \varepsilon_t = \sum_d \mathbb{1}\{d_{t-1}=d\}[\alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^* + \eta_t^d] + \varepsilon_t \\
&= \sum_d \mathbb{1}\{d_{t-1}=d\}[\alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1} + \eta_t^d - \gamma^d \varepsilon_{t-1}] + \varepsilon_t
\end{aligned}
$$

we obtain the following equalities for each d:

$$\begin{aligned}
\mathrm{E}[x_t \mid d_{t-1}=d] &= \alpha^d + \beta^d\,\mathrm{E}[w_{t-1} \mid d_{t-1}=d] + \gamma^d\,\mathrm{E}[x_{t-1} \mid d_{t-1}=d] \\
&\quad - \mathrm{E}[\gamma^d \varepsilon_{t-1} \mid d_{t-1}=d] + \mathrm{E}[\eta_t^d \mid d_{t-1}=d] + \mathrm{E}[\varepsilon_t \mid d_{t-1}=d] \\
&= \alpha^d + \beta^d\,\mathrm{E}[w_{t-1} \mid d_{t-1}=d] + \gamma^d\,\mathrm{E}[x_{t-1} \mid d_{t-1}=d] \\
\mathrm{E}[x_t w_{t-1} \mid d_{t-1}=d] &= \alpha^d\,\mathrm{E}[w_{t-1} \mid d_{t-1}=d] + \beta^d\,\mathrm{E}[w_{t-1}^2 \mid d_{t-1}=d] + \gamma^d\,\mathrm{E}[x_{t-1} w_{t-1} \mid d_{t-1}=d] \\
&\quad - \mathrm{E}[\gamma^d \varepsilon_{t-1} w_{t-1} \mid d_{t-1}=d] + \mathrm{E}[\eta_t^d w_{t-1} \mid d_{t-1}=d] + \mathrm{E}[\varepsilon_t w_{t-1} \mid d_{t-1}=d] \\
&= \alpha^d\,\mathrm{E}[w_{t-1} \mid d_{t-1}=d] + \beta^d\,\mathrm{E}[w_{t-1}^2 \mid d_{t-1}=d] + \gamma^d\,\mathrm{E}[x_{t-1} w_{t-1} \mid d_{t-1}=d] \\
\mathrm{E}[x_t w_t \mid d_{t-1}=d] &= \alpha^d\,\mathrm{E}[w_t \mid d_{t-1}=d] + \beta^d\,\mathrm{E}[w_{t-1} w_t \mid d_{t-1}=d] + \gamma^d\,\mathrm{E}[x_{t-1} w_t \mid d_{t-1}=d] \\
&\quad - \mathrm{E}[\gamma^d \varepsilon_{t-1} w_t \mid d_{t-1}=d] + \mathrm{E}[\eta_t^d w_t \mid d_{t-1}=d] + \mathrm{E}[\varepsilon_t w_t \mid d_{t-1}=d] \\
&= \alpha^d\,\mathrm{E}[w_t \mid d_{t-1}=d] + \beta^d\,\mathrm{E}[w_{t-1} w_t \mid d_{t-1}=d] + \gamma^d\,\mathrm{E}[x_{t-1} w_t \mid d_{t-1}=d]
\end{aligned}$$

by the independence and zero mean assumptions for η_t^d and ε_t. From these, we have the linear equation

$$\begin{bmatrix} \mathrm{E}[x_t \mid d_{t-1}=d] \\ \mathrm{E}[x_t w_{t-1} \mid d_{t-1}=d] \\ \mathrm{E}[x_t w_t \mid d_{t-1}=d] \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{E}[w_{t-1} \mid d_{t-1}=d] & \mathrm{E}[x_{t-1} \mid d_{t-1}=d] \\ \mathrm{E}[w_{t-1} \mid d_{t-1}=d] & \mathrm{E}[w_{t-1}^2 \mid d_{t-1}=d] & \mathrm{E}[x_{t-1} w_{t-1} \mid d_{t-1}=d] \\ \mathrm{E}[w_t \mid d_{t-1}=d] & \mathrm{E}[w_{t-1} w_t \mid d_{t-1}=d] & \mathrm{E}[x_{t-1} w_t \mid d_{t-1}=d] \end{bmatrix} \begin{bmatrix} \alpha^d \\ \beta^d \\ \gamma^d \end{bmatrix}$$

Provided that the matrix on the right-hand side is non-singular, we can identify the parameters $(\alpha^d, \beta^d, \gamma^d)$ by

$$\begin{bmatrix} \alpha^d \\ \beta^d \\ \gamma^d \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{E}[w_{t-1} \mid d_{t-1}=d] & \mathrm{E}[x_{t-1} \mid d_{t-1}=d] \\ \mathrm{E}[w_{t-1} \mid d_{t-1}=d] & \mathrm{E}[w_{t-1}^2 \mid d_{t-1}=d] & \mathrm{E}[x_{t-1} w_{t-1} \mid d_{t-1}=d] \\ \mathrm{E}[w_t \mid d_{t-1}=d] & \mathrm{E}[w_{t-1} w_t \mid d_{t-1}=d] & \mathrm{E}[x_{t-1} w_t \mid d_{t-1}=d] \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{E}[x_t \mid d_{t-1}=d] \\ \mathrm{E}[x_t w_{t-1} \mid d_{t-1}=d] \\ \mathrm{E}[x_t w_t \mid d_{t-1}=d] \end{bmatrix}$$

Next, we show identification of the distributions of ε_t and η_t^d for each d. Observe that

$$\begin{aligned}
& \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 x_t\right) | d_{t-1}=d\right] \\
=\;& \mathrm{E}\left[\exp\left(is_1\left(x_{t-1}^* + \varepsilon_{t-1}\right) + is_2\left(\alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^* + \eta_t^d + \varepsilon_t\right)\right) | d_{t-1}=d\right] \\
=\;& \mathrm{E}\left[\exp\left(i\left(s_1 x_{t-1}^* + s_2\alpha^d + s_2\beta^d w_{t-1} + s_2\gamma^d x_{t-1}^*\right)\right) | d_{t-1}=d\right] \\
& \times \mathrm{E}\left[\exp\left(is_1\varepsilon_{t-1}\right)\right] \mathrm{E}\left[\exp\left(is_2\left(\eta_t^d + \varepsilon_t\right)\right)\right]
\end{aligned}$$

follows from the independence assumptions for η_t^d and ε_t. Taking the derivative with respect to s_2 yields

$$\begin{aligned}
& \left[\frac{\partial}{\partial s_2} \ln \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 x_t\right)|d_{t-1}=d\right]\right]_{s_2=0} \\
= & \frac{\mathrm{E}\left[i(\alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^*)\exp\left(is_1 x_{t-1}^*\right)|d_{t-1}=d\right]}{\mathrm{E}\left[\exp\left(is_1 x_{t-1}^*\right)|d_{t-1}=d\right]} \\
= & i\alpha^d + \beta^d \frac{\mathrm{E}[iw_{t-1}\exp(is_1 x_{t-1}^*) \mid d_{t-1}=d]}{\mathrm{E}[\exp(is_1 x_{t-1}^*) \mid d_{t-1}=d]} + \gamma^d \frac{\partial}{\partial s_1} \ln \mathrm{E}\left[\exp\left(is_1 x_{t-1}^*\right)|d_{t-1}=d\right] \\
= & i\alpha^d + \beta^d \frac{\mathrm{E}[iw_{t-1}\exp(is_1 x_{t-1}) \mid d_{t-1}=d]}{\mathrm{E}[\exp(is_1 x_{t-1}) \mid d_{t-1}=d]} + \gamma^d \frac{\partial}{\partial s_1} \ln \mathrm{E}\left[\exp\left(is_1 x_{t-1}^*\right)|d_{t-1}=d\right]
\end{aligned}$$

where the switch of the differential and integral operators is permissible provided that there exists $h \in L^1(F_{w_{t-1}x_{t-1}^*|d_{t-1}=d})$ such that $\left|i(\alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^*)\exp\left(is_1 x_{t-1}^*\right)\right| < h(w_{t-1}, x_{t-1}^*)$ holds for all (w_{t-1}, x_{t-1}^*), which follows from the bounded conditional moment given in Assumption 10, and the denominators are nonzero as the conditional characteristic function of x_t^* given d_t does not vanish on the real line under Assumption 10. Therefore,

$$\begin{aligned}
\mathrm{E}\left[\exp\left(isx_{t-1}^*\right)|d_{t-1}=d\right] = & \exp\left[\int_0^s \left[\frac{1}{\gamma^d}\frac{\partial}{\partial s_2} \ln \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 x_t\right)|d_{t-1}=d\right]\right]_{s_2=0} ds_1 \right. \\
& \left. - \int_0^s \frac{i\alpha^d}{\gamma^d} ds_1 - \int_0^s \frac{\beta^d}{\gamma^d} \frac{\mathrm{E}[iw_{t-1}\exp(is_1 x_{t-1}) \mid d_{t-1}=d]}{\mathrm{E}[\exp(is_1 x_{t-1}) \mid d_{t-1}=d]} ds_1 \right] \\
= & \exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_t - \alpha^d - \beta^d w_{t-1})\exp\left(is_1 x_{t-1}\right)|d_{t-1}=d\right]}{\gamma^d\,\mathrm{E}\left[\exp\left(is_1 x_{t-1}\right)|d_{t-1}=d\right]} ds_1\right].
\end{aligned}$$

From the proxy model and the independence assumption for ε_t,

$$\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d\right] = \mathrm{E}\left[\exp\left(isx_{t-1}^*\right)|d_{t-1}=d\right]\mathrm{E}\left[\exp\left(is\varepsilon_{t-1}\right)\right].$$

We then obtain the following result using any d.

$$\begin{aligned}
\mathrm{E}\left[\exp\left(is\varepsilon_{t-1}\right)\right] = & \frac{\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d\right]}{\mathrm{E}\left[\exp\left(isx_{t-1}^*\right)|d_{t-1}=d\right]} \\
= & \frac{\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_t-\alpha^d-\beta^d w_{t-1})\exp(is_1 x_{t-1})|d_{t-1}=d\right]}{\gamma^d\,\mathrm{E}[\exp(is_1 x_{t-1})|d_{t-1}=d]} ds_1\right]}.
\end{aligned}$$

This argument holds for all t so that we can identify the characteristic function of ε_t with

$$\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right] = \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^d-\beta^d w_t)\exp(is_1x_t)|d_t=d\right]}{\gamma^d\,\mathrm{E}\left[\exp(is_1x_t)|d_t=d\right]}ds_1\right]} \tag{B.1}$$

using any d.

In order to identify $f_{\eta_t^d}$ for each d, consider

$$x_t+\gamma^d\varepsilon_{t-1}=\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1}+\varepsilon_t+\eta^d,$$

and thus

$$\begin{aligned}\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\mathrm{E}\left[\exp\left(is\gamma^d\varepsilon_{t-1}\right)\right] &= \mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1})\right)|d_{t-1}=d\right] \\ &\quad\times\mathrm{E}\left[\exp\left(is\eta_t^d\right)\right]\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]\end{aligned}$$

follows by the independence assumptions for η_t^d and ε_t. Therefore, by the formula (B.1), the characteristic function of η_t^d can be expressed by

$$\begin{aligned}\mathrm{E}\left[\exp\left(is\eta_t^d\right)\right] &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\cdot\mathrm{E}\left[\exp\left(is\gamma^d\varepsilon_{t-1}\right)\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1})\right)|d_{t-1}=d\right]\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]} \\ &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^d-\beta^d w_t)\exp(is_1x_t)|d_t=d\right]}{\gamma^d\,\mathrm{E}\left[\exp(is_1x_t)|d_t=d\right]}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1})\right)|d_{t-1}=d\right]\cdot\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]}\times \\ &\quad \frac{\mathrm{E}\left[\exp\left(is\gamma^d x_{t-1}\right)|d_{t-1}=d\right]}{\exp\left[\int_0^{s\gamma^d} \frac{\mathrm{E}\left[i(x_t-\alpha^d-\beta^d w_{t-1})\exp(is_1x_{t-1})|d_{t-1}=d\right]}{\gamma^d\,\mathrm{E}\left[\exp(is_1x_{t-1})|d_{t-1}=d\right]}ds_1\right]}.\end{aligned}$$

The denominator on the right-hand side is non-zero, as the conditional and unconditional characteristic functions do not vanish on the real line under Assumption 10. Letting $\mathcal{F}$ denote the operator defined by

$$\left(\mathcal{F}\phi\right)(\xi)=\frac{1}{2\pi}\int e^{-is\xi}\phi(s)ds \qquad \text{for all } \phi\in L^1(\mathbb{R}) \text{ and } \xi\in\mathbb{R},$$

we identify $f_{\eta_t^d}$ by

$$f_{\eta_t^d}(\eta)=\left(\mathcal{F}\phi_{\eta_t^d}\right)(\eta) \qquad \text{for all } \eta,$$

where the characteristic function $\phi_{\eta_t^d}$ is given by

$$\phi_{\eta_t^d}(s) = \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^d-\beta^d w_t)\exp(is_1x_t)|d_t=d\right]}{\gamma^d\,\mathrm{E}[\exp(is_1x_t)|d_t=d]}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1})\right)|d_{t-1}=d\right]\cdot\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]} \times \frac{\mathrm{E}\left[\exp\left(is\gamma^d x_{t-1}\right)|d_{t-1}=d\right]}{\exp\left[\int_0^{s\gamma^d} \frac{\mathrm{E}\left[i(x_t-\alpha^d-\beta^d w_{t-1})\exp(is_1x_{t-1})|d_{t-1}=d\right]}{\gamma^d\,\mathrm{E}[\exp(is_1x_{t-1})|d_{t-1}=d]}ds_1\right]}.$$

We can use this identified density in turn to identify the transition rule $f_3\left(x_t^*|d_{t-1},w_{t-1},x_{t-1}^*\right)$ with

$$f_3\left(x_t^*|d_{t-1},w_{t-1},x_{t-1}^*\right) = \sum_d \mathbb{1}\{d_{t-1}=d\} f_{\eta_t^d}\left(x_t^*-\alpha^d-\beta^d w_{t-1}-\gamma^d x_{t-1}^*\right).$$

In summary, we obtain the closed-form expression

$$\begin{aligned} f_3(x_t^* \mid d_{t-1},w_{t-1},x_{t-1}^*) &= \sum_d \mathbb{1}\{d_{t-1}=d\}\left(\mathcal{F}\phi_{\eta_t^d}\right)\left(x_t^*-\alpha^d-\beta^d w_{t-1}-\gamma^d x_{t-1}^*\right) \\ &= \sum_d \frac{\mathbb{1}\{d_{t-1}=d\}}{2\pi}\int \exp\left(-is(x_t^*-\alpha^d-\beta^d w_{t-1}-\gamma^d x_{t-1}^*)\right) \times \\ &\quad \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'}-\beta^{d'} w_t)\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1})\right)|d_{t-1}=d\right]\cdot\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]} \times \\ &\quad \frac{\mathrm{E}\left[\exp\left(is\gamma^d x_{t-1}\right)|d_{t-1}=d'\right]}{\exp\left[\int_0^{s\gamma^d} \frac{\mathrm{E}\left[i(x_t-\alpha^{d'}-\beta^{d'} w_{t-1})\exp(is_1x_{t-1})|d_{t-1}=d'\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_{t-1})|d_{t-1}=d']}ds_1\right]}\,ds. \end{aligned}$$

using any d'. This completes Step 1.

Step 2: Closed-form identification of the proxy model $f_4\left(x_t \mid x_t^*\right)$: Given (B.1), we can write the density of ε_t by

$$f_{\varepsilon_t}(\varepsilon) = \left(\mathcal{F}\phi_{\varepsilon_t}\right)(\varepsilon) \qquad \text{for all } \varepsilon,$$

where the characteristic function ϕ_{ε_t} is defined by (B.1) as

$$\phi_{\varepsilon_t}(s) = \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^d-\beta^d w_t)\exp(is'x_t)|d_t=d\right]}{\gamma^d\,\mathrm{E}[\exp(is'x_t)|d_t=d]}ds'\right]}.$$

Provided this identified density of ε_t, we nonparametrically identify the proxy model

$$f_4(x_t \mid x_t^*) = f_{\varepsilon_t}(x_t - x_t^*)$$

In summary, we obtain the closed-form expression

$$\begin{aligned} f_4(x_t \mid x_t^*) &= (\mathcal{F}\phi_{\varepsilon_t})(x_t - x_t^*) \\ &= \frac{1}{2\pi}\int \frac{\exp(-is(x_t - x_t^*)) \cdot \mathrm{E}[\exp(isx_t) | d_t = d]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1} - \alpha^d - \beta^d w_t)\exp(is_1 x_t) | d_t = d\right]}{\gamma^d\, \mathrm{E}[\exp(is_1 x_t) | d_t = d]} ds_1\right]} ds \end{aligned}$$

using any d. This completes Step 2.

Step 3: Closed-form identification of the transition rule $f_2\left(w_t | d_{t-1}, w_{t-1}, x_{t-1}^*\right)$**:** Consider the joint density expressed by the convolution integral

$$f\left(x_{t-1}, w_t \mid d_{t-1}, w_{t-1}\right) = \int f_{\varepsilon_{t-1}}\left(x_{t-1} - x_{t-1}^*\right) f\left(x_{t-1}^*, w_t \mid d_{t-1}, w_{t-1}\right) dx_{t-1}^*$$

We can thus obtain a closed-form expression of $f\left(x_{t-1}^*, w_t \mid d_{t-1}, w_{t-1}\right)$ by the deconvolution. To see this, observe

$$\begin{aligned} \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 w_t\right) | d_{t-1}, w_{t-1}\right] &= \mathrm{E}\left[\exp\left(is_1 x_{t-1}^* + is_1 \varepsilon_{t-1} + is_2 w_t\right) | d_{t-1}, w_{t-1}\right] \\ &= \mathrm{E}\left[\exp\left(is_1 x_{t-1}^* + is_2 w_t\right) | d_{t-1}, w_{t-1}\right] \mathrm{E}\left[\exp\left(is_1 \varepsilon_{t-1}\right)\right] \end{aligned}$$

by the independence assumption for ε_t, and so

$$\mathrm{E}\left[\exp\left(is_1 x_{t-1}^* + is_2 w_t\right) | d_{t-1}, w_{t-1}\right] = \frac{\mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 w_t\right) | d_{t-1}, w_{t-1}\right]}{\mathrm{E}\left[\exp\left(is_1 \varepsilon_{t-1}\right)\right]}$$

$$= \frac{\mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 w_t\right) | d_{t-1}, w_{t-1}\right] \cdot \exp\left[\int_0^{s_1} \frac{\mathrm{E}\left[i(x_t - \alpha^d - \beta^d w_{t-1})\exp\left(is_1' x_{t-1}\right) | d_{t-1} = d\right]}{\gamma^d\, \mathrm{E}\left[\exp\left(is_1' x_{t-1}\right) | d_{t-1} = d\right]} ds_1'\right]}{\mathrm{E}\left[\exp\left(is_1 x_{t-1}\right) | d_{t-1} = d\right]}$$

follows. Letting $\mathcal{F}_2$ denote the operator defined by

$$\left(\mathcal{F}_2 \phi\right)(\xi_1, \xi_2) = \frac{1}{4\pi^2}\int\int e^{-is_1\xi_1 - is_2\xi_2}\phi(s_1, s_2) ds_1 ds_2 \qquad \text{for all } \phi \in L^1(\mathbb{R}^2) \text{ and } (\xi_1, \xi_2) \in \mathbb{R}^2,$$

we can express the conditional density as

$$f\left(x_{t-1}^{*}, w_t|d_{t-1}, w_{t-1}\right) = \left(\mathcal{F}_2\phi_{x_{t-1}^{*},w_t|d_{t-1},w_{t-1}}\right)(w_t, x_{t-1}^{*})$$

where the characteristic function is defined by

$$\begin{aligned}&\phi_{x_{t-1}^{*},w_t|d_{t-1},w_{t-1}}(s_1,s_2)\\ =\ &\frac{\mathrm{E}\left[\exp\left(is_1x_{t-1}+is_2w_t\right)|d_{t-1},w_{t-1}\right]\cdot\exp\left[\int_0^{s_1}\frac{\mathrm{E}\left[i(x_t-\alpha^d-\beta^d w_{t-1})\exp\left(is_1'x_{t-1}\right)|d_{t-1}=d\right]}{\gamma^d\,\mathrm{E}\left[\exp\left(is_1'x_{t-1}\right)|d_{t-1}=d\right]}ds_1'\right]}{\mathrm{E}\left[\exp\left(is_1x_{t-1}\right)|d_{t-1}=d\right]}\end{aligned}$$

with any d. Using this conditional density, we can nonparametrically identify the transition rule $f_2(w_t \mid d_{t-1}, w_{t-1}, x_{t-1}^{*})$ with

$$f_2\left(w_t|d_{t-1},w_{t-1},x_{t-1}^{*}\right) = \frac{f\left(x_{t-1}^{*},w_t|d_{t-1},w_{t-1}\right)}{\int f\left(x_{t-1}^{*},w_t|d_{t-1},w_{t-1}\right)dw_t}.$$

In summary, we obtain the closed-form expression

$$\begin{aligned}f_2\left(w_t|d_{t-1},w_{t-1},x_{t-1}^{*}\right) &= \frac{\left(\mathcal{F}_2\phi_{x_{t-1}^{*},w_t|d_{t-1},w_{t-1}}\right)(x_{t-1}^{*},w_t)}{\int\left(\mathcal{F}_2\phi_{x_{t-1}^{*},w_t|d_{t-1},w_{t-1}}\right)(x_{t-1}^{*},w_t)dw_t}\\
= \sum_d \mathbb{1}\{d_{t-1}=d\}\int\int &\exp\left(-is_1w_t-is_2x_{t-1}^{*}\right)\cdot\mathrm{E}\left[\exp\left(is_1x_{t-1}+is_2w_t\right)|d_{t-1}=d,w_{t-1}\right]\times\\
&\frac{\exp\left[\int_0^{s_1}\frac{\mathrm{E}\left[i(x_t-\alpha^{d'}-\beta^{d'}w_{t-1})\exp\left(is_1'x_{t-1}\right)|d_{t-1}=d'\right]}{\gamma^{d'}\,\mathrm{E}\left[\exp\left(is_1'x_{t-1}\right)|d_{t-1}=d'\right]}ds_1'\right]}{\mathrm{E}\left[\exp\left(is_1x_{t-1}\right)|d_{t-1}=d'\right]}ds_1ds_2\Bigg/\\
\int\int\int &\exp\left(-is_1w_t-is_2x_{t-1}^{*}\right)\cdot\mathrm{E}\left[\exp\left(is_1x_{t-1}+is_2w_t\right)|d_{t-1}=d,w_{t-1}\right]\times\\
&\frac{\exp\left[\int_0^{s_1}\frac{\mathrm{E}\left[i(x_t-\alpha^{d'}-\beta^{d'}w_{t-1})\exp\left(is_1'x_{t-1}\right)|d_{t-1}=d'\right]}{\gamma^{d'}\,\mathrm{E}\left[\exp\left(is_1'x_{t-1}\right)|d_{t-1}=d'\right]}ds_1'\right]}{\mathrm{E}\left[\exp\left(is_1x_{t-1}\right)|d_{t-1}=d'\right]}ds_1ds_2dw_t\end{aligned}$$

using any d'. This completes Step 3.

Step 4: Closed-form identification of the CCP $f_1\left(d_t|w_t,x_t^{*}\right)$**:** Note that we have

$$\begin{aligned}\mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t\right)|w_t\right] &= \mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t^{*}+is\varepsilon_t\right)|w_t\right]\\
&= \mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t^{*}\right)|w_t\right]\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]\\
&= \mathrm{E}\left[\mathrm{E}\left[\mathbb{1}\{d_t=d\}|w_t,x_t^{*}\right]\exp\left(isx_t^{*}\right)|w_t\right]\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]\end{aligned}$$

by the independence assumption for ε_t and the law of iterated expectations. Therefore

$$\begin{aligned}\frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp\left(isx_t\right)|w_t\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]} &= \mathrm{E}\left[\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right]\exp\left(isx_t^*\right)|w_t\right] \\ &= \int \exp\left(isx_t^*\right)\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right] f\left(x_t^*|w_t\right) dx_t^*\end{aligned}$$

This is the Fourier inversion of $\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right] f\left(x_t^*|w_t\right)$. On the other hand, the Fourier inversion of $f\left(x_t^*|w_t\right)$ can be found as

$$\mathrm{E}\left[\exp\left(isx_t^*\right)|w_t\right] = \frac{\mathrm{E}\left[\exp\left(isx_t\right)|w_t\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]}.$$

Therefore, we find the closed-form expression for CCP $f_1\left(d_t|w_t, x_t^*\right)$ as follows.

$$\Pr\left(d_t = d|w_t, x_t^*\right) = \mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right] = \frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right] f\left(x_t^*|w_t\right)}{f\left(x_t^*|w_t\right)} = \frac{\left(\mathcal{F}\phi_{(d)x_t^*|w_t}\right)\left(x_t^*\right)}{\left(\mathcal{F}\phi_{x_t^*|w_t}\right)\left(x_t^*\right)}$$

where the characteristic functions are defined by

$$\begin{aligned}\phi_{(d)x_t^*|w_t}(s) &= \frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp\left(isx_t\right)|w_t\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]} \\ &= \frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp\left(isx_t\right)|w_t\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'}-\beta^{d'}w_t)\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right)|d_t = d'\right]}\end{aligned}$$

and

$$\begin{aligned}\phi_{x_t^*|w_t}(s) &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|w_t\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]} \\ &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|w_t\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'}-\beta^{d'}w_t)\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right)|d_t = d'\right]}\end{aligned}$$

by (B.1) using any d'. In summary, we obtain the closed-form expression

$$\begin{aligned}
\Pr\left(d_t = d | w_t, x_t^*\right) &= \frac{\left(\mathcal{F}\phi_{(d)x_t^*|w_t}\right)(x_t^*)}{\left(\mathcal{F}\phi_{x_t^*|w_t}\right)(x_t^*)} \\
&= \int \exp\left(-isx_t^*\right) \cdot \mathrm{E}\left[\mathbb{1}\{d_t = d\} \exp\left(isx_t\right) | w_t\right] \times \\
&\quad \frac{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'}-\beta^{d'} w_t) \exp(is_1 x_t) | d_t = d'\right]}{\gamma^{d'} \mathrm{E}[\exp(is_1 x_t) | d_t = d']} ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right) | d_t = d'\right]} ds \Bigg/ \\
&\quad \int \exp\left(-isx_t^*\right) \cdot \mathrm{E}\left[\exp\left(isx_t\right) | w_t\right] \times \\
&\quad \frac{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'}-\beta^{d'} w_t) \exp(is_1 x_t) | d_t = d'\right]}{\gamma^{d'} \mathrm{E}[\exp(is_1 x_t) | d_t = d']} ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right) | d_t = d'\right]} ds
\end{aligned}$$

using any d'. This completes Step 4. □

B.2 Proof of Theorem 3

Similarly to the baseline case, our closed-form identification includes four steps.

Step 1: Closed-form identification of the transition rule $f_3\left(x_t^* | d_{t-1}, w_{t-1}, x_{t-1}^*\right)$**:** First, we show the identification of the parameters and the distributions in transition of x_t^*. Since

$$\begin{aligned}
x_t &= \sum_d \mathbb{1}\{d_{t-1} = d\}[\delta^d x_t^* + \varepsilon_t^d] \\
&= \sum_d \mathbb{1}\{d_{t-1} = d\}[\alpha^d \delta^d + \beta^d \delta^d w_{t-1} + \gamma^d \delta^d x_{t-1}^* + \delta^d \eta_t^d + \varepsilon_t^d] \\
&= \sum_d \sum_{d'} \mathbb{1}\{d_{t-1} = d\} \mathbb{1}\{d_{t-2} = d'\} \left[\alpha^d \delta^d + \beta^d \delta^d w_{t-1} + \gamma^d \frac{\delta^d}{\delta^{d'}} x_{t-1} + \delta^d \eta_t^d + \varepsilon_t^d - \gamma^d \frac{\delta^d}{\delta^{d'}} \varepsilon_{t-1}^{d'}\right]
\end{aligned}$$

we obtain the following equalities for each d and d':

$$\begin{aligned}
\mathrm{E}[x_t \mid d_{t-1}=d, d_{t-2}=d'] &= \alpha^d\delta^d + \beta^d\delta^d\,\mathrm{E}[w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] \\
&\quad +\gamma^d\frac{\delta^d}{\delta^{d'}}\,\mathrm{E}[x_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] \\
\mathrm{E}[x_t w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] &= \alpha^d\delta^d\,\mathrm{E}[w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] \\
&\quad +\beta^d\delta^d\,\mathrm{E}[w_{t-1}^2 \mid d_{t-1}=d, d_{t-2}=d'] \\
&\quad +\gamma^d\frac{\delta^d}{\delta^{d'}}\,\mathrm{E}[x_{t-1}w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] \\
\mathrm{E}[x_t w_t \mid d_{t-1}=d, d_{t-2}=d'] &= \alpha^d\delta^d\,\mathrm{E}[w_t \mid d_{t-1}=d, d_{t-2}=d'] \\
&\quad +\beta^d\delta^d\,\mathrm{E}[w_{t-1}w_t \mid d_{t-1}=d, d_{t-2}=d'] \\
&\quad +\gamma^d\frac{\delta^d}{\delta^{d'}}\,\mathrm{E}[x_{t-1}w_t \mid d_{t-1}=d, d_{t-2}=d']
\end{aligned}$$

by the independence and zero mean assumptions for η_t^d and ε_t^d. From these, we have the linear equation

$$\begin{bmatrix} \mathrm{E}[x_t \mid d_{t-1}=d, d_{t-2}=d'] \\ \mathrm{E}[x_t w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] \\ \mathrm{E}[x_t w_t \mid d_{t-1}=d, d_{t-2}=d'] \end{bmatrix} =$$

$$\begin{bmatrix} 1 & \mathrm{E}[w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] & \mathrm{E}[x_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] \\ \mathrm{E}[w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] & \mathrm{E}[w_{t-1}^2 \mid d_{t-1}=d, d_{t-2}=d'] & \mathrm{E}[x_{t-1}w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] \\ \mathrm{E}[w_t \mid d_{t-1}=d, d_{t-2}=d'] & \mathrm{E}[w_{t-1}w_t \mid d_{t-1}=d, d_{t-2}=d'] & \mathrm{E}[x_{t-1}w_t \mid d_{t-1}=d, d_{t-2}=d'] \end{bmatrix} \begin{bmatrix} \alpha^d\delta^d \\ \beta^d\delta^d \\ \gamma^d\frac{\delta^d}{\delta^{d'}} \end{bmatrix}$$

Provided that the matrix on the right-hand side is non-singular, we can identify the composite parameters $\left(\alpha^d\delta^d, \beta^d\delta^d, \gamma^d\frac{\delta^d}{\delta^{d'}}\right)$ by

$$\begin{bmatrix} \alpha^d\delta^d \\ \beta^d\delta^d \\ \gamma^d\frac{\delta^d}{\delta^{d'}} \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{E}[w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] & \mathrm{E}[x_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] \\ \mathrm{E}[w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] & \mathrm{E}[w_{t-1}^2 \mid d_{t-1}=d, d_{t-2}=d'] & \mathrm{E}[x_{t-1}w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] \\ \mathrm{E}[w_t \mid d_{t-1}=d, d_{t-2}=d'] & \mathrm{E}[w_{t-1}w_t \mid d_{t-1}=d, d_{t-2}=d'] & \mathrm{E}[x_{t-1}w_t \mid d_{t-1}=d, d_{t-2}=d'] \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \mathrm{E}[x_t \mid d_{t-1}=d, d_{t-2}=d'] \\ \mathrm{E}[x_t w_{t-1} \mid d_{t-1}=d, d_{t-2}=d'] \\ \mathrm{E}[x_t w_t \mid d_{t-1}=d, d_{t-2}=d'] \end{bmatrix}.$$

Once the composite parameters $\gamma^d \frac{\delta^d}{\delta^0}$ and $\gamma^d = \gamma^d \frac{\delta^d}{\delta^d}$ are identified by the above formula, we can in turn identify

$$\delta^d = \frac{\gamma^d \frac{\delta^d}{\delta^0}}{\gamma^d \frac{\delta^d}{\delta^d}}$$

for each d by the normalization assumption $\delta^0 = 1$. It in turn can be used to identify $(\alpha^d, \beta^d, \gamma^d)$ for each d from the identified composite parameters $\left(\alpha^d \delta^d, \beta^d \delta^d, \gamma^d \frac{\delta^d}{\delta^0}\right)$ by

$$(\alpha^d, \beta^d, \gamma^d) = \frac{1}{\delta^d} \left(\alpha^d \delta^d, \beta^d \delta^d, \gamma^d \frac{\delta^d}{\delta^0}\right).$$

Next, we show identification of the distributions of ε_t^d and η_t^d for each d. Observe that

$$\begin{aligned}
& \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 x_t\right) | d_{t-1} = d, d_{t-2} = d'\right] \\
= \; & \mathrm{E}\left[\exp\left(is_1 \left(\delta^{d'} x_{t-1}^* + \varepsilon_{t-1}^{d'}\right) + is_2 \left(\alpha^d \delta^d + \beta^d \delta^d w_{t-1} + \gamma^d \delta^d x_{t-1}^* + \delta^d \eta_t^d + \varepsilon_t^d\right)\right) | d_{t-1} = d, d_{t-2} = d'\right] \\
= \; & \mathrm{E}\left[\exp\left(i \left(s_1 \delta^{d'} x_{t-1}^* + s_2 \alpha^d \delta^d + s_2 \beta^d \delta^d w_{t-1} + s_2 \gamma^d \delta^d x_{t-1}^*\right)\right) | d_{t-1} = d, d_{t-2} = d'\right] \\
& \quad \times \mathrm{E}\left[\exp\left(is_1 \varepsilon_{t-1}^{d'}\right)\right] \mathrm{E}\left[\exp\left(is_2 \left(\delta^d \eta_t^d + \varepsilon_t^d\right)\right)\right]
\end{aligned}$$

follows for each pair (d, d') from the independence assumptions for η_t^d and ε_t^d for each d. We may then use the Kotlarski's identity

$$\begin{aligned}
& \left[\frac{\partial}{\partial s_2} \ln \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 x_t\right) | d_{t-1} = d, d_{t-2} = d'\right]\right]_{s_2 = 0} \\
= \; & \frac{\mathrm{E}\left[i(\alpha^d \delta^d + \beta^d \delta^d w_{t-1} + \gamma^d \delta^d x_{t-1}^*) \exp\left(is_1 \delta^{d'} x_{t-1}^*\right) | d_{t-1} = d, d_{t-2} = d'\right]}{\mathrm{E}\left[\exp\left(is_1 \delta^{d'} x_{t-1}^*\right) | d_{t-1} = d, d_{t-2} = d'\right]} \\
= \; & i\alpha^d \delta^d + \beta^d \delta^d \frac{\mathrm{E}[iw_{t-1} \exp(is_1 \delta^{d'} x_{t-1}^*) \mid d_{t-1} = d, d_{t-2} = d']}{\mathrm{E}[\exp(is_1 \delta^{d'} x_{t-1}^*) \mid d_{t-1} = d, d_{t-2} = d']} \\
& + \gamma^d \frac{\delta^d}{\delta^{d'}} \frac{\partial}{\partial s_1} \ln \mathrm{E}\left[\exp\left(is_1 \delta^{d'} x_{t-1}^*\right) | d_{t-1} = d, d_{t-2} = d'\right] \\
= \; & i\alpha^d \delta^d + \beta^d \delta^d \frac{\mathrm{E}[iw_{t-1} \exp(is_1 x_{t-1}) \mid d_{t-1} = d, d_{t-2} = d']}{\mathrm{E}[\exp(is_1 x_{t-1}) \mid d_{t-1} = d, d_{t-2} = d']} \\
& + \gamma^d \frac{\delta^d}{\delta^{d'}} \frac{\partial}{\partial s_1} \ln \mathrm{E}\left[\exp\left(is_1 \delta^{d'} x_{t-1}^*\right) | d_{t-1} = d, d_{t-2} = d'\right]
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& \mathrm{E}\left[\exp\left(is\delta^{d'}x_{t-1}^*\right)|d_{t-1}=d,d_{t-2}=d'\right] \\
= \;& \exp\left[\int_0^s \left[\frac{\delta^{d'}}{\gamma^d\delta^d}\frac{\partial}{\partial s_2}\ln \mathrm{E}\left[\exp\left(is_1x_{t-1}+is_2x_t\right)|d_{t-1}=d,d_{t-2}=d'\right]\right]_{s_2=0} ds_1 \right.\\
& \left. -\int_0^s \frac{i\alpha^d\delta^{d'}}{\gamma^d}ds_1 - \int_0^s \frac{\beta^d\delta^{d'}}{\gamma^d}\frac{\mathrm{E}[iw_{t-1}\exp(is_1x_{t-1}) \mid d_{t-1}=d,d_{t-2}=d']}{\mathrm{E}[\exp(is_1x_{t-1}) \mid d_{t-1}=d,d_{t-2}=d']}ds_1\right] \\
= \;& \exp\left[\int_0^s \frac{\mathrm{E}\left[i(\frac{\delta^{d'}}{\delta^d}x_t-\alpha^d\delta^{d'}-\beta^d\delta^{d'}w_{t-1})\exp\left(is_1x_{t-1}\right)|d_{t-1}=d,d_{t-2}=d'\right]}{\gamma^d\,\mathrm{E}\left[\exp\left(is_1x_{t-1}\right)|d_{t-1}=d,d_{t-2}=d'\right]}ds_1\right].
\end{aligned}
$$

From the proxy model and the independence assumption for ε_t,

$$
\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d,d_{t-2}=d'\right] = \mathrm{E}\left[\exp\left(is\delta^{d'}x_{t-1}^*\right)|d_{t-1}=d,d_{t-2}=d'\right]\mathrm{E}\left[\exp\left(is\varepsilon_{t-1}^{d'}\right)\right].
$$

We then obtain the following result using any d.

$$
\begin{aligned}
\mathrm{E}\left[\exp\left(is\varepsilon_{t-1}^{d'}\right)\right] &= \frac{\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d,d_{t-2}=d'\right]}{\mathrm{E}\left[\exp\left(is\delta^{d'}x_{t-1}^*\right)|d_{t-1}=d,d_{t-2}=d'\right]} \\
&= \frac{\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d,d_{t-2}=d'\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(\frac{\delta^{d'}}{\delta^d}x_t-\alpha^d\delta^{d'}-\beta^d\delta^{d'}w_{t-1})\exp(is_1x_{t-1})|d_{t-1}=d,d_{t-2}=d'\right]}{\gamma^d\,\mathrm{E}[\exp(is_1x_{t-1})|d_{t-1}=d,d_{t-2}=d']}ds_1\right]}.
\end{aligned}
$$

This argument holds for all t so that we can identify the characteristic function of ε_t^d for each d with

$$
\mathrm{E}\left[\exp\left(is\varepsilon_t^d\right)\right] = \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d',d_{t-1}=d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(\frac{\delta^d}{\delta^{d'}}x_{t+1}-\alpha^{d'}\delta^d-\beta^{d'}\delta^d w_t)\exp(is_1x_t)|d_t=d',d_{t-1}=d\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d',d_{t-1}=d]}ds_1\right]}. \tag{B.2}
$$

using any d'.

In order to identify $f_{\eta_t^d}$ for each d, consider

$$
\begin{aligned}
& \mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d,d_{t-2}=d'\right]\mathrm{E}\left[\exp\left(is\gamma^d\frac{\delta^d}{\delta^{d'}}\varepsilon_{t-1}^{d'}\right)\right] \\
= \;& \mathrm{E}\left[\exp\left(is(\alpha^d\delta^d+\beta^d\delta^d w_{t-1}+\gamma^d\frac{\delta^d}{\delta^{d'}}x_{t-1})\right)|d_{t-1}=d,d_{t-2}=d'\right] \\
& \times \mathrm{E}\left[\exp\left(is\delta^d\eta_t^d\right)\right]\mathrm{E}\left[\exp\left(is\varepsilon_t^d\right)\right]
\end{aligned}
$$

by the independence assumptions for η_t^d and ε_t^d. Therefore,

$$\begin{aligned}
\mathrm{E}\left[\exp\left(is\delta^d\eta_t^d\right)\right] &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d, d_{t-2}=d'\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d\delta^d+\beta^d\delta^d w_{t-1}+\gamma^d\frac{\delta^d}{\delta^{d'}}x_{t-1})\right)|d_{t-1}=d, d_{t-2}=d'\right]} \\
&\quad\times\frac{\mathrm{E}\left[\exp\left(is\gamma^d\frac{\delta^d}{\delta^{d'}}\varepsilon_{t-1}^{d'}\right)\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t^d\right)\right]}
\end{aligned}$$

and the characteristic function of η_t^d can be expressed by

$$\begin{aligned}
\mathrm{E}\left[\exp\left(is\eta_t^d\right)\right] &= \frac{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^d}x_t\right)|d_{t-1}=d, d_{t-2}=d'\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d\frac{1}{\delta^{d'}}x_{t-1})\right)|d_{t-1}=d, d_{t-2}=d'\right]} \\
&\quad\times\frac{1}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^d}\varepsilon_t^d\right)\right]\mathrm{E}\left[\exp\left(-is\gamma^d\frac{1}{\delta^{d'}}\varepsilon_{t-1}^{d'}\right)\right]} \\
&= \frac{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^d}x_t\right)|d_{t-1}=d, d_{t-2}=d'\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d\frac{1}{\delta^{d'}}x_{t-1})\right)|d_{t-1}=d, d_{t-2}=d'\right]} \\
&\quad\times\frac{\exp\left[\int_0^{s/\delta^d}\frac{\mathrm{E}\left[i(\frac{\delta^d}{\delta^{d'}}x_{t+1}-\alpha^{d'}\delta^d-\beta^{d'}\delta^d w_t)\exp(is_1x_t)|d_t=d',d_{t-1}=d\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d',d_{t-1}=d]}ds_1\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^d}x_t\right)|d_t=d', d_{t-1}=d\right]} \\
&\quad\times\frac{\mathrm{E}\left[\exp\left(is\gamma^d\frac{1}{\delta^{d'}}x_{t-1}\right)|d_{t-1}=d, d_{t-2}=d'\right]}{\exp\left[\int_0^{s\gamma^d/\delta^{d'}}\frac{\mathrm{E}\left[i(\frac{\delta^{d'}}{\delta^d}x_t-\alpha^d\delta^{d'}-\beta^d\delta^{d'}w_{t-1})\exp(is_1x_{t-1})|d_{t-1}=d,d_{t-2}=d'\right]}{\gamma^d\,\mathrm{E}[\exp(is_1x_t)|d_{t-1}=d,d_{t-2}=d']}ds_1\right]}
\end{aligned}$$

by the formula (B.2). We can then identify $f_{\eta_t^d}$ by

$$f_{\eta_t^d}(\eta) = \left(\mathcal{F}\phi_{\eta_t^d}\right)(\eta) \qquad \text{for all } \eta,$$

where the characteristic function $\phi_{\eta_t^d}$ is given by

$$\begin{aligned}
\phi_{\eta_t^d}(s) &= \frac{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^d}x_t\right)|d_{t-1}=d, d_{t-2}=d'\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d\frac{1}{\delta^{d'}}x_{t-1})\right)|d_{t-1}=d, d_{t-2}=d'\right]} \\
&\quad\times\frac{\exp\left[\int_0^{s/\delta^d}\frac{\mathrm{E}\left[i(\frac{\delta^d}{\delta^{d'}}x_{t+1}-\alpha^{d'}\delta^d-\beta^{d'}\delta^d w_t)\exp(is_1x_t)|d_t=d',d_{t-1}=d\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d',d_{t-1}=d]}ds_1\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^d}x_t\right)|d_t=d', d_{t-1}=d\right]} \\
&\quad\times\frac{\mathrm{E}\left[\exp\left(is\gamma^d\frac{1}{\delta^{d'}}x_{t-1}\right)|d_{t-1}=d, d_{t-2}=d'\right]}{\exp\left[\int_0^{s\gamma^d/\delta^{d'}}\frac{\mathrm{E}\left[i(\frac{\delta^{d'}}{\delta^d}x_t-\alpha^d\delta^{d'}-\beta^d\delta^{d'}w_{t-1})\exp(is_1x_{t-1})|d_{t-1}=d,d_{t-2}=d'\right]}{\gamma^d\,\mathrm{E}[\exp(is_1x_t)|d_{t-1}=d,d_{t-2}=d']}ds_1\right]}.
\end{aligned}$$

We can use this identified density in turn to identify the transition rule $f_3\left(x_t^*|d_{t-1}, w_{t-1}, x_{t-1}^*\right)$ with

$$f_3\left(x_t^*|d_{t-1}, x_{t-1}, x_{t-1}^*\right) = \sum_d \mathbb{1}\{d_{t-1} = d\} f_{\eta_t^d}\left(x_t^* - \alpha^d - \beta^d w_{t-1} - \gamma^d x_{t-1}^*\right).$$

In summary, we obtain the closed-form expression

$$\begin{aligned} f_3\left(x_t^*|d_{t-1}, x_{t-1}, x_{t-1}^*\right) &= \sum_d \mathbb{1}\{d_{t-1} = d\} \left(\mathcal{F}\phi_{\eta_t^d}\right)\left(x_t^* - \alpha^d - \beta^d w_{t-1} - \gamma^d x_{t-1}^*\right) \\ &= \sum_d \frac{\mathbb{1}\{d_{t-1} = d\}}{2\pi} \int \exp\left(-is(x_t^* - \alpha^d - \beta^d w_{t-1} - \gamma^d x_{t-1}^*)\right) \times \\ & \quad \frac{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^d}x_t\right)|d_{t-1} = d, d_{t-2} = d'\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d + \beta^d w_{t-1} + \gamma^d \frac{1}{\delta^{d'}} x_{t-1})\right)|d_{t-1} = d, d_{t-2} = d'\right]} \times \\ & \quad \frac{\exp\left[\int_0^{s/\delta^d} \frac{\mathrm{E}\left[i(\frac{\delta^d}{\delta^{d'}}x_{t+1} - \alpha^{d'}\delta^d - \beta^{d'}\delta^d w_t)\exp(is_1 x_t)|d_t = d', d_{t-1} = d\right]}{\gamma^{d'}\mathrm{E}[\exp(is_1 x_t)|d_t = d', d_{t-1} = d]} ds_1\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^d}x_t\right)|d_t = d', d_{t-1} = d\right]} \times \\ & \quad \frac{\mathrm{E}\left[\exp\left(is\gamma^d\frac{1}{\delta^{d'}}x_{t-1}\right)|d_{t-1} = d, d_{t-2} = d'\right]}{\exp\left[\int_0^{s\gamma^d/\delta^{d'}} \frac{\mathrm{E}\left[i(\frac{\delta^{d'}}{\delta^d}x_t - \alpha^d\delta^{d'} - \beta^d\delta^{d'}w_{t-1})\exp(is_1 x_{t-1})|d_{t-1} = d, d_{t-2} = d'\right]}{\gamma^d\mathrm{E}[\exp(is_1 x_t)|d_{t-1} = d, d_{t-2} = d']} ds_1\right]} \end{aligned}$$

using any d'. This completes Step 1.

Step 2: Closed-form identification of the proxy model $f_4\left(x_t \mid d_{t-1}, x_t^*\right)$: Given (B.2), we can write the density of ε_t^d by

$$f_{\varepsilon_t^d}(\varepsilon) = \left(\mathcal{F}\phi_{\varepsilon_t^d}\right)(\varepsilon) \qquad \text{for all } \varepsilon,$$

where the characteristic function $\phi_{\varepsilon_t^d}$ is defined by (B.2) as

$$\phi_{\varepsilon_t^d}(s) = \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_t = d', d_{t-1} = d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(\frac{\delta^d}{\delta^{d'}}x_{t+1} - \alpha^{d'}\delta^d - \beta^{d'}\delta^d w_t)\exp(is_1 x_t)|d_t = d', d_{t-1} = d\right]}{\gamma^{d'}\mathrm{E}[\exp(is_1 x_t)|d_t = d', d_{t-1} = d]} ds_1\right]}.$$

Provided this identified density of ε_t^d, we nonparametrically identify the proxy model

$$f_4(x_t \mid d_{t-1} = d, x_t^*) = f_{\varepsilon_t^d|d_{t-1}=d}(x_t - \delta^d x_t^*) = f_{\varepsilon_t^d}(x_t - \delta^d x_t^*)$$

by the independence assumption for ε_t^d. In summary, we obtain the closed-form expression

$$\begin{aligned} f_4(x_t \mid d_{t-1}, x_t^*) &= \sum_d \mathbb{1}\{d_{t-1} = d\} \left(\mathcal{F}\phi_{\varepsilon_t^d}\right)(x_t - \delta^d x_t^*) \\ &= \sum_d \frac{\mathbb{1}\{d_{t-1} = d\}}{2\pi} \int \frac{\exp\left(-is(x_t - \delta^d x_t^*)\right) \cdot \mathrm{E}\left[\exp\left(isx_t\right)|d_t = d', d_{t-1} = d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(\frac{\delta^d}{\delta^{d'}} x_{t+1} - \alpha^{d'}\delta^d - \beta^{d'}\delta^d w_t)\exp(is_1 x_t)|d_t = d', d_{t-1} = d\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1 x_t)|d_t = d', d_{t-1} = d]} ds_1\right]} ds \end{aligned}$$

using any d'. This completes Step 2.

Step 3: Closed-form identification of the transition rule $f_2\left(w_t|d_{t-1}, w_{t-1}, x_{t-1}^*\right)$**:** Consider the joint density expressed by the convolution integral

$$f\left(x_{t-1}, w_t \mid d_{t-1}, w_{t-1}, d_{t-2} = d\right) = \int f_{\varepsilon_{t-1}^d}\left(x_{t-1} - \delta^d x_{t-1}^*\right) f\left(x_{t-1}^*, w_t \mid d_{t-1}, w_{t-1}, d_{t-2} = d\right) dx_{t-1}^*$$

We can thus obtain a closed-form expression of $f\left(x_{t-1}^*, w_t \mid d_{t-1}, w_{t-1}, d_{t-2}\right)$ by the deconvolution. To see this, observe

$$\begin{aligned} & \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 w_t\right)|d_{t-1}, w_{t-1}, d_{t-2} = d\right] \\ = \;& \mathrm{E}\left[\exp\left(is_1\delta^d x_{t-1}^* + is_1\varepsilon_{t-1}^d + is_2 w_t\right)|d_{t-1}, w_{t-1}, d_{t-2} = d\right] \\ = \;& \mathrm{E}\left[\exp\left(is_1\delta^d x_{t-1}^* + is_2 w_t\right)|d_{t-1}, w_{t-1}, d_{t-2} = d\right] \mathrm{E}\left[\exp\left(is_1\varepsilon_{t-1}^d\right)\right] \end{aligned}$$

by the independence assumption for ε_t^d, and so

$$\begin{aligned} \mathrm{E}\left[\exp\left(is_1\delta^d x_{t-1}^* + is_2 w_t\right)|d_{t-1}, w_{t-1}, d_{t-2} = d\right] &= \frac{\mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 w_t\right)|d_{t-1}, w_{t-1}, d_{t-2} = d\right]}{\mathrm{E}\left[\exp\left(is_1\varepsilon_{t-1}^d\right)\right]} \\ &= \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 w_t\right)|d_{t-1}, w_{t-1}, d_{t-2} = d\right] \\ &\quad \times \frac{\exp\left[\int_0^{s_1} \frac{\mathrm{E}\left[i(\frac{\delta^d}{\delta^{d'}} x_t - \alpha^{d'}\delta^d - \beta^{d'}\delta^d w_{t-1})\exp\left(is_1' x_{t-1}\right)|d_{t-1} = d', d_{t-2} = d\right]}{\gamma^{d'}\,\mathrm{E}\left[\exp\left(is_1' x_{t-1}\right)|d_{t-1} = d', d_{t-2} = d\right]} ds_1'\right]}{\mathrm{E}\left[\exp\left(is_1 x_{t-1}\right)|d_{t-1} = d', d_{t-2} = d\right]} \end{aligned}$$

follows with any choice of d'. Rescaling s_1 yields

$$\begin{aligned}
& \mathrm{E}\left[\exp\left(is_1 x^*_{t-1} + is_2 w_t\right)|d_{t-1}, w_{t-1}, d_{t-2}=d\right] \\
= \; & \mathrm{E}\left[\exp\left(is_1 \frac{1}{\delta^d} x_{t-1} + is_2 w_t\right)|d_{t-1}, w_{t-1}, d_{t-2}=d\right] \times \\
& \frac{\exp\left[\int_0^{s_1/\delta^d} \frac{\mathrm{E}\left[i(\frac{\delta^d}{\delta^{d'}} x_t - \alpha^{d'}\delta^d - \beta^{d'}\delta^d w_{t-1})\exp\left(is_1' x_{t-1}\right)|d_{t-1}=d', d_{t-2}=d\right]}{\gamma^{d'}\,\mathrm{E}\left[\exp\left(is_1' x_{t-1}\right)|d_{t-1}=d', d_{t-2}=d\right]} ds_1'\right]}{\mathrm{E}\left[\exp\left(is_1 \frac{1}{\delta^d} x_{t-1}\right)|d_{t-1}=d', d_{t-2}=d\right]}.
\end{aligned}$$

We can then express the conditional density as

$$f\left(x^*_{t-1}, w_t | d_{t-1}, w_{t-1}, d_{t-2}=d\right) = \left(\mathcal{F}_2 \phi_{x^*_{t-1}, w_t | d_{t-1}, w_{t-1}, d_{t-2}=d}\right)\left(w_t, x^*_{t-1}\right)$$

where the characteristic function is defined by

$$\begin{aligned}
\phi_{w_t, x^*_{t-1}|d_{t-1}, w_{t-1}, d_{t-2}=d}(s_1, s_2) \; = \; & \mathrm{E}\left[\exp\left(is_1 \frac{1}{\delta^d} x_{t-1} + is_2 w_t\right)|d_{t-1}, w_{t-1}, d_{t-2}=d\right] \times \\
& \frac{\exp\left[\int_0^{s_1/\delta^d} \frac{\mathrm{E}\left[i(\frac{\delta^d}{\delta^{d'}} x_t - \alpha^{d'}\delta^d - \beta^{d'}\delta^d w_{t-1})\exp\left(is_1' x_{t-1}\right)|d_{t-1}=d', d_{t-2}=d\right]}{\gamma^{d'}\,\mathrm{E}\left[\exp\left(is_1' x_{t-1}\right)|d_{t-1}=d', d_{t-2}=d\right]} ds_1'\right]}{\mathrm{E}\left[\exp\left(is_1 \frac{1}{\delta^d} x_{t-1}\right)|d_{t-1}=d', d_{t-2}=d\right]}.
\end{aligned}$$

Using this conditional density, we nonparametrically identify the transition rule

$$f_2\left(w_t | d_{t-1}, w_{t-1}, x^*_{t-1}\right) = \frac{\sum_d f\left(x^*_{t-1}, w_t | d_{t-1}, w_{t-1}, d_{t-2}=d\right)\Pr(d_{t-2}=d \mid d_{t-1}, w_{t-1})}{\int \sum_d f\left(x^*_{t-1}, w_t | d_{t-1}, w_{t-1}, d_{t-2}=d\right)\Pr(d_{t-2}=d \mid d_{t-1}, w_{t-1}) dw_t}.$$

In summary, we obtain the closed-form expression

$$
\begin{aligned}
f_2\left(w_t|d_{t-1},w_{t-1},x^*_{t-1}\right) &= \sum_d \mathbb{1}\{d_{t-1}=d\} \times \\
&\frac{\sum_{d'}\left(\mathcal{F}_2\phi_{x^*_{t-1},w_t|d_{t-1}=d,w_{t-1},d_{t-2}=d'}\right)(w_t,x^*_{t-1})\cdot \Pr(d_{t-2}=d' \mid d_{t-1}=d,w_{t-1})}{\int\sum_{d'}\left(\mathcal{F}_2\phi_{x^*_{t-1},w_t|d_{t-1}=d,w_{t-1},d_{t-2}=d'}\right)(w_t,x^*_{t-1})\cdot \Pr(d_{t-2}=d' \mid d_{t-1}=d,w_{t-1})dw_t} \\
= & \sum_d \mathbb{1}\{d_{t-1}=d\}\left\{\sum_{d'}\Pr(d_{t-2}=d' \mid d_{t-1}=d,w_{t-1})\int\int \exp\left(-is_1w_t-is_2x^*_{t-1}\right)\times\right. \\
&\frac{\mathrm{E}\left[\exp\left(is_1\frac{1}{\delta^{d'}}x_{t-1}+is_2w_t\right)|d_{t-1}=d,w_{t-1},d_{t-2}=d'\right]}{\mathrm{E}\left[\exp\left(is_1\frac{1}{\delta^{d'}}x_{t-1}\right)|d_{t-1}=d'',d_{t-2}=d'\right]}\times \\
&\left.\exp\left[\int_0^{s_1/\delta^{d'}}\frac{\mathrm{E}\left[i(\frac{\delta^{d'}}{\delta^{d''}}x_t-\alpha^{d''}\delta^{d'}-\beta^{d''}\delta^{d'}w_{t-1})\exp\left(is_1'x_{t-1}\right)|d_{t-1}=d'',d_{t-2}=d'\right]}{\gamma^{d''}\,\mathrm{E}\left[\exp\left(is_1'x_{t-1}\right)|d_{t-1}=d'',d_{t-2}=d'\right]}ds_1'\right]ds_1ds_2\right\}\Big/ \\
&\left\{\sum_{d'}\int\Pr(d_{t-2}=d' \mid d_{t-1}=d,w_{t-1})\int\int \exp\left(-is_1w_t-is_2x^*_{t-1}\right)\times\right. \\
&\frac{\mathrm{E}\left[\exp\left(is_1\frac{1}{\delta^{d'}}x_{t-1}+is_2w_t\right)|d_{t-1}=d,w_{t-1},d_{t-2}=d'\right]}{\mathrm{E}\left[\exp\left(is_1\frac{1}{\delta^{d'}}x_{t-1}\right)|d_{t-1}=d'',d_{t-2}=d'\right]}\times \\
&\left.\exp\left[\int_0^{s_1/\delta^{d'}}\frac{\mathrm{E}\left[i(\frac{\delta^{d'}}{\delta^{d''}}x_t-\alpha^{d''}\delta^{d'}-\beta^{d''}\delta^{d'}w_{t-1})\exp\left(is_1'x_{t-1}\right)|d_{t-1}=d'',d_{t-2}=d'\right]}{\gamma^{d''}\,\mathrm{E}\left[\exp\left(is_1'x_{t-1}\right)|d_{t-1}=d'',d_{t-2}=d'\right]}ds_1'\right]ds_1ds_2dw_t\right\}
\end{aligned}
$$

using any d' and d'' This completes Step 3.

Step 4: Closed-form identification of the CCP $f_1\left(d_t|w_t,x^*_t\right)$: Note that we have

$$
\begin{aligned}
\mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t\right)|w_t,d_{t-1}=d'\right] = &\ \mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(is\delta^{d'}x^*_t+is\varepsilon^{d'}_t\right)|w_t,d_{t-1}=d'\right] \\
= &\ \mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(is\delta^{d'}x^*_t\right)|w_t,d_{t-1}=d'\right]\mathrm{E}\left[\exp\left(is\varepsilon^{d'}_t\right)\right] \\
= &\ \mathrm{E}\left[\mathrm{E}\left[\mathbb{1}\{d_t=d\}|w_t,x^*_t,d_{t-1}=d'\right]\exp\left(is\delta^{d'}x^*_t\right)|w_t,d_{t-1}=d'\right]\mathrm{E}\left[\exp\left(is\varepsilon^{d'}_t\right)\right]
\end{aligned}
$$

by the independence assumption for $\varepsilon^{d'}_t$ and the law of iterated expectations. Therefore,

$$
\begin{aligned}
&\frac{\mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(isx_t\right)|w_t,d_{t-1}=d'\right]}{\mathrm{E}\left[\exp\left(is\varepsilon^{d'}_t\right)\right]} \\
= &\ \mathrm{E}\left[\mathrm{E}\left[\mathbb{1}\{d_t=d\}|w_t,x^*_t,d_{t-1}=d'\right]\exp\left(is\delta^{d'}x^*_t\right)|w_t,d_{t-1}=d'\right] \\
= &\int\exp\left(is\delta^{d'}x^*_t\right)\mathrm{E}\left[\mathbb{1}\{d_t=d\}|w_t,x^*_t,d_{t-1}=d'\right]f\left(x^*_t|w_t,d_{t-1}=d'\right)dx^*_t
\end{aligned}
$$

and rescaling s yields

$$\begin{aligned}
&\frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|w_t, d_{t-1} = d'\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}\varepsilon_t^{d'}\right)\right]} \\
= & \int \exp\left(isx_t^*\right)\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*, d_{t-1} = d'\right] f\left(x_t^*|w_t, d_{t-1} = d'\right) dx_t^*
\end{aligned}$$

This is the Fourier inversion of $\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*, d_{t-1} = d'\right] f\left(x_t^*|w_t, d_{t-1} = d'\right)$. On the other hand, the Fourier inversion of $f\left(x_t^*|w_t, d_{t-1}\right)$ can be found as

$$\mathrm{E}\left[\exp\left(isx_t^*\right)|w_t, d_{t-1} = d'\right] = \frac{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|w_t, d_{t-1} = d'\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}\varepsilon_t^{d'}\right)\right]}.$$

Therefore, we find the closed-form expression for CCP $f_1\left(d_t|w_t, x_t^*\right)$ as follows.

$$\begin{aligned}
\Pr\left(d_t = d|w_t, x_t^*\right) &= \sum_{d'} \Pr\left(d_t = d|w_t, x_t^*, d_{t-1} = d'\right)\Pr\left(d_{t-1} = d' \mid w_t, x_t^*\right) \\
&= \sum_{d'} \mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*, d_{t-1} = d'\right]\Pr\left(d_{t-1} = d' \mid w_t, x_t^*\right) \\
&= \sum_{d'} \frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*, d_{t-1} = d'\right] f\left(x_t^*|w_t, d_{t-1} = d'\right)}{f\left(x_t^*|w_t, d_{t-1} = d'\right)}\Pr\left(d_{t-1} = d' \mid w_t, x_t^*\right) \\
&= \sum_{d'} \frac{\left(\mathcal{F}\phi_{(d)x_t^*|w_t(d')}\right)\left(x_t^*\right)}{\left(\mathcal{F}\phi_{x_t^*|w_t(d')}\right)\left(x_t^*\right)}\Pr\left(d_{t-1} = d' \mid w_t, x_t^*\right)
\end{aligned}$$

where the characteristic functions are defined by

$$\begin{aligned}
\phi_{(d)x_t^*|w_t(d')}(s) &= \frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|w_t, d_{t-1} = d'\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}\varepsilon_t^{d'}\right)\right]} \\
&= \mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|w_t, d_{t-1} = d'\right] \\
&\times \frac{\exp\left[\int_0^{s/\delta^{d'}} \frac{\mathrm{E}\left[i\left(\frac{\delta^{d'}}{\delta^{d''}}x_{t+1} - \alpha^{d''}\delta^{d'} - \beta^{d''}\delta^{d'}w_t\right)\exp(is_1x_t)|d_t = d'', d_{t-1} = d'\right]}{\gamma^{d''}\,\mathrm{E}[\exp(is_1x_t)|d_t = d'', d_{t-1} = d']} ds_1\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|d_t = d', d_{t-1} = d''\right]}
\end{aligned}$$

and

$$\phi_{x_t^*|w_t(d')}(s) = \frac{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|w_t\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}\varepsilon_t^{d'}\right)\right]}$$

$$= \frac{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|w_t\right]\cdot\exp\left[\int_0^{s/\delta^{d'}}\frac{\mathrm{E}\left[i(\frac{\delta^{d'}}{\delta^{d''}}x_{t+1}-\alpha^{d''}\delta^{d'}-\beta^{d''}\delta^{d'}w_t)\exp(is_1x_t)|d_t=d'',d_{t-1}=d'\right]}{\gamma^{d''}\,\mathrm{E}[\exp(is_1x_t)|d_t=d'',d_{t-1}=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|d_t=d',d_{t-1}=d''\right]}$$

by (B.2) using any d''. In summary, we obtain the closed-form expression

$$\begin{aligned}
\Pr\left(d_t=d|w_t,x_t^*\right) &= \sum_{d'}\frac{\left(\mathcal{F}\phi_{(d)x_t^*|w_t(d')}\right)(x_t^*)}{\left(\mathcal{F}\phi_{x_t^*|w_t(d')}\right)(x_t^*)}\Pr\left(d_{t-1}=d' \mid w_t,x_t^*\right)\\
&= \sum_{d'}\Pr\left(d_{t-1}=d' \mid w_t,x_t^*\right)\int\exp\left(-isx_t^*\right)\times\\
&\quad \mathrm{E}\left[\mathbb{1}\{d_t=d\}\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|w_t,d_{t-1}=d'\right]\times\\
&\quad \frac{\exp\left[\int_0^{s/\delta^{d'}}\frac{\mathrm{E}\left[i(\frac{\delta^{d'}}{\delta^{d''}}x_{t+1}-\alpha^{d''}\delta^{d'}-\beta^{d''}\delta^{d'}w_t)\exp(is_1x_t)|d_t=d'',d_{t-1}=d'\right]}{\gamma^{d''}\,\mathrm{E}[\exp(is_1x_t)|d_t=d'',d_{t-1}=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|d_t=d',d_{t-1}=d''\right]}ds \Bigg/\\
&\quad \int\exp\left(-isx_t^*\right)\cdot\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|w_t\right]\times\\
&\quad \frac{\exp\left[\int_0^{s/\delta^{d'}}\frac{\mathrm{E}\left[i(\frac{\delta^{d'}}{\delta^{d''}}x_{t+1}-\alpha^{d''}\delta^{d'}-\beta^{d''}\delta^{d'}w_t)\exp(is_1x_t)|d_t=d'',d_{t-1}=d'\right]}{\gamma^{d''}\,\mathrm{E}[\exp(is_1x_t)|d_t=d'',d_{t-1}=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(is\frac{1}{\delta^{d'}}x_t\right)|d_t=d',d_{t-1}=d''\right]}ds.
\end{aligned}$$

This completes Step 4. □

B.3 Proof of Theorem 4

Our closed-form identification includes four steps.

Step 1: Closed-form identification of the transition rule $f_2\left(w_t,x_t^*|d_{t-1},w_{t-1},x_{t-1}^*\right)$: First, we show the identification of the parameters and the distributions in transition of x_t^*.

Since

$$\begin{aligned} x_t &= x_t^* + \varepsilon_t = \sum_d \mathbb{1}\{d_{t-1} = d\}[\alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^* + \eta_t^d] + \varepsilon_t \\ &= \sum_d \mathbb{1}\{d_{t-1} = d\}[\alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1} + \eta_t^d - \gamma^d \varepsilon_{t-1}] + \varepsilon_t \end{aligned}$$

we obtain the following equalities for each d:

$$\begin{aligned} \mathrm{E}[x_t \mid d_{t-1} = d] &= \alpha^d + \beta^d\, \mathrm{E}[w_{t-1} \mid d_{t-1} = d] + \gamma^d\, \mathrm{E}[x_{t-1} \mid d_{t-1} = d] \\ &\quad - \mathrm{E}[\gamma^d \varepsilon_{t-1} \mid d_{t-1} = d] + \mathrm{E}[\eta_t^d \mid d_{t-1} = d] + \mathrm{E}[\varepsilon_t \mid d_{t-1} = d] \\ &= \alpha^d + \beta^d\, \mathrm{E}[w_{t-1} \mid d_{t-1} = d] + \gamma^d\, \mathrm{E}[x_{t-1} \mid d_{t-1} = d] \\ \mathrm{E}[x_t d_{t-2} \mid d_{t-1} = d] &= \alpha^d\, \mathrm{E}[d_{t-2} \mid d_{t-1} = d] + \beta^d\, \mathrm{E}[w_{t-1} d_{t-2} \mid d_{t-1} = d] + \gamma^d\, \mathrm{E}[x_{t-1} d_{t-2} \mid d_{t-1} = d] \\ &\quad - \mathrm{E}[\gamma^d \varepsilon_{t-1} d_{t-2} \mid d_{t-1} = d] + \mathrm{E}[\eta_t^d d_{t-2} \mid d_{t-1} = d] + \mathrm{E}[\varepsilon_t d_{t-2} \mid d_{t-1} = d] \\ &= \alpha^d\, \mathrm{E}[d_{t-2} \mid d_{t-1} = d] + \beta^d\, \mathrm{E}[w_{t-1} d_{t-2} \mid d_{t-1} = d] + \gamma^d\, \mathrm{E}[x_{t-1} d_{t-2} \mid d_{t-1} = d] \\ \mathrm{E}[x_t d_{t-3} \mid d_{t-1} = d] &= \alpha^d\, \mathrm{E}[d_{t-3} \mid d_{t-1} = d] + \beta^d\, \mathrm{E}[w_{t-1} d_{t-3} \mid d_{t-1} = d] + \gamma^d\, \mathrm{E}[x_{t-1} d_{t-3} \mid d_{t-1} = d] \\ &\quad - \mathrm{E}[\gamma^d \varepsilon_{t-1} d_{t-3} \mid d_{t-1} = d] + \mathrm{E}[\eta_t^d d_{t-3} \mid d_{t-1} = d] + \mathrm{E}[\varepsilon_t d_{t-3} \mid d_{t-1} = d] \\ &= \alpha^d\, \mathrm{E}[d_{t-3} \mid d_{t-1} = d] + \beta^d\, \mathrm{E}[w_{t-1} d_{t-3} \mid d_{t-1} = d] + \gamma^d\, \mathrm{E}[x_{t-1} d_{t-3} \mid d_{t-1} = d] \end{aligned}$$

by the independence and zero mean assumptions for η_t^d and ε_t. From these, we have the linear equation

$$\begin{bmatrix} \mathrm{E}[x_t \mid d_{t-1} = d] \\ \mathrm{E}[x_t d_{t-2} \mid d_{t-1} = d] \\ \mathrm{E}[x_t d_{t-3} \mid d_{t-1} = d] \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{E}[w_{t-1} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} \mid d_{t-1} = d] \\ \mathrm{E}[d_{t-2} \mid d_{t-1} = d] & \mathrm{E}[w_{t-1} d_{t-2} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} d_{t-2} \mid d_{t-1} = d] \\ \mathrm{E}[d_{t-3} \mid d_{t-1} = d] & \mathrm{E}[w_{t-1} d_{t-3} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} d_{t-3} \mid d_{t-1} = d] \end{bmatrix} \begin{bmatrix} \alpha^d \\ \beta^d \\ \gamma^d \end{bmatrix}$$

Provided that the matrix on the right-hand side is non-singular, we can identify the parameters $(\alpha^d, \beta^d, \gamma^d)$ by

$$\begin{bmatrix} \alpha^d \\ \beta^d \\ \gamma^d \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{E}[w_{t-1} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} \mid d_{t-1} = d] \\ \mathrm{E}[d_{t-2} \mid d_{t-1} = d] & \mathrm{E}[w_{t-1} d_{t-2} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} d_{t-2} \mid d_{t-1} = d] \\ \mathrm{E}[d_{t-3} \mid d_{t-1} = d] & \mathrm{E}[w_{t-1} d_{t-3} \mid d_{t-1} = d] & \mathrm{E}[x_{t-1} d_{t-3} \mid d_{t-1} = d] \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{E}[x_t \mid d_{t-1} = d] \\ \mathrm{E}[x_t d_{t-2} \mid d_{t-1} = d] \\ \mathrm{E}[x_t d_{t-3} \mid d_{t-1} = d] \end{bmatrix}$$

Next, we show identification of the distributions of ε_t and η_t^d for each d. Observe that

$$\begin{aligned}
& \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 x_t\right)|d_{t-1} = d\right] \\
= \; & \mathrm{E}\left[\exp\left(is_1\left(x_{t-1}^* + \varepsilon_{t-1}\right) + is_2\left(\alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^* + \eta_t^d + \varepsilon_t\right)\right)|d_{t-1} = d\right] \\
= \; & \mathrm{E}\left[\exp\left(i\left(s_1 x_{t-1}^* + s_2\alpha^d + s_2\beta^d w_{t-1} + s_2\gamma^d x_{t-1}^*\right)\right)|d_{t-1} = d\right] \\
& \times \mathrm{E}\left[\exp\left(is_1\varepsilon_{t-1}\right)\right]\mathrm{E}\left[\exp\left(is_2\left(\eta_t^d + \varepsilon_t\right)\right)\right]
\end{aligned}$$

follows from the independence assumptions for η_t^d and ε_t. Taking the derivative with respect to s_2 yields

$$\begin{aligned}
& \left[\frac{\partial}{\partial s_2}\ln \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 x_t\right)|d_{t-1} = d\right]\right]_{s_2=0} \\
= \; & \frac{\mathrm{E}\left[i(\alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^*)\exp\left(is_1 x_{t-1}^*\right)|d_{t-1} = d\right]}{\mathrm{E}\left[\exp\left(is_1 x_{t-1}^*\right)|d_{t-1} = d\right]} \\
= \; & i\alpha^d + \beta^d\frac{\mathrm{E}[iw_{t-1}\exp(is_1 x_{t-1}^*) \mid d_{t-1} = d]}{\mathrm{E}[\exp(is_1 x_{t-1}^*) \mid d_{t-1} = d]} + \gamma^d\frac{\partial}{\partial s_1}\ln \mathrm{E}\left[\exp\left(is_1 x_{t-1}^*\right)|d_{t-1} = d\right] \\
= \; & i\alpha^d + \beta^d\frac{\mathrm{E}[iw_{t-1}\exp(is_1 x_{t-1}) \mid d_{t-1} = d]}{\mathrm{E}[\exp(is_1 x_{t-1}) \mid d_{t-1} = d]} + \gamma^d\frac{\partial}{\partial s_1}\ln \mathrm{E}\left[\exp\left(is_1 x_{t-1}^*\right)|d_{t-1} = d\right]
\end{aligned}$$

where the switch of the differential and integral operators is permissible provided that there exists $h \in L^1(F_{w_{t-1}x_{t-1}^*|d_{t-1}=d})$ such that $\left|i(\alpha^d + \beta^d w_{t-1} + \gamma^d x_{t-1}^*)\exp\left(is_1 x_{t-1}^*\right)\right| < h(w_{t-1}, x_{t-1}^*)$ holds for all (w_{t-1}, x_{t-1}^*), which follows from the bounded conditional moment given in Assumption 10, and the denominators are nonzero as the conditional characteristic function of x_t^* given d_t does not vanish on the real line under Assumption 10. Therefore,

$$\begin{aligned}
\mathrm{E}\left[\exp\left(isx_{t-1}^*\right)|d_{t-1} = d\right] \; = \; & \exp\left[\int_0^s \left[\frac{1}{\gamma^d}\frac{\partial}{\partial s_2}\ln \mathrm{E}\left[\exp\left(is_1 x_{t-1} + is_2 x_t\right)|d_{t-1} = d\right]\right]_{s_2=0} ds_1\right. \\
& \left. - \int_0^s \frac{i\alpha^d}{\gamma^d}ds_1 - \int_0^s \frac{\beta^d}{\gamma^d}\frac{\mathrm{E}[iw_{t-1}\exp(is_1 x_{t-1}) \mid d_{t-1} = d]}{\mathrm{E}[\exp(is_1 x_{t-1}) \mid d_{t-1} = d]}ds_1\right] \\
= \; & \exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_t - \alpha^d - \beta^d w_{t-1})\exp\left(is_1 x_{t-1}\right)|d_{t-1} = d\right]}{\gamma^d\,\mathrm{E}\left[\exp\left(is_1 x_{t-1}\right)|d_{t-1} = d\right]}ds_1\right].
\end{aligned}$$

From the proxy model and the independence assumption for ε_t,

$$\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1} = d\right] = \mathrm{E}\left[\exp\left(isx_{t-1}^*\right)|d_{t-1} = d\right]\mathrm{E}\left[\exp\left(is\varepsilon_{t-1}\right)\right].$$

We then obtain the following result using any d.

$$\begin{aligned}\mathrm{E}\left[\exp\left(is\varepsilon_{t-1}\right)\right] &= \frac{\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d\right]}{\mathrm{E}\left[\exp\left(isx_{t-1}^{*}\right)|d_{t-1}=d\right]} \\ &= \frac{\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_t-\alpha^d-\beta^d w_{t-1})\exp(is_1 x_{t-1})|d_{t-1}=d\right]}{\gamma^d\,\mathrm{E}[\exp(is_1 x_{t-1})|d_{t-1}=d]}ds_1\right]}.\end{aligned}$$

This argument holds for all t so that we can identify the characteristic function of ε_t with

$$\mathrm{E}\left[\exp\left(is\varepsilon_{t}\right)\right] = \frac{\mathrm{E}\left[\exp\left(isx_{t}\right)|d_{t}=d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^d-\beta^d w_{t})\exp(is_1 x_{t})|d_{t}=d\right]}{\gamma^d\,\mathrm{E}[\exp(is_1 x_{t})|d_{t}=d]}ds_1\right]} \tag{B.3}$$

using any d.

In order to identify f_{w_t,η_t^d} for each d, consider

$$x_t+\gamma^d\varepsilon_{t-1}=\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1}+\varepsilon_t+\eta^d,$$

and thus

$$\begin{aligned}\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d,w_t\right]\mathrm{E}\left[\exp\left(is\gamma^d\varepsilon_{t-1}\right)\right] &= \mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1})\right)|d_{t-1}=d,w_t\right] \\ &\quad\times\mathrm{E}\left[\exp\left(is\eta_t^d\right)\mid w_t\right]\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]\end{aligned}$$

follows by the independence assumptions for η_t^d and ε_t. Therefore, by the formula (B.3), the conditional characteristic function of η_t^d given w_t can be expressed by

$$\begin{aligned}\mathrm{E}\left[\exp\left(is\eta_t^d\right)\mid w_t\right] &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d,w_t\right]\cdot\mathrm{E}\left[\exp\left(is\gamma^d\varepsilon_{t-1}\right)\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1})\right)|d_{t-1}=d,w_t\right]\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]} \\ &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d,w_t\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^d-\beta^d w_t)\exp(is_1x_t)|d_t=d\right]}{\gamma^d\,\mathrm{E}[\exp(is_1x_t)|d_t=d]}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1})\right)|d_{t-1}=d,w_t\right]\cdot\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]}\times \\ &\quad \frac{\mathrm{E}\left[\exp\left(is\gamma^d x_{t-1}\right)|d_{t-1}=d\right]}{\exp\left[\int_0^{s\gamma^d} \frac{\mathrm{E}\left[i(x_t-\alpha^d-\beta^d w_{t-1})\exp(is_1x_{t-1})|d_{t-1}=d\right]}{\gamma^d\,\mathrm{E}[\exp(is_1x_{t-1})|d_{t-1}=d]}ds_1\right]}.\end{aligned}$$

The denominator on the right-hand side is non-zero, as the conditional and unconditional characteristic functions do not vanish on the real line under Assumption 10. Letting $\mathcal{F}$ denote the operator defined by

$$(\mathcal{F}\phi)(\xi) = \frac{1}{2\pi}\int e^{-is\xi}\phi(s)ds \qquad \text{for all } \phi \in L^1(\mathbb{R}) \text{ and } \xi \in \mathbb{R},$$

we identify f_{w_t,η_t^d} by

$$f_{w_t,\eta_t^d}(w,\eta) = f_{w_t}(w)\left(\mathcal{F}\phi_{\eta_t^d|w_t}(\cdot \mid w)\right)(\eta) \qquad \text{for all } \eta,$$

where the conditional characteristic function $\phi_{\eta_t^d|w_t}$ is given by

$$\phi_{\eta_t^d|w_t}(s \mid w) = \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d, w_t=w\right]\cdot \exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^d-\beta^d w_t)\exp(is_1x_t)|d_t=d\right]}{\gamma^d\,\mathrm{E}[\exp(is_1x_t)|d_t=d]}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1})\right)|d_{t-1}=d, w_t=w\right]\cdot \mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]} \times \frac{\mathrm{E}\left[\exp\left(is\gamma^d x_{t-1}\right)|d_{t-1}=d\right]}{\exp\left[\int_0^{s\gamma^d} \frac{\mathrm{E}\left[i(x_t-\alpha^d-\beta^d w_{t-1})\exp(is_1x_{t-1})|d_{t-1}=d\right]}{\gamma^d\,\mathrm{E}[\exp(is_1x_{t-1})|d_{t-1}=d]}ds_1\right]}.$$

We can use this identified joint density in turn to identify the transition rule $f_2\left(w_t, x_t^*|d_{t-1}, w_{t-1}, x_{t-1}^*\right)$ with

$$f_2\left(w_t, x_t^*|d_{t-1}, w_{t-1}, x_{t-1}^*\right) = \sum_d \mathbb{1}\{d_{t-1}=d\} f_{w_t,\eta_t^d}\left(w_t,\ x_t^* - \alpha^d - \beta^d w_{t-1} - \gamma^d x_{t-1}^*\right).$$

In summary, we obtain the closed-form expression

$$\begin{aligned} f_2(w_t, x_t^* \mid d_{t-1}, w_{t-1}, x_{t-1}^*) &= \sum_d \mathbb{1}\{d_{t-1}=d\} f_{w_t}(w_t)\left(\mathcal{F}\phi_{\eta_t^d|w_t}(\cdot \mid w_t)\right)(x_t^* - \alpha^d - \beta^d w_{t-1} - \gamma^d x_{t-1}^*) \\ &= \sum_d \frac{\mathbb{1}\{d_{t-1}=d\}}{2\pi} f_{w_t}(w_t)\int \exp\left(-is(x_t^* - \alpha^d - \beta^d w_{t-1} - \gamma^d x_{t-1}^*)\right) \times \\ &\quad \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d, w_t\right]\cdot \exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'}-\beta^{d'} w_t)\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^d+\beta^d w_{t-1}+\gamma^d x_{t-1})\right)|d_{t-1}=d, w_t\right]\cdot \mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]} \times \\ &\quad \frac{\mathrm{E}\left[\exp\left(is\gamma^d x_{t-1}\right)|d_{t-1}=d'\right]}{\exp\left[\int_0^{s\gamma^d} \frac{\mathrm{E}\left[i(x_t-\alpha^{d'}-\beta^{d'} w_{t-1})\exp(is_1x_{t-1})|d_{t-1}=d'\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_{t-1})|d_{t-1}=d']}ds_1\right]}\,ds. \end{aligned}$$

using any d'. This completes Step 1.

Step 2: Closed-form identification of the proxy model $f_3(x_t \mid x_t^*)$: Given (B.3), we can write the density of ε_t by

$$f_{\varepsilon_t}(\varepsilon) = (\mathcal{F}\phi_{\varepsilon_t})(\varepsilon) \qquad \text{for all } \varepsilon,$$

where the characteristic function ϕ_{ε_t} is defined by (B.3) as

$$\phi_{\varepsilon_t}(s) = \frac{\mathrm{E}\left[\exp(isx_t) \,|d_t = d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1} - \alpha^d - \beta^d w_t)\exp(is' x_t)|d_t = d\right]}{\gamma^d \,\mathrm{E}[\exp(is' x_t)|d_t = d]} ds'\right]}.$$

Provided this identified density of ε_t, we nonparametrically identify the proxy model

$$f_3(x_t \mid x_t^*) = f_{\varepsilon_t}(x_t - x_t^*)$$

In summary, we obtain the closed-form expression

$$\begin{aligned} f_3(x_t \mid x_t^*) &= (\mathcal{F}\phi_{\varepsilon_t})(x_t - x_t^*) \\ &= \frac{1}{2\pi}\int \frac{\exp(-is(x_t - x_t^*)) \cdot \mathrm{E}\left[\exp(isx_t)\,|d_t = d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1} - \alpha^d - \beta^d w_t)\exp(is_1 x_t)|d_t = d\right]}{\gamma^d \,\mathrm{E}[\exp(is_1 x_t)|d_t = d]} ds_1\right]} ds \end{aligned}$$

using any d. This completes Step 2.

Step 3: Closed-form identification of the CCP $f_1(d_t|w_t, x_t^*)$: Note that we have

$$\begin{aligned} \mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp(isx_t)\,|w_t\right] &= \mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp(isx_t^* + is\varepsilon_t)\,|w_t\right] \\ &= \mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp(isx_t^*)\,|w_t\right]\mathrm{E}\left[\exp(is\varepsilon_t)\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right]\exp(isx_t^*)\,|w_t\right]\mathrm{E}\left[\exp(is\varepsilon_t)\right] \end{aligned}$$

by the independence assumption for ε_t and the law of iterated expectations. Therefore

$$\begin{aligned} \frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp(isx_t)\,|w_t\right]}{\mathrm{E}\left[\exp(is\varepsilon_t)\right]} &= \mathrm{E}\left[\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right]\exp(isx_t^*)\,|w_t\right] \\ &= \int \exp(isx_t^*)\,\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right] f(x_t^*|w_t)\,dx_t^* \end{aligned}$$

This is the Fourier inversion of $\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right] f\left(x_t^*|w_t\right)$. On the other hand, the Fourier inversion of $f\left(x_t^*|w_t\right)$ can be found as

$$\mathrm{E}\left[\exp\left(isx_t^*\right)|w_t\right] = \frac{\mathrm{E}\left[\exp\left(isx_t\right)|w_t\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]}.$$

Therefore, we find the closed-form expression for CCP $f_1\left(d_t|w_t, x_t^*\right)$ as follows.

$$\Pr\left(d_t = d|w_t, x_t^*\right) = \mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right] = \frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}|w_t, x_t^*\right] f\left(x_t^*|w_t\right)}{f\left(x_t^*|w_t\right)} = \frac{\left(\mathcal{F}\phi_{(d)x_t^*|w_t}\right)\left(x_t^*\right)}{\left(\mathcal{F}\phi_{x_t^*|w_t}\right)\left(x_t^*\right)}$$

where the characteristic functions are defined by

$$\begin{aligned}\phi_{(d)x_t^*|w_t}(s) &= \frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp\left(isx_t\right)|w_t\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]} \\ &= \frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp\left(isx_t\right)|w_t\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'}-\beta^{d'}w_t)\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right)|d_t = d'\right]}\end{aligned}$$

and

$$\begin{aligned}\phi_{x_t^*|w_t}(s) &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|w_t\right]}{\mathrm{E}\left[\exp\left(is\varepsilon_t\right)\right]} \\ &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|w_t\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'}-\beta^{d'}w_t)\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right)|d_t = d'\right]}\end{aligned}$$

by (B.3) using any d'. In summary, we obtain the closed-form expression

$$\begin{aligned}\Pr\left(d_t = d|w_t, x_t^*\right) &= \frac{\left(\mathcal{F}\phi_{(d)x_t^*|w_t}\right)\left(x_t^*\right)}{\left(\mathcal{F}\phi_{x_t^*|w_t}\right)\left(x_t^*\right)} \\ &= \int \exp\left(-isx_t^*\right)\cdot\mathrm{E}\left[\mathbb{1}\{d_t = d\}\exp\left(isx_t\right)|w_t\right]\times \\ &\qquad \frac{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'}-\beta^{d'}w_t)\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right)|d_t = d'\right]}ds \Bigg/ \\ &\qquad \int \exp\left(-isx_t^*\right)\cdot\mathrm{E}\left[\exp\left(isx_t\right)|w_t\right]\times \\ &\qquad \frac{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{d'}-\beta^{d'}w_t)\exp(is_1x_t)|d_t=d'\right]}{\gamma^{d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(isx_t\right)|d_t = d'\right]}ds\end{aligned}$$

using any d'. This completes Step 3. □

B.4 Proof of Theorem 5

Our closed-form identification involves three steps.

Step 1: Closed-form identification of $f_{2x}\left(x_t^*|d_{t-1},x_{t-1}^*\right)$ **and** $f_{2y}\left(y_t^*|d_{t-1},y_{t-1}^*\right)$**:** First, we show the identification of the parameters and the distributions in the transition law of x_t^*. Since

$$\begin{aligned} x_t &= x_t^* + \varepsilon_t^x = \sum_d \mathbb{1}\{d_{t-1}=d\}[\alpha^{x,d} + \gamma^{x,d} x_{t-1}^* + \eta_t^{x,d}] + \varepsilon_t^x \\ &= \sum_d \mathbb{1}\{d_{t-1}=d\}[\alpha^{x,d} + \gamma^{x,d} x_{t-1} + \eta_t^{x,d} - \gamma^{x,d}\varepsilon_{t-1}^x] + \varepsilon_t^x \end{aligned}$$

is true under Assumption 17, we obtain the following equalities for each d:

$$\begin{aligned} \mathrm{E}[x_t \mid d_{t-1}=d] &= \alpha^{x,d} + \gamma^{x,d}\,\mathrm{E}[x_{t-1} \mid d_{t-1}=d] - \\ &\quad \mathrm{E}[\gamma^{x,d}\varepsilon_{t-1}^x \mid d_{t-1}=d] + \mathrm{E}[\eta_t^{x,d} \mid d_{t-1}=d] + \mathrm{E}[\varepsilon_t^x \mid d_{t-1}=d] \\ &= \alpha^{x,d} + \gamma^{x,d}\,\mathrm{E}[x_{t-1} \mid d_{t-1}=d] \end{aligned}$$

$$\begin{aligned} \mathrm{E}[x_t d_{t-2} \mid d_{t-1}=d] &= \alpha^{x,d}\,\mathrm{E}[d_{t-2} \mid d_{t-1}=d] + \gamma^{x,d}\,\mathrm{E}[x_{t-1}d_{t-2} \mid d_{t-1}=d] - \\ &\quad \mathrm{E}[\gamma^{x,d}\varepsilon_{t-1}^x d_{t-2} \mid d_{t-1}=d] + \mathrm{E}[\eta_t^{x,d} d_{t-2} \mid d_{t-1}=d] + \mathrm{E}[\varepsilon_t^x d_{t-2} \mid d_{t-1}=d] \\ &= \alpha^{x,d}\,\mathrm{E}[d_{t-2} \mid d_{t-1}=d] + \gamma^{x,d}\,\mathrm{E}[x_{t-1}d_{t-2} \mid d_{t-1}=d] \end{aligned}$$

The independence and zero mean assumptions for $\eta_t^{x,d}$ and ε_t^x stated in Assumption 17 are used above. We thus obtain the linear equation

$$\begin{bmatrix} \mathrm{E}[x_t \mid d_{t-1}=d] \\ \mathrm{E}[x_t d_{t-2} \mid d_{t-1}=d] \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{E}[x_{t-1} \mid d_{t-1}=d] \\ \mathrm{E}[d_{t-2} \mid d_{t-1}=d] & \mathrm{E}[x_{t-1}d_{t-2} \mid d_{t-1}=d] \end{bmatrix} \begin{bmatrix} \alpha^{x,d} \\ \gamma^{x,d} \end{bmatrix}.$$

By the non-singularity of the matrix on the right-hand side stated in Assumption 18, we can identify the parameters $(\alpha^{x,d}, \gamma^{x,d})$ by

$$\begin{bmatrix} \alpha^{x,d} \\ \gamma^{x,d} \end{bmatrix} = \begin{bmatrix} 1 & \mathrm{E}[x_{t-1} \mid d_{t-1}=d] \\ \mathrm{E}[d_{t-2} \mid d_{t-1}=d] & \mathrm{E}[x_{t-1}d_{t-2} \mid d_{t-1}=d] \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{E}[x_t \mid d_{t-1}=d] \\ \mathrm{E}[x_t d_{t-2} \mid d_{t-1}=d] \end{bmatrix}$$

Next, we show the identification of the distributions of ε_t^x and $\eta_t^{x,d}$ for each d. Observe that

$$\begin{aligned}
& \mathrm{E}\left[\exp\left(is_1x_{t-1}+is_2x_t\right)|d_{t-1}=d\right] \\
= \ & \mathrm{E}\left[\exp\left(is_1\left(x_{t-1}^*+\varepsilon_{t-1}^x\right)+is_2\left(\alpha^{x,d}+\gamma^{x,d}x_{t-1}^*+\eta_t^{x,d}+\varepsilon_t^x\right)\right)|d_{t-1}=d\right] \\
= \ & \mathrm{E}\left[\exp\left(i\left(s_1x_{t-1}^*+s_2\alpha^{x,d}+s_2\gamma^{x,d}x_{t-1}^*\right)\right)|d_{t-1}=d\right]\mathrm{E}\left[\exp\left(is_1\varepsilon_{t-1}^x\right)\right]\mathrm{E}\left[\exp\left(is_2\left(\eta_t^{x,d}+\varepsilon_t^x\right)\right)\right]
\end{aligned}$$

follows from the independence assumptions for $\eta_t^{x,d}$ and ε_t^x stated in Assumption 17. Taking the derivative with respect to s_2 yields

$$\begin{aligned}
\left[\frac{\partial}{\partial s_2}\ln\mathrm{E}\left[\exp\left(is_1x_{t-1}+is_2x_t\right)|d_{t-1}=d\right]\right]_{s_2=0} &= \frac{\mathrm{E}\left[i(\alpha^{x,d}+\gamma^{x,d}x_{t-1}^*)\exp\left(is_1x_{t-1}^*\right)|d_{t-1}=d\right]}{\mathrm{E}\left[\exp\left(is_1x_{t-1}^*\right)|d_{t-1}=d\right]} \\
&= i\alpha^{x,d}+\gamma^{x,d}\frac{\partial}{\partial s_1}\ln\mathrm{E}\left[\exp\left(is_1x_{t-1}^*\right)|d_{t-1}=d\right]
\end{aligned}$$

where the switch of the differential and integral operators is permissible provided that there exists $h\in L^1(F_{x_{t-1}^*|d_{t-1}=d})$ such that $\left|i(\alpha^{x,d}+\gamma^{x,d}x_{t-1}^*)\exp\left(is_1x_{t-1}^*\right)\right|<h(x_{t-1}^*)$ holds for all x_{t-1}^*, which follows from the bounded conditional moment condition provided in Assumption 19, and the denominator is nonzero as the conditional characteristic function of x_t^* given d_t does not vanish on the real line under Assumption 19. Therefore, we have

$$\mathrm{E}\left[\exp\left(isx_{t-1}^*\right)|d_{t-1}=d\right] = \exp\left[\int_0^s\frac{\mathrm{E}\left[i(x_t-\alpha^{x,d})\exp\left(is_1x_{t-1}\right)|d_{t-1}=d\right]}{\gamma^{x,d}\,\mathrm{E}\left[\exp\left(is_1x_{t-1}\right)|d_{t-1}=d\right]}ds_1\right].$$

On the other hand, from the proxy model and the independence conditions for ε_t stated in Assumption 17, we also have

$$\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d\right]=\mathrm{E}\left[\exp\left(isx_{t-1}^*\right)|d_{t-1}=d\right]\mathrm{E}\left[\exp\left(is\varepsilon_{t-1}^x\right)\right].$$

Combining the above two equations, we obtain the following identifying formula using any d.

$$\mathrm{E}\left[\exp\left(is\varepsilon_{t-1}^x\right)\right] = \frac{\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d\right]}{\mathrm{E}\left[\exp\left(isx_{t-1}^*\right)|d_{t-1}=d\right]} = \frac{\mathrm{E}\left[\exp\left(isx_{t-1}\right)|d_{t-1}=d\right]}{\exp\left[\int_0^s\frac{\mathrm{E}\left[i(x_t-\alpha^{x,d})\exp(is_1x_{t-1})|d_{t-1}=d\right]}{\gamma^{x,d}\,\mathrm{E}\left[\exp(is_1x_{t-1})|d_{t-1}=d\right]}ds_1\right]}.$$

This argument holds for all t so that we can identify the characteristic function of ε_t^x by

$$\phi_{\varepsilon_t^x}(s) = \mathrm{E}\left[\exp\left(is\varepsilon_t^x\right)\right] = \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]}{\exp\left[\int_0^s\frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{x,d})\exp(is_1x_t)|d_t=d\right]}{\gamma^{x,d}\,\mathrm{E}\left[\exp(is_1x_t)|d_t=d\right]}ds_1\right]} \tag{B.4}$$

using any d.

In order to identify the distribution of $\eta_t^{x,d}$ for each d, consider

$$x_t + \gamma^{x,d}\varepsilon_{t-1} = \alpha^{x,d} + \gamma^{x,d}x_{t-1} + \varepsilon_t^x + \eta^{x,d}$$

which holds under Assumption 17. From this equality,

$$\begin{aligned} \mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\mathrm{E}\left[\exp\left(is\gamma^{x,d}\varepsilon_{t-1}\right)\right] &= \mathrm{E}\left[\exp\left(is(\alpha^{x,d}+\gamma^{x,d}x_{t-1})\right)|d_{t-1}=d\right] \\ &\quad \times \mathrm{E}\left[\exp\left(is\eta_t^{x,d}\right)\right]\mathrm{E}\left[\exp\left(is\varepsilon_t^x\right)\right] \end{aligned}$$

follows by the independence assumptions for $\eta_t^{x,d}$ and ε_t^x stated in Assumption 17. Therefore, by the identifying formula (B.4) for $\phi_{\varepsilon_t^x}$, the characteristic function of $\eta_t^{x,d}$ can be expressed by

$$\begin{aligned} \phi_{\eta_t^{x,d}}(s) = \mathrm{E}\left[\exp\left(is\eta_t^{x,d}\right)\right] &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\cdot\mathrm{E}\left[\exp\left(is\gamma^{x,d}\varepsilon_{t-1}^x\right)\right]}{\mathrm{E}\left[\exp\left(is(\alpha^{x,d}+\gamma^{x,d}x_{t-1})\right)|d_{t-1}=d\right]\mathrm{E}\left[\exp\left(is\varepsilon_t^x\right)\right]} \\ &= \frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right]\cdot\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{x,d})\exp(is_1x_t)|d_t=d\right]}{\gamma^{x,d}\,\mathrm{E}\left[\exp(is_1x_t)|d_t=d\right]}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^{x,d}+\gamma^{x,d}x_{t-1})\right)|d_{t-1}=d\right]\cdot\mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]} \\ &\quad \times \frac{\mathrm{E}\left[\exp\left(is\gamma^{x,d}x_{t-1}\right)|d_{t-1}=d\right]}{\exp\left[\int_0^{s\gamma^d}\frac{\mathrm{E}\left[i(x_t-\alpha^{x,d})\exp(is_1x_{t-1})|d_{t-1}=d\right]}{\gamma^{x,d}\,\mathrm{E}\left[\exp(is_1x_{t-1})|d_{t-1}=d\right]}ds_1\right]}. \end{aligned} \tag{B.5}$$

The denominator on the right-hand side is non-zero, as the conditional and unconditional characteristic functions do not vanish on the real line under Assumption 19. Letting $\mathcal{F}$ denote the Fourier transform operator defined by

$$\left(\mathcal{F}\phi\right)(\xi) = \frac{1}{2\pi}\int e^{-is\xi}\phi(s)ds \qquad \text{for all } \phi \in L^1(\mathbb{R}) \text{ and } \xi \in \mathbb{R},$$

we identify $f_{\eta_t^{x,d}}$ by

$$f_{\eta_t^{x,d}}(\eta) = \left(\mathcal{F}\phi_{\eta_t^{x,d}}\right)(\eta) \qquad \text{for all } \eta,$$

under Assumption 19, where the characteristic function $\phi_{\eta_t^{x,d}}$ is identified in (B.5). We can use this identified density function $f_{\eta_t^{x,d}}$ in turn to identify the transition rule $f_{2x}\left(x_t^*|d_{t-1},x_{t-1}^*\right)$ with

$$f_{2x}\left(x_t^*|d_{t-1},x_{t-1}^*\right) = \sum_d \mathbb{1}\{d_{t-1}=d\}f_{\eta_t^{x,d}}\left(x_t^*-\alpha^{x,d}-\gamma^{x,d}x_{t-1}^*\right).$$

In summary, we obtain the closed-form identifying formula for the law of state transition $f_{2x}(x_t^* \mid d_{t-1}, x_{t-1}^*)$:

$$f_{2x}(x_t^* \mid d_{t-1}, x_{t-1}^*) = \sum_d \frac{\mathbb{1}\{d_{t-1}=d\}}{2\pi} \int \exp\left(-is(x_t^* - \alpha^{x,d} - \gamma^{x,d} x_{t-1}^*)\right) \times$$
$$\frac{\mathrm{E}\left[\exp\left(isx_t\right)|d_{t-1}=d\right] \cdot \exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{x,d'})\exp(is_1x_t)|d_t=d'\right]}{\gamma^{x,d'}\,\mathrm{E}[\exp(is_1x_t)|d_t=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^{x,d}+\gamma^{x,d}x_{t-1})\right)|d_{t-1}=d\right] \cdot \mathrm{E}\left[\exp\left(isx_t\right)|d_t=d\right]} \times$$
$$\frac{\mathrm{E}\left[\exp\left(is\gamma^{x,d}x_{t-1}\right)|d_{t-1}=d'\right]}{\exp\left[\int_0^{s\gamma^{x,d}} \frac{\mathrm{E}\left[i(x_t-\alpha^{x,d'})\exp(is_1x_{t-1})|d_{t-1}=d'\right]}{\gamma^{x,d'}\,\mathrm{E}[\exp(is_1x_{t-1})|d_{t-1}=d']}ds_1\right]}\, ds$$

using any d'. By similar lines of argument, we also obtain the closed-form identifying formula for the law of state transition $f_{2y}(y_t^* \mid d_{t-1}, y_{t-1}^*)$:

$$f_{2y}(y_t^* \mid d_{t-1}, y_{t-1}^*) = \sum_d \frac{\mathbb{1}\{d_{t-1}=d\}}{2\pi} \int \exp\left(-is(y_t^* - \alpha^{y,d} - \gamma^{y,d} y_{t-1}^*)\right) \times$$
$$\frac{\mathrm{E}\left[\exp\left(isy_t\right)|d_{t-1}=d\right] \cdot \exp\left[\int_0^s \frac{\mathrm{E}\left[i(y_{t+1}-\alpha^{y,d'})\exp(is_1y_t)|d_t=d'\right]}{\gamma^{y,d'}\,\mathrm{E}[\exp(is_1y_t)|d_t=d']}ds_1\right]}{\mathrm{E}\left[\exp\left(is(\alpha^{y,d}+\gamma^{y,d}y_{t-1})\right)|d_{t-1}=d\right] \cdot \mathrm{E}\left[\exp\left(isy_t\right)|d_t=d\right]} \times$$
$$\frac{\mathrm{E}\left[\exp\left(is\gamma^{y,d}y_{t-1}\right)|d_{t-1}=d'\right]}{\exp\left[\int_0^{s\gamma^{y,d}} \frac{\mathrm{E}\left[i(y_t-\alpha^{y,d'})\exp(is_1y_{t-1})|d_{t-1}=d'\right]}{\gamma^{y,d'}\,\mathrm{E}[\exp(is_1y_{t-1})|d_{t-1}=d']}ds_1\right]}\, ds$$

using any d'. This completes Step 1.

Step 2: Closed-form identification of $f_{3x}\left(x_t \mid x_t^*\right)$ and $f_{3y}\left(y_t \mid y_t^*\right)$: We can write the density function of ε_t^x by

$$f_{\varepsilon_t^x}(\varepsilon) = \left(\mathcal{F}\phi_{\varepsilon_t^x}\right)(\varepsilon^x) \qquad \text{for all } \varepsilon^x,$$

where the characteristic function $\phi_{\varepsilon_t^x}$ is identified in (B.4) with a closed-form formula. Provided this identified density function $f_{\varepsilon_t^x}$, we identify the proxy model $f_{3x}(x_t \mid x_t^*)$ by

$$f_{3x}(x_t \mid x_t^*) = f_{\varepsilon_t^x}(x_t - x_t^*).$$

In summary, we obtain the closed-form identifying formula for the proxy model $f_{3x}(x_t \mid x_t^*)$:

$$f_{3x}(x_t \mid x_t^*) \quad = \quad \frac{1}{2\pi} \int \frac{\exp\left(-is(x_t - x_t^*)\right) \cdot \mathrm{E}\left[\exp\left(isx_t\right) | d_t = d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(x_{t+1} - \alpha^{d,x}) \exp(is_1 x_t) | d_t = d\right]}{\gamma^{d,x}\, \mathrm{E}[\exp(is_1 x_t)|d_t = d]} ds_1\right]} ds$$

using any d. By similar lines of argument, we also obtain the closed-form identifying formula for the proxy model $f_{3y}(y_t \mid y_t^*)$:

$$f_{3y}(y_t \mid y_t^*) \quad = \quad \frac{1}{2\pi} \int \frac{\exp\left(-is(y_t - y_t^*)\right) \cdot \mathrm{E}\left[\exp\left(isy_t\right) | d_t = d\right]}{\exp\left[\int_0^s \frac{\mathrm{E}\left[i(y_{t+1} - \alpha^{d,y}) \exp(is_1 y_t) | d_t = d\right]}{\gamma^{d,y}\, \mathrm{E}[\exp(is_1 y_t)|d_t = d]} ds_1\right]} ds$$

using any d. This completes Step 2.

Step 3: Closed-form identification of the CCP $f_1\left(d_t | x_t^*, y_t^*\right)$: We can write

$$\begin{aligned} \mathrm{E}\left[\mathbb{1}\{d_t = d\} \exp\left(is^x x_t + is^y y_t\right)\right] &= \mathrm{E}\left[\mathbb{1}\{d_t = d\} \exp\left(is^x x_t^* + is^y y_t^* + is^x \varepsilon_t^x + is^y \varepsilon_t^y\right)\right] \\ &= \mathrm{E}\left[\mathbb{1}\{d_t = d\} \exp\left(is^x x_t^* + is^y y_t^*\right)\right] \mathrm{E}\left[\exp\left(is^x \varepsilon_t^x\right)\right] \mathrm{E}\left[\exp\left(is^y \varepsilon_t^y\right)\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\mathbb{1}\{d_t = d\} | x_t^*, y_t^*\right] \exp\left(is^x x_t^* + is^y y_t^*\right)\right] \mathrm{E}\left[\exp\left(is^x \varepsilon_t^x\right)\right] \mathrm{E}\left[\exp\left(is^y \varepsilon_t^y\right)\right] \end{aligned}$$

by the independence assumption for ε_t^x and ε_t^y stated in Assumption 17 and the law of iterated expectations. Therefore, we obtain

$$\begin{aligned} &\frac{\mathrm{E}\left[\mathbb{1}\{d_t = d\} \exp\left(is^x x_t + is^y y_t\right)\right]}{\mathrm{E}\left[\exp\left(is^x \varepsilon_t^x\right)\right] \mathrm{E}\left[\exp\left(is^y \varepsilon_t^y\right)\right]} = \mathrm{E}\left[\mathrm{E}\left[\mathbb{1}\{d_t = d\} | x_t^*, y_t^*\right] \exp\left(is^x x_t^* + is^y y_t^*\right)\right] \\ &= \int \int \exp\left(is^x x^* + is^y y^*\right) \mathrm{E}\left[\mathbb{1}\{d_t = d\} | (x_t^*, y_t^*) = (x^*, y^*)\right] f_{x_t^*, y_t^*}\left(x^*, y^*\right) dx^* dy^* \end{aligned}$$

This is the two-dimensional Fourier inversion of $\mathrm{E}\left[\mathbb{1}\{d_t = d\} | (x_t^*, y_t^*) = (\,\cdot\,,\,\cdot\,)\right] f_{x_t^*, y_t^*}\left(\,\cdot\,,\,\cdot\,\right)$. On the other hand, the two-dimensional Fourier inversion of $f_{x_t^*, y_t^*}$ can be found as

$$\mathrm{E}\left[\exp\left(is^x x_t^* + is^y y_t^*\right)\right] = \frac{\mathrm{E}\left[\exp\left(is^x x_t + is^y y_t\right)\right]}{\mathrm{E}\left[\exp\left(is^x \varepsilon_t^x\right)\right] \mathrm{E}\left[\exp\left(is^y \varepsilon_t^y\right)\right]}.$$

Letting $\mathcal{F}^2$ denote the two-dimensional Fourier transform operator defined by

$$\left(\mathcal{F}^2 \phi\right)(\xi_1, \xi_2) = \frac{1}{4\pi^2} \int \int e^{-is_1 \xi_1 - is_2 \xi_2} \phi(s_1, s_2) ds_1 ds_2 \qquad \text{for all } \phi \in L^1(\mathbb{R}^2) \text{ and } (\xi_1, \xi_2) \in \mathbb{R}^2,$$

we find the closed-form expression for CCP $f_1(d_t|x_t^*, y_t^*)$ as follows.

$$\Pr(d_t = d|x_t^*, y_t^*) = \frac{\mathrm{E}[\mathbb{1}\{d_t = d\}|x_t^*, y_t^*] f_{x_t^*,y_t^*}(x_t^*, y_t^*)}{f_{x_t^*,y_t^*}(x_t^*, y_t^*)} = \frac{\left(\mathcal{F}^2\phi_{(d)x_t^*,y_t^*}\right)(x_t^*, y_t^*)}{\left(\mathcal{F}^2\phi_{x_t^*,y_t^*}\right)(x_t^*, y_t^*)}$$

where the 'phi' functions in the last expression are

$$\begin{aligned}
\phi_{(d)x_t^*,y_t^*}(s^x, s^y) &= \frac{\mathrm{E}[\mathbb{1}\{d_t = d\}\exp(is^x x_t + is^y y_t)]}{\mathrm{E}[\exp(is\varepsilon_t^x)] + \mathrm{E}[\exp(is\varepsilon_t^y)]} \\
&= \mathrm{E}[\mathbb{1}\{d_t = d\}\exp(is^x x_t + is^y y_t)] \frac{\exp\left[\int_0^{s^x} \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{x,d'})\exp(is_1 x_t)|d_t=d'\right]}{\gamma^{x,d'}\,\mathrm{E}[\exp(is_1 x_t)|d_t=d']} ds_1\right]}{\mathrm{E}[\exp(is^x x_t)|d_t = d']} \\
&\quad \times \frac{\exp\left[\int_0^{s^y} \frac{\mathrm{E}\left[i(y_{t+1}-\alpha^{y,d'})\exp(is_1 y_t)|d_t=d'\right]}{\gamma^{y,d'}\,\mathrm{E}[\exp(is_1 y_t)|d_t=d']} ds_1\right]}{\mathrm{E}[\exp(is^y y_t)|d_t = d']}
\end{aligned}$$

and

$$\begin{aligned}
\phi_{x_t^*,y_t^*}(s^x, s^y) &= \frac{\mathrm{E}[\exp(is^x x_t + is^y y_t)]}{\mathrm{E}[\exp(is\varepsilon_t^x)]\,\mathrm{E}[\exp(is\varepsilon_t^y)]} = \mathrm{E}[\exp(is^x x_t + is^y y_t)] \times \\
&\frac{\exp\left[\int_0^{s^x} \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{x,d'})\exp(is_1 x_t)|d_t=d'\right]}{\gamma^{x,d'}\,\mathrm{E}[\exp(is_1 x_t)|d_t=d']} ds_1\right]}{\mathrm{E}[\exp(is^x x_t)|d_t = d']} \frac{\exp\left[\int_0^{s^y} \frac{\mathrm{E}\left[i(y_{t+1}-\alpha^{y,d'})\exp(is_1 y_t)|d_t=d'\right]}{\gamma^{y,d'}\,\mathrm{E}[\exp(is_1 y_t)|d_t=d']} ds_1\right]}{\mathrm{E}[\exp(is^y y_t)|d_t = d']}
\end{aligned}$$

from (B.4) using any d'. In summary, we obtain the closed-form identifying formula for the CCP $f_1(d_t \mid x_t^*, y_t^*)$:

$$\begin{aligned}
&\Pr(d_t = d|x_t^*, y_t^*) = \frac{\left(\mathcal{F}^2\phi_{(d)x_t^*,y_t^*}\right)(x_t^*, y_t^*)}{\left(\mathcal{F}^2\phi_{x_t^*,y_t^*}\right)(x_t^*, y_t^*)} \\
= &\int\int \exp(-is^x x_t^* - is^y y_t^*) \cdot \mathrm{E}[\mathbb{1}\{d_t = d\}\exp(is^x x_t + is^y y_t)] \times \\
&\frac{\exp\left[\int_0^{s^x} \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{x,d'})\exp(is_1 x_t)|d_t=d'\right]}{\gamma^{x,d'}\,\mathrm{E}[\exp(is_1 x_t)|d_t=d']} ds_1\right]}{\mathrm{E}[\exp(is^x x_t)|d_t = d']} \frac{\exp\left[\int_0^{s^y} \frac{\mathrm{E}\left[i(y_{t+1}-\alpha^{y,d'})\exp(is_1 y_t)|d_t=d'\right]}{\gamma^{y,d'}\,\mathrm{E}[\exp(is_1 y_t)|d_t=d']} ds_1\right]}{\mathrm{E}[\exp(is^y y_t)|d_t = d']} ds^x ds^y \Bigg/ \\
&\int\int \exp(-is^x x_t^* - is^y y_t^*) \cdot \mathrm{E}[\exp(is^x x_t + is^y y_t)] \times \\
&\frac{\exp\left[\int_0^{s^x} \frac{\mathrm{E}\left[i(x_{t+1}-\alpha^{x,d'})\exp(is_1 x_t)|d_t=d'\right]}{\gamma^{x,d'}\,\mathrm{E}[\exp(is_1 x_t)|d_t=d']} ds_1\right]}{\mathrm{E}[\exp(is^x x_t)|d_t = d']} \frac{\exp\left[\int_0^{s^y} \frac{\mathrm{E}\left[i(y_{t+1}-\alpha^{y,d'})\exp(is_1 y_t)|d_t=d'\right]}{\gamma^{y,d'}\,\mathrm{E}[\exp(is_1 y_t)|d_t=d']} ds_1\right]}{\mathrm{E}[\exp(is^y y_t)|d_t = d']} ds^x ds^y
\end{aligned}$$

using any d'. This completes Step 3. □

C Consistency of $\widehat{f}_2(x_t^* \mid d_{t-1}, x_{t-1}^*)$

This section discusses consistency of the analog estimator $\widehat{f}_2(x_t^* \mid d_{t-1}, x_{t-1}^*)$ for the law of state transition proposed in Section 2.3. For each time period t, consider the sequence $\{r_{N,t}\}_{N=1}^{\infty}$ of positive numbers such that

$$\sup_{s \in [-h_N^{-1}, h_N^{-1}]} \left| \widehat{\phi}_{\varepsilon_t}(s) - \phi_{\varepsilon_t}(s) \right| = o_p(r_{N,t}) \tag{C.1}$$

holds as $N \to \infty$. Li and Vuong (1998) derive admissible combinations of $\{r_{N,t}\}_N$ and $\{h_N\}_N$ under various cases of smoothness assumptions for the distributions of x_t^* and ε_t. Also let $\{c_{N,t}\}_{N=1}^{\infty}$ be the sequence of positive numbers defined by

$$c_{N,t} = \inf_{s \in [-h_N^{-1}, h_N^{-1}]} |\phi_{\varepsilon_t}(s)| \tag{C.2}$$

Note that this sequence also depends on smoothness assumptions for the distribution of ε_t. We make the following assumptions.

Assumption 20. *The following conditions hold. (i)* $\left|\gamma^d\right| < 1$. *(ii)* $\left|\widehat{\gamma}^d - \gamma^d\right| = O_p(N^{-1/2})$.
(iii) $\left|\widehat{E}_N[\exp(isx_t) \mid d_{t-1} = d] - E[\exp(isx_t) \mid d_{t-1} = d]\right| = O_p(N^{-1/2})$.
(iv) $\left|\widehat{E}_N[\exp(is(\widehat{\alpha}^d + \widehat{\gamma}^d x_{t-1})) \mid d_{t-1} = d] - E[\exp(is(\alpha^d + \gamma^d x_{t-1})) \mid d_{t-1} = d]\right| = O_p(N^{-1/2})$.
(v) $c_{N,t} N^{1/2} \to \infty$ *as* $N \to \infty$. *(vi)* $\frac{c_{N,t}}{r_{N,t}} \to \infty$ *as* $N \to \infty$. *(vii)* $\phi_K(s) = \frac{1}{2}\mathbb{1}[-1 \le s \le 1]$.

Part (i) requires that the dynamics of x_t^* follows a sub-unit-root process. Parts (ii), (iii), and (vi) require parametric convergence rates for the respective parametric estimators, which is satisfied under the standard assumptions for central limit theorems. Parts (v) and (vi) require h_N to be chosen to vanis slow enough to have $c_{N,t}$ dominate $r_{N,t}$ as well as $N^{-1/2}$. Part (v) and (vi) are consistent with the restrictions imposed by Li and Vuong (1998) under various smoothness cases. Finally, part (vii) assumes that ϕ_K is an indicator function to have the setting equivalent to that of Li and Vuong (1998). We obtain the following consistency results.

Lemma 1. *If Assumption 20 (i)–(vi) are satisfied, then*

$$\sup_{s\in[-h_N^{-1},h_N^{-1}]}\left|\widehat{\phi}_{\eta_t^d}(s)-\phi_{\eta_t^d}(s)\right| = O_P\left(\frac{r_{N,t}\vee r_{N,t-1}}{c_{N,t}^2}\right)$$

as $N\to\infty$.

Proof. By Assumption 20 (i) and (ii) and the uniform continuity of characteristic functions, we have

$$\sup_{s\in[-h_N^{-1},h_N^{-1}]}\left|\widehat{\phi}_{\varepsilon_{t-1}}(\widehat{\gamma}^d s)-\phi_{\varepsilon_{t-1}}(\gamma^d s)\right| \le \sup_{s\in[-h_N^{-1},h_N^{-1}]}\left|\widehat{\phi}_{\varepsilon_{t-1}}(s)-\phi_{\varepsilon_{t-1}}(s)\right|$$

with probability approaching one as $N\to\infty$, and hence

$$\sup_{s\in[-h_N^{-1},h_N^{-1}]}\left|\widehat{\phi}_{\varepsilon_{t-1}}(\widehat{\gamma}^d s)-\phi_{\varepsilon_{t-1}}(\gamma^d s)\right| = o_p(r_{N,t-1}) \tag{C.3}$$

holds as $N\to\infty$. Also, we can write

$$\begin{aligned}
\widehat{\phi}_{\eta_t^d}(s)-\phi_{\eta_t^d}(s) = \Big[\Big\{&\Big(\widehat{\mathrm{E}}_N[\exp(isx_t)\mid d_{t-1}=d]-\mathrm{E}[\exp(isx_t)\mid d_{t-1}=d]\Big)\cdot\Big(\widehat{\phi}_{\varepsilon_{t-1}}(\widehat{\gamma}^d s)-\phi_{\varepsilon_{t-1}}(\gamma^d s)\Big)\\
&+\Big(\widehat{\mathrm{E}}_N[\exp(isx_t)\mid d_{t-1}=d]-\mathrm{E}[\exp(isx_t)\mid d_{t-1}=d]\Big)\cdot\phi_{\varepsilon_{t-1}}(\gamma^d s)+\mathrm{E}[\exp(isx_t)\mid d_{t-1}=d]\\
&\times\Big(\widehat{\phi}_{\varepsilon_{t-1}}(\widehat{\gamma}^d s)-\phi_{\varepsilon_{t-1}}(\gamma^d s)\Big)\Big\}\cdot\mathrm{E}[\exp(is(\alpha^d+\gamma^d x_{t-1}))\mid d_{t-1}=d]\cdot\phi_{\varepsilon_t}(s)+\mathrm{E}[\exp(isx_t)\mid d_{t-1}=d]\\
&\times\phi_{\varepsilon_{t-1}}(\gamma^d s)\cdot\Big\{\Big(\widehat{\mathrm{E}}_N[\exp(is(\widehat{\alpha}^d+\widehat{\gamma}^d x_{t-1}))\mid d_{t-1}=d]-\mathrm{E}[\exp(is(\alpha^d+\gamma^d x_{t-1}))\mid d_{t-1}=d]\Big)\\
&\times\Big(\widehat{\phi}_{\varepsilon_t}(s)-\phi_{\varepsilon_t}(s)\Big)+\Big(\widehat{\mathrm{E}}_N[\exp(is(\widehat{\alpha}^d+\widehat{\gamma}^d x_{t-1}))\mid d_{t-1}=d]-\mathrm{E}[\exp(is(\alpha^d+\gamma^d x_{t-1}))\mid d_{t-1}=d]\Big)\\
&\times\phi_{\varepsilon_t}(s)+\mathrm{E}[\exp(is(\alpha^d+\gamma^d x_{t-1}))\mid d_{t-1}=d]\cdot\Big(\widehat{\phi}_{\varepsilon_t}(s)-\phi_{\varepsilon_t}(s)\Big)\Big\}\Big]\Big/\\
&\Big[\mathrm{E}[\exp(is(\alpha^d+\gamma^d x_{t-1}))\mid d_{t-1}=d]^2\cdot\phi_{\varepsilon_t}(s)^2+\mathrm{E}[\exp(is(\alpha^d+\gamma^d x_{t-1}))\mid d_{t-1}=d]\cdot\phi_{\varepsilon_t}(s)\\
&\times\Big\{\Big(\widehat{\mathrm{E}}_N[\exp(is(\widehat{\alpha}^d+\widehat{\gamma}^d x_{t-1}))\mid d_{t-1}=d]-\mathrm{E}[\exp(is(\alpha^d+\gamma^d x_{t-1}))\mid d_{t-1}=d]\Big)\cdot\Big(\widehat{\phi}_{\varepsilon_t}(s)-\phi_{\varepsilon_t}(s)\Big)\\
&+\Big(\widehat{\mathrm{E}}_N[\exp(is(\widehat{\alpha}^d+\widehat{\gamma}^d x_{t-1}))\mid d_{t-1}=d]-\mathrm{E}[\exp(is(\alpha^d+\gamma^d x_{t-1}))\mid d_{t-1}=d]\Big)\cdot\phi_{\varepsilon_t}(s)\\
&+\mathrm{E}[\exp(is(\alpha^d+\gamma^d x_{t-1}))\mid d_{t-1}=d]\cdot\Big(\widehat{\phi}_{\varepsilon_t}(s)-\phi_{\varepsilon_t}(s)\Big)\Big\}\Big]
\end{aligned}$$

Therefore, using (C.1), (C.2), (C.3), and Assumption 20 (iii)–(vi) yields

$$\sup_{s\in[-h_N^{-1},h_N^{-1}]}\left|\widehat{\phi}_{\eta_t^d}(s)-\phi_{\eta_t^d}(s)\right| = O_P\left(\frac{r_{N,t}\vee r_{N,t-1}}{c_{N,t}^2}\right)$$

as $N\to\infty$. □

Theorem 6. *If Assumption 20 is satisfied, then*

$$\sup_{(x^*,x^*_-)} \left|\widehat{f}_2(x^* \mid d, x^*_-) - f_2(x^* \mid d, x^*_-)\right| = O_P\left(\frac{r_{N,t} \vee r_{N,t-1}}{h_N \cdot c^2_{N,t}}\right) + O\left(\int_{h_N^{-1}}^{\infty} \left|\phi_{\eta_t^d}(s)\right| ds\right)$$

as $N \to \infty$.

Proof. Assumption 20 (vii) allows us to write

$$\begin{aligned}
\sup_{(x^*,x^*_-)} \left|\widehat{f}_2(x^* \mid d, x^*_-) - f_2(x^* \mid d, x^*_-)\right| &= O_P\left(\int_{-h_N^{-1}}^{h_N^{-1}} \left|\widehat{\phi}_{\eta_t^d}(s) - \phi_{\eta_t^d}(s)\right| ds\right) + O\left(\int_{h_N^{-1}}^{\infty} \left|\phi_{\eta_t^d}(s)\right| ds\right) \\
&= O_P\left(\frac{r_{N,t} \vee r_{N,t-1}}{h_N \cdot c^2_{N,t}}\right) + O\left(\int_{h_N^{-1}}^{\infty} \left|\phi_{\eta_t^d}(s)\right| ds\right)
\end{aligned}$$

as $N \to \infty$, where the last equality follows from Lemma 1. □